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Sujet de la thèse:

# Mesures de Poisson, infinie divisibilité et propriétés ergodiques

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CHAPITRE 1

# Introduction

Ce premier chapitre est une présentation succinte de nos principaux résultats, ils sont exposés de façon détaillée dans les chapitres correspondants.

Le point commun des objets mathématiques que nous avons étudié est leur caractère infiniment divisible (abr. ID). Dans le chapitre 2, correspondant à l'article "Infinitely divisible stationary processes", l'objectif a été d'approfondir la théorie ergodique des processus stationnaires ID en se reposant fortement sur l'étude d'autres objets ID, les mesures et suspensions de Poisson. L'article "Properties of a generic Poisson suspension and $\mathcal{P}$-entropy" (Chapitre 3) est l'étude des propriétés ergodiques d'une suspension "typique" et l'introduction d'un nouvel invariant pour les systèmes dynamiques par l'intermédiaire des suspensions de Poisson. Dans "Some non asymptotic tail estimates for Hawkes processes" (Chapitre 4), en collaboration avec Patricia Reynaud-Bouret (ENS Paris), nous avons effectué une étude fine d'un modèle de processus ponctuel ID sur $\mathbb{R}$ très utilisé dans les applications (sismologie, génétique...), dans le but de donner des outils théoriques pour leur simulation et l'étude de leur propriétés statistiques. Enfin, dans le chapitre 5, nous nous intéressons au spectre de Bartlett des mesures aléatoires stationnaires de carré intégrable en montrant notamment qu'il détermine l'ergodicité et le mélange dans le cas ID.

Avant de poursuivre la description de ces travaux, donnons d'abord la définition de la notion se trouvant au coeur de notre travail:

Une loi de probabilité $\mu$ sur $\left(\mathbb{R}^d, \mathcal{B}^d\right)$ est dite ID si, pour tout entier $k$ non nul, il existe $\mu_k$ telle que la $k$-ième puissance de convolution de $\mu_k$, $\mu_k^{*k}$, soit égale à $\mu$.

La théorie (voir [**57**]) nous indique alors que $\mu$ se factorise, à une multiplication par une masse de Dirac près, en $\mu_g * \mu_p$ où $\mu_g$ est une loi Gaussienne, et $\mu_p$ est une loi ID entièrement déterminée par une mesure $\nu$ sur $\mathbb{R}^d$, éventuellement infinie, appelée *mesure de Lévy*. La définition d'infinie divisibilité s'étend immédiatement à des lois sur des espaces plus généraux, munis d'une addition, et notamment $\mathbb{R}^{\mathbb{Z}}$, c'est à dire ici, à des lois de processus stochastiques indexés par $\mathbb{Z}$. Maruyama, dans [**38**], a montré qu'on pouvait étendre la notion de mesure de Lévy à des processus ID. Après ses travaux pionniers en 1970, l'étude systématique des processus ID stationnaires sans partie Gaussienne (notés IDp) n'a vraiment commencé que vers la fin des années 80 (voir par exemple [**11**]). Les auteurs ont cherché à, d'une part, déterminer des critères d'ergodicité, de mélange faible, de mélange, et à exhiber des exemples d'autre part (voir [**49, 22, 34, 48, 23, 21, 32, 12, 24, 53, 55, 54**]) en se concentrant principalement sur un cas particulier de processus IDp, les processus dit symétriques $\alpha$-stables ($S\alpha S$), pour lesquels ont été élaborées des représentations permettant une étude plus aisée de leurs propriétés ergodiques. Ces représentations ont permis d'organiser la classification de ces processus via des factorisations. Un des théorèmes principaux, dû à Rosiński [**51**], est le suivant:

La loi $\mathbb{P}$ d'un processus stationnaire $S\alpha S$ se factorise de manière unique en:

$$\mathbb{P}_1 * \mathbb{P}_2 * \mathbb{P}_3$$

où $\mathbb{P}_1$ est $S\alpha S$ et "mixed moving average", $\mathbb{P}_2$ est $S\alpha S$ et "harmonizable" et $\mathbb{P}_3$ est $S\alpha S$ et n'a pas de composante d'un des deux types précédents. Il est prouvé que $\mathbb{P}_1$ est mélangeant, $\mathbb{P}_2$ n'est pas ergodique et $\mathbb{P}_3$ peut être mélangeant aussi bien que non ergodique [**55**].

Cette décomposition a été raffinée récemment par Pipiras et Taqqu [**47**] qui ont montré la possibilité de factoriser le terme $\mathbb{P}_3$ en $\mathbb{P}_{3,1} * \mathbb{P}_{3,2}$ où $\mathbb{P}_{3,2}$ est non ergodique; enfin, dans [**56**], Samorodnitsky a réussi à extraire la plus grande composante ergodique.

Des travaux ont été menés pour construire explicitement des exemples de processus IDp avec des comportements particuliers, par exemple des processus faiblement mélangeants mais non mélangeants, ou mélangeants du type $\mathbb{P}_3$ (voir [**55**]), etc. Enfin, un théorème général a été établi: l'ergodicité implique le mélange faible pour un processus IDp [**54**].

Dans l'article "Infinitely divisible stationary processes", nous avons entrepris d'étendre ces résultats obtenus précédemment en nous basant sur les propriétés dynamiques du système associé à la mesure de Lévy de chaque processus IDp. Ceci nous a permis d'obtenir le résultat suivant:

La loi $\mathbb{P}$ d'un processus stationnaire IDp se factorise de manière unique en le produit de cinq lois de processus IDp stationnaires:

$$\mathbb{P}_1 * \mathbb{P}_2 * \mathbb{P}_3 * \mathbb{P}_4 * \mathbb{P}_5$$

où $\mathbb{P}_1$ a la propriété de Bernoulli, $\mathbb{P}_2$ est mélangeant, $\mathbb{P}_3$ est doucement mélangeant et non mélangeant, $\mathbb{P}_4$ est faiblement mélangeant et non doucement mélangeant, $\mathbb{P}_5$ est non ergodique.

Pour obtenir, entre autres, ce résultat, nous utilisons la représentation, due à Maruyama (voir [**38**]) des processus IDp comme intégrales contre une mesure de Poisson. Nous avons donc entrepris l'étude de la théorie ergodique de ces objets. Une mesure de Poisson sur un espace muni d'une mesure $\sigma$-finie $(\Omega, \mathcal{F}, \mu)$, est l'unique probabilité $\mathcal{P}_\mu$ sur $(M_\Omega, \mathcal{M}_\mathcal{F})$ (l'espace des mesures ponctuelles sur $\Omega$ munie de sa tribu naturelle) telle que pour tout entier $k$ et toute collection d'ensembles mesurables disjoints $A_1, A_2, \ldots, A_k$ de $\mu$-mesure finie et en notant $N$ l'identité sur $M_\Omega$, la famille de variables aléatoires $\{N(A_1), N(A_2), \ldots, N(A_k)\}$ est une famille indépendante dont tous les membres suivent une loi de Poisson dont les paramètres respectifs sont $\mu(A_1), \mu(A_2), \ldots, \mu(A_k)$. Si maintenant $(\Omega, \mathcal{F}, \mu)$ est muni d'une transformation bijective $T$ préservant $\mu$, alors la transformation $T^\star$ définie sur $M_\Omega$ par $T^\star(m) = m \circ T^{-1}$ préserve la probabilité $\mathcal{P}_\mu$. Le quadruplet $(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_\mu, T^\star)$ est appelé suspension de Poisson construite au dessus de la base $(\Omega, \mathcal{F}, \mu, T)$. Nous avons donc classé les propriétés ergodiques d'une suspension de Poisson en fonction des propriétés ergodiques de la base (dissipativité, conservativité, type positif, type nul, type $\mathbf{II}_1$, type $\mathbf{II}_\infty$...). De plus, ces liens étroits donnent un éclairage intéressant, il nous semble, sur les propriétés ergodiques de systèmes en mesure infinie par la correspondance, via la suspension, avec des propriétés ergodiques en mesure finie bien mieux connues. Pour élucider les propriétés ergodiques de nature spectrale, la stucture d'espace de Fock de $L^2(\mathcal{P}_\mu)$ est cruciale car elle rend accessible le type spectral maximal dont nous donnons la forme et que nous exploitons fortement. Nous déduisons les propriétés correspondantes pour les processus IDp vus comme des facteurs de la suspension construite au dessus de leur mesure de Lévy et nous donnons, dans le cas où ces processus sont positifs et de carré intégrable, des critères spectraux assez simples: Si $\sigma$ est la mesure spectrale de la coordonée en 0 du processus IDp positif $X$, alors $\sigma(0) = 0$ implique l'ergodicité et $\hat{\sigma}(k) \to 0$ implique le mélange. Le résultat d'ergodicité est en fait la traduction $L^2$ du critère suivant où l'intégrabilité suffit:

Si $X_0$ vérifie la loi des grands nombres, alors le processus est ergodique.

Ensuite, nous introduisons la notion de "couplage ID". Soit $\mathbb{P}_1$ et $\mathbb{P}_2$ les lois de deux processus stationnaires ID et soit $\nu$ la loi d'un processus stationnaire bivarié $(X_1, X_2)$ tel que $X_1$ a la loi $\mathbb{P}_1$ et $X_2$ la loi $\mathbb{P}_2$. Si $\nu$ est encore ID, $\nu$ est appelé couplage ID entre $\mathbb{P}_1$ et $\mathbb{P}_2$. Si la seule manière de réaliser un tel couplage est de choisir $X_1$ indépendant de $X_2$, on dit alors que $\mathbb{P}_1$ et $\mathbb{P}_2$ sont ID-disjointes. On montre notamment que les cinq familles identifiées dans le théorème ci-dessus sont mutuellement ID-disjointes, ce qui, d'une certaine manière, légitime notre factorisation. De plus, un processus IDp est ID-disjoint d'un processus Gaussien. Ces résultats reposent sur les propriétés de couplage et de disjonction forte des mesures de Lévy sous-jacentes. Ces notions s'étendent naturellement aux couplages ID de suspensions de Poisson. On montre notamment que deux systèmes ergodiques sont fortement disjoints si et seulement si leurs suspensions sont ID-disjointes. On prouve aussi que les transformations Poissoniennes (les transformations du type $T^\star$, pour une certaine transformation $T$ sur la base) sont exactement celles dont l'autocouplage porté par le graphe de cette transformation est ID.

Au cours du chapitre, nous produisons des exemples pour montrer, notamment, que les cinq classes de la factorisation ne sont pas vides. Par ailleurs, nous nous intéressons également à l'entropie de ces processus IDp en établissant une condition suffisante à l'obtention d'une entropie nulle.

Nous regardons aussi particulièrement l'effet de notre factorisation sur les processus stables en montrant que celle-ci préserve leur caractère stable, à savoir, chaque terme est stable. Ceci nous permet de replacer notre résultat dans le cadre des factorisations obtenues par Rosiński puis Pipiras et Taqqu et enfin Samorodnitsky décrites plus haut.

Enfin, grâce à la disjonction ID et aux systèmes dits "Gaussien-Kronecker", nous avons pu prouver qu'il n'existe pas de système dynamique ergodique en mesure infinie ayant une mesure du type $(FS)$ (voir [**37**]) comme mesure spectrale, ce qui constitue, à notre connaissance, la seule restriction, en dehors de l'absence d'atome, pour les mesures spectrales de tels systèmes.

Dans le chapitre 3, "Properties of a generic Poisson suspension", nous poursuivons l'étude ébauchée dans l'article précédent visant à connaitre les propriétés ergodiques d'une suspension "typique" construite au dessus d'une base $(\Omega, \mathcal{F}, \mu)$ non atomique où $\mu$ est infinie (c'est le seul cas non trivial). Les transformations préservant $\mu$ forment un groupe $G$ qui se plonge naturellement (via la correspondance $T \mapsto T^\star$) dans le groupe $G'$ des transformations sur $(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_\mu)$ préservant $\mathcal{P}_\mu$. En munissant ces groupes de transformations de leur topologie usuelle, dite faible, nous avons montré que la topologie de $G'$ restreinte à $G$ est bien la topologie faible d'origine sur $G$ avant le plongement. On peut donc regarder les propriétés dites *génériques* (c'est à dire ici, partagées par un ensemble dense contenant une intersection dénombrable d'ouverts denses de transformations dans $G$) d'une transformation du type $T^\star$ grâce aux propriétés génériques des transformations sur la base. Ainsi, on montre qu'une suspension générique est faiblement mélangeante, rigide et à spectre simple. Pour établir un critère de simplicité du spectre, nous nous basons sur un théorème d'Ageev (voir [**3, 4**]) destiné à l'origine à établir, notamment, la singularité mutuelle des puissances de convolution du type spectral maximal d'un automorphisme générique d'un espace de probabilité et la simplicité du spectre des puissances symétriques de l'opérateur unitaire associé. Ce théorème s'adapte complètement au cadre pourtant bien différent où nous l'appliquons. On démontre alors quelques resultats sur la structure de ces suspensions à spectre simple, notamment qu'elles sont non isomorphes à tout système Gaussien, que leur centre ne contient que des automorphismes Poissoniens et qu'aucun de leur facteur non trivial ne possède de complément indépendant. Enfin, nous introduisons un nouveau type d'entropie, que nous appelons $\mathcal{P}$-entropie, pour n'importe quel système dynamique et notamment en mesure infinie. C'est un invariant non trivial qui généralise l'entropie classique en mesure finie. Nous montrons que la $\mathcal{P}$-entropie nulle est générique.

Ainsi, les résultats sur les suspensions de Poisson obtenus dans le deuxième et le troisième chapitre illustrent le fait que l'on peut "plonger" la théorie ergodique en mesure infinie dans la théorie ergodique en mesure finie, via les suspensions, en tissant des liens entre la plupart des notions importantes.

Dans l'article "Some non asymptotic tail estimates for Hawkes processes" (chapitre 4) en collaboration avec Patricia Reynaud-Bouret (ENS Paris), nous étudions les propriétés fines de processus de Hawkes. Un processus de Hawkes stationnaire est un processus ponctuel ID formé de la manière suivante: un processus de Poisson dit d'ancêtres est généré, puis chaque ancêtre donne naissance à des enfants suivant un processus de Galton-Watson "spatial", le processus de Hawkes est l'ensemble des instants de naissance de tous ces individus). Nous obtenons, entre autres, des estimés non asymptotiques du nombre de points par intervalle, du temps d'extinction du processus des ancêtres arrivés avant le temps 0. Ceci nous permet de préciser les vitesses de convergence dans le théorème ergodique et d'évaluer de manière non asymptotique l'erreur commise dans différents types de simulation par approximation.

Enfin, dans le dernier chapitre, intitulé "Bartlett spectrum and ID random measure" (chapitre 5), nous nous concentrons sur des mesures aléatoires stationaires sur $\mathbb{R}^d$ pour lesquelles la mesure de tout ensemble borné est de carré intégrable. On a alors à notre disposition une mesure appelée spectre de Bartlett qui gère une partie importante des propriétés $L^2$ de ces mesures aléatoires. D'un point de vue ergodique, ce spectre de Bartlett nous permet d'avoir accès aux mesures spectrales d'une famille particulièrement intéressante de vecteurs. Dans le cas ID, que nous étudions précisément, nous montrons que ce spectre à une forme très spéciale qui nous permet d'élucider l'ergodicité et le mélange. Ces résultats reposent essentiellement sur la théorie de Palm des mesures aléatoires ainsi que sur les processus de Cox qui nous offrent le moyen de transférer des résultats sur des mesures aléatoires discrètes (des processus ponctuels) à leurs analogues pour des mesures aléatoires quelconques.

CHAPITRE 2

# Infinitely divisible stationary processes

Abstract. We show that an infinitely divisible (ID) stationary process without Gaussian part can be written as the independent sum of five ID stationary processes, each of them belonging to a different class characterized by its Lévy measure. The ergodic properties of each class are respectively: non ergodicity, weak mixing, mild mixing, mixing and Bernoullicity. To obtain these results, we use the representation of an ID process as an integral with respect to a Poisson suspension, which, more generally, has led us to study ergodic properties of these objects. We then introduce and study the notions of ID-joining, ID-disjointness and ID-similarity; we show in particular that the five classes of the above decomposition are ID-disjoint.

## 1. Introduction

A stochastic process is said to be *infinitely divisible* (abr. ID) if, for any positive integer $k$, it equals, in distribution, the sum of $k$ independent and identically distributed processes. These processes are fundamental objects in probability theory, the most popular being the intensively studied Lévy processes (see for example [**57**]). We will focus here on ID stationary processes $\{X_n\}_{n\in\mathbb{Z}}$. Stationary Gaussian processes have a particular place among stationary ID processes and have already been the subject of very deep studies (see [**37**] for recent results). Gaussian processes will constitute a very small part of this paper since we will concentrate on non Gaussian ID processes; Maruyama [**38**] first started their study. Since the late eighties, many authors are looking for criteria of ergodicity, weak mixing or mixing of a general ID process, exhibiting examples, studying particular subfamilies (mainly symmetric $\alpha$-stable ($S\alpha S$) processes). We mention the result of Rosiński and Żak [**54**] which shows the equivalence of ergodicity and weak mixing for general ID processes. Some *factorizations* have been obtained in the $S\alpha S$ case, in particular, Rosiński [**51**] has shown that a $S\alpha S$ process can be written in a unique way as the independent sum of three $S\alpha S$ processes, one being called *mixed moving average* (which is mixing), the second *harmonizable* (non ergodic) and the third not in the aforementioned categories and which is potentially the most interesting (see [**55**]) (Note that Rosiński has developed, in [**52**], a multidimensional version of this factorization). Recently, this third part has been split by Pipiras and Taqqu (see [**47**]) and by Samorodnitsky (see [**56**]). Factorizations already appeared in [**41**], where the ID objects were ID point processes.

The fundamental tool in the study of a non Gaussian ID process is its *Lévy measure*. In the stationary case, its existence has been shown by Maruyama in [**38**]: it is a stationary measure on $\mathbb{R}^{\mathbb{Z}}$, which might be infinite, related to the distribution of the ID process by the characteristic functions of its finite dimensional distributions through an extended *Lévy-Khintchine* formula. A general ID process is the independent sum of a Gaussian process, a constant process and a *Poissonian* (IDp) process, this last process being uniquely determined by its Lévy measure. Reciprocally, if we are given a (shift-)stationary measure on $\mathbb{R}^{\mathbb{Z}}$, under some mild conditions, it can be seen as the Lévy measure of a unique IDp stationary process.

Our main result consists in establishing the following factorization result: every IDp stationary process can be written in a unique way as the independent sum of five IDp processes which are respectively non ergodic, weakly mixing, mildly mixing, mixing and Bernoulli (Theorem 7.8 and Proposition 7.11).

The proof is divided in several steps which have their own interest. The first step is based on the following remark: if the support of the Lévy measure can be partitioned into invariant sets, then the restrictions to these sets of the measure are the Lévy measures of processes that form a factorization of the initial process. We point out here that, it may happen that a stationary ID process can be factorizable into infinitely many components, however, we only consider factorizations that make sense in term of ergodic behaviour of each class. It is remarkable that those distinct behaviours are naturally linked to those of the corresponding Lévy measures. Thus, it is essential to get a better understanding of general dynamical systems (particularly with infinite measure) and to study some canonical decomposition along their invariant sets. Section 2 presents some elements of ergodic theory. In particular, we recall a decomposition, mostly due to Hopf and Krengel and Sucheston (see [**36**]), of an invariant measure into the sum of four invariant measures which are the restrictions of the initial measure to as many invariant sets with distinctive properties (Proposition 2.11). Section 3 presents some basic facts of spectral theory that will be used later.

Then, back to Lévy measures, we have to link the different categories to the corresponding ergodic properties of the underlying ID process. To do so, we use the representation due to Maruyama [**38**] of any IDp process as a stochastic integral with respect to the Poisson measure with the Lévy measure as intensity. In ergodic terms, we will say that an IDp process is a *factor* of the Poisson suspension constructed above its Lévy measure. We thus are led to a specific study of Poisson suspensions built above dynamical systems, that is the subject of Section 4. This study is mostly based upon the particular structure of the associated $L^2$-space, which admits a *chaotic* decomposition, isometric to the Fock factorization of the $L^2$-space associated to the underlying dynamical system. In Section 5, we return briefly to the canonical decomposition presented in Section 2 to enrich it with a new class by breaking one part into two and isolate the five previously announced classes. This preliminary work allows us to elucidate, in Section 6, absence of ergodicity, weak mixing, mild mixing and mixing (of all order) of a Poisson suspension. We give a criterion for a suspension to be $K$ and a criterion for the Bernoulli property.

In Section 7, we first recall the basic facts on infinitely divisible processes and then apply the results of the preceding sections to their Lévy measure. Thanks to our factorization, ergodic properties can be easily derived. In cases where the process is square integrable, some spectral criteria for ergodic behaviours can be established (Section 8).

In Section 9, we introduce a natural definition when studying stationary ID processes. On the canonical space of bivariate processes, a distribution $\mathbb{P}^{biv}$, stationary with respect to the product-shift whose marginals $\mathbb{P}_1$ and $\mathbb{P}_2$ are the distributions of two stationary ID processes, is called *ID-joining* of $\mathbb{P}_1$ and $\mathbb{P}_2$ if it is itself ID. If the product measure, which is always an ID-joining, is the only way to achieve such a joining, $\mathbb{P}_1$ and $\mathbb{P}_2$ are said to be *ID-disjoint*. ID-disjointness is a common feature among ID-processes and one of the results is to show that the five classes of the decomposition are mutually ID-disjoint. Moreover, it is easy to see that Gaussian processes are ID-disjoint of IDp processes. The notion of ID-joining and ID-disjointness can be generalized to a Poisson suspension in a very natural way (joinings of Poisson suspensions have been considered in [**17**] without making reference to infinite divisibility, see the comment in the conclusion of this chapter). The main tools in this section, are the joining and strong disjointness properties of the underlying dynamical systems. We show (Theorem 9.21) that two ergodic dynamical systems are strongly disjoint if and only if their associated Poisson suspensions are ID-disjoint.

In Section 10, particular cases ($\alpha$-stable processes, squared Gaussian processes, etc...) and related results are treated. In the $\alpha$-stable case, we show that our factorization preserves the distributional properties, that is, each of the five components is $\alpha$-stable. We can thus replace in this context the previously obtained factorization of Rosiński [**51**], as well as the refinements of Pipiras and Taqqu [**47**] and Samorodnitsky [**56**]. We treat an example illustrating the contribution of the theory of ID processes to the general ergodic theory in infinite measure.

## 2. Elements of ergodic theory

Let $(\Omega, \mathcal{F}, \mu, T)$ be a Borel space endowed with a $\sigma$-finite measure $\mu$ preserved by a bijective measurable transformation $T$. Such a quadruplet is called *dynamical system*, or shortly, *system*.

The aim of this section is to introduce notions and terminology used in the study of dynamical systems. We first concentrate on the structure of a general dynamical system that will lead us to the decomposition in Proposition 2.11 which is a compilation of known results, it will be enriched at Section 5. The rest of the section is devoted to notions specific to dynamical systems with a probability measure. The book of Aaronson [**1**] covers most of the definitions and results exposed here.

In the following, if $\phi$ is a measurable map defined on $(\Omega, \mathcal{F}, \mu, T)$, the image measure of $\mu$ by $\phi$ is denoted $\phi^{\star}(\mu)$.

**2.1. Factors, extensions and isomorphic systems.** Consider another dynamical system $(\Omega', \mathcal{F}', \mu', T')$ and a real number $c > 0$.

DEFINITION 2.1. Call $(\Omega', \mathcal{F}', \mu', T')$ a *factor* of $(\Omega, \mathcal{F}, \mu, T)$ (or $(\Omega, \mathcal{F}, \mu, T)$ a *extension* of $(\Omega', \mathcal{F}', \mu', T')$) if there exists a map $\varphi$, measurable from $(\Omega, \mathcal{F})$ to $(\Omega', \mathcal{F}')$ such that $\varphi^{\star}(\mu) = \mu'$ and $\varphi \circ T = T' \circ \varphi$. If $\varphi$ is invertible, then $(\Omega, \mathcal{F}, \mu, T)$ and $(\Omega', \mathcal{F}', \mu', T')$ are said *isomorphic*.

A $T$-invariant $\sigma$-finite sub-$\sigma$-algebra $\mathcal{A}$ of $\mathcal{F}$ is also called a *factor*. It can be shown (see [**1**]) that there exists a factor $(\Omega', \mathcal{F}', \mu', T')$ with a map $\varphi$ such that $\varphi^{-1}\mathcal{F}' = \mathcal{A}$.

### 2.2. Ergodicity.

DEFINITION 2.2. The *invariant $\sigma$-field* of $(\Omega, \mathcal{F}, \mu, T)$ is the sub-$\sigma$-field $\mathcal{I}$ of $\mathcal{F}$ that contains the sets $A \in \mathcal{F}$ such that $T^{-1}A = A$ ($A$ is said *$T$-invariant*).

This definition leads to the following one:

DEFINITION 2.3. $(\Omega, \mathcal{F}, \mu, T)$ is said *ergodic* if, for all set $A \in \mathcal{I}$:

$$\mu(A) = 0 \text{ or } \mu(A^c) = 0$$

### 2.3. Dissipative and conservative transformations.

DEFINITION 2.4. A set $A \in \mathcal{F}$ is called a *wandering set* if the $\{T^{-n}A\}_{n\in\mathbb{Z}}$ are disjoint.

We denote by $\mathfrak{D}$, the (measurable) union of all the wandering sets for $T$, this set is $T$-invariant. Its complement is denoted by $\mathfrak{C}$.

DEFINITION 2.5. We call $(\Omega, \mathcal{F}, \mu, T)$ *dissipative if* $\mathfrak{D} = \Omega$ mod. $\mu$. If $\mathfrak{C} = \Omega$ mod. $\mu$, then $(\Omega, \mathcal{F}, \mu, T)$ is said *conservative*.

LEMMA 2.6. *There exists a wandering set $W$ such that $\mathfrak{D} = \cup_{n\in\mathbb{Z}} T^{-n}W$ mod. $\mu$.*

PROPOSITION 2.7. *Hopf decomposition.*
*The Hopf decomposition is the partition $\{\mathfrak{D}, \mathfrak{C}\}$.*
$\left(\Omega, \mathcal{F}, \mu_{|\mathfrak{D}}, T\right)$ *is dissipative and* $\left(\Omega, \mathcal{F}, \mu_{|\mathfrak{C}}, T\right)$ *is conservative.*

### 2.4. Type $\mathbf{II}_1$ and type $\mathbf{II}_\infty$.

PROPOSITION 2.8. *Let $(\Omega, \mathcal{F}, \mu, T)$ be a dynamical system. There exists a unique partition $\{\mathfrak{P}, \mathcal{N}\}$ of $\Omega$ in $T$-invariant sets such that there exists a $T$-invariant probability measure equivalent to $\mu_{|\mathfrak{P}}$ and that there doesn't exists a non zero $T$-invariant probability measure absolutely continuous with respect to $\mu_{|\mathcal{N}}$. We have $\mathfrak{P} \subset \mathfrak{C}$ and $\mathfrak{D} \subset \mathcal{N}$.* $\left(\Omega, \mathcal{F}, \mu_{|\mathfrak{P}}, T\right)$ *is said to be of* type $\mathbf{II}_1$, *and* $\left(\Omega, \mathcal{F}, \mu_{|\mathcal{N}}, T\right)$ *of* type $\mathbf{II}_\infty$.

### 2.5. Zero type and positive type.

DEFINITION 2.9. Let $(\Omega, \mathcal{F}, \mu, T)$ be a dynamical system.
$\mu$ (or $(\Omega, \mathcal{F}, \mu, T)$) is said to be of *zero type* if, for all $A \in \mathcal{F}$ such that $\mu(A) < +\infty$, $\mu\left(A \cap T^{-k}A\right) \to 0$ as $k$ tends to $+\infty$.
$\mu$ (or $(\Omega, \mathcal{F}, \mu, T)$) is said to be of *positive type* if, for all $A \in \mathcal{F}$ such that $\mu(A) > 0$, $\overline{\lim}_{k\to\infty} \mu\left(A \cap T^{-k}A\right) > 0$.

Krengel and Sucheston obtained the following decomposition (see [**36**], page 155):

PROPOSITION 2.10. *There exists a partition $\{\mathcal{N}_0, \mathcal{N}_+\}$ of $\Omega$ in $T$-invariant sets such that $\left(\Omega, \mathcal{F}, \mu_{|\mathcal{N}_0}, T\right)$ (resp. $\left(\Omega, \mathcal{F}, \mu_{|\mathcal{N}_+}, T\right)$) is of zero type (resp. of positive type). We have $\mathfrak{D} \subset \mathcal{N}_0$ and $\mathcal{N}_+ \subset \mathfrak{C}$.*

This terminology has been introduced by Hajian and Kakutani, note that Aaronson calls positive part, the part of type $\mathbf{II}_1$ and null part, the part of type $\mathbf{II}_\infty$.

We can group all these decompositions in the following proposition:

PROPOSITION 2.11. *Canonical decomposition*
*Let $(\Omega, \mathcal{F}, \mu, T)$ be a dynamical system. We can write, in a unique way:*

$$\mu = \mu_{|\mathfrak{D}} + \mu_{|\mathcal{N}_0 \cap \mathfrak{C}} + \mu_{|\mathcal{N}_+} + \mu_{|\mathfrak{P}}$$

*where:*
$\left(\Omega, \mathcal{F}, \mu_{|\mathfrak{D}}, T\right)$ *is dissipative.*
$\left(\Omega, \mathcal{F}, \mu_{|\mathcal{N}_0 \cap \mathfrak{C}}, T\right)$ *is conservative and of zero type.*
$\left(\Omega, \mathcal{F}, \mu_{|\mathcal{N}_+}, T\right)$ *is of type* $\mathbf{II}_\infty$ *and of positive type.*
$\left(\Omega, \mathcal{F}, \mu_{|\mathfrak{P}}, T\right)$ *is of type* $\mathbf{II}_1$.

Before stating the differences between finite and infinite measure preserving systems, we give the definition of a notion shared by these two types of systems:

DEFINITION 2.12. $(\Omega, \mathcal{F}, \mu, T)$ is said to be *rigid* if there exists a strictly increasing sequence $n_k$ such that, for all $f \in L^2(\mu)$, $f \circ T^{n_k} \to f$ in $L^2(\mu)$.

In any dynamical system $(\Omega, \mathcal{F}, \mu, T)$ and for any sequence $\{n_k\}$ there exists a factor algebra $\mathcal{A}_{n_k}$ such that a square integrable function $f$ is $\mathcal{A}_{n_k}$-measurable if and only if $f \circ T^{n_k} \to f$ in $L^2(\mu)$. The proof in the finite measure case can be found in [**59**], and can be translated, up to minor modifications, to the infinite measure context.

**2.6. The case of a probability measure.** We assume here that $\mu(\Omega) = 1$

THEOREM 2.13. *(Birkhoff ergodic theorem)*
*Let $f \in L^1(\mu)$, then, $\mu$-a.e. and in $L^1(\mu)$:*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f \circ T^k = \mu(f|\mathcal{I})$$

*where $\mu(f|\mathcal{I})$ is the conditional expectation of $f$ with respect to the invariant $\sigma$-algebra.*

DEFINITION 2.14. $(\Omega, \mathcal{F}, \mu, T)$ is said *weakly mixing* if, for all $A, B \in \mathcal{F}$:

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \left|\mu\left(A \cap T^{-k} B\right) - \mu(A)\mu(B)\right| = 0$$

$(\Omega, \mathcal{F}, \mu, T)$ is said *mildly mixing* if, for all $f \in L^2(\mu)$:

$$\int_\Omega f \circ T^{n_k} \bar{f} d\mu \to \int_\Omega |f|^2 d\mu \ (n_k \uparrow \infty) \ \Rightarrow f \text{ constant}$$

$(\Omega, \mathcal{F}, \mu, T)$ is said *mixing* if, for all $A, B \in \mathcal{F}$:

$$\lim_{n \to \infty} \left|\mu\left(A \cap T^{-n} B\right) - \mu(A)\mu(B)\right| = 0$$

DEFINITION 2.15. $(\Omega, \mathcal{F}, \mu, T)$ is called a *K-system* if there exists a sub-$\sigma$-algebra $\mathcal{G} \subset \mathcal{F}$ such that:

$$T^{-1}\mathcal{G} \subseteq \mathcal{G}$$

$$\vee_{n\in\mathbb{Z}} T^{-n}\mathcal{G} = \mathcal{F}$$

and

$$\cap_{n\in\mathbb{Z}} T^{-n}\mathcal{G} = \{\emptyset, \Omega\}$$

We now introduce a dynamical system that will constantly be used in the body of the paper. We consider here the space $\mathbb{R}^{\mathbb{Z}}$ of $\mathbb{Z}$-indexed sequences. The natural $\sigma$-algebra is the product algebra $\mathcal{B}^{\otimes\mathbb{Z}}$ where $\mathcal{B}$ is the natural Borel algebra on $\mathbb{R}$. The transformation is the shift $T$ that acts in the following way:

$$T\left\{x_i\right\}_{i\in\mathbb{Z}} = \left\{x_{i+1}\right\}_{i\in\mathbb{Z}}$$

The dynamical system $\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, \mu, T\right)$ is the canonical space of the stationary process of distribution $\mu$.

DEFINITION 2.16. The system associated to an i.i.d. process is called a *Bernoulli scheme*.

This definition is extended in the following abstract way:

DEFINITION 2.17. $(\Omega, \mathcal{F}, \mu, T)$ is said *Bernoulli* if it is isomorphic to a Bernoulli scheme.

We end this section by the following proposition:

PROPOSITION 2.18. *We have the implications:*

$$\textit{Bernoulli} \Rightarrow K \Rightarrow \textit{mixing} \Rightarrow \textit{mildly mixing}$$

$$\Rightarrow \textit{weakly mixing} \Rightarrow \textit{ergodic}$$

*Moreover, these six properties are shared by all the factors.*

## 3. Spectral theory

Here we only give results that will be needed in the rest of the paper.

**3.1. Hilbert space, unitary operator and spectral measure.** We consider a complex Hilbert space $(H, \langle . \rangle)$ endowed with a unitary operator $U$. To each vector $f \in H$, we can associate a finite measure $\sigma_f$ on $[-\pi, \pi[$, called the *spectral measure of* $f$ by the formula:

$$\hat{\sigma}_f(n) := \langle U^n f, f \rangle = \int_{[-\pi,\pi[} e^{inx} \sigma_f(dx)$$

Let $C(f)$ be the closure of the linear space generated by the family $\{U^n f\}_{n \in \mathbb{Z}}$, $C(f)$ is called the *cyclic space* of $f$. We summarize the following properties in the following proposition.

PROPOSITION 3.1. *Let $f$ and $g$ be in $H$. If $\sigma_f$ and $\sigma_g$ are mutually singular, then $f$ and $g$ are orthogonal.*

*There exists an isometry $\phi$ between $C(f)$ and $L^2(\sigma_f)$ with $\phi(f) = 1$ and such that the unitary operator $h \mapsto e^{i\cdot} h$ on $L^2(\sigma_f)$ is conjugate to $U$ by $\phi$.*

*If $g$ and $h$ belong to the same cyclic space $C(f)$, then $C(g) \perp C(h)$ implies that $\sigma_g$ and $\sigma_h$ are mutually singular.*

**3.2. Maximal spectral type.** On $(H, \langle . \rangle, U)$ there exists a finite measure $\sigma_M$ such that, for all $f \in H$, $\sigma_f \ll \sigma_M$. The (equivalence class of the) measure $\sigma_M$ is called the *maximal spectral type* of $(H, \langle . \rangle, U)$. Moreover, for all finite measure $\sigma \ll \sigma_M$, there exists a vector $g$ such that $\sigma_g = \sigma$.

**3.3. Application to ergodic theory.** A dynamical system $(\Omega, \mathcal{F}, \mu, T)$ induces a complex Hilbert space, the space $L^2(\mu)$ endowed with a unitary operator $U : f \mapsto f \circ T$.

3.3.1. *The case of a probability measure.* We restrict the study to the orthocomplement of the constant functions in $L^2(\mu)$. That is, we note $L^2_0(\mu) := L^2(\mu) \ominus \mathbb{C}\langle 1 \rangle$ and we speak of the maximal spectral type of $(\Omega, \mathcal{F}, \mu, T)$, the maximal spectral type of $(L^2_0(\mu), U)$. Nevertheless, if there is a risk of misunderstanding, we will call it the *reduced maximal spectral type.* One finds the following ergodic properties on the maximal spectral type $\sigma_M$:

PROPOSITION 3.2. *$(\Omega, \mathcal{F}, \mu, T)$ is ergodic if and only if $\sigma_M \{0\} = 0$.*

*$(\Omega, \mathcal{F}, \mu, T)$ is weakly mixing if and only if $\sigma_M$ is continuous.*

*$(\Omega, \mathcal{F}, \mu, T)$ is mildly mixing if and only if $\sigma_M(\Gamma) = 0$ for all weak Dirichlet sets (a set $\Gamma$ is a weak Dirichlet set if all finite measure $\nu$ supported by $\Gamma$ satisfies $\limsup |\hat{\nu}(n)| = \hat{\nu}(0)$).*

*$(\Omega, \mathcal{F}, \mu, T)$ is mixing if and only if $\sigma_M$ is a Rajchman measure (i.e., $\hat{\sigma}_f(n) \to 0$ as $|n|$ tends to $+\infty$).*

3.3.2. *The infinite measure case.* Since constant functions are not in $L^2(\mu)$, we don't impose the restriction made in the preceding section. We have the important proposition, see for example [**1**]:

PROPOSITION 3.3. *$(\Omega, \mathcal{F}, \mu, T)$ is of type $\mathbf{II}_\infty$ if and only if $\sigma_M$ is continuous.*

To be of zero type is a spectral property:

PROPOSITION 3.4. *$(\Omega, \mathcal{F}, \mu, T)$ is of zero type if and only if $\sigma_M$ is a Rajchman measure.*

## 4. Poisson Suspension

In this section, we introduce and study "dynamical systems over dynamical systems", namely, point processes, called Poisson suspensions, which are random discrete measures on the underlying dynamical system. This is a way to transfer the study of a system with a possibly infinite measure to a system with a probability measure. The particular form, in chaos, of the $L^2$-space associated to the Poisson suspension allows a useful spectral analysis. We will recall basic facts on the intensively studied Poisson measure. The particular case we are interested in, that is, when the distribution of the Poisson measure is preserved by a well chosen transformation (and then called Poisson suspension) has received much less attention.

**4.1. Definitions.** We consider a Borel space $(\Omega, \mathcal{F}, \mu)$ where $\mu$ is $\sigma$-finite and the space $(M_\Omega, \mathcal{M}_\mathcal{F})$ of measures $\nu$ on $(\Omega, \mathcal{F})$ satisfying $\nu(A) \in \mathbb{N}$ for all $A \in \mathcal{F}$ of finite $\mu$-measure. $\mathcal{M}_\mathcal{F}$ is the smallest $\sigma$-algebra on $M_\Omega$ such that the mappings $\nu \mapsto \nu(A)$ are measurable for all $A \in \mathcal{F}$ of finite $\mu$-measure. We denote by $N$ the identity on $(M_\Omega, \mathcal{M}_\mathcal{F})$.

DEFINITION 4.1. We call *Poisson measure* the triplet $(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_\mu)$ where $\mathcal{P}_\mu$ is the unique probability measure such that, for all finite collections $\{A_i\}$ of elements belonging to $\mathcal{F}$, disjoint and of finite $\mu$-measure, the $\{N(A_i)\}$ are independent and distributed as the Poisson law of parameter $\mu(A_i)$.

In the sequel, if $f$ is a measurable map defined from $(\Omega, \mathcal{F}, \mu)$ to $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mu}\right)$ (where $\tilde{\mu}$ is the image measure of $\mu$ by $f$, assumed $\sigma$-finite), then we denote by $f^\star$ the measurable map from $(M_\Omega, \mathcal{M}_\mathcal{F})$ to $(M_{\tilde{\Omega}}, \mathcal{M}_{\tilde{\mathcal{F}}})$ that associates $\nu$ to $f^\star(\nu)$, the image measure of $\nu$ by $f$. The following well known proposition is crucial, we omit the proof which is straightforward:

PROPOSITION 4.2. *The image of $\mathcal{P}_\mu$ by $f^\star$ is $\mathcal{P}_{\tilde{\mu}}$.*

That is, the image of a Poisson measure by a map of the form $f^\star$ is still a Poisson measure. We call such a map *Poissonian.*

Assume now that $(\Omega, \mathcal{F}, \mu)$ is endowed by an invertible measurable transformation $T$ that preserves $\mu$, then $T^\star$ is an invertible measurable transformation on $(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_\mu)$ that preserves $\mathcal{P}_\mu$.

DEFINITION 4.3. The dynamical system $(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_\mu, T^\star)$ is called the *Poisson suspension* above $(\Omega, \mathcal{F}, \mu, T)$. The suspension is said to be *pure* if $(\Omega, \mathcal{F}, \mu, T)$ is ergodic.

**4.2. Factors, isomorphisms and product structure.** As a direct consequence of Proposition 4.2, we obtain:

THEOREM 4.4. *Let* $(\Omega_2, \mathcal{F}_2, \mu_2, T_2)$ *be a c-factor of* $(\Omega_1, \mathcal{F}_1, \mu_1, T_1)$.

*Then* $(M_{\Omega_2}, \mathcal{M}_{\mathcal{F}_2}, \mathcal{P}_{c\mu_2}, T_2^\star)$ *is a factor of* $(M_{\Omega_1}, \mathcal{M}_{\mathcal{F}_1}, \mathcal{P}_{\mu_1}, T_1^\star)$ *and the factor map can be chosen to be Poissonian.*

*Assume that* $(\Omega_1, \mathcal{F}_1, \mu_1, T_1)$ *is c-isomorphic to* $(\Omega_2, \mathcal{F}_2, \mu_2, T_2)$.

*Then* $(M_{\Omega_1}, \mathcal{M}_{\mathcal{F}_1}, \mathcal{P}_{\mu_1}, T_1^\star)$ *is isomorphic to* $(M_{\Omega_2}, \mathcal{M}_{\mathcal{F}_2}, \mathcal{P}_{c\mu_2}, T_2^\star)$ *and the isomorphism can be chosen to be Poissonian.*

The independence properties of a Poisson suspension along invariant subsets imply:

LEMMA 4.5. *Let* $(\Omega, \mathcal{F}, \mu, T)$ *be a dynamical system and suppose there exists a countable partition* $\{\Omega_n\}_{n\in\mathbb{N}}$ *of* $\Omega$ *into* $T$*-invariant sets of non zero* $\mu$*-measure.*

*Then* $(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_\mu, T^\star)$ *is isomorphic to* $\left(M_\Omega^{\times\mathbb{N}}, \mathcal{M}_\mathcal{F}^{\otimes\mathbb{N}}, \mathcal{P}_{\mu_{|\Omega_0}} \otimes \cdots \otimes \mathcal{P}_{\mu_{|\Omega_n}} \otimes \ldots, T^{\star^{\times N}}\right)$

**4.3. General $L^2$ properties of a Poisson suspension.** In this section, we recall the basic facts on the Fock space structure of the $L^2$-space associated to a Poisson measure. [**45**] is a reference for this section.

4.3.1. *Fock factorization.*

DEFINITION 4.6. The *Fock factorization* of the Hilbert space $K$ is the Hilbert space $\mathbf{exp}K$ given by:

$$\mathbf{exp}K := \mathfrak{S}^0 K \oplus \mathfrak{S}^1 K \oplus \cdots \oplus \mathfrak{S}^n K \oplus \cdots$$

where $\mathfrak{S}^n K$ is the $n$-th symmetric tensor power of $K$ and is called the *$n$-th chaos*, with $\mathfrak{S}^0 K = \mathbb{C}$.

On $\mathbf{exp}K$, the set of *factorizable vectors* is particularly important:

$$\mathcal{E}_h := 1 \oplus h \oplus \frac{1}{\sqrt{2!}} h \otimes h \oplus \cdots \oplus \frac{1}{\sqrt{n!}} h \otimes \cdots \otimes h \oplus \cdots$$

for $h \in K$.

They form a total part in $\mathbf{exp}K$ and satisfy the identity:

$$\langle \mathcal{E}_h, \mathcal{E}_g \rangle_{\mathbf{exp}K} = \exp \langle h, g \rangle_K$$

Now suppose we are given an operator $U$ on $K$ with norm at most 1, it extends naturally to an operator $\tilde{U}$ on $\mathbf{exp}K$ called the *exponential* of $U$, by acting on each chaos via the formula:

$$\tilde{U}(h \otimes \cdots \otimes h) = (Uh) \otimes \cdots \otimes (Uh)$$

leading to the identity:

$$\tilde{U} \mathcal{E}_h = \mathcal{E}_{Uh}$$

We then have the fundamental well known property:

PROPOSITION 4.7. *Consider* $(\Omega, \mathcal{F}, \mu)$ *and* $(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_\mu)$*, the corresponding Poisson measure. There is a natural isometry between* $L^2(\mathcal{P}_\mu)$ *and the Fock factorization* $\mathbf{exp}\left[L^2(\mu)\right]$.

4.3.2. *Description of chaos.* Call $\Delta_n$ the diagonal in $\Omega^{\times n}$ (the $n$-uplets with identical coordinates). Multiple integrals (that will correspond, through the natural isometry, to tensor products), for $f$ in $L^1(\mu) \cap L^2(\mu)$ are defined by:
$J^{(n)}(f) :=$

$$\int \dots \int_{\Delta_n^c} f(x_1) \dots f(x_n) \left(N(dx_1) - \mu(dx_1)\right) \dots \left(N(dx_n) - \mu(dx_n)\right)$$

Then the $J^{(n)}(f)$ for $f$ in $L^1(\mu) \cap L^2(\mu)$ form a total part of the $n$-th chaos, $H^n$, of $L^2(\mathcal{P}_\mu)$ and we have the isometry formula:

$$\left\langle J^{(n)}(f), J^{(p)}(g)\right\rangle_{L^2(\mathcal{P}_\mu)} = n! \left(\langle f, g\rangle_{L^2(\mu)}\right)^n 1_{n=p}$$

Call $\mathfrak{H}$ the set of functions $h$, finite linear combination of indicator functions of elements of $\mathcal{F}$ with finite $\mu$-measure, through the natural isometry, the factorizable vectors $\mathcal{E}_h$ are:

$$\mathcal{E}_h(\nu) = \exp\left(-\int_\Omega h d\mu\right) \prod_{x \in \nu} (1 + h(x))$$

They form a total part in $L^2(\mathcal{P}_\mu)$, moreover, $\mathbb{E}_{\mathcal{P}_\mu}[\mathcal{E}_h] = 1$.

**4.4. Spectral properties of a Poisson suspension.** We now consider the case of a dynamical system $(\Omega, \mathcal{F}, \mu, T)$ and its associated Poisson suspension $(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_\mu, T^\star)$. It is obvious that the unitary operator $f \mapsto f \circ T^\star$ acting on $L^2(\mathcal{P}_\mu)$ is the exponential of the corresponding unitary operator on $L^2(\mu)$, $g \mapsto g \circ T$. From this simple remark and the above isometry identities between chaos, it can be deduced:

COROLLARY 4.8. *If* $\sigma_M$ *is the maximal spectral type of* $(\Omega, \mathcal{F}, \mu, T)$*, then on each chaos* $H^n$*, the maximal spectral type of* $U$ *is* $\sigma_M^{*n}$*. The (reduced) maximal spectral type the Poisson suspension* $(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_\mu, T^\star)$ *is* $e(\sigma_M) := \sum_{n \geq 1} \frac{1}{n!} \sigma_M^{*n}$.

We get also the following proposition (the proof is similar to the Gaussian case (see [**13**])):

PROPOSITION 4.9. *If the maximal spectral type of* $(\Omega, \mathcal{F}, \mu, T)$ *is absolutely continuous with respect to Lebesgue measure, then the (reduced) maximal spectral type of* $(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_\mu, T^\star)$ *is equivalent to Lebesgue measure.*

**4.5. Poissonian factors of a Poisson suspension.** To each sub-$\sigma$-algebra $\mathcal{A}$ of $\mathcal{F}$, we associate the sub-$\sigma$-algebra $\mathcal{A}_N$ of $\mathcal{M}_{\mathcal{F}}$ generated by the random variables $N(A)$ where $A$ describes $\mathcal{A}$.

DEFINITION 4.10. A *Poissonian factor* is a sub-$\sigma$-algebra of the kind $\mathcal{A}_N$ where $\mathcal{A}$ is a $T$-invariant sub-$\sigma$-algebra of $(\Omega, \mathcal{F}, \mu, T)$.

To a sub-$\sigma$-algebra $\mathcal{A}_N$ is associated $U_{\mathcal{A}_N}$ ($U_{\mathcal{A}_N}(f) = \mathbb{E}_{\mathcal{P}_\mu}[f|\mathcal{A}_N]$), the conditional expectation operator with respect to this sub-$\sigma$-algebra. It is easily checked that $U_{\mathcal{A}_N}$ is the exponential of the operator $U_{\mathcal{A}}$ on $L^2(\mu)$ "conditional expectation" on $\mathcal{A}$. Indeed, noting first that if $h$ is in $\mathfrak{H}$, $\mathcal{E}_{U_{\mathcal{A}}h}$ is $\mathcal{A}_N$-measurable and taking $g$ $\mathcal{A}$-measurable:

$$\mathbb{E}_{\mathcal{P}_\mu}[U_{\mathcal{A}_N}\mathcal{E}_h\mathcal{E}_g] = \mathbb{E}_{\mathcal{P}_\mu}[\mathcal{E}_h\mathcal{E}_g]$$

$$= \exp\langle h, g\rangle_{L^2(\mu)} = \exp\langle U_{\mathcal{A}}h, g\rangle_{L^2(\mu)} = \mathbb{E}_{\mathcal{P}_\mu}[\mathcal{E}_{U_{\mathcal{A}}h}\mathcal{E}_g]$$

Note that conditional expectation is between quotation marks to stress that we have to be careful in the infinite measure case; conditional expectation is usually reserved to sub-$\sigma$-algebra where the measure is still $\sigma$-finite, a hypothesis we don't require here. Note that in our case conditional expectation means orthogonal projection, and it hopefully coincides in the preceding case. This enables us to prove the next lemma that will be the key to derive a criterion for the $K$ property of a Poisson suspension.

LEMMA 4.11. *Let $\{\mathcal{A}^n\}_{n\in\mathbb{N}}$ be a collection of sub-$\sigma$-fields of $\mathcal{F}$ such that $\mathcal{A}^n \subset \mathcal{A}^m$ if $n \geq m$. We have:*

$$(\cap_{n\in\mathbb{N}}\mathcal{A}^n)_N = \cap_{n\in\mathbb{N}}\mathcal{A}^n_N$$

*and, with $\{\mathcal{A}^n\}_{n\in\mathbb{N}}$ a collection of $\sigma$-finite sub-$\sigma$-fields of $\mathcal{F}$ such that $\mathcal{A}^n \subset \mathcal{A}^m$ if $n \leq m$*

$$(\vee_{n\in\mathbb{N}}\mathcal{A}^n)_N = \vee_{n\in\mathbb{N}}\mathcal{A}^n_N$$

PROOF. The proof is based on the following fact:

In a Hilbert space $H$, if $\{E_n\}_{n\geq 0}$ form a decreasing collection of closed convex sets and if $E_\infty$, their intersection, is non-empty, then for all $x \in H$, the projections $P_{E_n}$ on $E_n$ satisfy $P_{E_n}(x) \to P_{E_\infty}(x)$ in $H$. If now $\{E_n\}_{n\geq 0}$ form an increasing collection of closed convex sets and if $E_\infty$ is the closure of their union, we still have $P_{E_n}(x) \to P_{E_\infty}(x)$ in $H$. Here the closed convex sets are the subspaces of measurable functions with respect to sub-$\sigma$-algebras and the projectors are the corresponding conditional expectations.

We will show $(\cap_{n\in\mathbb{N}}\mathcal{A}^n)_N = \cap_{n\in\mathbb{N}}\mathcal{A}^n_N$. Call $U_{\mathcal{A}^n}$, the conditional expectation on $L^2(\mu)$ with respect to $\mathcal{A}^n$, $U_{\mathcal{A}^n}$ tends weakly to $U_{\cap_{n\in\mathbb{N}}\mathcal{A}^n}$ on $L^2(\mu)$, this implies that, its exponential, $U_{\mathcal{A}^n_N}$ tends weakly to $U_{(\cap_{n\in\mathbb{N}}\mathcal{A}^n)_N}$ on $L^2(\mathcal{P}_\mu)$. But on $L^2(\mathcal{P}_\mu)$, $U_{\mathcal{A}^n_N}$ tends weakly to $U_{\cap_{n\in\mathbb{N}}\mathcal{A}^n_N}$, so by the unicity of the limit, $U_{(\cap_{n\in\mathbb{N}}\mathcal{A}^n)_N} = U_{\cap_{n\in\mathbb{N}}\mathcal{A}^n_N}$ and then $(\cap_{n\in\mathbb{N}}\mathcal{A}^n)_N = \cap_{n\in\mathbb{N}}\mathcal{A}^n_N$.

The proof of $(\vee_{n\in\mathbb{N}}\mathcal{A}^n)_N = \vee_{n\in\mathbb{N}}\mathcal{A}^n_N$ is identical. □

We have a spectral criterion to show that the entropy of a Poisson suspension is zero:

PROPOSITION 4.12. *Let $\sigma_M$ be the maximal spectral type of $(\Omega, \mathcal{F}, \mu, T)$, if $\sigma_M$ is singular with respect to Lebesgue measure, then the entropy of a $(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_\mu, T^\star)$ is zero.*

PROOF. The proof is completely similar to the analogous proposition for square integrable stationary processes, with the difference that, in general, we don't deal with a cyclic subspace here. Assume $\sigma_M$ is singular with respect to Lebesgue measure and let $\mathcal{A}$ be the Pinsker algebra of $(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_\mu, T^\star)$ and $L_0^2(\mathcal{P}_\mu)$ the square integrable vectors of zero expectation. In [**46**], it is proved that the spectral measures of the vectors that belong to the ortho-complement in $L_0^2(\mathcal{P}_\mu)$ of $L_0^2(\mathcal{P}_\mu|\mathcal{A})$ (the $\mathcal{A}$-measurable vectors of $L_0^2(\mathcal{P})$) are absolutely continuous with respect to Lebesgue measure. Thus, vectors of the form $N(A) - \mu(A)$, for all $A$ in $\mathcal{F}$ of finite $\mu$-measure, which possess a spectral measure absolutely continuous with respect to $\sigma_M$, and thus singular with respect to Lebesgue measure, belong to the ortho-complement of the ortho-complement of $L_0^2(\mathcal{P}_\mu|\mathcal{A})$, that is, belong to $L_0^2(\mathcal{P}_\mu|\mathcal{A})$ and as such are $\mathcal{A}$-measurable. Since the $\sigma$-field generated by the vectors $N(A) - \mu(A)$ is nothing other than the total $\sigma$-field $\mathcal{M}_\mathcal{F}$, $\mathcal{M}_\mathcal{F}$ is indeed the Pinsker algebra, that is, the entropy of $(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_\mu, T^\star)$ is zero. □

### 4.6. Topology on Poissonian transformations, Poissonian centralizer and self-similarity.

4.6.1. *The weak topology.* For the sake of simplicity, we consider here a Poisson measure constructed above $(\mathbb{R}, \mathcal{B}, \lambda)$ where $\lambda$ denotes Lebesgue measure and $\mathcal{B}$ the Borel sets (indeed, this is not really a restriction, for if our Poisson measure $(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_\mu)$ is constructed above a Borel space $(\Omega, \mathcal{F}, \mu)$ which doesn't posses atoms and where $\mu$ is infinite, there exists an isomorphism $\Phi$ between $(\Omega, \mathcal{F}, \mu)$ and $(\mathbb{R}, \mathcal{B}, \lambda)$ and thus a Poissonian isomorphism $\Phi^\star$ that transfers all the Poissonian structure (for example, this will be so in the interesting cases, when the underlying space is that of a dynamical system of type $\mathbf{II}_\infty$ and conservative).

The group $G$ on $(\mathbb{R}, \mathcal{B}, \lambda)$ is the group of all measure preserving automorphisms on $(\mathbb{R}, \mathcal{B}, \lambda)$. $G$ is a topological group when endowed with the topology $\mathcal{O}_G$ of weak convergence of bounded operators acting on $L^2(\lambda)$ (when an automorphism $T$ is considered as $U_T$, the unitary operator acting on $L^2(\lambda)$ by $f \mapsto f \circ T$), and this topology is metrizable (and then separable and complete) by the distance $d$ given by:

$$d(S,T) = \sum_{n=0}^{+\infty} \frac{\lambda(T(E_n) \triangle S(E_n)) + \lambda(T^{-1}(E_n) \triangle S^{-1}(E_n))}{a_n}$$

where the $\{E_n\}_{n\in\mathbb{N}}$ is any countable total family (with $\sup \lambda(E_n) < +\infty$), and $a_n$ is any sequence of positive numbers such that $\sum_{n=0}^{+\infty} \frac{1}{a_n} < +\infty$. We refer to [**1**], page 108, for details on this topology and their properties.

On the probability space $(M_{\mathbb{R}}, \mathcal{M}_{\mathcal{B}}, \mathcal{P}_\lambda)$, there is also the group, that we denote by $G'$, of all measure preserving automorphisms. Its weak topology $\mathcal{O}_{G'}$ is metrizable by a distance $\delta$, defined exactly in the same way as $d$, that makes it separable and complete.

As we have seen throughout this section, if $T$ is an automorphism on $(\mathbb{R}, \mathcal{B}, \lambda)$, $T^\star$ is a Poissonian automorphism on $(M_{\mathbb{R}}, \mathcal{M}_{\mathcal{B}}, \mathcal{P}_\lambda)$ leading to the associated Poisson suspension $(M_{\mathbb{R}}, \mathcal{M}_{\mathcal{B}}, \mathcal{P}_\lambda, T^\star)$. The group $G$ can thus be embedded into $G'$, we denote it by $G^\star$, this is the group of Poissonian automorphisms on $(M_{\mathbb{R}}, \mathcal{M}_{\mathcal{B}}, \mathcal{P}_\lambda)$. Evidently, the topology $\mathcal{O}_G$ of $G$ becomes a topology $\mathcal{O}_{G^\star}$ on $G^\star$, by defining a distance $d^\star$ in the most natural way:

$$d^\star(S^\star, T^\star) = d(S, T)$$

Now, we can see that $\mathcal{O}_{G^\star}$ is in fact the topology $\mathcal{O}_{G'}$ when restricted to $G^\star$, since the weak convergence of a sequence $U_{T_n}$ to $U_T$ in $L^2(\lambda)$ is equivalent to the weak convergence in $L^2(\mathcal{P}_\lambda)$ of the sequence $\tilde{U}_{T_n}$ of their exponential to $\tilde{U}_T$, the exponential of $U_T$. But, as we have already pointed out, $\tilde{U}_{T_n} = U_{T_n^\star}$ and $\tilde{U}_T = U_{T^\star}$ which ends the proof. We are now going to prove that $G^\star$ is closed in $G'$. For each positive integer $p$, define $\{E_{n,p}\}_{n\in\mathbb{N}}$ as the family of intervals $\left[e_n, e_n + 1 + \frac{1}{p}\right]$, where the $e_n$ form an enumeration of the rationals. It is clear that $\{E_{n,p}\}_{n\in\mathbb{N}, p\in\mathbb{N}^\star}$ is dense. We will now make use of the following identities:

$$\mathcal{P}_\lambda(\{N(A) = 0\} \triangle \{N(B) = 0\}) = e^{-\lambda(A)} + e^{-\lambda(B)} - 2e^{-\lambda(A\cup B)}$$

and, if $y \in [x, 2x]$:

$$e^{-2x}(y - x) \leq e^{-x} - e^{-y}$$

This leads to, if $\lambda(A) = \lambda(B)$, and noting that $2\lambda(A \cup B) - \lambda(A) - \lambda(B) = \lambda(A \triangle B)$:

$$e^{-2\lambda(A)}\lambda(A \triangle B) \leq \mathcal{P}_\lambda(\{N(A) = 0\} \triangle \{N(B) = 0\}) \tag{1}$$

Let $A_n$ be a dense family of sets in $(M_{\mathbb{R}}, \mathcal{M}_{\mathcal{B}}, \mathcal{P}_\lambda)$ and let $a_{n,p}$ be a sequence of positive numbers well chosen to satisfy $\sum_{n\in\mathbb{N}, p\in\mathbb{N}^\star} \frac{1}{a_{n,p}} < +\infty$. We define a distance $\delta$ that generates $\mathcal{O}_{G'}$ and $d^\star$ that generate $\mathcal{O}_{G^\star}$ by:

$$\delta(\theta, \beta) := \sum_{n\in\mathbb{N}, p\in\mathbb{N}^\star} \frac{\mathcal{P}_\lambda(\theta\{N(E_{n,p})=0\}\triangle\beta\{N(E_{n,p})=0\}) + \mathcal{P}_\lambda\left(\theta^{-1}\{N(E_{n,p})=0\}\triangle\beta^{-1}\{N(E_{n,p})=0\}\right)}{a_{n,p}}$$
$$+\sum_n \frac{\mathcal{P}_\lambda(\theta(A_n)\triangle\beta(A_n)) + \mathcal{P}_\lambda\left(\theta^{-1}(A_n)\triangle\beta^{-1}(A_n)\right)}{2^{-n}}$$

$$d^\star(T^\star, S^\star) := e^{-2} \sum_{n\in\mathbb{N}, p\in\mathbb{N}^\star} \frac{\lambda(T(E_{n,p})\triangle S(E_{n,p})) + \lambda\left(T^{-1}(E_{n,p})\triangle S^{-1}(E_{n,p})\right)}{a_{n,p}}$$

By the identity (1), it is clear that $d^\star(T^\star, S^\star) \leq \delta(T^\star, S^\star)$. Now suppose there is a sequence $T_n^\star \in G^\star$ that tends to $\theta \in G'$ for the distance $\delta$, this sequence is then $\delta$-Cauchy and thus $d^{\star}$-Cauchy, but since $(G^{\star}, d^{\star})$ is complete, $T_n^{\star}$ tends to a certain $T^{\star}$ in $G^{\star}$. This means that $\theta = T^{\star}$ and $\theta \in G^{\star}$. Finally $G^{\star}$ is meagre in $G'$, indeed, Katok and Stepin have shown that for a generic automorphism of a probability space, the maximal spectral type and its convolution powers are mutually singular, which is of course never the case for a Poisson suspension.

This topology allows us, as in the traditional case, to consider the relative abundance of different kind of Poisson suspensions in terms of Baire categories:

Definition 4.13. We say that **P** is a generic property in $G^{\star}$ if the set of Poissonian automorphisms that satisfy **P** contains a countable intersection of open and dense sets. With an abuse of terminology, we will speak of a "generic Poisson suspension" a Poisson suspension that shares a generic property **P**.

For example, it is shown in [**1**], page 108, that a generic automorphism of a non atomic dynamical system with an infinite measure is ergodic. The consequence of this fact is:

Proposition 4.14. *A generic Poisson suspension is pure.*

We will see in Remark 6.3 that a generic Poisson suspension is weakly mixing and rigid.

4.6.2. *Poissonian centralizer and self-similarity.* To any system $(\Omega, \mathcal{F}, \mu, T)$ is associated $C(T)$, the group of automorphisms of $(\Omega, \mathcal{F}, \mu)$ that preserve $\mu$ and commute with $T$, it is called the *centralizer*. The preceding discussion shows this group can be embedded as a subgroup $C(T^{\star})$ of automorphisms of $(M_{\Omega}, \mathcal{M}_{\mathcal{F}}, \mathcal{P}_{\mu}, T^{\star})$ that preserve $\mathcal{P}_{\mu}$ and commute with $T^{\star}$. We call this subgroup the *Poissonian centralizer* and denote it by $C^{p}(T^{\star})$. $C(T)$ is a closed subgroup of the group of all measure preserving automorphisms of $(\Omega, \mathcal{F}, \mu)$ then, the above topological considerations prove:

Proposition 4.15. *$C^{p}(T^{\star})$ is closed in $C(T^{\star})$.*

When the measure $\mu$ is infinite, it may happen that $(\Omega, \mathcal{F}, \mu, T)$ and $(\Omega, \mathcal{F}, c\mu, T)$, with $c \neq 1$, are isomorphic. We denote by $C_0(T)$ the group of automorphisms $S$ commuting with $T$ and such that there exists $c > 0$, $c < +\infty$ and $S^{\star}\mu = c\mu$ (Note that Aaronson in [**1**] denotes this group by $C(T)$). Any such $S$ induces a Poissonian isomorphism $S^{\star}$ such that $(M_{\Omega}, \mathcal{M}_{\mathcal{F}}, \mathcal{P}_{\mu}, T^{\star})$ and $(M_{\Omega}, \mathcal{M}_{\mathcal{F}}, \mathcal{P}_{c\mu}, T^{\star})$ are isomorphic. This self-similarity property leads to the following definition:

Definition 4.16. A Poisson suspension $(M_{\Omega}, \mathcal{M}_{\mathcal{F}}, \mathcal{P}_{\mu}, T^{\star})$ is said to be *self-similar of index c*, $c \neq 1$, if it is isomorphic to $(M_{\Omega}, \mathcal{M}_{\mathcal{F}}, \mathcal{P}_{c\mu}, T^{\star})$ by a Poissonian isomorphism. It is said to be *completely self-similar* if it is self-similar of index $c$, for all $c > 0$.

We will see examples of self-similar Poisson suspensions in Section 10. In particular, Poisson suspensions constructed above the Lévy measure of $\alpha$-stable processes are completely self-similar.

## 5. Refinement of the canonical decomposition of a dynamical system

We have seen in Lemma 4.5 that the decomposition of a dynamical system along its invariant sets allows us to consider the associated Poisson suspension as the direct product of the suspensions constructed above them. There can be an infinite number of disjoint invariant sets and as many independent factors in the Poisson suspension. The decomposition in Proposition 2.11 has enabled us to isolate four classes of dynamical systems with very distinct properties, we will see in this section that we can identify another important class which will lead us to the decomposition of a dynamical system into five classes (Proposition 5.4).

**5.1. Rigidity-free systems.** For this decomposition, we have first to introduce the following definition:

DEFINITION 5.1. $(\Omega, \mathcal{F}, \mu, T)$ is said to be *rigidity-free* if its maximal spectral type $\sigma_M$ satisfies: $\sigma_M(\Gamma) = 0$ for every weak Dirichlet set $\Gamma \subset [-\pi, \pi[$.

PROPOSITION 5.2. *There exists a partition* $\{\mathcal{N}_{rf}, \{\mathcal{N}_{rf}\}^c\}$ *into* $T$*-invariant sets such that* $\left(\Omega, \mathcal{F}, \mu_{|\mathcal{N}_{rf}}, T\right)$ *is rigidity-free and such that, for every* $T$*-invariant set* $B$ *of non-zero* $\mu$*-measure included in* $\{\mathcal{N}_{rf}\}^c$, $\left(\Omega, \mathcal{F}, \mu_{|B}, T\right)$ *is not rigidity-free.*

PROOF. Let $\mathfrak{K}$ be the collection of sets $A$ in $\mathcal{F}$ such that, $A_\infty$ defined by $A_\infty := \cup_{n\in\mathbb{Z}} T^{-k}A$, $\left(\Omega, \mathcal{F}, \mu_{|A_\infty}, T\right)$, is rigidity-free. If $B \subset A$, with $B$ of non-zero $\mu$-measure, $\left(\Omega, \mathcal{F}, \mu_{|B_\infty}, T\right)$ is still rigidity-free since the maximal spectral type of $\left(\Omega, \mathcal{F}, \mu_{|B_\infty}, T\right)$ is absolutely continuous with respect to the maximal spectral type of $\left(\Omega, \mathcal{F}, \mu_{|A_\infty}, T\right)$. That is, $\mathfrak{K}$ is hereditary; form the *measurable union* of the elements in $\mathfrak{K}$ and call it $\mathcal{N}_{rf}$. From Lemma 1.0.7 in [**1**], $\mathcal{N}_{rf}$ can be written as the disjoint union $\cup_{n\in\mathbb{N}} B_n$ of elements in $\mathfrak{K}$.

We will show that $\mathcal{N}_{rf}$ can be written as the disjoint union $\cup_{n\in\mathbb{N}} A_n$ where the $A_n$ are in $\mathfrak{K}$ and $T$-invariant. Indeed, we clearly have $\cup_{n\in\mathbb{N}} B_n = \cup_{n\in\mathbb{N}} B_{n,\infty}$, the $B_{n,\infty}$ are in $\mathfrak{K}$ and $T$-invariant. We then define $A_0 := B_{0,\infty}$, $A_1 := B_{1,\infty} \setminus B_{0,\infty}$, $A_2 := B_{2,\infty} \setminus (B_{0,\infty} \cup B_{1,\infty})$ and so on. The $A_i$ are in $\mathfrak{K}$, since $\mathfrak{K}$ is hereditary, and are disjoint. But the maximal spectral type of $\left(\Omega, \mathcal{F}, \mu_{|\mathcal{N}_{rf}}, T\right)$ is the sum (weighted to be finite) of the maximal spectral types of the rigidity-free systems $\left(\Omega, \mathcal{F}, \mu_{|A_n}, T\right)$, so $\left(\Omega, \mathcal{F}, \mu_{|\mathcal{N}_{rf}}, T\right)$ is effectively rigidity-free. By construction, there don't exist $T$-invariant sets $B \subset \{\mathcal{N}_{rf}\}^c$ such that $\left(\Omega, \mathcal{F}, \mu_{|B}, T\right)$ is rigidity-free. □

REMARK 5.3. Instead of partitioning the support of the measure with respect to these particular maximal spectral types, we could have chosen other families of spectral measures, and this would have led to other decompositions. However, we will see that our choice is justified by the notion of ID-disjointness (see Section 9).

We thus can specify the decomposition in Proposition 2.11, the notation anticipates the forthcoming section.

PROPOSITION 5.4. *Let $(\Omega, \mathcal{F}, \mu, T)$ be a dynamical system. Then we can write, in a unique way:*

$$\mu = \mu_B + \mu_m + \mu_{mm} + \mu_{wm} + \mu_{ne}$$

*where:*
$(\Omega, \mathcal{F}, \mu_B, T)$ *is dissipative,*
$(\Omega, \mathcal{F}, \mu_m, T)$ *is conservative of zero type,*
$(\Omega, \mathcal{F}, \mu_{mm}, T)$ *is rigidity-free of positive type,*
$(\Omega, \mathcal{F}, \mu_{wm}, T)$ *is of type* $\mathbf{II}_\infty$*, of positive type and, for every invariant set A of non zero* $\mu_{wm}$*-measure,* $\left(\Omega, \mathcal{F}, \mu_{wm|A}, T\right)$ *is not rigidity-free,*
$(\Omega, \mathcal{F}, \mu_{ne}, T)$ *is of type* $\mathbf{II}_1$.

PROOF. Thanks to Proposition 2.11, we can write:

$$\Omega = \mathfrak{D} \cup (\mathcal{C} \cap \mathcal{N}_0) \cup (\mathcal{N}_+ \cap \mathcal{N}_{rf}) \cup \left(\mathcal{N}_{rf}^c \cap \mathcal{N}\right) \cup \mathfrak{P}$$

by noting first that $\mathcal{N}_0 \subset \mathcal{N}_{rf}$, since a Rajchman measure annihilates the weak Dirichlet sets. We define then $\mu_B := \mu_{|\mathfrak{D}}$, $\mu_m := \mu_{|\mathcal{C}\cap\mathcal{N}_0}$, $\mu_{mm} := \mu_{|\mathcal{N}_+\cap\mathcal{N}_{rf}}$, $\mu_{wm} := \mu_{|\mathcal{N}_{rf}^c\cap\mathcal{N}}$, $\mu_{ne} := \mu_{|\mathfrak{P}}$. □

We finish this section by showing, with a classical argument, that the ergodic system constructed by Hajian and Kakutani in [**26**] is of type $(\Omega, \mathcal{F}, \mu_{mm}, T)$. The authors have shown that the system is not of zero type. Moreover, Aaronson and Nadkarni, in [**2**] have proved that this system is prime (it has no strict ($\sigma$-finite) factor)) and has trivial centralizer. It thus can't have a rigid factor (if it was not the case, the system would be rigid and would have an uncountable centralizer (the proof of this last fact is the same as in the finite measure case, see for example [**33**]).

Note that the fact that this system is prime implies that the corresponding Poisson suspension has no non trivial Poissonian factor.

## 6. Ergodic properties of a Poisson suspension

In this section we consider a system $(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_\mu, T^\star)$ where $\mu = \mu_B + \mu_m + \mu_{mm} + \mu_{wm} + \mu_{ne}$ from the decomposition in Proposition 5.4. Lemma 4.5 immediately implies:

PROPOSITION 6.1. $(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_\mu, T^\star)$ *is isomorphic to:*

$$\left(M^5, \mathcal{M}_\mathcal{F}^{\otimes 5}, \mathcal{P}_{\mu_B} \otimes \mathcal{P}_{\mu_m} \otimes \mathcal{P}_{\mu_{mm}} \otimes \mathcal{P}_{\mu_{wm}} \otimes \mathcal{P}_{\mu_{ne}}, T^\star \times \cdots \times T^\star\right)$$

We now look at the ergodic properties in each class:

PROPOSITION 6.2. $(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_{\mu_{ne}}, T^\star)$ *is not ergodic.*
$(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_{\mu_{wm}}, T^\star)$ *is weakly mixing and possesses rigid Poissonian factors.*
$(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_{\mu_{mm}}, T^\star)$ *is mildly mixing, not mixing.*
$(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_{\mu_m}, T^\star)$ *is mixing of all orders.*
$(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_{\mu_B}, T^\star)$ *is Bernoulli.*

PROOF. There exists a $T$-invariant probability $\nu$, equivalent to $\mu_{ne}$. Let $f := \sqrt{\frac{d\nu}{d\mu_{ne}}}$, $f$ is square integrable of norm 1 and $T$-invariant so its spectral measure is the Dirac mass at 0. Then, the maximal spectral type of $(\Omega, \mathcal{F}, \mu_{ne}, T)$ has an atom at 0. This atom is also, thanks to Corollary 4.8, part of the (reduced) maximal spectral type of $(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_{\mu_{ne}}, T^\star)$ so $(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_{\mu_{ne}}, T^\star)$ can't be ergodic.

The fact that $(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_{\mu_{wm}}, T^\star)$ is weakly mixing is a direct consequence of Proposition 3.3 and Corollary 4.8. By construction, there exists a weak Dirichlet set $\Gamma$ such that $\sigma_M$, the maximal spectral type $(\Omega, \mathcal{F}, \mu_{wm}, T)$ doesn't annihilate $\Gamma$. It follows that there exists a vector $f \in L^2(\mu_{wm})$ such that, for a certain sequence $n_k \uparrow \infty$, $f \circ T^{n_k} \to f$ in $L^2(\mu_{wm})$ and thus a rigid factor $\mathcal{A}$ for this sequence. This implies that $\mathcal{A}_N$ is a rigid factor for the suspension.

We now show that $(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_{\mu_{mm}}, T^\star)$ is mildly mixing and not mixing. Remark first that all translations of a weak Dirichlet set $\Gamma$ are weak Dirichlet sets. Indeed, let $s \in [-\pi, \pi[$, and let $\nu$ be a measure with support in $s + \Gamma$, the image of $\nu$ by the translation $-s$, noted $\nu_{-s}$ is a measure supported by $\Gamma$ and their Fourier coefficients are related by $\hat{\nu}(n) = e^{-ins}\hat{\nu}_{-s}(n)$. So, since $\limsup |\hat{\nu}_{-s}(n)| = \nu_{-s}(0)$, $\limsup |\hat{\nu}(n)| = \nu(0)$ and this ends the proof. We will show that, if $\sigma$ annihilates all weak Dirichlet sets, then the convolution powers $\sigma^{*n}$ annihilate all the weak Dirichlet sets as well. Let $\Gamma$ be a weak Dirichlet set, we can write $\sigma^{*n}(\Gamma) = \int_{[-\pi,\pi[} \left[ \int_{[-\pi,\pi[} 1_{-s+\Gamma}(t)\, \sigma(dt) \right] \sigma^{*(n-1)}(ds)$ and the part between brackets is zero for all $s \in [-\pi, \pi[$, thanks to the first part of the proof. Finally $\sigma^{*n}(\Gamma) = 0$. The maximal spectral type $\sigma_M$ of $(\Omega, \mathcal{F}, \mu_{mm}, T)$ annihilates all the weak Dirichlet sets, this property is conserved by the successive convolution powers of $\sigma_M$, thus, thanks to Corollary 4.8, the maximal spectral type of $(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_{\mu_{mm}}, T^\star)$ annihilates all the weak Dirichlet sets. The system is then mildly mixing. Since $\sigma_M$ is not a Rajchman measure, it cannot be mixing.

If now we consider $(\Omega, \mathcal{F}, \mu_m, T)$, this system is of zero type, that is to say, for all $A \in \mathcal{F}$, $B \in \mathcal{F}$ of finite $\mu$-measure, $\mu_m\left(A \cap T^{-k}B\right)$ tends to 0 as $k$ tends to infinity.

We are going to generalize the identity $\langle \mathcal{E}_h, \mathcal{E}_g \rangle_{L^2(\mathcal{P}_{\mu_m})} = \exp \langle h, g \rangle_{L^2(\mu_m)}$ :

$$\mathbb{E}_{\mathcal{P}_{\mu_m}}\left[\mathcal{E}_{h_1}\mathcal{E}_{h_2}\ldots\mathcal{E}_{h_n}\right]$$
$$= \exp \sum_{1 \le i_1 < i_2 \le n} \int h_{i_1} h_{i_2}\, d\mu_m + \ldots$$
$$\cdots + \sum_{1 \le i_1 < i_2 < \cdots < i_n \le n} \int h_{i_1} \ldots h_{i_n}\, d\mu_m$$

We show, more generally, the following formula for functions $h_1, \ldots, h_n$ of $\mathfrak{H}$ :

$$\mathcal{E}_{h_1}\mathcal{E}_{h_2}\ldots\mathcal{E}_{h_n}$$
$$= \mathcal{E}_{(1+h_1)(1+h_2)\ldots(1+h_n)-1} \exp \sum_{1 \le i_1 < i_2 \le n} \int h_{i_1} h_{i_2}\, d\mu_m + \ldots$$
$$\cdots + \sum_{1 \le i_1 < i_2 < \cdots < i_n \le n} \int h_{i_1} \ldots h_{i_n}\, d\mu_m$$

At rank 2, the computation is easy, let $n \geq 2$ and suppose that the formula is true at this rank.

Let $h_1, \ldots, h_n, h_{n+1}$ be functions in $\mathfrak{H}$.

We first evaluate $\mathcal{E}_{(1+h_1)(1+h_2)\ldots(1+h_n)-1}\mathcal{E}_{h_{n+1}}$. The formula, at rank 2 gives us:
$\mathcal{E}_{(1+h_1)(1+h_2)\ldots(1+h_n)-1}\mathcal{E}_{h_{n+1}}$
$= \exp \int h_{n+1} \left( (1+h_1)(1+h_2) \ldots (1+h_n) - 1 \right) d\mu_m$

$$\times \mathcal{E}_{(1+h_1)(1+h_2)\ldots(1+h_n)(1+h_{n+1})-1}$$

But $\exp \int h_{n+1} \left( (1+h_1)(1+h_2) \ldots (1+h_n) - 1 \right) d\mu_m$ equals:
$\exp \sum_{i=1}^{n} \int h_i h_{n+1} d\mu_m + \ldots$

$$\cdots + \sum_{1 \leq i_1 < i_2 < \cdots < i_n \leq n} \int h_{i_1} \ldots h_{i_n} h_{n+1} d\mu_m$$

Combining this result with the formula at rank $n$, we show that the formula is true at rank $n+1$ and this ends the proof by recurrence.

To show mixing of order $n$ with the functions $\mathcal{E}_{h_1}, \ldots, \mathcal{E}_{h_n}$ with $h_1, \ldots, h_n$ in $\mathfrak{H}$, take $k_1, \ldots, k_n$ tending towards infinity and such that $|k_j - k_i|$ tends to infinity too for all $i \neq j$. We have to show that:
$\mathbb{E}_{\mathcal{P}_{\mu_m}} \left[ \mathcal{E}_{h_1} \circ T^{\star k_1} \mathcal{E}_{h_2} \circ T^{\star k_2} \ldots \mathcal{E}_{h_n} \circ T^{\star k_n} \right]$ tends to

$$\mathbb{E}_{\mathcal{P}_{\mu_m}} \left[ \mathcal{E}_{h_1} \circ T^{\star k_1} \right] \ldots \mathbb{E}_{\mathcal{P}_{\mu_m}} \left[ \mathcal{E}_{h_n} \circ T^{\star k_n} \right] = 1$$

But:
$\mathbb{E}_{\mathcal{P}_{\mu_m}} \left[ \mathcal{E}_{h_1} \circ T^{\star k_1} \mathcal{E}_{h_2} \circ T^{\star k_2} \ldots \mathcal{E}_{h_n} \circ T^{\star k_n} \right]$

$$= \mathbb{E}_{\mathcal{P}_{\mu_m}} \left[ \mathcal{E}_{h_1 \circ T^{k_1}} \mathcal{E}_{h_2 \circ T^{k_n}} \ldots \mathcal{E}_{h_n \circ T^{k_n}} \right]$$

and then, from the preceding formula, we have to show that quantities of the kind $\int h_i \circ T^{k_i} \ldots h_j \circ T^{k_j} d\mu_m$ tend to 0.

The $h_i$ are finite linear combinations of indicator functions of sets of finite $\mu$-measure, then, expanding $\int h_i \circ T^{k_i} \ldots h_j \circ T^{k_j} d\mu_m$, we obtain a finite linear combination of quantities of the kind $\mu_m \left( T^{-k_l} A_l \cap \cdots \cap T^{-k_m} A_m \right)$. But these quantities tend to 0 since:

$$\mu_m \left( T^{-k_l} A_l \cap \cdots \cap T^{-k_m} A_m \right) \leq \mu_m \left( A_l \cap T^{-(k_m - k_l)} A_m \right)$$

We thus have the mixing of order $n$ on the factorizable vectors $\mathcal{E}_{h_1}, \ldots, \mathcal{E}_{h_n}$, and, by standard approximation arguments, taking advantages of the properties of these vectors, we get mixing of order $n$ for the suspension.

$(\Omega, \mathcal{F}, \mu_B, T)$ is dissipative, so, from Lemma 2.6, there exists a wandering set $W$ such that $\Omega = \cup_{n \in \mathbb{Z}} T^{-n} W$ mod. $\mu_B$. Denote by $\mathcal{W}$ the $\sigma$-field generated by $A \in \mathcal{F}$ such that $A \subset W$. Then $\mathcal{W}_N$ generates $\mathcal{M}_{\mathcal{F}}$ (i.e. $\mathcal{M}_{\mathcal{F}} = \vee_{n \in \mathbb{Z}} T^{\star -n} \mathcal{W}_N$) and, thanks to the independence properties of a Poisson measure, the $\sigma$-fields $T^{\star -n} \mathcal{M}_W$ are independent. Hence $(M_\Omega, \mathcal{M}_{\mathcal{F}}, \mathcal{P}_{\mu_B}, T^\star)$ is Bernoulli. $\square$

REMARK 6.3. In the proof, we have shown that the Poisson suspension constructed above a rigid dynamical system of type $\mathbf{II}_\infty$ is itself rigid (the converse being also true), Silva and Ageev have shown that rigidity is a generic property for these systems (see [**5**]). This proves that:

*A generic Poisson suspension is weakly mixing and rigid.*

We give a criterion for a suspension to be $K$, by introducing a particular class of systems, the definition is due to Krengel and Sucheston [**36**].

DEFINITION 6.4. $(\Omega, \mathcal{F}, \mu, T)$ is said to be *remotely infinite* if there exists a $\sigma$-finite sub-$\sigma$-field $\mathcal{G} \subset \mathcal{F}$ such that:

$$T^{-1}\mathcal{G} \subseteq \mathcal{G}$$

$$\vee_{n\in\mathbb{Z}} T^{-n}\mathcal{G} = \mathcal{F}$$

and

$$\mathcal{F}_f \cap_{n\in\mathbb{Z}} T^{-n}\mathcal{G} = \{\emptyset\}$$

where $\mathcal{F}_f$ are the elements of $\mathcal{F}$ of finite $\mu$-measure.

PROPOSITION 6.5. *Let $(\Omega, \mathcal{F}, \mu, T)$ be a remotely infinite system. The Poisson suspension $(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_\mu, T^\star)$ is a $K$-system.*

PROOF. In Definition 6.4, the $\sigma$-field $\mathcal{G}$ satisfies:

$$T^{-1}\mathcal{G} \subseteq \mathcal{G}$$

$$\vee_{n\in\mathbb{Z}} T^{-n}\mathcal{G} = \mathcal{F}$$

and

$$\mathcal{F}_f \cap_{n\in\mathbb{Z}} T^{-n}\mathcal{G} = \{\emptyset\}$$

Thus, from Lemma 4.11, the $\sigma$-field $\mathcal{G}_N$ satisfies:

$$T^{\star-1}\mathcal{G}_N \subseteq \mathcal{G}_N$$

and

$$\cap_{n\in\mathbb{Z}} T^{\star-n}\mathcal{G}_N = \{\emptyset, M_\Omega\}$$

This shows that $(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_\mu, T^\star)$ is a $K$-system. □

Conservative remotely infinite systems are given by invariant measures associated to null recurrent Markov chains (see for example [**36**]).

## 7. Infinitely divisible stationary processes

After some generalities on stationary processes, we next introduce the notion of infinite divisibility by rapidly recalling the finite dimensional case (the book of K. Sato [**57**] is a reference on this vast subject). Infinite divisibility for processes, our main concern, is then an immediate generalization of this notion, notably, the accompanying tools such as the Lévy measure find its equivalent notion for processes as shown by Maruyama in [**38**]. This measure is the key object that will allow to connect results of the preceding sections to prove Theorem 7.8, which was the motivation for this work, and to deduce their ergodic properties in Proposition 7.11.

**7.1. Dynamical system associated to a stationary stochastic process.** We consider $\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, \mu, T\right)$ introduced in Section 2.6, $\mu$ may be infinite. When we will deal with stationary processes, only the measure will change throughout the study and, to simplify, we will often use it to designate such a system. Affirmations such as "$\mu$ is ergodic" or "$\mu$ is dissipative" will be shortening of "$\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, \mu, T\right)$ is ergodic" or "$\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, \mu, T\right)$ is dissipative". We will try to keep the notation $X := \{X_0 \circ T^n\}_{n\in\mathbb{Z}}$ for the identity process, $X_0$ being the "coordinate at 0" map $\{x_i\}_{i\in\mathbb{Z}} \mapsto x_0$. $X$, $\{X_n\}_{n\in\mathbb{Z}}$, $\{X_0 \circ T^n\}_{n\in\mathbb{Z}}$, $\mu$ or $\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, \mu, T\right)$ is essentially the same object.

7.1.1. *Linear factors.* Denote by $a$ a sequence $\{a_i\}_{i\in\mathbb{Z}}$ where only a finite number of coordinates are non zero and call $\mathfrak{A}$ their union in $\mathbb{R}^{\mathbb{Z}}$. The process $\{\langle a, X\rangle \circ T^n\}_{n\in\mathbb{Z}}$ generated by $\langle a, X\rangle := \sum_{k=-\infty}^{+\infty} a_k X_k$ will be called a *simple linear factor*. If $a_\infty$ is any sequence $\{a_i^\infty\}_{i\in\mathbb{Z}}$, we consider the process $\{\langle a_\infty, X\rangle \circ T^n\}_{n\in\mathbb{Z}}$ generated by $\langle a_\infty, X\rangle := \sum_{k=-\infty}^{+\infty} a_k^\infty X_k$ as long as it is well defined $\mu$-a.e.. We will use the term *generalized linear factor* for such a process. In both cases, $\mu^a$ or $\mu^{a_\infty}$ will denote the measure associated to the process $\{\langle a, X\rangle \circ T^n\}_{n\in\mathbb{Z}}$ or $\{\langle a_\infty, X\rangle \circ T^n\}_{n\in\mathbb{Z}}$.

**7.2. Convolution on $\left(\mathbb{R}^d, \mathcal{B}^d\right)$.** On the product $\left(\mathbb{R}^d \times \mathbb{R}^d, \mathcal{B}^d \otimes \mathcal{B}^d\right)$, we consider the mapping "sum" with values in $\left(\mathbb{R}^d, \mathcal{B}^d\right)$ which associates $x+y$ to $(x, y)$. Given two probabilities $\lambda_1$ and $\lambda_2$ on $\left(\mathbb{R}^d, \mathcal{B}^d\right)$, we call $\lambda_1 * \lambda_2$ the "convolution of $\lambda_1$ with $\lambda_2$" . $\lambda_1 * \lambda_2$ is the image distribution of $\lambda_1 \otimes \lambda_2$ by the mapping already defined. This operation is clearly associative and we denote by $\lambda^{*k}$ the convolution of $k$ identical copies of $\lambda$.

Definition 7.1. Let $\lambda$ be a distribution on $\left(\mathbb{R}^d, \mathcal{B}^d\right)$, $\lambda$ is *infinitely divisible* (abr. ID) if, for all integer $k$, there exists a distribution $\lambda_k$ on $\left(\mathbb{R}^d, \mathcal{B}^d\right)$ such that $\lambda = \lambda_k^{*k}$.

7.2.1. *Lévy-Khintchine representation.* The characteristic function of an ID distribution $\lambda$ admits a remarkable representation:

$$\int_{\mathbb{R}^d} e^{i\langle z,x\rangle} \lambda(dx) = \exp\left[-\frac{1}{2}\langle z, Az\rangle + i\langle \gamma, z\rangle + \int_{\mathbb{R}^d} \left(e^{i\langle z,x\rangle} - 1 - i\langle z, c(x)\rangle\right) \nu(dx)\right]$$

where $A$ is a symmetric $d \times d$ matrix that corresponds to the *Gaussian* part, $\gamma$ belongs to $\mathbb{R}^d$ and is called the *drift*, and $\nu$ a measure on $\left(\mathbb{R}^d, \mathcal{B}^d\right)$, called *Lévy measure* such that $\nu\{0\} = 0$ and $\displaystyle\int_{\mathbb{R}^d} \left(1 \wedge \|x\|^2\right) \nu(dx) < +\infty$, this is called the *Poissonian* part.

$c$ is defined by:
$c(x)_i = -1$ if $x_i < -1$
$c(x)_i = x_i$ if $-1 \leq x_i \leq 1$
$c(x)_i = 1$ if $x_i > 1$
This representation is unique.

In the case where $\int_{\mathbb{R}^d} |x| \lambda(dx) < +\infty$ and $\int_{\mathbb{R}^d} x\lambda(dx) = 0$ , we have the more tractable representation:

$$\int_{\mathbb{R}^d} e^{i\langle z,x\rangle} \lambda(dx) = \exp\left[-\frac{1}{2}\langle z, Az\rangle + \int_{\mathbb{R}^d} \left(e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle\right) \nu(dx)\right]$$

$A$ and $\nu$ are unchanged with respect to the first representation and only the drift is affected.

**7.3. Convolution of processes.** These notions are absolutely similar to the preceding ones.

On the product $\left(\mathbb{R}^{\mathbb{Z}} \times \mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}} \otimes \mathcal{B}^{\otimes\mathbb{Z}}\right)$, we once again consider the mapping "sum" with values in $\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}\right)$ which associates $\{x_i + y_i\}_{i\in\mathbb{Z}}$ to $\left(\{x_i\}_{i\in\mathbb{Z}}, \{y_i\}_{i\in\mathbb{Z}}\right)$. Given two distributions $\mathbb{P}_1$ and $\mathbb{P}_2$ on $\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}\right)$, we call $\mathbb{P}_1 * \mathbb{P}_2$ the "convolution of $\mathbb{P}_1$ with $\mathbb{P}_2$" . $\mathbb{P}_1 * \mathbb{P}_2$ is the image distribution of $\mathbb{P}_1 \otimes \mathbb{P}_2$ by the mapping already defined. Since this operation is clearly associative, we can denote $\mathbb{P}^{*k}$ to be the convolution of $k$ identical copies of $\mathbb{P}$.

Definition 7.2. Let $\mathbb{P}$ be a distribution on $\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}\right)$, $\mathbb{P}$ is *infinitely divisible* (abr. ID) if, for all integer $k$, there exists a distribution $\mathbb{P}_k$ on $\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}\right)$ such that $\mathbb{P} = \mathbb{P}_k^{*k}$.

We remark that this definition forces the finite-dimensional distributions to be ID, in fact, this can be taken as an equivalent definition.

7.3.1. *Lévy measure of an ID stationary process.* We have, here again, an analogous representation, due to Maruyama (see [**38**]) of characteristic functions of the finite-dimensional distributions of an ID stationary process of distribution $\mathbb{P}$.

$$\mathbb{E}\left[\exp i\langle a, X\rangle\right] = \exp\left[-\frac{1}{2}\langle Ra, a\rangle + i\langle a, b_\infty\rangle + \int_{\mathbb{R}^{\mathbb{Z}}} e^{i\langle a,x\rangle} - 1 - i\langle c(x), a\rangle Q(dx)\right] \tag{2}$$

where $R$ is the covariance function of a centered stationary Gaussian process, $b_\infty \in \mathbb{R}^{\mathbb{Z}}$ is a sequence identically equal to $b$ and $Q$ is a $\sigma$-finite measure on $\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}\right)$ invariant with respect to the shift and such that $Q\{0\} = 0$ (where $\{0\}$ is the identically zero sequence) and $\displaystyle\int_{\mathbb{R}^{\mathbb{Z}}} (x_0^2 \wedge 1)\, Q(dx) < +\infty$.

$c(x)_i = -1$ if $x_i < -1$
$c(x)_i = x_i$ if $-1 \leq x_i \leq 1$
$c(x)_i = 1$ if $x_i > 1$
$\langle R, b, Q \rangle$ is called the *generating triplet* of $\mathbb{P}$.
The dynamical system $\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes \mathbb{Z}}, Q, T\right)$ will be our main concern in the sequel.

When the process is integrable and centered, we have the following representation, where $R$ and $Q$ are unchanged:

$$\mathbb{E}\left[\exp i \langle a, X \rangle\right] = \exp\left[-\frac{1}{2}\langle Ra, a\rangle + \int_{\mathbb{R}^{\mathbb{Z}}} e^{i\langle a,x\rangle} - 1 - i\langle a, x\rangle Q(dx)\right] \tag{3}$$

Finally, if the process only takes positive values (and then without Gaussian part), we can write down its finite-dimensional distribution through their Laplace transforms, with $a \in \mathfrak{A} \cap \mathbb{R}_+^{\mathbb{Z}}$:

$$\mathbb{E}\left[\exp - \langle a, X \rangle\right] = \exp\left[-\langle a, b_\infty \rangle - \int_{\mathbb{R}^{\mathbb{Z}}} 1 - e^{-\langle a,x\rangle} Q(dx)\right] \tag{4}$$

If, moreover, it is integrable, under this representation, we have:

$$\mathbb{E}\left[X_0\right] = b + \int_{\mathbb{R}^{\mathbb{Z}}} x_0 Q(dx)$$

REMARK 7.3. If we are given a covariance function $R$, a drift $b$, and a measure $Q$ satisfying the hypothesis specified above, it determines the distribution of an ID process of generating triplet $\langle R, b, Q \rangle$ by defining its finite-dimensional distribution through the representation 2. Then we can apprehend the extraordinary variety of the processes at our disposal.

DEFINITION 7.4. An ID process is said to be *Poissonian* (abr. IDp) if its generating triplet doesn't possess a Gaussian part.

In the sequel, when we will speak of IDp process with Lévy measure $Q$, we will consider a process whose generating triplet is $\langle 0, 0, Q \rangle$ under the representation (2). Of course, the drift has no impact in our study.

### 7.4. First examples and representation.

7.4.1. *Canonical example.* Maruyama in [**38**] has given the canonical example of an IDp stationary process:

We consider a Poisson suspension $(M_\Omega, \mathcal{M}_{\mathcal{F}}, \mathcal{P}_\mu, T^\star)$ above $(\Omega, \mathcal{F}, \mu, T)$ and a real function $f$ defined on $(\Omega, \mathcal{F}, \mu, T)$ such that $\int_\Omega \frac{f^2}{1+f^2} d\mu < +\infty$. We define the stochastic integral $I(f)$ by the limit in probability, as $n$ tends towards infinity, of:

$$\int_{|f| > \frac{1}{n}} f dN - \int_{|f| > \frac{1}{n}} c(f)\, d\mu$$

Then the process $X=\{I(f)\circ T^{\star n}\}_{n\in\mathbb{Z}}$ is IDp and its distribution is given by:

$$\mathbb{E}\left[\exp i\left\langle a,X\right\rangle\right]=\exp\left[\int_{\Omega}\exp\left(i\sum_{n\in\mathbb{Z}}a_n f\circ T^n\right)-1-i\sum_{n\in\mathbb{Z}}a_n c\left(f\circ T^n\right)d\mu\right]$$

for $a\in\mathfrak{A}$.

Maruyama has also shown in [**38**] that all the IDp processes can be represented this way. Now we consider $Q$, the Lévy measure of an IDp process of generating triplet $\langle 0,0,Q\rangle$. Let now $(M_{\mathbb{R}^{\mathbb{Z}}},\mathcal{M}_{\mathcal{B}^{\otimes\mathbb{Z}}},\mathcal{P}_Q,T^{\star})$ be the Poisson suspension constructed above $\left(\mathbb{R}^{\mathbb{Z}},\mathcal{B}^{\otimes\mathbb{Z}},Q,T\right)$. Then, in the preceding example, it suffices to take, as function $f$, the function $X_0:\{x_i\}_{i\in\mathbb{Z}}\mapsto x_0$.

THEOREM 7.5. *(Maruyama) The process* $\{I(X_0)\circ T^{\star n}\}_{n\in\mathbb{Z}}$ *admits* $\langle 0,0,Q\rangle$ *as generating triplet.*

This theorem is crucial since it allows us to consider an IDp process as a factor of a Poisson suspension, precisely the Poisson suspension constructed above its Lévy measure.

**7.5. Fundamental family and first factorization.** It is obvious that the convolution of two ID distributions is still ID, the class of this type of distributions being closed under convolution. Given a stationary ID distribution, we ask when it is *factorizable*, that is, can it be written as the convolution of two or more ID distributions ? An immediate factorization comes from the representation (2):

Suppose that $\mathbb{P}$ admits the triplet $\langle R,b,Q\rangle$, then if $\mathbb{P}_s$ admits the triplet $\langle sR,sb,sQ\rangle$ and $\mathbb{P}_{1-s}$ admits the triplet $\langle(1-s)R,(1-s)b,(1-s)Q\rangle$ with $0<s<1$, then $\mathbb{P}=\mathbb{P}_s*\mathbb{P}_{1-s}$.

DEFINITION 7.6. We call *fundamental family* associated with $\mathbb{P}$ having generating triplet $\langle R,b,Q\rangle$, the family of distributions $\mathbb{P}_s$ with generating triplet $\langle sR,sb,sQ\rangle$, for $s>0$.

The representation (2) allows another more interesting factorization. Let $\mathbb{P}_R$ of triplet $\langle R,0,0\rangle$, $\mathbb{P}_b$ of triplet $\langle 0,b,0\rangle$ and $\mathbb{P}_Q$ of triplet $\langle 0,0,Q\rangle$, we have:

$$\mathbb{P}=\mathbb{P}_R*\mathbb{P}_b*\mathbb{P}_Q$$

where $\mathbb{P}_R$ is the distribution of a stationary centered Gaussian process, $\mathbb{P}_b$ is a constant process and $\mathbb{P}_Q$ the distribution of an IDp process.

As in the Poisson suspension case, we introduce the following definition:

DEFINITION 7.7. An IDp process is said to be *pure* if its Lévy measure is ergodic.

**7.6. Factorization through invariant components of the Lévy measure.** We can apply to $Q$ the decomposition $Q = Q_B + Q_m + Q_{mm} + Q_{wm} + Q_{ne}$ along the five disjoint shift-invariant subsets as in Proposition 5.4. By considering (2), we get the following factorization result:

THEOREM 7.8. *Factorization of a stationary IDp process.*
*Let $\mathbb{P}$ be the distribution of a stationary IDp process. $\mathbb{P}$ can be written in the unique way:*

$$\mathbb{P} = \mathbb{P}_{Q_B} * \mathbb{P}_{Q_m} * \mathbb{P}_{Q_{mm}} * \mathbb{P}_{Q_{wm}} * \mathbb{P}_{Q_{ne}}$$

*where:*
$\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, Q_B, T\right)$ *is dissipative,*
$\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, Q_m, T\right)$ *is conservative of zero type,*
$\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, Q_{mm}, T\right)$ *is rigidity-free of positive type,*
$\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, Q_{wm}, T\right)$ *is of type* $\mathbf{II}_\infty$, *of positive type and, for every invariant set $A$ of non zero $Q_{wm}$-measure,* $\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, Q_{wm|A}, T\right)$ *is not rigidity-free,*
$\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, Q_{ne}, T\right)$ *is of type* $\mathbf{II}_1$.

Since we have shown that these classes were not empty for the corresponding Poisson suspensions, we deduce they are not empty for the IDp processes by considering stochastic integrals with respect to these Poisson suspensions.

We can ask whether the Poissonian part is irreducible. The answer is easy:

PROPOSITION 7.9. *Let $\mathbb{P}$ be the distribution of an IDp process of Lévy measure $Q$. The equation $\mathbb{P} = \mathbb{P}_1 * \mathbb{P}_2$ with $\mathbb{P}_1$ and $\mathbb{P}_2$ ID only admits solutions among the fundamental family of $\mathbb{P}$ if and only if $\mathbb{P}$ is pure.*

PROOF. Two $T$-invariant and ergodic measures are mutually singular, or one is a multiple of the other. If $Q$ is ergodic and $\mathbb{P} = \mathbb{P}_1 * \mathbb{P}_2$ with $Q_1$ and $Q_2$ the Lévy measure of $\mathbb{P}_1$ and $\mathbb{P}_2$, since $Q = Q_1 + Q_2$, we have $Q_1 \ll Q$ and thus $Q_1 = cQ$ so $Q_2 = (1-c)\,Q$. If $Q$ is not ergodic, its support can be written as the disjoint union of two $T$-invariant sets of non zero $Q$-measure, $A$ and $A^c$. We then define $Q_1 := Q_{|A}$ and $Q_2 := Q_{|A^c}$. $Q_1$ and $Q_2$ are then the Lévy measures associated to two IDp processes of distribution $\mathbb{P}_1$ and $\mathbb{P}_2$ such that $\mathbb{P} = \mathbb{P}_1 * \mathbb{P}_2$ and $\mathbb{P}_1$ and $\mathbb{P}_2$ are not in the same fundamental family. □

**7.7. Linear factors.** The fact that linear factors of an ID process are ID is immediate. But this is also true for generalized linear factors, using the fact that limit in distribution of ID distributions are ID and the fact that a process is ID if its finite dimensional distributions are ID.

The Lévy measure of simple linear factors is easily deduced. Indeed, we verify that the Lévy measure of $\mathbb{P}^a$, a simple linear factor of $\mathbb{P}$, is nothing other than $Q^a_{|\{0\}^c}$, the measure, restricted to $\{0\}^c$, of the simple linear factor coming from $Q$, the Lévy measure of $\mathbb{P}$.

**7.8. Ergodic properties of stationary IDp processes.** Before enunciating the properties of each class, we will need the following lemma which is the interpretation, in our framework, of a computation done by Rosiński and Żak in [**54**]. Their computation led to show that, if $X$ is an IDp process, the spectral measure of $e^{iX_0} - \mathbb{E}\left[e^{iX_0}\right]$ has the form $\left|\mathbb{E}\left[e^{iX_0}\right]\right|^2 e(m)$ (we still use the notation $e(m) := \sum_{k=1}^{+\infty} \frac{1}{k!} m^{*k}$, where $m$ is a finite measure on $[-\pi,\pi[$). We will see that $m$ is indeed itself a spectral measure, but for the system associated to the Lévy measure of $X$.

LEMMA 7.10. *Let $X$ be an IDp process of Lévy measure $Q$ and $a \in \mathfrak{A}$. The spectral measure of $e^{i\langle a,X\rangle} - \mathbb{E}\left[e^{i\langle a,X\rangle}\right]$ is $\left|\mathbb{E}\left[e^{i\langle a,X\rangle}\right]\right|^2 e(\sigma_a)$ where $\sigma_a$ is the spectral measure of $e^{i\langle a,X\rangle} - 1$ under $Q$.*

PROOF. In [**54**], the following formula is established:

$$\mathbb{E}\left[e^{iX_0}\overline{e^{iX_k}}\right] = \left|\mathbb{E}\left[e^{iX_0}\right]\right|^2 \left(\exp\left[\int_{\mathbb{R}^2} \left(e^{ix}-1\right)\left(\overline{e^{iy}-1}\right) Q_{0,k}(dx,dy)\right]\right)$$

where $Q_{0,k}$ is the Lévy measure of the bivariate ID vector $(X_0, X_k)$. But, since we make use of Lévy measure of processes, this formula can be written into:

$$\mathbb{E}\left[e^{iX_0}\overline{e^{iX_k}}\right] = \left|\mathbb{E}\left[e^{iX_0}\right]\right|^2 \left(\exp\left[\int_{\mathbb{R}^{\mathbb{Z}}} \left(e^{ix_0}-1\right)\left(\overline{e^{ix_k}-1}\right) Q(dx)\right]\right)$$

which equals:

$$\left|\mathbb{E}\left[e^{iX_0}\right]\right|^2 \left(\exp \hat{\sigma}_0(k)\right) = \left|\mathbb{E}\left[e^{iX_0}\right]\right|^2 \left(\sum_{n=0}^{+\infty} \frac{1}{n!} \left(\hat{\sigma}_0(k)\right)^n\right)$$

where $\sigma_0$ is the spectral measure of $e^{iX_0} - 1$ under $Q$. The conclusion follows.

For the general case, with $\langle a, X\rangle$, it is easily deduced from this last computation and the observations made in Section 7.7 about Lévy measures of simple linear factors. ☐

PROPOSITION 7.11. *$\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, \mathbb{P}_{Q_{ne}}, T\right)$ is not ergodic.*
*$\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, \mathbb{P}_{Q_{wm}}, T\right)$ is weakly mixing, not mildly mixing.*
*$\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, \mathbb{P}_{Q_{mm}}, T\right)$ is mildly mixing, not mixing.*
*$\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, \mathbb{P}_{Q_{m}}, T\right)$ is mixing of all order.*
*$\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, \mathbb{P}_{Q_{B}}, T\right)$ has the Bernoulli property.*

PROOF. There exists a probability measure $\nu$ which is $T$-invariant and equivalent to $Q_{ne}$. Let $f := \sqrt{\frac{dQ_{ne}}{d\nu}}$ (note that $\frac{dQ_{ne}}{d\nu}$ is just $\left(\frac{d\nu}{dQ_{ne}}\right)^{-1}$) and $\lambda \in \mathbb{R}$.

The spectral measure of $e^{i\lambda x_0} - 1$ under $Q_{ne}$ is the spectral measure of $fe^{i\lambda x_0} - f$ under $\nu$. The set $\{f < a\}$ is $T$-invariant since $f$ is $T$-invariant, moreover this set is of non zero measure if $a$ is large enough. Thus the spectral measure of $fe^{i\lambda x_0} - f$ under $\nu$ is the sum of the spectral measures of $\left(fe^{i\lambda x_0} - f\right) 1_{\{f<a\}}$ and $\left(fe^{i\lambda x_0} - f\right) 1_{\{f\geq a\}}$ under $\nu$.

If $\left(fe^{i\lambda x_0}-f\right)1_{\{f<a\}}$ is centered, we have

$$\int_{\mathbb{R}^{\mathbb{Z}}\cap\{f<a\}} f(x)\,e^{i\lambda x_0}\nu(dx)=\int_{\mathbb{R}^{\mathbb{Z}}\cap\{f<a\}} f(x)\,\nu(dx)\in\mathbb{R}$$

This implies

$$\int_{\mathbb{R}^{\mathbb{Z}}\cap\{f<a\}} f(x)\left[1-\cos(\lambda x_0)\right]\nu(dx)=0$$

Since $f$ is non negative on $\{f<a\}$ $\nu$-a.e., this implies that $\cos(\lambda x_0)=1$ on $\{f<a\}$ $\nu$-a.e. or that $\lambda x_0=0$ mod. $\pi$. But this is impossible for all $\lambda\in\mathbb{R}$ simultaneously.

That is, there exists a $\lambda\in\mathbb{R}$ such that $\left(fe^{i\lambda x_0}-f\right)1_{\{f<a\}}$ is not centered and this implies that the spectral measure of $e^{i\lambda x_0}-1$ under $Q_{ne}$ possesses an atom at 0. This atom is also in the spectral measure of $e^{i\lambda X_0}-\mathbb{E}\left[e^{i\lambda X_0}\right]$ by Lemma 7.10 and then in the maximal spectral type, which prevents ergodicity.

$\left(\mathbb{R}^{\mathbb{Z}},\mathcal{B}^{\otimes\mathbb{Z}},\mathbb{P}_{Q_{wm}},T\right)$ is a factor of $\left(M_{\mathbb{R}^{\mathbb{Z}}},\mathcal{M}_{\mathcal{B}^{\otimes\mathbb{Z}}},\mathcal{P}_{Q_{wm}},T^{\star}\right)$ which is weakly mixing. We will later show (Theorem 9.27) the absence of mild mixing.

$\left(\mathbb{R}^{\mathbb{Z}},\mathcal{B}^{\otimes\mathbb{Z}},\mathbb{P}_{Q_{mm}},T\right)$ is a factor of $\left(M_{\mathbb{R}^{\mathbb{Z}}},\mathcal{M}_{\mathcal{B}^{\otimes\mathbb{Z}}},\mathcal{P}_{Q_{mm}},T^{\star}\right)$ which is mildly mixing. We will use ID-joining properties to show in section 9 (Lemma 9.17) the absence of mixing.

The rest of the properties are proved in the same way by considering the system as a factor of the corresponding Poisson suspension whose properties such as mixing of all order, and Bernoullicity are inherited by its factors. □

Remark 7.12. The first part of the proof also shows that, if $\mathbb{P}*\mathbb{P}_{Q_{ne}}$ is assumed to be ergodic, $\mathbb{P}_{Q_{ne}}$ is necessarily trivial.

The properties of each member of the factorization together with this last remark lead to the following general theorem proved in a different way by Rosiński and Żak in [**54**].

Theorem 7.13. *If $\mathbb{P}$ is ID and ergodic, then $\mathbb{P}$ is weakly mixing.*

From Proposition 7.11, the hierarchy of "mixing" properties among ergodic IDp processes is explicit. Those processes with a dissipative Lévy measure possessing the strongest mixing behaviour. It is thus not surprising to find in this class the $m$-dependent IDp processes studied by Harrelson and Houdré in [**27**] (the fact that their Lévy measure is dissipative is a direct consequence of Lemma 2.1 page 7 in the same article). Nevertheless, since we have established that a generic Poisson suspension is weakly mixing and rigid, by looking at all the stochastic integrals produced by Poisson suspensions (and then covering all IDp processes), we can say that, in "general", an IDp process is weakly mixing and rigid; i.e. has very poor mixing properties.

## 8. Square integrable ID Processes

Here we consider (with the exception of Proposition 8.2), square integrable IDp processes. To motivate this section, note that if $Q$ is a (shift)-stationary measure on $\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes \mathbb{Z}}\right)$ such that $\int_{\mathbb{R}^{\mathbb{Z}}} x_0^2 Q\left(dx\right) < +\infty$ satisfies $Q\left\{0\right\} = 0$, $Q$ can be considered as the Lévy measure of an ID process which will prove to be square integrable. The family of Lévy measures of this type is hence quite large.

**8.1. Fundamental isometry.** We assume that the process is centered and we denote by $U$ (resp. $V$) the unitary operator associated to $T$ in $L^2\left(\mathbb{P}\right)$ (resp. $L^2\left(Q\right)$) and $\mathfrak{C}_{X_0}\left(\mathbb{P}\right)$ (resp. $\mathfrak{C}_{X_0}\left(Q\right)$) the cyclic subspace associated to $X_0$ in $L^2\left(\mathbb{P}\right)$ (resp. $L^2\left(Q\right)$). We establish the following result:

Proposition 8.1. *$\mathfrak{C}_{X_0}\left(\mathbb{P}\right)$ is unitarily isometric to $\mathfrak{C}_{X_0}\left(Q\right)$, the unitary operators $U$ and $V$ being conjugate.*

Proof. The property comes from the following identities:

$$\left\langle X_k, X_p \right\rangle_{L^2(\mathbb{P})} = \int_{\mathbb{R}^{\mathbb{Z}}} x_k x_p \mathbb{P}\left(dx\right) = \int_{\mathbb{R}^2} uv \mathbb{P}_{(X_k, X_p)}\left(du, dv\right)$$

$$= \int_{\mathbb{R}^2} uv Q_{(X_k, X_p)}\left(du, dv\right) = \int_{\mathbb{R}^{\mathbb{Z}}} x_k x_p Q\left(dx\right) = \left\langle X_k, X_p \right\rangle_{L^2(Q)}$$

That is, if we denote by $\Phi$ the mapping that associates $X_k$ in $L^2\left(\mathbb{P}\right)$ to $X_k$ in $L^2\left(Q\right)$ for all $k \in \mathbb{Z}$, then $\Phi$ can be extended linearly to an isometry between $\mathfrak{C}_{X_0}\left(\mathbb{P}\right)$ and $\mathfrak{C}_{X_0}\left(Q\right)$. The fact that $U$ and $V$ are conjugate relatively to $\Phi$ is obvious. If now, we denote by $\sigma$ the spectral measure associated to $X_0$ under $\mathbb{P}$, and $\Psi$ the unitary isometry between $\mathfrak{C}_{X_0}\left(\mathbb{P}\right)$ and $L^2\left(\sigma\right)$, $\Lambda := \Psi \circ \Phi^{-1}$ defines an unitary isometry between $\mathfrak{C}_{X_0}\left(Q\right)$ and $L^2\left(\sigma\right)$. $\square$

**8.2. Ergodic and mixing criteria.** We recall the Gaussian case, where ergodicity and mixing of the system is determined by the spectral measure of $X_0$:

- The system is ergodic if and only if $\sigma$ is continuous.
- The system is mixing if and only if $\sigma$ is a Rajchman measure.

We then observe that, thanks to Proposition 8.1, such criteria no longer apply for square integrable ID processes. Indeed, taking a probability $Q$ associated to a centered square integrable mixing process, the IDp process with Lévy measure $Q$ is not ergodic by Proposition 7.11, but the spectral measure $\sigma$ of $X_0$ satisfies $\hat{\sigma}\left(k\right) \to 0$ as $|k|$ tends towards infinity. We must then assume some restrictions on the trajectories of the process to draw conclusions on ergodicity and mixing by only looking at the spectral measure of $X_0 - \mathbb{E}\left[X_0\right]$.

We start by a result where integrability suffices.

Proposition 8.2. *Let $X$ be an IDp process of distribution $\mathbb{P}$ such that, up to a possible translation or a change of sign, $X_0$ is non-negative. Then $\mathbb{P}$ is ergodic if and only if $\frac{1}{n}\sum_{k=1}^{n} X_k \to \mathbb{E}\left[X_0\right]$ $\mathbb{P}$-a.s. as $n$ tends to infinity.*

Proof. We suppose that $X_0$ is non-negative and that we have the representation (4) through the Laplace transform, $a \in \mathfrak{A} \cap \mathbb{R}_+^{\mathbb{Z}}$:

$$\mathbb{E}\left[\exp - \langle a, X\rangle\right] = \exp - \left[\int_{\mathbb{R}^{\mathbb{Z}}} 1 - e^{-\langle a,x\rangle} Q\left(dx\right)\right]$$

Suppose that $\frac{1}{n}\sum_{k=1}^{n} X_k \to \mathbb{E}\left[X_0\right]$ as $n$ tends to infinity $\mathbb{P}$-a.s. without ergodicity of $\mathbb{P}$. The decomposition of $\mathbb{P}$ is of the type $\mathbb{P}_e * \mathbb{P}_{Q_{ne}}$ where $\mathbb{P}_e$ is ergodic. Let $X^{ne}$ be of distribution $\mathbb{P}_{Q_{ne}}$ and $X^e$ be of distribution $\mathbb{P}_e$, assumed independent, such that $X^{ne} + X^e$ is of distribution $\mathbb{P}$.

The fact that $\frac{1}{n}\sum_{k=1}^{n}\left[\left(X^{ne} + X^e\right)_n\right] \to \mathbb{E}\left[X_0^{ne}\right] + \mathbb{E}\left[X_0^e\right]$ implies:

$$\frac{1}{n}\sum_{k=1}^{n} X_n^{ne} \to \mathbb{E}\left[X_0^{ne}\right]$$

Hence, using

$$\mathbb{E}_{Q_{ne}}\left[\exp - \frac{1}{n}\sum_{k=1}^{n} X_k\right] = \exp - \left[\int_{\mathbb{R}^{\mathbb{Z}}} 1 - \exp\left[-\frac{1}{n}\sum_{k=1}^{n} x_k\right] Q_{ne}\left(dx\right)\right]$$

we note that the term of the left hand side tends to $\exp - \mathbb{E}_{Q_{ne}}\left[X_0\right]$ by dominated convergence and, by continuity of the exponential, we then have:

$$\int_{\mathbb{R}^{\mathbb{Z}}} 1 - \exp\left[-\frac{1}{n}\sum_{k=1}^{n} x_k\right] Q_{ne}\left(dx\right) \to \mathbb{E}_{Q_{ne}}\left[X_0\right] \tag{5}$$

Under this representation, we also know, by 4, that:

$$\mathbb{E}_{Q_{ne}}\left[X_0\right] = \int_{\mathbb{R}^{\mathbb{Z}}} x_0 Q_{ne}\left(dx\right)$$

Now consider, the probability $\nu$ which is $T$-invariant and equivalent to $Q_{ne}$ and let $f := \frac{dQ_{ne}}{d\nu}$ ($f$ is $T$-invariant).

$f x_0$ is $\nu$-integrable and we can apply the Birkhoff ergodic theorem to deduce that $\frac{1}{n}\sum_{k=1}^{n} f \circ T^k x_k = f\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)$ converges $\nu$-a.e. and in $L^1\left(\nu\right)$ to the conditional expectation of $f x_0$ with respect to the invariant $\sigma$-field which we denote by $\nu\left(f x_0 | \mathcal{I}\right)$. But, since $f$ is $T$-invariant and non negative, $\nu\left(f x_0|\mathcal{I}\right) = f \nu\left(x_0|\mathcal{I}\right)$ that is, by dividing by $f$, $\frac{1}{n}\sum_{k=1}^{n} x_k$ converges $\nu$-a.e. to $\nu\left(x_0|\mathcal{I}\right)$.

Since $\left(1-\exp\left[-\frac{1}{n}\sum_{k=1}^{n}x_k\right]\right)f \le f\left(\frac{1}{n}\sum_{k=1}^{n}x_k\right)$ and by using the fact that $f\left(\frac{1}{n}\sum_{k=1}^{n}x_k\right)$ converges in $L^1(\nu)$, the sequence $\left(1-\exp\left[-\frac{1}{n}\sum_{k=1}^{n}x_k\right]\right)f$ is uniformly integrable and, since it tends $\nu$-a.e. to $(1-\exp[-\nu(x_0|\mathcal{I})])$ we observe that

$$\int_{\mathbb{R}^{\mathbb{Z}}} 1-\exp\left[-\frac{1}{n}\sum_{k=1}^{n}x_k\right] Q_{ne}(dx) = \int_{\mathbb{R}^{\mathbb{Z}}}\left(1-\exp\left[-\frac{1}{n}\sum_{k=1}^{n}x_k\right]\right)f\nu(dx)$$

tends, as $n$ tends to infinity, to

$$\int_{\mathbb{R}^{\mathbb{Z}}}(1-\exp[-\nu(x_0|\mathcal{I})])\,f\nu(dx)$$

But since $x_0 \ge 0$ and $Q_{ne}\{0\} = 0$ (and then $\nu\{0\} = 0$), we have $\nu(x_0|\mathcal{I}) > 0$ $\nu$-a.e. thus:

$$\int_{\mathbb{R}^{\mathbb{Z}}}(1-\exp[-\nu(x_0|\mathcal{I})])\,f\nu(dx) < \int_{\mathbb{R}^{\mathbb{Z}}}\nu(x_0|\mathcal{I})\,f\nu(dx) = \int_{\mathbb{R}^{\mathbb{Z}}}x_0 f\nu(dx)$$

that is, the limit, as $n$ tends to infinity of $\int_{\mathbb{R}^{\mathbb{Z}}}1-\exp\left[-\frac{1}{n}\sum_{k=1}^{n}x_k\right]Q_{ne}(dx)$ is strictly less than $\int_{\mathbb{R}^{\mathbb{Z}}}x_0 Q_{ne}(dx)$. This contradicts (5), there is no term of the form $\mathbb{P}_{Q_{ne}}$ in the factorization of $\mathbb{P}$ and $\mathbb{P}$ is thus ergodic. □

We can now prove a proposition for square integrable processes:

PROPOSITION 8.3. *Let $X$ be an IDp process of distribution $\mathbb{P}$ such that, up to a possible translation or a change of sign, $X_0$ is non-negative. Let $\sigma$ be the spectral measure of $X_0 - \mathbb{E}[X_0]$.*
*$\mathbb{P}$ is ergodic if and only if $\sigma\{0\} = 0$.*
*$\mathbb{P}$ is mixing if and only if $\sigma$ is a Rajchman measure.*

PROOF. We know that $\sigma\{0\}$ equals the variance of $\mathbb{E}[X_0|\mathcal{I}]$. Moreover, the Birkhoff ergodic theorem tells us that $\frac{1}{n}\sum_{k=1}^{n}X_k \to \mathbb{E}[X_0|\mathcal{I}]$ $\mathbb{P}$-a.s.. Thus, if $\sigma\{0\} = 0$, $\mathbb{E}[X_0|\mathcal{I}]$ is constant and equals $\mathbb{E}[X_0]$, so we can apply Proposition 8.2 to conclude. Now if $\sigma$ is a Rajchman measure, by the isometry, $\sigma$ is also the spectral measure of $X_0$ under $Q$ and we get $\int_{\mathbb{R}^{\mathbb{Z}}}x_0x_nQ(dx) \to 0$ as $n$ tends to infinity and we can apply the mixing criterion established by Rosiński and Żak in [**53**] (Corollary 3 page 282). □

## 9. Infinitely divisible joinings

We study first bivariate ID processes $\{X_n^1, X_n^2\}_{n\in\mathbb{Z}}$.

**9.1. Bivariate ID processes.** Here we consider the product space of bivariate sequences $\left(\mathbb{R}^{\mathbb{Z}} \times \mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}} \otimes \mathcal{B}^{\otimes\mathbb{Z}}\right)$. Infinite divisibility on this space is defined in an obvious way. We endow this space with the product shift transformation $T \times T$ and we will say a bivariate process is stationary if its distribution is $T \times T$-invariant. The distribution of a bivariate stationary ID process is still determined by a generating triplet $\left\langle R^b, (c^1, c^2), Q^b \right\rangle$ identified by the characteristic functions of its finite-dimensional distributions:

$\mathbb{E}\left[\exp i\left[\langle a, X^1\rangle + \langle b, X^2\rangle\right]\right]$

$= \exp[-\frac{1}{2}\left\langle R^b(a,b), (a,b)\right\rangle + i\langle a, c_1\rangle + i\langle b, c_2\rangle$

$$+\int_{\mathbb{R}^{\mathbb{Z}}\times\mathbb{R}^{\mathbb{Z}}} e^{i[\langle a,x_1\rangle+\langle b,x_2\rangle]} - 1 - i\langle c(x), (a,b)\rangle\, Q^b\left(dx^1, dx^2\right)] \tag{6}$$

where $R^b$ is a covariance function of a bivariate Gaussian process, $(c^1, c^2)$ is an element of $\mathbb{R}^{\mathbb{Z}} \times \mathbb{R}^{\mathbb{Z}}$ whose coordinates all equal a constant $(d^1, d^2) \in \mathbb{R}^2$ and $Q^b$, a Lévy measure on $\left(\mathbb{R}^{\mathbb{Z}} \times \mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}} \otimes \mathcal{B}^{\otimes\mathbb{Z}}\right)$, invariant under the action of $T \times T$ and satisfying $Q^b\{0,0\} = 0$.

**9.2. ID-joinings and ID-disjointness.** First, we have to define the notion of joining between dynamical systems associated to possibly infinite measures (see [**1**], page 264).

Definition 9.1. Let $(\Omega_1, \mathcal{F}_1, \mu_1, T_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2, T_2)$ be two dynamical systems and let $0 < c_1 < +\infty$ and $0 < c_2 < +\infty$.

A $(c_1, c_2)$-*joining* is a dynamical system $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu, T_1 \times T_2)$ where $\mu$ is $T_1 \times T_2$-invariant, $\pi_1^\star(\mu) = c_1\mu_1$, $\pi_2^\star(\mu) = c_2\mu_2$ (where $\pi_1$ and $\pi_2$ are the canonical projections on $\Omega_1$ and $\Omega_2$).

Contrary to us, Aaronson allows one of $c_1$ and $c_2$ to be infinite.

To simplify, we will say that $\mu$ is a $(c_1, c_2)$-joining of $\mu_1$ and $\mu_2$. A joining is then a $(1,1)$-joining. If there doesn't exist a $(c_1, c_2)$-joining of $\mu_1$ and $\mu_2$, the systems are said to be *strongly disjoint*, and *similar* otherwise. If the systems considered are associated to probability measures, there are only $(1,1)$-joinings and the product measure is always available. If it is the only one, the systems are said to be *disjoint*.

In the particular case we are interested in, we look at joinings between $\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, \mathbb{P}_1, T\right)$ and $\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, \mathbb{P}_2, T\right)$ associated to $X^1$ and $X^2$, two ID processes such that the bivariate process $(X^1, X^2)$ is still ID under $\nu$. We thus have a first definition:

Definition 9.2. An *ID-joining* of $\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, \mathbb{P}_1, T\right)$ and $\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, \mathbb{P}_2, T\right)$, is any joining $\left(\mathbb{R}^{\mathbb{Z}} \times \mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}} \otimes \mathcal{B}^{\otimes\mathbb{Z}}, \nu, T \times T\right)$ such that $\nu$ is ID.

This is the generalization of Gaussian joinings (joinings of Gaussian processes such that the bivariate process is still Gaussian). Note that the notions of ID and Gaussian joinings coincide in the Gaussian case (immediate from (6)).

REMARK 9.3. We can endow the set of joinings of the two systems $\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, \mathbb{P}_1, T\right)$ and $\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, \mathbb{P}_2, T\right)$ with a topology given by the following distance that makes this set separable, complete and compact (see Glasner [**20**]) :

$$d\left(\nu, \mu\right) = \sum_{n,m\in\mathbb{N}} \frac{\left|\nu\left(A_n \times A_m\right) - \mu\left(A_n \times A_m\right)\right|}{a_{n,m}}$$

where the $A_n$ are Borel cylinder sets that generate $\mathcal{B}^{\otimes\mathbb{Z}}$ and such that $a_{n,m} > 0$ and $\sum_{n,m\in\mathbb{N}} \frac{1}{a_{n,m}} < +\infty$.

Since it is clear that the convergence of a sequence of joinings with respect to this topology implies convergence in distribution of the finite dimensional distributions, the set of ID-joinings is a compact subset (infinite divisibility is preserved under convergence in distribution).

DEFINITION 9.4. $\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, \mathbb{P}_1, T\right)$ and $\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, \mathbb{P}_2, T\right)$, two ID processes, are said to be *ID-disjoint* if their only ID-joining is the independent joining, if not, we say that they are *ID-similar*.

We have a first result:

PROPOSITION 9.5. *A Gaussian process and an IDp process are ID-disjoint.*

PROOF. This is a direct consequence of the following property:

Let $(V_1, V_2)$ be an ID vector in $\mathbb{R}^2$ where $V_1$ is Gaussian and $V_2$ IDp, then $V_1$ and $V_2$ are independent. Indeed, from the Lévy-Khintchine representation, there exists a Gaussian vector $(V_1^1, V_2^1)$ and an IDp vector $(V_1^2, V_2^2)$ independent of $(V_1^1, V_2^1)$ such that $(V_1^1 + V_1^2, V_2^1 + V_2^2)$ has the distribution of $(V_1, V_2)$. But the Lévy-Khintchine representation tells us that $V_2^1 + V_2^2$ has the distribution of $V_2$ which is possible if and only if $V_2^1$ is zero and, in the same way, $V_1^1 + V_1^2$ has the distribution of $V_1$ if and only if $V_1^2$ is zero. □

An ID process generates a whole family of ID-joinings. Indeed, let $X$ be an ID process and let $Y := \{\langle a_\infty, X\rangle \circ T^n\}_{n\in\mathbb{Z}}$ be a generalized linear factor of $X$. We already know that this factor was ID, but we have more: the bivariate process $(X, Y)$ is ID (immediate verification) and thus generates an ID-joining. From the preceding proposition, we have:

PROPOSITION 9.6. *A generalized linear factor of an IDp process is never Gaussian.*

PROOF. Assume that $X$ is an IDp process and $Y$ a generalized linear factor of $X$. If $Y$ is Gaussian, since $(X, Y)$ is ID, $Y$ is necessarily independent from $X$, from Proposition 9.5. Since $Y$ is measurable with respect to $X$, this implies that $Y$ is constant. □

Among Gaussian processes, we have:

PROPOSITION 9.7. *Two Gaussian processes are ID-disjoint if and only if their respective spectral measures are mutually singular.*

PROOF. If the spectral measures $\sigma_1$ and $\sigma_2$ of the Gaussian processes $X_1$ and $X_2$ are mutually singular and $(X_1, X_2)$ is Gaussian, then the cyclic spaces associated with $X_1$ and $X_2$ are orthogonal in the Gaussian space spanned by $(X_1, X_2)$. Orthogonality implies independence in a Gaussian space, so $X_1$ is independent of $X_2$.

Conversely if $\sigma_1$ and $\sigma_2$ are not mutually singular, then, constructing the Gaussian process with spectral measure $\sigma_1 + \sigma_2$, we easily construct, thanks to the unitary isometry between the Gaussian space and $L^2\left([-\pi, \pi[\,, \sigma_1 + \sigma_2\right)$, two Gaussian processes with respective spectral measures $\sigma_1$ and $\sigma_2$ in such a way that the bivariate process is still Gaussian. However, they are not independent because their Gaussian spaces are both included in the Gaussian space of the initial process, and, as cyclic space, orthogonality implies singularity of the spectral measures (see Proposition 3.1). □

**9.3. IDp-joinings.** We concentrate now on IDp-processes. We prove a very simple lemma that will help us in understanding IDp-joinings.

LEMMA 9.8. *Canonical decomposition of the Lévy measure of a bivariate IDp process: Let $Q^b$ be a Lévy measure on $\left(\mathbb{R}^{\mathbb{Z}} \times \mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}} \otimes \mathcal{B}^{\otimes\mathbb{Z}}\right)$. Then there exist two Lévy measures $Q'$ and $Q''$ on $\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}\right)$ and a Lévy measure $Q^b_r$ on $\left(\mathbb{R}^{\mathbb{Z}} \times \mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}} \otimes \mathcal{B}^{\otimes\mathbb{Z}}\right)$ such that $Q^b = Q' \otimes \delta_{\{0\}} + Q^b_r + \delta_{\{0\}} \otimes Q''$ and $Q^b_r\left(\mathbb{R}^{\mathbb{Z}} \times \{0\} \cup \{0\} \times \mathbb{R}^{\mathbb{Z}}\right) = 0$. $Q^b_r$ will be called the* reduction *of $Q^b$. Denoting by $Q^{b,1}$ and $Q^{b,2}$ (resp. $Q^{b,1}_r$ and $Q^{b,2}_r$) the marginals of $Q^b$ (resp. $Q^b_r$) on $\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}\right)$, we have $Q^{b,1}_{|\{0\}^c} = Q' + Q^{b,1}_r$ and $Q^{b,2}_{|\{0\}^c} = Q'' + Q^{b,2}_r$ .*

*In terms of processes, if $(X, Y)$ is a bivariate ID process with Lévy measure $Q^b$. Then its distribution is the distribution of the independent sum of $(X', \{0\})$, $(X^1, X^2)$ and $(\{0\}, X'')$ where $Q'$ is the Lévy measure of $X'$, $Q''$ is the Lévy measure of $X''$ and $Q^b_r$ is the Lévy measure of $(X^1, X^2)$.*

PROOF. This follows by decomposing $\left(\mathbb{R}^{\mathbb{Z}} \times \mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}} \otimes \mathcal{B}^{\otimes\mathbb{Z}}\right)$ along $T \times T$-invariant sets $\mathbb{R}^{\mathbb{Z}} \times \{0\}$, $\{0\} \times \mathbb{R}^{\mathbb{Z}}$ and $\left(\mathbb{R}^{\mathbb{Z}} \times \{0\} \cup \{0\} \times \mathbb{R}^{\mathbb{Z}}\right)^c$. □

REMARK 9.9. In the above decomposition $Q^b_r$ is a joining between $Q^{b,1}_r$ and $Q^{b,2}_r$ but $Q^b$ is not a joining between $Q' + Q^{b,1}_r$ and $Q'' + Q^{b,2}_r$ unless $Q'$ and $Q''$ are zero measures.

Now consider two Lévy measures $Q_1$ and $Q_2$ and suppose there exists a $(c_1, c_2)$-joining denoted by $Q^b$. Then $Q^b$ is the Lévy measure of an ID bivariate process, indeed:
$Q^b\{0,0\} = 0$ since $Q^b\{0,0\} \leq Q^b\left(\{0\} \times \mathbb{R}^{\mathbb{Z}}\right) = c_1 Q_1\{0\} = 0$
$\int_{\mathbb{R}^{\mathbb{Z}} \times \mathbb{R}^{\mathbb{Z}}} \left(x_0^2 + y_0^2\right) \wedge 1 Q^b(dx, dy) < +\infty$ since
$\int_{\mathbb{R}^{\mathbb{Z}} \times \mathbb{R}^{\mathbb{Z}}} \left(x_0^2 + y_0^2\right) \wedge 1 Q^b(dx, dy) \leq \int_{\mathbb{R}^{\mathbb{Z}} \times \mathbb{R}^{\mathbb{Z}}} \left(x_0^2\right) \wedge 1 Q^b(dx, dy) + \int_{\mathbb{R}^{\mathbb{Z}} \times \mathbb{R}^{\mathbb{Z}}} \left(y_0^2\right) \wedge 1 Q^b(dx, dy)$
and finally:
$\int_{\mathbb{R}^{\mathbb{Z}} \times \mathbb{R}^{\mathbb{Z}}} \left(x_0^2\right) \wedge 1 Q^b(dx, dy) = c_1 \int_{\mathbb{R}^{\mathbb{Z}}} \left(x_0^2\right) \wedge 1 Q_1(dx) < +\infty$
and $\int_{\mathbb{R}^{\mathbb{Z}} \times \mathbb{R}^{\mathbb{Z}}} \left(y_0^2\right) \wedge 1 Q^b(dx, dy) = c_2 \int_{\mathbb{R}^{\mathbb{Z}}} \left(y_0^2\right) \wedge 1 Q_2(dy) < +\infty$.

DEFINITION 9.10. An ID-joining will be called *reduced*, if, in the decomposition in Lemma 9.8, $Q'$ and $Q''$ are zero. In particular, a pure ID-joining is necessarily reduced.

**9.4. ID-joinings of Poisson suspensions.** Inspired by the structure of IDp-joining for processes, we can propose a definition of ID-joinings of Poisson suspensions. We first investigate the ID nature of a Poisson suspension.

9.4.1. *ID nature of a Poisson suspension.* Let $(\Omega, \mathcal{F})$ be a Borel space and consider the space $(M_\Omega, \mathcal{M}_\mathcal{F})$ of counting Radon measures on $(\Omega, \mathcal{F})$. The sum of two elements is well defined as the classical sum of two measures. The convolution operation $*$ is thus well defined for distribution on $(M_\Omega, \mathcal{M}_\mathcal{F})$, and so is infinite divisibility. Obviously, if $\mu$ is a $\sigma$-finite measure on $(\Omega, \mathcal{F})$, the distribution of the suspension $\mathcal{P}_\mu$ is ID since for all $k \geq 1$ $\mathcal{P}_\mu = \mathcal{P}_{\frac{1}{k}\mu}^{*k}$. This makes the intrinsic ID nature of a Poisson measure precise. As such, it admits a Lévy measure $Q_\mu$ (see [**41**]), that is, a measure on $(M_\Omega, \mathcal{M}_\mathcal{F})$ and it is not difficult to see that this is the image measure of $\mu$ by the application $x \mapsto \delta_x$ by looking at the Laplace transform of this point process:

$$\mathbb{E}\left[\exp\left(-\lambda \int_\Omega f dN\right)\right] = \exp\left[\int_\Omega \left(e^{-\lambda f} - 1\right) d\mu\right]$$

$$= \exp\left[\int_{M_\Omega} \left(\exp -\lambda \int_\Omega f dN\right) - 1 Q_\mu \left(dN\right)\right]$$

This Lévy measure $Q_\mu$ puts mass only on sets containing one-point Dirac measures, which are "randomized" according to $\mu$. By considering the subset $(M_\Omega^1, \mathcal{M}_\mathcal{F}^1)$ of $(M_\Omega, \mathcal{M}_\mathcal{F})$ that supports the Lévy measure, the Dirac measures on points of $(\Omega, \mathcal{F})$, the application $x \mapsto \delta_x$ defines a bijective map between $(\Omega, \mathcal{F})$ and $(M_\Omega^1, \mathcal{M}_\mathcal{F}^1)$. We can form the Poisson measure $\left(M_{M_\Omega^1}, \mathcal{M}_{\mathcal{M}_\mathcal{F}^1}, \mathcal{P}_{Q_\mu}\right)$ above $(M_\Omega, \mathcal{M}_\mathcal{F}, Q_\mu)$. If we assume that $(\Omega, \mathcal{F}, \mu, T)$ is a dynamical system, $(\Omega, \mathcal{F}, \mu, T)$ is isomorphic to $(M_\Omega, \mathcal{M}_\mathcal{F}, Q_\mu, T^\star)$ and, thanks to Theorem 4.4, we can see that $(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_\mu, T^\star)$ is isomorphic to $\left(M_{M_\Omega}, \mathcal{M}_{\mathcal{M}_\mathcal{F}}, \mathcal{P}_{Q_\mu}, [T^\star]^\star\right)$ by a Poissonian isomorphism.

9.4.2. *ID-joinings.* We now look at infinite divisibility on $(M_{\Omega_1} \times M_{\Omega_2}, \mathcal{M}_{\mathcal{F}_1} \otimes \mathcal{M}_{\mathcal{F}_2})$. The sum is defined in the obvious way by the formula $(\mu_1, \mu_2) + (\nu_1, \nu_2) = (\mu_1 + \nu_1, \mu_2 + \nu_2)$. Thus the definition of ID-joinings and ID-disjointness of Poisson suspensions arises naturally:

Definition 9.11. Let $(M_{\Omega_1}, \mathcal{M}_{\mathcal{F}_1}, \mathcal{P}_{\mu_1}, T_1^\star)$ and $(M_{\Omega_2}, \mathcal{M}_{\mathcal{F}_2}, \mathcal{P}_{\mu_2}, T_2^\star)$ be two Poisson suspensions.

We call a joining $(M_{\Omega_1} \times M_{\Omega_2}, \mathcal{M}_{\mathcal{F}_1} \otimes \mathcal{M}_{\mathcal{F}_2}, \mathfrak{P}, T_1^\star \times T_2^\star)$ an ID-joining if $\mathfrak{P}$ is ID. If the only ID-joining is the independent joining, we say that the systems $(M_{\Omega_1}, \mathcal{M}_{\mathcal{F}_1}, \mathcal{P}_{\mu_1}, T_1^\star)$ and $(M_{\Omega_2}, \mathcal{M}_{\mathcal{F}_2}, \mathcal{P}_{\mu_2}, T_2^\star)$ are *ID-disjoint*. Otherwise they are said to be *ID-similar*. An ID-joining of Poisson suspension is said to be *pure* if its Lévy measure is ergodic.

As in Remark 9.3, it can be noted that the sets of ID-joinings between suspensions is a compact subset of joinings for the weak topology.

We now elucidate the structure of such a joining which is completely similar to ID-joinings of ID-processes.

$\mathfrak{Q}$, the Lévy measure of $\mathfrak{P}$ is a measure on $(M_{\Omega_1} \times M_{\Omega_2}, \mathcal{M}_{\mathcal{F}_1} \otimes \mathcal{M}_{\mathcal{F}_2})$ invariant with respect to $T_1^\star \times T_2^\star$ which can thus be decomposed on the following three invariant subsets $A := M_{\Omega_1} \times \{0\}$, $B := \{0\} \times M_{\Omega_2}$ and $C := \{M_{\Omega_1} \times \{0\} \cup \{0\} \times M_{\Omega_2}\}^c$, where $\{0\}$ denotes the zero measure on both $M_{\Omega_1}$ and $M_{\Omega_2}$. There exists $\nu_1$ and $\nu_2$ such that $\mathfrak{Q}_{|A} = Q_{\nu_1} \otimes \delta_{\{0\}}$ and $\mathfrak{Q}_{|B} = \delta_{\{0\}} \otimes Q_{\nu_2}$ and if $\sigma_1 := \mu_1 - \nu_1$ and $\sigma_2 := \mu_2 - \nu_2$, $\mathfrak{Q}_{|C}$ is a $(1,1)$-joining between $Q_{\sigma_1}$ and $Q_{\sigma_2}$. This joining is canonically isomorphic to a joining $\sigma$ of $\sigma_1$ and $\sigma_2$ by the application $(\delta_x, \delta_y) \mapsto (x, y)$ from $\left(M^1_{\Omega_1} \times M^1_{\Omega_2}, \mathcal{M}^1_{\mathcal{F}_1} \otimes \mathcal{M}^1_{\mathcal{F}_2}\right)$ to $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$.

This discussion shows that we can describe an ID-joining of two Poisson suspension without making reference to their Lévy measure. We summarize it in a lemma:

Lemma 9.12. *Let $(M_{\Omega_1}, \mathcal{M}_{\mathcal{F}_1}, \mathcal{P}_{\mu_1}, T_1^\star)$ and $(M_{\Omega_2}, \mathcal{M}_{\mathcal{F}_2}, \mathcal{P}_{\mu_2}, T_2^\star)$ be two Poisson suspensions. Let $(M_{\Omega_1} \times M_{\Omega_2}, \mathcal{M}_{\mathcal{F}_1} \otimes \mathcal{M}_{\mathcal{F}_2}, \mathfrak{P}, T_1^\star \times T_2^\star)$ be an ID-joining between them. Consider the bivariate Poisson measure $\left(N, \tilde{N}\right)$ of distribution $\mathfrak{P}$. This distribution is the distribution of the independent sum of $(N^1, \{0\})$, $\left(N', \tilde{N}''\right)$ and $\left(\{0\}, \tilde{N}^2\right)$, where $N^1$ (resp. $\tilde{N}^2$) is a Poisson suspension associated with $(\Omega_1, \mathcal{F}_1, \mu_1, T_1)$ (resp. $(\Omega_2, \mathcal{F}_2, \mu_2, T_2)$), and if $\mathfrak{N}$ is a Poisson suspension associated with $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \sigma, T_1 \times T_2)$, $\left(N', \tilde{N}''\right)$ has the distribution of $\left(\mathfrak{N} \circ \pi_1^{-1}(.), \mathfrak{N} \circ \pi_2^{-1}(.)\right)$, i.e. the bivariate Poisson measure coming from the marginals of the Poisson measure $\mathfrak{N}$. We denote this distribution by $\tilde{\mathcal{P}}_\sigma$. We can write $\mathfrak{P} = (\mathcal{P}_{\nu_1} \otimes \mathcal{P}_0) * \tilde{\mathcal{P}}_\sigma * (\mathcal{P}_0 \otimes \mathcal{P}_{\nu_2})$ where $\mathcal{P}_0$ denotes the distribution associated with the zero measure.*

Corollary 9.13. *Keeping the notations of the preceding lemma, an ID-joining between $(M_{\Omega_1}, \mathcal{M}_{\mathcal{F}_1}, \mathcal{P}_{\mu_1}, T_1^\star)$ and $(M_{\Omega_2}, \mathcal{M}_{\mathcal{F}_2}, \mathcal{P}_{\mu_2}, T_2^\star)$ is pure if and only if $\sigma$ is ergodic.*

It is now clear that ID-joinings of Poisson suspensions generalize ID-joinings of processes. Each ID-joining is a natural factor of the ID-joining of the Poisson suspensions constructed above the Lévy measure of the two processes.

Definition 9.14. Let $(X_1, X_2)$ be an IDp process with a Lévy measure $Q^b$ that decomposes into $Q_1 \otimes \delta_{\{0\}} + Q_r^b + \delta_{\{0\}} \otimes Q_2$. By Lemma 9.8, the *Poissonian extension* of $(X_1, X_2)$ is the ID-joining of Poisson suspensions described by the preceding lemma, i.e. whose distribution is the independent sum of $(N^1, \{0\})$, $\left(N', \tilde{N}''\right)$ and $\left(\{0\}, \tilde{N}^2\right)$, where $N^1$ (resp. $\tilde{N}^2$) is a Poisson suspension associated with $\left(\mathbb{R}^\mathbb{Z}, \mathcal{B}^{\otimes\mathbb{Z}}, Q_1, T\right)$ (resp. $\left(\mathbb{R}^\mathbb{Z}, \mathcal{B}^{\otimes\mathbb{Z}}, Q_2, T\right)$), and $\left(N', \tilde{N}''\right)$ has the distribution $\tilde{\mathcal{P}}_{Q_r^b}$.

Corollary 9.15. *Two IDp processes with respective Lévy measures $Q_1$ an $Q_2$ are ID-disjoint (resp. ID-similar) if and only if the corresponding suspensions constructed above $Q_1$ and $Q_2$ are ID-disjoint (resp. ID-similar).*

and also:

Corollary 9.16. *An ID-joining of two IDp processes is pure if and only if its Poissonian extension is pure.*

The close link between an ID-joining of two IDp processes and its Poissonian extension is illustrated by the following lemma, which has been announced in the proof of Proposition 7.11:

LEMMA 9.17. *An IDp process is mixing if and only if the Poisson suspension built above the Lévy measure of the process is mixing.*

PROOF. Let $(M_{\mathbb{R}^{\mathbb{Z}}}, \mathcal{M}_{\mathcal{B}^{\otimes\mathbb{Z}}}, \mathcal{P}_Q, T^\star)$ be the Poisson suspension associated to the Lévy measure $Q$ of a mixing IDp process. Consider the sequence $\{\mu_n\}_{n\in\mathbb{N}}$ of off-diagonal joinings defined by $\mu_n(A\times B) = \mathcal{P}_Q(A\cap T^{\star-n}B)$. $\{\mu_n\}_{n\in\mathbb{N}}$ is thus a sequence of ID-self-joinings of the suspension. Consider a subsequence of $\{\mu_n\}_{n\in\mathbb{N}}$. Since the set of ID-self-joinings is compact, this subsequence possesses a sub-subsequence which tends to an ID-self-joinings of the suspension. This ID-self-joining is the Poissonian extension of an ID-self-joining of the IDp process with Lévy measure $Q$. But, since this process is mixing, the self-joining is the independent joining, that is the ID-joining with Lévy measure $Q\otimes\delta_{\{0\}}+\delta_{\{0\}}\otimes Q$. This implies that the corresponding Poissonian extension is also the independent joining. This proves that $\{\mu_n\}_{n\in\mathbb{N}}$ tends to the independent joining that is, the Poisson suspension $(M_{\mathbb{R}^{\mathbb{Z}}}, \mathcal{M}_{\mathcal{B}^{\otimes\mathbb{Z}}}, \mathcal{P}_Q, T^\star)$ is mixing (we have used the fact that, if there exist a sequence $\{x_n\}_{n\in\mathbb{N}}$ and $y$ such that each subsequence of $\{x_n\}_{n\in\mathbb{N}}$ possesses a sub-subsequence which tends to $y$, then $\{x_n\}_{n\in\mathbb{N}}$ tends to $y$). □

Finally, we show that Poissonian automorphisms can be defined in terms of ID-joinings:

PROPOSITION 9.18. *Let $(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_\mu)$ be a Poisson measure constructed above the space $(\Omega, \mathcal{F}, \mu)$ and $S$ an automorphism of $(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_\mu)$ such that the graph joining given by $G(A\times B) = \mathcal{P}_\mu(A\cap S^{-1}B)$ is an ID-joining. Then $S$ is a Poissonian automorphism, that is, there exists an automorphism $T$ of $(\Omega, \mathcal{F}, \mu)$ such that $S = T^\star$ mod. $\mathcal{P}_\mu$.*

PROOF. Consider the ID-joining $G$ and use the notations of Lemma 9.12 (where $(\Omega_1, \mathcal{F}_1, \mu_1) = (\Omega_2, \mathcal{F}_2, \mu_2) = (\Omega, \mathcal{F}, \mu)$). On factorizable vectors $\mathcal{E}_h$ and $\mathcal{E}_g$ we have the relation:

$$\mathbb{E}_{\mathcal{P}_\mu}[\mathcal{E}_h\mathcal{E}_g\circ S] = \mathbb{E}_{\mathcal{P}_{\nu_1}}[\mathcal{E}_h]\,\mathbb{E}_{\mathcal{P}_\sigma}[\mathcal{E}_{h\otimes 1}\mathcal{E}_{1\otimes g}]\,\mathbb{E}_{\mathcal{P}_{\nu_2}}[\mathcal{E}_g]$$

We can define a positive operator $J$ on $L^2(\mu)$ through the identity:

$$\langle Jh, g\rangle_{L^2(\mu)} = \int_{\Omega\times\Omega} h(x)\,g(y)\,\sigma(dx,dy)$$

Now it is easy to see that the exponential of $J$, $\tilde{J}$, is indeed $U_S$:

$\left\langle \tilde{J}\mathcal{E}_h, \mathcal{E}_g\right\rangle_{L^2(\mathcal{P}_\mu)} = \langle\mathcal{E}_{Jh}, \mathcal{E}_g\rangle_{L^2(\mathcal{P}_\mu)} = \exp\int_{\Omega\times\Omega} h(x)\,g(y)\,\sigma(dx,dy)$

$= \mathbb{E}_{\mathcal{P}_{\nu_1}}[\mathcal{E}_h]\,\mathbb{E}_{\mathcal{P}_\sigma}[\mathcal{E}_{h\otimes 1}\mathcal{E}_{1\otimes g}]\,\mathbb{E}_{\mathcal{P}_{\nu_2}}[\mathcal{E}_g]$

Now, since $\tilde{J} = U_S$ is an invertible isometry, its restriction to the first chaos, identified to $J$, is an invertible isometry. But a positive invertible isometry of this kind is necessarily induced by an automorphism. So, there exists $T$ such that $Jh = h\circ T$. Finally, it implies that $T^\star = S$. □

**9.5. Joinings/strong disjointness and ID-joinings/ID-disjointness.** The main result of this section is to show that the five classes of IDp processes identified by the canonical factorization of Theorem 7.8 are ID-disjoint. To achieve that, we need general results concerning the strong disjointness of dynamical systems, which, as we will see, is closely linked to ID-disjointness. We will need the following lemma:

LEMMA 9.19. *Let* $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu, T_1 \times T_2)$ *be a* $(c_1, c_2)$*-joining of* $(\Omega_1, \mathcal{F}_1, \mu_1, T_1)$ *with* $(\Omega_2, \mathcal{F}_2, \mu_2, T_2)$*, both of type* $\mathbf{II}_\infty$*. Then* $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu, T_1 \times T_2)$ *is also of type* $\mathbf{II}_\infty$*. In particular, if two Poisson suspensions are ergodic, the suspension constructed above a joining of the two initial dynamical systems is ergodic.*

PROOF. Assume there exists a $T_1 \times T_2$-invariant probability $p$ such that $p \ll \mu$ and consider the image of this probability by the first projection, that is $\pi_1^*(p)$ defined on $(\Omega_1, \mathcal{F}_1, T_1)$. $\pi_1^*(p)$ is $T_1$-invariant, moreover, let $A \in \mathcal{F}_1$ have zero $\mu_1$-measure, then $\pi_1^*(p)(A) = p\left(\pi_1^{-1}(A)\right) = p(A \times \Omega_2) = 0$ since $p$ is absolutely continuous with respect to $\mu$ and since $\mu(A \times \Omega_2) = c_1\mu_1(A) = 0$. Thus, $\pi_1^*(p)$ is absolutely continuous with respect to $\mu_1$ which contradicts the fact that $(\Omega_1, \mathcal{F}_1, \mu_1, T_1)$ is of type $\mathbf{II}_\infty$. ☐

The content of this lemma will be further developed, in Theorem 9.27. We have the following links between strong disjointness and ID-disjointness. The first proposition is immediate:

PROPOSITION 9.20. *If two dynamical systems are similar, their associated Poisson suspensions are ID-similar.*

We have the partial but crucial converse:

THEOREM 9.21. *Two ergodic dynamical systems are strongly disjoint if and only if their associated Poisson suspensions are ID-disjoint.*

PROOF. Suppose $(\Omega_1, \mathcal{F}_1, \mu_1, T_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2, T_2)$ are two strongly disjoint ergodic systems and suppose there exists an ID joining between the suspensions $(M_{\Omega_1}, \mathcal{M}_{\mathcal{F}_1}, \mathcal{P}_{\mu_1}, T_1^\star)$ and $(M_{\Omega_2}, \mathcal{M}_{\mathcal{F}_2}, \mathcal{P}_{\mu_2}, T_2^\star)$. From the discussion before Lemma 9.12, we can write $\mu_1 = \sigma_1 + \nu_1$ and $\mu_2 = \sigma_2 + \nu_2$, and $\sigma$ is a joining between $\sigma_1$ and $\sigma_2$. But the ergodicity implies the existence of $c_1$ and $c_2$ such that $\sigma_1 = c_1\mu_1$ and $\sigma_2 = c_2\mu_2$. So there exists a $(c_1, c_2)$-joining between $\mu_1$ and $\mu_2$ which is fact impossible by strong disjointness. ☐

In [**1**], page 89, it is shown:

PROPOSITION 9.22. *Similarity is an equivalence relation.*

We have a corollary:

COROLLARY 9.23. *ID-similarity is an equivalence relation among pure Poisson suspensions (or IDp processes).*

There is a spectral criterion of strong disjointness:

THEOREM 9.24. *Let* $\lambda_{M,1}$ *(resp.* $\lambda_{M,2}$*) be the maximal spectral type of* $(\Omega_1, \mathcal{F}_1, \mu_1, T_1)$ *(resp.* $(\Omega_2, \mathcal{F}_2, \mu_2, T_2)$*). Then, if* $\lambda_{M,1}$ *and* $\lambda_{M,2}$ *are mutually singular, those systems are strongly disjoint.*

PROOF. Assume there exists a $(c_1, c_2)$-joining $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu, T_1 \times T_2)$ between them. Let $\{A_n\}_{n\in\mathbb{N}} \in \mathcal{F}_1$ be a collection of subsets of non zero and finite $\mu_1$-measure such that $\cup_{n\in\mathbb{Z}} A_n = \Omega_1$ and consider $B \in \mathcal{F}_2$ of non zero and finite $\mu_2$-measure. Denote $\lambda_{1,n}$ and $\lambda_2$ the spectral measure of $1_{A_n}$ and $1_B$ in their respective systems. Then $1_{A_n\times\Omega_2}$ and $1_{\Omega_1\times B}$ belong to $L^2(\mu)$ and have $c_1\lambda_{1,n}$ and $c_2\lambda_2$ as spectral measures which are mutually singular since $c_1\lambda_{1,n} \ll \lambda_{M,1}$ and $c_2\lambda_2 \ll \lambda_{M,2}$. These vectors are thus orthogonal, that is, $\int_{\Omega_1\times\Omega_2} 1_{A_n\times\Omega_2} 1_{\Omega_1\times B} d\mu = 0$, so $A_n \times \Omega_2$ and $\Omega_1 \times B$ are disjoint mod. $\mu$ for all $n \in \mathbb{N}$. This implies that $\cup_{n\in\mathbb{N}} A_n \times \Omega_2$ and $\Omega_1 \times B$ are disjoint mod. $\mu$ but this is impossible since $\cup_{n\in\mathbb{N}} A_n \times \Omega_2 = \Omega_1 \times \Omega_2$. □

This last condition is, more generally, a criterion for ID-disjointness:

PROPOSITION 9.25. *Let $\lambda_{M,1}$ (resp. $\lambda_{M,2}$) be the maximal spectral type of the system $(\Omega_1, \mathcal{F}_1, \mu_1, T_1)$ (resp. $(\Omega_2, \mathcal{F}_2, \mu_2, T_2)$). Then, if $\lambda_{M,1}$ and $\lambda_{M,2}$ are mutually singular, $(M_{\Omega_1}, \mathcal{M}_{\mathcal{F}_1}, \mathcal{P}_{\mu_1}, T_1^\star)$ and $(M_{\Omega_2}, \mathcal{M}_{\mathcal{F}_2}, \mathcal{P}_{\mu_2}, T_2^\star)$ are ID-disjoint.*

PROOF. Assume there is a non-independent ID-joining. From Lemma 9.12, there exist $\nu_1$, $\sigma_1$, $\nu_2$, $\sigma_2$, and $\sigma$ such that $\mu_1 = \nu_1 + \sigma_1$, $\mu_2 = \nu_2 + \sigma_2$ and $\sigma$ is a $(1, 1)$-joining of $\sigma_1$ and $\sigma_2$. Since the maximal spectral type of $(\Omega_1, \mathcal{F}_1, \sigma_1, T_1)$ (resp. $(\Omega_2, \mathcal{F}_2, \sigma_2, T_2)$) is absolutely continuous with respect to $\lambda_{M,1}$ (resp. $\lambda_{M,2}$), such a joining between those systems cannot exist by Theorem 9.24. □

REMARK 9.26. This criterion is analogous to the necessary and sufficient condition for ID-disjointness among Gaussian processes, that is, mutual singularity of the maximal spectral type in the first chaos.

We will now show that the factorization of Theorem 7.8 is compatible with ID-disjointness, by first proving, more generally, strong disjointness among dynamical systems of the canonical decomposition in Proposition 5.4.

THEOREM 9.27. *Dynamical systems of the category $\mu_B$, $\mu_m$, $\mu_{mm}$, $\mu_{wm}$ and $\mu_{ne}$ are mutually strongly disjoint. A joining between dynamical systems of the same category stay in this category.*

PROOF. Consider $(\Omega_1, \mathcal{F}_1, \mu_{ne}, T_1)$ and let $(\Omega_2, \mathcal{F}_2, \mu_2, T_2)$ be a dynamical system of type $\mathbf{II}_\infty$ and suppose they are not strongly disjoint, then consider a $(c_1, c_2)$-joining, $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu, T_1 \times T_2)$. There exists a $T_1$-invariant probability measure $p$ equivalent to $\mu_{ne}$. Let $f := \frac{dp}{d\mu_{ne}}$ and define $f'(x, y) := f(x)$. We compute $\int_{\Omega_1\times\Omega_2} f' d\mu$:

$$\int_{\Omega_1\times\Omega_2} f' d\mu = c_1 \int_{\Omega_1} f d\mu_{ne} = c_1 \int_{\Omega_1} dp = c_1 < +\infty$$

But $f'$ is non-negative $\mu$-a.e. and $T_1 \times T_2$-invariant, thus the measure $\nu$ with density $f'$ with respect to $\mu$ is a finite measure which proves that $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu, T_1 \times T_2)$ is of type $\mathbf{II}_1$. Consider the image measure $\nu \circ \pi_2^{-1}$ where $\pi_2$ is the canonical projection on $\Omega_2$ and let $A \in \mathcal{F}_2$ have zero $\mu_2$-measure. Then $\mu \circ \pi_2^{-1}(A) = \mu_2(A) = 0$, since $\nu$ is absolutely continuous with respect to $\mu$. This implies that $\nu \circ \pi_2^{-1}(A) = 0$ so $\nu \circ \pi_2^{-1}$ is absolutely continuous with respect to $\mu_2$. But $\nu \circ \pi_2^{-1}$ is a finite $T_2$-invariant measure and this is not compatible with the fact that $(\Omega_2, \mathcal{F}_2, \mu_2, T_2)$ is of type $\mathbf{II}_\infty$. We thus have the strong disjointness of the two systems.

Consider $(\Omega_1, \mathcal{F}_1, \mu_{wm}, T_1)$ and let $(\Omega_2, \mathcal{F}_2, \mu_2, T_2)$ be a rigidity-free system. Note that we can find a non zero function $f \in L^2(\mu_{wm})$, such that its spectral measure is rigid (i.e. there exists $n_k \uparrow \infty$ such that $\hat{\sigma}_f(n_k) \to \hat{\sigma}_f(0)$). Suppose now that $(\Omega_1, \mathcal{F}_1, \mu_{wm}, T_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2, T_2)$ are not strongly disjoint and consider $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu, T_1 \times T_2)$, a $(c_1, c_2)$-joining. Consider also the Poisson suspension $(M_{\Omega_1 \times \Omega_2}, \mathcal{M}_{\mathcal{F}_1 \otimes \mathcal{F}_2}, \mathcal{P}_\mu, (T_1 \times T_2)^\star)$, for all $\Omega_1 \times A$ such that $A$ is of finite $\mu_2$-measure, $I(1_{\Omega_1 \times A})$ is independent of $I(f')$ (where $f'(x_1, x_2) = f(x_1)$), indeed , the $I(1_{\Omega_1 \times A})$ generate a mildly mixing factor (by isomorphism with $(M_{\Omega_2}, \mathcal{M}_{\mathcal{F}_2}, \mathcal{P}_{\mu_2}, T_2^\star)$) and the $I(f' \circ (T_1 \times T_2)^n)$ generate a rigid factor, such systems are disjoint (see for example [**37**]), and these factors are thus independent. But the support of $f'$ is included in $\Omega_1 \times \Omega_2$ which is the union of the $\Omega_1 \times A$ for $A$ in $\mathcal{F}_2$. This is impossible from a lemma due to Maruyama [**38**] that says that, for a Poisson suspension, two stochastic integrals $I(h_1)$ and $I(h_2)$ are independent if and only if the supports of $h_1$ and $h_2$ are disjoint. These two systems are thus strongly disjoint.

Note that the same arguments prove that an IDp process whose Lévy measure is of type $Q_{wm}$ is never mildly mixing (Proposition 7.11), otherwise, by considering this process as a stochastic integral against the Poisson suspension constructed above $Q_{wm}$, we would end up in the preceding case where a mildly mixing factor meets a rigid factor also generated by stochastic integrals. This would imply independence and disjointness of their supports which is not possible.

Consider $(\Omega_1, \mathcal{F}_1, \mu_{mm}, T_1)$ and a system $(\Omega_2, \mathcal{F}_2, \mu_2, T_2)$ where $\mu_2$ is of zero type, suppose once again that they are not strongly disjoint and consider a $(c_1, c_2)$-joining between them, $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu, T_1 \times T_2)$. For every $A \subset \Omega_2$ of finite $\mu_2$-measure:

$$\mu\left[(\Omega_1 \times A) \cap (T_1 \times T_2)^{-n}(\Omega_1 \times A)\right] = c_2 \mu_2\left(A \cap T_2^{-n} A\right)$$

which tends to 0 as $n$ tends towards infinity.

The "zero type" part of $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu, T_1 \times T_2)$ is the measurable union of sets $C$ such that $\mu\left(C \cap (T_1 \times T_2)^{-n} C\right)$ tends to 0 as $n$ tends towards infinity (see the proof of Proposition 2.10 in [**36**]). However the measurable union of sets $\Omega_1 \times A$ equals $\Omega_1 \times \Omega_2$. Thus $\mu$ is of zero type. Moreover, in the mean time, taking $B \subset \Omega_1$ of finite $\mu_{mm}$-measure:

$$\mu\left[(B \times \Omega_2) \cap (T_1 \times T_2)^{-n}(B \times \Omega_2)\right] = c_1 \mu_{mm}\left(B \cap T_1^{-n} B\right)$$

which doesn't tend to 0 as $n$ tends towards infinity since $\mu_{mm}$ is of positive type and this is not compatible with the preceding statement.

Finally, Proposition 3.1.2 page 88 in [**1**] shows that two similar transformations are conservative if one of the two is conservative. Dissipative systems are therefore strongly disjoint from conservative ones.

The last statement of the theorem comes from the following fact:

If we take, for example, a system of type $(\Omega_1, \mathcal{F}_1, \mu_{wm}, T_1)$, a system $(\Omega_2, \mathcal{F}_2, \mu_2, T_2)$ and a $(c_1, c_2)$-joining $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \nu, T_1 \times T_2)$ between the two. $\nu$ can be split into $\nu_B + \nu_m + \nu_{mm} + \nu_{wm} + \nu_{ne}$ and, denoting by $\pi_1$ and $\pi_2$ the projections onto $\Omega_1$ and $\Omega_2$, we have:

$\pi_1^\star(\nu_B + \nu_m + \nu_{mm} + \nu_{wm} + \nu_{ne})$
$= c_1(\pi_1^\star(\nu_B) + \pi_1^\star(\nu_m) + \pi_1^\star(\nu_{mm}) + \pi_1^\star(\nu_{fm}) + \pi_1^\star(\nu_{ne}))$

Thus $\mu_{wm} = \pi_1^\star(\nu_B) + \pi_1^\star(\nu_m) + \pi_1^\star(\nu_{mm}) + \pi_1^\star(\nu_{wm}) + \pi_1^\star(\nu_{ne})$. It is clear that each term in the sum is of the same type as $\mu_{wm}$ and this implies that $\pi_1^\star(\nu_B)$, $\pi_1^\star(\nu_m)$, $\pi_1^\star(\nu_{mm})$ and $\pi_1^\star(\nu_{ne})$ are zero because if it were not the case, we could construct joinings between strongly disjoint systems ($\pi_1^\star(\nu_B)$ with $\nu_B$, etc...). Finally, $\nu = \nu_{wm}$ and $\mu_2 = \pi_2^\star(\nu_{wm})$, that is $\mu_2$ is of the same type as $\mu_{wm}$. □

COROLLARY 9.28. *Poisson suspensions above systems of the kind $\mu_B$, $\mu_m$, $\mu_{mm}$, $\mu_{wm}$ and $\mu_{ne}$ are mutually ID-disjoint.*

PROOF. Take for example $\mu_{mm}$ and $\mu_{wm}$ and suppose that the associated suspensions are not ID-disjoint. From Lemma 9.12, there exist $\nu_1$, $\sigma_1$, $\nu_2$, $\sigma_2$, and $\sigma$ such that $\mu_{mm} = \nu_1 + \sigma_1$, $\mu_{wm} = \nu_2 + \sigma_2$ and $\sigma$ is a $(1,1)$-joining of $\sigma_1$ and $\sigma_2$. But $\sigma_1$ is necessarily of the same type as $\mu_{mm}$ and $\sigma_2$ of the same type as $\mu_{wm}$, and as such, they are strongly disjoint, and the joining $\sigma$ cannot take place. □

COROLLARY 9.29. *An ID-joining between two weakly mixing (resp. mildly mixing, resp. mixing) Poisson suspensions is always weakly mixing (resp. mildly mixing, resp. mixing).*

## 9.6. Illustrations.

9.6.1. *ID-joinings in the dissipative and type* $\mathbf{II}_1$ *cases.* Corollary 9.28 explains, in a certain way, that each component of the canonical factorization is isolated from the others, as far as ID is concerned. We will see various possible behaviour, by showing first that the dissipative class and the class of type $\mathbf{II}_1$ are not the most interesting.

PROPOSITION 9.30. *Two dissipative systems are similar.*
*Two systems associated to finite measures are similar.*

PROOF. In every dissipative system, there exists, up to a multiplicative coefficient, a wandering set $A$ of measure 1 and the factor generated by the sequence $\left\{1_A \circ T^k\right\}_{k\in\mathbb{Z}}$ is common to each of those systems and this implies their similarity.

For the second point, the Cartesian product is available. □

COROLLARY 9.31. *Two non-ergodic Poisson suspensions (resp. non-ergodic IDp processes) are ID-similar.*

PROOF. We do the proof for Poisson suspensions. Let us consider $(\Omega_1, \mathcal{F}_1, \mu_1, T_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2, T_2)$, two systems such that $(M_{\Omega_1}, \mathcal{M}_{\mathcal{F}_1}, \mathcal{P}_{\mu_1}, T_1^\star)$ and $(M_{\Omega_2}, \mathcal{M}_{\mathcal{F}_2}, \mathcal{P}_{\mu_2}, T_2^\star)$ are not ergodic. This is equivalent to saying that these systems are not of type $\mathbf{II}_\infty$. We can thus find two sets $A$ and $B$ which are $T_1$ and $T_2$ invariant respectively such that $\mu_{1|A}$ and $\mu_{2|B}$ are finite measures of respective total mass $c_A$ and $c_B$. We can form the following ID-joining of Poisson suspensions:

$$\left(\left(\mathcal{P}_{\mu_{1|A^c}} * \mathcal{P}_{\frac{c_A}{c_A+c_B}\mu_{1|A}}\right) \otimes \left(\mathcal{P}_{\mu_{2|B^c}} * \mathcal{P}_{\frac{c_B}{c_A+c_B}\mu_{1|B}}\right)\right) * \tilde{\mathcal{P}}_{\frac{1}{c_A+c_B}\left(\mu_{1|A}\otimes\mu_{2|B}\right)}$$

□

The situation changes radically as soon as we look at the other classes of dynamical systems. For example, among remotely infinite systems, it is shown in [**1**], page 194, that there exists a continuum of mutually strongly disjoint ergodic systems (given by invariant measures of null recurrent aperiodic Markov chains). Thanks to Proposition 9.21, there exists a continuum of Poisson suspensions or IDp processes with the $K$ property and which are mutually ID-disjoint.

9.6.2. *Relatively independent joining above a Poissonian factor.* This section deals with Poisson suspensions exclusively.

PROPOSITION 9.32. *The relatively independent joining of a Poisson suspension above one of its Poissonian factor is an ID-self joining. In particular, an ergodic Poisson suspension is relatively weakly mixing over its Poissonian factors.*

PROOF. Let $(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_\mu, T^\star)$ be the suspension built above $(\Omega, \mathcal{F}, \mu, T)$, we recall that a Poissonian factor is a sub-$\sigma$-algebra of the kind $\mathcal{A}_N$ where $\mathcal{A}$ is a $T$-invariant sub-$\sigma$-algebra of $\mathcal{F}$. There is no loss of generality to assume that $\mathcal{A}$ is a factor of $(\Omega, \mathcal{F}, \mu, T)$ (that is $\mu$ is $\sigma$-finite on $\mathcal{A}$). If it is not the case, $\Omega$ splits in two $T$-invariant sets $A_1$ and $A_2$, where $\mu$ restricted to $A_1$ is $\sigma$-finite on $\mathcal{A}$ and there is not set of finite non-zero measure in $\mathcal{A}$ that belongs to $A_2$. The suspension thus splits in a direct product and we return to the initial situation.

We are going to show that the relatively independent joining above $\mathcal{A}_N$ comes from the relatively independent joining of $(\Omega, \mathcal{F}, \mu, T)$ above $\mathcal{A}$.

Consider the application $\varphi$ defined on $M_{\Omega\times\Omega}$ by $\varphi(\nu) \mapsto (\pi_1^\star(\nu), \pi_2^\star(\nu))$ with values in $M_\Omega \times M_\Omega$, we will prove: $\varphi^\star(\mathcal{P}_{\mu\otimes_\mathcal{A}\mu}) = \tilde{\mathcal{P}}_{\mu\otimes_\mathcal{A}\mu} = \mathcal{P}_\mu \otimes_{\mathcal{A}_N} \mathcal{P}_\mu$.

We have:

$$\mathbb{E}_{\mathcal{P}_\mu\otimes_{\mathcal{A}_N}\mathcal{P}_\mu}[\mathcal{E}_h \otimes 11 \otimes \mathcal{E}_g] = \mathbb{E}_{\mathcal{P}_\mu}\left[\mathbb{E}_{\mathcal{P}_\mu}[\mathcal{E}_h|\mathcal{A}_N]\,\mathbb{E}_{\mathcal{P}_\mu}[\mathcal{E}_g|\mathcal{A}_N]\right]$$

but $\mathbb{E}_{\mathcal{P}_\mu}[\mathcal{E}_h|\mathcal{A}_N] = \mathcal{E}_{\mathbb{E}_\mu[h|\mathcal{A}]}$ and $\mathbb{E}_{\mathcal{P}_\mu}[\mathcal{E}_g|\mathcal{A}_N] = \mathcal{E}_{\mathbb{E}_\mu[g|\mathcal{A}]}$. Hence:

$$\mathbb{E}_{\mathcal{P}_\mu}\left[\mathbb{E}_{\mathcal{P}_\mu}[\mathcal{E}_h|\mathcal{A}_N]\,\mathbb{E}_{\mathcal{P}_\mu}[\mathcal{E}_g|\mathcal{A}_N]\right] = \mathbb{E}_{\mathcal{P}_\mu}\left[\mathcal{E}_{\mathbb{E}_\mu[h|\mathcal{A}]}\mathcal{E}_{\mathbb{E}_\mu[g|\mathcal{A}]}\right]$$

and then

$$\mathbb{E}_{\mathcal{P}_\mu}\left[\mathcal{E}_{\mathbb{E}_\mu[h|\mathcal{A}]}\mathcal{E}_{\mathbb{E}_\mu[g|\mathcal{A}]}\right] = \exp\int_\Omega \mathbb{E}_\mu[h|\mathcal{A}]\,\mathbb{E}_\mu[g|\mathcal{A}]\,d\mu$$

Now

$$\mathbb{E}_{\varphi^\star\left(\mathcal{P}_{\mu\otimes_\mathcal{A}\mu}\right)}[\mathcal{E}_h \otimes 11 \otimes \mathcal{E}_g] = \mathbb{E}_{\mathcal{P}_{\mu\otimes_\mathcal{A}\mu}}[\mathcal{E}_{h\otimes 1}\mathcal{E}_{1\otimes g}]$$

and finally:

$$\mathbb{E}_{\mathcal{P}_{\mu\otimes_{\mathcal{A}}\mu}}\left[\mathcal{E}_{h\otimes 1}\mathcal{E}_{1\otimes g}\right] = \exp\int_{\Omega\times\Omega} h\otimes 11\otimes g d\mu\otimes_{\mathcal{A}}\mu = \exp\int_{\Omega}\mathbb{E}_{\mu}\left[h|\mathcal{A}\right]\mathbb{E}_{\mu}\left[g|\mathcal{A}\right]d\mu$$

which gives the desired equality $\tilde{\mathcal{P}}_{\mu\otimes_{\mathcal{A}}\mu} = \mathcal{P}_{\mu}\otimes_{\mathcal{A}_N}\mathcal{P}_{\mu}$.

If $(M_\Omega, \mathcal{M}_{\mathcal{F}}, \mathcal{P}_\mu, T^\star)$ is ergodic, any ID-self joining is ergodic by Lemma 9.19 and so are the relatively independent joinings above Poissonian factors. □

9.6.3. *Minimal ID-self joinings.* We will see that there exist Poisson suspensions (or IDp-processes) whose ID-selfjoinings are reduced to the minimum.

Definition 9.33. A Poisson suspension $(M_\Omega, \mathcal{M}_{\mathcal{F}}, \mathcal{P}_\mu, T^\star)$ is said to have *minimal ID-self joinings* if, considering any ID-self-joining $(M_\Omega\times M_\Omega, \mathcal{M}_{\mathcal{F}}\otimes\mathcal{M}_{\mathcal{F}}, \mathfrak{P}, T^\star\times T^\star)$, $\mathfrak{P}$ has the following form:

$$\mathfrak{P} = \left(\mathcal{P}_{a\mu}\otimes\mathcal{P}_{a\mu}\right) * \cdots * \tilde{\mathcal{P}}_{c_{-k}\nu_{-k}} * \cdots * \tilde{\mathcal{P}}_{c_0\nu_0} * \cdots * \tilde{\mathcal{P}}_{c_k\nu_k} * \dots$$

where $\nu_k(A\times B) = \mu\left(A\cap T^{-k}B\right)$ and $\sum_{k\in\mathbb{Z}} c_k = 1-a$ with $a\geq 0$, $c_k\geq 0$, $k\in\mathbb{Z}$.

The existence of such suspensions is guaranteed by the existence of dynamical systems $(\Omega, \mathcal{F}, \mu, T)$ (said to have *minimal self joinings*) whose only ergodic self-joinings are of the form $\mu\left(A\cap T^{-k}B\right)$. Indeed, the example investigated by Aaronson and Nadkarni in [**2**] that we have already considered at the end of section 5 has this property.

In this example, the Poissonian centralizer $C^p(T^\star)$ is trivial; that is it consists only of the powers of $T^\star$.

## 10. Applications

**10.1. $\alpha$-stable and $\alpha$-semi-stable processes.** We recall the definition of an $\alpha$-semi-stable (resp. $\alpha$-stable) distribution on $(\mathbb{R}, \mathcal{B})$. Denote by $D_b$ the application which associates $x\in\mathbb{R}$ to $bx\in\mathbb{R}$. Assume that $0<\alpha<2$.

Definition 10.1. An *$\alpha$-semi-stable distribution of span $b$* $(b>0)$ is an IDp distribution on $(\mathbb{R}, \mathcal{B})$ whose Lévy measure $\nu$ satisfies

$$\nu = b^{-\alpha} D_b^\star(\nu)$$

A distribution is said to be *$\alpha$-stable* if it is $\alpha$-semi-stable of span $b$ for all $b>0$.

We will now discuss $\alpha$-semi-stable and $\alpha$-stable processes by introducing the application $S_b$ which associates $\{x_n\}_{n\in\mathbb{Z}}\in\mathbb{R}^{\mathbb{Z}}$ to $\{bx_n\}_{n\in\mathbb{Z}}$.

Definition 10.2. A stationary process is said to be *$\alpha$-semi-stable of span $b$* if it is IDp and its Lévy measure $Q$ satisfies:

$$Q = b^{-\alpha} S_b^\star(Q) \tag{7}$$

A stationary process is said to be *$\alpha$-stable* if it is $\alpha$-semi-stable of span $b$ for all $b>0$.

In particular, $S_b$ is non-singular and commutes with $T$. Remark that an $\alpha$-semi-stable distribution of span $b$ or an $\alpha$-semi-stable process of span $b$ is also $\alpha$-semi-stable of span $\frac{1}{b}$. More generally, the set of $b$ such that a distribution or a process is $\alpha$-semi-stable with span $b$ is a subgroup of the multiplicative group $\mathbb{R}_+^*$.

Poisson suspensions constructed above such Lévy measure are examples of self-similar (completely self-similar in the stable case) Poisson suspensions.

PROPOSITION 10.3. *The canonical factorization of Theorem 7.8 of an $\alpha$-semi-stable process of span $b$ is exclusively made of $\alpha$-semi-stable processes of span $b$.*

PROOF. It suffices to show that the $T$-invariant subsets of the partition given in the canonical decomposition in Proposition 5.4 are also $S_b$-invariant.

Consider $\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, Q, T\right)$ where $Q$ satisfies (7). Let $\mathfrak{P}$ be the part of type $\mathbf{II}_1$ of the system, then there exists a $T$-invariant function $f$ such that $\mathfrak{P} = \{f > 0\}$ and $\int_{\mathbb{R}^{\mathbb{Z}}} f dQ = 1$. Let $b > 0$. The function $f \circ S_b$ is $T$-invariant since $f \circ S_b \circ T = f \circ T \circ S_b = f \circ S_b$. Thus, from (7), $\int_{\mathbb{R}^{\mathbb{Z}}} f \circ S_b dQ = \int_{\mathbb{R}^{\mathbb{Z}}} f dS_b^{\star}(Q) = b^{\alpha} \int_{\mathbb{R}^{\mathbb{Z}}} f dQ = b^{\alpha}$, so the probability measure with density $b^{-\alpha} f \circ S_b$ with respect to $Q$ is $T$-invariant. Thus $S_b^{-1}\mathfrak{P} = \{f \circ S_b > 0\} \subset \mathfrak{P}$. By the same arguments $S_{\frac{1}{b}}^{-1}\mathfrak{P} \subset \mathfrak{P}$ and thus, $S_b^{-1}\left(S_{\frac{1}{b}}^{-1}\mathfrak{P}\right) \subset S_b^{-1}\mathfrak{P}$ and this shows $S_b^{-1}\mathfrak{P} = \mathfrak{P}$.

Now consider the $T$-invariant set $\mathcal{N}_+$ of Proposition 2.10. Let $A \subset \mathcal{N}_+$ be such that $0 < Q(A) < +\infty$. Then

$$Q\left(\left(S_b^{-1}A\right) \cap T^{-k}\left(S_b^{-1}A\right)\right) = Q\left(\left(S_b^{-1}A\right) \cap S_b^{-1}\left(T^{-k}A\right)\right)$$

$$= Q\left(S_b^{-1}\left(A \cap T^{-k}A\right)\right) = b^{-\alpha} Q\left(A \cap T^{-k}A\right)$$

and thus $\overline{\lim}_{k\to\infty} Q\left(\left(S_b^{-1}A\right) \cap T^{-k}\left(S_b^{-1}A\right)\right) = \overline{\lim}_{k\to\infty} b^{-\alpha} Q\left(A \cap T^{-k}A\right) > 0$. Then $S_b^{-1}A \subset \mathcal{N}_+$ so we have $S_b^{-1}\mathcal{N}_+ \subset \mathcal{N}_+$, and, by symmetric arguments $S_b^{-1}\mathcal{N}_+ = \mathcal{N}_+$.

Consider the set $\mathcal{N}_{rf}$, and let $f$ be a vector in $L^2\left(\mu_{|S_b^{-1}\mathcal{N}_{rf}}\right)$ with spectral measure $\sigma$, the maximal spectral type of $L^2\left(\mu_{|\mathcal{N}_{rf}}\right)$. Then $f \circ S_{\frac{1}{b}}$ belongs to $L^2\left(\mu_{|\mathcal{N}_{rf}}\right)$ and admits $b^{\alpha}\sigma$ as spectral measure since

$$\int_{\mathcal{N}_{rf}} f \circ S_{\frac{1}{b}} \circ T^k \overline{f \circ S_{\frac{1}{b}}} dQ = \int_{\mathcal{N}_{rf}} f \circ T^k \circ S_{\frac{1}{b}} \bar{f} \circ S_{\frac{1}{b}} dQ$$
$$= \int_{S_b^{-1}\mathcal{N}_{wD}} f \circ T^k \bar{f} dS_{\frac{1}{b}}^{\star} Q = b^{\alpha} \int_{S_b^{-1}\mathcal{N}_{wD}} f \circ T^k \bar{f} dQ$$

Thus $b^{\alpha}\sigma$, and of course, $\sigma$, annihilates every weak Dirichlet set. This proves $S_b^{-1}\mathcal{N}_{rf} \subset \mathcal{N}_{rf}$, and then $S_b^{-1}\mathcal{N}_{rf} = \mathcal{N}_{rf}$.

Consider $\mathfrak{D}$, the dissipative part of the system. From Lemma 2.6, there exists a wandering set $W$ such that $\mathfrak{D} = \cup_{n\in\mathbb{Z}} T^{-n}W$. Let $b > 0$ and consider the set $S_b^{-1}W$ (which is of non-zero $Q$-measure from (7)). We have, if $n \neq m$, $T^{-n}\left(S_b^{-1}W\right) \cap T^{-m}\left(S_b^{-1}W\right) = \emptyset$; indeed, using the non-singularity of $S_b$:

$T^{-n}\left(S_b^{-1}W\right) \cap T^{-m}\left(S_b^{-1}W\right)$

$$= S_b^{-1}\left(T^{-n}W\right) \cap S_b^{-1}\left(T^{-m}W\right) = S_b^{-1}\left(T^{-n}W \cap T^{-m}W\right) = \emptyset$$

Thus $S_b^{-1}W$ is a wandering set, so $S_b^{-1}\mathfrak{D} \subset \mathfrak{D}$ since:

$$S_b^{-1}\mathfrak{D} = S_b^{-1}\left(\cup_{n\in\mathbb{Z}}T^{-n}W\right) = \cup_{n\in\mathbb{Z}}S_b^{-1}\left(T^{-n}W\right) = \cup_{n\in\mathbb{Z}}T^{-n}\left(S_b^{-1}W\right)$$

and $\mathfrak{D}$ is, by definition, the union of all the wandering sets. We conclude $S_b^{-1}\mathfrak{D} = \mathfrak{D}$.

It is now easy to finish the proof by looking at the invariance of complements, intersections, etc... and show the invariance of each set in the partition:

$$\mathfrak{D} \cup (\mathcal{C} \cap \mathcal{N}_0) \cup (\mathcal{N}_+ \cap \mathcal{N}_{rf}) \cup \left(\mathcal{N}_{rf}^c \cap \mathcal{N}\right) \cup \mathfrak{P}$$

□

COROLLARY 10.4. *The canonical factorization of an $\alpha$-stable process is exclusively made of $\alpha$-stable processes.*

10.1.1. *$\alpha$-semi-stable joinings.*

DEFINITION 10.5. An *$\alpha$-semi-stable joining of span $b$* is a joining of two $\alpha$-semi-stable process of span $b$ which, as a bivariate process, is an $\alpha$-semi-stable process of span $b$, i.e. the bivariate Lévy measure $Q^{biv}$ of the joining satisfies:

$$Q^{biv} = b^{-\alpha}\left(S_b \times S_b\right)^{\star}\left(Q^{biv}\right)$$

where $S_b \times S_b\left(\{x_n\}_{n\in\mathbb{Z}}, \{y_n\}_{n\in\mathbb{Z}}\right) = \left(S_b\{x_n\}_{n\in\mathbb{Z}}, S_b\{y_n\}_{n\in\mathbb{Z}}\right)$

An $\alpha$-stable joining is obviously an ID-joining. Some natural ID-joinings are $\alpha$-stable joinings, for example, joinings with linear factors.

DEFINITION 10.6. Let $\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, Q, T\right)$ be a system where $Q$ is the Lévy measure of an $\alpha$-semi-stable process of span $b$. A factor $\mathcal{A}$ is called $\alpha$-semi-stable of span $b$ if $\mathcal{A}$ is $S_b$-invariant.

It is easily verified that the relatively independent joining above an $\alpha$-semi-stable factor of span $b$ gives way to an $\alpha$-semi-stable joining of span $b$.

10.1.2. *$S\alpha S$-processes and factorizations.* The most frequently studied stationary $\alpha$-stable processes are the so-called $S\alpha S$-processes, where the distribution is preserved under the change of sign. In our framework, this means that the involution $S$ that changes the sign belongs to $C(T)$, i.e. $S$ commutes with the shift $T$ and preserves the Lévy measure (note that $S$ commutes also with $S_b$). It is easy to see that the canonical factorization of an $S\alpha S$ process is only made of $S\alpha S$ processes. Moreover, we can define $S\alpha S$-joinings and $S\alpha S$-factors in the obvious way by also requiring the invariance of the joinings by $S \times S$ and the invariance of the factors by $S$.

We now show some connections existing between the decomposition in Theorem 7.8 and decompositions of an $S\alpha S$ process previously established respectively by Rosiński [**51**], Pipiras and Taqqu [**47**], and Samorodnitsky [**56**]. We first recall their results (we refer to these papers for precise definitions), the symbol "=" means "equality in distribution":

THEOREM 10.7. *(Rosiński) A stationary $S\alpha S$ process $X$ admits the unique decomposition into independent $S\alpha S$ processes:*

$$X = X_r^1 + X_r^2 + X_r^3$$

$X_r^1$ *is a mixed moving average process*
$X_r^2$ *is harmonizable*
$X_r^3$ *cannot be decomposed as the sum of a mixed moving average (or harmonizable) process and an independent $S\alpha S$ process.*

Then the refinement due to Pipiras and Taqqu:

THEOREM 10.8. *(Pipiras and Taqqu) A stationary $S\alpha S$ process $X$ admits the unique decomposition into independent $S\alpha S$ processes:*

$$X = X_{pt}^1 + X_{pt}^2 + X_{pt}^3 + X_{pt}^4$$

$X_{pt}^1$ *is a mixed moving average process*
$X_{pt}^2$ *is harmonizable*
$X_{pt}^3$ *is associated to a cyclic flow without harmonizable component*
$X_{pt}^4$ *cannot be decomposed as the sum of a mixed moving average, or a harmonizable process or a process associated to a cyclic flow, and an independent $S\alpha S$ process.*

and finally the most recent decomposition due to Samorodnitsky:

THEOREM 10.9. *(Samorodnitsky) A stationary $S\alpha S$ process $X$ admits the unique decomposition into independent $S\alpha S$ processes:*

$$X = X_s^1 + X_s^2 + X_s^3$$

$X_s^1$ *is a mixed moving average process*
$X_s^2$ *is associated to a conservative null flow*
$X_s^3$ *is associated to a positive flow*

These authors study both discrete and continuous time in the same framework and, to avoid unnecessary different terminology, use "flow" to designate both an action of $\mathbb{R}$ and of $\mathbb{Z}$. There is a confusing terminology in the literature about null and positive flows (see the remark after Proposition 2.10) and here, Samorodnitsky uses the one found in Aaronson's book [**1**].

We mention that these decompositions could also be obtained by the procedure we used throughout this paper, that is, cutting the support of Lévy measure along shift invariant subsets. Here we recall, as we already pointed out, that, in general, there can be an infinity of components in the decomposition, our criteria were mostly chosen with respect to the ergodic properties of the components and also their ID-disjointness. In that way, our decomposition is closer to Samorodnitsky's: if we write our decomposition (see Theorem 7.8) of an $S\alpha S$ process as $X = Y^1 + Y^2 + Y^3 + Y^4 + Y^5$ ($Y^5$ being the non-ergodic component), $Y^5$ is exactly $X_s^3$ since it is, roughly speaking, the "biggest" non-ergodic component of the process.

Indeed, the decomposition of Samorodnitsky is made in such a way that there don't exist two independent $S\alpha S$ processes $Z_1$ and $Z_2$, one of them being ergodic and such that $X_s^3 = Z_1 + Z_2$. Our decomposition allows us to make more precise this statement: there don't exist two independent IDp processes $Z_1$ and $Z_2$, one of them being ergodic and such that $X_s^3 = Z_1 + Z_2$.

From the paper of Samorodnitsky, it is clear that $X_{pt}^2$ and $X_{pt}^3$ can be extracted from $X_s^3$ (that is, we can find an $S\alpha S$ process $Y$ independent of $X_{pt}^3$ and $X_{pt}^4$ such that $X_s^3 = Y + X_{pt}^3 + X_{pt}^4$).

We will show that $X_s^1$ (which has the same distribution as $X_r^1$ and $X_{pt}^1$) can be extracted from $Y^1$ (which is the component with a dissipative Lévy measure) by showing that its Lévy measure is dissipative.

Rosiński has shown in [**51**] that $X_s^1$ admits the representation:

$$\left\{ \int_\Omega \sum_{p\in\mathbb{Z}} K\left(x, p+n\right) M_\alpha^p\left(dx\right) \right\}_{n\in\mathbb{Z}}$$

where $(\Omega, \mathcal{F}, \mu)$ is a $\sigma$-finite measure space, $\{M_\alpha^p\}_{p\in\mathbb{Z}}$ is an i.i.d sequence of $S\alpha S$ (generalized) random measures on $(\Omega, \mathcal{F})$ with control measure $\mu$. As an i.i.d sequence, the Lévy measure of $\{M_\alpha^p\}_{p\in\mathbb{Z}}$ is easily obtained:

Call $q$ the Lévy measure of $M_\alpha^0$, then the Lévy measure of $\{M_\alpha^p\}_{p\in\mathbb{Z}}$ is the measure on $M_\Omega^{\mathbb{Z}}$ (we denote by $M_\Omega$ the canonical space of random measures on $\Omega$) given by $Q := \sum_{k\in\mathbb{Z}} j_k$ where $j_k$ is the sequence which is identically the zero measure on $M_\Omega$ except on the $k$-th place where it equals $q$. It is clear that this measure is dissipative with respect to the shift transformation and this implies the dissipativity of the Lévy measure of $\left\{ \int_\Omega \sum_{p\in\mathbb{Z}} K\left(x, p+n\right) M_\alpha^p\left(dx\right) \right\}_{n\in\mathbb{Z}}$.

**10.2. Embedding a Poisson suspension in a flow.** If $(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_\mu, T^\star)$ is such that the transformation $T$ of the dynamical system $(\Omega, \mathcal{F}, \mu, T)$ can be embedded in a measurable flow $T_t$; that is, there exists a measurable flow $\{T_t\}_{t\in\mathbb{R}}$ on $(\Omega, \mathcal{F}, \mu)$ such that $T = T_1$, then the transformation $T^\star$ can be embedded in the measurable flow $(T_t)^\star$ consisting in Poissonian automorphisms (the fact that this flow is measurable follows from Maruyama's results, see [**38**]).

**10.3. Entropy.** For Gaussian processes, the situation is easily described: the entropy is zero if and only if the spectral measure of $X_0$ is singular with respect to Lebesgue measure and infinite otherwise (see [**15**]).

Since IDp processes can be seen as a factor of the Poisson suspension constructed above their Lévy measure and thanks to Proposition 4.12, we have the following sufficient condition for the zero entropy:

PROPOSITION 10.10. *If the maximal spectral type of the Lévy measure of an IDp process is singular then the entropy of the process is zero.*

**10.4. Type *multG* processes as a source of differences with Gaussian processes.** The aim of this section is to show it is impossible, in general, to split a Poisson suspension (as in Lemma 4.5) as a direct product of two suspensions, one being a $K$-system and the other a zero-entropy system. This is a major difference with the Gaussian process (where the splitting is achieved by cutting the spectral measure of the coordinate at 0 according to the Lebesgue part and the singular part). Our example will be obtained by constructing a so-called type *multG* process (see [**8**]). Denote by $X$ the standard i.i.d. Gaussian process and $Y$ an ergodic rigid IDp process, independent of $X$, whose coordinates follow a Poisson distribution (this can be achieved by constructing a rigid Poisson suspension $N$, considering a finite measure subset $A$ and defining $Y := \{N(A) \circ T^n\}_{n\in\mathbb{Z}}$). Then the process $X\sqrt{Y}$ is a stationary IDp process, said to be of type *multG*. We make some easy observations:

$X\sqrt{Y}$ is not independent of $X$ since $X_0\sqrt{Y_0} > 1$ implies $X_0 > 0$.

$X\sqrt{Y}$ is not independent of $Y$ since they are zero simultaneously.

Suppose $X\sqrt{Y} = Z^1+Z^2$, with $Z^1$ and $Z^2$ being factors of $X\sqrt{Y}$, both ID and mutually independent. $Z^1$ nor $Z^2$ is independent of $Y$ since $Y_0 = 0$ implies $Z_0^1 = 0$ and $Z_0^2 = 0$.

Now assume that the distribution of $X\sqrt{Y}$ can be seen as the sum of $U_1$ and $U_2$, both ID, $U_1$ being $K$ and $U_2$ with zero entropy. Since a $K$-system and a zero entropy system are disjoint, $U_1$ and $U_2$ are independent and can be filtered from (are factors of) their sum by a theorem of Furstenberg (see [**19**]). But $U_1$ plays the role of $Z^1$ and this is impossible since $U_1$ must be independent of $Y$ which has zero entropy. Moreover $X\sqrt{Y}$ has positive entropy since it is not independent of $X$, which is a $K$-system, and it is not mildly mixing since it is not independent of $Y$ which is rigid.

By considering the Poisson suspension constructed over the Lévy measure of $X\sqrt{Y}$, we easily obtain the result announced at the beginning of this section.

**10.5. The square and exponential of Gaussian processes.** If $X$ is a Gaussian random variable, $X^2$ and $\exp X$ are respectively Gamma and log-normal random variables, they both are IDp. It is natural to ask whether a process obtained by taking the square or the exponential of a Gaussian process is ID.

The square case has been subject of interest for a long time but comprehensive results have just emerged recently (see [**18**]). In particular a strong link has been established between Green functions of Markov processes and covariance functions of Gaussian processes with an ID square. In the stationary case, it gives us access to a large family of ID squared Gaussian processes.

THEOREM 10.11. *Let $X$ be a centered Gaussian process with a covariance equal to the Green function of a transient symmetric $\mathbb{Z}$-valued random walk, then $X^2$ is ID. The spectral density of $X_0$ is given by $\frac{1}{1-\hat{\mu}}$ where $\hat{\mu}$ is the Fourier transform (on $[-\pi,\pi[$) of the symmetric $\mathbb{Z}$-valued distribution that governs the walk.*

The achievement of the ergodic theory of Gaussian processes tells us that those squared Gaussian processes are factors of Bernoulli systems, and are therefore Bernoulli themselves.

Concerning the exponential case, results are pretty scarce, nevertheless, the following proposition shows that the most important class of Gaussian processes doesn't possess the property of infinite divisibility of their exponential.

PROPOSITION 10.12. *Let $\{X_n\}_{n\in\mathbb{Z}}$ be a Gaussian process with simple spectrum. The process $\{\exp X_n\}_{n\in\mathbb{Z}}$ is not ID.*

PROOF. Suppose $\{\exp X_n\}_{n\in\mathbb{Z}}$ ID. If $\sigma$ is the spectral measure of $X_0$, it is easy to see that the spectral measure of $\exp X_0$ is equivalent to $\sum_{k=0}^{+\infty}\frac{\sigma^{*k}}{k!}$. Thus, since $\sigma \ll \sum_{k=0}^{+\infty}\frac{\sigma^{*k}}{k!}$ and since $\{X_n\}_{n\in\mathbb{Z}}$ has simple spectrum, $X_0$ is in the cyclic space of $\exp X_0$.

Thus $\{X_n\}_{n\in\mathbb{Z}}$ is a generalized linear factor of $\{\exp X_n\}_{n\in\mathbb{Z}}$. But this is excluded thanks to Proposition 9.6. □

**10.6. The Foïaş-Strătilă property: application to the spectral theory of infinite measure preserving systems.** In Section 8, we have seen that any symmetric spectral measure coming from a dynamical system $(\Omega,\mathcal{F},\mu,T)$ of type $\mathbf{II}_\infty$ can be the spectral measure of $X_0$ where $\{X_n\}_{n\in\mathbb{Z}}$ is an ergodic square integrable and centered IDp process. We don't know the restrictions for such measures but the theory of Gaussian processes and ID-disjointness will allow us to exclude a whole family of measures.

DEFINITION 10.13. (See [**37**]) A symmetric measure $\sigma$ on $[-\pi,\pi[$, possesses the $(FS)$ property (for Foïaş and Strătilă) if, for any ergodic probability preserving dynamical system $(Y,\mathcal{C},\nu,S)$, every $f\in L_0^2(\nu,S)$ with spectral measure $\sigma$, is a Gaussian random variable.

The terminology comes from the "Foïaş and Strătilă" theorem: if the support of a symmetric measure $\sigma$ on $[-\pi,\pi[$ is $A\cup(-A)$ where $A$ is a Kronecker set, then $\sigma$ has the $(FS)$ property (see [**13**]).

PROPOSITION 10.14. *Let $(\Omega,\mathcal{F},\mu,T)$ be a dynamical system and $f\in L^2(\mu)$ with spectral measure $\sigma$ that possesses the $(FS)$ property, then $f=f1_{\mathfrak{P}}$ mod. $\mu$ where $\mathfrak{P}$ is the part of $(\Omega,\mathcal{F},\mu,T)$ of type $\mathbf{II}_1$.*

PROOF. First assume there exists a system $(\Omega,\mathcal{F},\mu,T)$ of type $\mathbf{II}_\infty$ with a vector $g$ in $L^2(\mu)$ with spectral measure $\sigma$ that possesses the $(FS)$ property. Let $\{g\circ T^n\}_{n\in\mathbb{Z}}$ the process, factor of $(\Omega,\mathcal{F},\mu,T)$ generated by $g$. By eventually removing a $T$-invariant set, we can suppose that $\Omega$ is the union of the supports of $\{g\circ T^n\}_{n\in\mathbb{Z}}$. Denote by $\mu_g$ its measure on $\mathbb{R}^{\mathbb{Z}}$ and take $X$ the centered IDp square integrable process, with Lévy measure $\mu_g$. $X$ is ergodic since $\mu_g$ is of type $\mathbf{II}_\infty$ and, thanks to the unitary isometry established in Proposition 8.1, the spectral measure of $X_0$ is $\sigma$. But, since $\sigma$ possesses the $(FS)$ property, $X_0$ is Gaussian, which is not possible since $X_0$ is IDp.

From now on, we don't make the hypothesis that the system $(\Omega,\mathcal{F},\mu,T)$ is of type $\mathbf{II}_\infty$. We only assume there exists a vector $f\in L^2(\mu)$ with spectral measure $\sigma$ that possesses the $(FS)$ property. Let $\mu_\infty:=\mu_{|\mathfrak{P}^c}$ and $\mu_f:=\mu_{|\mathfrak{P}}$. We have:

$$\begin{aligned}\hat{\sigma}(k) &= \int_{\Omega} f f \circ T^k d\mu \\ &= \int_{\Omega} (f1_{\mathfrak{P}})(f1_{\mathfrak{P}}) \circ T^k d\mu_f + \int_{\Omega} (f1_{\mathfrak{P}^c})(f1_{\mathfrak{P}^c}) \circ T^k d\mu_{\infty} = \hat{\sigma}_f(k) + \hat{\sigma}_{\infty}(k)\end{aligned}$$

where $\sigma_f + \sigma_\infty = \sigma$.

This yields to the relation of absolute continuity $\sigma_\infty \ll \sigma$ and as such, $\sigma_\infty$ possesses the $(FS)$ property too (see [**37**]) unless it is the zero measure. But the system $(\Omega, \mathcal{F}, \mu_\infty, T)$, of type $\mathbf{II}_\infty$, would have a vector, $f1_{\mathfrak{P}^c}$, with a spectral measure that possesses the $(FS)$ property. This is not possible from the first part of the proof. Hence $\sigma_\infty$ is zero and so is $f1_{\mathfrak{P}^c}$, that is $f = f1_{\mathfrak{P}}$ mod. $\mu$. □

## 11. Conclusion

For the sake of simplicity, we have mostly dealt with a single transformation, but many of the techniques used here can be applied more generally to the study of infinitely divisible objects whose Lévy measure is preserved by any kind of transformations, for example, the continuous time versions of our results are mostly straightforward, as are the multidimensional or the complex valued ones. The use of Poisson suspensions seems "natural" in some way.

We now ask some questions raised by our results:

Is a Poisson suspension with the Bernoulli property necessarily associated to a dissipative system ?

Is the Pinsker algebra of a suspension a Poissonian factor (this would lead to the possibility of refining the canonical decomposition once again, with the existence of a biggest invariant set where the suspension is $K$. The ID-disjointness of this new part with the others would be guaranteed by the disjointness between $K$ and zero entropy systems and the technique used in Theorem 9.27, the Pinsker algebra would play the role of the rigid Poissonian factor) ?

Do there exist semi-stable processes with an ergodic Lévy measure ?

An ergodic system $\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, Q, T\right)$ where $Q$ is the Lévy measure of a semi-stable processes is an example of *squashable* system (*completely squashable* in the stable case), in particular it doesn't admit a *law of large numbers* (see [**1**]). Then semi-stable processes with an ergodic Lévy measure would be ID-disjoint from ID processes whose ergodic Lévy measure admits a law of large numbers. Moreover, it is shown in [**1**] that a "generic" dynamical system in infinite measure admits a law of large numbers. We thus can deduce that semi-stable processes with an ergodic Lévy measure would be "rare" among ID processes. What special ergodic properties can we expect with such processes, for example, in terms of semi-stable joinings ?

Does there exist a class of Poisson suspension whose ergodic self-joinings are necessarily ID ?

We end this paper by mentioning that Poisson joinings of Poisson suspensions have been considered in [**17**] with another point of view than ours, that is, without considering the infinite divisibility of their distribution, but by looking at the action in the chaos of the

Markov operator describing the joining. As a result, Poisson joinings are exactly our ID joinings of Poisson suspensions. They also proved, as us, that these joinings are ergodic as soon as the suspensions are ergodic. This was to prove that ergodic Poisson suspensions have the so-called ELF property, that is, the weak closure of the off-diagonal joinings are ergodic. In the same paper, they do a deep analysis of the ELF property in general and prove many disjointness results that, thus, can be applied to Poisson suspensions. They also prove the ELF property for ergodic $S\alpha S$ processes (which, by Maruyama's result, follows from the fact that they are factors of their Poisson suspensions associated to their Levy measure (these suspensions remain ergodic by our results)) by showing that self-joinings of an ergodic $S\alpha S$ processes that remain $S\alpha S$ are ergodic (which is, in our framework, a consequence of the fact that they are ID joinings).

CHAPITRE 3

# Generic properties of Poisson suspensions and $\mathcal{P}$-entropy

ABSTRACT. We show that a generic Poisson suspension built above a dynamical system associated to a non atomic $\sigma$-finite infinite measure space has simple spectrum. To obtain this result, we prove generic properties of infinite measure preserving automorphisms that are similar to the finite measure case. Then we show some structure properties of such suspensions, notably some drastic differences with Gaussian systems, despite the similarity of their spectral structure. Finally, with the help of their associated Poisson suspension we define the $\mathcal{P}$-entropy of a measure (finite or infinite) preserving transformation. This gives a new non trivial invariant with some nice properties. When the measure is a probability measure, $\mathcal{P}$-entropy equals classical entropy.

## 1. Introduction

We consider a non atomic Borel space $(\Omega, \mathcal{F}, \mu)$ where $\mu$ is a $\sigma$-finite infinite measure and the space $(M_\Omega, \mathcal{M}_\mathcal{F})$ of measures $\nu$ on $(\Omega, \mathcal{F})$ satisfying $\nu(A) \in \mathbb{N}$ for all $A \in \mathcal{F}$ of finite $\mu$-measure. $\mathcal{M}_\mathcal{F}$ is the smallest $\sigma$-algebra on $M_\Omega$ such that the mappings $\nu \mapsto \nu(A)$ are measurable for all $A \in \mathcal{F}$ of finite $\mu$-measure. We denote by $N$ the identity on $(M_\Omega, \mathcal{M}_\mathcal{F})$. A *Poisson measure* is the triplet $(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_\mu)$ where $\mathcal{P}_\mu$ is the unique probability measure such that, for all finite collections $\{A_i\}$ of elements belonging to $\mathcal{F}$, disjoint and of finite $\mu$-measure, the $\{N(A_i)\}$ are independent and distributed as the Poisson law of parameter $\mu(A_i)$. If $T$ is a measure preserving automorphism of $(\Omega, \mathcal{F}, \mu)$, then $T^\star$ defined on $(M_\Omega, \mathcal{M}_\mathcal{F})$ by $T^\star(\nu)\{A\} = \nu\{T^{-1}(A)\}$ for all $A \in \mathcal{F}$ is a probability preserving automorphism of $(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_\mu)$, also called *Poissonian automorphism*. The quadruplet $(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_\mu, T^\star)$ is called the *Poisson suspension* constructed above the *base* $(\Omega, \mathcal{F}, \mu, T)$. The suspension is said to be *pure* if the base is ergodic, and *self-similar of index* $c$, $c \neq 1$, if there exists an automorphism $\varphi$ of $(\Omega, \mathcal{F}, \mu, T)$, that commutes with $T$ and such that $\varphi^\star\mu = c\mu$. Suppose $(\Omega, \mathcal{F}, \mu)$ is measurably isomorphic, through $\psi$, to $(X, \mathcal{X}, m)$ such that $\psi^\star\mu = m$ then $\psi^\star$ defines a measurable isomorphism between $(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_\mu)$ and $(M_X, \mathcal{M}_\mathcal{X}, \mathcal{P}_m)$ such that $(\psi^\star)^\star \mathcal{P}_\mu = \mathcal{P}_m$. Thus, in this context, we can always take $(\mathbb{R}, \mathcal{B}, \lambda)$ as base where $\lambda$ is the Lebesgue measure. Denote by $\mathcal{G}_\lambda$ (resp. $\mathcal{G}_{\mathcal{P}_\lambda}$) the group of all measure preserving automorphism of $(\mathbb{R}, \mathcal{B}, \lambda)$ (resp. $(M_\mathbb{R}, \mathcal{M}_\mathcal{B}, \mathcal{P}_\lambda)$) endowed with the coarse topology (also called the weak topology), $G_\lambda$ is seen as a subgroup of $\mathcal{G}_{\mathcal{P}_\lambda}$ through the correspondence between an automorphism $T$ and the corresponding Poissonian automorphism $T^\star$. It is shown in Chapter 2, that the topology induced by $\mathcal{G}_{\mathcal{P}_\lambda}$ on $\mathcal{G}_\lambda$ is precisely the initial coarse topology of $\mathcal{G}_\lambda$, moreover $G_\lambda$ is closed and meagre in $\mathcal{G}_{\mathcal{P}_\lambda}$. We have considered in Chapter 2 generic properties for a Poissonian automorphism: if a property concerns a dense $G_\delta$ set of Poissonian automorphisms in $\mathcal{G}_\lambda$, then this property is said *generic*. Thus, through the correspondence of topologies, finding generic properties for Poissonian automorphisms of $(M_\mathbb{R}, \mathcal{M}_\mathcal{B}, \mathcal{P}_\lambda)$ is equivalent to finding generic properties for automorphisms of the base $(\mathbb{R}, \mathcal{B}, \lambda)$. This has enabled to prove in Chapter 2 that a generic Poisson suspension is pure, not self-similar, weakly mixing and rigid. By $C(T^\star)$, we denote the subgroup of $\mathcal{G}_{\mathcal{P}_\lambda}$ of measure preserving automorphisms commuting with $T^\star$, in other terms $C(T^\star)$ is the *centralizer*. $C^p(T^\star)$, the *Poissonian centralizer*, is defined by $C^p(T^\star) := C(T^\star) \cap \mathcal{G}_\lambda$. The space $L^2(\mathcal{P}_\lambda)$ has a remarkable Fock space structure, that is, $L^2(\mathcal{P}_\lambda)$ can be seen as the orthogonal sum $\mathfrak{S}^0 L^2(\lambda) \oplus \mathfrak{S}^1 L^2(\lambda) \oplus \cdots \oplus \mathfrak{S}^n L^2(\lambda) \oplus \cdots$ where $\mathfrak{S}^n L^2(\lambda)$ is the $n$-th symmetric tensor power of $L^2(\lambda)$ and called the $n$-th *chaos*. Thus (see Chapter 2), if we denote by $\sigma$ the maximal spectral type of $(\mathbb{R}, \mathcal{B}, \lambda, T)$, since the maximal spectral of the $n$-th chaos is the $n$-th convolution power $\sigma^{*n}$, the (reduced) maximal spectral type of $(M_\mathbb{R}, \mathcal{M}_\mathcal{B}, \mathcal{P}_\lambda, T^\star)$ is $\sum_{k=1}^{+\infty} \frac{1}{k!}\sigma^{*k}$.

Our first aim is to prove that a generic Poisson suspension has simple spectrum. To do so we first prove, in Section 2, two generic properties for the base, the simplicity of the spectrum which will give us the simplicity in the first chaos of the suspension, and a property which, thanks to a theorem of Ageev ([**3**] and [**4**]), will control what is going on for the successive chaos. In particular, it will give us the simplicity of the spectrum in each chaos and the mutual singularity of their maximal spectral type, which gives the desired result. In section 3, we show some properties shared by a generic Poisson suspension (these results are a bit more general), namely, it is non isomorphic to any Gaussian system, none of its factor has an independent complement and its centralizer is exactly its Poissonian centralizer. We also give some sufficient condition for an infinitely divisible process to be isomorphic to its Poissonian extension. At last, in Section 5, we introduce a new invariant, the $\mathcal{P}$-entropy for any dynamical system, and we show that it generalizes classical entropy. We prove that zero $\mathcal{P}$-entropy is generic for an infinite measure preserving system.

## 2. Two generic properties on a non atomic $\sigma$-finite infinite measure space

We consider the space $(\mathbb{R}, \mathcal{B}, \lambda)$ and the group $\mathcal{G}_\lambda$ endowed with the coarse topology generated by the distance $d$ which makes this group separable and complete:

$$d(S,T) := \sum_{n=1}^{+\infty} \frac{\lambda(SA_n \triangle TA_n) + \lambda(S^{-1}A_n \triangle T^{-1}A_n)}{2^n}$$

where the $A_n$ are the dyadic intervals of length bounded by 1.

### 2.1. Simplicity of the spectrum.

THEOREM 2.1. *The set of measure preserving automorphisms on* $(\mathbb{R}, \mathcal{B}, \lambda)$ *with simple spectrum is a dense* $G_\delta$ *subset.*

PROOF. We consider the set $\mathcal{U}$ of all unitary operators acting on the Hilbert space $L^2(\lambda)$. $\mathcal{U}$ is endowed with its coarse topology given by the distance $\delta$:

$$\delta(U,V) := \sum_{n=1}^{+\infty} \frac{1}{2^n} \|U1_{A_n} - V1_{A_n}\|$$

Thus, the map $\Phi : T \mapsto U_T$ is continuous from $\mathcal{G}_\lambda$ into $\mathcal{U}$. Now, the set $\mathcal{W}_1$ of unitary operators with simple spectrum is a $G_\delta$ set in $\mathcal{U}$ (see [**44**]), and so is $\Phi^{-1}\mathcal{W}_1$. We have to show it is dense in $\mathcal{G}_\lambda$, but is is sufficient to exhibit one ergodic element $T \in \Phi^{-1}\mathcal{W}_1$ , since then, the set $\{S^{-1}TS,\ S \in \mathcal{G}_\lambda\}$ is dense in $\mathcal{G}_\lambda$ (see [**44**] again) and any such $S^{-1}TS$ is in $\Phi^{-1}\mathcal{W}_1$. We can refer to [**30**] for such a transformation $T$. □

**2.2. Another generic property.** Now we consider automorphisms $T$ with the following property:

For $\theta$, $0 < \theta < 1$,

(8) $$\theta Id + (1-\theta)\, U_T \text{ is in the weak closure of } \{U_{T^n}\}_{n\in\mathbb{Z}}$$

THEOREM 2.2. *The set of measure preserving automorphisms $T$ on $(\Omega, \mathcal{F}, \mu)$ that verifies property (8) contains a dense $G_\delta$ subset.*

PROOF. We follow the ideas and notation of the proof of Theorem 3.1 in [**5**]. Fix $0 < \theta < 1$. We consider the following $k$-indexed dyadic partitions of $\mathbb{R}$:

On $\left[-2^k, 2^k\right)$, we consider the half-open intervals of equal length, $\left[0, \frac{1}{2^k}\right)$ and its $2^{2k+1}$ translates, and any transformation $\tilde{T}$ that cyclically permutes these intervals and is the identity on $\mathbb{R} \setminus \left[-2^k, 2^k\right)$, $\tilde{T}$ is said of rank $k$. We now define a new automorphism $T$ by only modifying $\widetilde{T}$ on the sets $\widetilde{T}^{2^{2k+1}-1}\left[0, \frac{1}{2^k}\right)$ and $\left[2^k, 2^k + \frac{1-\theta}{2^k}\right)$ in the following way:

$Tx = \widetilde{T}x$ if $x \in \widetilde{T}^{2^{2k+1}-1}\left[0, \frac{\theta}{2^k}\right)$

$Tx = \widetilde{T}^{-2^{2k+1}+1}x + 2^k - \frac{\theta}{2^k}$ if $x \in \widetilde{T}^{2^{2k+1}-1}\left[\frac{\theta}{2^k}, \frac{1}{2^k}\right)$

$Tx = x - 2^k + \frac{\theta}{2^k}$ if $x \in \left[2^k, 2^k + \frac{1-\theta}{2^k}\right)$

that is, we can represent the automorphism $T$ into two columns of size differing by one level and base of length $\frac{\theta}{2^k}$ and $\frac{1-\theta}{2^k}$ respectively, $T$ being the identity on the rest of the space. Let $\mathcal{O}_k$ denote the set of such automorphisms. For an automorphism $S$, let $B_\delta(S)$ denote the open ball of radius $\delta > 0$ centered at $S$. For any sequence of positive functions $\delta_k(S)$ defined on $\mathcal{O}_k$, define the sets:

$$C_n = \bigcup_{k>n} \bigcup_{S\in\mathcal{O}_k} B_{\delta_k}(S)$$

$$C(\delta_1, \delta_2, \dots) = \bigcap_{n\geq 1} C_n$$

For any $n$, $C_n$ is open, we will show it is dense. Indeed, if $R$ is an automorphism, there exists a sequence $\widetilde{T}_{n_k}$ of cyclic permutations of rank $n_k$ ($n_k \uparrow \infty$) converging to $R$ in the uniform topology. But it is clear that the sequence of $T_{n_k}$ which are the $\widetilde{T}_{n_k}$ modified as above are closer to $\widetilde{T}_{n_k}$ in the uniform topology (since they only differ on an increasingly small set) as $k$ increases, that is, $T_{n_k}$ tends to $R$. Thus $C(\delta_1, \delta_2, \dots)$ is a dense $G_\delta$ set for any sequence of positive functions $\delta_k$. Now for $S \in \mathcal{O}_k$, define $\delta_k(S)$ such that, $\delta_k(S) < \frac{1}{k}$ and if $T \in B_{\delta_k}(S)$ then $T^{2^{2k+1}+1} \in B_{\frac{1}{k}}\left(S^{2^{2k+1}+1}\right)$ (this choice is possible since the mapping $S \mapsto S^n$ is continuous in the coarse topology).

We want to show that if $T \in C(\delta_1, \delta_2, \dots)$ for this choice of $\delta_k$ then $\theta Id + (1-\theta) U_T$ is in the weak closure of the $U_T^n$. Take $A$ and $B$ two distinct intervals with dyadic extremities and $k$ large enough for that $A$ and $B$ are included in $\left[-2^k, 2^k\right)$ and such that $A$ and $B$ are union of intervals of length $\frac{1}{2^k}$. It is now easily seen that for all $n > k$, we have:

$$m\left(S^{2^{2n+1}+1} A \cap B\right) = \theta m(A \cap B) + (1-\theta) m(SA \cap B)$$

for any $S$ in $\mathcal{O}_n$. We can write the following inequalities:

$$\begin{aligned}
&\left|m\left(T^{2^{2n+1}+1} A \cap B\right) - \theta m(A \cap B) + (1-\theta) m(TA \cap B)\right| \\
&\leq \left|m\left(T^{2^{2n+1}+1} A \cap B\right) - m\left(S^{2^{2n+1}+1} A \cap B\right)\right| \\
&\qquad\qquad + (1-\theta)\left|m(TA \cap B) - m(SA \cap B)\right| \\
&\leq m\left(T^{2^{2n+1}+1} A \triangle S^{2^{2n+1}+1} A\right) + (1-\theta) m(TA \triangle SA)
\end{aligned}$$

Now, there exists a sequence $S_{n_k}$ $(n_k \uparrow \infty)$ such that $S_{n_k}$ is in $\mathcal{O}_{n_k}$ and $T \in B_{\delta_{n_k}}(S_{n_k})$. This implies that $d\left(T^{2^{n_k+1}+1}, S_{n_k}^{2^{2n_k+1}+1}\right)$ tends to 0 and thus, $m\left(T^{2^{2n_k+1}+1} A \triangle S_{n_k}^{2^{2n_k+1}+1} A\right)$, $m(TA \triangle S_{n_k} A)$ and thus $\left|m\left(T^{2^{2n_k+1}+1} A \cap B\right) - \theta m(A \cap B) + (1-\theta) m(TA \cap B)\right|$ tends to 0, and this gives the desired property for the set $A$ and $B$. But this is enough to verify that $U_T^{2^{2n_k+1}+1}$ tends weakly to $\theta Id + (1-\theta) U_T$.

Finally, the desired property will be generic for countably many $\theta$, $0 < \theta < 1$. □

## 3. The results applied to Poisson suspensions

It is easily deduced that a Poisson suspension will have simple spectrum if and only if it has simple spectrum on each chaos and if, for all $m \neq n$, $\sigma^{*m} \perp \sigma^{*n}$. We thus are led to seek criteria under which these conditions are satisfied.

Ageev in [**3**] (see also [**4**]) has proved the following result (we present a reduced version of it):

Proposition 3.1. *Let $U$ be a unitary operator acting on a complex Hilbert space $H$. Assume $U$ has simple and continuous spectrum and moreover, that, for countably many $\theta$, $0 < \theta < 1$, $\theta Id + (1-\theta) U$ belongs to the weak closure of powers of $U$.*

*Then, for all $n \geq 1$ $U^{\otimes n}$ has simple spectrum on $\mathfrak{S}^n H$ (the $n$-th symmetric tensor power of $H$) and the corresponding maximal spectral types are all pairwise mutually singular.*

This leads immediately to:

PROPOSITION 3.2. *Let* $(\Omega, \mathcal{F}, \mu, T)$ *be a dynamical system such that* $U_T$ *has simple continuous spectrum on* $L^2(\mu)$ *and possesses property (8) for countably many* $\theta$, $0 < \theta < 1$, *then* $(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_\mu, T^\star)$ *has simple spectrum.*

As a corollary of the results obtained in Section 2, we obtain the desired theorem:

THEOREM 3.3. *A generic Poisson suspension has simple spectrum.*

## 4. Some consequences

In this section, we consider a base $(\Omega, \mathcal{F}, \mu, T)$ such that $\sigma$, its maximal spectral type, is singular with respect to all its successive convolution powers, and $(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_\mu, T^\star)$, the corresponding Poisson suspension. From the results of the preceding section, it concerns the generic Poisson suspension.

We begin with a result similar to the Gaussian case:

PROPOSITION 4.1. $C(T^\star) = C^p(T^\star)$

PROOF. Take $S \in C(T^\star)$ and any $f$ in the first chaos. $f \circ S$ has the same spectral measure as $f$, thus, since it is absolutely continuous with respect to $\sigma$ and that $\sigma \perp \sum_{k=2}^{+\infty} \frac{1}{k!}\sigma^{*k}$, $f \circ S$ also belongs to the first chaos. It is thus clear that the graph joining given by $\mathcal{P}_\mu(A \cap S^{-1}B)$ is an ID joining (see Chapter 2 for the definition). As a consequence of Proposition 9.18 in Chapter 2, $S$ in indeed a Poissonian automorphism, that is, $S \in C^p(T^\star)$. □

We now show some drastic differences with Gaussian systems (the following proposition has been proved by François Parreau without been published (Mariusz Lemańczyk, personnal communication):

PROPOSITION 4.2. $(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_\mu, T^\star)$ *is not isomorphic to a Gaussian system.*

PROOF. Suppose there exists a Gaussian system $(X, \mathcal{X}, \nu, S)$ which is isomorphic to $(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_\mu, T^\star)$ by an isomorphism $\varphi$. Call $m$ the maximal spectral type of the first chaos of $L^2(\nu)$. Due to the assumptions on the spectral structure of $L^2(\mathcal{P}_\mu)$, we can't have $m \perp \sigma$. So, consider $\rho$ such that $\rho \ll m$ and $\rho \ll \sigma$. Take $f$, a vector in the first chaos of $L^2(\nu)$ with spectral measure $\rho$, $f$ is thus a Gaussian random variable. Now the vector $f \circ \varphi^{-1}$ belongs to $L^2(\mathcal{P}_\mu)$, has spectral measure $\rho$ and is a Gaussian random variable under $\mathcal{P}_\mu$. But since $\rho \ll \sigma$, we have $\rho \perp \sum_{k=2}^{+\infty} \frac{1}{k!}\sigma^{*k}$ which implies that $f \circ \varphi^{-1}$ belongs to the first chaos of $L^2(\mathcal{P}_\mu)$. But, due to ID-disjointness (see Chapter 2), there is no Gaussian random variable in the first chaos of $L^2(\mathcal{P}_\mu)$ which leads to a contradiction. □

PROPOSITION 4.3. *Assume now that* $(\Omega, \mathcal{F}, \mu, T)$ *is ergodic with simple spectrum, then no nontrivial factor of* $(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_\mu, T^\star)$ *has an independent complement.*

PROOF. Let $\mathcal{A}$ be a non trivial $T^{\star}$-invariant sub-$\sigma$-algebra of $\mathcal{M}_{\mathcal{F}}$ and suppose there exist another $T^{\star}$-invariant sub-$\sigma$-algebra $\mathcal{B}$ independent of $\mathcal{A}$ and such that $\mathcal{M}_{\mathcal{F}} = \mathcal{A} \vee \mathcal{B}$. Denote by $\nu_a$ (resp. $\nu_b$) the (reduced) maximal spectral type of $(M_\Omega, \mathcal{A}, \mathcal{P}_\mu, T^\star)$ (resp. $(M_\Omega, \mathcal{B}, \mathcal{P}_\mu, T^\star)$). We must have, up to equivalence of measures, $\sum_{k=1}^{+\infty} \frac{1}{k!}\sigma^{*k} = \nu_a + \nu_b + \nu_a * \nu_b$; the singularity of $\sigma$ from its successive convolution powers, the non triviality of $\mathcal{A}$ and the simplicity of the spectrum of $(\Omega, \mathcal{F}, \mu, T)$ necessarily implies that $\sigma \perp \nu_a$ nor $\sigma \perp \nu_b$ takes place. But this means that there exist two stochastic integrals in the first chaos $I(f_a)$ and $I(f_b)$ measurable with respect to $\mathcal{A}$ and $\mathcal{B}$ respectively. Then $\{I(f_a) \circ T^{\star n}\}_{n\in\mathbb{Z}}$ and $\{I(f_b) \circ T^{\star n}\}_{n\in\mathbb{Z}}$ are independent which is in fact impossible because this would imply the disjunction of the support of the functions $\{f_a \circ T^n\}_{n\in\mathbb{Z}}$ with the support of the functions $\{f_b \circ T^n\}_{n\in\mathbb{Z}}$ (see [**38**]) and these two $T$-invariant sets cannot be disjoint since $(\Omega, \mathcal{F}, \mu, T)$ is ergodic. □

We will study factors of a Poisson suspension with simple spectrum generated by processes defined by stochastic integrals.

PROPOSITION 4.4. *Let $\{X_n\}_{n\in\mathbb{Z}}$ be an ID stationary process of Lévy measure $Q$. Assume that $\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\otimes\mathbb{Z}}, Q, T\right)$ has simple spectrum and its maximal spectral type $\sigma$ verifies $\sigma \perp \sum_{k=2}^{+\infty} \frac{1}{k!}\sigma^{*k}$, assume moreover that the Lévy measure of the coordinate $X_0$ is a finite measure. Then $\{X_n\}_{n\in\mathbb{Z}}$ is isomorphic to its Poissonian extension.*

PROOF. Let $(M_{\mathbb{R}^{\mathbb{Z}}}, \mathcal{M}_{\mathcal{B}^{\otimes\mathbb{Z}}}, \mathcal{P}_Q, T^\star)$ be the Poissonian extension of $\{X_n\}_{n\in\mathbb{Z}}$, that is, the suspension built above the Lévy measure of $X$. $X$ admits the representation as a stochastic integral $\{I(X_0) \circ T^{\star n}\}_{n\in\mathbb{Z}}$ with respect to the suspension as proved by Maruyama in [**38**]. We will prove that the sub-$\sigma$-algebra of $\mathcal{M}_{\mathcal{B}^{\otimes\mathbb{Z}}}$ generated by $X$ is in fact the whole $\sigma$-algebra $\mathcal{M}_{\mathcal{B}^{\otimes\mathbb{Z}}}$. We consider vectors of the kind $\exp i\langle a, X\rangle - \mathbb{E}[\exp i\langle a, X\rangle]$ for $a \in \mathfrak{A}$. We have shown in Chapter 2 that the spectral measure of such vector is $|\mathbb{E}[\exp i\langle a, X\rangle]|^2 e(\sigma_a)$ where $\sigma_a$ is the spectral measure of the vector $\exp i\langle a, X\rangle - 1$ under $Q$. Call $C_a$ the cyclic space associated to $\exp i\langle a, X\rangle - \mathbb{E}[\exp i\langle a, X\rangle]$, since $\sigma_a \ll e(\sigma_a)$, this cyclic space contains vectors with $\sigma_a$ as spectral measure. Thanks to the mutual singularity of the $\sigma^{*n}$, this vectors necessarily belong to the first chaos, moreover, since $Q$ has simple spectrum, $I(\exp i\langle a, X\rangle - 1)$ and its iterates stand among them. We now show that linear combinations of vectors $\exp i\langle a, X\rangle - 1$ for $a \in \mathfrak{A}$ span the whole space $L^2(Q)$.

Consider the increasing sequence $\mathcal{B}_n$ of sub-$\sigma$-algebra of $\mathcal{B}^{\otimes\mathbb{Z}}$ consisting of finite dimensional vectors between $-n$ and $n$ ($\mathcal{B}_n := \sigma\{X_{-n}, \ldots, X_0, \ldots, X_n\}$) and $L^2(Q, \mathcal{B}_n)$, the closed subspace of $\mathcal{B}_n$-measurable and square integrable vectors. If we denote by $\mathfrak{A}_n$ the sequences $\{a_k\}_{k\in\mathbb{Z}}$ whose coordinates are 0 if $|k| > n$, then, for all $a \in \mathfrak{A}_n$, $\exp i\langle a, X\rangle - 1$ belongs to $L^2(Q, \mathcal{B}_n)$. Denote by $f$ a vector in $L^2(Q, \mathcal{B}_n)$ which is orthogonal to $\exp i\langle a, X\rangle - 1$ for all $a \in \mathfrak{A}_n$. If $Q_n$ is the image measure of $Q$ under the projection $\{x_k\}_{k\in\mathbb{Z}} \mapsto \{x_{-n}, \ldots, x_0, \ldots, x_n\}$ on $\mathbb{R}^{2n+1}$, with a slight abuse of notation, we can write, for all $a \in \mathfrak{A}_n$:

$$\int_{\mathbb{R}^{2n+1}} (\exp i \langle a, X \rangle - 1) f dQ_n = 0$$

Since $\exp i \langle a, X \rangle - 1$ is 0 on $\{0, \ldots, 0, \ldots, 0\} \in \mathbb{R}^{2n+1}$, we can also write:

$$\int_{\mathbb{R}^{2n+1}} (\exp i \langle a, X \rangle - 1) f dQ_n^0 = 0$$

where $Q_n^0$ is $Q_n$ restricted to $\mathbb{R}^{2n+1} \setminus \{0, \ldots, 0, \ldots, 0\}$. But $Q_n^0$ is precisely the Lévy measure of the vector $\{X_{-n}, \ldots, X_0, \ldots, X_n\}$ and as such, since the Lévy measure of $X_0$ is a finite measure, $Q_n^0$ is also a finite measure. This also means that $Q_n \{0, \ldots, 0, \ldots, 0\} = +\infty$, since $Q$ is an infinite measure. Then, if $f \in L^2(Q, \mathcal{B}_n)$, $f$ is also integrable under $Q_n^0$, $f\{0, \ldots, 0, \ldots, 0\} = 0$ and we can write, for all $a \in \mathfrak{A}_n$:

$$\int_{\mathbb{R}^{2n+1}} \exp i \langle a, X \rangle f dQ_n^0 = \int_{\mathbb{R}^{2n+1}} f dQ_n^0$$

but this means that the complex measure $f dQ_n^0$ is a multiple of the Dirac mass at $\{0, \ldots, 0, \ldots, 0\}$ which is impossible unless $f$ is 0 on $\mathbb{R}^{2n+1} \setminus \{0, \ldots, 0, \ldots, 0\}$. Then $f = 0$ and this proves that linear combinations of vectors $\exp i \langle a, X \rangle - 1$ for $a \in \mathfrak{A}_n$ span $L^2(Q, \mathcal{B}_n)$. Since $L^2(Q)$ is the closure of $\bigcup_{n \in \mathbb{N}} L^2(Q, \mathcal{B}_n)$, we can conclude that linear combinations of vectors $\exp i \langle a, X \rangle - 1$ for $a \in \mathfrak{A}$ span $L^2(Q)$ which is the desired result.

Then, thanks to the isometry between $L^2(Q)$ and the first chaos, we can see that any vector in the first chaos is measurable with respect to $X$ and, since the smallest $\sigma$-algebra generated by the first chaos is nothing else than the whole $\sigma$-algebra $\mathcal{M}_{\mathcal{B} \otimes \mathbb{Z}}$, the proposition is proved. □

This proposition solves the problem of classifying a vast class of ID processes since they inherit the properties of a Poisson suspension whose study is much more tractable. Moreover it allows to identify factors generated by stochastic integrals as Poissonian factors:

PROPOSITION 4.5. *Let* $(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_\mu, T^\star)$ *a Poisson suspension such that* $\mu$ *has simple spectrum and whose maximal spectral type* $\sigma$ *verifies* $\sigma \perp \sum_{k=2}^{+\infty} \frac{1}{k!} \sigma^{*k}$, *then a factor generated by a stochastic integral* $I(f)$ *such that the Lévy measure of* $I(f)$ *is a finite measure, is the Poissonian factor* $\mathcal{A}_N$ *where* $\mathcal{A}$ *is the sub-*$\sigma$*-algebra of* $\mathcal{F}$ *generated by the process* $\{f \circ T^n\}_{n \in \mathbb{Z}}$.

## 5. A new invariant

There has been some discussion to give a reasonable definition for the entropy of a dynamical system $(\Omega, \mathcal{F}, \mu, T)$ of type $\mathbf{II}_\infty$(in the conservative case, see [**35**]). Through Poisson suspension, we propose a definition of a kind of entropy which gives a new invariant and coincides with usual entropy for a probability preserving transformation.

DEFINITION 5.1. Consider a system $(\Omega, \mathcal{F}, \mu, T)$ and its associated Poisson suspension $(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_\mu, T^\star)$. Denote by $h(T^\star)$ the (classical) entropy of $T^\star$. We define the number $h_\mathcal{P}(T)$ by $h_\mathcal{P}(T) := h(T^\star)$ and call it the *$\mathcal{P}$-entropy* of $T$.

### 5.1. Basic properties.

PROPOSITION 5.2. *If $S$ is a factor of $T$ then $h_\mathcal{P}(S) \leq h_\mathcal{P}(T)$*
*If $S$ is isomorphic to $T$ then $h_\mathcal{P}(S) = h_\mathcal{P}(T)$*
$h_\mathcal{P}(T^n) = |n| \, h_\mathcal{P}(T)$
*If $T$ is the time 1 map of a measurable flow $\{T_t\}_{t\in\mathbb{R}}$, $h_\mathcal{P}(T_t) = |t| \, h_\mathcal{P}(T)$*
*Suppose $A$ is a $T$-invariant set, denote by $T_A$ the transformation restricted to $A$, then $h_\mathcal{P}(T) = h_\mathcal{P}(T_A) + h_\mathcal{P}(T_{A^c})$*

PROOF. The proof of the first two facts is a consequence of the following (see Chapter 2):
Suppose $\varphi$ is a factor map from $(\Omega, \mathcal{F}, \mu, T)$ to $(\Omega', \mathcal{F}', \mu', S)$ then $\varphi^\star$ defines a factor map from $(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_\mu, T^\star)$ to $(M_{\Omega'}, \mathcal{M}_{\mathcal{F}'}, \mathcal{P}_{\mu'}, S^\star)$.
The last one comes from the fact that the system $(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_\mu, T^\star)$ is isomorphic to $\left(M_A \times M_{A^c}, \mathcal{M}_{\mathcal{F}_{|A}} \otimes \mathcal{M}_{\mathcal{F}_{|A^c}}, \mathcal{P}_{\mu_{|A}} \otimes \mathcal{P}_{\mu_{|A^c}}, T_A^\star \times T_{A^c}^\star\right)$. □

**5.2. The case of probability preserving transformations.** We consider here the particular case where $\mu$ is a probability measure. The non-ergodic Poisson suspension $(M_\Omega, \mathcal{M}_\mathcal{F}, \mathcal{P}_\mu, T^\star)$ has the following non-trivial $T^\star$-invariant sets:

$$\{N(\Omega) = k\}$$

Moreover $\mathcal{P}_\mu \{N(\Omega) = k\} = \frac{e^{-1}}{k!}$. Now it is easy to see that $T^\star$ restricted on $\{N(\Omega) = k\}$ is isomorphic to $\left(\Omega^k, \mathfrak{S}^k\mathcal{F}, \mu^{\otimes k}, T \times \cdots \times T\right)$, where $\mathfrak{S}^k\mathcal{F}$ is the sub-$\sigma$-algebra of $\mathcal{F}^{\otimes k}$ invariant by the group of permutations. Denote by $h^{(k)}(T)$ the (classical) entropy of $\left(\Omega^k, \mathfrak{S}^k\mathcal{F}, \mu^{\otimes k}, T \times \cdots \times T\right)$, we can give a formula for the $\mathcal{P}$-entropy of $T$, namely:

$$h_\mathcal{P}(T) = e^{-1} \sum_{k=1}^{+\infty} \frac{h^{(k)}(T)}{k!}$$

But, since $h^{(k)}(T) = kh(T)$,

$$h_\mathcal{P}(T) = h(T)$$

Thus the $\mathcal{P}$-entropy for probability preserving transformations is not a new concept as it reduces to the study of classical entropy. The interesting aspect is for infinite measure preserving systems.

### 5.3. The infinite measure case.

PROPOSITION 5.3. *Suppose the maximal spectral type of $(\Omega, \mathcal{F}, \mu, T)$ is singular, then $h_\mathcal{P}(T) = 0$.*

PROOF. This is a consequence of Proposition 4.12 in Chapter 2. □

COROLLARY 5.4. *Rigidity implies* $h_{\mathcal{P}}(T) = 0$

Thus, a generic transformation has zero $\mathcal{P}$-entropy.
We recall what a remotely infinite system is:

DEFINITION 5.5. $(\Omega, \mathcal{F}, \mu, T)$ is said to be *remotely infinite* if there exists a sub-$\sigma$-field $\mathcal{G} \subset \mathcal{F}$ such that:

$$T^{-1}\mathcal{G} \subseteq \mathcal{G}$$

$$\vee_{n\in\mathbb{Z}} T^{-n}\mathcal{G} = \mathcal{F}$$

and

$$\mathcal{F}_f \cap_{n\in\mathbb{Z}} T^{-n}\mathcal{G} = \{\emptyset\}$$

These systems have been introduced by Krengel and Sucheston in [**36**], but with the additional requirement that the measure $\mu$ restricted to $\mathcal{G}$ must be $\sigma$-finite. For example, $K$ automorphisms with type $\mathbf{II}_\infty$ are remotely infinite systems. We have shown in Chapter 2 that Poisson suspensions build above those systems are $K$. Thus remotely infinite systems have completely positive $\mathcal{P}$-entropy, that is, if $S$ is a ($\sigma$-finite) factor of $T$, $h_{\mathcal{P}}(S) > 0$.

PROPOSITION 5.6. *A remotely infinite system is strongly disjoint from a system with a non trivial zero $\mathcal{P}$-entropy factor.*

PROOF. It is basically the same proof as in Theorem 9.27 in Chapter 2 were it is proved the strong disjointness between a rigidity-free system and a system with a rigid factor. The role of the rigid factor being played by the zero $\mathcal{P}$-entropy factor. □

To show that the $\mathcal{P}$-entropy is not a trivial quantity, we point out the following observations:

- It can take all values between 0 and $+\infty$ (0 and $+\infty$ included) (our examples of ergodic systems with finite and non zero $\mathcal{P}$-entropy are dissipative)
- Strictly positive entropy can occur among conservative ergodic systems as there are conservative remotely infinite systems (we don't know the value of the $\mathcal{P}$-entropy)

We end this section with a natural question: Does it exist a biggest factor where the $\mathcal{P}$-entropy is zero ?

We believe it is so. To achieve this, one must prove that the Pinsker factor of the corresponding suspension is in fact a Poissonian factor. This would enable to show that remotely infinite systems are exactly those systems with completely positive $\mathcal{P}$-entropy.

At last, we don't know if our notion coincides with the entropy of conservative systems defined through induced transformation (see [**35**]).

CHAPITRE 4

# Some non asymptotic tail estimates for Hawkes processes

*Joint work with Patricia Reynaud-Bouret (ENS Paris)*

ABSTRACT. We use the Poisson cluster process structure of a Hawkes process to derive non asymptotic estimates of the tail of the extinction time, of the coupling time or of the number of points per interval. This allows us to define a time cutting of the process into approximating independent pieces. Then we can easily derive exponential inequalities for Hawkes processes which can precise the ergodic theorem.

## 1. Introduction

The Hawkes processes have been introduced by Hawkes in [**28**]. Since then they are especially applied to earthquake occurrences (see [**58**]), but have recently found applications to DNA modeling (see [**25**]). In particular, an assumption which was not very realistic for earthquakes has become quite evident in this framework: the support of the reproduction measure is known and bounded. The primary work is motivated by getting non asymptotic concentration inequalities for the Hawkes process, using intensively the bounded support assumption. Those concentration inequalities are fundamental to construct adaptive estimation procedure as the penalized model selection [**40, 50**]. To do so, we study intensively in this paper the link between cluster length, extinction time and time cutting into approximating independent pieces. Doing the necessary computations, we find out that other possible assumptions are also giving nice estimates of those quantities. Those estimates allow us to answer non asymptotically to some problems studied by Brémaud, Nappo and Torrisi in [**10**] on approximate simulation. But first, let us start by presenting the model and giving the main notations.

A point process $N$ is a countable random set of points on $\mathbb{R}$ without accumulation. In an equivalent way, $N$ denotes the point measure, i.e. the sum of the Dirac measures in each point of $N$. Consequently, $N(\mathcal{A})$ is the number of points of $N$ in $\mathcal{A}$, $N_{|\mathcal{A}}$ represents the points of $N$ in $\mathcal{A}$; if $N'$ is another point process, $N + N'$ is the set of points that are both in $N$ and $N'$. The Hawkes process $N^h$ is a point process whose intrinsic stochastic intensity is defined by:

$$\Lambda(t) = \lambda + \int_{-\infty}^{t^-} h(t-u)N(du) \tag{9}$$

where $\lambda$ is a positive constant and $h$ is a positive function with support in $\mathbb{R}_+$ such that $\int_0^{+\infty} h < 1$. We refer to [**14**] for the basic definitions of intensity and point process. We call $h$ the reproduction function. The reproduction measure is $\mu(dt) = h(t)dt$, where $dt$ represents the Lebesgue measure on the real line.

Hawkes and Oakes prove in [**29**] that $N^h$ can be seen as a generalized branching process and admits a cluster structure. The structure is based on inductive constructions of the points of $N^h$ on the real line, which can be interpreted, for a more visual approach, as births in different families. In this setup, the reproduction measure $\mu$ (with support in $\mathbb{R}_+$) is not necessarily absolutely continuous with respect to the Lebesgue measure. However, to avoid multiplicities on points (which would means simultaneous births at the same date), we make the additional assumption that the measure is continuous.

**The basic cluster process.** Shortly speaking, considering the birth of an ancestor at time 0, the cluster associated to this ancestor is the set of births of all descendants of all generations of this ancestor, where the ancestor is included.

To fix the notations, let us consider an i.i.d. sequence $\{P_{i,j}\}_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ of Poisson variables with parameter $p = \mu([0,\infty))$. Let us consider independently an i.i.d. sequence $\{X_{i,j,k}\}_{(i,j,k)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$ of positive variables with law given by $\mu/p$. Let $m = \mathbb{E}(X_{i,j,k})$, $v = \mathrm{Var}(X_{i,j,k})$ and $\ell(t) = \log\left[\mathbb{E}(e^{tX_{i,j,k}})\right]$ if they exist.

We construct now the successive generations which constitute the Hawkes process. The 0th generation is given by the ancestor $\{0\}$. The number of births in this generation is $K_0 = 1$, the total number of births in the family until the 0th generation is $W_0 = 1$. The successive births in this generation are given by $\{X_1^0 = 0\}$.

By induction, let us assume that we have already constructed the $(n-1)$th generation, i.e. we know the following quantities: $K_{n-1}$, the number of births in the $(n-1)$th generation, $W_{n-1}$, the total number of births in the family until the $(n-1)$th generation with the addition of the successive births in the $(n-1)$th generation $\{X_1^{n-1}, \dots, X_{K_{n-1}}^{n-1}\}$.

Then the $n$th generation is constructed as follows:

- if $K_{n-1} = 0$ then the $(n-1)$th generation is empty and the $n$th generation does not exist. We set $K_n = 0$ and $W_n = W_{n-1}$.
- if $K_{n-1} > 0$ then
  - $K_n = P_{n,1} + \dots + P_{n,K_{n-1}}$ is the number of births in the $n$th generation,
  - $W_n = W_{n-1} + K_n$ is the total number of births until the $n$th generation,
  - the births of the $n$th generations are given by:

$$\left\{X_1^{n-1} + X_{n,1,1}, \dots, X_1^{n-1} + X_{n,1,P_{n,1}}\right\}$$

which are the births of the children of the parent born at $X_1^{n-1}$,

$$\left\{X_2^{n-1} + X_{n,2,1}, \dots, X_2^{n-1} + X_{n,2,P_{n,2}}\right\}$$

which are the births of the children of the parent born at $X_2^{n-1}$,

$$\dots$$

$$\left\{X_{K_{n-1}}^{n-1} + X_{n,K_{n-1},1}, \dots, X_{K_{n-1}}^{n-1} + X_{n,K_{n-1},P_{n,K_{n-1}}}\right\}$$

which are the births of the children of the parent born at $X^{n-1}_{K_{n-1}}$.

All these points are the births in the $n$th generation. We arrange them by increasing order to obtain $\{X_1^n, \dots, X_{K_n}^n\}$, the successive births in the $n$th generation.

To make the notations clearer, $X_{i,j,k}$ is the time that the $j$th parent in the $(i-1)$th generation has waited before giving birth to his $k$th child (the children are not ordered by age).

The sequence $(K_n)_{n\in\mathbb{N}}$ is a Galton-Watson process (see [**6**]) from an initial population of one individual and with a Poisson distribution of parameter $p$ as reproduction law. Since $p < 1$, the Galton-Watson process is sub-critical and the construction reaches an end almost surely, i.e. almost surely, there exists $\mathcal{N}$ such that $K_{\mathcal{N}} = 0$. The cluster is then given by $\cup_{n=0}^{\mathcal{N}}\{X_1^n, \dots, X_{K_n}^n\}$. We denote this point process by $N^c$.

**Hawkes process as Poisson cluster process.** We are now considering the general case where numerous ancestors coexist and produce, independently of each others their own family. Let $N^a$ be a Poisson process on $\mathbb{R}$ of intensity measure $\nu$, which corresponds to the births of the different ancestors. Let us call the successive births of the ancestors $-\infty \le \dots < T_{-1} < T_0 \le 0 < T_1 < \dots \le +\infty$ where the eventual unnecessary points are rejected at infinity (this happens if there is a finite number of points).

Let us consider now independently an i.i.d. collection $\{N_n^c\}_{n\in\mathbb{Z}}$ of cluster processes constructed as previously according to the reproduction measure $\mu$. Let us denote by $\{T_j^n, j \in \mathbb{N}\}$ the successive births in the cluster process $N_n^c$.

The Hawkes process $N^h$ with ancestor measure $\nu$ and reproduction measure $\mu$ is given by $\cup_{n\in\mathbb{Z}} \cup_{j\in\mathbb{N}} \{T_n + T_j^n\}$, $T_n \in \mathbb{R}$. Heuristically, the points of $N^h$ can be seen as the births in the different families: a family corresponding to one ancestor and all his progeny.

The case $\nu(dt) = \lambda dt$ corresponds to the stationary version of the Hawkes process. The intensity of $N^h$ is given by (9) when $\nu(dt) = \lambda dt$ and $\mu(dt) = h(t)dt$ where $dt$ is the Lebesgue measure on the real line.

When there is no possible confusion, $N^h$ will always denote the Hawkes process with ancestor measure $\nu$ and reproduction measure $\mu$. When several measures may coexist, we will denote the law of $N^h$, seen as a random variable on the point measures, by $\mathrm{H}(\nu, \mu)$.

One of the major interest of the Poisson cluster process structure of the Hawkes process is the superposition property (a straightforward consequence of (16)).

PROPOSITION 1.1. *[Superposition property] Let $N_1^h$ and $N_2^h$ be two independent Hawkes processes, respectively with distributions $\mathrm{H}(\nu_1, \mu)$ and $\mathrm{H}(\nu_2, \mu)$. Then $N^h = N_1^h + N_2^h$ is a Hawkes process with distribution $\mathrm{H}(\nu_1 + \nu_2, \mu)$.*

In the first section, we intensively study the cluster process and we obtain some tail estimates for various quantities. In the second section, we apply these results to the Hawkes process. In the third section, we use the previous results to get a time cutting of the Hawkes process in approximating independent pieces and we apply this to get some non asymptotic estimates of the speed in the ergodic theorem.

## 2. Study of one cluster

**2.1. Length of a cluster.** Let $N^c$ be a cluster process constructed as before. Let us denote by $H$ the length of a cluster (i.e. the latest birth in the family), then $H$ is given by

$$H = \sup_{j \leq K_n, n \in \mathbb{N}} X_j^n.$$

If $H$ is quite naturally a.s. finite by construction, the question of integrability is not that clear. First of all, let us remark that if the $X_{i,j,k}$'s are not integrable, then of course $H$ is not integrable, as soon as $p > 0$. Now let us assume that the $X_{i,j,k}$'s are integrable. Let us define

$$U_n = \sum_{k=1}^{K_{n-1}} \sum_{j=1}^{P_{n,k}} X_{n,k,j}.$$

Clearly, $U_1$ is an upper bound of the latest birth in the first generation; $U_1 + U_2$ is an upper bound of the latest birth until the second generation and by induction, $\mathcal{U} = \sum_{n=1}^{\infty} U_n$ is clearly an upper bound for $H$. By independence between the $X_{i,j,k}$ and the $P_{i,j}$, one has that $\mathbb{E}(U_n) = m\mathbb{E}(K_n)$. But, by induction (see [**6**]), $\mathbb{E}(K_n) = p\mathbb{E}(K_{n-1}) = p^n$. Thus, $\mathbb{E}(\mathcal{U}) \leq m/(1-p) < \infty$, as soon as $p < 1$. We can then easily get the following proposition.

Proposition 2.1. *Assume that* $0 < p < 1$. *The length of the cluster,* $H$, *is integrable if and only if* $m$ *is finite.*

But if we need a good estimate for the tail of $H$, we have to look closer. We already remarked that the sequence $(K_n)_{n \in \mathbb{N}}$ is a sub-critical Galton-Watson process. Consequently the Laplace transform of $W_\infty$ exists and verifies this well known equation, see [**6, 31**]:

$$L_W(t) = t + p\left(e^{L_W(t)} - 1\right) \tag{10}$$

for all $t$ such that $L_W(t) = \log[\mathbb{E}(e^{tW_\infty})]$ is finite. Let us denote $g_p(u) = u - p(e^u - 1)$ for all $u > 0$. Then it is easy to see that for all $0 \leq t \leq (p - \log p - 1)$,

$$L_W(t) = g_p^{-1}(t), \tag{11}$$

where $g_p^{-1}$ is the reciprocal function of $g_p$ and that if $t > (p - \log p - 1)$, $L_W(t)$ is infinite. We can now apply this to derive tail estimates for $H$.

Proposition 2.2.

- *If* $v < +\infty$ *then for all positive* $x$,

$$\mathbb{P}(H > x) \leq \frac{1}{x^2}\left(\frac{p}{1-p}v + \left(\frac{p}{(1-p)^3} + \frac{p^2}{1-p}\right)m^2\right).$$

- *If there exists an interval* $I$ *such that for all* $t \in I$, $l(t) \leq p - \log p - 1$, *then for all positive* $x$,

$$\mathbb{P}(H > x) \leq \exp\left(-\sup_{t \in I}\left[xt + l(t) - g_p^{-1}(l(t))\right]\right).$$

*In particular if there exists* $t > 0$ *such that* $l(t) \leq p - \log p - 1$ *then*

$$\mathbb{P}(H > x) \leq \exp\left[-xt + 1 - p\right].$$

- *If* $\mathrm{Supp}(\mu) \subset [0, A]$*, then*

$$\forall x \geq 0, \quad \mathbb{P}(H > x) \leq \exp\left[-\frac{x}{A}(p - \log p - 1) + 1 - p\right]$$

PROOF. Let $(Y_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. variables with law $\mu/p$, independent of the Hawkes process. Then $\sum_{n=1}^{W_\infty - 1} Y_n$, with the $Y_n$'s independent of $W_\infty$, has the same law as $\mathcal{U}$ which is an upper bound for $H$. But for all $t \in I$, conditioning in $W_\infty$:

$$\mathbb{E}\left[\exp\left(t\sum_{n=1}^{W_\infty-1} Y_n\right)\right] = \mathbb{E}\left[\exp(l(t)(W_\infty - 1))\right] = \exp(-l(t))\exp[g_p^{-1}(l(t))],$$

and differentiating $L_W$ to get the moment of $W_\infty$, one gets also that

$$\mathbb{E}\left[\left(\sum_{n=1}^{W_\infty-1} Y_n\right)^2\right] = \frac{p}{1-p}v + \left(\frac{p}{(1-p)^3} + \frac{p^2}{1-p}\right)m^2.$$

It is now sufficient to use Chebyshev's inequality to conclude the proof for the first two results. For the last result it is sufficient to note that $\ell(t) \leq tA$. □

**2.2. Exponential decreasing of the number of points in the cluster.** Now we would like to understand more precisely the repartition of the number of points of $N^c$. More precisely we would like to prove that $N^c([a; +\infty))$ is exponentially decreasing in $a$ in some sense. The probability generating functional of a point process is a well known tool which is equivalent to the log-Laplace transform and which helps us here. For any bounded function $f$, let us define

$$L(f) = \log\left[\mathbb{E}\left(\exp\left[\int f(u)N^c(du)\right]\right)\right].$$

Then Daley and Vere-Jones in [**14**] gives for the Hawkes process that:

$$L(f) = f(0) + \int_0^\infty \left[e^{L(f(t+.))} - 1\right]\mu(dt),$$

where

$$L(f(t+.)) = \log\left[\mathbb{E}\left(\exp\left[\int f(t+u)N^c(du)\right]\right)\right].$$

Let $z > 0$ and $a \geq 0$ and let us apply this formula to $f = z\mathbb{1}_{[a;+\infty)}$. Then

$$U(a,z) = L(f) = \log\left[\mathbb{E}\left(e^{zN^c([a;+\infty))}\right)\right]$$

is the log-Laplace transform of the number of births after $a$. We are assuming in this section that $\mathrm{Supp}(\mu) \subset [0, A]$, then for all $a > 0$

$$U(a,z) = \int_0^A (e^{U(a-t,z)} - 1)\mu(dt). \tag{12}$$

Let us remark that the function $U(a,z)$ is decreasing in $a$, since the number of remaining births is decreasing. Moreover, $U(0,z)$ is the log-Laplace transform of $W_\infty$. The previous computations give that for all $0 \le z \le (p - \log p - 1)$, $U(0,z) = g_p^{-1}(z)$. Moreover if we define $U(+,z) = \log\left[\mathbb{E}\left(e^{zN^c((0;+\infty))}\right)\right]$, since the ancestor is always in 0, this quantity verifies for all $0 \le z \le (p - \log p - 1)$,

$$U(+,z) = g_p^{-1}(z) - z.$$

Hence, for all $0 < a < A$ and for all $0 \le z \le (p - \log p - 1)$,

$$U(a,z) \le U(+,z) = g_p^{-1}(z) - z. \tag{13}$$

Let us prove by induction the following result which gives a sense to "the number of births after $a$ is exponentially decreasing in $a$".

PROPOSITION 2.3. *Assume that* $\operatorname{Supp}(\mu) \subset [0,A]$. *For all* $a > 0$, *let* $k = \lfloor a/A \rfloor$. *Then for all* $0 \le z \le (p - \log p - 1)$,

$$U(a,z) \le \left(g_p^{-1}(z) - z\right) e^{-kz}. \tag{14}$$

PROOF. We already checked this fact for $k = 0$. Let us assume that the second inequality holds for $k$ and let us prove it for $k+1$. As $U(a,z)$ is decreasing in $a$, one has that $U(a,z) \le U((k+1)A,z)$. Applying (12) and (14), since $\mu$ is continuous, one has for all $0 \le z \le (p - \log p - 1)$,

$$U((k+1)A,z) \le p\left(\exp\left[(g_p^{-1}(z) - z)e^{-kz}\right] - 1\right).$$

But for all $a \le 1$ and $x \ge 0$,

$$e^{ax} - 1 \le a(e^x - 1). \tag{15}$$

Moreover one has $g_p^{-1}(z) \ge z$, since their inverses are in the inverse order. Consequently for all $0 \le z \le (p - \log p - 1)$,

$$U((k+1)A,z) \le pe^{-kz}\left(\exp\left[g_p^{-1}(z) - z\right] - 1\right).$$

But

$$e^{g_p^{-1}(z)} = 1 + \frac{g_p^{-1}(z) - z}{p}.$$

Hence for all $0 \le z \le (p - \log p - 1)$,

$$\begin{aligned} U((k+1)A,z) &\le pe^{-kz}\left(\left[1 + \frac{g_p^{-1}(z) - z}{p}\right]e^{-z} - 1\right) \\ &\le (g_p^{-1}(z) - z)e^{-(k+1)z} + pe^{-kz}(e^{-z} - 1). \end{aligned}$$

Since the last term is negative, this achieves the proof. □

## 3. Consequences for the Hawkes process

Let us now look at the consequences for the Hawkes process of these results.

**3.1. Application to the number of points per interval.** One of the first application is really straightforward. It is based on the link between the different probability generating functionals. Let us define for all bounded function $f$,

$$\mathcal{L}(f) = \log\left[\mathbb{E}\left(\exp\left[\int f(u)N^h(du)\right]\right)\right].$$

Then Vere-Jones proves in [**58**] that

$$\mathcal{L}(f) = \int_{-\infty}^{+\infty} \left(e^{L(f(t+.))} - 1\right)\nu(dt). \tag{16}$$

Let $z > 0$ and $T > 0$. Let us apply this formula to $f = z\mathbb{1}_{[0,T]}$. Then $\mathcal{L}(f)$ is the log-Laplace of the number of points of the Hawkes process between 0 and $T$. But $L(f(t+.))$ is the log-Laplace of the number of births of the cluster $N^c$ between $-t$ and $T-t$. Consequently

- if $t > T$, $L(f(t+.)) = 0$,
- if $T \geq t \geq 0$, $L(f(t+.))$ can be upper bounded by the log-Laplace of $W_\infty$, i.e. $U(0,z)$.
- if $0 > t$, $L(f(t+.))$ can be upper bounded by the log-Laplace of the number of births of the cluster $N^c$ after $-t$, i.e. $U(-t,z)$.

This leads to

$$\mathcal{L}(f) \leq \int_{-\infty}^{0} \left(e^{U(-t,z)} - 1\right) d\nu_t + \left(\int_0^T d\nu_t\right)\left(e^{U(0,z)} - 1\right).$$

If we assume that the ancestors are "uniformly" distributed, one can prove the following fact.

Proposition 3.1. *Let us assume that* $\nu(dt) = \lambda dt$ *and* $\mathrm{Supp}(\mu) \subset [0,A]$. *Let* $0 \leq z \leq (p - \log p - 1)$ *and* $T > 0$. *Then*

$$\log\left[\mathbb{E}\left(e^{zN^h([0,T])}\right)\right] \leq \lambda T \ell_0(z) + \lambda A \ell_1(z)$$

*where* $\ell_0(z) = e^{g_p^{-1}(z)} - 1$ *and* $\ell_1(z) = \dfrac{e^{g_p^{-1}(z)-z}}{1-e^{-z}} - 1$. *Moreover for all integer* $n$

$$\mathbb{P}(N^h([0,T]) \geq n) \leq \exp\left[-nz + \lambda T \ell_0(z) + \lambda A \ell_1(z)\right]. \tag{17}$$

Proof. We know that $U(0,z) = g_p^{-1}(z)$. Now let us split the integral into pieces of length $A$ and use the fact that $U(a,z)$ is decreasing in $a$. This gives

$$\log\left[\mathbb{E}\left(e^{zN^h([0,T])}\right)\right] \leq \lambda T\left[e^{g_p^{-1}(z)} - 1\right] + \sum_{k=0}^{\infty} \int_{kA}^{(k+1)A} \lambda(e^{U(t,z)} - 1)dt.$$

Let us apply Proposition 2.3. This gives, using (15),

$$\log\left[\mathbb{E}\left(e^{zN^h([0,T])}\right)\right] \leq \lambda T\left[e^{g_p^{-1}(z)}-1\right]+\sum_{k=0}^{\infty}\lambda A\left(\exp\left[(g_p^{-1}(z)-z)e^{-kz}\right]-1\right)$$
$$\leq \lambda T\left[e^{g_p^{-1}(z)}-1\right]+\sum_{k=0}^{\infty}\lambda A e^{-kz}\left(\exp[g_p^{-1}(z)-z]-1\right).$$

This easily concludes the proof. □

**3.2. Application to the extinction time.** Another important quantity on the Hawkes process is the extinction time $T_e$. Let us define a Hawkes process $N^h$ with reproduction measure $\mu$ and ancestor measure $\nu = \lambda \mathbb{1}_{\mathbb{R}_-}dt$. i.e. the ancestors appear homogeneously before 0 but not after. The latest birth in this process is the extinction time $T_e$. How fast does $\mathbb{P}(T_e > a)$ decrease in $a$ ?

We keep the notations given in the introduction and we define $H_n$ the length of the cluster $N_n^c$. Then $T_e = \sup_{n\in\mathbb{Z}_-}\{T_n + H_n\}$. So one can easily compute $\mathbb{P}(T_e \leq a)$ for any positive $a$. By conditioning with respect to the ancestors and using Proposition 2.1, one gets the following result, which seems to be known for a while (see for instance [**42**]).

Proposition 3.2. *Let $0<p<1$. For all $a \geq 0$, one has*

$$\mathbb{P}(T_e \leq a) = \exp\left(-\lambda\int_a^{+\infty}\mathbb{P}(H > x)dx\right).$$

*Moreover, the extinction time, $T_e$, is finite if and only if the reproduction measure, $\mu$, verifies $\int_0^{+\infty} t\mu(dt) < \infty$.*

Since we have now good estimates for the cluster length, we get the following bounds under various assumptions, simply using Chebyshev's inequality.

Proposition 3.3.

- *Assume that* $\left(\frac{p}{1-p}v+\left(\frac{p}{(1-p)^3}+\frac{p^2}{1-p}\right)m^2\right) = c$ *is finite, then*

$$\mathbb{P}(T_e > a) \leq 1-\exp\left[-\lambda\min\left(2\sqrt{c}-a,\frac{c}{a}\right)\right] \leq \lambda\min\left(2\sqrt{c}-a,\frac{c}{a}\right)$$

- *Assume that there exists $t>0$ such that $l(t) \leq p-\log p-1$, then*

$$\mathbb{P}(T_e > a) \leq 1-\exp\left[-\frac{\lambda}{t}e^{-at+1-p}\right] \leq \frac{\lambda}{t}e^{-at+1-p}$$

- *Assume that* $\mathrm{Supp}(\mu) \subset [0,A]$, *then*

$$\mathbb{P}(T_e > a) \leq 1-\exp\left[-\frac{\lambda A e^{-\frac{a}{A}(p-\log p-1)+1-p}}{p-\log p-1}\right] \leq \frac{\lambda A e}{p-\log p-1}e^{-\frac{a}{A}(p-\log p-1)}. \tag{18}$$

**3.3. Superposition property and approximate simulation.** As it has been said in the introduction, the Hawkes processes are modeling a lot of different problems. It is so natural to search for theoretical validation of simulation procedures. To simulate a stationary Hawkes process on $\mathbb{R}_+$ (that is, the restriction of $\mathrm{H}(\lambda dt, \mu)$ to $\mathbb{R}_+$), it is classical to use the superposition property (Proposition 1.1): a stationary Hawkes process is the independent superposition of $\mathrm{H}(\lambda \mathbb{1}_{\mathbb{R}_-} dt, \mu)$ and $\mathrm{H}(\lambda \mathbb{1}_{\mathbb{R}_+} dt, \mu)$. This means that we have to simulate first a Hawkes process with ancestors after time 0, which is easy, and then make the correct adjustment by artificially adding, independently, the points coming from ancestors born before time 0, that is, points coming from the the restriction of $\mathrm{H}(\lambda \mathbb{1}_{\mathbb{R}_-} dt, \mu)$ to $\mathbb{R}_+$. But to create these points, one needs, a priori, the knowledge of the whole past. However, we know they are a.s. in finite number if and only if $\int_0^\infty t\mu\,(dt) < +\infty$ by Proposition 3.2 (this result can also be found in [**43**]). Under this assumption, it is not surprising we will get a good approximation of the restriction of $\mathrm{H}(\lambda \mathbb{1}_{\mathbb{R}_-} dt, \mu)$ to $\mathbb{R}_+$ by using the restriction of $\mathrm{H}(\lambda \mathbb{1}_{[-a,0)} dt, \mu)$ to $\mathbb{R}_+$ for a large $a$. Finally, putting things together, we can approximate a stationary Hawkes process on $\mathbb{R}_+$ by looking at the restriction of $\mathrm{H}(\lambda \mathbb{1}_{[-a,+\infty)} dt, \mu)$ to $\mathbb{R}_+$. We can see that doing this, the error is easy to evaluate non asymptotically by means of the variation distance which, here, is less than $\mathbb{P}\,(T_e > a)$ where $T_e$ still denotes the extinction time of the previous section. Proposition 3.3 then gives some explicit and non asymptotic values in various useful cases. This answers a question asked to the second author by Brémaud who previously, together with Nappo and Torrisi in [**10**] gave some asymptotic results for this error. In particular, they give in the exponential unmarked case, an asymptotic exponential rate of decreasing for the extinction time (see Proposition 3.3, result 2) which is larger than ours. It seems to us that the results of Proposition 3.3 are probably non sharp, but are giving answers in a non asymptotic way, that can be really useful in practice. The question of approximate and perfect simulation has also been considered in [**43, 42**], however their setup is quite different and makes the comparison with our results very difficult.

## 4. Applications of the superposition property

**4.1. Construction of approximating i.i.d. sequences.** A Poisson process $N^p$ is said to be completely independent, that is for instance, $N^p_{|\mathcal{A}}$, the set of points of $N^p$ in $\mathcal{A}$, is independent of $N^p_{|\mathcal{B}}$, the set of points of $N^p$ in $\mathcal{B}$ as soon as $\mathcal{A}$ and $\mathcal{B}$ are disjoint. For $N^h$, a Hawkes process with distribution $\mathrm{H}(\lambda dt, \mu)$, despite of a hidden independent structure explained earlier, the clusters overlap each others and such independence cannot happen. Nevertheless by looking at very disjoint intervals we are very close to independence.

Let us assume that the reproduction measure (Proposition 3.2) is such that the extinction time is almost surely finite. Our aim is to build an independent sequence $\left\{M_q^x\right\}_{q\in\mathbb{N}}$ such that $M_q^x$ has the distribution of $H(\lambda dt, \mu)$ restricted to $[2qx - a, 2qx + x)$, for $0 < a < x$ and the variation distance between the distribution of $M_q^x$ and $N_{|[2qx-a,2qx+x)}$ is controlled. The form of the interval (( ] or [ ], etc...) has no impact since there is a.s. no point of the process at a given site (this is a consequence of stationarity which implies that the measure that counts the mean number of points on Borel sets is indeed a multiple of Lebesgue measure and thus, non atomic). Let $\left\{N^h_{q,n}\right\}_{(q,n)\in\mathbb{N}\times\mathbb{Z}}$ be independent Hawkes processes $\mathrm{H}(\lambda \mathbb{1}_{[-x+2nx,x+2nx)}dt,\mu)$ which means that the ancestors appears homogeneously only on the interval $[-x+2nx,x+2nx)$. We now form the following point processes:

$$N^h := \sum_{n=-\infty}^{n=+\infty} N^h_{0,n}, \text{ and for all } q \geq 1,\ N^h_q := \sum_{n=-\infty}^{n=q-1} N^h_{q,n} + N^h_{0,q}.$$

It is clear, from the superposition property (Proposition 1.1) that, for each $q \geq 1$, $N^h_q$ is a Hawkes process with distribution $\mathrm{H}(\lambda \mathbb{1}_{(-\infty,2qx+x)}dt,\mu)$ and that $N^h$ a Hawkes process with distribution $\mathrm{H}(\lambda dt,\mu)$. It is also clear that all the $N^h_q$'s are independent, for $q \geq 1$. We now take $M^x_q$ to be $N^h_q{}_{|[2qx-a,2qx+x)}$, the points of $N^h_q$ in $[2qx-a,2qx+x)$. It is clear from the construction that the $M^x_q$'s are independent and that they all have the stationary distribution $\mathrm{H}(\lambda dt,\mu)$ restricted to an interval of length $x+a$.

Let $q \geq 1$. Let $S = N^h_{0,q}{}_{|[2qx-a,2qx+x)}$,

$$S_1 = \sum_{n=-\infty}^{n=q-1} N^h_{q,n}{}_{|[2qx-a,2qx+x)} \text{ and } S'_1 = \sum_{n=-\infty}^{n=q-1} N^h_{0,n}{}_{|[2qx-a,2qx+x)}.$$

To evaluate the variation distance between $M^x_q$ and $N^h_{|(2qx-a,2qx+x]}$, we can write $M^x_q = S+S_1$ and $N^h_{|(2qx-a,2qx+x]} = S+S'_1$. We have for all measurable subset $\mathcal{A}$ of the set of point measures:

$$\begin{aligned}
\left|\mathbb{P}\left(M^x_q \in \mathcal{A}\right) - \mathbb{P}\left(N^h_{|[2qx-a,2qx+x)} \in \mathcal{A}\right)\right| &= \left|\mathbb{E}\left[1_{\{S+S_1\in\mathcal{A}\}} - 1_{\{S+S'_1\in\mathcal{A}\}}\right]\right| \\
&= \left|\mathbb{E}\left[\left(1_{\{S+S_1\in\mathcal{A}\}} - 1_{\{S+S'_1\in\mathcal{A}\}}\right)\left(1-\left(1_{S_1=\emptyset}1_{S'_1=\emptyset}\right)\right)\right]\right| \\
&\leq \mathbb{E}\left[\left(1-\left(1_{S_1=\emptyset}1_{S'_1=\emptyset}\right)\right)\right] = \left(1-\mathbb{P}\left[S_1=\emptyset\right]^2\right).
\end{aligned}$$

Now we can remark that $S_1 = \emptyset$ if $\sum_{n=-\infty}^{q-1} N^h_{q,n}$ is extinct before $2qx-a$. By stationarity, this probability is larger than $\mathbb{P}\left(T_e \leq x-a\right)$. Consequently the variation distance is less than $\left[1-\mathbb{P}\left(T_e \leq x-a\right)^2\right]$. It is then very easy to prove the following result.

Proposition 4.1. *Let $0 < a < x$. Let $N^h$ be a Hawkes process with distribution $\mathrm{H}(\lambda dt,\nu)$. There exists an i.i.d. sequence $M^x_q$ of Hawkes processes with distribution $\mathrm{H}(\lambda dt,\nu)$ restricted to $[2qx-a,2qx+x)$ such that for all $q$, the variation distance between $M^x_q$ and $N^h_{|[2qx-a,2qx+x)}$ is less than $2\mathbb{P}(T_e > x-a)$ as soon as the extinction time $T_e$ of $N^h$ is an almost surely finite random variable.*

**4.2. Example of application.** Let $f$ be a measurable function of $N^h_{|[-a,0)}$. For instance, the intensity $\Lambda$ of the process in 0 is a possible $f$ with $a = A$, if $\mathrm{Supp}(\mu) = \mathrm{Supp}(hdt) \subset [0,A]$ (see (9)). Let $\{\theta_s\}_{s\in\mathbb{R}}$ be the flow induced by the stationarity of the Hawkes process. This implies that for instance if $f = \Lambda(0)$, $f\circ\theta_s = \Lambda(s)$ is the intensity in $s$. The Hawkes process is ergodic since it is a Poisson cluster process (page 347 of [**14**]), this means that for $f \in L^1$

$$\frac{1}{T}\int_0^T f\circ\theta_s ds \to_{T\to\infty} \mathbb{E}(f) \text{ a.s.}$$

We are interested in this subsection in majorizing quantities such as:

$$\mathbb{P}\left(\frac{1}{T}\int_0^T f\circ\theta_s ds \geq \mathbb{E}(f)+u\right), \tag{19}$$

for any positive $u$, in order to get a "non asymptotic ergodic theorem".

Let $T>0$, $k\in\mathbb{N}$ and $x>0$ such that $T=2kx$. Let us assume now that $f$ has zero mean for care of simplicity. First let us remark by stationarity that:

$$\mathbb{P}\left(\frac{1}{T}\int_0^T f\circ\theta_s ds \geq u\right) \leq 2\mathbb{P}\left(\sum_{q=0}^{k-1}\int_{2qx}^{2qx+x} f\circ\theta_s ds \geq \frac{uT}{2}\right)$$

But $G_q = \int_{2qx}^{2qx+x} f\circ\theta_s ds$ is a measurable function of the points of $N^h$ appearing in $[2qx-a, 2qx+x)$, denoted by $\mathfrak{F}(N_{|[2qx-a,2qx+x)})$. Let us now pick a sequence $\left\{M_q^x\right\}_{0,\dots,(k-1)}$ of i.i.d. stationary Hawkes processes restricted to an interval of length $a+x$ and let $F_q = \mathfrak{F}(M_q^x)$. We have consequently constructed an i.i.d sequence $\{F_q\}_{0,\dots,(k-1)}$ with the same law as the $G_q$'s. Moreover, by Proposition 4.1, the sequence $\left\{M_q^x\right\}_{0,\dots,(k-1)}$ can be chosen such that $\mathbb{P}(F_q\neq G_q)$ is less $2\mathbb{P}(T_e > x-a)$. By using Proposition 4.1, one gets

$$\mathbb{P}\left(\frac{1}{T}\int_0^T f\circ\theta_s ds \geq u\right) \leq 2\left[\mathbb{P}\left(\frac{1}{k}\sum_{q=0}^{k-1} F_q \geq ux\right) + \mathbb{P}(\exists q, F_q\neq G_q)\right].$$

This leads to the following result.

THEOREM 4.2. *Let $N^h$ be a stationary Hawkes process with distribution* $\mathrm{H}(\lambda dt,\mu)$. *Let $T, a>0$ and $k$ a positive integer such that $0<a<T/2k$. Let $f$ be a measurable function of $N^h_{|[-a,0)}$ with zero mean and $\theta_s$ be the flow induced by $N^h$.*
*Then there exists an i.i.d. sequence $F_k$ with distribution $\int_0^{T/2k} f\circ\theta_s ds$ such that*

$$\mathbb{P}\left(\frac{1}{T}\int_0^T f\circ\theta_s ds \geq u\right) \leq 2\mathbb{P}\left(\frac{1}{k}\sum_{q=0}^{k-1} F_q \geq \frac{uT}{2k}\right) + 4k\mathbb{P}\left(T_e > \frac{T}{2k} - a\right),$$

*where $T_e$ is the extinction time of a Hawkes process with law* $\mathrm{H}(\lambda \mathbb{1}_{\mathbb{R}_-}dt,\mu)$.

Now to get precise estimates, we need extra-assumptions. Here are just a few examples of the possible applications of our construction.

PROPOSITION 4.3. *Let $N^h$ be a stationary Hawkes process with distribution* $\mathrm{H}(\lambda dt,\mu)$ *such that* $\mathrm{Supp}(\mu)\subset[0,A]$. *Let $\theta_s$ be the flow induced by $N^h$.*

*Let $a > 0$ and $f$ be a measurable function of $N^h_{|[-a,0)}$ with zero mean. Let $u, T > 0$ such that*

$$a \leq A(u + \log T)/(p - \log p - 1) \text{ and } 4A(u + \log T) \leq T(p - \log p - 1).$$

*Then with probability larger than $1 - \left(2 + \frac{\lambda e}{u+\log T}\right) e^{-u}$,*

(1) *(Hoeffding) if there exist $B, b > 0$ such that $B \geq f \geq b$,*

$$\frac{1}{T}\int_0^T f \circ \theta_s ds \leq (B - b)\sqrt{\frac{4A(u + \log T)u}{T(p - \log p - 1)}}$$

(2) *(Bernstein) if there exist $V, C > 0$ such that $\forall n \geq 2, \mathbb{E}(f^n) \leq \frac{n!}{2} V C^{n-2}$,*

$$\frac{1}{T}\int_0^T f \circ \theta_s ds \leq \sqrt{\frac{16VA(u + \log T)u}{T(p - \log p - 1)}} + \frac{8CAu(u + \log T)}{T(p - \log p - 1)}$$

(3) *(Weak Bernstein) if there exists $V, B > 0$ such that $V \geq \mathbb{E}(f^2)$ and $-B \leq f \leq B$,*

$$\frac{1}{T}\int_0^T f \circ \theta_s ds \leq \sqrt{\frac{16VA(u + \log T)u}{T(p - \log p - 1)}} + \frac{8BAu(u + \log T)}{3T(p - \log p - 1)}.$$

Proof. First we need to apply Hoeffding or Bernstein inequalities [**39**] to the first term in Proposition 4.1. It remains then to bound the extinction time using Equation (18). The only remaining problem is then to choose $k$. With the assumption on $T$ and $u$ there exists always an integer $k$ such that $a \leq T/4k$ and

$$\frac{T(p - \log p - 1)}{8A(u + \log T)} \leq k \leq \frac{T(p - \log p - 1)}{4A(u + \log T)},$$

which concludes the proof. □

First let us remark that the condition on $T$ are fulfilled as soon as $T$ is large enough. One can also see that, as usual, under the same assumptions the "weak Bernstein" inequality is sharper than the "Hoeffding inequality". The construction and proof of these time cutting and application to concentration inequalities is mainly inspired by the work of Baraud, Comte and Viennet in [**7**] on autoregressive sequence. In particular the $\log T$ factor seems to be, by analogy, a weak loss with respect to the independent case.

Finally, we would like to give a nice estimate for an unbounded function $f$, which naturally appears: the intensity. First let us suppose that the reproduction measure $\mu(du)$ is given by $h(u)du$. Then the intensity of $N^h$ with distribution $\mathrm{H}(\lambda dt, \mu)$ is given by (9). Let us assume that $h$ has support in $[0, A]$ and that $h$ is bounded by a positive constant $H$. Let us first remark that $f = \Lambda(0) \leq \lambda + HN^h((-A, 0])$. So bounding the intensity $\Lambda(s) = f \circ \theta_s$ can be done if we bound the number of points per interval of length $A$.

Let $K = \lceil (T + A)/A \rceil$. Let $\mathcal{N}$ be a positive number and

$$\Omega = \left\{\forall k \in \{0, \ldots, K - 1\}, N^h((-A + kA, kA]) \leq \mathcal{N}\right\}.$$

Then by Proposition 17, $\mathbb{P}(\Omega^c) \leq K e^{\lambda A[\ell_0(p - \log p - 1) + \ell_1(p - \log p - 1)]} e^{-\mathcal{N}(p - \log p - 1)}$.

Now let us apply Proposition 4.3 (Weak Bernstein) to $f = \Lambda(0) \wedge M - \mathbb{E}(\Lambda(0) \wedge M)$, where $M = \lambda + 2H\mathcal{N}$. As on $\Omega$, $f = \Lambda(0)$, we get the following result.

Proposition 4.4.
*Let $N^h$ be a Hawkes process with distribution $\mathrm{H}(\lambda dt, h(t)dt)$ where $h$ the reproduction function is bounded by $H$ and has a support included in $[0, A]$. Let $\Lambda$ be its intensity given by (9). Let $u > 0$. There exists a $T_0 > 0$ depending on $A, u$ and $p$ such that for all $T \geq T_0$, with probability larger than $1 - \left(3 + \frac{\lambda e}{u + \log T}\right) e^{-u}$,*

$$\frac{1}{T}\int_0^T \Lambda(s)ds \leq \mathbb{E}(\Lambda(0)) + \sqrt{\frac{16\mathbb{E}(\Lambda(0)^2)A(u+\log T)u}{T(p-\log p-1)}} + \frac{8u(\lambda+2H\mathcal{N})(u+\log T)}{3T(p-\log p-1)}$$

*where*

$$\mathcal{N} = \frac{\lambda A[\ell_0(p-\log p-1) + \ell_1(p-\log p-1)] + \log T + u}{p - \log p - 1}.$$

In view of the ergodic theorem, this result explains very precisely and non asymptotically, how far $\frac{1}{T}\int_0^T \Lambda(s)ds$ is from its expectation. This result may partially and non asymptotically answer to a question asked to us by P. Brémaud on the existence of a C.L.T. for those quantities.

CHAPITRE 5

# Bartlett spectrum and ID random measures

ABSTRACT. We study some aspects of the Bartlett spectrum of square integrable stationary random measures. When restricting our attention to infinitely divisible random measures, we show that the Bartlett spectrum determines ergodicity and mixing. In this context, the Bartlett spectrum plays the same role as does the spectral measure of the coordinate at 0 of a stationary Gaussian process.

## 1. Introduction

Random measures, and particularly, point processes, are widely used in applied mathematics. A point process in $\mathbb{R}$ is often used to represent the arrival dates of customers in a queueing system or, in seismology, the times where earthquakes occur. In $\mathbb{R}^2$, point processes can represent positions of trees in a forest, or positions of base stations in a wireless network, etc.

To compute useful quantities and do statistics, hypothesis of stationarity is often assumed, as are square integrable properties. Thus, one can hope to have some tools close to those coming from the $L^2$ theory of square integrable stochastic processes. The Bartlett spectrum plays this role and allows to obtain nice covariance formulas. From an ergodic point of view, it gives us access to a whole family of spectral measures. Although the ergodic information contained in the Bartlett spectrum is partial in general, we will see that, in the infinitely divisible case, it determines ergodicity and mixing. Infinitely divisible point processes are maybe the most commonly found point processes in the applications. Poisson point processes, cluster Poisson processes, Cox processes with ID directing measure, Hawkes processes without ancestors (see [**9**]) among others, are ID point processes.

We first recall in Sections 2 and 3 basic facts of random measure and Bartlett spectrum, and we traduce some existing results in a more ergodic oriented way for our purposes and, at last, in Section 4, we study the ID case more accurately.

## 2. Random measures

Call $M$ the space of non-negative Radon measures on $\left(\mathbb{R}^d, \mathcal{B}^d\right)$, $\mathcal{M}$ the smallest $\sigma$-algebra such that the sets $\{\mu, \mu(A) \in B\}$ are measurable for all bounded $A$ in $\mathcal{B}^d$ and $B$ in $\mathcal{B}^d$. A *random measure* is then a random variable whose distribution is a distribution on $(M, \mathcal{M})$. Let us denote by $M_p$ (resp. $M_p^s$) , the measurable subset of $M$ whose elements are discrete and allocate only integer values (resp. 1 or 0) to singletons. A *point process* is a random measure whose distribution, say $\mathbb{P}$, verifies $\mathbb{P}(M_p) = 1$. A *simple point process* is a point process whose distribution, say $\mathbb{P}$, verifies $\mathbb{P}\left(M_p^s\right) = 1$. In the following, we will assume that the probabilities involved verify $\mathbb{P}(\{\mu_0\}) = 0$, where $\mu_0$ denotes the null measure.

There is a natural measurable flow $\{\theta_t\}_{t\in\mathbb{R}^d}$ acting on $(M, \mathcal{M})$, the translations of measures:

$$\theta_t\mu(A) = \mu(t + A)$$

A random measure will be said *stationary* if its distribution is preserved under the action of the flow. The dynamical system we will consider here is a quadruplet $\left(M, \mathcal{M}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}^d}\right)$ where $\mathbb{P}$ is a distribution of a stationary random measure. The subset $M_p$ (resp. $M_p^s$) being invariant under the action of the flow, we can consider stationary point processes (resp. stationary simple point processes). Following the usual convention, $N$ will denote the random measure and in our case, it will be the identity on $(M, \mathcal{M})$.

$\{U_t\}_{t\in\mathbb{R}^d}$ is the group of unitary operators associated to this dynamical system and acting on $L^2(\mathbb{P})$ by:

$$U_t f = f \circ \theta_t$$

**2.1. Integrable and square-integrable random measures.** A stationary random measure $N$ is said integrable, or $L^1$, or of finite intensity if $\lambda := \mathbb{E}[N(\mathbb{U})] < +\infty$. $\lambda$ is called the *intensity* of $N$. It is easy to see that $\mathbb{E}[N(A)] = \lambda m(A)$ for every $A$ in $\mathcal{B}^d$

In the same way, a stationary random measure $N$ is said square-integrable, or $L^2$, if $\mathbb{E}\left[N(\mathbb{U})^2\right] < +\infty$, $\mathbb{U}$ being the hypercube $\left[-\frac{1}{2}, \frac{1}{2}\right]^d$. A $L^2$ stationary random measure is off course integrable and for every bounded $A$ in $\mathcal{B}^d$, $\mathbb{E}\left[N(A)^2\right] < +\infty$

**2.2. Laplace functional.** To identify the distribution of a random measure, a practical tool is the so-called Laplace functional:

$$L(f) := \mathbb{E}[\exp - N(f)]$$

for all non-negative function $f$.

**2.3. Two important classes of point processes.**

2.3.1. *The Poisson process.* Let $\nu$ be a Radon measure on $\left(\mathbb{R}^d, \mathcal{B}^d\right)$.

A *Poisson process of intensity measure* $\nu$ is a point process (non stationary in general) $N$ on $\left(\mathbb{R}^d, \mathcal{B}^d\right)$ such that for any $k \in \mathbb{N}$ and any collection $\{A_1, \ldots, A_k\}$ of disjoint Borel sets of finite $\nu$-measure, the joint law of $\{N(A_1), \ldots, N(A_k)\}$ is $p[\nu(A_1)] \otimes \cdots \otimes p[\nu(A_k)]$ where $p[\nu(A_i)]$ denotes the Poisson distribution of parameter $\nu(A_i)$.

Its Laplace functional is:

$$L(f) = \exp - \int ((\exp - f(x)) - 1)\, \nu(dx) \tag{20}$$

This point process is simple if and only if $\nu$ is a continuous measure.

In the special case where $\nu$ is $\lambda m$, we get the so-called *homogeneous Poisson process of parameter* $\lambda$. This process is then stationary.

2.3.2. *Cox processes.* Cox processes are the first natural generalization of Poisson processes. The deterministic intensity measure is now made random, that is to say: from each realisation of a random measure, form the corresponding Poisson process. This construction can be made rigorous (see [**41**] chap. 7 or [**14**] chap. 8).

Note $\mathbb{P}_\mu$ the distribution of a Poisson process of intensity measure $\mu$. The distribution $\mathbb{P}_c$ of the Cox process directed by a random measure of distribution $\mathbb{P}$ is given, for all $A \in \mathcal{M}$ by:

$$\mathbb{P}_c(A) = \int_M \mathbb{P}_\mu(A)\,\mathbb{P}(d\mu) \tag{21}$$

The Laplace functionals of the Cox process and of the random measure are related by:

$$L_c(f) = L\left[1 - \exp(-f)\right] \tag{22}$$

It is easy to see that if random measure is stationary then the corresponding Cox process is also stationary. We can summarize results concerning stationary Cox processes in the following proposition:

PROPOSITION 2.1. *Let* $\left(M, \mathcal{M}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}^d}\right)$ *be the dynamical system associated to a stationary random measure* $\Lambda$ *and* $\left(M, \mathcal{M}, \mathbb{P}_c, \{\theta_t\}_{t\in\mathbb{R}^d}\right)$ *the corresponding system associated to the corresponding Cox process* $N$*. Then:*

$\Lambda$ *is of finite intensity* $\lambda$ *if and only if* $N$ *is, with the same intensity.*

$\Lambda$ *is* $L^2$ *if and only if* $N$ *is.*

$\Lambda$ *is ergodic (resp. weakly mixing, resp. mixing) if and only if* $N$ *is ergodic (resp. weakly mixing, resp. mixing).*

PROOF. See [**14**] prop 10.3.VII. for the last point. □

Finally if $\mathbb{P}$ has an ergodic decomposition $\mathbb{P}(A) := \int_{\mathfrak{M}} p(A)\,\mathfrak{P}(dp)$ we can deduce, by equation 21:

$$\mathbb{P}_c(A) = \int_M \mathbb{P}_\mu(A)\,\mathbb{P}(d\mu) = \int_{\mathfrak{M}}\int_M \mathbb{P}_\mu(A)\,p(d\mu)\,\mathfrak{P}(dp) = \int_{\mathfrak{M}} p_c(A)\,\mathfrak{P}(dp) \tag{23}$$

From last proposition, each $p_c$ is ergodic, so we get a decomposition of $\mathbb{P}_c$ into ergodic components. By unicity of the ergodic decomposition, we can deduce that the ergodic components of $\mathbb{P}_c$ are the Cox processes directed by the ergodic components of $\mathbb{P}$.

## 3. Palm calculus and Bartlett spectrum

The *Palm probability* of an integrable stationary random measure $N$ of intensity $\lambda$ is the unique probability $\mathbb{P}_N^0$ on $(M, \mathcal{M})$ defined by:

$$\mathbb{P}_N^0(C) = \frac{1}{\lambda}\mathbb{E}\left[\int_{\mathbb{U}} (1_C \circ \theta_t)\,N(dt)\right] \tag{24}$$

If $N$ is an $L^2$ stationary random measure, then we can show that $\sigma(A) := \mathbb{E}_N^0[N(A)]$ for all $A$ in $\mathcal{B}^d$, defines a Radon measure, the *Palm intensity measure.* This measure is also positive-definite and thus *transformable* which means that, for every $f$ in the Schwarz space $\mathcal{S}$:

$$\int_{\mathbb{R}^d} \hat{f} d\sigma = \int_{\mathbb{R}^d} f d\hat{\sigma}$$

where $\hat{\sigma}$, the Fourier transform of $\sigma$, seen as a tempered distribution, is also a positive-definite measure. We can express a formula for the covariance in terms of the measure $\sigma$ by:

$$Cov\left[N(f)\,N(g)\right] = \lambda \int_{\mathbb{R}^d} \left(f * \check{\bar{g}}\right) d\sigma - \lambda^2 \int_{\mathbb{R}^d} \left(f * \check{\bar{g}}\right) dm \tag{25}$$

where $f$ and $g$ are $\mathbb{C}$-valued functions in $\mathcal{S}$ .

Using the transformability of $\sigma$ and $m$, we can write:

$$Cov\left[N(f)\,N(g)\right] = \lambda \int_{\mathbb{R}^d} \hat{f}\bar{\hat{g}} d\hat{\sigma} - \lambda^2 \int_{\mathbb{R}^d} \hat{f}\bar{\hat{g}} d\delta_0 = \int_{\mathbb{R}^d} \hat{f}\bar{\hat{g}} d\Gamma \tag{26}$$

$\Gamma := \lambda\hat{\sigma} - \lambda^2\delta_0$ is called the *Bartlett spectrum* of $N$.

This formula gives us access to a whole family of spectral measures by:

$$Cov\left[N(f) \circ \theta_t N(f)\right] = \int_{\mathbb{R}^d} e^{itx} \hat{f}(x)\, \bar{\hat{f}}(x)\, \Gamma(dx) = \int_{\mathbb{R}^d} e^{itx} \left|\hat{f}(x)\right|^2 \Gamma(dx)$$

Remark 3.1. Contrary to the spectral measure of a stationary square integrable stochastic process, the measure $\Gamma$ is not finite in general, for example this is never the case for point processes:

$$\lambda\left(\frac{a}{2}\right)^d \int_{\mathbb{R}^d} \exp -a\sum_{i=1}^d |x_i|\, d\sigma - \lambda^2 \left(\frac{a}{2}\right)^d \int_{\mathbb{R}^d} \exp -a\sum_{i=1}^d |x_i|\, dm$$
$$= \int_{\mathbb{R}^d} \Pi_{i=1}^d \frac{1}{1+\left(\frac{x_i}{a}\right)^2} \Gamma(dx)$$

By monotone convergence, the right-hand side tends to $\Gamma\left(\mathbb{R}^d\right)$as $a$ tends to infinity. And the left hand side is greater than $\lambda\left(\frac{a}{2}\right)^d \sigma\{0\} - \lambda^2$, so if $\sigma\{0\} > 0$ (this is the case for point processes), this quantity tends to infinity with $a$ and $\Gamma$ is not finite. In fact, the function $x \mapsto \frac{1}{1+\|x\|^2}$ is always $\Gamma$-integrable as $\Gamma$ is a positive definite measure.

The main characteristics of the Bartlett spectrum are summarized in the following theorem:

Theorem 3.2. *Denote by* $L_\Gamma$ *the closed subspace of* $L^2(\mathbb{P})$ *generated by the vectors of the form* $N(A) - \lambda m(A)$ *where* $A$ *is a bounded Borel set then:*

- There exist an isometry of Hilbert spaces between $L_\Gamma$ and $L^2\left(\frac{\Gamma}{1+\|x\|^2}\right)$ explicitly defined for those $f$ in $\mathcal{S}$ by:
$$\left(N(f)-\lambda\int_{\mathbb{R}^d} f dm\right)\mapsto \hat{f}\sqrt{1+\|.\|^2}$$
- To the action of $U_t$ on $L_\Gamma$ corresponds the multiplication by $e^{it\cdot}$ on $L^2\left(\frac{\Gamma}{1+\|x\|^2}\right)$.

In other terms, $L_\Gamma$ is a cyclic subspace of spectral measure $\frac{\Gamma}{1+\|x\|^2}$ associated to the vector $N(e)-\lambda\int_{\mathbb{R}^d} edm$ where $e(x):=\exp-\sum_{i=1}^{d}|x_i|$, i.e. the vectors $\left(N(e)-\lambda\int_{\mathbb{R}^d} edm\right)\circ\theta_t$, for $t\in\mathbb{R}^d$ span $L_\Gamma$.

Proof. The first point is proved in details in [**14**] Chapter 11 with a slight adaptation, using $\Gamma$ instead of $\frac{\Gamma}{1+\|x\|^2}$. For the second point, it is enough to remark that, writing $\tau_t f(u)=f(u-t)$:

$$U_t\left(N(f)-\lambda\int_{\mathbb{R}^d} f dm\right)=\left(N(f)-\lambda\int_{\mathbb{R}^d} f dm\right)\circ\theta_t=N(\tau_t f)-\lambda\int_{\mathbb{R}^d}\tau_t f dm$$

and $\hat{\tau_t f}(x)=e^{itx}\hat{f}(x)$. □

In the sequel, we denote $\frac{\Gamma}{1+\|x\|^2}$ by $\Gamma'$.

Let's illustrate this by the following notions and results.

**3.1.** ***N*-mixing.** Delasnerie, in [**16**], introduced a result attributed to Neveu using the notion of $N$-mixing. A $L^2$ stationary random measure $N$ is said $N$*-mixing* if, for all bounded measurable real $f$ and $g$ with compact support, the following convergence holds as $|t|$ tends to infinity:

$$\mathbb{E}\left[\left(N(f)-\lambda\int_{\mathbb{R}^d} f dm\right)\circ\theta_t\left(N(g)-\lambda\int_{\mathbb{R}^d} g dm\right)\right]\to 0$$

Proposition 3.3. *(Renewal Theorem) Let $N$ be an $L^2$ stationary random measure and $\sigma$ its Palm intensity measure. If $N$ is $N$-mixing then $\lambda\theta_t\sigma$ tends vaguely to $\lambda^2 m$ as $|t|$ tends to infinity.*

The name of this proposition comes from the fact that, when the point process is a non-lattice renewal process, this result is exactly the renewal theorem.

For our purposes we will precise these notions in terms of the Bartlett spectrum, the proofs of the two following propositions are straightforward:

Proposition 3.4. *An $L^2$ stationary random measure $N$ is $N$-mixing if and only if $\hat{\Gamma}'(t)$ tends to $0$ as $|t|$ tends to infinity.*

And the renewal Theorem is in fact an equivalent characterization of $N$-mixing:

Proposition 3.5. *(Renewal Theorem) Let $N$ be an $L^2$ stationary random measure and $\sigma$ its Palm intensity measure. $N$ is $N$-mixing if and only if $\lambda\theta_t\sigma$ converges vaguely to $\lambda^2 m$ as $|t|$ tends to infinity.*

The following simple lemma concern the possible atom at 0 of the Bartlett spectrum. $\mathcal{I}$ denotes the invariant $\sigma$-algebra.

Lemma 3.6. *$\Gamma$ doesn't put a mass at 0 if and only if $\mathbb{E}\left[N\left(\mathbb{U}\right)|\mathcal{I}\right] = \lambda$ $\mathbb{P}$-a.s.*

Proof. It can be easily deduced by use of 24, but a little computation shows also:

$$\Gamma\left(\{0\}\right) = Var\left[\mathbb{E}\left[N\left(\mathbb{U}\right)|\mathcal{I}\right]\right]$$

as outlined in [**14**]. □

Finally, for a square integrable Cox process, its Bartlett spectrum is naturally linked to the Bartlett spectrum of the directing measure, see [**14**].

Lemma 3.7. *If the random measure $\Lambda$ on $\mathbb{R}^d$ of intensity $\lambda$ admits $\Gamma_\Lambda$ as Bartlett spectrum, the corresponding Cox process admits $\Gamma_\Lambda + \frac{\lambda}{(2\pi)^d}m$ as Bartlett spectrum.*

## 4. The infinitely divisible case

We introduce first this important class of random measures that often arises as powerful models in the applications. An amazingly rich theory exists on the notion of infinite divisibility and it's not surprising that the random measure case has allowed a very deep study of these objects. The point process case has been much more studied than the general random measure one, as many constructions arise naturally when Dirac masses are considered as points. For the interested reader [**41**] presents a detailed analysis and [**14**] a much more compact but still very efficient presentation. Nevertheless our results are given in full generality for random measures. We point out that the results that we will give here can be recovered by the abstract machinery developed in Chapter 2, but we believe it is interesting to get very straightforward proofs by only using tools specific to the theory of random measure such as Palm theory and Cox processes theory which have nice behaviours interesting in their own in the ID case.

**4.1. The convolution in $(M, \mathcal{M})$.** The application defined on $(M \times M, \mathcal{M} \otimes \mathcal{M})$, that takes a pair of measures $(\mu, \nu)$ and gives the sum $\mu + \nu$ on $(M, \mathcal{M})$ is measurable. The convolution of two probability measure $\mathbb{P}_1$ and $\mathbb{P}_2$ on $(M, \mathcal{M})$ is thus defined as the image of the product $\mathbb{P}_1 \otimes \mathbb{P}_2$ by this application, we denote the result by $\mathbb{P}_1 * \mathbb{P}_2$.

An *infinitely divisible (ID) random measure* is a random measure whose distribution $\mathbb{P}$ can be written, for every integer $k$, as $\mathbb{P}_k * \cdots * \mathbb{P}_k$ ($k$ terms) for a certain random measure distribution $\mathbb{P}_k$ (necessarily unique). The best known example is the Poisson process whose infinite divisibility is easily deduced by its Laplace functional 20.

In the following, we will need the following fact, we give as a remark:

Remark 4.1. A Cox process directed by an ID random measure is an ID point process as can be checked directly using Laplace functional at 22.

**4.2. The KLM measure for point processes.** We introduce a measure associated to a distribution of any ID point process.

Let $\tilde{\mathbb{Q}}$ be a measure on $(M_p, \mathcal{M})$ (possibly infinite) such that $\tilde{\mathbb{Q}}(\mu_0) = 0$.

The formula $\exp \int_{M_p} (1 - \exp(-\mu(f))) \tilde{\mathbb{Q}}(d\mu)$ for every Lebesgue-integrable and non-negative function $f$, defines the Laplace functional $L(f)$ of an ID point process. Moreover, the Laplace functional of any ID point process possesses has this form for a uniquely determined measure $\tilde{\mathbb{Q}}$.

$\tilde{\mathbb{Q}}$ is called the KLM measure, it was introduced in the point process case by Kerstan, Lee and Matthes. It has exactly the same meaning as the Lévy measure.

**4.3. The Palm measures.** Although $\tilde{\mathbb{Q}}$ is an infinite measure we can, just as the Palm formula 24 associate its Palm measure $\tilde{\mathbb{Q}}_N^0$ and when the underlying point process is of finite intensity, this Palm measure can be renormalized to form the probability measure $\tilde{\mathbb{P}}_N^0$.

$$\tilde{\mathbb{P}}_N^0(C) = \frac{1}{\lambda} \int_{M_p} \int_{\mathbb{U}} 1_C(\theta_t \mu) \mu(dt) \tilde{\mathbb{Q}}(d\mu)$$

We then have this remarkable property (see [**41**]):

$$\mathbb{P}_N^0 = \tilde{\mathbb{P}}_N^0 * \mathbb{P} \tag{27}$$

This fact is well known in the Poisson case, as $\tilde{\mathbb{P}}_N^0$ reduces to $\delta_{\delta_{\{0\}}}$(the probability concentrated on the measure consisting on a Dirac mass at 0).

In the square-integrable case, 27 allows further simplification that, apparently, has never been pointed out before, namely:

$$\lambda\sigma - \lambda^2 m = \lambda\tilde{\sigma} \tag{28}$$

where, as above, $\tilde{\sigma}(A) = \tilde{\mathbb{E}}_N^0[N(A)]$

The Bartlett spectrum $\Gamma = \hat{\tilde{\sigma}}$ is thus a positive definite measure.

**4.4. Applications.** Before stating our results, we will need criteria for ergodicity and mixing found in [**41**].

The first is to be found at Proposition 6.4.10 in [**41**]:

Lemma 4.2. *A distribution $\mathbb{P}$ of a stationary ID point process of finite intensity $\lambda$ is ergodic if and only if* $\mathbb{E}[N(\mathbb{U})|\mathcal{I}] = \lambda$ $\mathbb{P}$-*a.s.*

We can immediately extend this result to general integrable stationary ID random measures:

LEMMA 4.3. *A distribution* $\mathbb{P}$ *of a stationary ID random measure of finite intensity* $\lambda$ *is ergodic if and only if* $\mathbb{E}\left[N\left(\mathbb{U}\right)|\mathcal{I}\right]=\lambda$ $\mathbb{P}$*-a.s.*

PROOF. It follows from 23 that the intensities of the ergodic components of the random measure are the same as their corresponding Cox processes, ergodic components of the Cox process. By remark 4.1, we can apply Lemma 4.2 to say that if $\mathbb{E}\left[N\left(\mathbb{U}\right)|\mathcal{I}\right]=\lambda$ $\mathbb{P}$-a.s., then $\mathbb{E}_c\left[N\left(\mathbb{U}\right)|\mathcal{I}\right]=\lambda$ $\mathbb{P}_c$-a.s. and then $\mathbb{P}_c$ is ergodic which in turns implies that $\mathbb{P}$ is ergodic by Proposition 2.1. □

We will use another criteria found in [**41**] (Proposition 9.2.1.) concerning mixing:

PROPOSITION 4.4. $\mathbb{P}$ *is mixing if and only if* $lim_{|t|\to\infty}\tilde{\mathbb{P}}_N^0\left[N\left(t+A\right)>0\right]=0$

Let's turn to the results on the Bartlett spectrum:

THEOREM 4.5. *The dynamical system* $\left(M,\mathcal{M},\mathbb{P},\{\theta_t\}_{t\in\mathbb{R}^d}\right)$*, where* $\mathbb{P}$ *is the distribution of an ID* $L^2$ *stationary random measure is ergodic if and only if* $\Gamma'\{0\}=0$ *and mixing if and only if* $\hat{\Gamma'}(t)$ *tends to* 0 *as* $|t|$ *tends to infinity.*

PROOF. The combination of Lemmas 3.6 and 4.3 suffices to prove the result for ergodicity. To prove the theorem in the mixing case, as before we start with point processes. Given Proposition 3.4, we only have to prove, with our assumptions, that $N$-mixing imply mixing. But $N$-mixing implies, by Proposition 3.3 and use of characterisation 28, the vague convergence to 0 of the measure $\lambda\theta_t\tilde{\sigma}$ to 0 as $|t|$ tends to infinity. Finally, an application of the Markov inequality shows, for every $A$ in $\mathcal{B}_b^d$:

$$\tilde{\mathbb{P}}_N^0\left[N\left(t+A\right)>0\right]=\tilde{\mathbb{P}}_N^0\left[N\left(t+A\right)\geq 1\right]\leq\tilde{\mathbb{E}}_N^0\left[N\left(t+A\right)\right]=\lambda\tilde{\sigma}\left(t+A\right)$$

which tends to 0 as $|t|$ tends to infinity.

It is now just an application of the criteria of Proposition 4.4 to prove the mixing of the dynamical system. For the general random measure case, we use the same arguments and Proposition 2.1, looking at the associated Cox process. Now we remark that $\hat{\Gamma'}(t)$ tends to 0 if and only if $\hat{\Gamma'}(t)+\frac{\lambda}{(2\pi)^d}\hat{m'}(t)$ tends to 0, where $m'$ denotes any finite measure equivalent to $m$. □

4.4.1. *Example: a singular continuous Bartlett spectrum.* In [**41**] is given an example, attributed to Herrmann, of an ergodic non mixing ID point process on $\mathbb{R}^2$, let us describe it.

We define an infinite stationary measure $\tilde{\mathbb{Q}}$ on $M$ in the following way:

First, let $\nu_0:=\sum_{n\in\mathbb{Z}}\delta_{(n,0)}$ and note $\mathbb{P}_{\nu_0}:=\delta_{\nu_0}$, a probability on $M$.

Let now randomize it with respect to the Lebesgue measure on $\mathbb{R}^2$, namely:

$$\tilde{\mathbb{Q}}\left(A\right):=\int_{\mathbb{R}^2}\mathbb{P}_{\nu_0}\left(\theta_t^{-1}A\right)dt$$

This measure is clearly stationary, moreover $\tilde{\mathbb{Q}}(\mu_0) = 0$, indeed:

$$\tilde{\mathbb{Q}}(\mu_0) = \int_{\mathbb{R}^2} \mathbb{P}_{\nu_0}\left(\theta_t^{-1}\{\mu_0\}\right) dt$$
$$= \int_{\mathbb{R}^2} \mathbb{P}_{\nu_0}(\{\mu_0\})\, dt = 0$$

So, this measure determines uniquely the distribution of an ID point process, we can now compute its intensity:

$$\int_M \mu(\mathbb{U})\, \tilde{\mathbb{Q}}(d\mu) = \int_{\mathbb{R}^2}\int_M \mu(\mathbb{U}) \circ \theta_t \mathbb{P}_{\nu_0}(d\mu)\, dt$$
$$= \int_{\mathbb{R}^2} \nu_0\left(\theta_t^{-1}\mathbb{U}\right) dt = 1$$

We thus express its Palm probability $\tilde{\mathbb{P}}_N^0$ by showing it is indeed $\mathbb{P}_{\nu_0}$:

$$\tilde{\mathbb{P}}_N^0(\{\nu_0\}) = \int_M \int_{\mathbb{U}} 1_{\{\nu_0\}}(\theta_t\mu)\, \mu(dt)\, \tilde{\mathbb{Q}}(d\mu)$$
$$= \int_{\mathbb{R}^2}\int_M \left[\int_{\mathbb{U}} 1_{\{\nu_0\}}(\theta_t\mu)\, \mu(dt)\right] \circ \theta_s \mathbb{P}_{\nu_0}(d\mu)\, ds$$
$$= \int_{\mathbb{R}^2} \left[\int_{s+\mathbb{U}} 1_{\{\nu_0\}}(\theta_t\nu_0)\, \nu_0(dt)\right] ds$$

We remark that $\int_{s+\mathbb{U}} 1_{\{\nu_0\}}(\theta_t\nu_0)\, \nu_0(dt)$ is equal to 1 if $s$ is in $\mathbb{U}$ and 0 elsewhere.
So we get $\tilde{\mathbb{P}}_N^0(\{\nu_0\}) = 1$.
We can determine easily its Bartlett spectrum as $\tilde{\sigma} = \nu_0$.
Let $f$ be a function in $\mathcal{S}$.

$$\int_{\mathbb{R}^2} \hat{f}\, d\tilde{\sigma} = \sum_{n\in\mathbb{Z}} \hat{f}(n,0) = \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-inx} f(x,y)\, dxdy$$
$$= \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} e^{-inx}\left(\int_{\mathbb{R}} f(x,y)\, dy\right) dx = \sum_{n\in\mathbb{Z}} \left(\int_{\mathbb{R}} f\left(\frac{n}{2\pi}, y\right) dy\right)$$

by the Poisson formula

$$= \int_{\mathbb{R}^2} f\, d\nu^{2\pi} \otimes m$$

where $\nu^{2\pi}$ denotes the measure $\sum_{n\in\mathbb{Z}} \delta_{\frac{n}{2\pi}}$.
So $\Gamma = \nu^{2\pi} \otimes m$ is a continuous singular Bartlett spectrum.

Consequently, it is ergodic by Theorem 4.5, non mixing because it is easily checked it is not $N$-mixing by taking $A = \mathbb{U}$ and looking at the sequence $\tilde{\sigma}(\mathbb{U} + (n, 0))$ which equals 1 for all $n$. We could also check that this system has zero entropy since the cyclic space associated to the Bartlett spectrum (which is singular here) generates the whole $\sigma$-algebra.

**4.5. Equivalence between ergodicity and weak mixing.** The next result is proved in the point process case in [**41**].

The extension to the general case is immediate:

PROPOSITION 4.6. *A distribution $\mathbb{P}$ of a stationary ID random measure is weakly mixing if and only if it is ergodic*

PROOF. If $\mathbb{P}$ is ergodic, $\mathbb{P}_c$, the distribution of the associated Cox process is ID by remark 4.1 and ergodic by Proposition 2.1. But as mentioned, for ID point process, ergodicity implies weak mixing and another use of Proposition 2.1 shows that $\mathbb{P}$ is weakly mixing. □

**4.6. A note on short and long range dependence.**

DEFINITION 4.7. A square integrable point process on $\mathbb{R}$ is said to be long-range dependent (LRD) if $limsup\frac{1}{t}VarN(0,t] = +\infty$.

For ID point processes, it is very easy to characterize LRD.

PROPOSITION 4.8. *A square integrable ID point process $N$ is LRD if and only if $\tilde{\sigma}$ is an infinite measure.*

PROOF. Using 25 with the function $g_t : x \mapsto 1_{(0,t]}(x)$ yields, as $t1_{[-t,t]}(x) \geq g_t * \check{g}_t(x) \geq \frac{t}{2}1_{\left[-\frac{t}{2},\frac{t}{2}\right]}(x)$:

$$t\tilde{\sigma}[-t,t] \geq VarN(0,t] \geq \frac{t}{2}\tilde{\sigma}\left[-\frac{t}{2},\frac{t}{2}\right]$$

by use of 26 and 28.

The conclusion follows. □

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