Final Exam

Curves on an Algebraic Surface

Master 2, Paris

Friday April 17, 2026

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Instructions. Email to me your solutions as a unique pdf file before 10pm of Sunday April 19.

You may use the theory developed in the two Scheme Theory courses and in this course. Whenever you apply a theorem that was discussed in class, please say it explicitly and either state the theorem or give a precise reference to Mumford's book.

In order to get full marks, it is necessary to write very detailed proofs. You may write your solutions in English or French.

Honor Code. By signing below, you certify that:

- You have completed this exam independently.
- You have not used unauthorized resources, including but not limited to online materials (Mathoverflow, Stacks Project, ...), communication with others, or AI-based tools. You are allowed to use your notes from class and Mumford's book.
- You have not copied from any other student, nor allowed your work to be copied.
- You agree to abide by the academic integrity standards of your University.

Date: ______________________________

Name: ______________________________

University: ______________________________

Signature: ______________________________

Problem 1. Let k be a field.

1. Recall (without proof) the description of the functor of points of $\mathbb{P}^1_k$ in terms of invertible sheaves that we discussed in class.

2. Let X be the blowup of $\mathbb{A}^2_k$ at the origin. Determine (with proof) the functor of points of X. More precisely, for every k-scheme S, give a functorial bijection between $h_X(S)$ and the set of 4-tuples (x, y, L, φ), where $x, y \in \Gamma(S, \mathcal{O}_S)$, L is an invertible sheaf on S, and $\varphi\colon \mathcal{O}_S^{\oplus 2} \to L$ is a surjection such that ... modulo the equivalence relation ... . (You may use (1).)

Problem 2. Let k be an algebraically closed field, and let $n \geq 1$ be an integer.

1. Give an example (with proof) of a k-variety S (an integral separated k-scheme of finite type) and a coherent sheaf F on S with support equal to S and whose flattening stratification consists of exactly n strata $S_1, \ldots, S_n$.
2. Let S be a k-scheme of finite type, and let S_{red} be the underlying reduced scheme of S. Determine (with proof) the flattening stratification for the coherent $\mathcal{O}_S$-module $\mathcal{O}_{S_{\mathrm{red}}}$.
3. Let $S = \mathbb{A}^2_k$, and let $\pi \colon T \to S$ be the blowup of S at the origin. Determine (with proof) the flattening stratification of the coherent $\mathcal{O}_S$-module $\pi_* \mathcal{O}_T$ over S.

Problem 3. Let t be a coordinate on $\mathbb{A}^1_k$, that is, $\mathbb{A}^1_k = \mathrm{Spec}(k[t])$.

1. We saw in class a locally free sheaf $\mathcal{E}$ on $\mathbb{P}^1_k \times_k \mathbb{A}^1_k$ such that $\mathcal{E}|_t \cong \mathcal{O} \oplus \mathcal{O}$ if $t \neq 0$ and $\mathcal{E}|_0 \cong \mathcal{O}(-1) \oplus \mathcal{O}(1)$. Reprove this with all the details.
2. Prove that there is no locally free sheaf $\mathcal{F}$ on $\mathbb{P}^1_k \times_k \mathbb{A}^1_k$ such that $\mathcal{F}|_t \cong \mathcal{O}(-1) \oplus \mathcal{O}(1)$ if $t \neq 0$ and $\mathcal{F}|_0 \cong \mathcal{O} \oplus \mathcal{O}$.

Problem 4. Let k be an algebraically closed field, let E be an elliptic curve over k, and let $O \in E(k)$ be the origin of E. The goal of this problem is to determine the Picard scheme of E.

1. Define the Picard functor $\underline{\mathrm{Pic}}_E$ of E. (As I mentioned in class, the definition of p. 88 of Mumford's book is given for a surface but is the correct definition in general.)

2. Construct a morphism $\varphi \colon E \to \underline{\mathrm{Pic}}_E$ which, at the level of k-points, sends $P \in E(k)$ to the invertible sheaf $\mathcal{O}_E(P - O)$.

3. Prove that, if R is a k-algebra of finite type and L is an invertible sheaf on E_R, of degree 1 on the fibers of the projection $p \colon E_R \to \mathrm{Spec}(R)$, then $R^1 p_* L = 0$ and $p_* L$ is an invertible sheaf on $\mathrm{Spec}(R)$ whose formation commutes with arbitrary base change.

4. Show that the map φ induces an isomorphism $E \xrightarrow{\sim} \underline{\mathrm{Pic}}^0_E$, where the target is by definition the identity component of the Picard scheme of E. (Hint 1: Use (3) to construct an inverse. Hint 2: Consider relative effective Cartier divisors on E_R.)

Problem 5. Let k be an algebraically closed field, let E be an elliptic curve over k, and consider the surface $F = E \times_k \mathbb{P}^1_k$. In this problem, we fill in the missing details in Example 3 p. 2 of Mumford's book.

1. Determine the abelian group $\mathrm{Num}(F)$.
2. For every $\xi \in \mathrm{Num}(F)$, determine the dimension of $P(\xi)$ (the scheme representing the Picard functor of line bundles with numerical class ξ). You are allowed to use Problem 4 even if you have not completed its solution.
3. For every $\xi \in \mathrm{Num}(F)$, determine the dimension of $C(\xi)$ (the scheme representing the functor of curves with numerical class ξ).

Problem 6. Let k be an algebraically closed field, let F be a smooth projective surface, let S be a (separated) k-scheme of finite type, let $s \in S$ be a closed point, and consider a family of curves $\mathcal{D} \subset F \times S$. We defined in class the *characteristic map* $\rho\colon T_{S,s} \to H^0(F, N_{\mathcal{D}_s})$ (cf. p. 154 of Mumford's book).

1. Recall the definitions of $T_{S,s}$, $N_{\mathcal{D}_s}$ and ρ.
2. Prove that the map ρ is a linear transformation.