The mod (p, v_1, v_2) Syntomic Cohomology of *ku*

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Introduction

Algebraic K-theory with input a ring or a ring spectrum, is an algebro-homotopic invariant. First introduced by Grothendieck in [Gro57], as a group completion of a monoid of finitely generated projective modules over a ring. This is now known as K_0, and later extended to higher K-groups by Quillen in [Qui73],

$$K(R) = BGL(R)^+ \tag{1}$$

While highly powerful, its computations are notoriously difficult. Trace methods approximate K-theory by comparing it to more accessible invariants like topological Hochschild homology (THH), topological cyclic homology (TC), and its variants TC^- and TP.

$$\begin{array}{ccc} K(R) & \xrightarrow{tr} & TC(R) = Eq(can, \varphi) \\ & \swarrow \quad \searrow & \\ THH(R)^{hS^1} = TC^-(R) & \overset{can}{\underset{\varphi}{\rightrightarrows}} & TP(R) = THH(R)^{tS^1} \\ & \nwarrow \quad \nearrow & \\ & THH(R) & \end{array} \tag{2}$$

Since ku is an $\mathbb{E}_\infty$-ring, one can simply define THH for such rings as $R \otimes_{R \otimes R} R$. The Dundas–Goodwillie–McCarthy theorem in [DGM13], shows that the cyclotomic trace map is often close to an equivalence.

Theorem. *Let $A \to B$ be a map of connective ring spectra inducing a surjection on $\pi_0(A) \to \pi_0(B)$ with nilpotent kernel. Then the square is homotopy cartesian*

$$\begin{array}{ccc} K(A) & \longrightarrow & K(B) \\ {\scriptstyle tr}\downarrow & & \downarrow{\scriptstyle tr} \\ TC(A) & \longrightarrow & TC(B) \end{array} \tag{3}$$

Building on this, Nikolaus and Scholze [NS18] developed a modern framework for cyclotomic spectra, simplifying the understanding of TC, in terms of an equalizer of the canonical and Frobenius maps. Recently, Hahn, Wilson, and Raksit [HRW22] constructed a motivic filtration on THH for 'nice enough' rings, which descends to TC, and recovers syntomic cohomology in the discrete p-adic case, and extending syntomic cohomology to higher algebra.

$$\mathrm{fil}^*_{mot} F(R) = colim_{\Delta^{op}} \left(\tau_{\geq 2*} \left(F(R/MU^{\otimes \bullet+1}) \right) \right) \tag{4}$$

In other words, the décalage of the motivic cobar complex.

$$F(R) \longrightarrow F(R) \otimes_{F(MU)} MU \longrightarrow F(R) \otimes_{F(MU)^{\otimes 2}} MU^{\otimes 2} \longrightarrow F(R) \otimes_{F(MU)^{\otimes 3}} MU^{\otimes 3} \longrightarrow \ldots \tag{5}$$

with $F \in \{THH, TC^-, TP, TC\}$. Computationally, the motivic filtration enables us to recover results of Ausoni and Rognes [Aus10] with much less difficulty than their computations, and extends their results for primes where Smith-Toda complex does not exist.

Main Theorem

Theorem. *The mod (p, v_1, v_2) syntomic cohomology of topological complex K-theory has homotopy groups*

$$\begin{aligned} V(2)_* \mathrm{gr}^*_{mot} TC(ku) = {} & P_{p-1}(b) \otimes \Lambda(\partial, a_1, \lambda_1) \\ & \oplus P_{p-1}(b) \otimes \Lambda(a_1) \otimes \{t^i \lambda_1 \mid 0 < i < p\} \\ & \oplus P_{p-1}(b) \otimes \Lambda(\lambda_1) \otimes \{u^i du, t^{p^2-p} \lambda_2 \mid 0 \leq i < p-2\} \end{aligned} \tag{6}$$

where the generators have total degree $|b| = 2p+2$, $|a_1| = 2p+3$, $|\lambda_1| = 2p-1$, $|t| = -2$, $|u| = 2$, $|du| = 3$, $|\lambda_2| = 2p^2 - 1$.

There are interesting consequences to deduce from this. One observes chromatic redshift, as conjectured by Ausoni and Rognes in [Aus10]. Meaning TC and so K, shift chromatic height of ku, which is 1, to height 2 commutative spectrum. We also observe a higher Bott periodicity, analogous to Bott class in degree 2 having $u^{p-1} = v_1$, we get $b^{p-1} = v_2$ with higher Bott class b in degree $2p+2$. This suggests that $K(ku)$ is a form of elliptic cohomology theory, represented via two-vector bundles [Aus08].

Computing $THH(ku, \mathbb{F}_p)$

Recall ku has homotopy groups $\mathbb{Z}[u]$, where u represents the Bott class of degree 2. We p-complete ku, and note that $\mathbb{F}_p \cong ku_p/(p, u)$. We have

$$THH_*(ku, \mathbb{F}_p) = \mathbb{F}_p[\mu_1] \otimes \Lambda(du, \lambda_1) \tag{7}$$

The filtration to use is $THH(-, \mathbb{F}_p)$ applied to the Adams $\mathbb{F}_p$-filtration;

$$ku_p \longrightarrow ku_p \otimes \mathbb{F}_p \longrightarrow ku_p \otimes \mathbb{F}_p^{\otimes 2} \longrightarrow ku_p \otimes \mathbb{F}_p^{\otimes 3} \longrightarrow \ldots \tag{8}$$

Taking the homotopy groups we get an E_2-page that collapses after a single differential, $d_2(\mu_0) = dv_0$, detected from $THH(\mathbb{Z}_p, \mathbb{F}_p)$.

$$\begin{aligned} THH_*(\mathbb{F}_p[v_0, u]) &= THH_*(\mathbb{F}_p) \otimes_{\mathbb{F}_p} HH_*(\mathbb{F}_p[v_0, u]/\mathbb{F}_p) \\ &= \mathbb{F}_p[\mu_0] \otimes \Lambda(dv_0, du) \end{aligned} \tag{9}$$

Computing $V(1)_* THH(ku)$

By this we mean $THH(ku)/(p, v_1)$. There is a multiplication relation $u^{p-1} = v_1$ inducing the Adams splitting. So we run u-Bockstein spectral sequence that has $(p-1)$-length;

$$\begin{array}{ccccccc} \ldots & \xrightarrow{\times u} & THH(ku)/p, v_1 & \xrightarrow{\times u} & THH(ku)/p, v_1 & \xrightarrow{\times u} & \ldots \\ & & \downarrow & & \downarrow & & \\ & & THH(ku)/p, u & & THH(ku)/p, u & & \end{array} \tag{10}$$

The spectral sequence collapses after a single differential $d_{p-2}(\mu_1) = u^{p-2} du$, which can be deduced from analysis of the Hopf algebroid (MU_*, MU_*MU).

Collapsing of the Motivic Spectral Sequence

To utilise the motivic spectral sequence we need to see that the homotopy groups above lie on the E_2-page of the motivic spectral sequence, which collapses without extensions. Consider the filtration 5, but for $THH(ku, \mathbb{F}_p)$ and apply the Postnikov filtration $\tau_{\geq *}(-)$ to each of the tensor terms. This constructs Tor-spectral sequence converging to $\mathrm{gr}^*_{mot} THH(ku, \mathbb{F}_p)$. Note,

$$THH_*(ku/MU, \mathbb{F}_p) = \mathbb{F}_p[\sigma^2 x_2, \sigma^2 x_3, \ldots][\sigma^2 b_1, \sigma^2 b_2, \ldots] \tag{11}$$

is concentrated in even terms, and so the spectral sequence collapses. To check for $V(1)_* \mathrm{gr}^*_{mot} THH(ku)$, one observes that u-Bockstein does not change the motivic filtration. Hence, we associate motivic cobar elements to the previously computed generators of $V(1)_* THH(ku)$ and use theory in [HRW22].

$V(1)_* \mathrm{gr}^*_{mot} THH^{tC_p}(ku) \cong V(2)_* \mathrm{gr}^*_{mot} TP(ku)$

We compute TP and TC^- now, doing this via the useful isomorphism of Nikolaus-Scholze. For any BP-module with S^1-action M, recall that $[p](t) = \sum_i v_i t^{p^i}$ is the p-series defined iteratively using BP-formal group law. One has,

$$M^{tS^1}/[p](t) \cong M^{tC_p} \tag{12}$$

The first term in the series modulo $(p = v_0, v_1)$ is v_2, and the rest is a unit in $V(1)_* \mathrm{gr}^*_{mot} TP(ku)$, giving the equivalence in the title. This simplifies the computations greatly, as computing $THH(ku)^{tC_p}$ and the Frobenius map from $THH(ku)$ can be deduced using Bokstedt's result in [Bok]. Thus obtain the following version of the Segal conjecture.

Theorem. *The mod (p, v_1) Frobenius map is identified as an inversion of μ_2*

$$V(1)_* \mathrm{gr}^*_{mot} THH(ku) \xrightarrow{\varphi} V(1)_* \mathrm{gr}^*_{mot} THH^{tC_p}(ku) \tag{13}$$

And as a corollary after a generator change $b_i \mapsto b_1 t^{pi}$, $a_j \mapsto a_1 t^{pj}$;

$$\begin{aligned} V(2)_* \mathrm{gr}^*_{mot} TP(ku) \cong {} & \mathbb{F}_p[t^{\pm p^2}] \otimes \Lambda(\lambda_1, a_1) \otimes P_{p-1}(b) \\ & \oplus \mathbb{F}_p[t^{\pm p^2}] \otimes \Lambda(\lambda_1) \otimes P_{p-1}(b) \\ & \otimes \{a_1 t^{pi}, bt^{pi} \mid 0 < i < p\} \end{aligned} \tag{14}$$

Computing $V(2)_* \mathrm{gr}^*_{mot} TC^-(ku)$

There are tS^1 and hS^1 spectral sequences constructed by taking $\tau_{\geq *}$-filtration of the spectrum and applying either the homotopy fixed points or Tate construction. Their E_2-pages have the form,

$$\mathrm{gr}^*_{mot} THH(ku)[t] \implies \mathrm{gr}^*_{mot} TC^-(ku) \tag{15}$$

$$\mathrm{gr}^*_{mot} THH(ku)[t^{\pm}] \implies \mathrm{gr}^*_{mot} TP(ku) \tag{16}$$

On the E_2-pages, (p, v_1) are modded out internally to $\mathrm{gr}^*_{mot} THH(ku)$ as a ku-module. Referring to Hahn and Wilson [HW20], we determine the circle t action on elements of motivic cobar complex; $t \cdot \sigma^2 b_1 = b_1$, and $t \cdot \sigma x_i = x_i$, so mod v_2 is just mod $t\mu_2$.

Above we have the abutment of $V(2)_* \mathrm{gr}^*_{mot} TP(ku)$, and so one can determine the pattern of differentials going into the tS^1 spectral sequence. Via the canonical map from $TC^-(ku) \to TP(ku)$, we see that hS^1 spectral sequence sits inside the tS^1-one with only positive powers of t. So the differentials are also the same,

- $d_2(b_i) = (1-i) a_i t$
- $d_{2p}(t) = t^{p+1} \lambda_1$
- $d_{2p^2}(t^{p^2}) = t^{p^2+p} b^{p-2} a_1$

These compute hS^1 spectral sequence to E_∞-page with no multiplicative extensions.

$$\begin{aligned} & \mathbb{F}_p[t^{p^2}, \mu_2]/(t^{p^2} \mu_2) \otimes \Lambda(\lambda_1, a_1) \otimes P_{p-1}(b_1) \\ & \oplus \mathbb{F}_p[t^{p^2}] \otimes \Lambda(\lambda_1) \otimes P_{p-2}(b_1) \otimes \{a_1 t^{p^i}, bt^{p^i} \mid 0 < i < p\} \\ & \oplus \mathbb{F}_p[\mu_2] \otimes \Lambda(\lambda_1) \otimes P_{p-2}(b_1) \otimes \{a_1 \mu_1^i, b_1 \mu_1^{p+i} \mid 0 < i < p\} \\ & \oplus P_{p-1}(b_1) \otimes \Lambda(a_1) \otimes \{t^i \lambda_1 \mid 0 < i < p\} \\ & \oplus P_{p-2}(b_1) \otimes \Lambda(\lambda_1) \otimes \{u^i du \mid 0 \leq i < p-1\} \\ & \oplus \Lambda(\lambda_1) \otimes \{t^{pk} b_1^{p-2} a_1 \mid 0 < k < p\} \end{aligned} \tag{17}$$

Equalizer of can and φ

It remains to evaluate the Frobenius and canonical maps and take their equalizer. On homotopy groups, so for dgas, this is just evaluating the kernel of $can_* - \varphi_*$, fitting inside the long exact homotopy fiber sequence. To evaluate the maps, one utilises the following commutative diagram, derived from the useful isomorphism g and canonical inclusion on the 0-line denoted as map f.

$$\begin{array}{ccc} V(2)_* \mathrm{gr}^*_{mot} TC^-(ku) & \xrightarrow{can \text{ or } \varphi} & V(2)_* \mathrm{gr}^*_{mot} TP(ku) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ V(1)_* \mathrm{gr}^*_{mot} THH(ku) & \xrightarrow{can \text{ or } \varphi} & V(1)_* \mathrm{gr}^*_{mot} THH^{tC_p}(ku) \end{array} \tag{18}$$

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