M2 Seminar on Infinity Categories

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1 Introduction

In classical category theory, a category is a structure consisting of objects with a set of morphisms between any pair of objects. These can be composed uniquely and are subject to the usual axioms governing functions and their composition. But why stop at just morphisms between objects, what about morphisms between morphisms *ad infinitum*. ∞-categories provide a robust framework to deal with categories with 'higher morphisms' by combining category theory with homotopy theory. ∞-categories are already widely used in algebraic topology, K-theory, algebraic geometry, mathematical physics, and more.

In the seminar we will learn the basics of $\infty-$categories, finding analogues to the known notions and theorems from classical category theory. The last three talk will provide an introduction to stable homotopy theory, first invented to investigate stable homotopy groups of spheres, but can be show to underlie topics such as the classification of smooth structures on manifolds, and homological algebra. The goal of this seminar is for students to obtain a working knowledge of ∞-categories, understanding the main concept and be ready to further specialise in its branches. After reflecting on the content of last year's edition of this seminar, we will focus less on technical details and more on how to 'think ∞-categorically'.

Each participant is **expected to prepare a talk** with our help, and present it to the class. We expect **each talk to be roughly 90 minutes long**, accounting for questions and comments, and students can pair-up to divide each talk in half. In addition all should attend the each of the talks and participate actively.

It is expected that you have prior knowledge in classical category theory, acquaintance with simplicial sets and classical homotopy theory. We review basics on simplicial sets on the first week. More concretely, we encourage the students to be familiar with the M2 courses (or equivalent to) from Mathématiques fondamentales '**Homologie, cohomologie et faisceaux**' and '**Homotopie I**'.

2 Preliminary Plan

First a brief note on the sources we will follow. Based on the practical approach we will take, we will use a variety of recent lecture notes. The courses that these notes compliment adopt a similar philosophy to ours and so will provide a nice template on which to base talks.

For a rigorous treatment, speakers are also encouraged to consult additional sources. The book of Markus Land [5] provides a relatively gentle albeit rigorous introduction to the theory of ∞-categories. This is complimented nicely by the set of notes written by Charles Rezk [8] which comes much of the same material in a similar style. Finally, no course on ∞-categories can be conducted without mentioning the seminal book *Higher Topos*

Theory by Jacob Lurie [6] which provides a nearby encyclopedic treatment of the theory. Though Higher Topos Theory is long and at times difficult, it contains a wealth of material, including expository notes, that the shorter texts omit. Finally, Lurie is in the process of compiling an online resource, *Kerodon* [7], which contains many more details than even Higher Topos Theory.

1. **Motivation for ∞-categories, Basic Definitions and Constructions:**
   22.01.2025 : Antoine Dubreuil and Roy Nehme
   The first lecture will serve as an introduction to the homotopy theory with many concepts already defined, or to be defined in Homotopie I& II classes. We suggest to go quickly through the construction of homotopy colimits and limits of spaces, suspension and loop as adjoints. Then, moving to generalised cohomology theories, and spectra, giving examples and defining the corresponding properties that correspond to the Eilenberg-Steenrod axioms. Finish with stating the Brown representability theorem and sketch its proof.

   References: [1, Chapter 1]

2. **Definition of ∞-categories:**
   29.01.2025 : Raphael de Maleprade and Tomás Fernández
   Give the definition of an ∞-category, explain its terms recalling basic constructions in simplicial sets and explain its objects, morphisms, and compositions. Discuss some of the basic philosophy and informal 'rules' that govern the manipulation of ∞-categories. Discuss the geometric realisation in simplicial sets. Construct the nerve functor and and explain how discrete categories are an example of ∞-categories. Then, using the singular simplicial complex show that spaces are also ∞-categories. Define Kan complexes and ∞-groupoids and state their equivalence.

   References: [4, Chapter 1], [2, Section 1.1-1.3 until Definition 1.3.11].

3. **Functors and Mapping Spaces:**
   05.02.2025 : Julien Ketabi and Lucas Jacquier
   Define functors, functor categories, and natural transformations. State that the functor category is an ∞-category and without proving it, illustrate the technical ideas going into the proof, namely fibrations, anodyne maps and Kan fibrations. Then, define the mapping spaces, their construction and examples. Give definitions of fully faithfulness and essential surjectivity and outline the proof of functors being equivalent $\iff$ they are fully faithful and essentially surjective.

   References: [4, Section 2.1-2.2]

4. **Spc and Cat$_\infty$, and Limits and Colimits:**
   12.02.2025 : Dorian Boully and Pierre Nouhaud
   Construct the ∞-category of spaces using the homotopy coherent nerve. Define the (large) ∞-category of small ∞-categories, without dwelling too much on the size issues. Then define limits and colimits via the usual (co)limit cone construction and give explicit description of (co)products, (co)equalizers, mapping telescopes, initial/terminal objects, pushouts and pullbacks. Give the definition of (co)completeness and state that ∞-categories has finite (co)limits $\iff$ has (co)equalizers and finite (co)products $\iff$ has (pullbacks)pushout and finite coproducts, and give examples of such ∞-categories.

   References: [4, Section 2.3, 3.1 and 3.2]

5. **Slice categories, alternative description of (co)limits and Adjuntions:**
   05.03.2025 : Quentin Schroeder and Jingsong Feng
   Give description of slice categories via joins, and thus give an alternative description of limits and colimits. Then give definition of adjunctions, show that the inclusion

Spc $\hookrightarrow$ Cat_∞ admits left and right adjoint. Define the ∞-category BG in Spc, and show that homotopy orbits and homotopy fixed points are adjoint functors, motivating the example of I-shaped (co)limits and adjunction between constant functor and I-shaped colimit. Finally, mention that Kan extensions can be made sense of for ∞-categories.

References: [4, Section 3.3, and 3.4, Appendix A.1, and A.2], [2, Section 1.6-1.9].

6. **Straightening-Unstraightening and the Yoneda's lemma:**
12.03.2025 : Quentin Schroeder and Newsha Karimi
Building on the theory of adjunctions from before, state and sketch the proof that the left (resp. right) functor preserve all colimits (resp limits) that exist in the source ∞-category. Then, define Yoneda embedding, the ∞-category of presheaves and sketch the proof for Yoneda's lemma. Then introduce the ideas and definitions of the Grothendieck construction and state the Straightening-Unstraightening theorem, explain the concepts. Motivate this result by defining the twisted arrow category and providing an alternative definition of the Yoneda embedding.

References: [4, Section 2.4, 3.4, and Appendix A.3]

7. **Alternative models for ∞-categories:**
19.03.2025 : Ishan Dasgupta Samarendra
This lecture will serve as a break from the previous lectures, where traditional categorical notions were generalized. We will survey some other approaches to ∞-categories such as simplicial categories, topological categories, and perhaps recent work with Denis-Charles Cisinski, Kim Nguyen and Tashi Walde [3], on axiomatizing the language of ∞-categories.

References: [1, Chapter 2]

8. **Presentable Categories:**
26.03.2025 : Antoine Rodrigues and Tomás Fernández
Define an accessible and a (locally) presentable ∞-category and show that Spc is presentable. Prove the adjoint functor theorem for (locally) presentable ∞-categories.

References: [4, Section 4.1-4.4]

9. **Stable Categories and ∞-category of Spectra:**
02.04.2025 : Tianjian Tan and Hongxiao Bu
Motivate the need for stable categories using homological algebra - the derived categories of rings - and stable homotopy theory - categories of cohomology theories. Define stable ∞-categories and prove alternative characterizations. Provide intuition for what it means for a category to be stable. Define the functor $\mathrm{Sp}(-)$ and construct the ∞-category of spectra. Demonstrate that $\mathrm{Sp}(-)$ participates in a variety of adjunctions and in particular allows us to lift the Yoneda embedding to spectra. Discuss how $\mathcal{D}(-)$ is stable.

References: [1, Section 3.1-3.3]

10. **Spectra and Outlook**
09.04.2025 : El mokhtar Mokkedem
In this final very informal seminar we will look at the analogy of spectra with Abelian groups. Discuss the tensor product of spectra and (very briefly) rings and modules. Indicate that in stable homotopy theory we obtain a hierarchy of commutativity of rings. The topics presented in this talk are not at all final so you are encouraged to find a part of mathematics that interests you, provided it can be discussed in the language of stable homotopy theory just presented.

References:

References

[1] Bastiaan Cnossen. *Introduction to stable homotopy theory*. December 8, 2024. URL: https://sites.google.com/view/bastiaan-cnossen.

[2] Jack Morgan Davies. *V4D2 - Algebraic Topology II So24 (Stable and chromatic homotopy theory)*. July 3, 2024. URL: https://sites.google.com/view/jackmdavies/teaching.

[3] Kim Nguyen Denis-Charles Cisinski Bastiaan Cnossen and Tashi Walde. *Formalization of Higher Categories*. URL: https://drive.google.com/file/d/1lKaq7watGGl3xvjqw9qHjm6SDPFJ2-0o/view.

[4] Martin Gallauer. *infinity-categories: a first course*. December 26, 2023. URL: https://homepages.warwick.ac.uk/staff/Martin.Gallauer/teaching.html.

[5] Markus Land. *Introduction to infinity-categories*. English. Compact Textb. Math. Cham: Birkhäuser, 2021. ISBN: 978-3-030-61523-9; 978-3-030-61524-6. DOI: 10.1007/978-3-030-61524-6.

[6] Jacob Lurie. *Higher topos theory*. English. Vol. 170. Ann. Math. Stud. Princeton, NJ: Princeton University Press, 2009. ISBN: 978-0-691-14049-0; 978-0-691-14048-3. DOI: 10.1515/9781400830558.

[7] Jacob Lurie. *Kerodon*. https://kerodon.net. 2018.

[8] Charles Rezk. *Introduction to Quasicategories*. 2022. URL: https://rezk.web.illinois.edu/quasicats.pdf (visited on 09/11/2022).