Scientific Report of the node of Saint Petersburg

This concerns tasks **4.3.5.1, 4.3.5.2, 4.3.5.3**

In a series of papers [P1], [P2], [P3], [P4] Poincaré duality for oriented cohomology theories was proven, multiplicative structure on Voevodsky K-theory spectra was constructed, a motivic variant of Quillen's universality theorem for algebraic cobordism was proven, a motivic version of a theorem due to Conner and Floyd was proven. The last theorem expresses the higher Quillen K-groups in terms of algebraic cobordism of Voevodsky.

[P1] Panin I. Yagunov S. T-spectra and Poincaré Duality. — J. reine angew. Math. (accepted by Feb. 2007).

[P2] Panin I, Pimenov K. Roendigs O. On Voevodsky's algebraic K-theory P^1-spectrum BGL, http://www.math.uiuc.edu/K-theory/0838

[P3] Panin I, Pimenov K. Roendigs O. A universality theorem for Voevodsky's algebraic cobordism spectrum, http://www.math.uiuc.edu/K-theory/0846

[P3] Panin I, Pimenov K. Roendigs O. On the relation of Voevodsky's algebraic cobordism to Quillen's K-theory, http://www.math.uiuc.edu/K-theory/0847

Chow motives were introduced by Grothendieck and Manin as a universal cohomology theory on the category of smooth varieties. A motivic decomposition gives rise to the respective decomposition at the level of any oriented cohomology theory. We are interested in motivic decompositions of projective homogeneous varieties and their relations to various invariants of algebraic groups (Chow groups, splitting patterns, cohomological invariants, canonical dimensions).
We succeeded in computing motivic decompositions of generically split G-homogeneous varieties of inner type. Namely, it turns out that the Chow motive of such a variety X modulo prime p decomposes into a direct sum of shifted copies of some indecomposable motive $R_p(G)$ depending only on G and p. Its shape over an algebraic closure depends on some discrete invariant $J_p(G)$ first introduced by Vishik in the case of orthogonal groups. We show that the p-canonical dimension $cd_p(X)$ can be expressed in terms of this invariant, too.
As an application we investigate splitting behavior of exceptional groups. Namely, define the higher Tits index of G to be the set of Tits indices of G_E, where E runs over all extensions (possibly infinite) of the ground field. We show that the higher Tits index of groups of types G_2, F_4, E_6 and of strongly inner type E_7 depends only on their J-invariant.

[CPSZ] Calmès B., Petrov V., Semenov N., Zainoulline K. Chow motives of twisted flag varieties. — *Compositio Math.* **142** (2006), no. 4, 1063–1080.

[PSZ] Petrov V., Semenov N., Zainoulline K. J-invariant of linear algebraic groups. — preprint, available at `http://www.arxiv.org/math.AG/0607476`

[PS] Petrov V., Semenov N. Higher Tits indices of algebraic groups. — preprint, available at `http://www.arxiv.org/abs/0706.2827`
E.Sopkina has successfully completed classification of all subgroup schemes in a Chavelley group over any field, containing a split maximal torus [S1], [S2]. This very important result generalises a number of preceeding results, by several leading experts in the field, and constituted the core of her Ph.D. thesis.

[S1] Sopkina E., Classification of all connected subgroup schemes of a reductive group containing a split maximal torus. - POMI preprint, 2006, 17p.

[S2] Sopkina E., Subgroup schemes of a reductive group containing a split maximal torus. - POMI preprint, 2006, 10p.

In the papers by Vavilov, M.Gavrilovich and S.Nikolenko, see, in particular, [VGN], a new geometric approach towards calculations in exceptional groups has been proposed. This approach allows to carry many proofs in exactly the same way, as for classical groups, with the same bounds on the ranks of subsystems, where we search unipotent elements. At the level of K_1, in particular in description of overgroups of semi-simple subgroups, this approach depends only on the possibility to embed every root in a subsystem of type A_2. One could mention the two recent advances in this direction: complete analysis of the difficult case F_4 over a ring, where 2 is not invertible [VN], and a further development of the method, using unipotents of type A_3 to stabilise two columns of E_6 and E_7 in minimal representations [V5]. This allows to apply the method at the level of K_2, in particular, to construct other models of Steinberg groups, for which K_2 is automatically central.
In a closely related direction N.Vavilov, A.Luzgarev and I.Pevzner have studied minimal representations of exceptional groups, and, in particular, explicitly listed equations for E_6 and E_7 in minimal representations, and obtained many similar results, such as the calculation of the normaliser of elementary Chevalley groups, etc., see [VGN], [V3], [VL], [V8]. Other papers of related interest are [VPn], [VSt], [VS]. The surveys [V6] and [V9] by N.Vavilov contain a detailed description of the current state of the structure theory of exceptional algerbaic groups over commutative rings.

[VGN] N.Vavilov, M.Yu.Gavrilovich, S.I.Nikolenko Structure of Chevalley groups: the proof from the Book. – *Zapiski Nauch. Semin. POMI*, 2006, vol.330, p.30–71. (English transl. to appear in J. Math. Sci.).

[VLP] N.Vavilov, A.Yu.Luzgarev, I.M.Pevzner, Chevalley group of type E_6 in the 27-dimensional representation. – *Zapiski Nauch. Semin. POMI*, 2006, vol.338, p.5–68 (English transl. to appear in J. Math. Sci.).

[VPn] N.Vavilov, E.Ya.Perelman, Polyvector representations of GL_n. – *Zapiski Nauch. Semin. POMI*, 2006, vol.338, p.69–97 (English transl. to appear in J. Math. Sci.).

[V3] N.Vavilov, Can one see the signs of the structure constants? – *Algebra and Analysis*, 2007, vol. 19, N.4, p.34–68 (English transl. to appear in St.-Petersburg Math. J.)

[VL] N.Vavilov, A.Yu.Luzgarev, Normaliser of the Chevalley group of type E_6. – *Algebra and Analysis*, 2007, vol. 19, N.5, p.35–62 (English transl. to appear in St.-Petersburg Math. J.)

[VSt] N.Vavilov, A.K.Stavrova, Reduction theorems in description of normal subgroups. – *Zapiski Nauch. Semin. POMI*, 2007, vol.347, (English transl. to appear in J. Math. Sci.).

[V5] N.A.Vavilov, An A_3-proof of the main structure theorems for Chevalley groups of types E_6 and E_7. – Int. J. Algebra Comput., 2007, vol.17, N.5-6, p.1283–1298.

[V6] N.A.Vavilov, Calculations in exceptional groups. – accepted to *Vestnik Samara Univ.*, 2008.

[VS] N.A.Vavilov, A.V.Stepanov, Standard commutator formulae. – accepted to *Vestnik St.-Petersburg Univ.*, 2008.

[V8] N.A.Vavilov, Numerology of square equations. – accepted to *Algebra and Analysis*, 2008, vol. 20, N.2, (English transl. to appear in St.-Petersburg Math. J.)

[VN] N.A.Vavilov, S.I.Nikolenko, An A_2-proof of the structure theorems for Chevalley groups of type F_4. – accepted to *Algebra and Analysis*, 2008, vol. 20, N.3 (English transl. to appear in St.-Petersburg Math. J.)

[V8] N.A.Vavilov, Structure of exceptional groups over rings. – submitted to *Proc. 2nd Intern. Congress Algebra and Combinatorics*, World Sci., 2008.

Stepanov [S2] obtained a complete description of overgroups of the classical groups $\mathrm{Sp}(2l, R)$ and $\mathrm{SO}(2l + 1, R)$ over a commutative ring in another classical group of the same type over a different ring. From the viewpoint of the Maximal Subgroup Classification Project, this result (almost) completely describes overgroups for Aschbacher class C_5. This is in fact a very striking result, since for the general linear group [S1] and other simply-laced systems [S3] the standard description holds only for some very special classes of rings, say 1-dimensional ones. At the same time, for a (seamingly) much more difficult case of multiply laced root systems, the standard description holds over an aribtrary commutative ring.

The papers by N.Vavilov, A.Stepanov, V.Petrov and Hong You, written over the last years, brought the study of subgroups of classical groups over rings to an entirely new technological level. Namely, in these papers geometric ideas and techniques are effectively merged with the method of localisation-completion, see the description of this direction in [SVY]. In particular, they succeeded in giving a complete description of overgroups for Aschbacher class C_8, see [VP]. Other papers of related interest are [VKh], [V2], [V4].

On a slightly different slope, N.Vavilov, I.Pevzner and V.Nesterov studied applications of Bruhat-type decomposition to problems of generation of algebraic groups by various small elements or subgroups. While the case of unipotent subgroups has been widely studied, and [VPz] is essentially just another approach to the known results, for semisimple elements and tori this approach is novel. The results of [V1], [V7], [VNe] constitute the first steps towards what (in analogy with the "geometry of root subgroups") could be called the "geometry of tori".

[S1] A.V.Stepanov, Nonstandard subgroups between $E_n(R)$ and $\mathrm{GL_n(A)}$. – Algebra Colloquium, 2004, vol.10, N.3, p.321-334.

[S2] A.V.Stepanov, Subring subgroups in symplectic and odd orthogonal groups. – 2007, 10pp.

[S3] A.V.Stepanov, Free product subgroups between Chevalley groups $G(\Phi, R)$ and $G(\Phi, R[t])$. – 2007, 6pp.

[SVY], A.V.Stepanov, N.A.Vavilov, You Hong, The method of localisation-completion in the description of intermediate subgroups. – submitted to *Algebra Colloquium*, 2008.

[VKh] N.Vavilov, N.P.Kharchev, Orbits of subsystem stabilisers. – *Zapiski Nauch. Semin. POMI*, 2006, vol.338, p.98–124 (English transl. to appear in J. Math. Sci.).

[VP] N.Vavilov, V.A.Petrov, Overgroups of $\mathrm{EO(n, R)}$. – *Algebra and Analysis*, 2007, vol. 19, N.2, p.10–53. (English transl. to appear in St.-Petersburg Math. J.)

[V1] N.Vavilov, Geometry of 1-tori in $\mathrm{GL_n}$. – *Algebra and Analysis*, 2007, vol. 19, N.3, p.120–151 (English transl. to appear in St.-Petersburg Math. J.)

[V2] N.Vavilov, Subgroups of $\mathrm{SL_n}$ over a semi-local ring. – *Zapiski Nauch. Semin. POMI*, 2007, vol.343, p.33–53 (English transl. to appear in J. Math. Sci.).

[VPz] N.Vavilov, I.M.Pevzner, Triples of long root subgroups in Chevalley groups. – *Zapiski Nauch. Semin. POMI*, 2007, vol.343, p.54–83 (English transl. to appear in J. Math. Sci.).

[V4] N.Vavilov, Subgroups of symplectic group containing a subsystem subgroup. – *Zapiski Nauch. Semin. POMI*, 2007, vol.347, (English transl. to appear in J. Math. Sci.).

[V7] N.A.Vavilov, Weight elements of Chevalley groups. – accepted to *Algebra and Analysis*, 2008, vol. 20, N.1, (English transl. to appear in St.-Petersburg Math. J.)

[VNe] N.A.Vavilov, V.V.Nesterov, Pairs of microweight tori in Chevalley groups of type $\mathrm{E_6}$. – accepted to *Algebra and Analysis*, 2008, vol. 20, N.4, (English transl. to appear in St.-Petersburg Math. J.)

In [Ya1] we proved that there exist natural isomorphisms between generalized cohomology groups of a closed point on an algebraic variety and its étale neighborhood. This generalizes the result obtained below for K-groups by A.Suslin, O.Gabber, and H.Gillet—R.Thomason. Besides orientable theories this result also holds (in some special degrees) for generalized Witt groups.
In paper [Ya2] we showed that the classical Poincaré duality theorem also holds for all orientable cohomology theories on projective algebraic varieties. This fundamental classical result is as well important in the algebro-geometrical context. For example, it implies the generalization of Friedlander—Voevodsky duality theorem on the case of arbitrary characteristic (Suslin, unpublished)
It is also interesting to mention that our method gives a completely new proof of the statement in the topological case.

[Ya1] J.Hornbostel, S. Yagunov, Rigidity for Henselian local rings and A^1-representable theories. – Math. Z., vol. 255, No. 2 (2007) pp. 437-449.

[Ya3] S. Yagunov, Oriented cohomology theories over a field. – Oberwolfach Rep. 2, No. 2, Report 16, 889-892 (2005).

[Ya4] S. Yagunov, On some differentials in the Motivic cohomology spectral sequence. – Preprint MPI fuer Mathematik (Bonn) (to appear)

In January 2007 Viktor Petrov has obtained his second Ph.D. from the Univ. of Bielefeld for the Thesis.

[P] Petrov V., J-invariant of semisimple algebraic groups. – Dissertation, Univ. Bielefeld, Nov. 2006, 38p.

There are two further Ph.D Theses submitted this year by the junior members of the group Alexander Luzgarev and Igor Pevzner (defense scheduled for the Winter 2007/2008):

[L] A.Yu.Luzgarev, Overgroups of exceptional groups in minimal representations. – St-Petersburg State Univ., 2007.

[P] I.M.Pevzner, Generation of exceptional groups. – St-Petersburg State Univ., 2007.

For their work both of them have won prestigious RFFI–DAAD stipends "Mikhail Lomonosov" for prolonged research stays at the Univ. Bielefeld, starting Fall 2007.