Scaling Limit of the Abelian Sandpile in $\mathbb{Z}^2$

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Joint work with Wesley Pegden (Carnegie Mellon)
and Charles Smart (MIT)

Talk Outline

- The abelian sandpile as a growth model
  - origins in physics: Bak-Tang-Wiesenfeld 1987, Ostojic 2002, Dhar-Sadhu-Chandra 2008.
- Least Action Principle
- Existence of the scaling limit (Pegden-Smart 2011)
- The set $\Gamma(\mathbb{Z}^2)$

The Abelian Sandpile as a Growth Model

- Start with a pile of n chips at the origin in $\mathbb{Z}^d$.
- Each site $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d$ has $2d$ neighbors

$$x \pm e_i, \qquad i = 1, \ldots, d.$$

- Any site with at least $2d$ chips is unstable, and topples by sending one chip to each neighbor.

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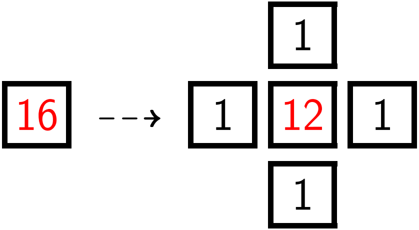
- Any site with at least $2d$ chips is unstable, and topples by sending one chip to each neighbor.
- This may create further unstable sites, which also topple.
- Continue until there are no more unstable sites.

Toppling to Stabilize A Sandpile

- Example: n=16 chips in $\mathbb{Z}^2$.
- Sites with 4 or more chips are unstable.



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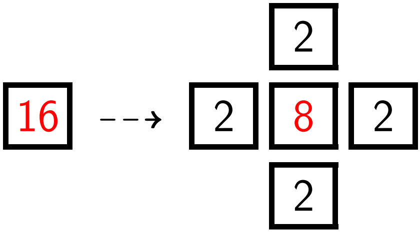

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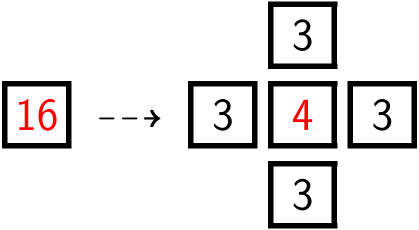

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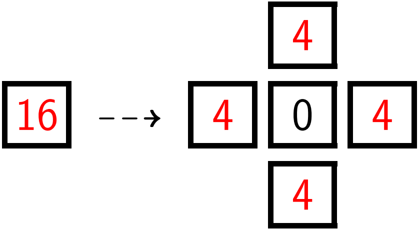

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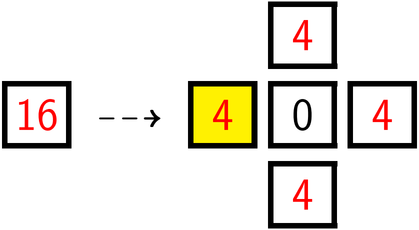

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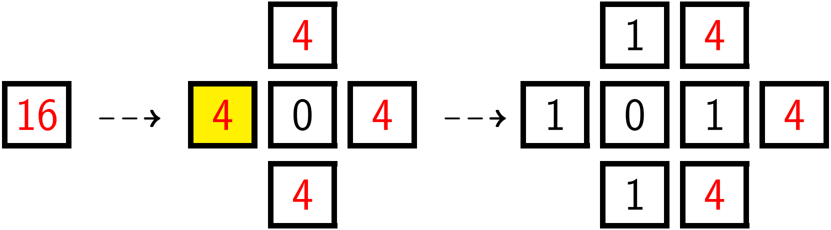

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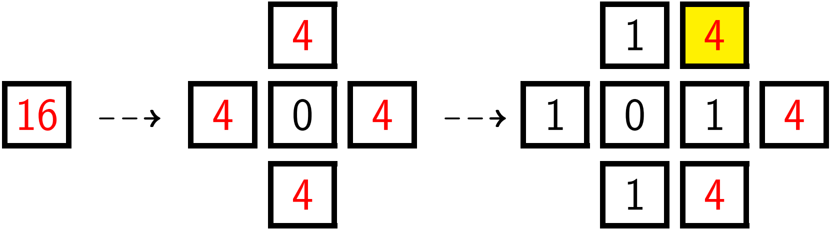

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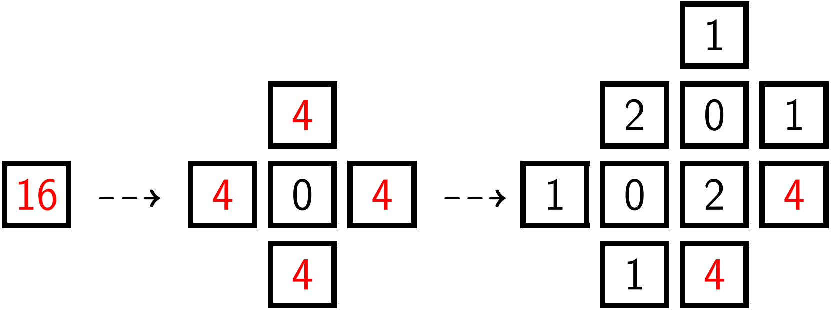

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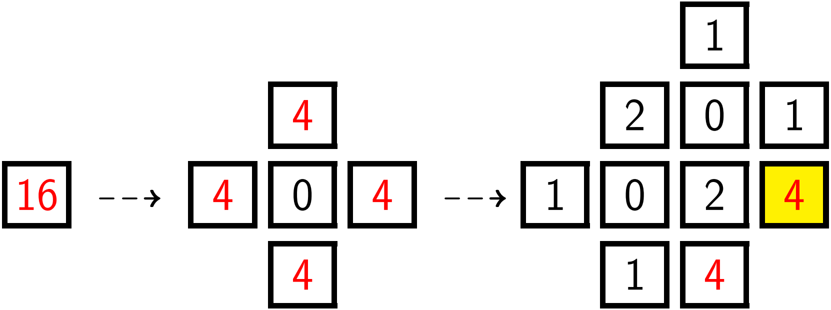

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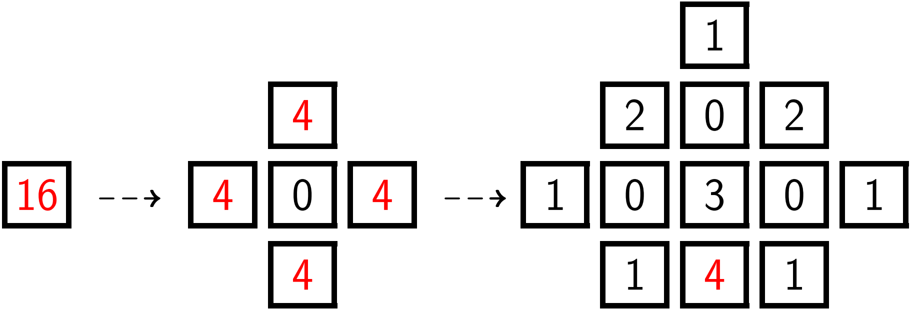

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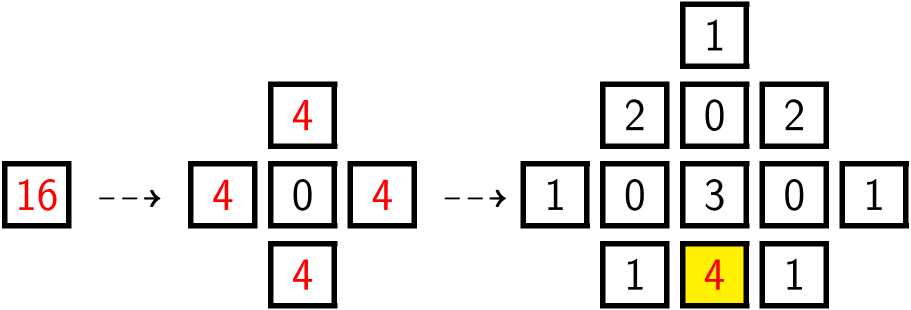

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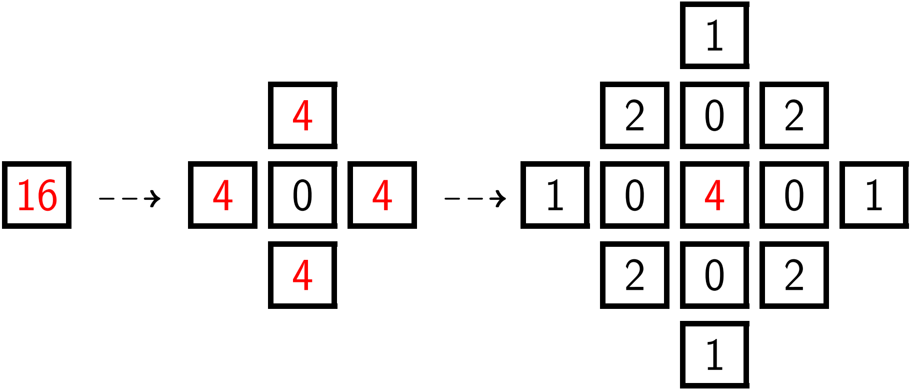

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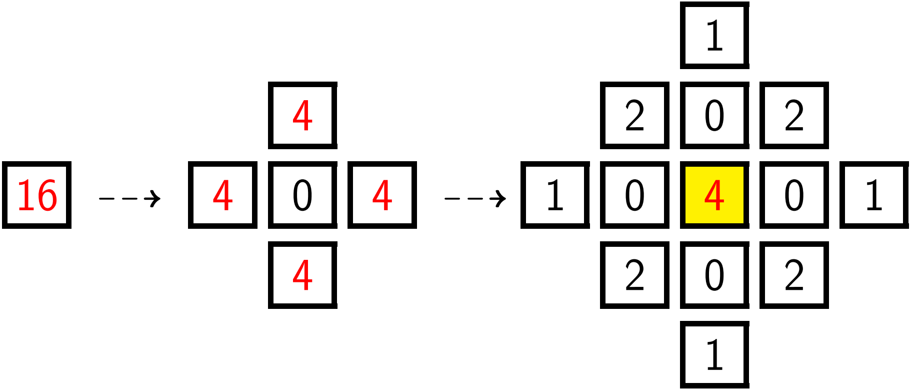

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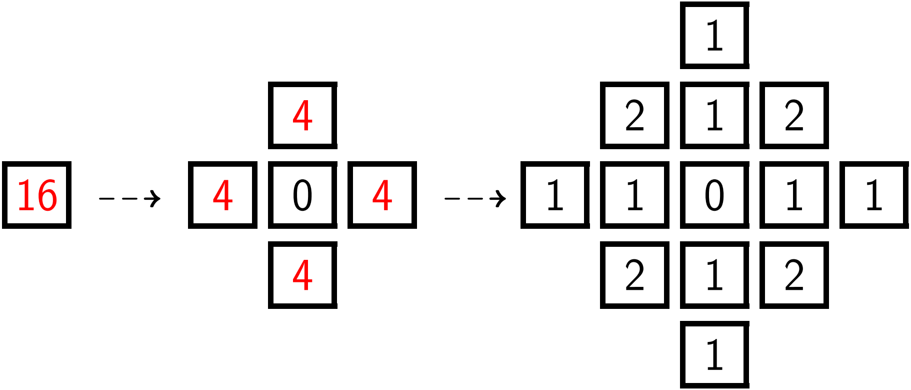
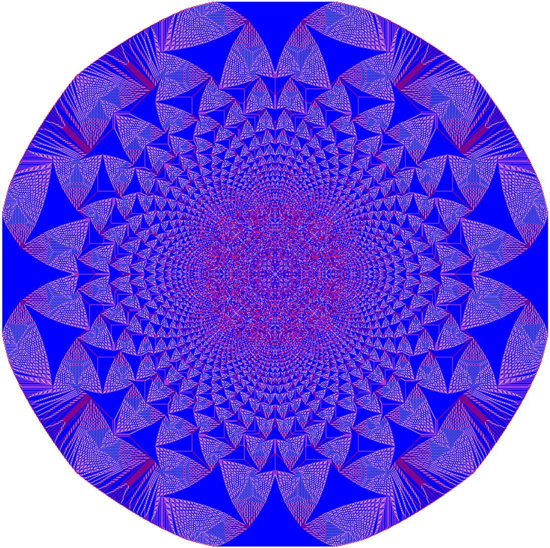


Stable.

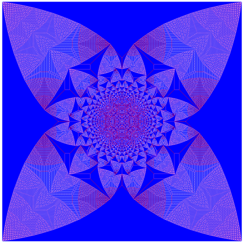
Abelian Property

- The final stable configuration does not depend on the order of topplings.
- Neither does the number of times a given vertex topples.

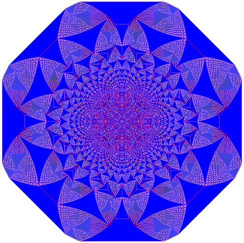
Sandpile of $1,000,000$ chips in $\mathbb{Z}^2$



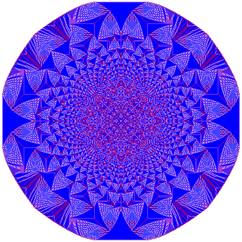
Sandpiles of the form $h + n\delta_0$



$h = 2$



$h = 1$



$h = 0$

What about $h = 3$?

3	3	3	3	3	3	3
3	3	3	3	3	3	3
3	3	3	3	3	3	3
3	3	3	4	3	3	3
3	3	3	3	3	3	3
3	3	3	3	3	3	3
3	3	3	3	3	3	3

3	3	3	3	3	3	3
3	3	3	3	3	3	3
3	3	3	4	3	3	3
3	3	4	0	4	3	3
3	3	3	4	3	3	3
3	3	3	3	3	3	3
3	3	3	3	3	3	3

3	3	3	3	3	3	3
3	3	3	4	3	3	3
3	3	5	0	5	3	3
3	4	0	4	0	4	3
3	3	5	0	5	3	3
3	3	3	4	3	3	3
3	3	3	3	3	3	3

3	3	3	4	3	3	3
3	3	5	0	5	3	3
3	5	1	4	1	5	3
4	0	4	0	4	0	4
3	5	1	4	1	5	3
3	3	5	0	5	3	3
3	3	3	4	3	3	3

3	3	5	0	5	3	3
3	5	1	4	1	5	3
5	1	5	0	5	1	5
0	4	0	4	0	4	0
5	1	5	0	5	1	5
3	5	1	4	1	5	3
3	3	5	0	5	3	3

. . . Never stops toppling!

3	5	1	4	1	5	3
5	1	5	0	5	1	5
1	5	1	4	1	5	1
4	0	4	0	4	0	4
1	5	1	4	1	5	1
5	1	5	0	5	1	5
3	5	1	4	1	5	3

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5	1	5	0	5	1	5
1	5	1	4	1	5	1
5	1	5	0	5	1	5
0	4	0	4	0	4	0
5	1	5	0	5	1	5
1	5	1	4	1	5	1
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1	5	1	4	1	5	1
5	1	5	0	5	1	5
1	5	1	4	1	5	1
4	0	4	0	4	0	4
1	5	1	4	1	5	1
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1	5	1	4	1	5	1

. . . Never stops toppling!

A dichotomy

Any sandpile $\tau : \mathbb{Z}^d \to \mathbb{N}$ is either

- *stabilizing*: every site topples finitely often
- or *exploding*: every site topples infinitely often

An open problem

- Given a probability distribution μ on $\mathbb{N}$, decide whether the i.i.d. sandpile $\tau \sim \prod_{x \in \mathbb{Z}^2} \mu$ is stabilizing or exploding.
- For example, find the smallest λ such that i.i.d. Poisson(λ) is exploding.

How to prove an explosion

- **Claim**: If every site in $\mathbb{Z}^d$ topples at least once, then every site topples infinitely often.

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- **Claim**: If every site in $\mathbb{Z}^d$ topples at least once, then every site topples infinitely often.
- Otherwise, let x be the first site to finish toppling.
- Each neighbor of x topples at least one more time, so x receives at least $2d$ additional chips.
- So x must topple again. $\Rightarrow\Leftarrow$

The Odometer Function

- $u(x) =$ number of times x topples.

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$$\begin{aligned}\Delta u(x) &= \sum_{y \sim x} u(y) - 2d\, u(x) \\ &= \text{chips received} - \text{chips emitted} \\ &= \tau_\infty(x) - \tau(x)\end{aligned}$$

where τ is the initial unstable chip configuration and τ_∞ is the final stable configuration.

Stabilizing Functions

- Given a chip configuration τ on $\mathbb{Z}^d$ and a function $u_1 : \mathbb{Z}^d \to \mathbb{Z}$, call u_1 stabilizing for τ if

$$\tau + \Delta u_1 \leq 2d - 1.$$

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- If u_1 and u_2 are stabilizing for τ, then

$$\begin{aligned} \tau + \Delta \min(u_1, u_2) &\leq \tau + \max(\Delta u_1, \Delta u_2) \\ &\leq 2d - 1 \end{aligned}$$

  so $\min(u_1, u_2)$ is also stabilizing for τ.

Least Action Principle

- Let τ be a sandpile on $\mathbb{Z}^d$ with odometer function u.
- Least Action Principle:

  If $v : \mathbb{Z}^d \to \mathbb{Z}_{\geq 0}$ is stabilizing for τ, then $u \leq v$.

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- So the odometer is minimal among all nonnegative stabilizing functions:

$$u(x) = \min\{v(x) \,|\, v \geq 0 \text{ is stabilizing for } \tau\}.$$

- Interpretation: "Sandpiles are lazy."

The Green function of $\mathbb{Z}^d$

- $G : \mathbb{Z}^d \to \mathbb{R}$ and $\Delta G = -\delta_0$.
- In dimensions $d \geq 3$,

$$G(x) = \mathbb{E}_0 \#\{k | X_k = x\}$$

  is the expected number of visits to x by simple random walk started at 0.
- As $|x| \to \infty$,

$$G(x) \sim g(x) = \begin{cases} c_d |x|^{2-d} & d \geq 3 \\ c_2 \log |x| & d = 2. \end{cases}$$

An integer obstacle problem

- The odometer function for n chips at the origin is given by

$$u = nG + w$$

where G is the Green function of $\mathbb{Z}^d$, and w is the pointwise smallest function on $\mathbb{Z}^d$ satisfying

$$w \geq -nG$$
$$\Delta w \leq 2d - 1$$
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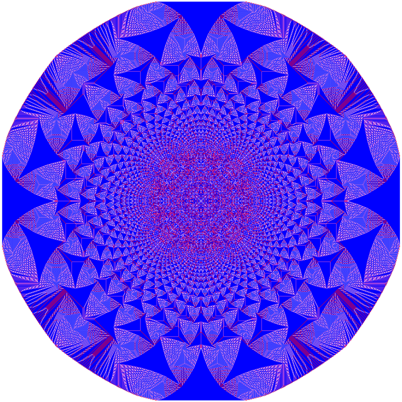
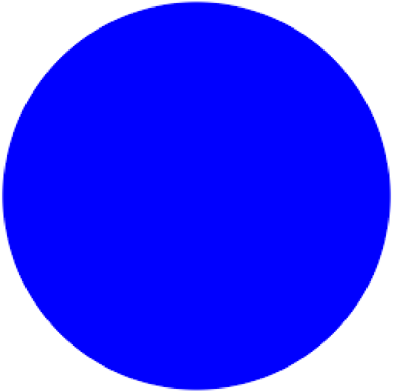
- What happens if we replace $\mathbb{Z}$ by $\mathbb{R}$?



Abelian sandpile
(Integrality constraint)



Divisible sandpile
(No integrality constraint)

Scaling limit of the abelian sandpile in $\mathbb{Z}^d$

- Consider $s_n = n\delta_0 + \Delta u_n$, the sandpile formed from n chips at the origin.
- Let $r = n^{1/d}$ and

$$
\begin{aligned}
\bar{s}_n(x) &= s_n(rx) && \text{(rescaled sandpile)} \\
\bar{w}_n(x) &= r^{-2} u_n(rx) - ng(rx) && \text{(rescaled odometer)}
\end{aligned}
$$

Theorem (Pegden-Smart, 2011)

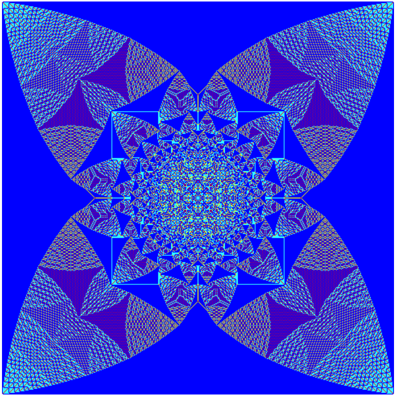
- There are functions $w, s : \mathbb{R}^d \to \mathbb{R}$ such that as $n \to \infty$,

$$\begin{aligned} \bar{w}_n \to w \qquad & \text{locally uniformly in } C(\mathbb{R}^d) \\ \bar{s}_n \to s \qquad & \text{weakly-}* \text{ in } L^\infty(\mathbb{R}^d). \end{aligned}$$

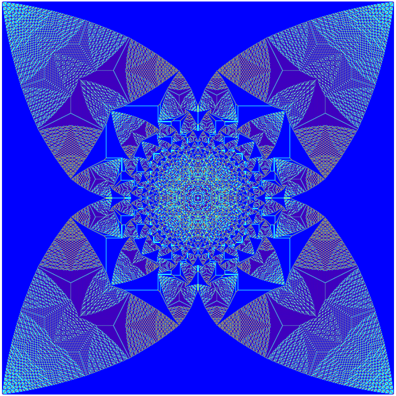
  Moreover s is a weak solution to $\Delta w = s$.

Two Sandpiles of Different Sizes

$n = 100,000$

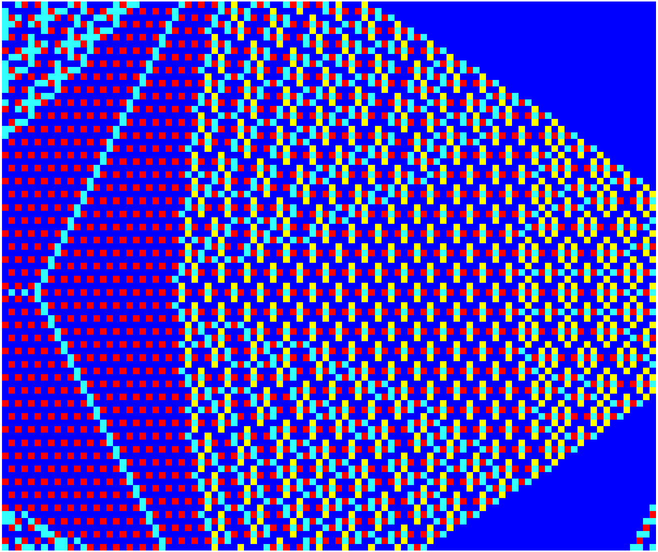


$n = 200,000$



(scaled down by $\sqrt{2}$)

Locally constant "steps" of s correspond to periodic patterns:



Limit of the least action princpile

$$w = \min\{v \in C(\mathbb{R}^d) \mid v \geq -g \text{ and } D^2(v+g) \in \Gamma\}.$$

- g encodes the initial condition (rotationally symmetric!)
- Γ is a set of symmetric $d \times d$ matrices, to be described. It encodes the sandpile "dynamics."

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- g encodes the initial condition (rotationally symmetric!)
- Γ is a set of symmetric $d \times d$ matrices, to be described. It encodes the sandpile "dynamics."
- $D^2 u \in \Gamma$ is interpreted in the sense of viscosity:

  $$D^2\phi(x) \in \Gamma$$

  whenever ϕ is a C^∞ function touching u from below at x (that is, $\phi(x) = u(x)$ and $\phi - u$ has a local maximum at x).

The set Γ of stabilizable matrices

- $\Gamma = \Gamma(\mathbb{Z}^d)$ is the set of $d \times d$ real symmetric matrices A for which there exists a slope $b \in \mathbb{R}^d$ and a function $v : \mathbb{Z}^d \to \mathbb{Z}$ such that

$$\Delta v(x) \leq 2d - 1 \quad \text{and} \quad v(x) \geq \tfrac{1}{2} x \cdot Ax + b \cdot x$$

for all $x \in \mathbb{Z}^d$.

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  for all $x \in \mathbb{Z}^d$.

- How to test for membership in Γ?
  - Start with $v(x) = \left\lceil \frac{1}{2} x \cdot Ax + b \cdot x \right\rceil$.
  - For each $x \in \mathbb{Z}^d$ such that $\Delta v(x) \geq 2d$, increase $v(x)$ by 1. Repeat.

Testing for membership in Γ

- $A \in \Gamma$ if and only if there exists b such that the sandpile

$$s_{A,b} = \Delta \lceil q_{A,b} \rceil$$

  is stabilizable, where $q_{A,b}(x) = \frac{1}{2} x \cdot Ax + b \cdot x$.
- if A, b have rational entries, then $s_{A,b}$ is periodic.
- Topple until stable, or until every site has toppled at least once.

The structure of $\Gamma(\mathbb{Z}^2)$

Parameterize 2×2 real symmetric matrices by

$$M(a, b, c) = \frac{1}{2} \begin{bmatrix} c + a & b \\ b & c - a \end{bmatrix}.$$

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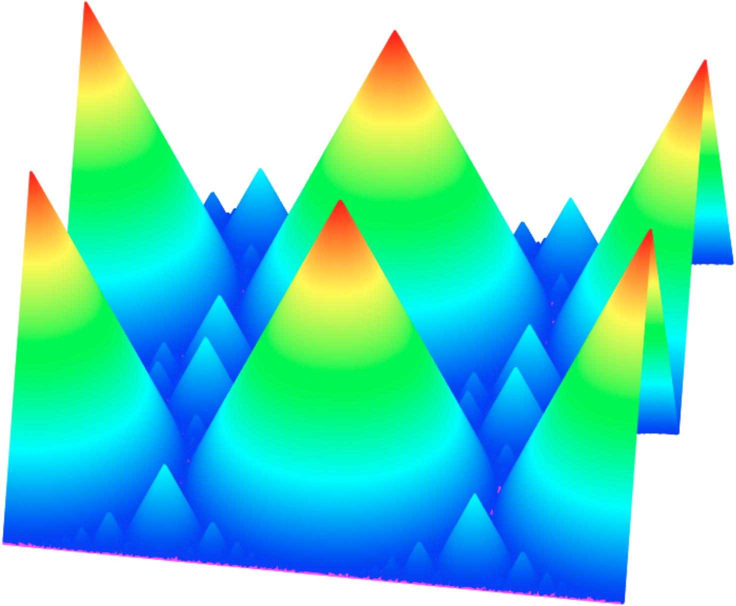
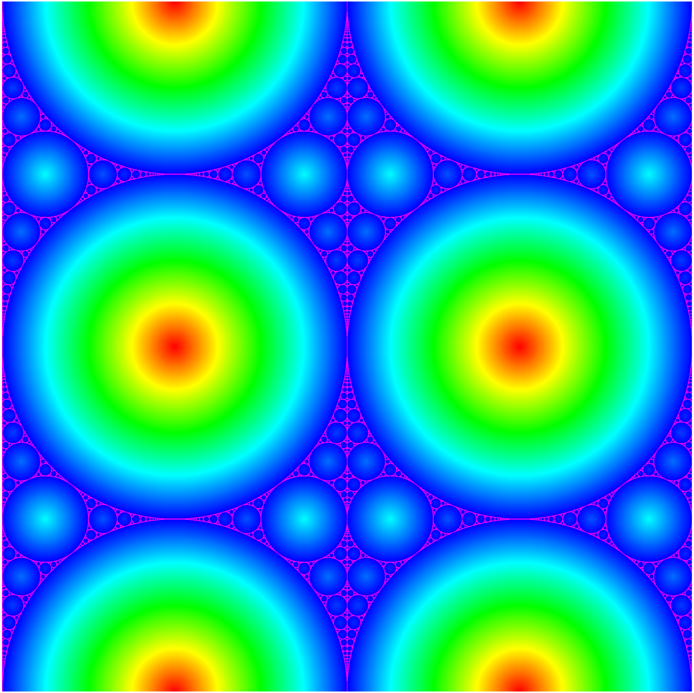
Note that if $A \leq B$ (that is, $B - A$ is positive semidefinite) and $B \in \Gamma$ then $A \in \Gamma$. In particular,

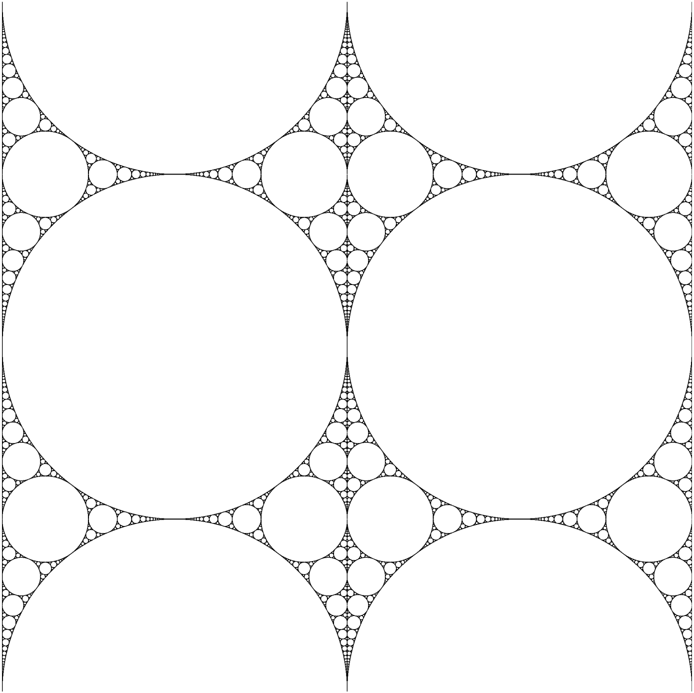
$$\Gamma = \{M(a, b, c) \mid c \leq \gamma(a, b)\}$$

for some function $\gamma : \mathbb{R}^2 \to \mathbb{R}$.

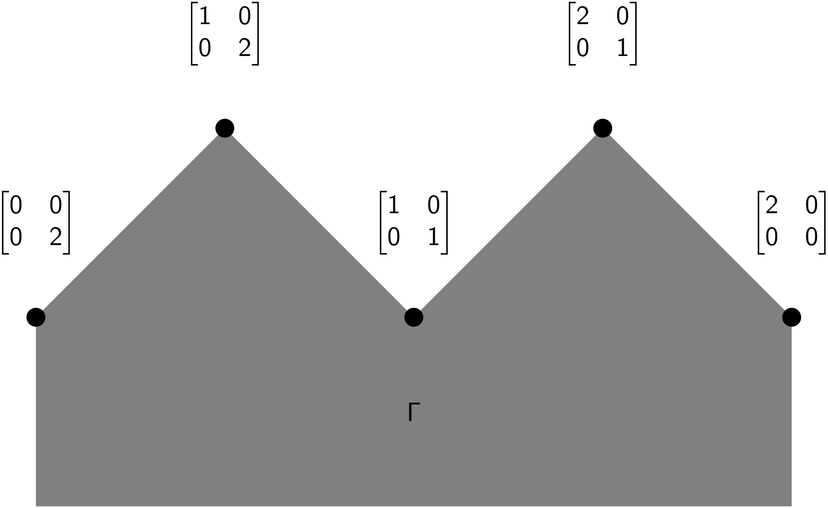
Graph of $\gamma(a, b)$







Cross section



Cross section

The Laplacian of

$$v(x) = \tfrac{1}{2}x_1(x_1 + 1) + \tfrac{1}{2}x_2(x_2 + 1)$$

is

2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2

Cross section

The Laplacian of

$$v(x) = \tfrac{1}{2}x_1(x_1+1) + \tfrac{1}{2}x_2(x_2+1) + \lceil \varepsilon x_2^2 \rceil$$

is

3	3	3	3	3	3	3	3	3	3	3	3
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3	3	3	3
2	2	2	2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3	3	3	3
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3	3	3	3
2	2	2	2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3	3	3	3
2	2	2	2	2	2	2	2	2	2	2	2

Cross section

The Laplacian of

$$v(x) = \tfrac{1}{2}x_1(x_1+1) + x_2(x_2+1)$$

is

3	3	3	3	3	3	3	3	3	3	3	3
3	3	3	3	3	3	3	3	3	3	3	3
3	3	3	3	3	3	3	3	3	3	3	3
3	3	3	3	3	3	3	3	3	3	3	3
3	3	3	3	3	3	3	3	3	3	3	3
3	3	3	3	3	3	3	3	3	3	3	3
3	3	3	3	3	3	3	3	3	3	3	3
3	3	3	3	3	3	3	3	3	3	3	3
3	3	3	3	3	3	3	3	3	3	3	3
3	3	3	3	3	3	3	3	3	3	3	3
3	3	3	3	3	3	3	3	3	3	3	3
3	3	3	3	3	3	3	3	3	3	3	3

Cross section

The Laplacian of

$$v(x) = \tfrac{1}{2}x_1(x_1+1) + \tfrac{1}{2}x_2(x_2+1) + \lceil \varepsilon x_1^2 \rceil$$

is

2	3	2	2	2	3	2	2	2	3	2	2
2	3	2	2	2	3	2	2	2	3	2	2
2	3	2	2	2	3	2	2	2	3	2	2
2	3	2	2	2	3	2	2	2	3	2	2
2	3	2	2	2	3	2	2	2	3	2	2
2	3	2	2	2	3	2	2	2	3	2	2
2	3	2	2	2	3	2	2	2	3	2	2
2	3	2	2	2	3	2	2	2	3	2	2
2	3	2	2	2	3	2	2	2	3	2	2
2	3	2	2	2	3	2	2	2	3	2	2
2	3	2	2	2	3	2	2	2	3	2	2
2	3	2	2	2	3	2	2	2	3	2	2

Cross section

The Laplacian of

$$v(x) = \tfrac{1}{2}x_1(x_1+1) + \tfrac{1}{2}x_2(x_2+1) + \lceil \varepsilon x_1^2 + \varepsilon x_2^2 \rceil$$

is

2	2	2	2	2	2	2	2	3	2	2	2
2	2	2	2	2	2	2	2	3	2	2	2
2	2	2	2	2	2	2	2	3	2	2	2
2	2	2	2	2	2	2	2	3	2	2	2
2	2	2	2	2	2	2	2	3	2	2	2
3	3	3	3	3	3	3	3	4	3	3	3
2	2	2	2	2	2	2	2	3	2	2	2
2	2	2	2	2	2	2	2	3	2	2	2
2	2	2	2	2	2	2	2	3	2	2	2
2	2	2	2	2	2	2	2	3	2	2	2
2	2	2	2	2	2	2	2	3	2	2	2
2	2	2	2	2	2	2	2	3	2	2	2

Another example

We have $\frac{1}{4}\left[\begin{smallmatrix}5 & 2\\ 2 & 4\end{smallmatrix}\right] \in \partial\Gamma$ because

$$v(x) = \left\lceil \frac{1}{8}(5x_1^2 + 4x_1x_2 + 4x_2^2 + 2x_1 + 4x_2) \right\rceil$$

has Laplacian

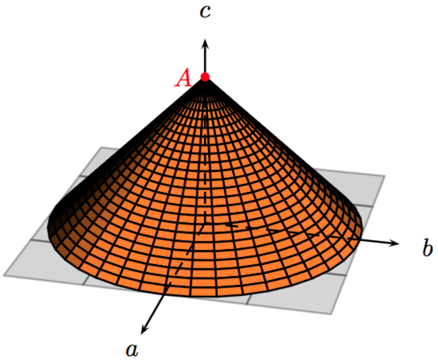
3	3	3	0	3	3	3	0	3	3	3	0
3	0	3	3	3	0	3	3	3	0	3	3
3	3	3	0	3	3	3	0	3	3	3	0
3	0	3	3	3	0	3	3	3	0	3	3
3	3	3	0	3	3	3	0	3	3	3	0
3	0	3	3	3	0	3	3	3	0	3	3
3	3	3	0	3	3	3	0	3	3	3	0
3	0	3	3	3	0	3	3	3	0	3	3
3	3	3	0	3	3	3	0	3	3	3	0
3	0	3	3	3	0	3	3	3	0	3	3
3	3	3	0	3	3	3	0	3	3	3	0
3	0	3	3	3	0	3	3	3	0	3	3

Rank-1 cones

The set $\Gamma(\mathbb{Z}^2)$ is a union of downward cones

$$\{B \mid B \leq A\},$$

for a set of *peaks* $A \in \mathcal{P}$.



Periodicity

Since the matrices

$$M(2,0,0) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad M(0,2,0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

have integer valued discrete harmonic quadratic forms

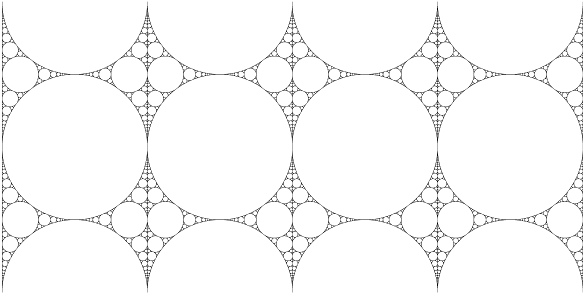
$$u(x) = \tfrac{1}{2}x_1(x_1+1) - \tfrac{1}{2}x_2(x_2+1) \quad \text{and} \quad u(x) = x_1 x_2,$$

we see that γ is $2\mathbb{Z}^2$-periodic.

Associating a matrix to each circle

If C is a circle of radius r centered at (a, b), define

$$A_C := \frac{1}{2}\begin{bmatrix} a+2+r & b \\ b & -a+2+r \end{bmatrix}.$$

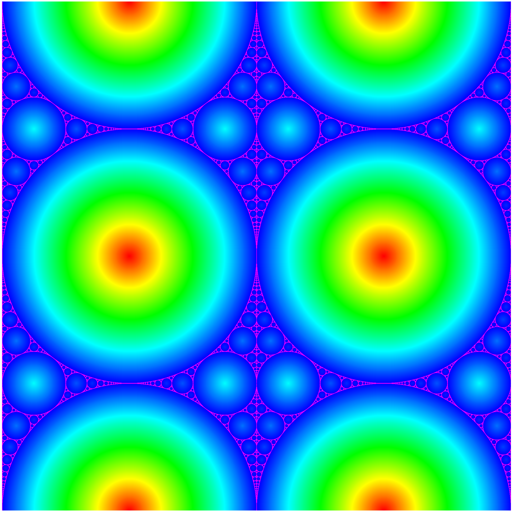


Let $\mathcal{A}$ be the circle packing in the (a, b)-plane generated by the vertical lines $a = 0$, $a = 2$ and the circle $(a-1)^2 + b^2 = 1$, repeated horizontally so it is $2\mathbb{Z}^2$-periodic.

The Apollonian structure of Γ

Theorem (L-Pegden-Smart 2013)

$B \in \Gamma$ if and only if $B \leq A_C$ for some $C \in \mathcal{A}$.



Analysis of the peaks

Theorem (L-Pegden-Smart 2013)

$B \in \Gamma$ if and only if $B \leq A_C$ for some $C \in \mathcal{A}$.

Proof idea: It is enough to show that each peak matrix A_C lies on the boundary of Γ.

Analysis of the peaks

Theorem (L-Pegden-Smart 2013)

$B \in \Gamma$ if and only if $B \leq A_C$ for some $C \in \mathcal{A}$.

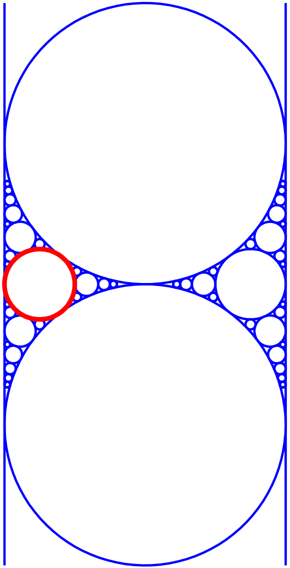
Proof idea: It is enough to show that each peak matrix A_C lies on the boundary of Γ.

For each A_C we must find $v_C : \mathbb{Z}^2 \to \mathbb{Z}$ and $b_C \in \mathbb{R}^2$ such that

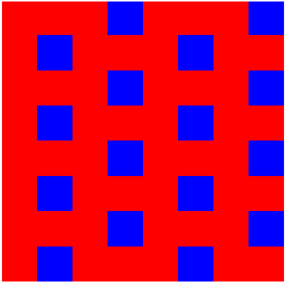
$$\Delta v_C(x) \leq 3 \quad \text{and} \quad v_C(x) \geq \tfrac{1}{2} x \cdot A_C x + b_C \cdot x$$

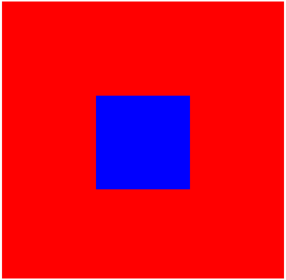
for all $x \in \mathbb{Z}^2$.

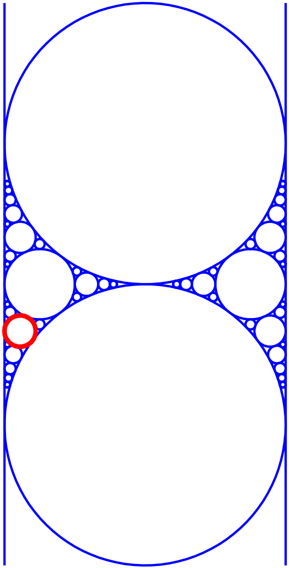
We use the recursive structure of the circle packing to construct v_C and b_C.



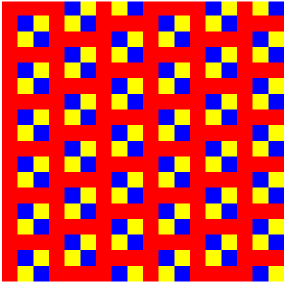
(4,1,4)

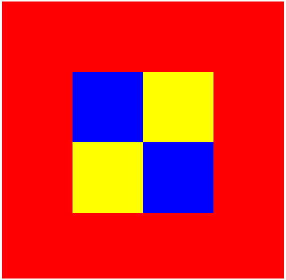


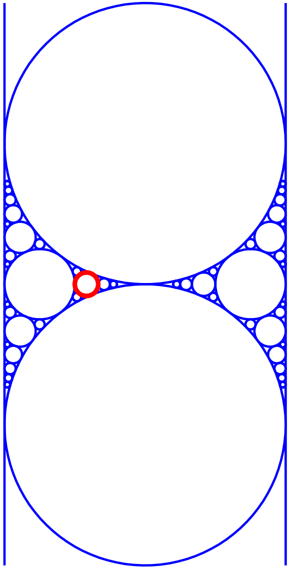




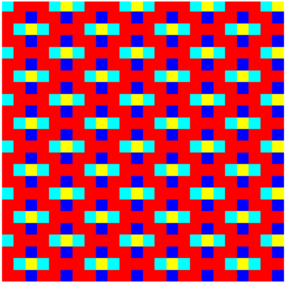
(9,1,6)

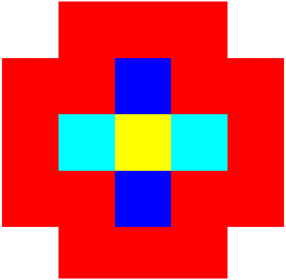


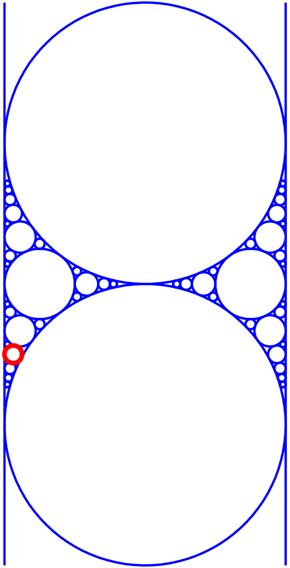




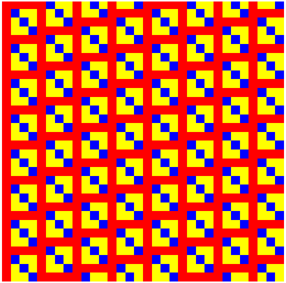
(12,7,12)

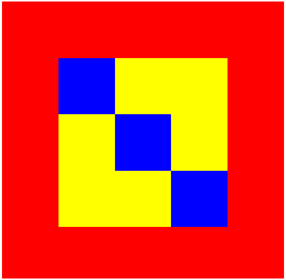


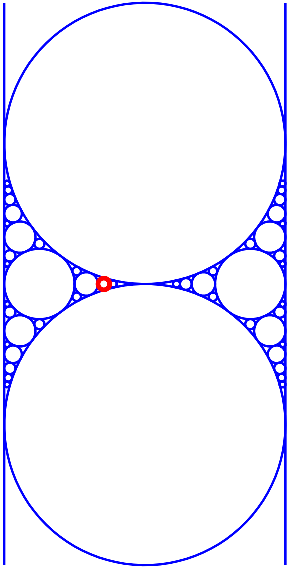




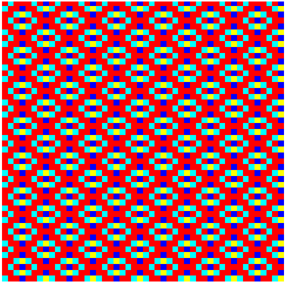
(16,1,8)

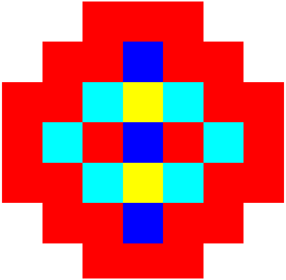






(24,17,24)



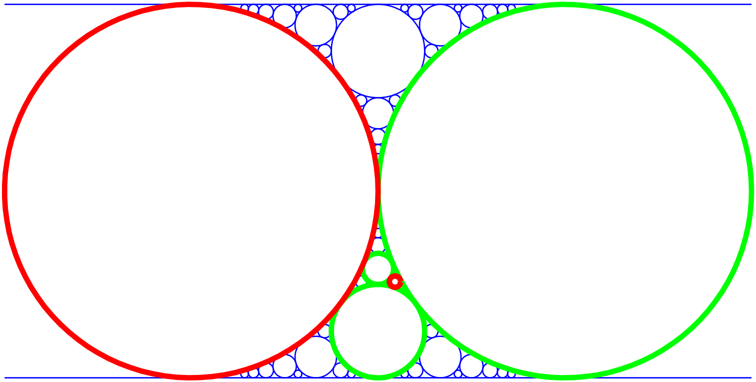


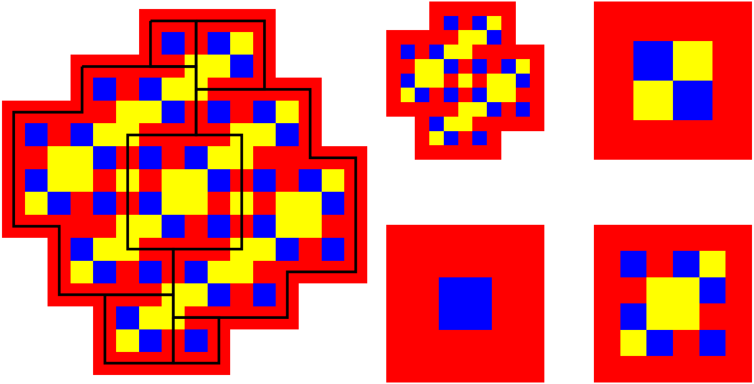
Curvature coordinates
(Descartes 1643; Lagarias-Mallows-Wilks 2002)

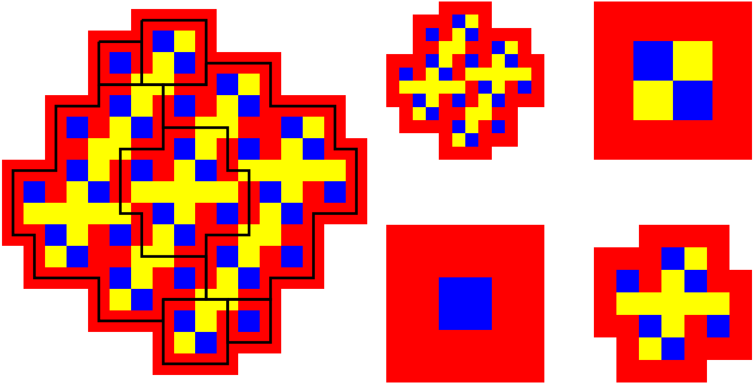
If C_0 has parents C_1, C_2, C_3 and grandparent C_4, then

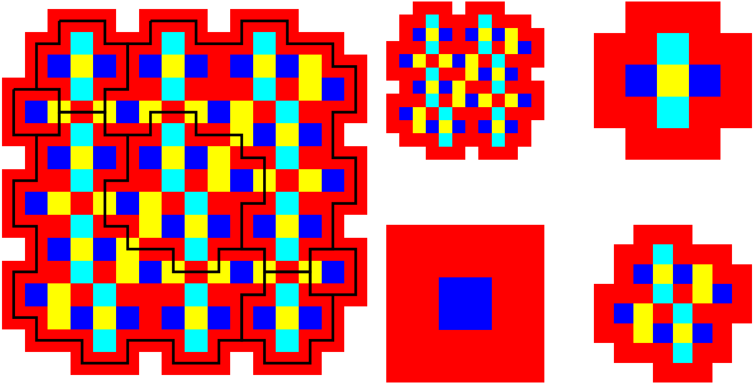
$$C_0 = 2(C_1 + C_2 + C_3) - C_4$$

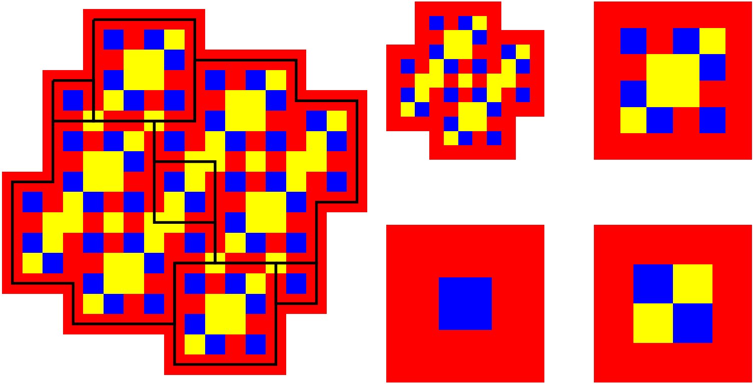
in curvature coordinates $C = (c, cx, cy)$.





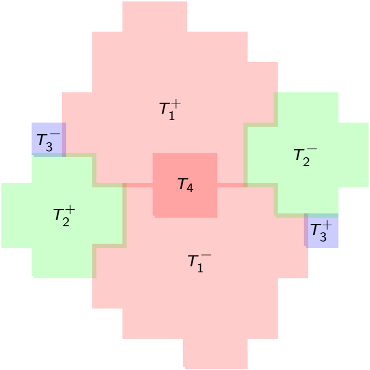




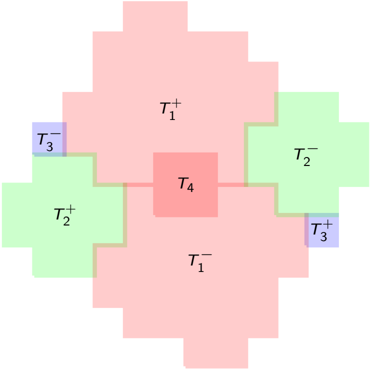


Inductive tile construction

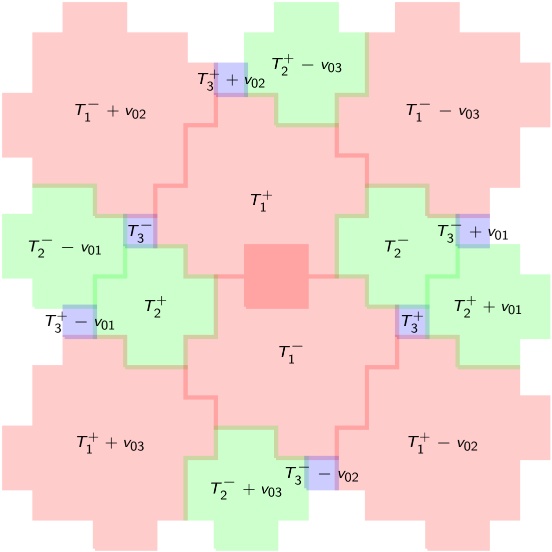
We build tiles from copies of earlier tiles, using ideas from Stange 2012 “The Sensual Appolonian Circle Packing” to keep track of the tile interfaces.



Inductive tile construction



Inductive tile construction



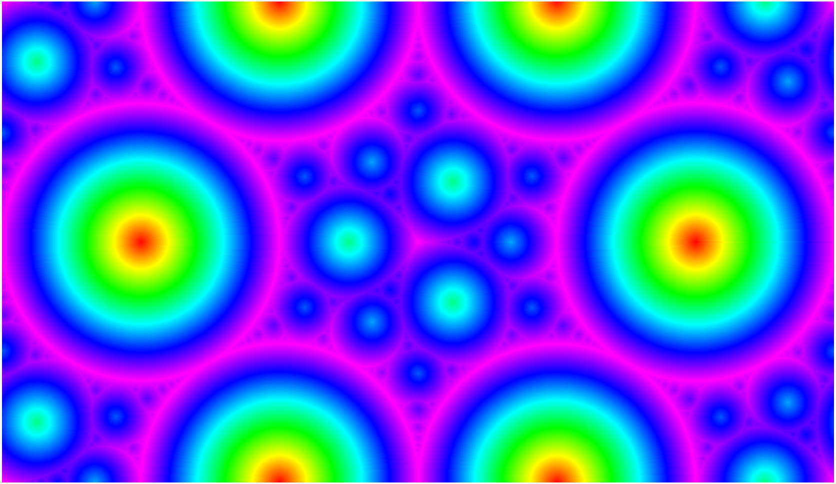
Other lattices, higher dimensions

We have described the set $\Gamma(\mathbb{Z}^2)$ in terms of an Appolonian circle packing of $\mathbb{R}^2$.

What about $\Gamma(\mathbb{Z}^d)$ for $d \geq 3$?

In general any periodic graph G embedded in $\mathbb{R}^d$ has an associated set of $d \times d$ symmetric matrices $\Gamma(G)$, which captures some aspect of the infinitessimal geometry of $\frac{1}{n}G$ as $n \to \infty$.

Γ for the triangular lattice



Thank you!

Reference:

L.-Pegden-Smart, `arXiv:1309.3267`
The Apollonian structure of integer superharmonic matrices