Random Convex Hulls: Applications to Ecology and Animal Epidemics

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- Random Convex Hull $\Longrightarrow$ definition

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- Convex Hull of two-dimensional stochastic processes

  simple random walks, branching random walks etc.

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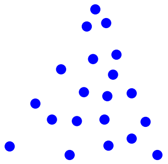
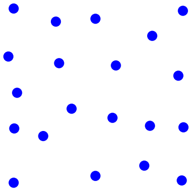
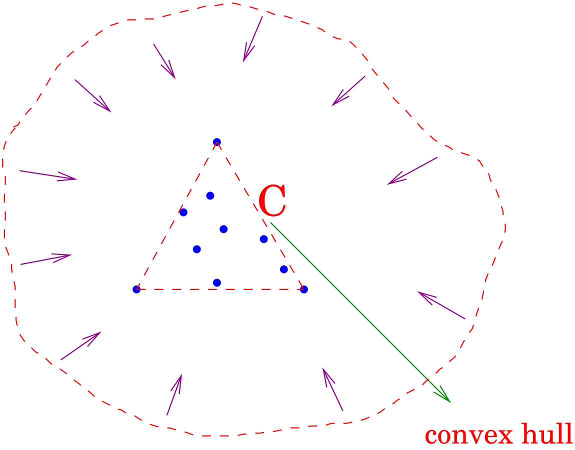
- Summary and Conclusion

Shape of a set of Points

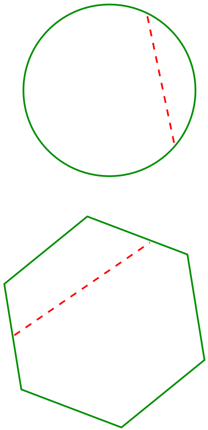




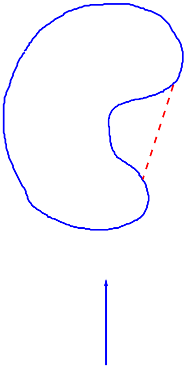
Shape of a set of Points: Convex Hull



Closed Convex Curves

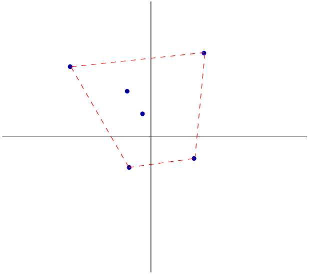


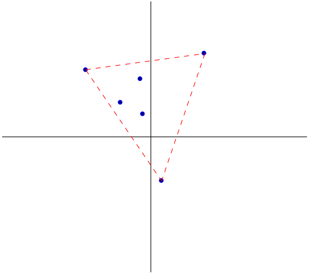
CONVEX



NON–CONVEX

Random Convex Hull in a Plane





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Random Convex Hull in a Plane

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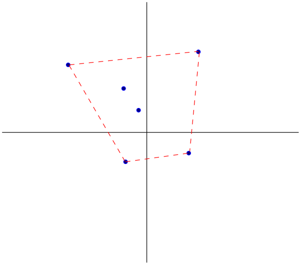
Random Convex Hull in a Plane

- Convex Hull $\Longrightarrow$ Minimal convex polygon enclosing the set
- The shape of the convex hull $\rightarrow$ different for each sample
- Points drawn from a distribution $P(\vec{r_1}, \vec{r_2}, \ldots, \vec{r_N})$
  $\rightarrow$ Independent or Correlated
- Question: Statistics of observables: perimeter, area and no. of vertices

Independent Points in a Plane



Each point chosen independently from the same distribution

$$P\left(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N\right) = \prod_{i=1}^{N} p(\vec{r}_i)$$

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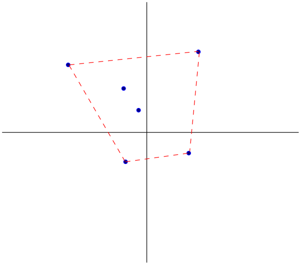

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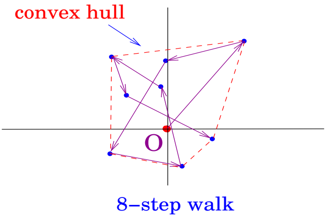
Associated Random Convex Hull $\rightarrow$ well studied by diverse methods

Lévy ('48), Geffroy ('59), Spitzer & Widom ('59), Baxter ('59)
Rényi & Sulanke ('63), Efron ('65), Molchanov ('07)....many others

Correlated Points: Open Random Walk



Discrete-time random Walk of N steps

$x_k = x_{k-1} + \xi_x(k)$
$y_k = y_{k-1} + \xi_y(k)$

$\xi_x(k), \xi_y(k) \to$ Independent jump lengths

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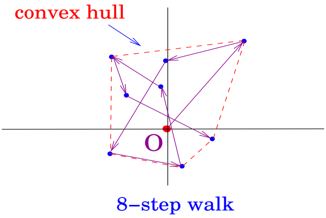

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Continuous-time limit: Brownian path of duration T

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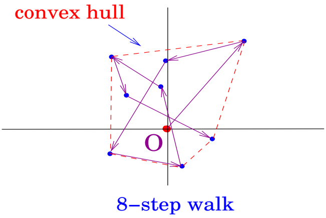

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$\langle \eta_x(\tau)\eta_x(\tau') \rangle = 2\, D\, \delta(\tau - \tau')$

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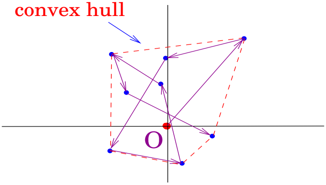

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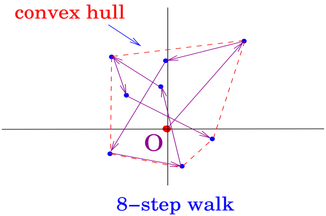

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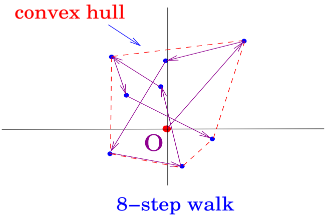
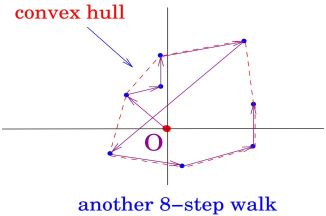
$$\langle \eta_x(\tau)\eta_y(\tau') \rangle = 0$$

Correlated Points: Open Random Walk





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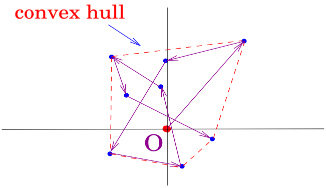
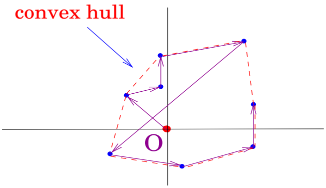
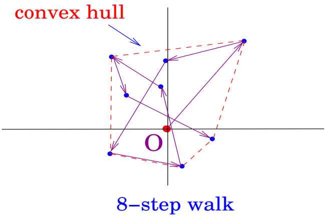


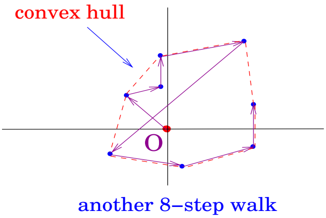
8–step walk



another 8–step walk

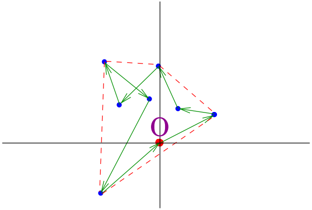
- Continuous-time limit: Brownian path of duration T
- mean perimeter and mean area of the associated Convex hull?



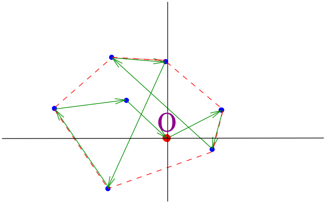


- Continuous-time limit: Brownian path of duration T
- mean perimeter and mean area of the associated Convex hull?
- mean perimeter: $\langle L_1 \rangle = \sqrt{8\pi}\,\sqrt{2\,D\,T}$ (Takács, '80)
- mean area: $\langle A_1 \rangle = \frac{\pi}{2}\,(2\,D\,T)$ (El Bachir, '83, Letac '93)

Correlated Points: Closed Random Walk



8 step random bridge



another 8 step bridge

- Continuous-time limit: Brownian bridge of duration T : starting at O and returning to it after time T

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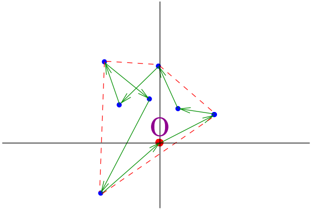

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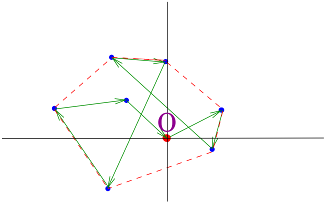

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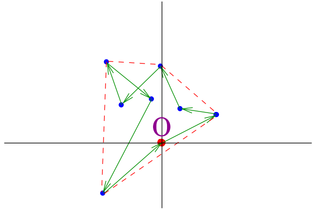

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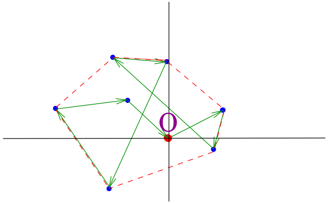

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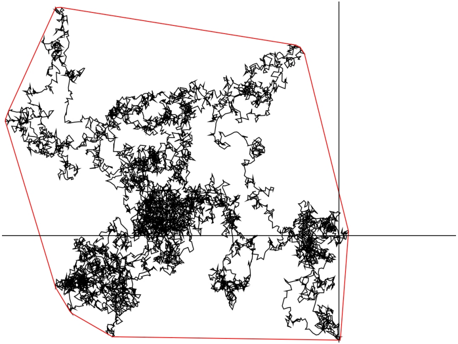
- mean area: $\langle A_1 \rangle = (?)\,(2\,D\,T)$

Home Range Estimate via Convex Hull



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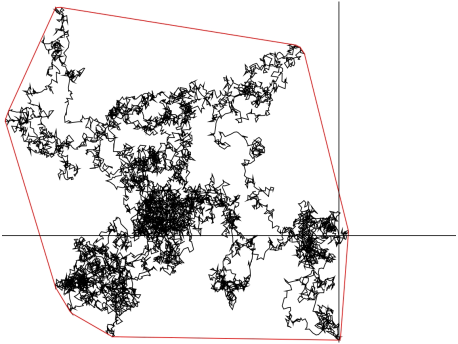

Models of home range for animal movement, Worton (1987)
Integrating Scientific Methods with Habitat Conservation Planning, Murphy and Noon (1992)
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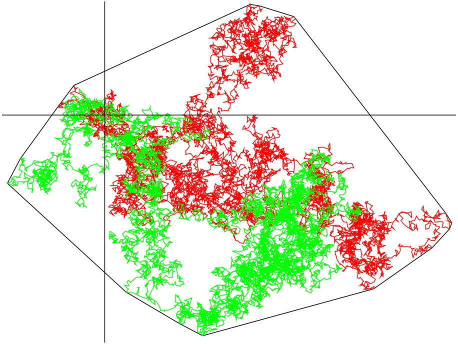

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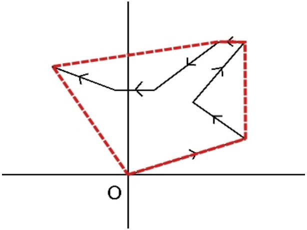
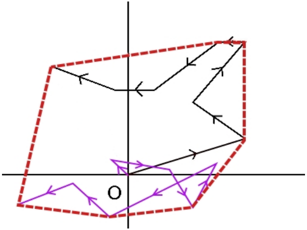
Global Convex Hull of n Independent Brownian Paths





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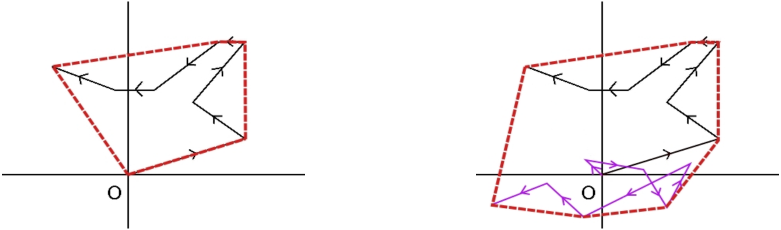
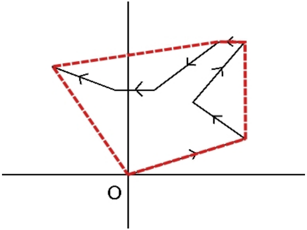
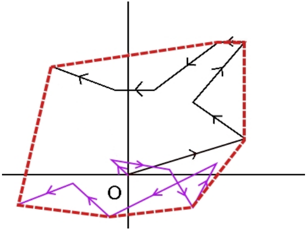


• Mean perimeter $\langle L_n \rangle$ and mean area $\langle A_n \rangle$ of n independent Brownian paths (bridges) each of duration T?

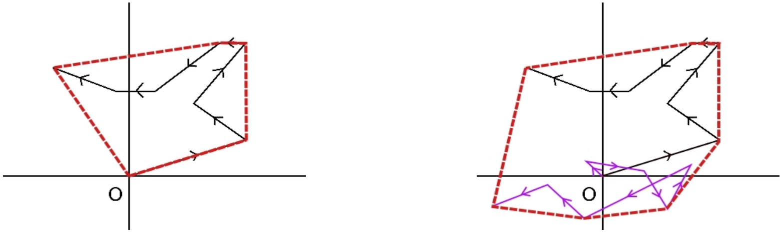
Global Convex Hull of n Independent Brownian Paths





- Mean perimeter $\langle L_n \rangle$ and mean area $\langle A_n \rangle$ of n independent Brownian paths (bridges) each of duration T?
- $\langle L_n \rangle = \alpha_n \sqrt{2\,D\,T}$; $\langle A_n \rangle = \beta_n\,(2\,D\,T)$

Global Convex Hull of n Independent Brownian Paths



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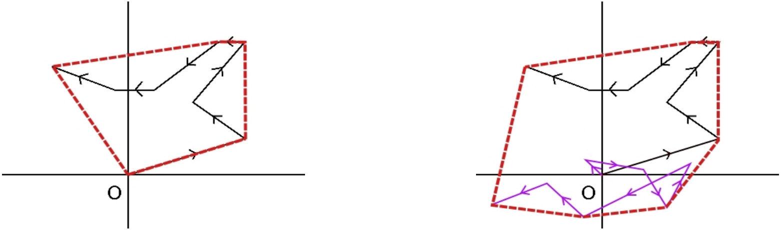
- Recall $\boxed{\alpha_1 = \sqrt{8\pi}, \quad \beta_1 = \pi/2}$ (open path)

$\boxed{\alpha_1 = \sqrt{\pi^3/2}, \quad \beta_1 = ?}$ (closed path)

Global Convex Hull of n Independent Brownian Paths



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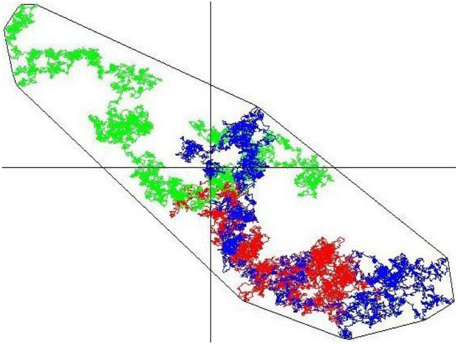
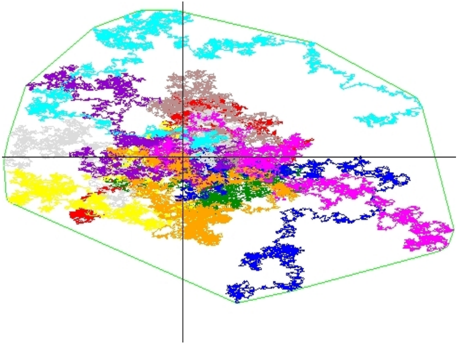
  $\boxed{\alpha_1 = \sqrt{\pi^3/2}, \quad \beta_1 = ?}$ (closed path)
- $\boxed{\alpha_n, \quad \beta_n = ?}$ $\rightarrow$ both for open and closed paths $\rightarrow$ n-dependence?

Global Convex Hull of n Independent Brownian Paths



$n = 3$ closed paths



$n = 10$ open paths

Outbreak of animal epidemics

Preventive Veterinary Medicine 92 (2009) 60–70

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PREVENTIVE VETERINARY MEDICINE

The equine influenza epidemic in Australia: Spatial and temporal descriptive analyses of a large propagating epidemic

Brendan Cowled[a,*], Michael P. Ward[b], Samuel Hamilton[a], Graeme Garner[a]

[a] *Office of the Chief Veterinary Officer, Department of Agriculture, Fisheries and Forestry, GPO Box 858, Canberra, ACT 2601, Australia*
[b] *Faculty of Veterinary Science, The University of Sydney, Private Mail Bag 3, Camden, NSW 2570, Australia*

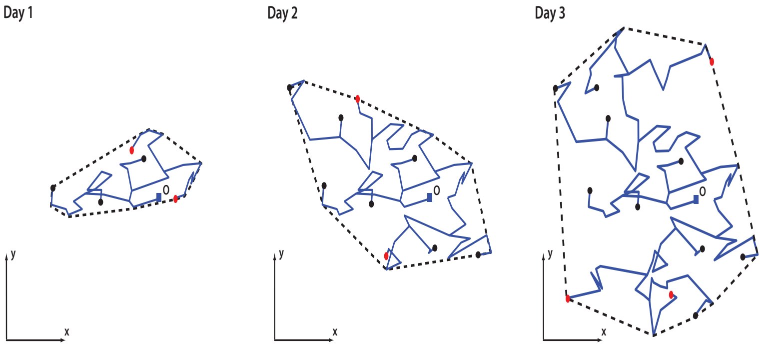
Outbreak of animal epidemics

B. Cowled et al. / Preventive Veterinary Medicine 92 (2009) 60–70

2.4. Spatial and temporal overview of the epidemic

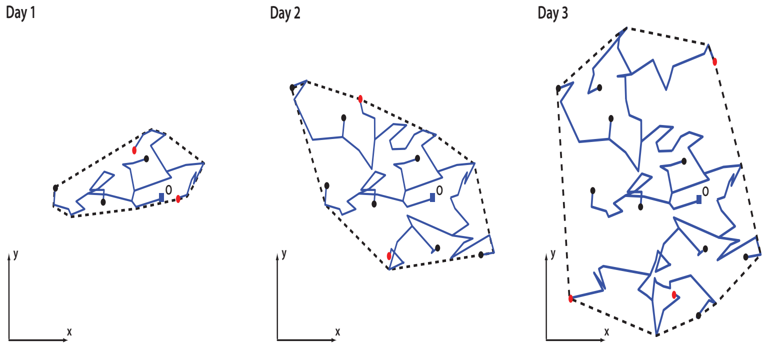
An epidemic curve was created by plotting the number of premises on which equids first displayed clinical signs for each day of the epidemic on the *Y* axis, and day of the epidemic on the *X* axis. The day clinical signs were first reported was used, rather than the date of confirmation of diagnosis to avoid reporting bias (Gibbens and Wilesmith, 2002). The change in the size of the infected area over time was also calculated. The area infected was estimated by creating a minimum convex hull (MCH) that encompassed all infected premises for every 2 days of the epidemic. A MCH is a convex polygon which encompasses a collection of point locations (Mapinfo Coorporation, 2006). The increase in the infected area from one 2-day time period

Outbreak of animal epidemics



animal epidemic spread $\Longrightarrow$ branching (infection) [with rate b] Brownian motion with death (recovery) [with rate a]

Outbreak of animal epidemics



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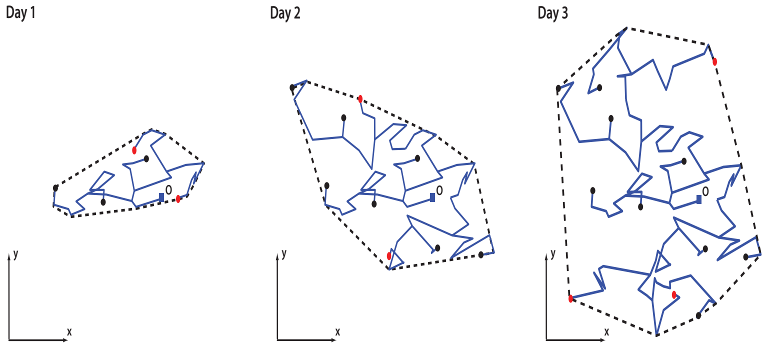
- supercritical: $b > a \to$ epidemic explodes
- subcritical: $b < a \to$ epidemic becomes extinct
- critical: $b = a \to$ epidemic critical

Outbreak of animal epidemics



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Q: how does the perimeter and area of the convex hull grow with time t?

Two problems in 2-d

Perimeter and Area of the convex hull in two different 2-d problems:

Two problems in 2-d

Perimeter and Area of the convex hull in two different 2-d problems:

I. convex hull of n independent Brownian motions of duration T

$\Longrightarrow$ home range estimate (dependence on the population size n)

Two problems in 2-d

Perimeter and Area of the convex hull in two different 2-d problems:

I. convex hull of n independent Brownian motions of duration T

$\Longrightarrow$ home range estimate (dependence on the population size n)

II. convex hull of a branching Brownian motion with death

$\Longrightarrow$ spread of animal epidemics (dependence on time T)

Two problems in 2-d

Perimeter and Area of the convex hull in two different 2-d problems:

I. convex hull of n independent Brownian motions of duration T

$\Longrightarrow$ home range estimate (dependence on the population size n)

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- Numerical simulations $\longrightarrow$ relatively easy

Two problems in 2-d

Perimeter and Area of the convex hull in two different 2-d problems:

I. convex hull of n independent Brownian motions of duration T

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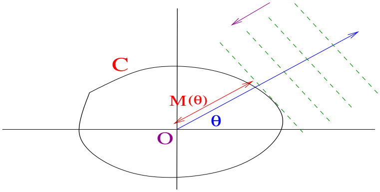
$\Longrightarrow$ spread of animal epidemics (dependence on time T)

- Numerical simulations $\longrightarrow$ relatively easy
- Analytical computation $\longrightarrow$ how to proceed?

Cauchy's Formulae for a Closed Convex Curve

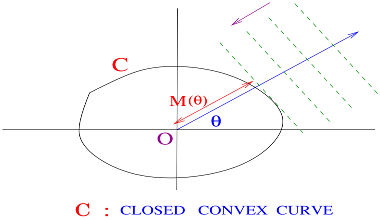


C : CLOSED CONVEX CURVE

- For any point $[X(s), Y(s)]$ on C define:

  Support function: $M(\theta) = \max_{s \in C} [X(s)\cos(\theta) + Y(s)\sin(\theta)]$

Cauchy's Formulae for a Closed Convex Curve



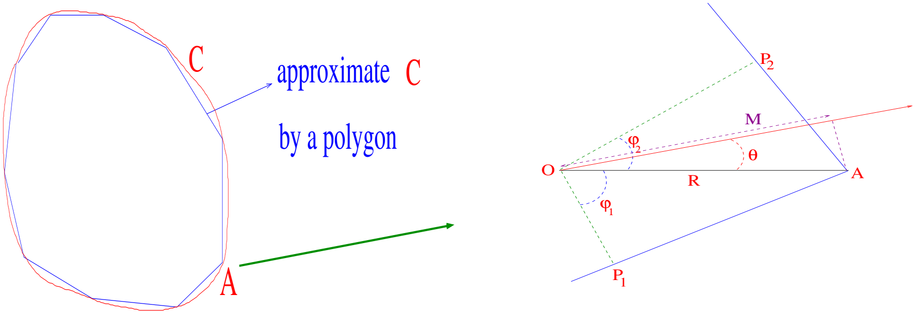
- For any point $[X(s), Y(s)]$ on C define:

  Support function: $M(\theta) = \max_{s \in C} [X(s)\cos(\theta) + Y(s)\sin(\theta)]$

- Perimeter: $$L = \int_0^{2\pi} d\theta \; M(\theta)$$ (A. Cauchy, 1832)

- Area: $$A = \frac{1}{2}\int_0^{2\pi} d\theta \; \left[M^2(\theta) - [M'(\theta)]^2\right]$$

A simple physicist's proof of Cauchy's formula



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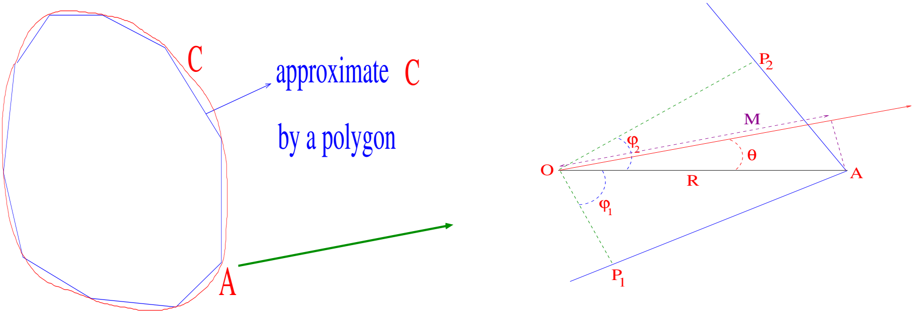
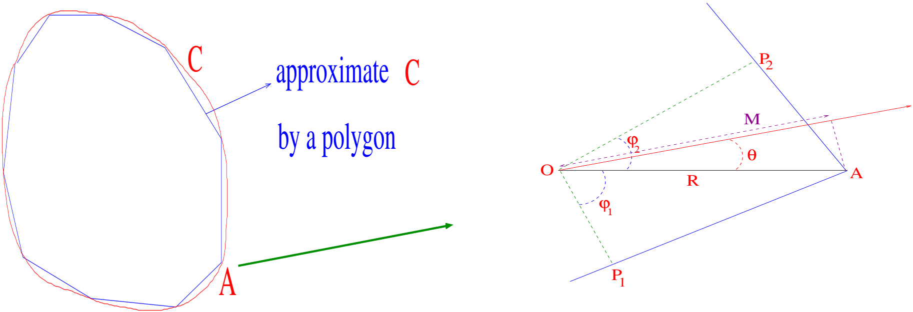


$$M(\theta) = R\cos\theta$$

A simple physicist's proof of Cauchy's formula



$$M(\theta) = R\cos\theta$$

Perimeter: $\int_{-\phi_1}^{\phi_2} M(\theta) d\theta = R\left[\sin(\phi_1) + \sin(\phi_2)\right] = L_{P_1 A P_2}$

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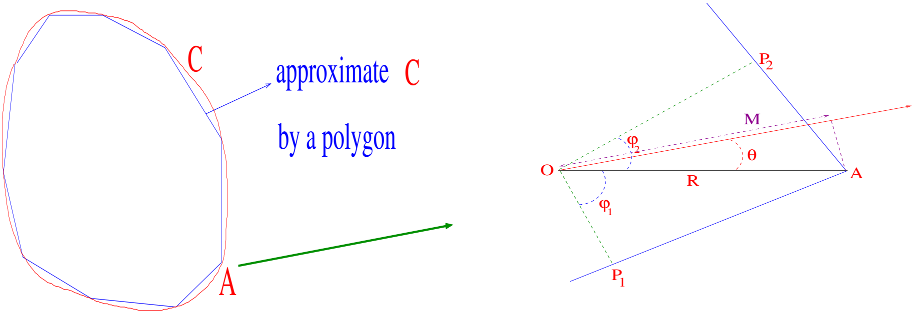

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Area: $\frac{1}{2}\int_{-\phi_1}^{\phi_2}\left[M^2(\theta) - (M'(\theta))^2\right]d\theta$

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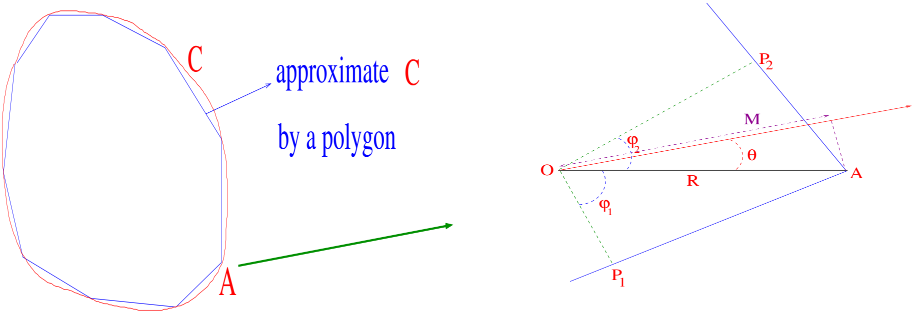


$$\boxed{M(\theta) = R\cos\theta}$$

Perimeter: $\int_{-\phi_1}^{\phi_2} M(\theta)d\theta = R\left[\sin(\phi_1) + \sin(\phi_2)\right] = L_{P_1AP_2}$

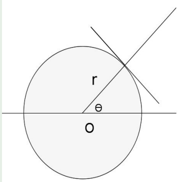
Area: $\frac{1}{2}\int_{-\phi_1}^{\phi_2}\left[M^2(\theta) - (M'(\theta))^2\right]d\theta$

$$= \frac{R^2}{2}\left[\sin(\phi_2)\cos(\phi_2) + \sin(\phi_1)\cos(\phi_1)\right] = A_{OP_1AP_2}$$

Examples

a circle centered at the origin:

$$M(\theta) = r$$



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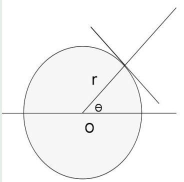

$$L = \int_0^{2\pi} d\theta \; M(\theta) = 2\pi r$$

$$A = \frac{1}{2}\int_0^{2\pi} d\theta \; \left[M^2(\theta) - [M'(\theta)]^2\right] = \pi r^2$$

Examples

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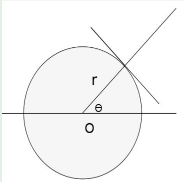
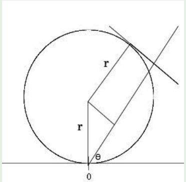


a circle touching the origin:

$$M(\theta) = r(1 + \sin\theta)$$



Examples

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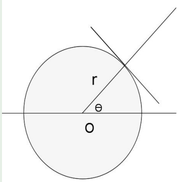

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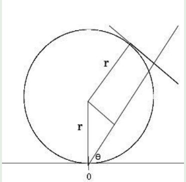

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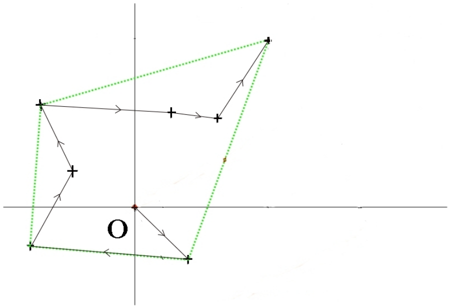
Cauchy's formulae Applied to Convex Polygon

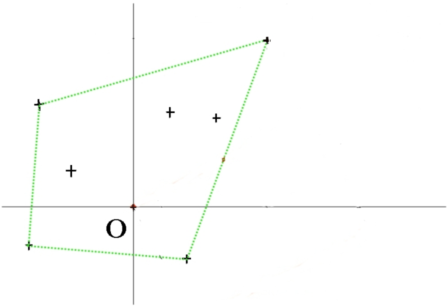


Consider an arbitrary stochastic process starting at O

Let $(x_k, y_k) \Longrightarrow$ vertices of the N-step walk

Let C (green) be the associated Convex Hull

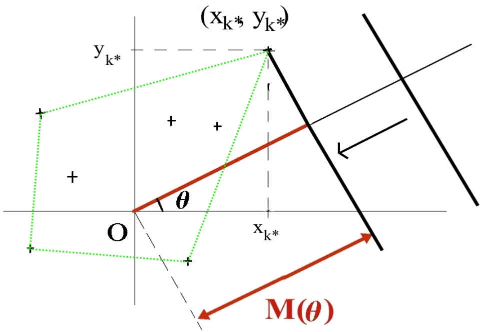
Cauchy's formulae Applied to Convex Polygon



$(x_k, y_k) \Longrightarrow$ vertices of the walk

$C \to$ Convex Hull with coordinates $\{X(s), Y(s)\}$ on C

Cauchy's formulae Applied to Convex Polygon



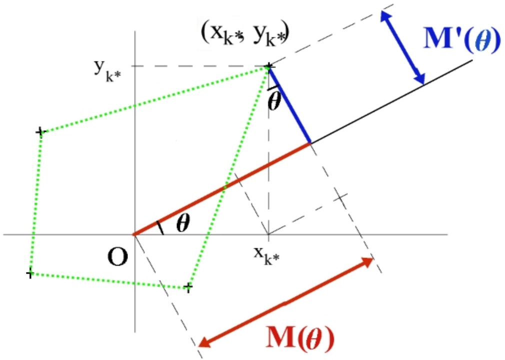
$$M(\theta) = \max_{s \in C} [X(s)\cos\theta + Y(s)\sin\theta]$$

$$= \max_{k \in I} [x_k \cos\theta + y_k \sin\theta]$$

$$= x_{k^*}\cos\theta + y_{k^*}\sin\theta$$

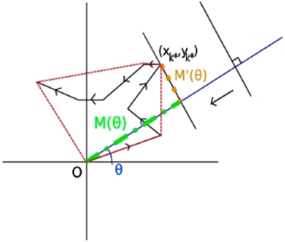
$k^* \rightarrow$ label of the point with largest projection along θ

Support Function of a Convex Hull



$$
\begin{aligned}
M(\theta) &= x_{k^*} \cos\theta + y_{k^*} \sin\theta \\
M'(\theta) &= -x_{k^*} \sin\theta + y_{k^*} \cos\theta
\end{aligned}
$$

Cauchy's Formulae Applied to Random Convex Hull



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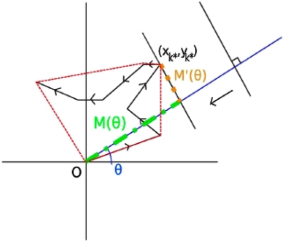


Mean perimeter of a random convex polygon

$$\langle L \rangle = \int_0^{2\pi} d\theta \; \langle M(\theta) \rangle$$

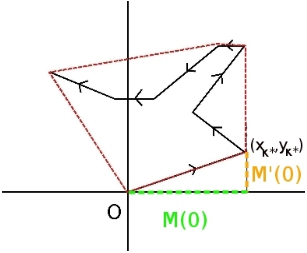
with $M(\theta) = x_{k^*} \cos\theta + y_{k^*} \sin\theta$

Mean area of a random convex polygon

$$\langle A \rangle = \frac{1}{2} \int_0^{2\pi} d\theta \left[\langle M^2(\theta) \rangle - \langle [M'(\theta)]^2 \rangle \right]$$

with $M'(\theta) = -x_{k^*} \sin\theta + y_{k^*} \cos\theta$

Isotropically Distributed Vertices

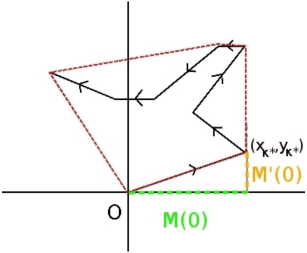


Mean Perimeter

$$\langle L \rangle = 2\pi \langle M(0) \rangle$$

with $M(\theta = 0) = \max_{k \in I} \{x_k\} = x_{k^*}$

Isotropically Distributed Vertices



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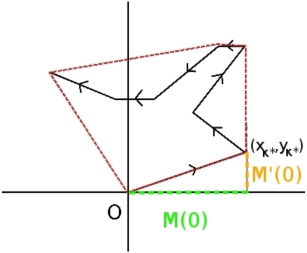
Isotropically Distributed Vertices



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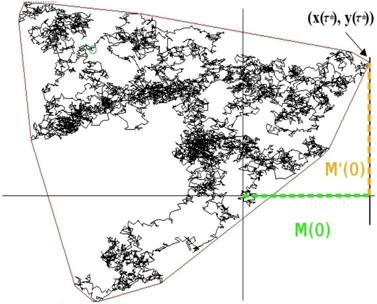
$\Longrightarrow$ Link to Extreme Value Statistics

Continuum limit



$x(\tau)$, $y(\tau)$ $\rightarrow$ a pair of independent one-dimensional processes: $0 \leq \tau \leq T$

Mean Perimeter

$$\langle L \rangle = 2\pi \langle M(0) \rangle$$

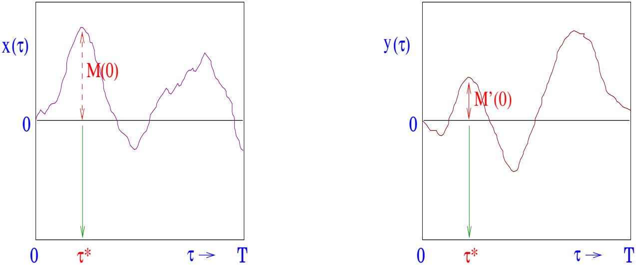
with $M(0) = \max_{0 \leq \tau \leq T} \{x(\tau)\} \equiv x(\tau^*)$

Mean Area

$$\langle A \rangle = \pi \left[\langle M^2(0) \rangle - \langle [M'(0)]^2 \rangle \right]$$

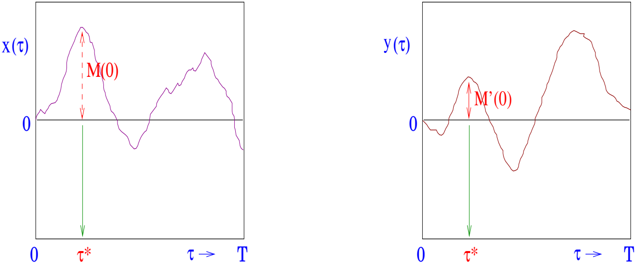
with $M'(0) = y(\tau^*)$

Interpretation of $M'(0)$



- $M(0) \to$ global maximum of $x(\tau)$ in $[0, T]$
- $M'(0) \equiv y(\tau^*)$ where $\tau^* \to$ time at which $x(\tau)$ is maximal in $[0, T]$

Interpretation of $M'(0)$



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$$\Longrightarrow \langle [M'(0)]^2 \rangle = \langle y^2(\tau^*) \rangle$$

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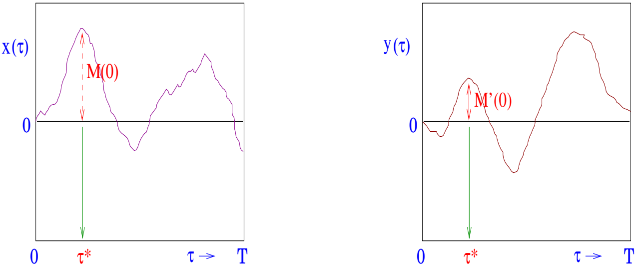

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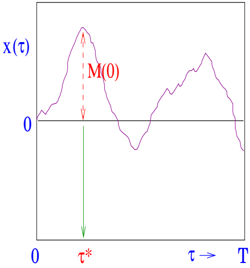
$$\Longrightarrow \langle [M'(0)]^2 \rangle = \langle y^2(\tau^*) \rangle$$

For diffusive processes: $\langle [M'(0)]^2 \rangle = 2\, D\, \langle \tau^* \rangle$

Reduction to 1-d extreme value problem



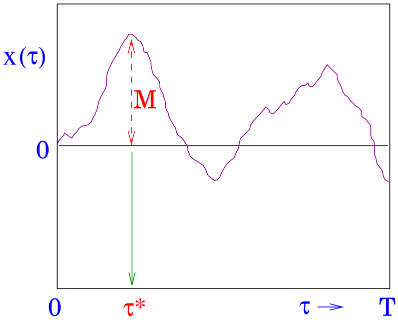
For arbitrary 2-d isotropic stochastic (diffusive) process

- Mean perimeter: $\langle L \rangle = 2\pi \langle M(0) \rangle$
- Mean area: $\langle A \rangle = \pi \left[\langle M^2(0) \rangle - 2\, D \, \langle \tau^* \rangle \right]$

$\Longrightarrow$ Need only to know the statistics of $M(0)$ and τ^* for the 1-d component process $x(\tau)$

[Randon-Furling, S.M., Comtet, PRL, 103, 140602 (2009)]

Distribution of M and τ^* for a single Brownian Path



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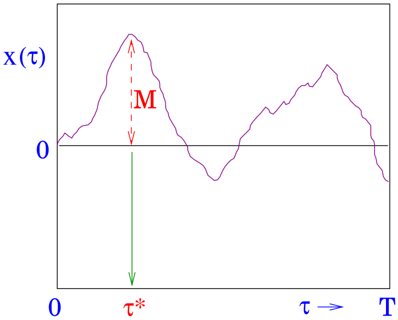
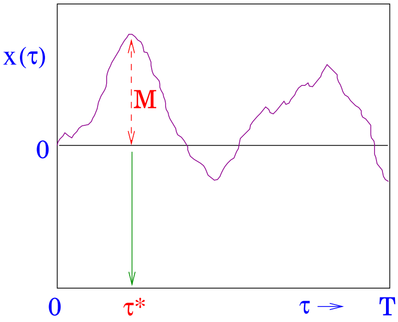


Joint Distribution: $P_1(M, \tau^* | T) = \frac{M}{\pi \tau^{*3/2}\sqrt{T-\tau^*}}\, e^{-M^2/2\tau^*}$ $\quad (D = 1/2)$

Distribution of M and τ^* for a single Brownian Path

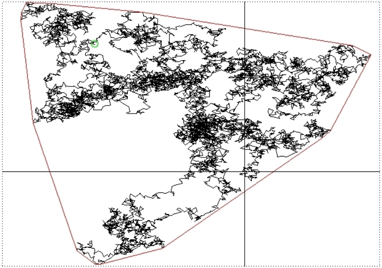


Joint Distribution: $P_1(M, \tau^* | T) = \frac{M}{\pi \tau^{*3/2} \sqrt{T - \tau^*}} \, e^{-M^2/2\tau^*}$ $\quad (D = 1/2)$

$\Longrightarrow$

$$\langle M \rangle = \sqrt{2T/\pi}$$
$$\langle M^2 \rangle = T$$
$$\langle \tau^* \rangle = T/2$$

Results for $n = 1$ Open Brownian Path



$x(\tau)$, $y(\tau)$ $\rightarrow$ a pair of independent one-dimensional Brownian motions over $0 \leq \tau \leq T$

Mean Perimeter

$$\langle L \rangle = \sqrt{8\pi T}$$

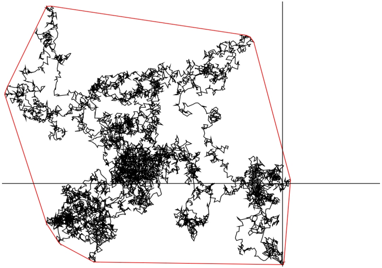
Mean Area

$$\langle A \rangle = \frac{\pi T}{2}$$

Takács, *Expected perimeter length*, Amer. Math. Month., **87** (1980)

El Bachir, (1983)

Results for $n = 1$ Closed Brownian Path

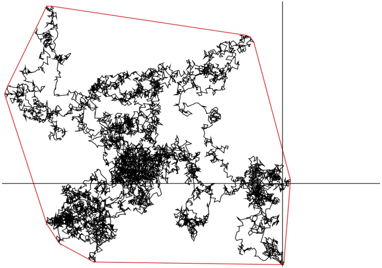


$x(\tau)$, $y(\tau)$ $\rightarrow$ a pair of independent one-dimensional Brownian bridges over $0 \leq \tau \leq T$

Mean Perimeter

$\langle L \rangle = \sqrt{\frac{\pi^3 T}{2}}$ Goldman, '96

Results for $n = 1$ Closed Brownian Path



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Mean Perimeter

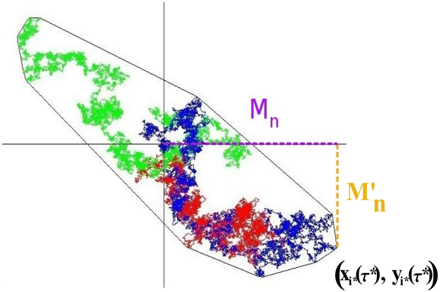
$$\langle L \rangle = \sqrt{\frac{\pi^3 T}{2}}$$ Goldman, '96

Mean Area

$$\boxed{\langle A \rangle = \frac{\pi T}{3}}$$ $\rightarrow$ New Result

[Randon-Furling, S.M., Comtet, PRL, 103, 140602 (2009)]

Convex Hull of n Independent Brownian Paths



$x_i(\tau)$, $y_i(\tau) \to 2\,n$ independent one-dimensional Brownian paths each of duration T

Mean Perimeter

$\langle L_n \rangle = 2\pi \langle M_n \rangle$

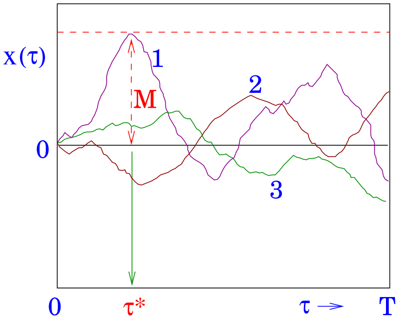
with $M_n = \max_{\tau,i} \{x_i(\tau)\} \equiv x_{i^*}(\tau^*)$

Mean Area

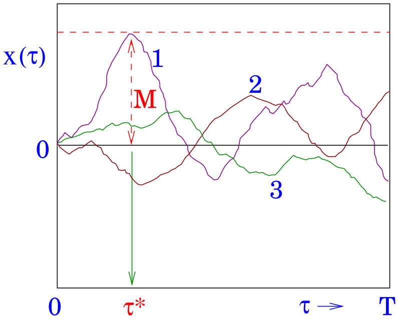
$\langle A_n \rangle = \pi \left[\langle M_n^2 \rangle - \langle [M_n']^2 \rangle \right]$

with $M_n' = y_{i^*}(\tau^*)$

Distribution of the global maximum M and τ^* for n paths



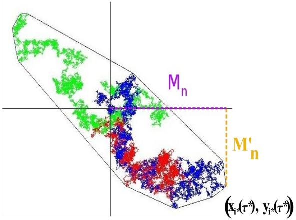
Distribution of the global maximum M and τ^* for n paths



Joint Distribution: $P_n(M, \tau^*|T) = n\, P_1(M, \tau^*|T)\, \left[\mathrm{erf}\left(\frac{M}{\sqrt{2T}}\right)\right]^{n-1}$

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z du\ e^{-u^2}$$

Results for n Open Brownian Paths



$x_i(\tau)$, $y_i(\tau) \to 2\,n$ independent one-dimensional Brownian paths over $0 \le \tau \le T$

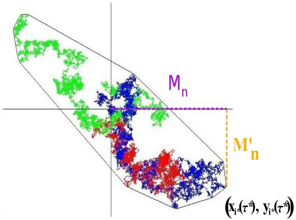
Mean Perimeter (open paths)

$$\langle L_n \rangle = 2\pi \langle M_n \rangle$$

with $M_n = \max_{\tau,i}\{x_i(\tau)\} \equiv x_{i^*}(\tau^*)$

[Randon-Furling, S.M., Comtet, PRL, 103, 140602 (2009)]

Results for n Open Brownian Paths



$x_i(\tau)$, $y_i(\tau) \to 2\,n$ independent one-dimensional Brownian paths over $0 \le \tau \le T$

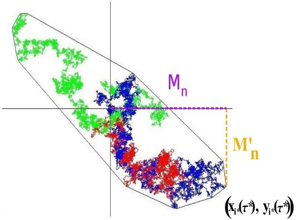
Mean Perimeter (open paths)

$$\langle L_n \rangle = \alpha_n \sqrt{T}$$

$$\alpha_n = 4n\sqrt{2\pi} \int_0^\infty du\ u\ e^{-u^2}\ [\mathrm{erf}(u)]^{n-1}$$

[Randon-Furling, S.M., Comtet, PRL, 103, 140602 (2009)]

Results for n Open Brownian Paths



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Mean Perimeter (open paths)

$$\langle L_n \rangle = \alpha_n \sqrt{T}$$

$$\alpha_n = 4n\sqrt{2\pi} \int_0^\infty du \; u \; e^{-u^2} \; [\mathrm{erf}(u)]^{n-1}$$

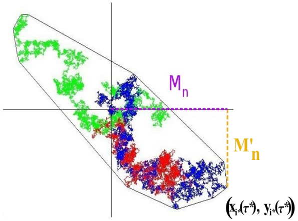
$$\alpha_1 = \sqrt{8\pi} = 5,013..$$

$$\alpha_2 = 4\sqrt{\pi} = 7,089..$$

$$\alpha_3 = 24 \frac{\tan^{-1}\left(1/\sqrt{2}\right)}{\sqrt{\pi}} = 8,333..$$

[Randon-Furling, S.M., Comtet, PRL, 103, 140602 (2009)]

Results for n Open Brownian Paths



$x_i(\tau)$, $y_i(\tau) \to 2\,n$ independent one-dimensional Brownian paths over $0 \leq \tau \leq T$

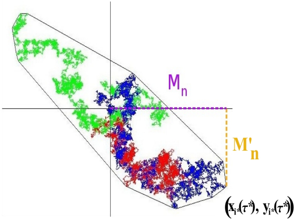
Mean Area (open paths)

$$\langle A_n \rangle = \pi \left[\langle M_n^2 \rangle - \langle [M_n']^2 \rangle \right]$$

with $M_n' = y_{i^*}(\tau^*)$

[Randon-Furling, S.M., Comtet, PRL, 103, 140602 (2009)]

Results for n Open Brownian Paths



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$$\beta_n = 4n\sqrt{\pi}\int_0^\infty du\ u\ [\mathrm{erf}(u)]^{n-1}\left(ue^{-u^2} - h(u)\right)$$

$$h(u) = \frac{1}{2\sqrt{\pi}}\int_0^1 \frac{e^{-u^2/t}\ dt}{\sqrt{t(1-t)}}$$

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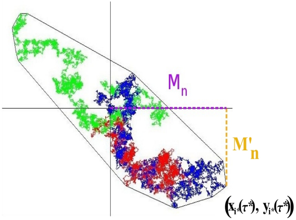

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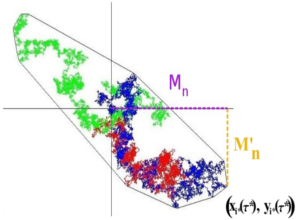
$$\beta_1 = \frac{\pi}{2} = 1,570..$$
$$\beta_2 = \pi = 3,141..$$
$$\beta_3 = \pi + 3 - \sqrt{3} = 4,409..$$

[Randon-Furling, S.M., Comtet, PRL, 103, 140602 (2009)]

Results for n Closed Brownian Paths



$x_i(\tau)$, $y_i(\tau) \to 2\,n$ independent one-dimensional Brownian bridges over $0 \le \tau \le T$

Mean Perimeter (Closed Paths)

$\langle L_n^c \rangle = \alpha_n^c \sqrt{T}$

$$\alpha_n^c = \frac{\pi^{3/2}}{\sqrt{2}} \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k+1}}{\sqrt{k}}$$

[Randon-Furling, S.M., Comtet, PRL, 103, 140602 (2009)]

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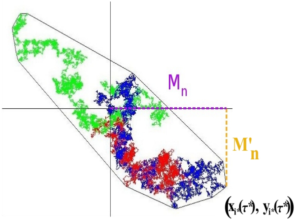

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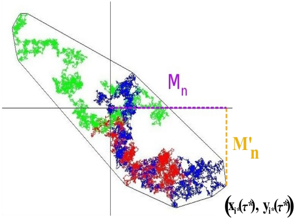
$$\begin{aligned} \alpha_1^c &= \sqrt{\pi^3/2} = 3,937. \\ \alpha_2^c &= \sqrt{\pi^3}(\sqrt{2} - 1/2) = 5,090.. \\ \alpha_3^c &= \sqrt{\pi^3}\left(\frac{3}{\sqrt{2}} - \frac{3}{2} + \frac{1}{\sqrt{6}}\right) = 5,732.. \end{aligned}$$

[Randon-Furling, S.M., Comtet, PRL, 103, 140602 (2009)]

Results for n Closed Brownian Paths



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$$w(k) = \binom{n}{k}\,(k-1)^{-3/2}\left(k\tan^{-1}(\sqrt{k-1}) - \sqrt{k-1}\right)$$

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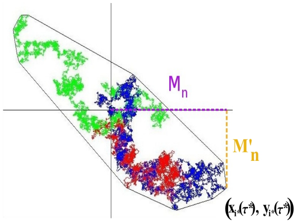

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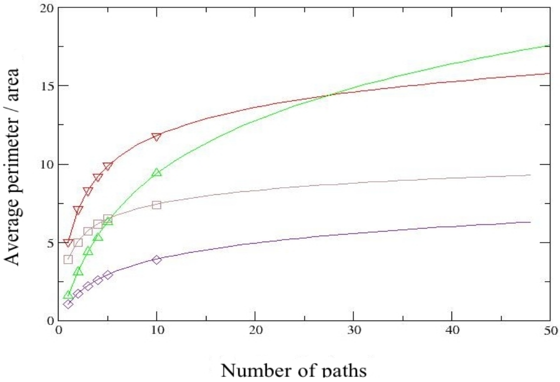
$$\beta_1^c = \frac{\pi}{3} = 1,047..$$

$$\beta_2^c = \frac{\pi(4+3\pi)}{24} = 1,757..$$

$$\beta_3^c = 2,250..$$

[Randon-Furling, S.M., Comtet, PRL, 103, 140602 (2009)]

Numerical Check



The coefficients α_n (mean perimeter) (lower triangle), β_n (mean area) (upper triangle) of n open paths and similarly α_n^c (square) and β_n^c (diamond) for n closed paths, plotted against n. The symbols denote numerical simulations (up to $n = 10$, with 10^3 realisations for each point)

Asymptotics for large n

For n open paths:

$$\langle L_n \rangle \simeq \left(2\,\pi\sqrt{2\ln n}\right)\ \sqrt{T}$$

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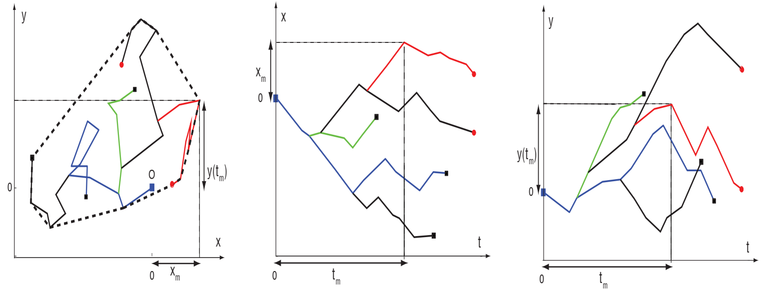
- Very slow growth with $n \Longrightarrow$ good news for conservation

Convex hull for animal epidemics: Exact results



For the critical case $(b = a)$

- mean perimeter: $\langle L(T)\rangle \xrightarrow[T\to\infty]{} 2\pi\sqrt{\frac{6D}{a}} + O\left(T^{-1/2}\right)$
- mean area: $\langle A(T)\rangle \xrightarrow[T\to\infty]{} \frac{24\pi D}{5a} \ln T + O(1)$

[E. Dumonteil, S.M., A. Rosso, A. Zoia, PNAS, 110, 4239-4244 (2013)]

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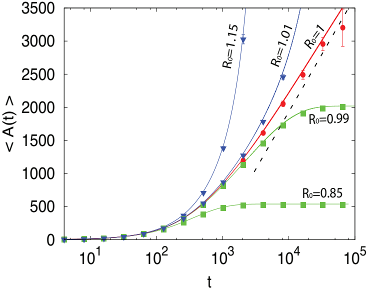

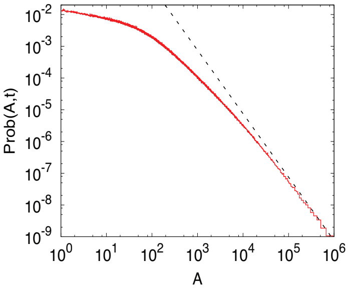

For the critical case $(b = a)$

- distribution of perimeter: $P(L,T) \xrightarrow[T\to\infty]{} P(L) \sim L^{-3}$ for large L
- distribution of area: $P(A,T) \xrightarrow[T\to\infty]{} P(A) \to \frac{24\,\pi\,D}{5\,a}\, A^{-2}$ for large A

[E. Dumonteil, S.M., A. Rosso, A. Zoia, PNAS, 110, 4239-4244 (2013)]

Summary and Conclusion

- Unified approach adapting Cauchy's formulae

$\Longrightarrow$ Mean Perimeter and Area of Random Convex Hull

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• Effect of Interactions between trajectories on convex hull?

• Non-Brownian paths $\rightarrow$ anomalous diffusion, e.g., Lévy flights, external potential ?

Convex Hull of Random Acceleration Process

- Convex hull of a 2-d random acceleration process: $\boxed{\frac{d^2\vec{r}}{dt^2} = \vec{\eta}(t)}$

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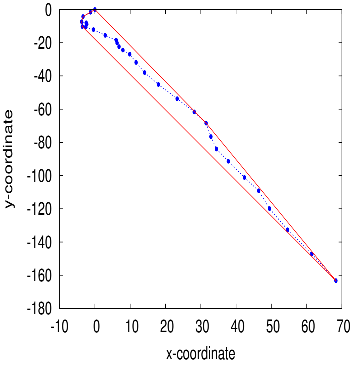

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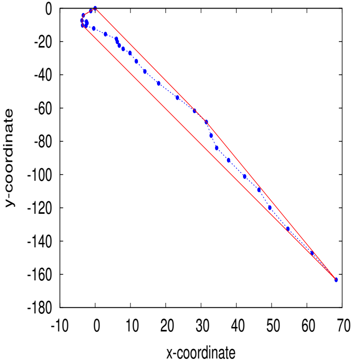

- Exact results for the mean perimeter and mean area

mean perimeter:

$$\langle L_1 \rangle = \frac{3\pi}{2}\, T^{3/2}$$

mean area:

$$\langle A_1 \rangle = \frac{5\pi}{192}\sqrt{\frac{3}{2}}\, T^3$$

A. Reymbaut, S.M. and A. Rosso, J. Phys. A: Math. Theor. 44, 415001 (2011)

Collaborators and References

Collaborators:

- A. Comtet (LPTMS, Orsay, France)
- E. Dumonteil (CEA Saclay, France)
- J. Randon-Furling (Univ. Paris-1, France)
- A. Reymbaut (Master student at Orsay, France)
- A. Rosso (LPTMS, Orsay, France)
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