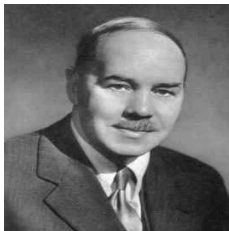


Top eigenvalue of a random matrix: Large deviations

Satya N. Majumdar

Laboratoire de Physique Théorique et Modèles Statistiques, CNRS,
Université Paris-Sud, France

First Appearance of Random Matrices



Biometrika, 20, 32-52 (1928)

THE GENERALISED PRODUCT MOMENT DISTRIBUTION IN SAMPLES FROM A NORMAL MULTIVARIATE POPULATION.

By **JOHN WISHART**, M.A., B.Sc. Statistical Department, Rothamsted Experimental Station.

1. Introduction.

For some years prior to 1915, various writers struggled with the problems that arise when samples are taken from uni-variate and bi-variate populations, assumed in most cases for simplicity to be normal. Thus "Student," in 1908*, by considering the first four moments, was led by K. Pearson's methods to infer the distribution of standard deviations, in samples from a normal population. His results, for comparison with others to be deduced later, will be stated in the form

$$dp = \frac{1}{\Gamma\left(\frac{N-1}{2}\right)} A^{\frac{N-1}{2}} \cdot e^{-Aa} \cdot a^{\frac{N-3}{2}} da \dots\dots\dots(1),$$

where N is the size of the sample, and

$$A = \frac{N}{2\sigma^2}, \quad a = s^2,$$

σ being the standard deviation of the sampled population, and s that estimated from the sample. Thus, if x_1, x_2, \dots, x_N are the sample values,

$$N\bar{x} = \sum_1^N (x),$$

and

$$Ns^2 = \sum_1^N (x - \bar{x})^2.$$

When bi-variate populations were considered, other problems arose, such as the distribution of the correlation coefficient and of the regression coefficient in samples. These problems, taken by themselves, were found to be difficult, and only approximative results had been reached, when, in 1915, R. A. Fisher† gave a formula for the simultaneous distribution of the three quadratic statistical derivatives, namely the two variances (squared standard deviations) and the product moment coefficient. Thus, let x_1, x_2, \dots, x_N represent the sample values of the

Covariance Matrix

$$\mathbf{X} = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \text{phys.} \\ \text{math} \end{array} \left| \begin{array}{cc} X_{11} & X_{12} \\ X_{21} & X_{22} \\ X_{31} & X_{32} \end{array} \right| \quad \begin{array}{l} \text{in general} \\ (M \times N) \end{array}$$

$$\mathbf{X}^t = \left| \begin{array}{ccc} X_{11} & X_{21} & X_{31} \\ X_{12} & X_{22} & X_{32} \end{array} \right| \quad \begin{array}{l} \text{in general} \\ (N \times M) \end{array}$$

$$\mathbf{W} = \mathbf{X}^t \mathbf{X} = \left| \begin{array}{cc} X_{11}^2 + X_{21}^2 + X_{31}^2 & X_{11}X_{12} + X_{21}X_{22} + X_{31}X_{32} \\ X_{12}X_{11} + X_{22}X_{21} + X_{32}X_{31} & X_{12}^2 + X_{22}^2 + X_{32}^2 \end{array} \right|$$

→ (N x N) COVARIANCE MATRIX (unnormalized)

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Null model → random data: $\mathbf{X} \rightarrow$ random $(M \times N)$ matrix

→ $\mathbf{W} = \mathbf{X}^t \mathbf{X} \rightarrow$ random $N \times N$ matrix (Wishart, 1928)

RMT in Nuclear Physics: Eugene Wigner



SESSION IIB

INTERPRETATION OF LOW ENERGY NEUTRON SPECTROSCOPY

CHAIRMAN—W. W. Havens, Jr.

IIB1. DISTRIBUTION OF NEUTRON RESONANCE LEVEL SPACING.

E. P. WIGNER, *Princeton University*
Presented by E. P. Wigner

The problem of the spacing of levels is neither a terribly important one nor have I solved it. That is really the point which I want to make very definitely. As we go up in the energy scale it is evident that the detailed analyses which we have seen for low energy levels is not possible, and we can only make

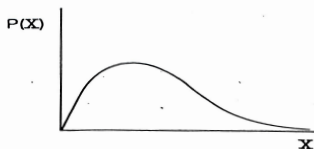


Fig. IIB1-1. Probability of a level spacing X.

our, that is a much more serious deviation and much less probable statistically.

Let me say only one more word. It is very likely that the curve in Figure 1 is a universal function. In other words, it doesn't depend on the details of the model with which you are working. There is one particular model in which the probability of the energy levels can be written down exactly. I mentioned this distribution already in Gatlinburg. It is called the Wishart distribution. Consider a set of symmetric matrices in such a way that the diagonal element m_{11} has a distribution $\exp(-m_{11}^2/4)$. In other words, the probability that this diagonal element shall assume the value m_{11} is proportional to $\exp(-m_{11}^2/4)$. Then as I mentioned, and this was shown a long time ago by Wishart, the probability for the characteristic roots to be $\lambda_1, \lambda_2, \lambda_3 \dots \lambda_n$, if it is an n dimensional matrix, is given by the expression:

$$\frac{1}{2^n} \frac{\lambda_1^{n-1} \lambda_2^{n-2} \dots \lambda_n}{\prod_{i < j} (\lambda_i - \lambda_j)^2} (\lambda_1 - \lambda_2) \dots (\lambda_{n-1} - \lambda_n).$$

probability that two successive roots have a distance X , then you have to integrate over all of them except two. This is very easy to do for the first integration, possible to do for the second integration, but when you get to the third, fourth and fifth, etc., integrations you have the same problem as in statistical mechanics, and presumably the solution of the problem will be accomplished by one of the methods of statistical mechanics. Let me only mention that I did integrate over all of them except one, and the result is $\frac{1}{2\pi} \sqrt{4n - \lambda^2}$. This is the probability that the root shall be λ . All I have to do is to integrate over one less variable than I have integrated over, but this I have not been able to do so far.

DISCUSSION

W. HAVENS: Where does one find out about a Wishart distribution?

E. WIGNER: A Wishart distribution is a

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DISCUSSION

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W. HAVENS: Where does one find out about a Wishart distribution?

E. WIGNER: A Wishart distribution is given in S. S. Wilks book about statistics and I found it just by accident.

Random Matrices in Nuclear Physics

spectra of heavy nuclei



WIGNER ('50) : replace complex H by random matrix
DYSON, GAUDIN, MEHTA,

Applications of Random Matrices

Physics: nuclear physics, quantum chaos, disorder and localization, mesoscopic transport, optics/lasers, quantum entanglement, neural networks, gauge theory, QCD, matrix models, cosmology, string theory, statistical physics (growth models, interface, directed polymers...),

Mathematics: Riemann zeta function (number theory), free probability theory, combinatorics and knot theory, determinantal points processes, integrable systems, ...

Statistics: multivariate statistics, principal component analysis (PCA), image processing, data compression, Bayesian model selection, ...

Information Theory: signal processing, wireless communications, ..

Biology: sequence matching, RNA folding, gene expression network ...

Economics and Finance: time series analysis,....

Recent Ref: [The Oxford Handbook of Random Matrix Theory](#)
ed. by G. Akemann, J. Baik and P. Di Francesco (2011)

Spectral Statistics in Random Matrix Theory (RMT)

Working model: real, symmetric $N \times N$ Gaussian random matrix

$$J = \begin{pmatrix} J_{11} & J_{12} & \dots & J_{1N} \\ J_{12} & J_{22} & \dots & J_{2N} \\ \dots & \dots & \dots & \dots \\ J_{1N} & J_{2N} & \dots & J_{NN} \end{pmatrix}$$

$$\begin{aligned} \text{Prob.}[J] &\propto \exp \left[-\frac{1}{2} \sum_{i,j} J_{ij}^2 \right] \\ &= \exp \left[-\frac{1}{2} \text{Tr} (J^2) \right] \end{aligned}$$

→ invariant under rotation

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N real eigenvalues: $\lambda_1, \lambda_2, \dots, \lambda_N \rightarrow$ strongly correlated

Spectral statistics in RMT \Rightarrow statistics of $\{\lambda_1, \lambda_2, \dots, \lambda_N\}$

Top Eigenvalue of a random matrix λ_{\max}

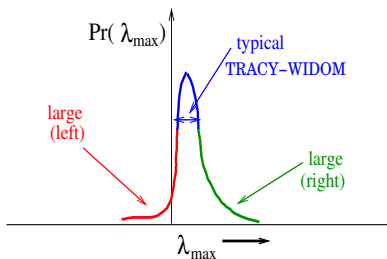
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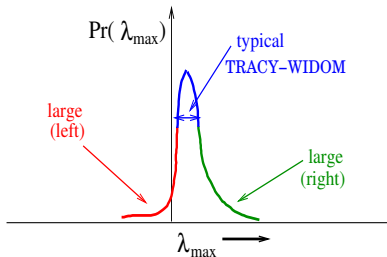
Typical fluctuations (**small**)
 \Rightarrow Tracy-Widom distribution
 \rightarrow **ubiquitous**

[directed polymer, random permutation,
growth models, KPZ equation, sequence
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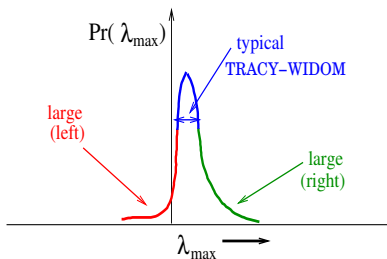
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- Summary and Generalizations

I. Why λ_{\max} ?

Stability of a Large Complex System



R.M. May, *Nature*, 238, 413 (1972)

GENERAL

Will a Large Complex System be Stable?

Gardner and Ashby¹ have suggested that large complex systems which are assembled (connected) at random may be expected to be stable up to a certain critical level of connectance, and then, as this increases, to suddenly become unstable. Their conclusions were based on the trend of computer studies of systems with 4, 7 and 10 variables.

Here I complement Gardner and Ashby's work with an analytical investigation of such systems in the limit when the number of variables is large. The sharp transition from stability to instability which was the essential feature of their paper is confirmed, and I go further to see how this critical transition point scales with the number of variables n in the system, and with the average connectance C and interaction magnitude α between the various variables. The object is to clarify the relation between stability and complexity in ecological systems with many interacting species, and some conclusions bearing on this question are drawn from the model. But, just as in Gardner and Ashby's work, the formal development of the problem is a general one, and thus applies to the wide range of contexts spelled out by these authors.

Specifically, consider a system with n variables (in an ecological application these are the populations of the n interacting species) which in general may obey some quite nonlinear set of first-order differential equations. The stability of the possible equilibrium or time-independent configurations of such a system may be studied by Taylor-expanding in the neighbourhood of the equilibrium point, so that the stability of the possible equilibrium is characterized by the equation

$$dx/dt = Ax \quad (1)$$

Here in an ecological context x is the $n \times 1$ column vector of the disturbed populations x_j , and the $n \times n$ interaction matrix A has elements a_{jk} which characterize the effect of species k on species j near equilibrium^{2,3}. A diagram of the trophic web immediately determines which a_{jk} are zero (no web link), and the type of interaction determines the sign and magnitude of a_{jk} .

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$$dx_i/dt = -x_i + \alpha \sum_{j=1}^N J_{ij} x_j$$

$J_{ij} \rightarrow (N \times N)$ random **interaction** matrix

$\alpha \rightarrow$ **interaction** strength

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- **Question:** What is the probability that the system remains **stable** once the **interaction** is switched on?

(R.M. May, Nature, 238, 413, 1972)

Stability Criterion

- linear stability: $\frac{d}{dt}[x] = [\alpha J - I][x]$ ($J \rightarrow$ random interaction matrix)

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- **Stable** if $\alpha\lambda_i < 1$ for all $i = 1, 2, \dots, N$

$$\Rightarrow \lambda_{\max} < \frac{1}{\alpha} = w \rightarrow \text{stability criterion}$$

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- Prob.(the system is stable) = Prob. $[\lambda_{\max} < w] = P(w, N)$

Cumulative distribution of the top eigenvalue

Stable-Unstable Phase Transition as $N \rightarrow \infty$

- Assuming that the **interaction** matrix $J_{ij} \rightarrow$ Real Symmetric Gaussian

$$\text{Prob.}[J_{ij}] \propto \exp \left[-\frac{N}{2} \sum_{i,j} J_{ij}^2 \right] \propto \exp \left[-\frac{N}{2} \text{Tr}(J^2) \right]$$

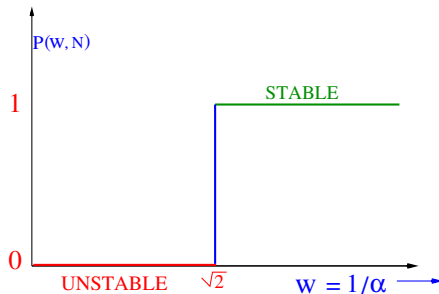
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- May observed a **sharp** phase transition as $N \rightarrow \infty$:
 - $w = \frac{1}{\alpha} > \sqrt{2} \Rightarrow$ **Stable** (weakly interacting)
 - $w = \frac{1}{\alpha} < \sqrt{2} \Rightarrow$ **Unstable** (strongly interacting)

$$\text{Prob.}(\text{the system is stable}) = \text{Prob.}[\lambda_{\max} < w] = P(w, N)$$



Finite but Large N :

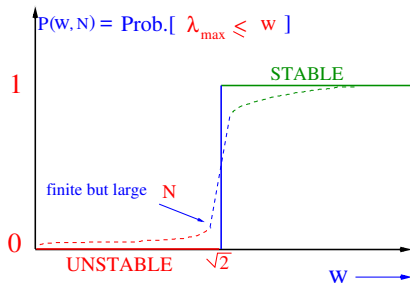
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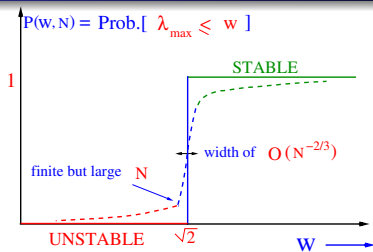
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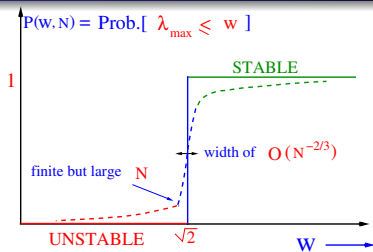
- Is there any thermodynamic sense to this phase transition?
- What is the analogue of free energy?
- What is the order of this phase transition?

II. Summary of Results

For Large but Finite N : Summary of Results

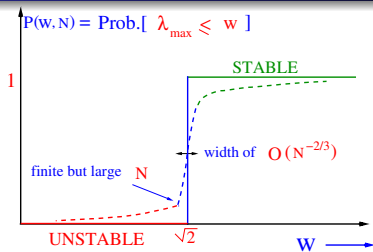


For Large but Finite N : Summary of Results



$$\begin{aligned}
 P(w, N) &\sim \exp[-N^2 \Phi_-(w) + \dots] && \text{for } \sqrt{2} - w \sim O(1) \\
 &\sim F_1 \left[\sqrt{2} N^{2/3} (w - \sqrt{2}) \right] && \text{for } |w - \sqrt{2}| \sim O(N^{-2/3}) \\
 &\sim 1 - \exp[-N \Phi_+(w) + \dots] && \text{for } w - \sqrt{2} \sim O(1)
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Crossover function: $F_1(z) \rightarrow$ Tracy-Widom (1994)

Exact rate functions:

$\Phi_-(w) \rightarrow$ Dean & S.M. 2006

$\Phi_+(w) \rightarrow$ Ben Arous, Dembo & Guionnet 2001, S.M. & Vergassola 2009

Higher order corrections: (Borot, Eynard, S.M., & Nadal 2011, Nadal & S.M., 2011)

Exact Left and Right Large Deviation Function

Using Coulomb gas + Saddle point method for large N :

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Using Coulomb gas + Saddle point method for large N :

- Left large deviation function:

$$\begin{aligned}\Phi_-(w) &= \frac{1}{108} \left[36w^2 - w^4 - (15w + w^3)\sqrt{w^2 + 6} \right. \\ &+ \left. 27 \left(\ln(18) - 2 \ln(w + \sqrt{6 + w^2}) \right) \right] \quad \text{where } w < \sqrt{2}\end{aligned}$$

[D. S. Dean & S.M., PRL, 97, 160201 (2006)]

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- Right large deviation function:

$$\Phi_+(w) = \frac{1}{2} w \sqrt{w^2 - 2} + \ln \left[\frac{w - \sqrt{w^2 - 2}}{\sqrt{2}} \right] \quad \text{where } w > \sqrt{2}$$

[Ben Arous, Dembo & Guionnet 2001, S.M. & Vergassola, PRL, 102, 060601 (2009)]

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Using Coulomb gas + Saddle point method for large N :

- Left large deviation function:

$$\Phi_-(w) = \frac{1}{108} \left[36w^2 - w^4 - (15w + w^3)\sqrt{w^2 + 6} + 27 \left(\ln(18) - 2 \ln(w + \sqrt{6 + w^2}) \right) \right] \quad \text{where } w < \sqrt{2}$$

[D. S. Dean & S.M., PRL, 97, 160201 (2006)]

In particular, as $w \rightarrow \sqrt{2}$ (from left), $\Phi_-(w) \rightarrow \frac{1}{6\sqrt{2}} (\sqrt{2} - w)^3$

- Right large deviation function:

$$\Phi_+(w) = \frac{1}{2} w \sqrt{w^2 - 2} + \ln \left[\frac{w - \sqrt{w^2 - 2}}{\sqrt{2}} \right] \quad \text{where } w > \sqrt{2}$$

[Ben Arous, Dembo & Guionnet 2001, S.M. & Vergassola, PRL, 102, 060601 (2009)]

As $w \rightarrow \sqrt{2}$ (from right), $\Phi_+(w) \rightarrow \frac{2^{7/4}}{3} (w - \sqrt{2})^{3/2}$

Large Deviation Functions

These large deviation functions $\Phi_{\pm}(w)$ have been found useful in a large variety of problems:

[Fyodorov 2004, Fyodorov & Williams 2007, Bray & Dean 2007, Auffinger, Ben Arous & Cerny 2010, Fyodorov & Nadal 2012.... — stationary points on **random Gaussian surfaces** and **spin glass landscapes**]

[Cavagna, Garrahan, Giardinà 2000,... — **Glassy systems**]

[Susskind 2003, Douglas et. al. 2004, Aazami & Easter 2006, Marsh et. al. 2011, ... — **String theory & Cosmology**]

[Beltrani 2007, Dedieu & Malajovich, 2007, Houdre 2011... — **Random Polynomials, Random Words (Young diagrams)**]

3-rd Order Phase Transition

$$P(w, N) \approx \begin{cases} \exp\{-N^2\Phi_-(w) + \dots\} & \text{for } w < \sqrt{2} & \text{(unstable)} \\ 1 - \exp\{-N\Phi_+(w) + \dots\} & \text{for } w > \sqrt{2} & \text{(stable)} \end{cases}$$

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$$\lim_{N \rightarrow \infty} -\frac{1}{N^2} \ln [P(w, N)] = \begin{cases} \Phi_-(w) \sim (\sqrt{2} - w)^3 & \text{as } w \rightarrow \sqrt{2}^- \\ 0 & \text{as } w \rightarrow \sqrt{2}^+ \end{cases}$$

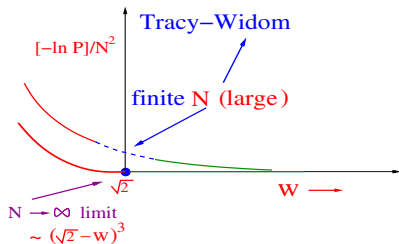
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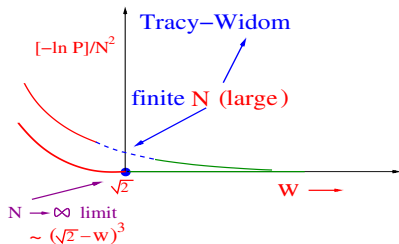


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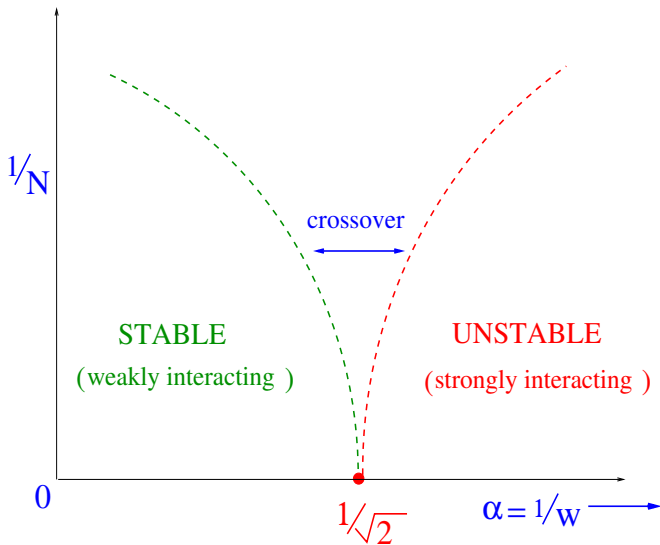
3-rd derivative → **discontinuous**

Crossover: $N \rightarrow \infty$, $w \rightarrow \sqrt{2}$ keeping
 $(w - \sqrt{2}) N^{2/3}$ fixed

$$P(w, N) \rightarrow F_1 [\sqrt{2} N^{2/3} (w - \sqrt{2})]$$

→ Tracy-Widom

Large N Phase Transition: Phase Diagram



Possible third-order phase transition in the large- N lattice gauge theory

David J. Gross

Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540

Edward Witten

Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138

(Received 10 July 1979)

The large- N limit of the two-dimensional $U(N)$ (Wilson) lattice gauge theory is explicitly evaluated for all fixed $\lambda = g^2 N$ by steepest-descent methods. The λ dependence is discussed and a third-order phase transition, at $\lambda = 2$, is discovered. The possible existence of such a weak- to strong-coupling third-order phase transition in the large- N four-dimensional lattice gauge theory is suggested, and its meaning and implications are discussed.

Volume 93B, number 4

PHYSICS LETTERS

30 June 1980

 **$N = \infty$ PHASE TRANSITION IN A CLASS OF EXACTLY SOLUBLE
MODEL LATTICE GAUGE THEORIES ***

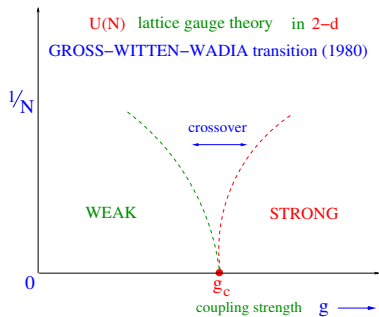
Spenta R. WADIA

The Enrico Fermi Institute, University of Chicago, Chicago, IL 60637, USA

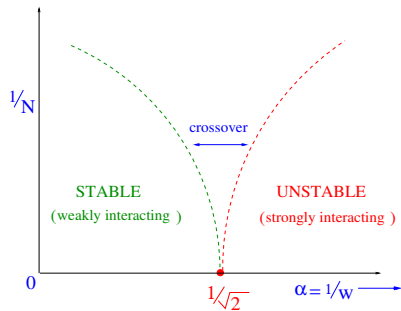
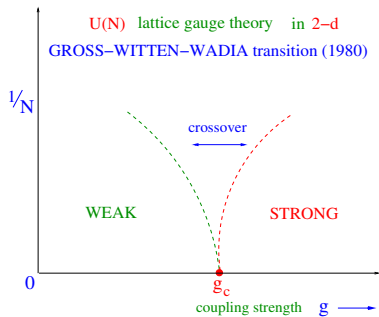
Received 27 March 1980

A nice review of large- N gauge theory: M. Marino, arXiv:1206.6272

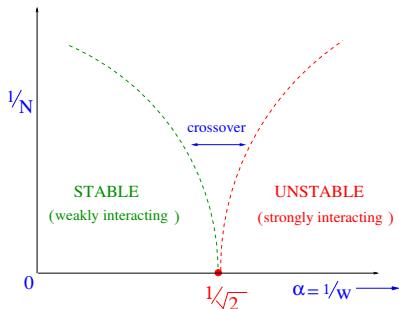
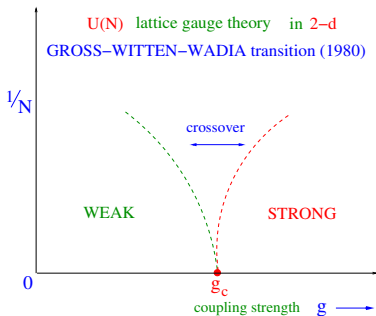
Large N Phase Transition: Phase Diagram



Large N Phase Transition: Phase Diagram



Large N Phase Transition: Phase Diagram



Similar 3-rd order phase transition in $U(N)$ lattice-gauge theory in 2-d

Unstable phase \equiv Strong coupling phase of Yang-Mills gauge theory

Stable phase \equiv Weak coupling phase of Yang-Mills gauge theory

Tracy-Widom \Rightarrow crossover function in the double scaling regime
(for finite but large N)

III.

Coulomb Gas

Gaussian Random Matrices

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- Ensembles: Orthogonal (GOE), Unitary (GUE) or Symplectic (GSE)

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- Joint distribution of eigenvalues (Wigner, 1951)

$$P(\lambda_1, \lambda_2, \dots, \lambda_N) = \frac{1}{Z_N} \exp \left[-\frac{\beta}{2} N \sum_{i=1}^N \lambda_i^2 \right] \prod_{j < k} |\lambda_j - \lambda_k|^\beta$$

where the Dyson index $\beta = 1$ (GOE), $\beta = 2$ (GUE) or $\beta = 4$ (GSE)

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- $Z_N =$ Partition Function

$$= \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \left\{ \prod_i d\lambda_i \right\} \exp \left[-\frac{\beta}{2} N \sum_{i=1}^N \lambda_i^2 \right] \prod_{j < k} |\lambda_j - \lambda_k|^\beta$$

Coulomb Gas Interpretation

• $Z_N =$

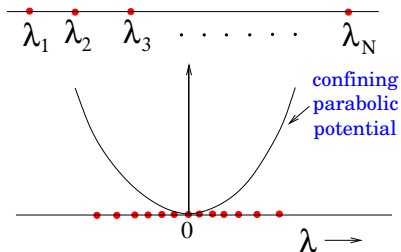
$$\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \left\{ \prod_i d\lambda_i \right\} \exp \left[-\frac{\beta}{2} \left\{ \sum_{i=1}^N N \lambda_i^2 - \sum_{j \neq k} \log |\lambda_j - \lambda_k| \right\} \right]$$

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- 2-d Coulomb gas confined to a line (Dyson) with $\beta \rightarrow$ inverse temp.

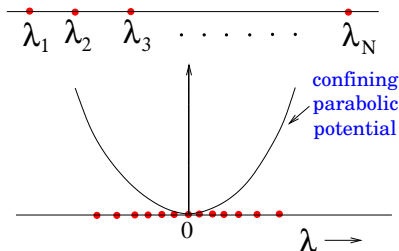


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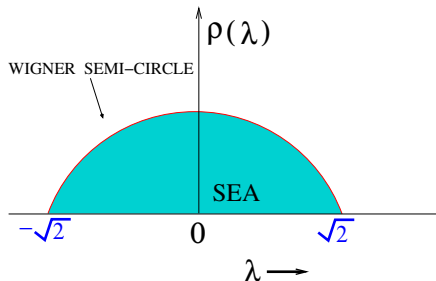
- Balance of energy $\Rightarrow N^2 \lambda^2 \sim N^2$
- Typical eigenvalue: $\lambda_{\text{typ}} \sim O(1)$ for large N

Spectral Density: Wigner's Semicircle Law

- Av. density of states: $\rho(\lambda, N) = \left\langle \frac{1}{N} \sum_{i=1}^N \delta(\lambda - \lambda_i) \right\rangle$

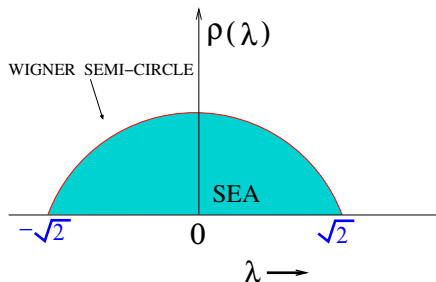
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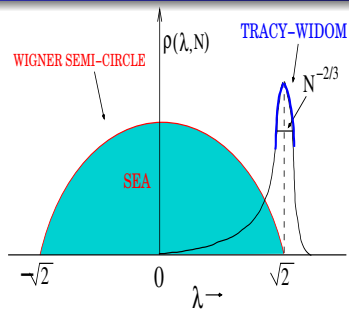
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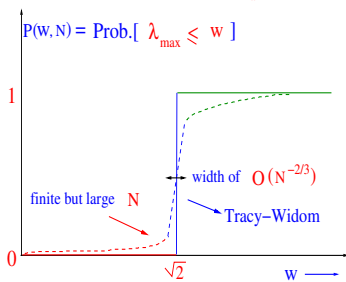


- $\langle \lambda_{\max} \rangle = \sqrt{2}$ for large N .
- λ_{\max} fluctuates from one sample to another. $\text{Prob}[\lambda_{\max}, N] = ?$

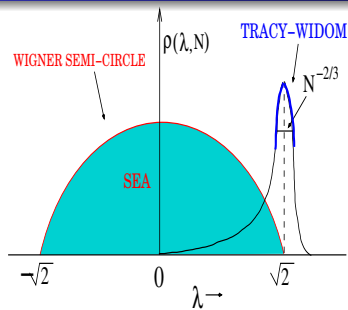
Tracy-Widom distribution for λ_{\max}



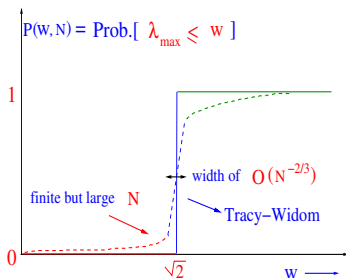
cumulative distribution of λ_{\max}



Tracy-Widom distribution for λ_{\max}



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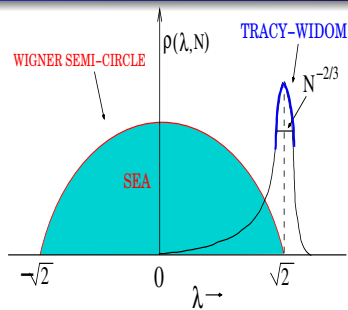


- $\langle \lambda_{\max} \rangle = \sqrt{2}$; typical fluctuation: $\int_{\lambda_{\max}}^{\infty} \rho(\lambda) d\lambda \sim 1/N$

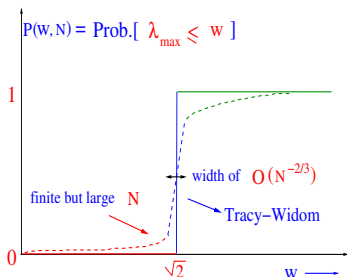
Using $\rho(\lambda) \sim (\sqrt{2} - \lambda)^{1/2} \Rightarrow |\lambda_{\max} - \sqrt{2}| \sim N^{-2/3} \rightarrow \text{small}$

[Bowick & Brezin '91, Forrester '93]

Tracy-Widom distribution for λ_{\max}



cumulative distribution of λ_{\max}



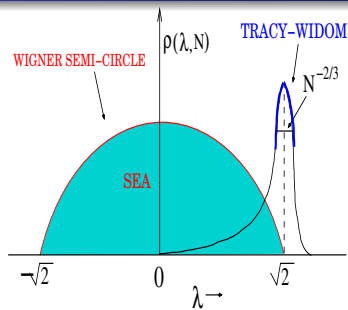
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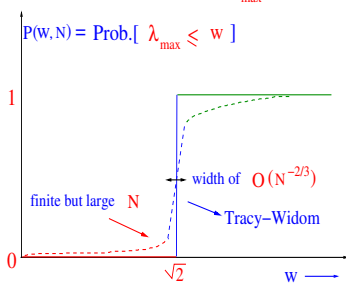
[Bowick & Brezin '91, Forrester '93]

- typical fluctuations are distributed via Tracy-Widom ('94):
- cumulative distribution: $\text{Prob}[\lambda_{\max} \leq w, N] \rightarrow F_{\beta}(\sqrt{2}N^{2/3}(w - \sqrt{2}))$

Tracy-Widom distribution for λ_{\max}



cumulative distribution of λ_{\max}



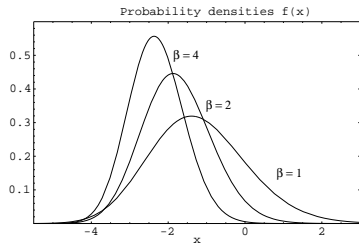
- $\langle \lambda_{\max} \rangle = \sqrt{2}$; **typical** fluctuation: $\int_{\lambda_{\max}}^{\infty} \rho(\lambda) d\lambda \sim 1/N$

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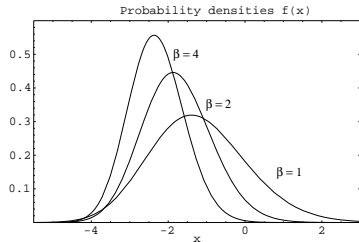
[Bowick & Brezin '91, Forrester '93]

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- cumulative distribution: $\text{Prob}[\lambda_{\max} \leq w, N] \rightarrow F_{\beta}(\sqrt{2}N^{2/3}(w - \sqrt{2}))$
- Prob. density (pdf): $f_{\beta}(x) = dF_{\beta}(x)/dx$; $F_{\beta}(x) \rightarrow$ Painlevé-II

Tracy-Widom Distribution for λ_{\max}

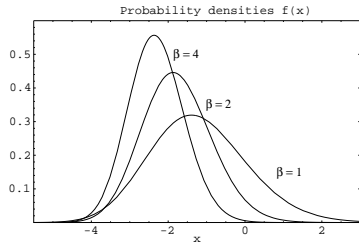


Tracy-Widom Distribution for λ_{\max}



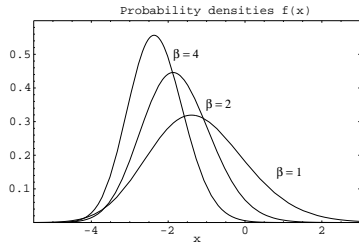
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Tracy-Widom Distribution for λ_{\max}



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- **Asymptotics:** $f_{\beta}(x) \sim \exp\left[-\frac{\beta}{24}|x|^3\right]$ as $x \rightarrow -\infty$
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Tracy-Widom Distribution for λ_{\max}

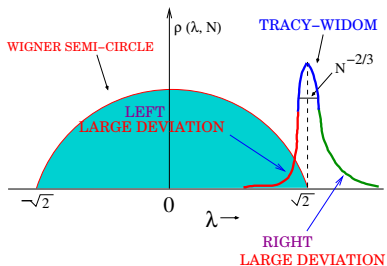


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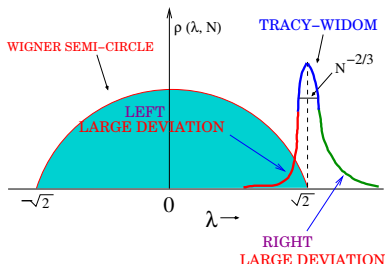
Applications: Growth models, Directed polymer, Sequence Matching

(Baik, Borodin, Calabrese, Comtet, Corwin, Deift, Dotsenko, Dumitriu, Edelman, Ferrari, Forrester, Johansson, Johnstone, Le doussal, Nadal, Nechaev, O'Connell, P ech e, Pr ahofer, Quastel, Rains, Rambeau, Rosso, Sano, Sasamoto, Schehr, Spohn, Takeuchi, Virag, ...)

Probability of Large Deviations of λ_{\max} :

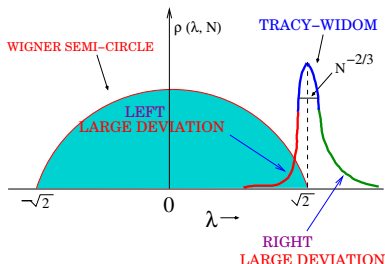


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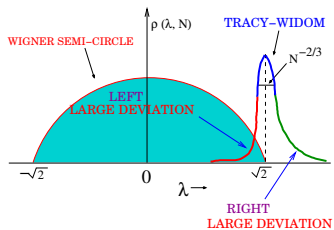
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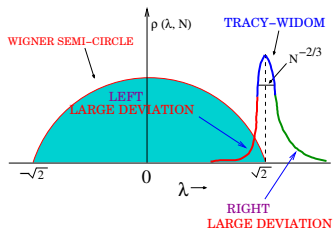


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- **Q**: How to describe the prob. of **large (atypical)** fluctuations when $|\lambda_{\max} - \sqrt{2}| \sim O(1) \rightarrow$ **Large** deviations from mean

Large Deviation Tails of λ_{\max}



Large Deviation Tails of λ_{\max}



Prob. density of the top eigenvalue: $\text{Prob.}[\lambda_{\max} = w, N]$ behaves as:

$$\sim \exp[-\beta N^2 \Phi_-(w)] \quad \text{for } \sqrt{2} - w \sim O(1)$$

$$\sim N^{2/3} f_\beta \left[\sqrt{2} N^{2/3} (w - \sqrt{2}) \right] \quad \text{for } |w - \sqrt{2}| \sim O(N^{-2/3})$$

$$\sim \exp[-\beta N \Phi_+(w)] \quad \text{for } w - \sqrt{2} \sim O(1)$$

IV. Saddle Point Method

Distribution of λ_{\max} : Saddle Point Method

$$\text{Prob}[\lambda_{\max} \leq w, N] = \text{Prob}[\lambda_1 \leq w, \lambda_2 \leq w, \dots, \lambda_N \leq w] = \frac{Z_N(w)}{Z_N(\infty)}$$

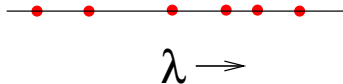
$$Z_N(w) = \int_{-\infty}^w \dots \int_{-\infty}^w \left\{ \prod_i d\lambda_i \right\} \exp \left[-\frac{\beta}{2} \left\{ N \sum_{i=1}^N \lambda_i^2 - \sum_{j \neq k} \log |\lambda_j - \lambda_k| \right\} \right]$$

Distribution of λ_{\max} : Saddle Point Method

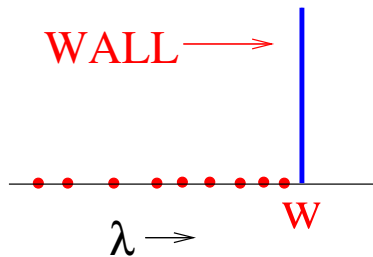
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denominator



numerator

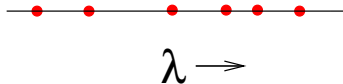


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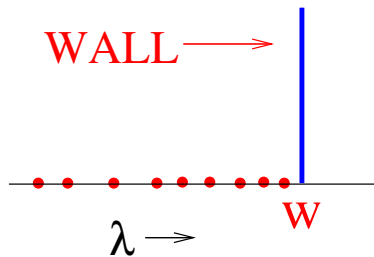
$$\text{Prob}[\lambda_{\max} \leq w, N] = \text{Prob}[\lambda_1 \leq w, \lambda_2 \leq w, \dots, \lambda_N \leq w] = \frac{Z_N(w)}{Z_N(\infty)}$$

$$Z_N(w) = \int_{-\infty}^w \dots \int_{-\infty}^w \left\{ \prod_i d\lambda_i \right\} \exp \left[-\frac{\beta}{2} \left\{ N \sum_{i=1}^N \lambda_i^2 - \sum_{j \neq k} \log |\lambda_j - \lambda_k| \right\} \right]$$

denominator



numerator



Setting up the Saddle Point Method



$$Z_N(w) \propto \int_{-\infty}^w \prod_i d\lambda_i \exp[-\beta N^2 E(\{\lambda_i\})]$$

$$E(\{\lambda_i\}) = \frac{1}{2N} \sum_i \lambda_i^2 - \frac{1}{2N^2} \sum_{j \neq k} \log |\lambda_j - \lambda_k|$$

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$$Z_N(w) \propto \int \mathcal{D}\rho(\lambda) \exp \left[-\beta N^2 \left\{ E[\rho(\lambda)] + C \left(\int \rho(\lambda) d\lambda - 1 \right) \right\} + O(N) \right]$$

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- saddle point $\frac{\delta S}{\delta \rho} = 0 \Rightarrow$

$$\lambda^2 - 2 \int_{-\infty}^w \rho_w(\lambda') \ln |\lambda - \lambda'| d\lambda' + C = 0$$

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$$\lambda = \mathcal{P} \int_{-\infty}^w \frac{\rho_w(\lambda') d\lambda'}{\lambda - \lambda'} \quad \text{for } \lambda \in [-\infty, w] \rightarrow \text{Semi-Hilbert transform}$$

→ force balance condition

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Exact solution for all w :

[D. S. Dean & S.M., PRL, 97, 160201 (2006); PRE, 77, 041108 (2008)]

Exact Saddle Point Solution

- Exact solution (D. Dean and S.M., 2006, 2008):

$$\rho_w(\lambda) = \begin{cases} \frac{1}{\pi} \sqrt{2 - \lambda^2} & \text{for } w \geq \sqrt{2} \\ \frac{\sqrt{\lambda + L(w)}}{2\pi\sqrt{w - \lambda}} [w + L(w) - 2\lambda] & \text{for } w < \sqrt{2} \end{cases}$$

where $L(w) = [2\sqrt{w^2 + 6} - w]/3$

Exact Saddle Point Solution

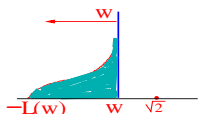
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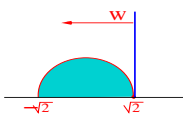
charge density $\rho_w(\lambda)$ vs. λ for different W

$W < \sqrt{2}$



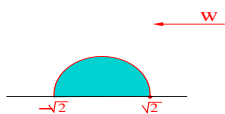
pushed
(UNSTABLE)

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critical

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unpushed
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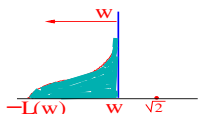
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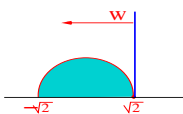
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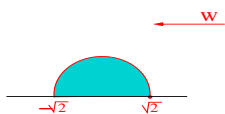
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Left Large Deviation Function

$$\begin{aligned}\text{Prob}[\lambda_{\max} \leq w, N] &= \frac{Z_N(w)}{Z_N(\infty)} \sim \exp[-\beta N^2 \{S[\rho_w(\lambda)] - S[\rho_\infty(\lambda)]\}] \\ &\sim \exp[-\beta N^2 \Phi_-(w)]\end{aligned}$$

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$\lim_{N \rightarrow \infty} -\frac{1}{N^2} \ln [P(w, N)] = \Phi_-(w) \rightarrow$ left large deviation function

physically $\Phi_-(w) \rightarrow$ energy cost in **pushing** the Coulomb gas

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physically $\Phi_-(w) \rightarrow$ energy cost in **pushing** the Coulomb gas

$$\begin{aligned}\Phi_-(w) &= \frac{1}{108} \left[36w^2 - w^4 - (15w + w^3)\sqrt{w^2 + 6} \right. \\ &\quad \left. + 27 \left(\ln(18) - 2 \ln(w + \sqrt{6 + w^2}) \right) \right] \quad \text{for } w < \sqrt{2}\end{aligned}$$

(Dean & S.M., 2006,2008)

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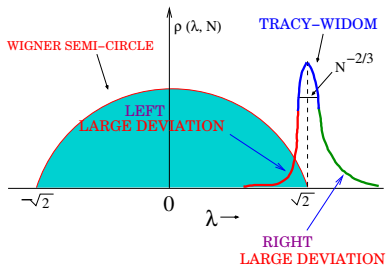
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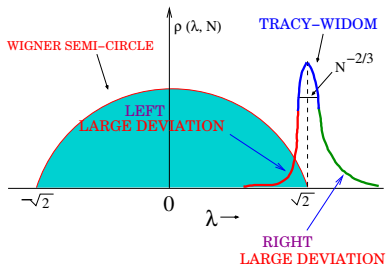
(Dean & S.M., 2006,2008)

Note also that $\Phi_-(w) \approx \frac{1}{6\sqrt{2}}(\sqrt{2} - w)^3$ as $w \rightarrow \sqrt{2}$ from below

Matching with the left tail of Tracy-Widom:



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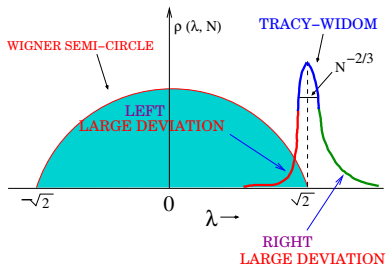


As $w \rightarrow \sqrt{2}$ from below, $\Phi_-(w) \rightarrow \frac{(\sqrt{2}-w)^3}{6\sqrt{2}}$

\rightarrow matches with the **left** tail of the Tracy-Widom distribution

$$\begin{aligned}\text{Prob.}[\lambda_{\max} = w, N] &\sim \exp[-\beta N^2 \Phi_-(w)] \\ &\sim \exp\left[-\frac{\beta}{24} \left|\sqrt{2} N^{2/3} (w - \sqrt{2})\right|^3\right]\end{aligned}$$

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recovers the **left** tail of TW: $f_\beta(x) \sim \exp[-\frac{\beta}{24} |x|^3]$ as $x \rightarrow -\infty$

Right Large Deviation Function: $w > \sqrt{2}$

- For $w \geq \sqrt{2}$, saddle point solution of the charge density $\rho_w(\lambda)$ sticks to the semi-circle form: $\rho_{sc}(\lambda) = \frac{1}{\pi} \sqrt{2 - \lambda^2}$ for all $w \geq \sqrt{2}$

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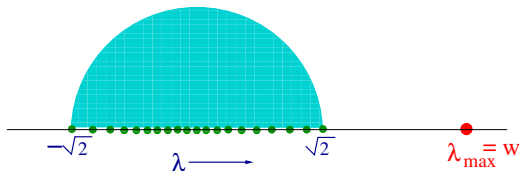
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$\rightarrow (N - 1)$ -fold integral

WIGNER SEMI-CIRCLE



Pulled Coulomb gas

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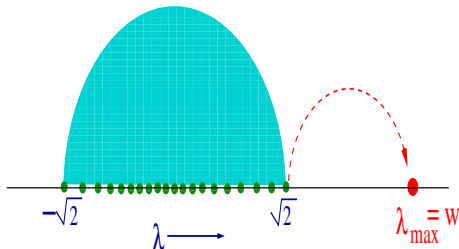
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$$\sim \exp[-\beta N \Phi_+(w)]$$

WIGNER SEMI-CIRCLE



$$N \Phi_+(w) = \Delta E(w)$$

$$= \frac{w^2}{2} - \int_{-\sqrt{2}}^{\sqrt{2}} \ln(w - \lambda) \rho_{\text{sc}}(\lambda) d\lambda$$

\Rightarrow energy cost in pulling a charge out of the Wigner sea

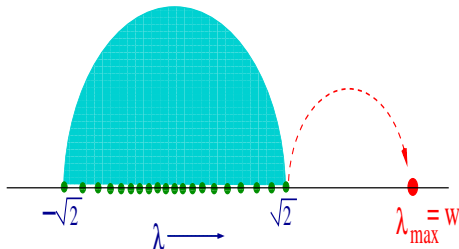
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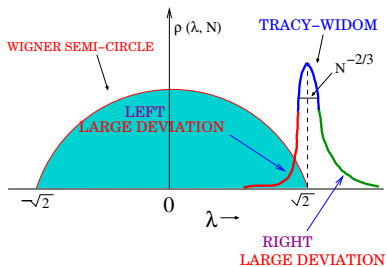
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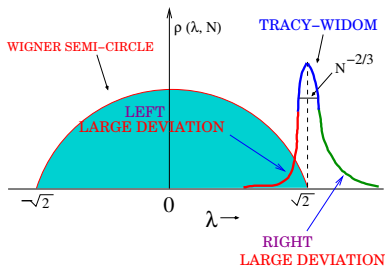
$$\Rightarrow \Phi_+(w) = \frac{1}{2} w \sqrt{w^2 - 2} + \ln \left[\frac{w - \sqrt{w^2 - 2}}{\sqrt{2}} \right] \quad (w > \sqrt{2})$$

[S.M. & Vergassola, PRL, 102, 160201 (2009)]

Matching with the right tail of Tracy-Widom



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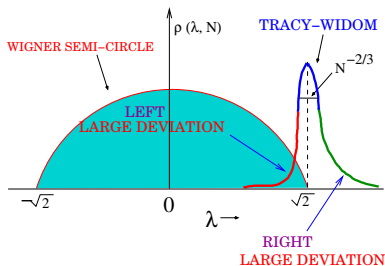


As $w \rightarrow \sqrt{2}$ from above, $\Phi_+(w) \rightarrow \frac{2^{7/4}}{3} (w - \sqrt{2})^{3/2}$

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$$\begin{aligned} \text{Prob.}[\lambda_{\max} = w, N] &\sim \exp[-\beta N \Phi_+(w)] \\ &\sim \exp\left[-\frac{2\beta}{3} \left|\sqrt{2} N^{2/3} (w - \sqrt{2})\right|^{3/2}\right] \end{aligned}$$

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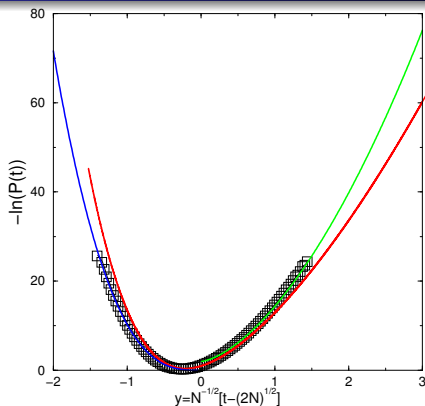
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⇒ recovers the right tail of TW: $f_\beta(x) \sim \exp\left[-\frac{2\beta}{3} |x|^{3/2}\right]$ as $x \rightarrow \infty$

Comparison with Simulations:



$N \times N$ real Gaussian matrix ($\beta = 1$): $N = 10$

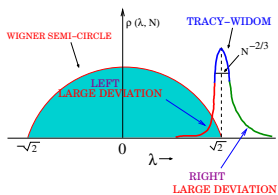
squares \rightarrow simulation points

red line \rightarrow Tracy-Widom

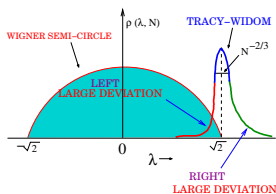
blue line \rightarrow left large deviation function ($\times N^2$)

green line \rightarrow right large deviation function ($\times N$).

Summary and Generalizations



Summary and Generalizations



Prob. density of the top eigenvalue: $\text{Prob.} [\lambda_{\max} = w, N]$ behaves as:

$$\sim \exp[-\beta N^2 \Phi_-(w)] \quad \text{for} \quad \sqrt{2} - w \sim O(1)$$

$$\sim N^{2/3} f_\beta \left[\sqrt{2} N^{2/3} (w - \sqrt{2}) \right] \quad \text{for} \quad |w - \sqrt{2}| \sim O(N^{-2/3})$$

$$\sim \exp[-\beta N \Phi_+(w)] \quad \text{for} \quad w - \sqrt{2} \sim O(1)$$

3-rd Order Phase Transition

Cumulative prob. of λ_{\max} :

$$P(\lambda_{\max} \leq w, N) \approx \begin{cases} \exp\{-\beta N^2 \Phi_-(w) + \dots\} & \text{for } w < \sqrt{2} \\ 1 - A \exp\{-\beta N \Phi_+(w) + \dots\} & \text{for } w > \sqrt{2} \end{cases}$$

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3-rd derivative \rightarrow discontinuous

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3-rd derivative \rightarrow discontinuous

- Left \rightarrow strong-coupling phase \rightarrow perturbative

higher order corrections ($1/N$ expansion) to free energy

[Borot, Eynard, S.M., & Nadal 2011]

- Right \rightarrow weak-coupling phase \rightarrow non-perturbative

higher order corrections

[Nadal & S.M. 2011, Borot & Nadal, 2012]

3-rd order transition → ubiquitous

- λ_{\max} for other matrix ensembles: **Wishart**: $W = X^\dagger X \rightarrow (N \times N)$
→ covariance matrix

Typical: Tracy-Widom

[Johansson 2000, Johnstone 2001]

Large deviations: Exact rate functions

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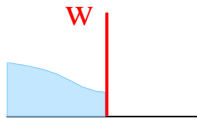
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- **Conductance** and Shot Noise in Mesoscopic Cavities
- **Entanglement entropy** of a random pure state in a bipartite system
- Maximum displacement in **Vicious** walker problem
- Distribution of **Wigner time-delay** ...

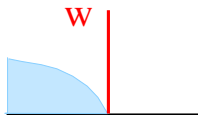
[Bohigas, Comtet, Forrester, Nadal, Schehr, Texier, Vergassola, Vivo,...+S.M. (2008-2014)]

Basic mechanism for 3-rd order transition

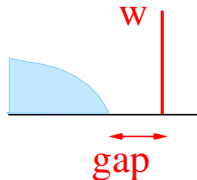
strong



critical



weak



Gap between the soft edge (square-root singularity) of the Coulomb droplet and the hard wall vanishes as a control parameter g goes through a critical value g_c :

$$\text{gap} \longrightarrow 0 \text{ as } g \longrightarrow g_c$$

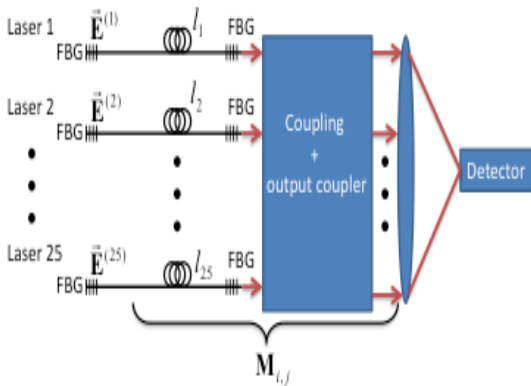
Experimental Verification with Coupled Lasers

Measuring maximal eigenvalue distribution of Wishart random matrices with coupled lasers

Moti Fridman, Rami Pugatch, Micha Nixon, Asher A. Friesem, and Nir Davidson^{*}
Weizmann Institute of Science, Dept. of Physics of Complex Systems, Rehovot 76100, Israel
(Dated: May 30, 2011)

We determined the probability distribution of the combined output power from twenty five coupled fiber lasers and show that it agrees well with the Tracy-Widom, Majumdar-Vergassola and Vivo-Majumdar-Bohigas distributions of the largest eigenvalue of Wishart random matrices with no fitting parameters. This was achieved with 500,000 measurements of the combined output power from the fiber lasers, that continuously changes with variations of the fiber lasers lengths. We show experimentally that for small deviations of the combined output power over its mean value the Tracy-Widom distribution is correct, while for large deviations the Majumdar-Vergassola and Vivo-Majumdar-Bohigas distributions are correct.

Experimental Verification with Coupled Lasers

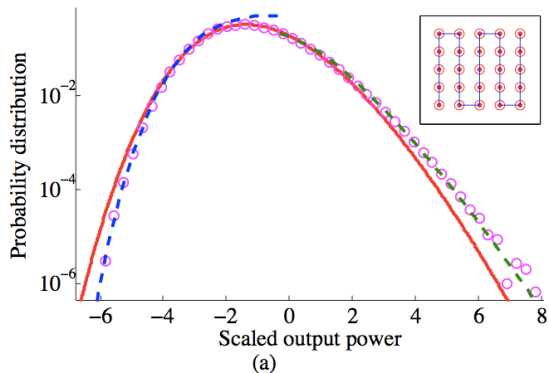


combined output power from fiber lasers $\propto \lambda_{\max}$

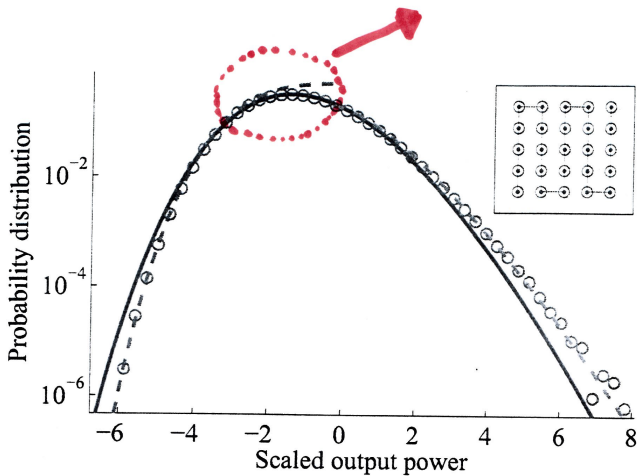
$\lambda_{\max} \rightarrow$ top eigenvalue of the Wishart matrix $W = X^t X$

where $X \rightarrow$ real symmetric Gaussian matrix ($\beta = 1$)

Experimental Verification with Coupled Lasers



Experimental Verification with coupled lasers

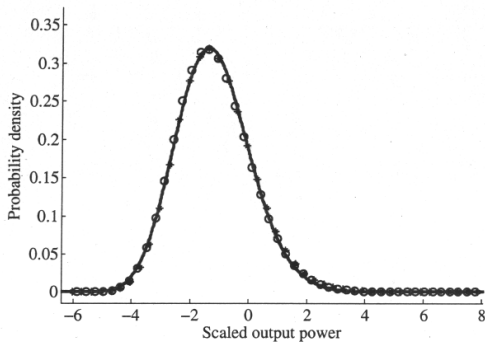


(a)

Tracy-Widom density with $\beta = 1$

Fridman et. al. arXiv:1012.1282

2



Collaborators

Students: **C. Nadal** (Oxford Univ., UK)

J. Randon-Furling (Univ. Paris-1, France)

R. Marino, J. Bun (LPTMS, Univ. Paris-Sud, France)

Postdoc: **D. Villamaina** (ENS, Paris, France)

Collaborators:

- **O. Bohigas, A. Comtet, G. Schehr, C. Texier, P. Vivo** (LPTMS, Orsay, France)
- R. Allez (Berlin, Germany)
- G. Borot (Geneva, Switzerland)
- J.-P. Bouchaud, M. Potters (CFM, Paris, France)
- B. Eynard (Saclay, France)
- K. Damle, V. Tripathi (Tata Institute, Bombay, India)
- **D.S. Dean** (Bordeaux, France)
- P. J. Forrester (Melbourne, Australia)
- A. Lakshminarayan (IIT Madras, India)
- A. Scardichio (ICTP, Trieste, Italy)
- S. Tomsovic (Washington State Univ., USA)
- **M. Vergassola** (UCSD, USA)

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J. Stat. Mech. P01012 (2014)

Tracy-Widom distributions (1994)

The scaling function $F_\beta(x)$ has the expression:

- $\beta = 1$: $F_1(x) = \exp \left[-\frac{1}{2} \int_x^\infty [(y-x)q^2(y) + q(y)] dy \right]$

- $\beta = 2$: $F_2(x) = \exp \left[-\int_x^\infty (y-x)q^2(y) dy \right]$

- $\beta = 4$: $F_4(x) = \exp \left[-\frac{1}{2} \int_x^\infty (y-x)q^2(y) dy \right] \cosh \left[\frac{1}{2} \int_x^\infty q(y) dy \right]$

$$\frac{d^2 q}{dy^2} = 2q^3(y) + yq(y) \text{ with } q(y) \rightarrow \text{Ai}(y) \text{ as } y \rightarrow \infty \rightarrow \text{Painlevé-II}$$

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- Probability density: $f_\beta(x) = dF_\beta(x)/dx$

Left Large Deviation: Beyond the Leading Order

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where explicit expressions for $\Phi_1(w)$, ϕ_β and $\Phi_2(\beta, w)$ were obtained recently (Borot, Eynard, S.M., & Nadal, JSTAT, P11024 (2011))

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(Borot, Eynard, S.M., & Nadal, 2011)

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For $\beta = 1, 2$ and 4

→ agrees with Baik, Buckingham and DiFranco (2008)

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- For general β , precise **right tail** of TW → obtained recently
(Dumaz and Virag, 2011)
- As a bonus, our method also provides a '**simpler**' derivation of TW distribution for $\beta = 2$ (Nadal and S.M., 2011)

A simple example of large deviation tails

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- $\Phi(x) \rightarrow$ symmetric with a minimum at $x = 1/2$ and for small arguments $|x - 1/2| \ll 1$, $\Phi(x) \approx 2(x - 1/2)^2$
 \rightarrow recovers the Gaussian form near the peak

Covariance Matrix

$$\mathbf{X} = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{array}{cc} \text{phys.} & \text{math} \\ \left| \begin{array}{cc} X_{11} & X_{12} \\ X_{21} & X_{22} \\ X_{31} & X_{32} \end{array} \right| \end{array}$$

in general

(MxN)

$$\mathbf{X}^t = \left| \begin{array}{ccc} X_{11} & X_{21} & X_{31} \\ X_{12} & X_{22} & X_{32} \end{array} \right|$$

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$$\mathbf{W} = \mathbf{X}^t \mathbf{X} = \left| \begin{array}{cc} X_{11}^2 + X_{21}^2 + X_{31}^2 & X_{11}X_{12} + X_{21}X_{22} + X_{31}X_{32} \\ X_{12}X_{11} + X_{22}X_{21} + X_{32}X_{31} & X_{12}^2 + X_{22}^2 + X_{32}^2 \end{array} \right|$$

(NxN) COVARIANCE MATRIX (unnormalized)

Principal Component Analysis

Consider N students and $M = 2$ subjects (phys. and math.)

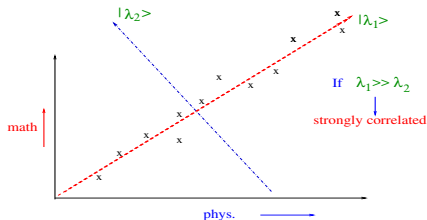
$X \rightarrow (N \times 2)$ matrix and $W = X^t X \rightarrow 2 \times 2$ matrix

Principal Component Analysis

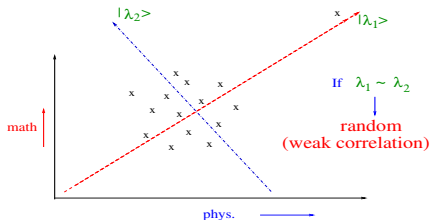
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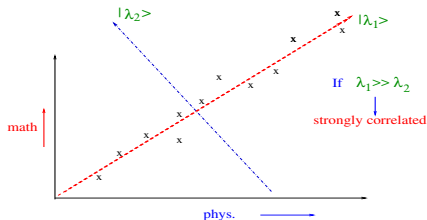


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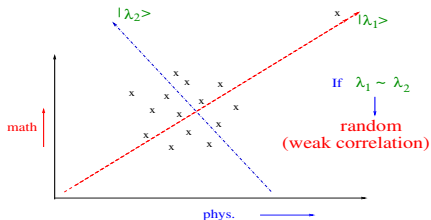
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data compression via 'Principal Component Analysis' (PCA)

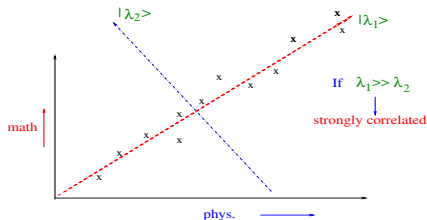
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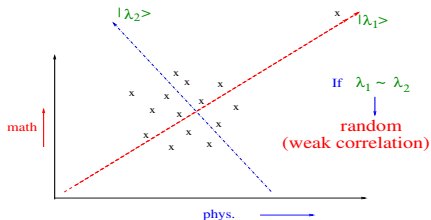
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Null model \rightarrow random data: $X \rightarrow$ random $(M \times N)$ matrix

$\rightarrow W = X^t X \rightarrow$ random $N \times N$ matrix (Wishart, 1928)

Generalization to Wishart Matrices

- $W = X^\dagger X \rightarrow (N \times N)$ square **covariance** matrix (Wishart, 1928)
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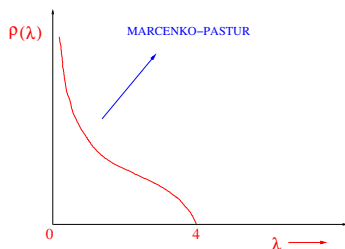
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$$\rho(\lambda, N) = \left\langle \frac{1}{N} \sum_{i=1}^N \delta(\lambda - \lambda_i) \right\rangle \xrightarrow{N \rightarrow \infty} \rho(\lambda) = \frac{1}{2\pi} \sqrt{\frac{4 - \lambda}{\lambda}}$$

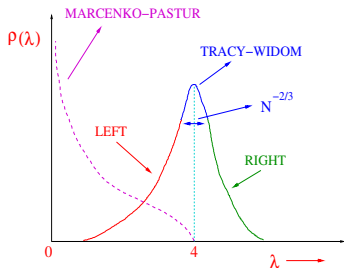
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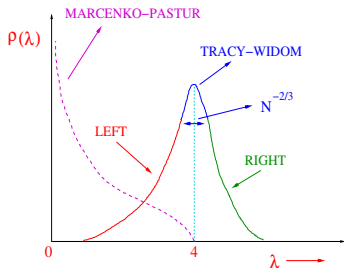


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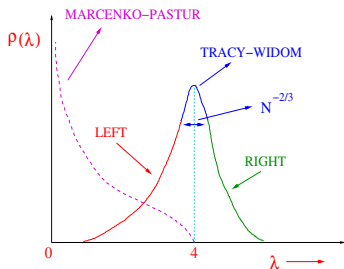


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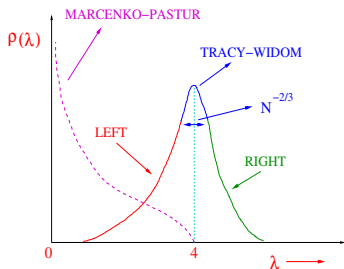
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$$P(\lambda_{\max} = w, N) \approx \begin{cases} \exp\{-\beta N^2 \Psi_-(w)\} & \text{for } w < 4 \\ \exp\{-\beta N \Psi_+(w)\} & \text{for } w > 4 \end{cases}$$

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- $\Psi_-(w)$ and $\Psi_+(w) \rightarrow$ computed exactly

(Vivo, S.M. & Bohigas 2007, S.M. & Vergassola 2009)

Exact Left and Right Large Deviation Functions

Using Coulomb gas + Saddle point method for large N :

- Left large deviation function:

$$\Psi_-(w) = \ln \left[\frac{2}{w} \right] - \frac{w-4}{8} - \frac{(w-4)^2}{64} \quad w \leq 4$$

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- **Right** large deviation function:

$$\Psi_+(w) = \sqrt{\frac{w(w-4)}{4}} + \ln \left[\frac{w-2-\sqrt{w(w-4)}}{2} \right] \quad w \geq 4$$

(S.M. and Vergassola, 2009)

Other Problems with 3-rd Order Phase Transitions

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- Bipartite Entanglement of a Random Pure State

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- Non-Intersecting Brownian Motions and Random Matrices

→ relation to 2-d Yang-Mills gauge theory

Schehr, S.M., Comtet, Randon-Furling, PRL, 101, 150601 (2008)

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