ANNEALED SCALING FOR A CHARGED POLYMER

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§ MOTIVATION

DNA and proteins are polyelectrolytes whose monomers are in a charged state that depends on the pH of the solution in which they are immersed. The charges may fluctuate in space and in time.

In this talk we consider the charged polymer chain introduced in Kantor & Kardar (1991). Our goal is to study the scaling properties of the polymer chain as its length tends to infinity.

We focus on the annealed version of the model in $d = 1$, which turns out to exhibit a very rich scaling behavior. We have no results for $d \geq 2$.

§ MODEL

1. Let $S = (S_i)_{i \in \mathbb{N}_0}$ be simple random walk on $\mathbb{Z}$ starting at 0. The path S models the configuration of the polymer chain, i.e., S_i is the location of monomer i. We use the letter P for probability with respect to S.

2. Let $\omega = (\omega_i)_{i \in \mathbb{N}}$ be i.i.d. random variables taking values in $\mathbb{R}$. The sequence ω models the electric charges along the polymer chain, i.e., ω_i is the charge of monomer i. We use the letter $\mathbb{P}$ for probability with respect to ω, and assume that

$$\mathbb{E}(\omega_1) = 0, \qquad \mathbb{V}\mathrm{ar}(\omega_1) = 1.$$

To allow for biased charges, we use a tilting parameter $\delta \in \mathbb{R}$ and write $\mathbb{P}^\delta$ for the i.i.d. law of ω with marginal

$$\mathbb{P}^\delta(\mathrm{d}\omega_1) = \frac{e^{\delta\omega_1}\,\mathbb{P}(\mathrm{d}\omega_1)}{M(\delta)}, \qquad M(\delta) = \mathbb{E}(e^{\delta\omega_1}).$$

W.l.o.g. we may take $\delta \in [0,\infty)$. Throughout the paper we assume that $M(\delta) < \infty$ for all $\delta \in [0,\infty)$.

3. Let Π denote the set of nearest-neighbor paths starting at 0. Given $n \in \mathbb{N}$, we associate with each $(\omega, S) \in \mathbb{R}^{\mathbb{N}} \times \Pi$ an energy given by the Hamiltonian

$$H_n^\omega(S) = \sum_{1 \le i < j \le n} \omega_i \omega_j \, 1_{\{S_i = S_j\}}.$$

4. Let β denote the inverse temperature. Throughout the sequel the relevant space for the pair of parameters (δ, β) is the quadrant

$$\mathcal{Q} = [0, \infty) \times (0, \infty).$$

5. Given $(\delta, \beta) \in \mathcal{Q}$, the annealed polymer measure of length n is the Gibbs measure $\mathbb{P}_n^{\delta,\beta}$ defined as

$$\frac{\mathrm{d}\mathbb{P}_n^{\delta,\beta}}{\mathrm{d}(\mathbb{P}^\delta \times \mathsf{P})}(\omega, S) = \frac{1}{\mathbb{Z}_n^{\delta,\beta}}\, e^{-\beta H_n^\omega(S)}, \qquad (\omega, S) \in \mathbb{R}^{\mathbb{N}} \times \Pi,$$

where

$$\mathbb{Z}_n^{\delta,\beta} = (\mathbb{E}^\delta \times \mathsf{E}) \left[e^{-\beta H_n^\omega(S)} \right]$$

is the annealed partition function of length n.

Remark: In what follows we will actually work with the Hamiltonian

$$H_n^\omega(S) = \sum_{1 \le i,j \le n} \omega_i \omega_j \, 1_{\{S_i = S_j\}} = \sum_{x \in \mathbb{Z}^d} \left(\sum_{i=1}^{n} \omega_i \, 1_{\{S_i = x\}} \right)^2 .$$

This choice amounts to replacing β by 2β and adding a charge bias.

Literature: The charged polymer with binary disorder interpolates between

simple random walk	$\beta = 0$
self-avoiding walk	$\beta = \delta = \infty$
weakly self-avoiding walk	$\beta \in (0, \infty), \delta = \infty$

PART I: general properties

§ FREE ENERGY

1. Let $Q(i,j)$ be the probability matrix defined by

$$Q(i,j) = \begin{cases} 1_{\{j=0\}}, & \text{if } i = 0,\ j \in \mathbb{N}_0, \\ \binom{i+j-1}{i-1} \left(\frac{1}{2}\right)^{i+j}, & \text{if } i \in \mathbb{N},\ j \in \mathbb{N}_0, \end{cases}$$

which is the transition kernel of a critical Galton-Watson branching process with a geometric offspring distribution.

2. For $(\delta,\beta) \in \mathcal{Q}$, let $G^*_{\delta,\beta}$ be the function defined by

$$G^*_{\delta,\beta}(\ell) = \log \mathbb{E}\left[e^{\delta\Omega_\ell - \beta\Omega_\ell^2}\right], \quad \Omega_\ell = \sum_{k=1}^{\ell} \omega_k, \quad \ell \in \mathbb{N}_0.$$

3. For $(\mu,\delta,\beta) \in [0,\infty) \times \mathcal{Q}$, define the $\mathbb{N}_0 \times \mathbb{N}_0$ matrices

$$A_{\mu,\delta,\beta}(i,j) = e^{-\mu(i+j+1)+G^*_{\delta,\beta}(i+j+1)}\, Q(i+1,j), \quad i,j \in \mathbb{N}_0,$$

and

$$\tilde{A}_{\mu,\delta,\beta}(i,j) = \begin{cases} 0, & \text{if } i = 0,\ j \in \mathbb{N}_0, \\ A_{\mu,\delta,\beta}(i-1,j), & \text{if } i \in \mathbb{N},\ j \in \mathbb{N}_0. \end{cases}$$

4. Let $\lambda_{\delta,\beta}(\mu)$ and $\tilde{\lambda}_{\delta,\beta}(\mu)$ be the spectral radius of $A_{\mu,\delta,\beta}$ and $\tilde{A}_{\mu,\delta,\beta}$ in $\ell^2(\mathbb{N}_0)$. For every $(\delta,\beta) \in \mathcal{Q}$, both $\mu \mapsto \lambda_{\delta,\beta}(\mu)$ and $\mu \mapsto \tilde{\lambda}_{\delta,\beta}(\mu)$ are continuous, strictly decreasing and log-convex on $[0,\infty)$, are analytic on $(0,\infty)$, have a finite strictly negative right-slope at 0, and satisfy

$$\tilde{\lambda}_{\delta,\beta}(\mu) < \lambda_{\delta,\beta}(\mu) \quad \forall\ \mu \in [0,\infty).$$

5. Let

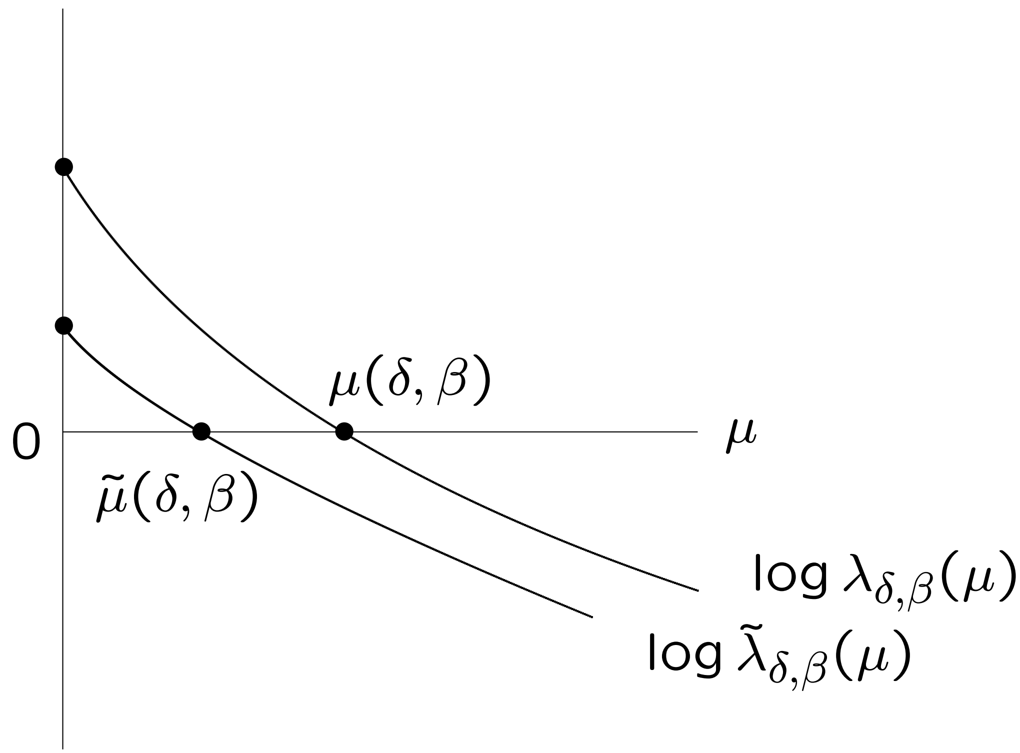
- $\mu(\delta,\beta)$ be the unique solution of the equation $\lambda_{\delta,\beta}(\mu) = 1$ when it exists and $\mu(\delta,\beta) = 0$ otherwise,

- $\tilde{\mu}(\delta,\beta)$ be the unique solution of the equation $\tilde{\lambda}_{\delta,\beta}(\mu) = 1$ when it exists and $\tilde{\mu}(\delta,\beta) = 0$ otherwise,

which satisfy

$$\tilde{\mu}(\delta,\beta) \le \mu(\delta,\beta),$$

with strict inequality as soon as $\mu(\delta,\beta) > 0$.



$\mu(\delta, \beta)$
0
μ
$\tilde{\mu}(\delta, \beta)$
$\log \lambda_{\delta,\beta}(\mu)$
$\log \tilde{\lambda}_{\delta,\beta}(\mu)$

THEOREM 1

(1) *For every* $(\delta, \beta) \in \mathcal{Q}$*, the annealed free energy per monomer*

$$F(\delta, \beta) = \lim_{n \to \infty} \frac{1}{n} \log \mathbb{Z}_n^{\delta,\beta}$$

exists, takes values in $(-\infty, 0]$*, and satisfies the inequality*

$$F(\delta, \beta) \geq f(\delta) = -\log M(\delta) \in (-\infty, 0].$$

(2) *The excess free energy*

$$F^*(\delta, \beta) = F(\delta, \beta) - f(\delta)$$

is convex in (δ, β) *and has the spectral representation*

$$F^*(\delta, \beta) = \mu(\delta, \beta).$$

The generating function for the excess annealed partion function is

$$\mathcal{Z}(\mu,\delta,\beta) = \sum_{n\in\mathbb{N}_0} e^{-\mu n}\, e^{-f(\delta)n}\, \mathbb{Z}_n^{\delta,\beta}, \qquad \mu \in [0,\infty).$$

The key to the spectral representation of the excess free energy is the formula

$$\mathcal{Z}(\mu,\delta,\beta) =^{*} \left[\frac{1}{1-\tilde{A}^{\intercal}_{\mu,\delta,\beta}}\,\frac{1+A_{\mu,\delta,\beta}}{1-A_{\mu,\delta,\beta}}\,\frac{1}{1-\tilde{A}_{\mu,\delta,\beta}}\right](0,0)$$

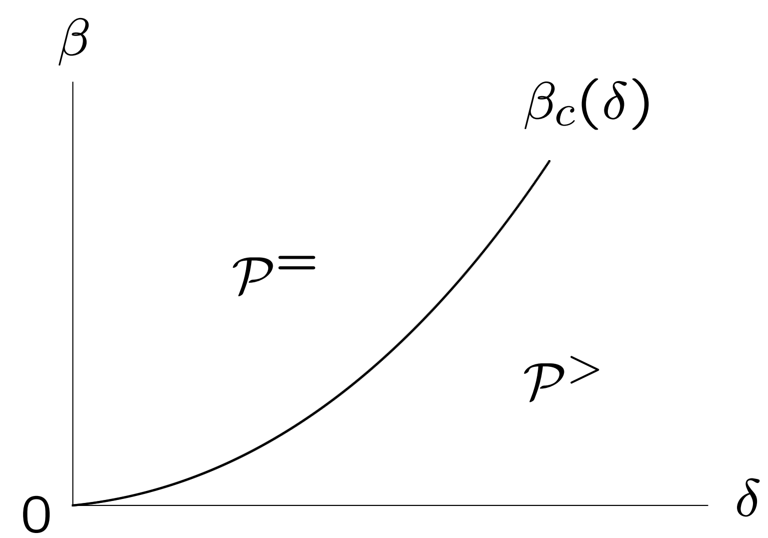
in terms of the two matrices introduced before. This formula arises from the Ray-Knight representation of the local times of simple random walk on the stretches $(-\infty,0)$, $[0,S_n]$ and (S_n,∞).

§ PHASE TRANSITION

The inequality $F^*(\delta,\beta) \geq 0$ leads us to define two phases:

$$\mathcal{P}^{>} = \Big\{(\delta,\beta) \in \mathcal{Q}\colon\ F^*(\delta,\beta) > 0\Big\},$$

$$\mathcal{P}^{=} = \{(\delta,\beta) \in \mathcal{Q}\colon\ F^*(\delta,\beta) = 0\Big\}.$$



THEOREM 2

(1) *There exists a critical curve* $\delta \mapsto \beta_c(\delta)$ *such that*

$$\mathcal{P}^{>} = \Big\{(\delta,\beta) \in \mathcal{Q} \colon\ 0 < \beta < \beta_c(\delta)\Big\},$$
$$\mathcal{P}^{=} = \Big\{(\delta,\beta) \in \mathcal{Q} \colon\ \beta \geq \beta_c(\delta)\Big\}.$$

(2) *For every* $\delta \in [0,\infty)$, $\beta_c(\delta)$ *is the unique solution of the equation* $\lambda_{\delta,\beta}(0) = 1$.

(3) $\delta \mapsto \beta_c(\delta)$ *is continuous, strictly increasing and convex on* $[0,\infty)$, *is analytic on* $(0,\infty)$, *and satisfies* $\beta_c(0) = 0$.

(4) $(\delta,\beta) \mapsto F^*(\delta,\beta)$ *is analytic on* $\mathcal{P}^{>}$.

§ LAWS OF LARGE NUMBERS

We proceed by stating a LLN for the empirical speed $n^{-1}S_n$ and the empirical charge $n^{-1}\Omega_n$, where

$$S_n = \sum_{i=1}^{n} X_i, \qquad \Omega_n = \sum_{i=1}^{n} \omega_i.$$

Let

$$\mathcal{B} = \Big\{(\delta,\beta) \in \mathcal{Q} \colon\ 0 < \beta \leq \beta_c(\delta)\Big\}, \qquad \mathcal{S} = \mathcal{Q}\backslash\mathcal{B}.$$

The set $\mathcal{B}$ will be referred to as the ballistic phase, the set $\mathcal{S}$ as the subballistic phase, for reason that become apparent in the next theorem.

THEOREM 3

(1) *For every* $(\delta,\beta) \in \mathcal{Q}$ *there exists a* $v(\delta,\beta) \in [0,1]$ *such that*

$$\lim_{n\to\infty} \mathbb{P}_n^{\delta,\beta}\Big(\big|n^{-1}S_n - v(\delta,\beta)\big| > \varepsilon \,\Big|\, S_n > 0\Big) = 0 \qquad \forall\,\varepsilon > 0,$$

where

$$v(\delta,\beta) \begin{cases} > 0, & (\delta,\beta) \in \mathcal{B}, \\ = 0, & (\delta,\beta) \in \mathcal{S}. \end{cases}$$

(2) *For every* $(\delta,\beta) \in \mathcal{B}$*,*

$$\frac{1}{v(\delta,\beta)} = \Big[-\frac{\partial}{\partial\mu} \log \lambda_{\delta,\beta}(\mu) \Big]_{\mu=\mu(\delta,\beta)}.$$

THEOREM 4

(1) *For every* $(\delta, \beta) \in \mathcal{Q}$, *there exists a* $\rho(\delta, \beta) \in [0, \infty)$ *such that*

$$\lim_{n \to \infty} \mathbb{P}_n^{\delta,\beta} \left(\left| n^{-1} \Omega_n - \rho(\delta, \beta) \right| > \epsilon \right) = 0 \qquad \forall \, \epsilon > 0,$$

where

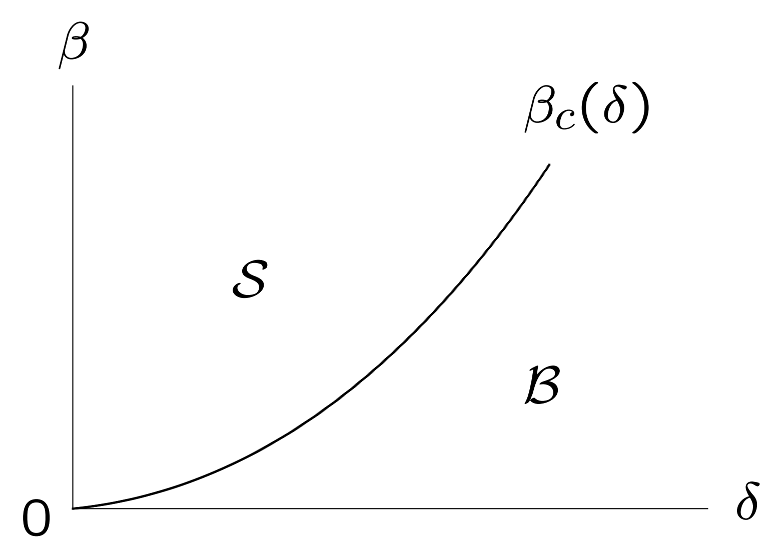
$$\rho(\delta, \beta) \begin{cases} > 0, & (\delta, \beta) \in \mathcal{B}, \\ = 0, & (\delta, \beta) \in \mathcal{S}. \end{cases}$$

(2) *For every* $(\delta, \beta) \in \mathcal{B}$,

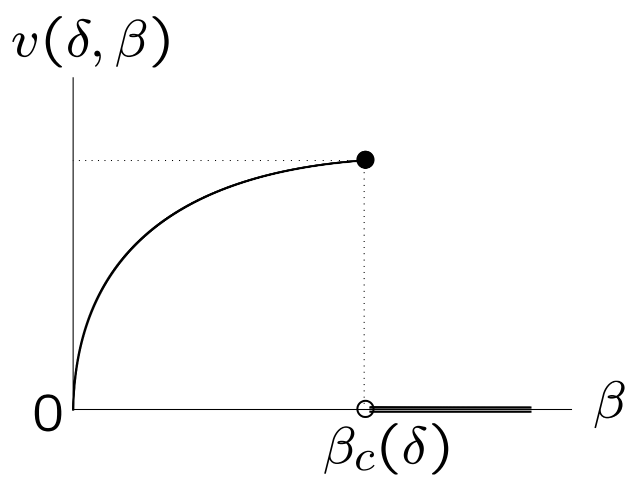
$$\rho(\delta, \beta) = \frac{\partial}{\partial \delta} \mu(\delta, \beta).$$

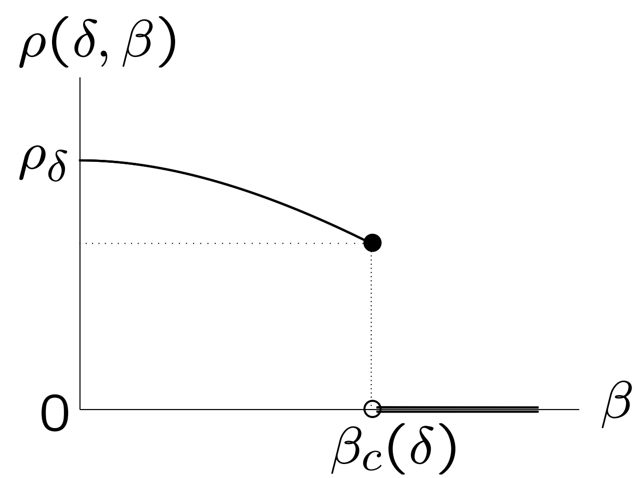
§ PHASE DIAGRAM

Picture of the ballistic phase $\mathcal{B}$ and the subballistic phase $\mathcal{S}$. The critical curve is part of $\mathcal{B}$, which implies that the phase transition is first order.



Conjectured qualitative plots of $\beta \mapsto v(\delta,\beta)$ and $\beta \mapsto \rho(\delta,\beta)$ for fixed δ:



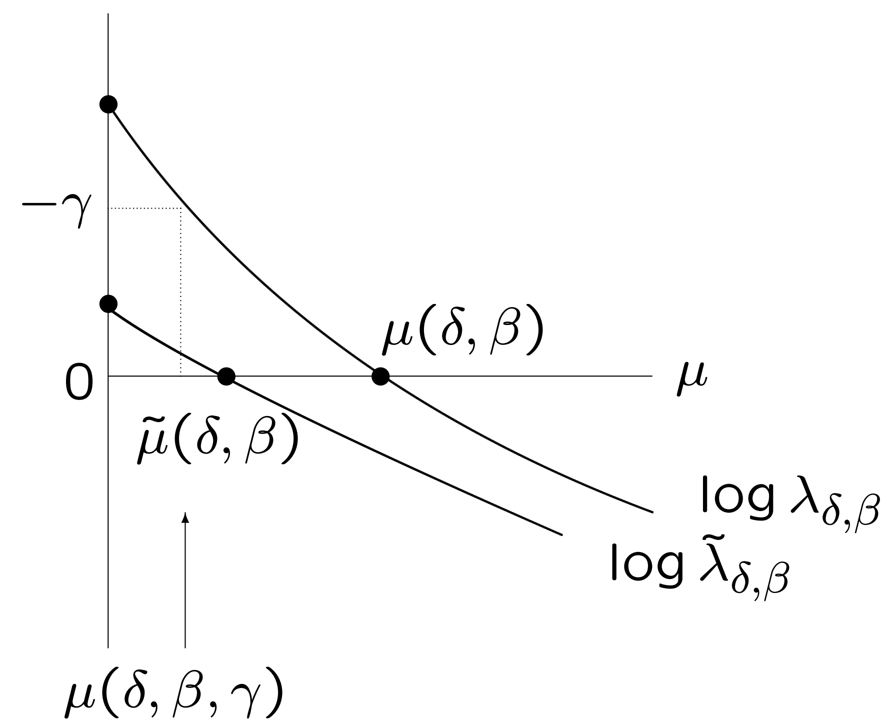


Heuristically, when the polymer moves ballistically the repulsion wins from the attraction. An increase of β tilts this unbalance even further. This causes a larger speed, which is partially compensated by a smaller charge.

§ LARGE DEVIATION PRINCIPLES

Let

- $\mu(\delta,\beta,\gamma)$ be the solution of the equation $\lambda_{\delta,\beta}(\mu) = e^{-\gamma}$ when it exists and $\mu(\delta,\beta,\gamma) = 0$ otherwise.



THEOREM 5

For every $(\delta, \beta) \in \mathcal{Q}$*:*

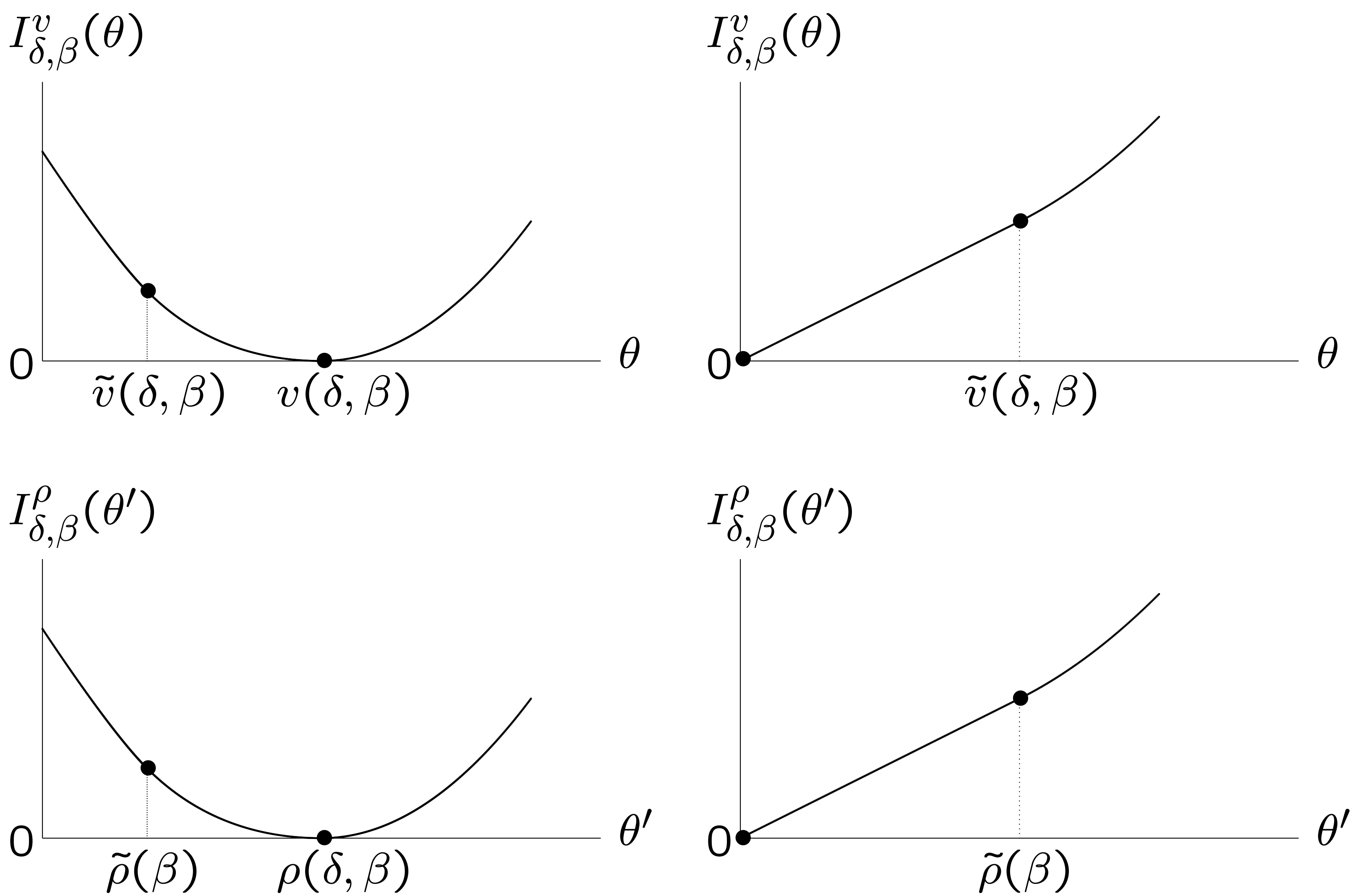
(1) *The sequence* $(n^{-1}S_n)_{n\in\mathbb{N}}$ *conditionally on* $\{S_n > 0\}_{n\in\mathbb{N}}$ *satisfies the LDP on* $[0,\infty)$ *with rate function* $I^{v}_{\delta,\beta}$ *given by*

$$I^{v}_{\delta,\beta}(\theta) = \mu(\delta,\beta) + \sup_{\gamma\in\mathbb{R}} \Big[\theta\gamma - \{\mu(\delta,\beta,\gamma) \vee \tilde{\mu}(\delta,\beta)\}\Big], \quad \theta \in [0,\infty).$$

(2) *The sequence* $(n^{-1}\Omega_n)_{n\in\mathbb{N}}$ *satisfies the LDP on* $[0,\infty)$ *with rate function* $I^{\rho}_{\delta,\beta}$ *given by*

$$I^{\rho}_{\delta,\beta}(\theta') = \mu(\delta,\beta) + \sup_{\gamma'\in\mathbb{R}} \Big[\theta'\gamma' - \mu(\delta+\gamma',\beta)\Big], \quad \theta' \in [0,\infty).$$

Pictures for $(\delta, \beta) \in \mathrm{int}(\mathcal{B})$ and $(\delta, \beta) \in \mathcal{S}$:



§ RATE FUNCTIONS

The two rate functions are strictly convex, except for linear pieces on $[0,\tilde{v}(\delta,\beta)]$ and $[0,\tilde{\rho}(\beta)]$ with

$$\frac{1}{\tilde{v}(\delta,\beta)} = \Big[-\frac{\partial}{\partial\mu}\log\lambda_{\delta,\beta}(\mu)\Big]_{\mu=\tilde{\mu}(\delta,\beta)}, \qquad \tilde{\rho}(\beta) = \rho(\delta_c(\beta),\beta),$$

where $\beta \mapsto \delta_c(\beta)$ is the inverse of $\delta \mapsto \beta_c(\delta)$.

While $v(\delta,\beta)$ and $\rho(\delta,\beta)$ jump from a strictly positive value to zero when (δ,β) moves from $\mathcal{B}$ to $\mathcal{S}$ inside $\mathrm{int}(\mathcal{Q})$, $\tilde{v}(\delta,\beta)$ and $\tilde{\rho}(\beta)$ are strictly positive throughout $\mathrm{int}(\mathcal{Q})$.

The flat pieces in the rate functions for the speed and the charge correspond to an inhomogeneous strategy for the polymer to realise a large deviation.

For instance, if the speed is $\theta < \tilde{v}(\delta, \beta)$, then the charge

- makes a large deviation on a stretch of the polymer of length $\theta / \tilde{v}(\delta, \beta)$ times the total length, so as to allow it to move at speed $\tilde{v}(\delta, \beta)$ at zero cost,

- makes a large deviation on the remaining stretch, so as to allow it to be subballistic at zero cost.

§ CENTRAL LIMIT THEOREMS

THEOREM 6*

For every $(\delta, \beta) \in \mathrm{int}(\mathcal{B})$ *the scaled quantities*

$$\frac{S_n - nv(\delta,\beta)}{\sigma_v(\delta,\beta)\sqrt{n}}, \qquad \frac{\Omega_n - n\rho(\delta,\beta)}{\sigma_\rho(\delta,\beta)\sqrt{n}},$$

converge in distribution to the standard normal law with

$$\frac{1}{\sigma_v^2(\delta,\beta)} = \left[\frac{\partial^2}{\partial\theta^2} I^v_{\delta,\beta}(\theta)\right]_{\theta=v(\delta,\beta)},$$

$$\frac{1}{\sigma_\rho^2(\delta,\beta)} = \left[\frac{\partial^2}{\partial\theta'^2} I^\rho_{\delta,\beta}(\theta')\right]_{\theta'=\rho(\delta,\beta)}.$$

PART II: asymptotic properties

§ SCALING OF THE CRITICAL CURVE

1. For $a \in \mathbb{R}$, let $\mathcal{L}^a$ be the Sturm-Liouville operator defined by

$$(\mathcal{L}^a g)(x) = (2ax - 4x^2)g(x) + g'(x) + xg''(x),$$
$$g \in C^\infty((0,\infty)).$$

The largest eigenvalue problem

$$\mathcal{L}^a g = \chi g, \qquad \chi \in \mathbb{R},\ g \in L^2((0,\infty)) \cap C^\infty((0,\infty)),$$
$$\|g\|_2 = 1,\ g > 0,\ \int_0^\infty \left[x^2 g(x)^2 + xg'(x)^2\right] dx < \infty,$$

has a unique solution $(g^a, \chi(a))$ with the following properties:

- $a \mapsto \chi(a)$ is analytic, strictly increasing and strictly convex on $\mathbb{R}$.

- $\chi(0) < 0$, $\lim_{a\to\infty} \chi(a) = \infty$, $\lim_{a\to-\infty} \chi(a) = -\infty$.

- $a \mapsto g^a$ is analytic as a map from $\mathbb{R}$ to $L^2((0,\infty))$.

Let $a^* \in (0,\infty)$ denote the unique solution of the equation

$$\chi(a) = 0.$$

THEOREM 7

(1) *As* $\delta \downarrow 0$*,*

$$\beta_c(\delta) - \tfrac{1}{2}\delta^2 \sim -a^*(\tfrac{1}{2}\delta^2)^{4/3}.$$

(2) *As* $\delta \to \infty$*,*

$$\beta_c(\delta) \sim \frac{\delta}{T}$$

with

$$T = \sup\left\{t > 0\colon\ \mathbb{P}(\omega_1 \in t\mathbb{Z}) = 1\right\}$$

Either $T > 0$ *(lattice case) or* $T = 0$ *(non-lattice case). If* $T = 0$ *and* ω_1 *has a bounded density with respect to the Lebesgue measure, then*

$$\beta_c(\delta) \sim \frac{1}{4}\frac{\delta^2}{\log \delta}.$$

Note that the scaling behavior of the critical curve is anomalous for small charge bias. This implies that the critical curve is not analytic at the origin.

Note that the scaling is also delicate for large charge bias. Heuristically, it is easier to build small absolute values of $\Omega_\ell = \sum_{k=1}^{\ell} \omega_k$ for small values of ℓ when the charge distribution is non-lattice rather than lattice.

§ CRITICAL SCALING OF THE EXCESS FREE ENERGY

The scaling behaviour of the excess free energy near the critical curve fits with the phase transition being first order.

THEOREM 8

For every $\delta \in (0,\infty)$,

$$F^*(\delta,\beta) \sim C_\delta[\beta_c(\delta)-\beta], \qquad \beta \uparrow \beta_c(\delta),$$

where $C_\delta \in (0,\infty)$ *is given by*

$$C_\delta = \left[-\frac{\frac{\partial}{\partial\beta}\log\lambda_{\delta,\beta}(\mu)}{\frac{\partial}{\partial\mu}\log\lambda_{\delta,\beta}(\mu)} \right]_{\beta=\beta_c(\delta),\mu=0}.$$

§ WEAK INTERACTION SCALING

To identify the scaling behaviour of the free energy for small inverse temperature, we need a two-parameter variant of the Sturm-Liouville operator defined above, namely, for $a \in \mathbb{R}$ and $b \in (0,\infty)$,

$$(\mathcal{L}^{a,b}g)(x) = (2ax - 4bx^2)g(x) + g'(x) + xg''(x),$$
$$g \in C^\infty((0,\infty)).$$

For this operator the largest eigenvalue problem has a unique solution $(g^{a,b}, \chi(a,b))$ with properties similar as before. In particular, for every $b \in (0,\infty)$ there is a unique $a^*(b)$ solving the equation

$$\chi(a,b) = 0.$$

THEOREM 9

(1) *For every* $\delta \in (0,\infty)$*,*

$$F(\delta,\beta) \sim -A_\delta \beta^{2/3}, \quad v(\delta,\beta) \sim B_\delta \beta^{1/3},$$
$$\rho(\delta,\beta) \sim -f'(\delta), \qquad \beta \downarrow 0,$$

where $A_\delta, B_\delta \in (0,\infty)$ *are given by*

$$A_\delta = a^*(\rho_\delta), \qquad \frac{1}{B_\delta} = \left[\frac{\partial}{\partial a}\chi(a,b)\right]_{a=a^*(\rho_\delta),\, b=\rho_\delta},$$

with $\rho_\delta = \mathbb{E}^\delta(\omega_1)$*.*

(2) *For every* $\epsilon > 0$*,*

$$F(\delta,\beta) \sim \beta_c(\delta) - \beta, \qquad \delta,\beta \downarrow 0$$

subject to the constraint $\beta_c(\delta) - \beta \geq \epsilon\beta^{4/3}$.

§ EPI-CONVERGENCE

The weak interaction scaling asymptotics are derived as follows. The Rayleigh formula gives

$$\lambda_{\delta,\beta}(\mu) - 1 = \sup_{\substack{v\in\ell^2(\mathbb{N}_0)\\ v\geq 0,\,\|v\|_2=1}} \Big\{ \sum_{i,j\in\mathbb{N}_0} A_{\mu,\delta,\beta}(i,j)v(i)v(j) - \sum_{i\in\mathbb{N}_0} v(i)^2 \Big\}.$$

The supremum can be rewritten as

$$\lambda_{\delta,\beta}(\mu) - 1 = \sup_{\substack{f\in L_2(0,\infty)\\ f\geq 0,\,\|f\|_2=1}} \left\{ \frac{1}{\beta\eta} \int_0^\infty dx \int_0^\infty dy\, f(x)f(y)\, A_{\mu,\delta,\beta}\Big(\frac{x}{\beta\eta}, \frac{y}{\beta\eta}\Big) - \int_0^\infty dx\, f(x)^2 \right\},$$

where $\eta > 0$ is an arbitrary scaling factor.

Decompose the variational formula as

$$\lambda_{\delta,\beta}(\mu) - 1 = \sup_{\substack{f \in L_2(0,\infty) \\ f \geq 0, \|f\|_2 = 1}} [(I) - (II)],$$

where

$$(I) = \int_0^\infty dx \int_0^\infty dy \, f(x)^2 \left[\frac{1}{\beta\eta} A_{\mu,\delta,\beta}\left(\frac{x}{\beta\eta} + 1, \frac{y}{\beta\eta}\right) - \frac{1}{\beta\eta} Q\left(\frac{x}{\beta\eta}, \frac{y}{\beta\eta}\right)\right],$$

$$(II) = \frac{1}{2} \int_0^\infty dx \int_0^\infty dy \, [f(x) - f(y)]^2 \frac{1}{\beta\eta} A_{\mu,\delta,\beta}\left(\frac{x}{\beta\eta}, \frac{y}{\beta\eta}\right).$$

For $\beta \downarrow 0$ this leads to approximation

$$\lambda_{\delta,\beta}(\mu) - 1 \sim \sup_{\substack{f \in L_2(0,\infty) \\ f \geq 0,\, \|f\|_2 = 1}} \Big\{ A_1(f) \Big[(\tfrac{1}{2}\delta^2 - \beta - \mu) \frac{1}{\beta^\eta} \Big] - A_2(f)\beta^{1-2\eta}\delta^2 - A_3(f)\beta^\eta \Big\},$$

where

$$A_1(f) = \int_0^\infty \mathrm{d}x\,(2x) f(x)^2, \quad A_2(f) = \int_0^\infty \mathrm{d}x\,(2x)^2 f(x)^2,$$
$$A_3(f) = \int_0^\infty \mathrm{d}x\, x[f'(x)]^2.$$

The choice

$$\eta = \tfrac{2}{3}, \qquad \mu = B\beta^{4/3}, \qquad \beta = \tfrac{1}{2}\delta^2 - C(\tfrac{1}{2}\delta^2)^{4/3},$$
$$B \geq 0, \qquad C \in \mathbb{R},$$

leads to

$$\lim_{\delta \downarrow 0} \frac{\lambda_{\delta,\beta}(\mu) - 1}{\beta^{2/3}} = \chi(C - B).$$

where $\chi(a)$ is the largest eigenvalue of the Sturm-Liouville operator with parameter $a \in \mathbb{R}$. By tuning B, C, part of the weak interaction scaling asymptotics are obtained.

§ DISCUSSION

1. It can be shown that for every $(\delta,\beta) \in \mathcal{S}$,

$$\lim_{n\to\infty} \frac{(\alpha_n)^2}{n} \log \mathbb{Z}_n^{*,\delta,\beta} = -\chi, \qquad \mathbb{Z}_N^{*,\delta,\beta} = e^{-f(\delta)n}\,\mathbb{Z}_n^{\delta,\beta},$$

with $\alpha_n = (n/\log n)^{1/3}$ and with $\chi \in (0,\infty)$ a constant that is explicitly computable.

The idea behind this scaling is that for $(\delta,\beta) \in \mathcal{S}$ the empirical charge makes a large deviation under the disorder measure $\mathbb{P}^\delta$ so that it becomes zero. The price for this large deviation is

$$e^{-nH(\mathbb{P}^0 \,|\, \mathbb{P}^\delta) + o(n)},$$

where $H(\mathbb{P}^0 \,|\, \mathbb{P}^\delta)$ denotes the specific relative entropy of $\mathbb{P}^0 = \mathbb{P}$ with respect to $\mathbb{P}^\delta$. Since the latter equals $\log M(\delta)$

$= -f(\delta)$, this accounts for the leading term in the free energy.

Conditional on the empirical charge being zero, the attraction between charged monomers with the same sign wins from the repulsion between charged monomers with opposite sign, making the polymer chain contract to a sub-diffusive scale α_n. This accounts for the correction term in the free energy.

It can be shown that under the annealed polymer measure,

$$\left(\frac{1}{\alpha_n} S_{\lfloor nt \rfloor}\right)_{0 \le t \le 1} \to^{\mathsf{D}} (U_t)_{0 \le t \le 1}, \qquad n \to \infty,$$

where $\to^{\mathsf{D}}$ denotes convergence in distribution and $(U_t)_{t \ge 0}$ is a Brownian motion on $\mathbb{R}$ conditioned not to leave an interval of a certain length and a certain randomly shifted center.

2. It would be interesting to deal with charges whose interaction extends beyond the on-site interaction used in our model, e.g.

- Coulomb potential (polynomial decay);
- Yukawa potential (exponential decay).

A Yukawa potential arises from a Coulomb potential via screening of the charges when the polymer chain is immersed in an ionic fluid.

3. Biskup and König (2001), Ioffe and Velenik (2012), Kosygina and Mountford (2013) deal with annealed versions of various models of simple random walk in a random potential.

In all these models the interaction is either attractive or repulsive, meaning that the annealed partition function is the expectation of the exponential of a functional of the local times of simple random walk that is either subadditive or superadditive.

Our annealed charged polymer model is neither attractive nor repulsive. However, our spectral representation is flexible so as to include such models.