Université Sorbonne Paris–Nord, Paris, 2022–23
Licence de Mathématiques 2$^{\text{nd}}$ year
Probabilities

Exercise sheet n°1

Counting

Exercise 1 Show that for any positive natural number $n \in \mathbb{N}^*$:

1. $$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

2. $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

These formulas should be known by heart.

Exercise 2 Let $E = \{1, 2, 3, 4\}$ be a set containing 4 elements.

1. List all possible permutations of E and count them. Check that the result matches the formula presented in class.
2. Similarly, list all possible permutations of subsets of E, count them, and verify that the result corresponds to the formula you saw in class.
3. Finally, list all combinations of elements of E, count them, and verify that the result matches the formula you discovered in class.

Exercise 3

1. An trainline company serves n different train stations. Direct trips between any two stations are possible. How many different tickets does the company sell ?
2. At a party (held before Covid), n people shake hands. How many handshakes were exchanged ?

Exercise 4 A competition takes place between 20 participants, one of which is named Émilie. We call a "podium" the ranking of the first three highest-scoring participants (who we reward with medals of gold, silver, and bronze).

1. How many possible podiums are there ?
2. How many possible podiums are there such that Émilie wins gold ?
3. How many possible podiums are there such that Émilie wins a medal ?
4. We would like to offer identical prizes to the three highest-ranked participants. How many possible distributions of prizes are there ?

Exercise 5 (Pascal's triangle)

1. Show that for all positive natural numbers $n \in \mathbb{N}^*$, $p \in \mathbb{N}$, $p \leq n$, the following formulas hold :

$$\binom{n}{p} = \binom{n}{n-p}, \quad p\binom{n}{p} = n\binom{n-1}{p-1} \quad \text{and} \quad \binom{n}{p} + \binom{n}{p+1} = \binom{n+1}{p+1}$$

   (for the second identity, we suppose that $p \geq 1$, and for the third that $p \leq n-1$).

2. Rewrite the following as a simple expression of n :

$$\sum_{p=0}^{n} p\binom{n}{p}.$$

Exercise 6 Let E be a set of 12 elements $\{a, b, c, d, e, f, g, h, i, j, k, l\}$.

1. How many different subsets of E of cardinality 5 can we obtain such that they contain :
   (a) a and b ;
   (b) a but not b ;
   (c) b but not a ;
   (d) neither a, nor b.

   Write your results using binomial coefficients.

2. Prove the following indentity :

$$\binom{12}{5} = \binom{10}{3} + 2\binom{10}{4} + \binom{10}{5}$$

3. We begin to study a set of n elements, n being a natural number greater than 2. We fix two of its elements, a and b. Conclude from the previous question that for all integers p, $2 \leq p \leq n$,

$$\binom{n}{p} = \binom{n-2}{p-2} + 2\binom{n-2}{p-1} + \binom{n-2}{p}$$

4. Prove the previous formula again, this time using Pascal's triangle.

Exercise 7 Let A be the set of 7-digit numbers (in base 10) that do not contain any "1". Find the cardinality of the following sets :

1. A
2. A_1, the subset of A containing numbers written using 7 different digits
3. A_2, the subset of even numbers in A
4. A_3, the subset of A defined such that the digits of each number increase strictly from left to right.

Exercise 8 (Examples of applications of Newton's binomial theorem)

1. (a) Let $E = \{1, 2, 3, 4, 5\}$. Write down all subsets of E and check that $\mathrm{Card}(\mathcal{P}(E)) = 2^5$.

   (b) Let $k \leq n$ be two natural numbers, where $n \geq 1$. Recall the meaning of $\binom{n}{k}$ (or C_n^k) and its expression. Using Newton's binomial theorem, show that if F is a set of n elements, then $\mathrm{Card}(\mathcal{P}(F)) = 2^n$.
2. Find the answer to Question 2 of Exercice 5 using Newton's binomial theorem, by writing the algebraic expansion of powers of the polynomial $(1 + x)^n$.

Probabilities. Spaces and events.

Exercise 9 Three balls are drawn in succession from an urn containing only white and red balls. We define the following events :

A : "the first ball is white"
B : "the second ball is white"
C : "the third ball is white"

Write the following events as functions of A, B, and C :

D : "all balls are white"
E : "the first two balls are white"
F : "at least one ball is white"
G : "only the third ball is white"
H : "exactly one ball is white"
I : "at least two balls are white"
J : "none of the balls are white".

Exercise 10 Let Ω be a set and A, B, C three subsets of Ω. Write each of the following events with the help of intersections, unions and complements :

"A and B hold, but not C" ;
"at least one of the events A, B, C holds" ;
"at most one of the events A, B, C holds" ;
"none of the three events A, B, C holds" ;
"all three events A, B, C hold".

Exercise 11 Imagine we could roll a standard 6-sided die an infinite number of times. For all $i \in \mathbb{N}^*$, we write :

$$A_i := \{\text{The } i\text{th die roll reads } 6\}$$

1. Without using any maths language, explain what each of the following events describe :
$$A_{10}^c, \qquad A_1 \cup A_2, \qquad \cap_{i=1}^{20} A_i^c.$$
2. Without using any maths language, explain what each of the following events describe :
$$\cap_{i=5}^{\infty} A_i, \qquad \left(\cap_{i=1}^{4} A_i^c\right) \cap \left(\cap_{i=5}^{\infty} A_i\right), \qquad \cup_{i\geq 4} A_i.$$
3. Write using A_i's the following event : "we obtain at least one 6 after n rolls".
4. Let $C_n := \cup_{i\geq n} A_i$. Show that the sequence $(C_n)_{n\geq 1}$ is decreasing (*i.e.* for all $n \geq 1$, C_{n+1} is contained in C_n).
5. Without using any maths language, give a definition of the event
$$C := \cap_{n\geq 1} C_n.$$

Exercise 12 Write down Ω the sample space (or set of possible outcomes) of the experiments described below. When the set Ω is finite, mention its cardinality.

1. Flipping a coin.
2. Successively flipping a coin N times ($N \in \mathbb{N}^*$).
3. Two successive rolls of a fair 6-sided die, where the outcome is the sum of the numbers obtained.
4. Rolling a die an infinite number of times.
5. Random walk on $\mathbb{Z}$: we start at 0 and at each step either go once forward or once backward. The total number of steps counts 5.

Exercise 13 Exercise from the Contrôle Continu n°1, 2021-2022 We roll a fair 6-sided die 5 times in a row.

1. Define a probability $(\Sigma, \mathbb{P})$ that models the experiment above.

2. Determine the probabilities of the following events (you may express them as fractions) :
   (a) A : "no 6 is ever obtained"
   (b) B : "we obtain at least one 6"
   (c) C : "we obtain 6 exactly two times"
   (d) D : "any 6 is immediately followed by another 6."

Exercise 14 Let $n \geq 1$. Define a probability on $\{1, \ldots, n\}$ such that the probability of $\{1, \ldots, k\}$ is proportional to k^2, for all $k \in \{1, \ldots, n\}$.

Exercise 15 Let $(\Omega, \mathbb{P})$ be a probability space, and A and B two events.

1. Write the probabilities $\mathbb{P}(A^c)$, $\mathbb{P}(A \cap B^c)$, $\mathbb{P}(A \cup B)$, $\mathbb{P}(A \cup B^c)$ as functions of $\mathbb{P}(A)$, $\mathbb{P}(B)$ and $\mathbb{P}(A \cap B)$.
2. In this question we suppose that $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$. In other words, A and B are independent. Show that the pairs A and B^c, then A^c are B^c are as well.
3. Let C be a different event. Show that

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cup B) - \mathbb{P}(B \cup C) - \mathbb{P}(C \cup A) + \mathbb{P}(A \cup B \cup C).$$

Exercise 16 Let $(\Omega, \mathbb{P})$ be a probability space, and $(A_i)_{i \geq 1}$ the events such that $\mathbb{P}(A_i) = 1$, $\forall i \geq 1$. Compute $\mathbb{P}(\cap_{i=1}^{\infty} A_i)$.

Conditional Probability. Independence.

Exercise 17 (Exercise from the Contrôle Continu n°1 of 2021-22) A student answers a *multiple choice question* with 5 suggested answers. Only one answer is correct. The student knows (and therefore gives) the correct answer with a probability $p \in [0, 1]$. If they do not know the answer, they choose exactly one of the 5 suggested answers at random with uniform probability. We denote K the event "the student *knows* the correct answer" and G the event "the student *gives* the correct answer".

1. Determine $\mathbb{P}(G)$ (as a function of p).
2. The professor reads the student's copy and sees that the question has been answered correctly. What is the probability that the student really knows the correct answer ?

Justify your answer and quote the precise results you use.

Exercise 18 The 2022 men's football cup takes place in Qatar and France is paricipating. We suppose that the probability of France winning its first match is 1/4. If France wins the first match, the probability that it wins the second match is 1/2. If the first match is lost, the French team can win the second match with a probability of 1/10. To simplify our model, we assume that no match can end in a tie.

1. What is the probability that France wins the second match ?
2. If France wins its second match, what is the probability that it has won its first ?
3. What is the probability that France wins both matches ?

Exercise 19 We consider n urns, numbered 1 to n. For all k, $1 \leq k \leq n$, the urn n°k contains k white balls and $n-k$ black balls. We choose an urn at random and we draw two balls with replacement (*i.e.* when a ball is drawn, it is then put back into the urn before the second draw takes place).

1. Determine the probability of drawing two white balls.
2. Given that the two balls which were drawn were white, determine the probability that the draw was made from urn n°k.

Exercise 20 Let $(\Omega, \mathbb{P})$ be a probability space and A and B two events such that $\mathbb{P}(A) = 1/5$ and $\mathbb{P}(A \cup B) = 1/2$.

1. Determine the probability $\mathbb{P}(B)$ in each of the following cases :
   (a) A and B are mutually exclusive,
   (b) A implies B,
   (c) A and B are independent.
2. Construct an example of a probability space $(\Omega, \mathbb{P})$ and events A, B such that $\mathbb{P}(A) = 1/5$, $\mathbb{P}(A \cup B) = 1/2$ and A and B are mutually exclusive.

Exercise 21 A couple has 3 children. We assume that each child is either a girl or a boy, independently of the other children's designation. Consider the events

A : "all children are the same gender",

B : "there is at least one child who is boy",

C : "there is at least one child who is a boy, and one child who is girl".

Are the events A and B independent ? How about B and C ? Or A and C ?

Exercise 22 Let $(\Omega, \mathbb{P})$ be a probability space and A, B, C three independent events. Show that A is independent of $B \cup C$.

Exercise 23 (Birthday problem) N students born in 2001 take the same Probability class. Assume that birthdays are randomly distributed among the 365 days. Write a model of the corresponding experiment and show that the probability of at least two students sharing the same birthday is equal to

$$1 - \frac{365!}{(365-N)!365^N} \quad \text{if } N \leq 365.$$

Exercise 24 We roll a random number $N \in \mathbb{N}^*$ of fair 6-sided dice. We call A_i the event $\{N = i\}$ and suppose that $\mathbb{P}(A_i) = 2^{-i}$, $\forall i \geq 1$. We denote by S the sum of the obtained results.

1. Show that $(A_i, i \geq 1)$ is a partition of Ω (which we do not need to define explicitly) and that $\sum_{i \geq 1} \mathbb{P}(A_i) = 1$.
2. Determine the probability that $S = 4$.
3. Determine the probability that $N = 2$, given that $S = 4$.
4. Determine the probability that $S = 4$, given that N is even.
5. Determine the probability that $N = 2$, given that $S = 4$ and that the first die roll reads 1.

Exercise 25 (Exercise from the Partiel 1 of 2021-22) We consider an infinite sequence of *fair* coin flips, each flip being independant of all the rest. For all $n \in \mathbb{N}^*$, we denote by F_n the event "we get Heads on the nth launch". These events are therefore independent by the previous hypothesis.

1. We introduce the event $B_n := F_{n-2}^C \cap F_{n-1}^C \cap F_n^C$, for all $n \geq 3$.
   (a) Describe, in one phrase, the events B_3, B_4, and B_n for all $n \geq 3$.
   (b) Determine $\mathbb{P}(B_n)$.
   (c) Show that the events B_n, B_{n+1} and B_{n+2} are all pairwise disjoint.
2. Consider the events $U_n := \cup_{i=3}^n B_i$, $n \geq 3$, and their probabilities $u_n := \mathbf{P}(U_n)$, $n \geq 3$, as well as $U_\infty := \cup_{i=3}^\infty B_i$ and $u_\infty := \mathbb{P}(U_\infty)$.
   (a) Describe, in one phrase, the event U_∞.
   (b) Determine u_3, u_4 and u_5.
   (c) Show that the sequence $(u_n)_{n \geq 3}$, is monotone and converges to u_∞.
   (d) In the following, $n \geq 5$.
      (i) Show that $U_n \cap B_{n+1} = U_{n-2} \cap B_{n+1}$. Determine $\mathbb{P}(U_n \cap B_{n+1})$ as a function of u_{n-2}.
      (ii) Express U_{n+1} as a function of U_n and B_{n+1}. Show that

$$u_{n+1} = u_n + \frac{1}{8}(1 - u_{n-2}).$$

      (iii) Determine u_∞.