Université Sorbonne Paris–Nord, 2022–23
Licence de mathématiques 2^{nd} year
Probabilities

Exercise sheet n°2 : Discrete random variables

Random variables with finite range

Exercice 1 We consider a tricked 6-sided die, its faces numbered from 1 to 6. The probability of rolling any number from 1 to 6 is proportional to the number itself. We roll the die and define X as the random variable corresponding to the result.

1. Find the probability density function of X.
2. Calculate $\mathbb{E}[X]$.
3. Calculate $\mathbb{E}[1/X]$.

(We recall that finding the probability density function of X means : 1) determining the range of X and 2) calculating $\mathbb{P}(X = x)$ for all elements x in the range of X.)

Exercice 2 Let X be a uniform-distributed random variable taking values in $\{0, 1, \ldots, k\}$ where $k \in \mathbb{N}$. Suppose that $\mathbb{E}[X] = 6$. Find k.

Exercice 3 Let A and B be two planes with 4 and 2 engines, respectively. We suppose that the engines function independetly of one another. Each engine has can break down with probability $p \in [0, 1]$. A plane can reach its destination if at least half its engines are still functional. Which plane do you choose ? (Your argument should depend on the values of p.)

To start, denote by X the number of engines which break down for plane A, and Y the correspoding number of engines for B. What is the name of the probability density functions of X and Y ?

Exercice 4 Draw a card randomly from a deck of 32 cards ; the deck contains the cards numbered 7, 8, 9, Jack, Queen, King, Ace, of all four suits (spade, heart, diamond, clubs). Let X be a random variable defined as :

- $X = -1$ if the card you draw is 7, 8, 9 or 10 ;
- $X = 1$ if the card you draw is a Jack, Queen or King ;
- $X = 2$ if the card you draw is an Ace.

1. Find the probability density function of X.
2. Calculate the expected value and the variance of X.
3. Calculate $\mathbb{P}(X \geq 0)$, then $\mathbb{P}(X = 2 | X \geq 0)$.
4. Set $Y = 2X - 1$. Calculate the expected value and the variance of Y.

Exercice 5 A survey comprises 10 questions, each of which can be answered with "True" or "False". A student answers all of the 10 questions randomly (to each question, they answer "True" with a probability of 1/2 and "False" with probability 1/2, independently). We write X the total number of correct answers.

1. What is the probability density function of X ?
2. Calculate the probability of the following events :

   A="the student correctly answers all questions",

   B="the student correctly answers exactly 6 questions",

   C="the student correctly answers at least 2 questions".
3. Each correct answer is worth 1 point and each wrong answer is penalised by $-1/2$ points. Let Y be the grade obtained by the student. Write Y as a function of X and find the expected value and variance of Y.

Exercice 6

1. Let $n \in \mathbb{N}$. For all real numbers x, y, we take

$$F(x,y) := \sum_{k=0}^{[n/2]} \binom{n}{2k} x^{2k}(1-y)^{n-2k} \text{ and } G(x,y) := \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} x^{2k+1}(1-y)^{n-2k-1}$$

   (where $[r]$ denotes the floor function r).

   Calculate $F(x,y) + G(x,y)$ and $F(x,y) - G(x,y)$ and find a simplified expression of $F(x,y)$.
2. Let X be a binomially distributed random variable $\mathrm{Bin}(n,p)$. Calculate the probability of the event "X is even".

Exercice 7 Let X be a binomially distributed random variable $\mathrm{Bin}(n,p)$, where $n \in \mathbb{N}^*$, $p \in [0,1]$. We define the random variable Y by : $Y = X$ if $X \neq 0$ and Y takes (uniformly) random value in $\{0, 1, \ldots, n\}$ if $X = 0$.

1. Recall the expression of $\mathbb{P}(X = k)$ for $k \in \{0, 1, \ldots, n\}$, then the value of $\mathbb{E}[X]$.
2. Calculate $\mathbb{P}(Y = i | X = k)$ for all pairs of integers $(i,k) \in \{0, 1, \ldots, n\}^2$.
3. Find the probability density function of Y.
4. Calculate the expected value of Y.

Random variables with countable range

Exercice 8 Let X be geometric distribution of parameter $2/3$.

1. Compute $\mathbb{P}(X > n)$, for all $n \in \mathbb{N}$.
2. Determine $\mathbb{E}[X]$ (show how you compute it).
3. Give the value of $Var(X)$ (without additional calculations; it suffices to use the formula seen in class).
4. Compute the probability that X takes an even value.

Exercice 9 (Memoryless process) Let $G(p)$ be a geometric distribution of parameter $p \in]0, 1[$.

1. Show that

$$\mathbb{P}(G(p) > k + n \mid G(p) > n) = \mathbb{P}(G(p) > k), \quad \forall k, n \in \mathbb{N}.$$

   Interpret this result.

2. Conversely, show that if X is a random variable taking values in $\mathbb{N}^*$ such that $\mathbb{P}(X > n) > 0$ for all $n \in \mathbb{N}$ and

$$\mathbb{P}(X > k + n \mid X > n) = \mathbb{P}(X > k), \quad \forall k, n \in \mathbb{N}$$

   then X is a geometric distribution. Start by considering a function $H(n) = \mathbb{P}(X > n)$ and establish a relationship between $H(n + k)$, $H(n)$ and $H(k)$.

Exercice 10 Let a be a real number and X be a random variable with values in $\mathbb{N}$ such that : for all $k \in \mathbb{N}$,

$$\mathbb{P}(X = k) = \frac{a}{2^k k!}.$$

1. Determine a. What is the name of this random variable?
2. What is the most probable value of X?
3. Calculate the expected value of X. (Just as during the lecture)
4. Calculate the expected value of $X(X - 1)$.
5. Calculate the variance of X from the previous two questions.
6. Calculate $\mathbb{E}[1/(X + 1)]$.

Exercice 11 We flip a tricked coin. Each time, the probability of getting tails is 2/3 and the probability of face is 1/3. The coin flips are independent. Let X be a random variable equal to the number of flips necessary to obtaining tails for the first time, and Y the random variable equal to the number of flips necessary to obtaining face two consecutive times.
Let n be a positive natural number. Let p_n be the probability of the event $\{Y = n\}$.

1. What is the name of the random variable X ? What is its expected value ?
2. Find the values of p_1, p_2, p_3, p_4 and p_5.
3. Using the law of total probabilities and distinguishing two cases depending on the result of the first flip, show that

$$\forall n \geq 3, \quad p_n = \frac{2}{9}p_{n-2} + \frac{1}{3}p_{n-1}.$$

4. Calculate $\mathbb{E}[Y]$ (you do not need to explicitely compute the values of p_n, $n \geq 6$).

Exercice 12 The number of times the website of the Université Sorbonne Paris-Nord is accessed during a period T (in hours) is a Poisson distribution of parameter $\lambda(T)$. We know that $\lambda(1) = 12,2$.

1. Calculate the probability that the website is not accessed at all during one hour.
2. Calculate the probability that there are at least two connections during an hour.
3. Let X_1 be a number of connections during a day between 14h and 15h and let X_2 be the number of connections on the same day between 15h and 16h.
   (a) Calculate $\mathbb{E}[X_1 + X_2]$.
   (b) Find $\lambda(2)$.
   (c) Are the events $\{X_1 = 0\}$ and $\{X_2 = 0\}$ independent ?

Problems

Problem 1 (Partiel 1 2019-2020). An urn initially contains a white ball and a black ball.

Part I. We draw one ball from the urn (each ball has the same probability of being drawn), we take a note of its colour, and we return it to the urn. We repeat this process n times ($n \geq 2$), and we thus draw n times from the urn, with replacement.

1. Let X be a random variable equal to the number of white balls obtained during the n draws. What is the name of the density function of X ? (Justify your answer.) Find its expected value and variance.

2. Let $Y := 0$ if the n balls that are drawn are black and $Y := k$ if we obtain a white ball for the first time on the kth draw, $1 \leq k \leq n$. Show that $\mathbb{P}(Y = k) = (1/2)^k$ for $1 \leq k \leq n$, and calculate $\mathbb{P}(Y = 0)$.
3. For $x \neq 1$, recall the result of $\sum_{k=0}^{n} x^k$ and show that

$$\sum_{k=1}^{n} kx^k = \frac{nx^{n+2} - (n+1)x^{n+1} + x}{(1-x)^2}.$$

4. Deduce that $\mathbb{E}[Y]$.

Part II. We start again. This time, every time we draw a ball, we note its colour, put it back, **and add another ball of the same colour to the urn**. We repeat this process n times, $n \geq 2$ (at the end of the n steps we have $2 + n$ balls in the urn). For $1 \leq i \leq n$, we consider the random variable X_i defined by

$$X_i = 1 \text{ if we obtain a white ball at the } i\text{th draw} \quad \text{and} \quad X_i = 0 \text{ otherwise.}$$

1. Give the probability density function of X_1.
2. Find the conditional probabilities $\mathbb{P}(X_2 = 1|X_1 = 0)$, $\mathbb{P}(X_2 = 0|X_1 = 0)$, $\mathbb{P}(X_2 = 1|X_1 = 1)$, and $\mathbb{P}(X_2 = 0|X_1 = 1)$.
3. Find the probability density function of X_2.
4. Let $1 \leq m \leq n$. We introduce the random variable $Z_m := \sum_{i=1}^{m} X_i$. What does this random variable represent ? Calculate $\mathbb{P}(X_{m+1} = 1|Z_m = k)$ for all $0 \leq k \leq m$.
5. Using the law of total probabilities, show that

$$\mathbb{P}(X_{m+1} = 1) = \frac{1 + \mathbb{E}[Z_m]}{2 + m}, \qquad \text{for all } 1 \leq m \leq n-1.$$

6. Find the probability density function of X_m for all $1 \leq m \leq n$.

Problem 2. A breakfast cereal company sells cereal boxes which contain puzzle pieces. The puzzle has n pieces. The pieces are assigned through a uniform, independent law, one piece for every box. Amélie decides to buy this specific brand of cereal boxes until she gets all the puzzle pieces. For all $k \geq 2$, we write Y_k the number of boxes she should buy to obtain the kth new piece, counted from the moment she found the $(k-1)$th piece. Let X_n be the total number of boxes she purchased.

1. What is the probability density function of Y_k ? Express X_n as a function of Y_k. Write $\mathbb{E}[X_n]$ as a sum.
2. Show that

$$\int_k^{k+1} \frac{\mathrm{d}x}{x} \leq \frac{1}{k} \leq \int_{k-1}^{k} \frac{\mathrm{d}x}{x}, \quad \forall k \geq 2.$$

Find the equivalent of $\mathbb{E}[X_n]$ when $n \to +\infty$.