Université Sorbonne Paris-Nord, 2022–23
Licence de mathématiques 2ème année
Probabilities

Exercise sheet n°3 : Discreet random variables

Random variables

Exercise 1 Let X and Y be two random variables such that X takes values in $\{1,2,3,4\}$ and Y takes values in $\{1,2\}$. We define the joint probability distribution (X,Y) by

$$\mathbb{P}(X=i, Y=j) = \frac{1}{7}\left(1 - \frac{ij}{30}\right), \quad \forall (i,j) \in \{1,2,3,4\} \times \{1,2\}.$$

1. Check that $\mathbb{P}$ is well-defined as a probability distribution function.
2. Find the marginal probability distributions.
3. Find the conditional law of X given $Y=1$ and of Y given $X=2$.

Exercise 2 Let X and Y be two random variables. Recall the necessary conditions for the existence of the linear correlation coefficient $\rho(X,Y)$. We set $U = aX+b$ and $V = cY+d$, where a,b,c,d are real numbers. Compute $\rho(U,V)$ as a function of $\rho(X,Y)$.

Exercise 3 Suppose a group of n people split themselves at random between the 3 hotels of a village, knowing that each hotel has at least n vacancies. Let X, Y, Z be the number of people choosing hotel A, B and C, respectively.

1. What are the probability distributions of X, Y and Z?
2. Express $X+Y$ as a function of n and Z. Deduce the value of $\mathrm{Var}(X+Y)$.
3. Without computing $\mathbb{E}[XY]$, find the value of $\rho(X,Y)$.

Exercise 4 Let (X,Y) be a joint random variable that acts uniformly on $\{0,\dots,n\}^2$.

1. Recall the definition of this probability distribution.
2. Find the probability distribution laws of X and Y.

3. Are X and Y independent?

Exercise 5 The probability distribution function of the joint random variable (X, Y) is described in the following table:

$X \backslash Y$	-1	1
0	$1/4$	$p + 1/8$
1	$1/2$	$-p + 1/8$

1. What conditions does p need to verify?
2. Compute the probability distribution function of X, $\mathbb{E}[X]$ and $\mathrm{Var}(X)$ as functions of p.
3. Compute the probability distribution function of Y, as well as $\mathbb{E}[Y]$ and $\mathrm{Var}(Y)$.
4. Find the conditional probability distribution function of X given $Y = -1$.
5. Find the covariance of (X, Y) as a function of p.
6. For which value(s) of p does the couple (X, Y) have null correlation?
7. Are there multiple values of p such that X and Y are independent?

Exercise 6 We roll two fair 6-sided dice. Let X be the larger result, and Y the smaller one.

1. Find the probability distribution function of the couple (X, Y), then of X and Y.
2. Compute $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.
3. Compute $\mathrm{Var}(X)$, $\mathrm{Var}(Y)$ and $\mathrm{Cov}(X, Y)$.
4. Find the probability distribution of $X + Y$.
5. Compute $\mathrm{Var}(X + Y)$ in two different ways.

Exercise 7 (Exercise from Partiel 2 of 2020–21) Let (X, Y) be a couple such that the joint distribution function is given by: $(X, Y)(\Omega) = \mathbb{N}^* \times \mathbb{N}^*$ and

$$\mathbb{P}(X = i, Y = j) = a^{i+j-2}(1-a)^2, \quad \forall (i, j) \in (X, Y)(\Omega),$$

where $a \in]0, 1[$ is a fixed parameter.

1. Show that the probability density function is well defined on $\mathbb{N}^* \times \mathbb{N}^*$.
2. Find the probability distribution functions of X and Y.

3. Are the random variables X and Y independent?
4. Find the probability density function of $X+Y$.
5. Let $n \in \mathbb{N}^*$. Find the conditional probability density function of X given $\{X+Y = n\}$.

Exercise 8 Let (X, Y) be a couple taking values in $\mathbb{N} \times \mathbb{N}$, where the probability density function is given by:

$$\mathbb{P}(X = i, Y = j) = \frac{1}{e} \frac{(i+j)\lambda^{i+j}}{i!j!}, \forall i \in \mathbb{N}, j \in \mathbb{N}.$$

1. Compute λ.
2. Find the marginal distributions X and Y.
3. Are X and Y independent?
4. Find the probability distribution law of $X+Y$, then compute $\mathbb{E}[2^{X+Y}]$.

Exercise 9 (Exercice from Partiel 2 of 2019–20) Let (X, Y) be a couple of discrete random variables such that : $X(\Omega) \times Y(\Omega) = \mathbb{N} \times \mathbb{N}$ and

$$\mathbb{P}(X = i, Y = j) = \frac{\alpha}{(i+j+1)!}, \qquad \forall (i, j) \in \mathbb{N} \times \mathbb{N},$$

where α is a strictly positive real number which we will sepcify later.

1. Let $S = X+Y$. Write $S(\Omega)$ and for all $k \in S(\Omega)$, compute the probability $\mathbb{P}(S = k)$ as a function of α and k.
2. Name the distribution S and deduce the value of α. What is the expected value of S?
3. Explain why the marginal distributions X and Y are the same, *without explicitely computing them.* Find the expected value of X.
4. Compute $\mathbb{P}(X = 0)$.
5. Are the random variables X and Y independent?

   (You can use the approximation $e^{-1} \sim 0,368$.)
6. Compute $\mathbb{P}(X = Y)$, then $\mathbb{P}(X > Y)$.

Exercise 10 (Exercise, based on the Partiel 2 of 2021–22.) Let $(X_i)_{i\geq 1}$ be a sequence of independent random variables, such that for all $i \geq 1$, X_i is a Bernoulli trial of parameter $p \in]0, 1[$. For all $i \geq 1$, let $Y_i := X_i X_{i+1}$.

1. What is the probability distribution Y_i?
2. Let $S_n = \sum_{i=1}^{n} Y_i$. Compute $\mathbb{E}[S_n]$.
3. Find $\mathrm{Var}(S_n)$.

Exercise 11 We flip infinitely many times a coin made in such a way that we obtain Tails with a probability p ($0 < p < 1$) and Heads with a probaility $q = 1 - p$. We call a "sequence" a succession of Tails (or Heads) that is interrupted by the opposite outcome. The "length" of the sequence is the number of Tails (or Heads) of which it is composed. For example, in the event TTHHHTHHH..., the first sequence is a Tails sequence of length 2, the second sequence is a Heads sequence of length 3, etc. We denote by X the random variable equal to the length of the first sequence if its length is finite, and equal to 0 otherwise. In the same way, let Y be the random variable associated to the length of the second sequence.

1. Find the probability distribution of X.
2. Find the joint probability distribution of (X, Y).
3. Deduce the probability distribution of Y.

Two classical inequalities

Exercise 12 Let $(X_i)_{i\geq 1}$ be a sequence of independent random variables sharing the same probability distribution, given by $X_i(\Omega) = \{-1, 1\}$, $\mathbb{P}(X_i = 1) = \frac{1}{4}$ and $\mathbb{P}(X_i = -1) = \frac{3}{4}$, $\forall i \geq 1$. For all $n \in \mathbb{N}^*$, let $S_n = \sum_{i=1}^{n} X_i$.

1. Compute $\mathbb{E}[X_1]$ and $\mathrm{Var}(X_1)$.
2. Deduce $\mathbb{E}[X_n]$ and $\mathrm{Var}(X_n)$.
3. Show that for all $\delta > 0$,

$$\mathbb{P}\left(\left|S_n + \frac{n}{2}\right| \geq n\delta\right) \to 0, \quad \text{when } n \to 0.$$

   How do we interpret this result?

Exercise 13 (Exercise from the Partiel 2 of 2020–21.) Consider a family of urns indexed by strictly positive integers: for $k \in \mathbb{N}^*$, the urn number k containts k balls numbered 1 to k. The player draws in succession (and randomly, uniformly and independently) one ball from the urn number 1, then one ball from the urn number 2, then one ball from the urn number 3, etc.
We denote X_k the number of the ball drawn from the urn number k: all the random variables $X_k, k \geq 1$ are independent, with X_k the uniform distribution on $[\![1;k]\!]$.
We recall that $\sum_{i=1}^k i = \frac{k(k+1)}{2}$ and $\sum_{i=1}^k i^2 = \frac{k(k+1)(2k+1)}{6}$.

1. Compute $\mathbb{E}[X_k]$ for all $k \in \mathbb{N}^*$.
2. Compute $\mathrm{Var}(X_k)$ for all $k \in \mathbb{N}^*$.

For $n \in \mathbb{N}^*$, let $S_n := \sum_{k=1}^n X_k$.

3. Show that $\mathbb{E}[S_n] = \frac{n(n+3)}{4}$.
4. Show that $\mathrm{Var}(S_n) \leq \frac{n^3}{12}$.
5. (i) Recall the inequality of Bienaymé-Tchébychev.
   (ii) Let $\varepsilon > 0$. Show that $\mathbb{P}\left(\frac{S_n}{n^2} \in \left]\frac{1}{4} - \varepsilon; \frac{1}{4} + \varepsilon\right[\right) \underset{n\to\infty}{\longrightarrow} 1$.

Exercise 14 Let X be a positive random variable such that $0 < \mathbb{E}[X^2] < \infty$. Show that

$$\mathbb{P}(X > 0) \geq \frac{\mathbb{E}[X]^2}{\mathbb{E}[X^2]}.$$

Indication : note that $\mathbb{E}[X] = \mathbb{E}[X\mathbf{1}_{\{X>0\}}]$.