RECOLLECTIONS ON TOPOLOGY

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CONTENTS

1. TOPOLOGICAL SPACES

Definition 1.1 (Topological space)**.** A *topology* on a set X is a collection $\mathcal{T} \subset \mathcal{P}(X)$ of subsets of X satisfying

(1) $\varnothing, X \in \mathcal{T}$,
(2) $\bigcup_{i\in I} O_i \in \mathcal{T}$, for any set I and for $O_i \in \mathcal{T}$, $i \in I$,
(3) $\bigcap_{i\in I} O_i \in \mathcal{T}$, for any finite set I and for $O_i \in \mathcal{T}$, $i \in I$.

The elements O of $\mathcal{T}$ are called the *open sets* and their complements are called the *closed sets*. The data $(X, \mathcal{T})$ is called a *topological space*. (When the topology is obvious, we will use just the notation X of the underline set to refer to the topological space.)

REMARK 1.2. One can equivalently define a topology by the data of the closed subsets satisfying the axiom (1), the stability under any intersection and the stability under any finite union.

EXAMPLES 1.3.

- ◇ Normed vector spaces,
- ◇ Metric spaces,
- ◇ The *trivial* topology $(X, \mathcal{P}(X))$,
- ◇ The *discrete* topology $(X, \{\varnothing, X\})$.

The inclusion defines a partial order on the set of topologies of a given set X.

Definition 1.4 (Finer topology)**.** A topology $\mathcal{T}_2$ on a set X is said to be *finer* than another one $\mathcal{T}_1$ when $\mathcal{T}_1 \subset \mathcal{T}_2$. (In the other way round, the topology $\mathcal{T}_1$ is *coarse* than the topology $\mathcal{T}_2$.)

EXAMPLE 1.5. The discrete topology $\mathcal{P}(X)$ is finer than the trivial topology $\{\varnothing, X\}$.

Definition 1.6 (Continuous map)**.** A set theoretical map $f\colon X \to Y$ between two topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ is *continuous* when

$$f^{-1}(O) \in \mathcal{T}, \text{ for any } O \in \mathcal{S} .$$

It is equivalent to ask that the pre-image of any closed set of the target is a closed set of the source.

Proposition 1.7 (Stability under composition)**.** *Continuous maps are stable under composition.*

Date: September 22, 2024.

Proof. The proof is straightforward and thus left to the reader. □

Definition 1.8 (Homeomorphism)**.** A continuous map $f\colon (X, \mathcal{T}) \to (Y, \mathcal{S})$ between two topological spaces is an *homeomorphism* when it is invertible and when its inverse is also continuous.

Two topological spaces are *homeomorphic* when there exists an homeomorphism between them. This defines an equivalence relation on topological spaces.

Definition 1.9 (Compact space)**.** A topological space $(X, \mathcal{T})$ is called *compact* when every open cover of X

$$X = \bigcup_{i\in I} O_i \ , \ \text{for } O_i \in \mathcal{T} \ ,$$

admits a finite subcover

$$X = \bigcup_{j=1}^{n} O_{i_j} \ , \ \text{for } i_j \in I \ .$$

This axiom is called the *Borel-Lebesgue property*. In France, for instance in Bourbaki, a compact space is moreover required to be *separated* (or *Hausdorff*): for any pair of different points $x \neq y \in X$, there exists two disjoint open sets O and U such that $x \in O$ and $y \in U$.

2. Product topology

Definition 2.1 (Product topology)**.** The *product topology* on the cartesian product $X \times Y$ of two topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ is the coarse topology among the ones for which the canonical projections $\pi_X\colon X\times Y \to X$ and $\pi_Y\colon X \times Y \to Y$ continuous.

This topology is well-defined: for instance, it is given by the intersection of all the topologies for which the canonical projections are continuous.

Proposition 2.2 (Explicit description)**.** *The product topology is given by the unions of products of open sets:*

$$\left\{ \bigcup_{i\in I} O_i \times U_i,\ O_i \in \mathcal{T}, U_i \in \mathcal{S},\ \text{for any } i \in T \right\} \ .$$

Proof. The proof was performed in class; one can do it by "Analyse-SynthŔse". □

Corollary 2.3. *Let X, Y, and Z be three topological spaces. Any map $f\colon X \to Y \times Z$ is continuous if and only if the two composites $\pi_Y \circ f\colon X \to Y$ and $\pi_Z \circ f\colon X \to Z$ with the canonical projections are continuous.*

Proof.

- **(⇒):** This is a direct corollary of the stability of continuity under composition (Proposition 1.7).
- **(⇐):** The pre-image under the composite $\pi_Y \circ f$ of any open set $\bigcup_{i\in I} O_i \times U_i$ of $Y \times Z$ is equal to

$$f^{-1}\left(\bigcup_{i\in I} O_i \times U_i\right) = \bigcup_{i\in I} f^{-1}\left(O_i \times U_i\right) = \bigcup_{i\in I} \underbrace{\left(\pi_Y \circ f\right)^{-1}\left(O_i\right)}_{\text{open set of } X} \cap \underbrace{\left(\pi_Z \circ f\right)^{-1}\left(U_i\right)}_{\text{open set of } X},$$

which is an open set of X since the two composites $\pi_Y \circ f\colon X \to Y$ and $\pi_Z \circ f\colon X \to Z$ are continuous.

□

Theorem 2.4 (Product of compact sets, Tychonoff 1935)**.** *The product of any collection of compact topological spaces is compact.*

Proof. The proof of this statement is more involved than the previous ones, so we skip it. □

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