WORKSHEET

Definition (Euler characteristic of a simplicial complex). The *Euler characteristic* of a finite 2-dimensional simplicial complex X is equal to

$$\chi(X) := |J_0| - |J_1| + |J_2| ,$$

where J_0, J_1, J_2 stand respectively for the sets of vertices, edges, and triangles of X.

Exercise 1. Let X and Y be two finite 2-dimensional simplicial complexes that are surfaces. Show that they have the same Euler characteristic $\chi(X) = \chi(Y)$ when they are homeomorphic.

Definition (Euler characteristic of a surface). The *Euler characteristic* of a compact connected surface S is defined by the Euler characteristic of any of its triangulation X:

$$\chi(S) := \chi(X) .$$

The previous question shows that this notion is well defined.

Exercise 2.

(1) Compute the Euler characteristic of the sphere $\mathbb{S}^2$, the torus $\mathbb{T}$, and the real projective surface $\mathbb{P}^2\mathbb{R}$.

(2) Prove that $\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2$.

(3) Compute the Euler characteristics of the connected sums $\mathbb{T}^g := (\mathbb{T})^{\# g}$ and $\left(\mathbb{P}^2(\mathbb{R})\right)^{\# k}$, for $g, k \geqslant 1$.

(4) What can you conclude regarding the proof of the classification of compact connected surfaces?
