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Introduction by Geoffroy

Étale fundamental group

X space, $x \in X$

$$\begin{tikzcd} \mathrm{Cov}(X) \arrow[rr, "\text{fibre at } x"] \arrow[dr, "\cong"'] & & \mathrm{Set} \\ & \mathrm{Set}^{\pi_1(X,x)} \arrow[ur, "\text{forget}"'] & \end{tikzcd}$$

covering spaces over X

Let G = group of automorphisms of F

$$\begin{tikzcd} \mathrm{Cov}(X) \arrow[rr, "\text{fibre at } x"] \arrow[dr, "\cong"'] & & \mathrm{Set} \\ & \mathrm{Set}^{G} \arrow[ur, "\text{forget}"'] & \end{tikzcd}$$

actually $G \cong \pi_1(X,x)$

Variant. finite cov. spaces

$$\begin{tikzcd} \mathrm{Cov}_f(X) \arrow[rr, "\text{fibre at } x"] \arrow[dr, "\cong"'] & & \mathrm{Set}_f \\ & \mathrm{Set}_f^{\widehat{\pi_1(X,x)}} \arrow[ur, "\text{forget}"'] & \end{tikzcd}$$

(finite sets)

(continuous action)

Recall. If G a group

$$\hat{G} = \varprojlim_{\substack{H \trianglelefteq G \\ \text{s.t. } G/H \text{ finite}}} G/H$$

give it inverse limit topology.

X algebraic variety over $\mathbb{C}$

Theorem [Serre] $\mathrm{Cov}_f(X_{an}) \simeq \mathrm{Et}(X)$

only depends on X as an algebraic variety

Can construct a profinite group $\pi_1^{ét}(X,x)$

such that $\pi_1^{ét}(X,x) \cong \widehat{\pi_1(X,x)}$

Now: if X is defined over $\mathbb{Q}$ then $\pi_1^{ét}(X \times_{\mathbb{Q}} \mathbb{C}, x) \cong \pi_1^{ét}(X \times_{\mathbb{Q}} \overline{\mathbb{Q}}, x)$

(and $x : \mathrm{Spec}(\mathbb{Q}) \to X$)

$\circlearrowleft\ \mathrm{Aut}_{\mathbb{Q}\text{-alg.}}\,\mathbb{C}$ $\qquad$ $\circlearrowleft\ \mathrm{Gal}\,\overline{\mathbb{Q}}/\mathbb{Q}$

example 1) $X = \mathbb{A}^1 \setminus \{0\}$
$X_{an} = \mathbb{C} \setminus \{0\}$ $\Rightarrow$ $\pi_1^{et}(X \times_{\mathbb{Q}} \overline{\mathbb{Q}}) \cong \hat{\mathbb{Z}}$

Fact. $\chi : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \twoheadrightarrow \underbrace{\hat{\mathbb{Z}}}_{\text{automorphisms of the group of roots of unity}}$ (Surjectivity Thm)

$\longrightarrow$ the action $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\hat{\mathbb{Z}}$ is by mult. with $\chi(-)$.

example 2) $X = \mathbb{A}^1 \setminus \{0,1\}$
$X_{an} = \mathbb{C} \setminus \{0,1\}$ $\Rightarrow$ $\pi_1^{et}(X \times_{\mathbb{Q}} \overline{\mathbb{Q}}) \cong \hat{\mathbb{F}}_2$

Theorem [Belyi] The action of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\hat{\mathbb{F}}_2$ is faithful.

Variant: $X = Conf_3(\mathbb{A}^1)$. Then the action of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ on
$\hat{P}_3 = \pi_1(Conf_3 \mathbb{A}^1)^\wedge$
pure braid group on 3 strands.
is also faithful.

Namely: $\mathbb{C} \setminus \{a,b\} \longrightarrow Conf_3\mathbb{C} \longrightarrow Conf_2\mathbb{C}$
gives SES $1 \to \mathbb{F}_2 \to P_3 \to \mathbb{Z} \to 1$

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the work of Ihara.

Def. Let $\widehat{GT}$ = the monoid of automorphisms of $\hat{\mathbb{F}}_2 = \widehat{\mathbb{F}(x,y)}$ of the form

$$x \longmapsto x^\lambda \qquad y \longmapsto f^{-1} y^\lambda f$$

where $(\lambda, f) \in \hat{\mathbb{Z}}^\times \times \hat{\mathbb{F}}_2$

such that

1) $f(x,y)\, f(y,x) = 1$

2) $f(z,x)\, z^m f(y,z)\, y^m f(x,y) = 1$ where $m = \frac{\lambda - 1}{2}$

3) ...

Def Let $\widehat{GT}$ be the group of units of $\underline{\widehat{GT}}$.

Theorem [Ihara] The map $Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow Aut(\hat{\mathbb{F}}_2)$ factors through $\widehat{GT}$.

$\Rightarrow$ There is an injection $Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow \widehat{GT}$ open: surjective or not ?

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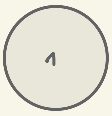
the work of Drinfeld.

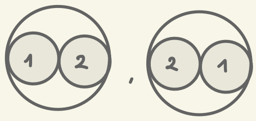
Operad of little disks D_2 has $D_2(n) = K(P_n, 1)$ (pure braids on n-strands.)

Corresponding operad in groups ? basepoint issues.

Instead $\pi_{\leq 1} D_2$ operad in groupoids s.t. $|N\pi_{\leq 1} D_2| \simeq D_2$

Assume $D_2(0) = \emptyset$

Basepts: $D_2(1) \ni x :=$ 

$D_2(2)$ $\{$  $\}$

$D_2(n) \supseteq \left\{ \begin{array}{l} \text{rooted binary planar trees} \\ \text{with leaves labelled} \\ \text{with } \{1, \ldots, n\} \end{array} \right\}$

Def. $PaB = \pi_{\leq 1}(D_2, \text{rooted binary trees})$ get $|NPaB| \simeq D_2$.

another definition. PaB is the operad that captures the definition of a braided monoidal cat. $(\mathcal{C}, \otimes)$

(namely, have assoc. isos $\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$ and braiding $\beta_{A,B} : A \otimes B \longrightarrow B \otimes A$ sat. pentagon 2 hexagon)

Def [Drinfeld] Let $\underline{GT}$ = the monoid of automorphisms of PaB fixing the objects.

($\Leftrightarrow$ fix $\otimes$ in $(\mathcal{C}, \otimes)$ but change α, β)

Let GT = the group of units of $\underline{GT}$.

For example $\forall A,B \in \mathcal{C}$ $\tau^{new}_{A,B} = \tau^{-1}_{B,A}$ $\Rightarrow$ $(\mathcal{C}^{new}, \otimes)$ a new braided monoidal cat.

This induces an automorphism of PaB of order 2.

Gives $\mathbb{Z}/2 \hookrightarrow GT$.

Theorem [Drinfeld] This is an iso.

Why care about braided monoidal cats?

Theorem [Tangle hypothesis]

The cat. of tangles is the free braided monoidal cat with duals on one object.

Prounipotent completion

Def. A $\mathbb{Q}$-unipotent group is a group that can be built from finite-dim $\mathbb{Q}$-vector spaces using finitely many central extensions

Equivalently, it is a nilpotent group s.t. the associated graded pieces in the l.c.s filtration are fin. dim. $\mathbb{Q}$-vector spaces.

Given a group G consider $\mathrm{Hom}_{grp}(G,-) : \mathrm{Grp}^{uni} \longrightarrow \mathrm{Sets}$. $*$

This functor is pro-representable:

there is a filtered cat. I and a diagram $I^{op} \longrightarrow \mathrm{Grp}^{uni}$, $i \longmapsto G_i$

s.t. $\mathrm{Hom}(G,-) \cong \mathrm{colim}_i \mathrm{Hom}(G_i,-)$

This pro-object is by definition the pro-unipotent completion of G. (terms of it as $*$)

Denote it by $\hat{G}_{\mathbb{Q}}$

Example. $(\hat{\mathbb{Z}})_{\mathbb{Q}} = \mathbb{Q}$

For fin.gen gp G explicit construction: $G^{\wedge}_{\mathbb{Q}} \cong \mathrm{Gplike}(\mathbb{Q}[G]^{\wedge}_{I})$ (Malcev completion).

For G a group or a groupoid have: $\hat{G}$ profinite completion

$\hat{G}_{\mathbb{Q}}$ pro-unipotent completion

So have: $\widehat{PaB}$ and $\widehat{PaB}_{\mathbb{Q}}$

Def. $\widehat{GT} := \mathrm{Aut}_{\text{fixing objects}}(\widehat{PaB})$ and $\widehat{GT}_{\mathbb{Q}} := \mathrm{Aut}_{\text{fixing objects}}(\widehat{PaB}_{\mathbb{Q}})$

Theorem. $\widehat{GT}$ coincides with Ihara's group.

Homotopy theory

non-completed version: $\mathrm{Aut}_{Ho(Op)}(D_2) \cong \underset{(cx\ conj)}{\mathbb{Z}/2} \cong \mathrm{Aut}_{Op}(PaB)$

profinite version: $\widehat{GT} = \mathrm{Aut}_{Op}(\widehat{PaB}) \longrightarrow \mathrm{Aut}_{HoOp}(\hat{D}_2)$

profinitely completed

Theorem [Horel]
This is an equivalence.

pro-unipotent version: $\widehat{GT}_{\mathbb{Q}} \longrightarrow \mathrm{Aut}_{HoOp}(\hat{D}_2)_{\mathbb{Q}}$

$$\mathrm{Aut}_{HoOp}(\hat{D}_2)_{\mathbb{Q}} = \mathrm{Aut}_{Ho(Hopf\ coOp)}(\Omega_{sul} D_2)$$

Theorem [Fresse]
This is an equivalence.

Drinfeld associators

Formality Theorem [Kontsevich]

$\Omega_{sul}(D_2)$ is weakly equivalent to $H^*(D_2)$.

Theorem. $\mathrm{Iso}_{Ho(Hopf\ coop)}(\Omega_{sul}(D_2), H^*(D_2))$ $\Big\}$ set of Drinfeld associators.

$\circlearrowleft \widehat{GT}_{\mathbb{Q}}$