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Lie Models in Topology

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Introduction

Rational homotopy theory is a branch of topology which studies the "non-torsion" behaviour of the homotopy type of topological spaces. This area was born about fifty years ago in pursuit, among others and broadly speaking, of the following objectives:

One of these aims was to associate functorially to each topological space another which keeps only the rational information of its homotopy type. This was outlined already by D. Sullivan in [126] and then presented by several authors for simply connected spaces or more generally, for nilpotent ones[1], see for instance [75, 107, 113, 127]. Indeed, an application of localization methods in homotopy theory gave rise to the *rationalization functor*, which assigns to each nilpotent space X the space $X_{\mathbb{Q}}$ which is characterized by being *rational* from the homotopical, or equivalently, homological point of view. That is,

$$\pi_*(X_{\mathbb{Q}}) \cong \pi_*(X) \otimes \mathbb{Q}, \qquad \text{or equivalently,} \qquad \widetilde{H}_*(X_{\mathbb{Q}}) \cong \widetilde{H}_*(X;\mathbb{Q}).$$

The notion of *completion* introduced in [13] generalizes that of localization in two ways: it can be defined for any simplicial set (and hence, for any topological space, not necessarily nilpotent) and it coincides with localization when the completion is taken on any subring of the rational numbers. This applies in particular to the *rational completion* or $\mathbb{Q}$-*completion*, a functor which assigns to a given simplicial set X another simplicial set $\mathbb{Q}_\infty X$ which is characterized by the property that a map of simplicial sets $f\colon X \to Y$ induces an isomorphism in rational homology if and only if the map

$$\mathbb{Q}_\infty f\colon \mathbb{Q}_\infty X \xrightarrow{\simeq} \mathbb{Q}_\infty Y$$

is a homotopy equivalence.

Another fundamental aim of rational homotopy theory was to characterize the homotopy type of the rationalization or $\mathbb{Q}$-completion of a space by means of algebraic models. This goal was attained simultaneously by D. Sullivan and D. Quillen in two different ways which are now classical in the theory.

[1] Recall that a space is nilpotent if it has a nilpotent fundamental group which operates nilpotently in the higher homotopy groups.

The Sullivan "commutative" approach, see [12, 128], is based on the discovery of a pair of adjoint functors

$$\mathbf{sset} \underset{\langle\,\cdot\,\rangle^{\mathrm{S}}}{\overset{A_{\mathrm{PL}}}{\rightleftarrows}} \mathbf{cdga} \tag{1}$$

that connect the categories of simplicial sets and commutative differential graded algebras (cdga's henceforth).

On the one hand, for any simplicial set X, $A_{\mathrm{PL}}(X)$ is the cdga of piecewise linear differential forms on X. Such a form assigns to each n-simplex of X a polynomial differential form in the topological n-simplex Δ^n, in a fashion compatible with the faces and degeneracies of X. Given a connected simplicial set X, the cdga of polynomial forms $A_{\mathrm{PL}}(X)$ may be replaced by a much simpler cdga, which still keeps all the "rational topological information" of X. This is the so-called *(Sullivan) minimal model of* X, which is a cdga of the form $(\wedge V, d)$, where $\wedge V$ denotes the free commutative graded algebra generated by the graded vector space V, and the differential d satisfies a certain recursive condition. The link between the minimal Sullivan model and the functor A_{PL} is established by a quasi-isomorphism (a cdga morphism inducing isomorphisms in homology),

$$(\wedge V, d) \xrightarrow{\simeq} A_{\mathrm{PL}}(X).$$

On the other hand, the realization functor is introduced, like in many other algebraic categories, as the morphisms to a "simplicial universal" object of the given category. Indeed, the realization of a given cdga A is the simplicial set

$$\langle A \rangle^{\mathrm{S}} = \mathrm{Hom}_{\mathbf{cdga}}(A, \Omega_\bullet)$$

of cdga morphisms from A to $\Omega_\bullet = A_{\mathrm{PL}}(\underline{\Delta}^\bullet)$, which is the simplicial cdga of PL-forms on the standard simplices. The simplicial structure on $\langle A \rangle$ is induced by that on $\Omega_\bullet$. In other words, we may say that the Sullivan realization functor $\langle\,\cdot\,\rangle^S$ is *representable* by the simplicial cdga $\Omega_\bullet$.

It turns out that the pair of adjoint functors in (1) induces equivalences between the homotopy categories of nilpotent simplicial sets (which are rational in the sense above and with finite type homology) and connected cdga's (these are non-negatively graded cdga's $A = \sum_{p\geq 0} A^p$ for which $A^0 = \mathbb{Q}$) with finite type minimal models.

Moreover, the techniques and ideas arising from this approach can be successfully applied to general, not simply connected, nor nilpotent spaces, see [51]. We mention an illustrative and key result in this direction: whenever X is of finite type, the Sullivan realization of its minimal model has the homotopy type of the $\mathbb{Q}$-completion of X:

$$\langle \wedge V, d \rangle^{\mathrm{S}} \simeq \mathbb{Q}_\infty X.$$

In other words, one of the adjunction maps of the above pair of functors is, up to homotopy, the $\mathbb{Q}$-completion

$$X \longrightarrow \mathbb{Q}_\infty X$$

for finite-type connected spaces.

On the other side, in his seminal paper [115], D. Quillen constructed a pair of functors,

$$\mathbf{sset}_1 \underset{\langle\,\cdot\,\rangle^Q}{\overset{\lambda}{\rightleftarrows}} \mathbf{dgl}_1 \tag{2}$$

between the category of simplicial sets with only one simplex in dimensions 0 and 1, and the category of reduced or 1-connected differential graded Lie algebras (dgl's for short), i.e., concentrated in positive degrees. Both the *Quillen model* functor λ and the *Quillen realization* functor $\langle\,\cdot\,\rangle^{\mathrm{Q}}$ are the result of a composition of other functors between several simplicial categories. In the end, each of these compositions induces an equivalence between the homotopy category of rational simply connected topological spaces of the homotopy type of CW-complexes, and the homotopy category of 1-connected dgl's.

However, in contrast to the Sullivan approach, all the functors whose composition determines the pair above need 1-connectivity in their corresponding domain and codomain categories to exist. Moreover, their extension to a more general setting is not possible for most of the functors involved.

With all of the above in mind, the present text is mainly devoted to a self-contained presentation of the Lie approach to rational homotopy for general, not necessarily simply connected, nor even connected, spaces. In this way we overcome the main restriction imposed by the classical Quillen approach.

From the topological point of view, the extension of the Quillen model functor to general spaces will have many useful application and thus, needs no further motivation. Think for instance of all possible variations of mapping spaces, which are key examples of non-connected spaces whose global algebraic modelling is even desirable without restricting to path components which, in any case, might not be even nilpotent.

On the other hand, there are many situations in a wide range of mathematics, from algebraic geometry to mathematical physics, where a suitable notion of geometrical realization of differential graded Lie algebras, not necessarily positively graded, would be most welcome. An illustrative and ubiquitous example in which unbounded dgl's are useful is given by a fundamental principle of deformation theory which we now briefly explain.

Let R be a local commutative algebra with maximal ideal $\mathfrak{M}$, let $\Bbbk = R/\mathfrak{M}$ be the residue field and let A be a graded $\Bbbk$-vector space endowed with some additional structure. An R-deformation of A is another such structure in $A \otimes_{\Bbbk} R$ that, modulo $\mathfrak{M}$, reduces to the original one in A. For instance, if A is a cdga and we choose $R = \Bbbk[[t]]$ to be the local ring of the formal power series in $\Bbbk$, an R-deformation of A is just a cdga structure on $A[[t]]$, the power series with coefficients in A, such that the constant part of a product of two series is precisely the multiplication in the original cdga.[2] In [110], A. Nijenhuis and R. Richardson outlined what is currently known as the *Deligne Principle* for deformations. As stated in a letter of Deligne to J. Millson [38], this principle asserts that, whenever $\Bbbk$ is of characteristic zero,

"Every deformation functor is governed by a dgl."

In the elaboration of this principle, essential concepts in our theory will come to light: for any given differential graded Lie algebra L, the set $\mathrm{MC}(L)$ of *Maurer–Cartan elements of* L, or simply MC elements, is the set of elements a degree -1 satisfying the classical Maurer–Cartan equation

$$da + \frac{1}{2}[a, a] = 0.$$

The vector space L_0 of degree-0 elements has a group structure with the classical *Baker–Campbell–Hausdorff product*. Moreover, this group acts as a "group of gauge transformations" on the Maurer–Cartan set $\mathrm{MC}(L)$ via the so-called *gauge action*, defined by

$$x \,\mathcal{G}\, a = \sum_{i\geq 0} \frac{\mathrm{ad}_x^i(a)}{i!} - \sum_{i\geq 0} \frac{\mathrm{ad}_x^i(dx)}{(i+1)!}, \quad \text{for} \quad x \in L_0 \quad \text{and} \quad a \in \mathrm{MC}(L),$$

in which ad_x is the usual adjoint operator, $\mathrm{ad}_x(a) = [x, a]$. The first summand is precisely $e^{\mathrm{ad}_x}(a)$, while the second can be written as $\frac{e^{\mathrm{ad}_x}-1}{\mathrm{ad}_x}(dx)$, in view of the equality $\sum_{n\geq 0} \frac{t^n}{(n+1)!} = \frac{e^t-1}{t}$. Hence, the gauge action takes the standard form,

$$x \,\mathcal{G}\, a = e^{\mathrm{ad}_x}(a) - \frac{e^{\mathrm{ad}_x} - 1}{\mathrm{ad}_x}(dx), \quad \text{for} \quad x \in L_0 \quad \text{and} \quad a \in \mathrm{MC}(L).$$

We then define

$$\widetilde{\mathrm{MC}}(L) = \mathrm{MC}(L)/\mathcal{G}$$

to be the set of Maurer–Cartan elements modulo the gauge action.

In this context, if we denote by $\mathrm{Def}(A; R)$ = the set of (isomorphisms classes of) R-deformations of A, the Deligne principle asserts that there exists a differential graded Lie algebra L such that

$$\mathrm{Def}(A; R) \cong \widetilde{\mathrm{MC}}(L).$$

Thus, an accurate notion of geometric or topological realization of any dgl will allow us to consider from a homotopy point of view the moduli spaces of deformations of some structure.

[2] This particular example is of crucial importance when A is taken to be the smooth functions on a manifold M in view of the Kontsevich Quantization Theorem [89], which asserts that every Poisson structure on A arises from a certain R-deformation of A as above.

After motivating our main objective, it is appropriate to emphasize that our approach is closely related to recent important results and constructions, including the following ones, which will be explained in some detail later in this introduction: the study of the *Deligne–Getzler–Hinich functor* [60] and its relation with rational homotopy [2, 8, 93]; the construction of the *Lawrence–Sullivan interval* [91]; or the use of transfer methods for building Lie models of spaces [2, 118].

Also, we remark that to avoid excessive technicalities on one side, and to give specific tools for computations on the other, we have decided to avoid the category of L_∞-algebras and the operadic framework. Rather, the fundamental algebraic category in the text is that of complete differential graded Lie algebras, cdgl's from now on. Its objects are dgl's L endowed with a filtration $\{F^p\}_{p\geq 0}$ compatible with the Lie bracket and for which

$$L = \varprojlim_p L/F^p.$$

Morphisms in this category are dgl morphisms which respect the corresponding filtrations.

Having said that, the starting point of our theory is the following observation: it is well known that, although conceptually different, the Quillen and Sullivan approaches to rational homotopy theory, briefly presented above, follow almost perfectly Eckmann–Hilton dual (or more generally Koszul dual) paths.

For instance, just as in the Sullivan setting, for any simply connected space X, the dgl $\lambda(X)$ can be replaced by a simpler dgl of the same homotopy type. This is the (*Quillen*) *minimal model of* X, denoted by $(\mathbb{L}(W), d)$, in which $\mathbb{L}(W)$ stands for the free graded Lie algebra generated by a graded vectors space W, and where the differential d again satisfies a certain recursive condition. Once more, the link between the minimal Quillen model and the functor λ is established by a quasi-isomorphism,

$$(\mathbb{L}(W), d) \xrightarrow{\simeq} \lambda(X).$$

Moreover, if $(\land V, d)$ denotes the Sullivan minimal model, the classical examples below show that the Sullivan and the Quillen approaches faithfully follow Eckmann–Hilton dual paths:

$$H^*(\land V, d) \cong H^*(X; \mathbb{Q}), \qquad \text{while} \qquad H_*(\mathbb{L}(W), d) \cong \pi_*(\Omega X) \otimes \mathbb{Q}.$$

Also,

$$V \cong \pi_*(X) \otimes \mathbb{Q}, \qquad \text{while} \qquad W \cong H_{*-1}(X; \mathbb{Q}).$$

However, this duality, of which we could present much more evidence, seems to fail precisely at the fundamental level of the pairs of functors in (1) and (2), which give rise to both approaches. Indeed, it quickly becomes quite clear how the complexity of the functors λ and Quillen realization $\langle \cdot \rangle^{\mathrm{Q}}$ strongly contrasts with the conceptual simplicity of the pair of adjoint functors A_{PL} and $\langle \cdot \rangle^S$ on which the Sullivan approach to rational homotopy theory is based.

For this reason, the lack of an Eckmann–Hilton dual of the simplicial cdga $\Omega_\bullet$ to construct a dgl realization functor has puzzled homotopy theorists since the birth of the theory. We then ask: does there exist such a universal cosimplicial object in the category of dgl's on which we may base the topological realization of any dgl?

The first step for a positive answer to that question was given in [91] by the construction of the *Lawrence–Sullivan interval*: in what follows, and given a graded vector space W, we denote by $\widehat{\mathbb{L}}(W)$ the *free complete Lie algebra* generated by W, which is defined as

$$\widehat{\mathbb{L}}(W) = \varprojlim_n \mathbb{L}(W)/\mathbb{L}^{\geq n}(W).$$

Here, $\mathbb{L}^{\geq n}(W)$ denotes the ideal generated by Lie brackets of length at least n. In other words, an element of $\widehat{\mathbb{L}}(W)$ consists of a series whose nth term is a sum of Lie brackets of length exactly n. With this notation, and using ideas from dynamical systems, R. Lawrence and D. Sullivan construct in [91] a free complete differential graded Lie algebra of the form,

$$\mathfrak{L}_1 = (\widehat{\mathbb{L}}(a,b,x), d),$$

in which a and b are Maurer–Cartan elements, and hence of degree -1, x is a degree-0 element, and

$$dx = \operatorname{ad}_x b + \sum_{n=0}^{\infty} \frac{B_n}{n!} \operatorname{ad}_x^n(b-a),$$

where the B_n's are the Bernoulli numbers.

We may think of the vector space generated by a, b and x as the desuspension of the simplicial chains on Δ^1, which we denote by $s^{-1}\Delta^1$. Moreover, the linear part of the differential d, which we denote by d_1, is 0 on a and b, and is $b-a$ on x. That is, d_1 is precisely the desuspension of the differential on the simplicial chains of Δ^1. In other words,

$$\mathfrak{L}_1 = (\widehat{\mathbb{L}}(s^{-1}\Delta^1), d),$$

in which d makes the vertices Maurer–Cartan elements and extends the usual chain operator on $s^{-1}\Delta^1$.

We extend this and build, for each $n \geq 0$, a free cdgl

$$\mathfrak{L}_n = (\widehat{\mathbb{L}}(s^{-1}\Delta^n), d)$$

in which $s^{-1}\Delta^n$, together with the linear part of the differential d, is again the (desuspension) of the rational simplicial chain complex of the standard n-simplex Δ^n, and the vertices still correspond to Maurer–Cartan elements.

Moreover, the cofaces and codegeneracies of the natural cosimplicial structure in the graded vector space $s^{-1}\Delta^\bullet$ are extended to cdgl morphisms on $(\widehat{\mathbb{L}}(s^{-1}\Delta^\bullet), d)$ so that the family

$$\mathfrak{L}_\bullet = \{\mathfrak{L}_n\}_{n\geq 0}$$

is a cosimplicial cdgl which becomes our coveted "cosimplicial universal object" in the category of cdgl's, dual of the simplicial cdga $\Omega_\bullet$.

Hence, we are able to define the realization of any cdgl L as the simplicial set

$$\langle L\rangle = \operatorname{Hom}_{\mathbf{cdgl}}(\mathfrak{L}_\bullet, L),$$

with the simplicial structure induced by the cosimplicial one in $\mathfrak{L}_\bullet$.

On the other hand, $\mathfrak{L}_\bullet$ also permits us to construct a model functor by defining the (global) model of any given simplicial set X as

$$\mathfrak{L}_X = \varinjlim_{\sigma\in X} \mathfrak{L}_{|\sigma|}.$$

In this way we obtain a pair of adjoint functors, *global model* and *realization*,

$$\mathbf{sset} \underset{\langle\,\cdot\,\rangle}{\overset{\mathfrak{L}}{\rightleftarrows}} \mathbf{cdgl},$$

whose main features we list below.

Before enumerating the main properties of the realization functor, we first need to introduce two fundamental notions: given any cdgl L and any MC element $a\in \mathrm{MC}(L)$, the differential d on L can be *perturbed* by a to produce another differential

$$d_a = d + \mathrm{ad}_a\,.$$

We then define the *component* of L at a as the connected (i.e., non-negatively graded) sub-cdgl of (L, d_a) consisting of

$$L^a_p = \begin{cases} \ker d_a, & \text{if } p = 0,\\ L_p, & \text{if } p > 0.\end{cases}$$

Then, we first prove that the realization functor "preserves path-connected components". That is, given any cdgl L,

$$\langle L\rangle = \amalg_{\,a\in\widetilde{\mathrm{MC}}(L)}\langle L^a\rangle.$$

In other words, the realization of L has as many path components as classes of MC elements modulo the gauge action, and each of these components is precisely the realization of the component of L at the corresponding class of MC elements.

We also compute the homotopy groups of each of these components and show that, for any connected cdgl L and any $n\geq 1$,

$$\pi_n\langle L\rangle \cong H_{n-1}(L).$$

It is important to remark that for $n = 1$ the group structure on $H_0(L)$ is given by the Baker–Campbell–Hausdorff product.

Via this isomorphism, the action $\bullet$ of $\pi_1\langle L\rangle$ on each $\pi_n\langle L\rangle$ is given by the "exponential morphism":

$$\alpha \bullet \beta = e^{\mathrm{ad}_\alpha}\beta, \quad \text{where} \quad \alpha \in H_0(L) \quad \text{and} \quad \beta \in H_{n-1}(L).$$

Moreover, there is an isomorphism of graded Lie algebras,

$$H_{\geq 1}(L) \cong \pi_{\geq 1}\Omega\langle L\rangle$$

where in the latter, the Lie algebra structure is, as usual, induced by the Whitehead product.

On the other hand, concerning the main characteristics of the global model functor, the cdgl $\mathfrak{L}_X$ is shown to be the only one, up to isomorphism, satisfying the following properties:

- As a graded Lie algebra, $\mathfrak{L}_X = \widehat{\mathbb{L}}(s^{-1}X)$ is the free complete Lie algebra generated by the desuspension of the non-degenerate simplicial chains on X.
- The generators corresponding to 0-simplices are Maurer–Cartan elements.
- The linear part of the differential in $\mathfrak{L}_X$ is the desuspension of the differential of the non-degenerate simplicial chain complex.
- If $Y \hookrightarrow X$ is a sub-simplicial set, then the map $\mathfrak{L}_Y \to \mathfrak{L}_X$ induced by the inclusion on simplicial chains is a cdgl morphism.

Moreover, we completely determine the Deligne groupoid of the global model by showing that there are as many classes of MC elements, modulo the gauge action, as path components (plus one) of X. That is,

$$\widetilde{\mathrm{MC}}(\mathfrak{L}_X) = \pi_0(X^+),$$

in which $X^+ = X \amalg *$ denotes the disjoint union of the given simplicial set X with an external base point. In particular, if X is connected and a is any 0-simplex,

$$\widetilde{\mathrm{MC}}(\mathfrak{L}_X) = \{0, a\}.$$

Another particularly interesting property of the model functor is its homological behaviour. Indeed, for any simplicial set X it is easy to see that

$$H(\mathfrak{L}_X) = 0.$$

The reader may be initially puzzled by this result, especially if $\mathfrak{L}_X$ is designed to contain all the information about the rational homotopy type of X. That is in fact the case, but one has to focus on each component of $\mathfrak{L}_X$ to obtain the corresponding data.

More concretely, given a 0-simplex a of the simplicial set X, which we may assume connected, the homology of the component $\mathfrak{L}_X^a$ of the global model $\mathfrak{L}_X$ at the MC element represented by a is far from being trivial. In fact, it gives precisely the homotopy groups of the realization of $\mathfrak{L}_X^a$,

$$H_n(\mathfrak{L}_X^a) \cong \pi_{n+1}\langle \mathfrak{L}_X^a \rangle, \quad \text{for} \quad n \geq 0.$$

Moreover, it turns out that both the inclusion and the projection (over the ideal of $\mathfrak{L}_X$ generated by a)

$$\mathfrak{L}_X^a \xhookrightarrow{\simeq} (\mathfrak{L}_X, d_a) \xrightarrow{\simeq} \mathfrak{L}_X/(a)$$

are quasi-isomorphisms. This shows how the homology of a cdgl drastically changes when its differential is perturbed by an MC element.

Another crucial property of the global model functor lies in its relation with the functor A_{PL} which we now briefly explain: given any simplicial set X, denote by $N^*(X)$ the complex of non-degenerate simplicial cochains on X. Then, there is a *transfer diagram* of the form,

$$k \circlearrowright \; A_{PL}(X) \underset{i}{\overset{p}{\rightleftarrows}} N^*(X).$$

This means that i and p are cochain maps for which $pi = \mathrm{id}$, and ip is chain homotopic to the identity via the chain homotopy k, which also satisfies $k^2 = ki = pk = 0$. The classical *homotopy transfer theorem* endows $N^*(X)$ with a structure of commutative A_∞-algebra and extends i to a quasi-isomorphism of A_∞-algebras. This amounts to having a quasi-isomorphism of differential graded coalgebras,

$$\big(T^c\big(sN^*(X)\big), d\big) \xrightarrow{\simeq} B^u\, A_{\mathrm{PL}}(X),$$

where $\big(T^{\mathrm{c}}\big(sN^*(X)\big), d\big)$ is the tensor coalgebra on the suspension of non-degenerate cochains on X, with the differential corresponding to the inherited A_∞ structure, and $B^{\mathrm{u}}\, A_{\mathrm{PL}}(X)$ is the unreduced bar construction on $A_{\mathrm{PL}}(X)$.

Whenever X is of finite type, a technical but straightforward procedure allows us to obtain from $\big(T^{\mathrm{c}}\big(sN^*(X)\big), d\big)$ a free differential graded Lie coalgebra, whose dual is a free cdgl of the form $(\widehat{\mathbb{L}}(s^{-1}X), d)$, which is isomorphic to $\mathfrak{L}_X$. In particular, choosing X to be Δ^n, for each $n \geq 0$, we obtain the cdgl $\mathfrak{L}_n$ and therefore we recover in this way the cosimplicial cdgl $\mathfrak{L}_\bullet$.

This characterization is of vital importance in relating our Lie models with the "commutative world" and in particular with Sullivan models. Moreover, this identification will also let us show that all known geometrical realizations of cdgl's coincide.

Recall that the *Deligne–Getzler–Hinich* ∞*-groupoid* is a functor

$$\mathrm{MC}_\bullet\colon \mathbf{sset} \longrightarrow \mathbf{cdgl}, \quad \text{defined by} \quad \mathrm{MC}_\bullet(L) = \mathrm{MC}(\Omega_\bullet \widehat{\otimes} L).$$

The simplicial set $\mathrm{MC}_\bullet(L)$ is a generalization of the classical way by which a nilpotent Lie algebra L integrates to its group G via the Baker–Campbell–Hausdorff product. Indeed, if $L = L_0$ is a finitely generated cdgl, then $\mathrm{MC}_\bullet(L)$ is equivalent to BG, the classifying space of G.

Then, the description of $\mathfrak{L}$ via transfer allows us to prove that, for any connected cdgl L,

$$\mathrm{MC}_\bullet(L) \simeq \langle L \rangle.$$

Also, these techniques permit us to finally assert that our model and the realization functors recover the classical Quillen pair (2) in the simply connected case: for any 1-connected dgl of finite type,

$$\langle L \rangle^{\mathrm{Q}} \simeq \langle L \rangle,$$

that is, the Quillen realization functor $\langle\,\cdot\,\rangle^{\mathrm{Q}}$ is "co-representable" by the cosimplicial cdgl $\mathfrak{L}_\bullet$. On the other hand, for any simply connected simplicial set X of finite type and any 0-simplex of X,

$$\mathfrak{L}_X^a \simeq \lambda(X).$$

Moreover, in the non-simply connected case, given a connected, finite type simplicial set X, and any 0-simplex a of X we have homotopy equivalences,

$$\langle \mathfrak{L}_X^a \rangle \simeq \mathbb{Q}_\infty X \simeq \langle \wedge V, d \rangle^S,$$

where $\mathbb{Q}_\infty X$ is again the $\mathbb{Q}$-completion of X and $\langle \wedge V, d\rangle^S$ the Sullivan realization of the Sullivan minimal model $(\wedge V, d)$ of X. In particular,

$$\langle \mathfrak{L}_X \rangle \simeq \mathbb{Q}_\infty X^+.$$

Another immediate consequence is that $H_0(\mathfrak{L}_X^a)$, with the group structure given by the Baker–Campbell–Hausdorff product, recovers the Malcev completion of the fundamental group $\pi_1(X)$:

$$H_0(\mathfrak{L}_X^a) \cong \mathbb{Q}_\infty \pi_1(X).$$

As a synopsis, we draw the attention of the reader to the following general picture. The category **sset** is fully embedded in the pointed category **sset*** by means of the functor

$$\iota\colon \mathbf{sset} \hookrightarrow \mathbf{sset}^*,$$

which sends X to X^+ and the map $f\colon X \to Y$ to the pointed map $f^+\colon X^+ \to Y^+$ preserving the external point, and being f on X. Then, what $\mathfrak{L}$ faithfully models is the rational homotopy category of $\operatorname{Im}\iota$. Indeed, given a map $f\colon X \to Y$ between finite type simplicial sets, there is a homotopy commutative square

$$\begin{array}{ccc} \mathbb{Q}_\infty X^+ & \xrightarrow{\mathbb{Q}_\infty \iota(f)} & \mathbb{Q}_\infty Y^+ \\ \simeq\uparrow & & \uparrow\simeq \\ \langle \mathfrak{L}_X \rangle & \xrightarrow{\langle \mathfrak{L}_f \rangle} & \langle \mathfrak{L}_Y \rangle. \end{array}$$

In summary, all of the above constitutes an answer, not just "cellularly" but "simplicialy", to the following problem posed by R. Lawrence and D. Sullivan in [91], which we quote:

> *"If X is a cell complex with one 0-cell and only $2, 3, 4, \dots$ cells, the rational theory of Quillen assigns a free differential graded Lie algebra L with one generator in degree k for each $(k+1)$-cell $(k \geq 0) \dots$. One imagines that enlarging this discussion to allow cells in degree 1 would be related to some Lie algebras associated to non-trivial fundamental groups, but little is known here, to our knowledge".*

To attain all the results listed up to this point, it is absolutely necessary to embed the global model and the realization functor in a suitable homotopy theoretical framework. We do that in such a way that the category **cdgl** reflects as accurately as possible the geometric properties of the category **sset**. It is well known that the latter category has a cofibrantly generated model structure (in the sense of Quillen) in which fibrations are Kan fibrations, cofibrations are just simplicial injections, and weak equivalences are simplicial maps inducing isomorphisms in all homotopy groups.

We then use the *transfer principle* [2, 32] to transport this model structure to **cdgl** through the model and realization functor. This process guarantees the existence of a cofibrantly generated model structure on the category of cdgl's in which a morphism f is a fibration or a weak equivalence if $\langle f \rangle$ is a fibration or a weak equivalence, respectively, of simplicial sets. We then algebraically characterize the class of fibrations and weak equivalences in **cdgl** as follows:

A cdgl morphism $f\colon A \to B$ is a fibration if it is surjective in non-negative degrees.

On the other hand, f is a weak equivalence if

$$\widetilde{\mathrm{MC}}(f)\colon \widetilde{\mathrm{MC}}(A) \xrightarrow{\cong} \widetilde{\mathrm{MC}}(B)$$

is a bijection and

$$f^a\colon A^a \xrightarrow{\simeq} B^{f(a)}$$

is a quasi-isomorphism for every $a \in \widetilde{\mathrm{MC}}(A)$.

An immediate consequence of endowing a given category with a model structure via the transfer principle through a pair of adjoint functors is that they automatically become a Quillen pair. This is then the case for the global model and realization functor. In particular, they induce adjoint functors in the homotopy categories,

$$\mathrm{Ho}\,\mathbf{sset} \underset{\langle\,\cdot\,\rangle}{\overset{\mathfrak{L}}{\rightleftarrows}} \mathrm{Ho}\,\mathbf{cdgl},$$

and both preserve weak equivalences and homotopies.

We analyze this model structure in sufficient detail to provide explicit path, cylinder and cone objects, as well as computable cofibrant replacements of a given cdgl. All of this confirms that in fact the homotopy category of cdgl's mimics the behaviour of the homotopy category of simplicial sets.

As an illustrative example, we show that the Lawrence–Sullivan interval $\mathfrak{L}_1$ is just the cylinder of $\mathfrak{L}_0 = (\mathbb{L}(a), d)$, where a is an MC element, or more generally,

$$\mathfrak{L}_n \cong \text{Cone}\ \mathfrak{L}_{n-1}, \quad \text{for} \quad n \geq 1.$$

By means of this homotopy theoretical setting we also prove that the gauge relation among MC elements corresponds quite simply to the existence of a path in this model structure, connecting gauge equivalent elements! Explicitly, the following are equivalent for any cdgl L:

- The MC elements a and b are gauge related, that is, there exists $x \in L_0$ such that $x \mathcal{G} a = b$.
- There exists a "path" in L joining a and b, that is, a cdgl morphism $\varphi\colon \mathfrak{L} \to L$ such that $\varphi(a) = a$ and $\varphi(b) = b$.
- There exists an MC element Φ in the path object L^I of L whose "endpoints" are precisely a and b, that is, $\varepsilon_0(\Phi) = a$ and $\varepsilon_1(\Phi) = b$, where $\varepsilon_0, \varepsilon_1\colon L^I \to L$ are the endpoint cdgl morphisms of the path object.

Of crucial importance in this setting is the notion of *minimal* (*Lie*) *model* of a connected simplicial set X, which is a particular cofibrant replacement of the non trivial component of the global model of X:

Given any 0-simplex a of X, the minimal Lie model of X is a free cdgl $(\widehat{\mathbb{L}}(V), d)$, where d has no linear term, together with a quasi-isomorphism,

$$(\widehat{\mathbb{L}}(V), d) \xrightarrow{\simeq} \mathfrak{L}_X^a.$$

This object is unique up to cdgl isomorphism, does not depend on the chosen 0-simplex, and is an invariant of the homotopy type of X containing all its rational data. For instance,

$$H_{q-1}(\widehat{\mathbb{L}}(V), d) = \begin{cases} \pi_q(X) \otimes \mathbb{Q}, & \text{if } q \geq 2, \\ \mathbb{Q}_\infty \pi_1(X), & \text{if } q = 1, \end{cases} \quad \text{and} \quad V_{q-1} \cong H_q(X, \mathbb{Q}), \quad q \geq 1.$$

We also give explicit procedures to construct a Sullivan model of any connected simplicial set of finite type starting from its minimal Lie model, and vice versa.

All of the above constitute the core of this text, which culminates with selected applications which show how the theory can be implemented in situations lying beyond the classical theory for simply connected or nilpotent complexes.

For example, a particularly illustrative application is given by the Lie models of 2-dimensional complexes (including surfaces): let X be obtained by attaching a family of 2-cells $\{e_j\}_{j\in J}$ to a wedge of circles $\bigvee_{i\in I} S^1_i$ along the maps

$$\omega_j = y_{j_1}^{r_{j_1}} \cdots y_{j_{q_j}}^{r_{q_j}}, \quad \text{for} \quad j \in J,$$

where, for $i \in I$, each y_i denotes a generator of $\pi_1(S^1_i)$. Then, in clear analogy with the presentation of $\pi_1(X)$, a Lie model of X is given by

$$(\widehat{\mathbb{L}}(y_i, e_j), d), \quad \text{where each } y_i \text{ is a 0-cycle and} \quad de_j = y_{j_1}^{r_{j_1}} * \cdots * y_{j_{q_j}}^{r_{q_j}}, \quad j \in J.$$

An immediate consequence exhibits an explicit description of the Malcev completion of a finitely presented group as follows: let

$$G = \langle\, a_1, \dots, a_p \mid b_1, \dots, b_k \,\rangle, \quad \text{with} \quad b_j = a_{j_1}^{r_{j_1}} \dots a_{j_{q_j}}^{r_{q_j}}, \quad j = 1, \dots, k,$$

be a finitely presented group. Then, the Malcev completion of G is the group

$$\mathbb{Q}_\infty G = \widehat{\mathbb{L}}(a_1, \dots, a_p)/(b_1, \dots, b_k), \quad \text{with} \quad b_k = a_{i_1}^{r_1} * \cdots * a_{i_q}^{r_q}, \quad j = 1, \dots, k.$$

We also show that for any finite 2-dimensional complex X, its rational completion is an Eilenberg–MacLane space,

$$\mathbb{Q}_\infty X \simeq K(\mathbb{Q}_\infty \pi_1(X), 1).$$

Another specially interesting application of our theory, with which we close this introduction, is the modeling of mapping spaces in full generality: let X and Y be connected simplicial sets. Let A be any cdga of the homotopy type of $A_{\mathrm{PL}}(X)$ and let L be any cdgl of the homotopy type of $\mathfrak{L}_Y$. Then, $A\widehat{\otimes}L$ is a Lie model of the simplicial mapping space $\mathrm{Map}(X, \mathbb{Q}_\infty Y)$. That is,

$$\langle A\widehat{\otimes}L\rangle \simeq \mathrm{Map}(X, \mathbb{Q}_\infty Y).$$

****

Although each chapter begins with a concise summary of the material developed in it, we now briefly outline how the content of this text is organized.

In the first chapter, we compile all the background we need with the main goal of setting the notation and recalling basic facts, mainly from three different areas: simplicial objects, algebraic categories (most of them related to rational homotopy) and some facts on model category theory.

The second chapter is devoted to a careful analysis of two pairs of adjoint functors and their close relations. The first is formed by the classical *chains* and *Lie* Quillen functors

$$\mathbf{cdgc} \underset{\mathscr{C}}{\overset{\mathscr{L}}{\rightleftarrows}} \mathbf{dgl},$$

between the categories of cocommutative differential graded coalgebras and of differential graded Lie algebras. The second, less familiar to experts, and which can be thought as a topological dual of the former, consists of the pair

$$\mathbf{cdga} \underset{\mathscr{E}}{\overset{\mathscr{A}}{\leftrightarrows}} \mathbf{dglc},$$

running, without any restrictions, between the categories of commutative differential graded algebras and of differential graded Lie coalgebras. The extension of the functor $\mathscr{E}$ to the category $\mathbf{cdga}_\infty$ of commutative A_∞-algebras is also presented. These functors turn out to be indispensable in relating, as mentioned earlier, the model functor $\mathfrak{L}$ with the Sullivan and Quillen models of a given simplicial set.

In the third chapter we introduce the category **cdgl** of complete differential graded Lie algebras and the completion procedure for any dgl. Of particular importance is, as previously remarked, the completion $(\widehat{\mathbb{L}}(V), d)$ of a dgl, free as a Lie algebra, generated by a given graded vector space V. As some classical features of free dgl's are not inherited by their complete counterparts, we provide a detailed analysis of these objects. In particular, as the Quillen functor $\mathscr{L}$ takes values in the category of free dgl's, we show how the completion of this functor overcomes certain restrictions in the classical case. We finish the chapter by comparing the completion of dgl's with their so-called profinite completion.

Chapter 4 contains a detailed description of the Deligne groupoid associated to a cdgl, that is, the set $\widetilde{\mathrm{MC}}(L)$ of Maurer–Cartan elements modulo the gauge action. A fundamental result in this chapter is a general form of the classical *Goldman–Millson theorem*, which gives sufficient conditions for a cdgl morphism to induce an equivalence between the corresponding Deligne groupoids.

In Chapter 5 we proceed to a detailed study of the *Lawrence–Sullivan interval* $\mathfrak{L}_1$, LS interval for short: we first introduce its original conception. We then show how this cdgl can also be constructed from a flow associated to the differential equation

$$u' = dx + \mathrm{ad}_x\, u,$$

similar to others on some principal bundles whose flows define the so-called *gauge transformations*. Then, from a totally different perspective, we also prove that $\mathfrak{L}_1$ can also be obtained as the classical dgl cylinder of a point. The main characteristics and properties of this fundamental cdgl are studied in depth in this chapter.

Chapter 6 is devoted to the construction of the cosimplicial cdgl

$$\mathfrak{L}_\bullet = \{\mathfrak{L}_n\}_{n\geq 0},$$

by means of an inductive procedure. Of particular importance is showing that in the unique isomorphism class of such a cosimplicial cdgl, each $\mathfrak{L}_n$ can be chosen to be equivariant for the natural action of Σ_{n+1}.

In Chapter 7 we first introduce the global model $\mathfrak{L}$ and the realization functor $\langle\,\cdot\,\rangle$, demonstrate their adjoint character, and prove most of their main features listed above.

A careful study of the homotopy framework in which the category **cdgl** is located is developed in Chapter 8. As described before, this is attained by transferring the usual closed model structure on the category **sset** of simplicial sets to a new model structure on **cdgl**. As special cofibrant replacements, the minimal Lie models of connected spaces are introduced and carefully investigated in this chapter.

As previously indicated, the global model can be obtained by a transfer procedure from the simplicial cdga $\Omega_\bullet$. This is developed in Chapter 9, which in particular contains a detailed presentation of the classical Dupont simplicial transfer diagram

$$s_\bullet \circlearrowright \; A_{\mathrm{PL}}(\underline{\Delta}^\bullet) \underset{i_\bullet}{\overset{p_\bullet}{\rightleftarrows}} C^*(\Delta^\bullet),$$

from which $\mathfrak{L}_\bullet$ can be extracted.

In Chapter 10, we show that the global model functor recovers the classical Quillen model of any simply connected simplicial set of finite type, and more generally, the Neisendorfer model of any nilpotent simplicial set of finite type. As remarked above, this is based on both the characterization via transfer of the global model functor, and essential properties of the pair of adjoint functors $\mathscr{A}$ and $\mathscr{E}$ introduced in Chapter 2. Moreover, we also show in this chapter how to obtain a Sullivan model of a connected finite type simplicial set from its minimal Lie model, and vice versa. This link between the Sullivan and the Lie models allows us to introduce the notion of a coformal space in the general, non-nilpotent case.

The Deligne–Getzler–Hinich ∞-groupoid functor $\mathrm{MC}_\bullet\colon$ **sset** $\to$ **cdgl** is studied in Chapter 11 and is shown to coincide, up to homotopy, with our realization functor. From this, we also prove in this chapter that the realization functor extends that of Quillen in the classical setting. We finish by proving that

$$\mathbb{Q}_\infty X \simeq \langle \wedge V, d\rangle \simeq \mathrm{MC}_\bullet(\mathfrak{L}_X^a) \simeq \langle \mathfrak{L}_X^a \rangle,$$

for any finite connected simplicial set of finite type.

Chapter 12 includes a selected set of examples and applications of our theory. We include Lie models of 2-complexes, paying special attention to those of surfaces. From this, as previously observed, one easily deduces the Malcev $\mathbb{Q}$-completion of a finitely presented group. The same $\mathbb{Q}$-completion of an Artin group is also explicitly described by a similar method using the Lie model of the associated *Salvetti complex*. We then show how to obtain a Lie model of a product of simplicial sets from Lie models of each factor. The solution turns out to be an extension of the classical simply connected setting. After that, we give a detailed presentation of Lie models of mapping spaces, both in the free and pointed version. Special particular cases, like self equivalences or free loops are also treated. Finally, we show how to read, algebraically, some homotopy invariants in the realization of a given cdgl. For instance, and as indicated above, the action of the fundamental group on higher homotopy groups, the Whitehead product and the Postnikov decomposition of the realization of a connected cdgl are explicitly described.

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Chapter 1

Background

We inform the reader that both the writing style and structure of this chapter are noticeably different from the rest. In fact, we do not intend the content of this chapter to be exhaustive. Rather, it is mainly and purposely prepared to set the notation to be used in the text and to highlight the facts we will assume to be known by the reader. Some of them are indeed well-known results and part of the folklore in closely related subjects. Others may be less familiar to the non-expert but, being of general nature, we will not enter into specific details and instead provide appropriate references. Following this expositional treatment, and contrary to the self-contained and detailed presentation of the rest of the text, almost no proofs are given in this chapter, but again we provide the pertinent and most standard references.

We begin by listing the main definitions and general facts concerning simplicial sets, and more generally, simplicial objects in a given category. For our purposes, the simplicial chains, cochains (with their corresponding non-degenerate versions) and homology of a given simplicial object are of particular interest and therefore carefully introduced.

We then move on to describe the main algebraic categories to be used throughout the text. For the category of commutative differential graded algebras we briefly present the bridge given by the Sullivan approach to rational homotopy theory, connecting this category with that of simplicial sets by means of the $A_{\rm PL}$ and Sullivan realization functors. Analogously, we recount the Quillen approach to this theory by briefly outlining the λ functor on simplicial sets and the Quillen realization functor of positively graded differential Lie algebras. In particular, the notions of Sullivan and Quillen models are presented.

Finally, we highlight the most general facts concerning model categories, from their definition to the induced homotopy categories, passing through the particularities of cofibrantly generated model categories. We will pay special attention to those model category structures with which the previously presented algebraic structures are endowed.

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From now on and throughout the text, we fix the field $\mathbb{Q}$ of rational numbers as the ground field of coefficients for any algebraic object considered. Also, for any category $\mathcal{C}$, we will abuse the notation and write $A \in \mathcal{C}$ whenever A is an object of $\mathcal{C}$. We denote by $\mathrm{Hom}_{\mathcal{C}}$ the set of morphisms in $\mathcal{C}$, except for the category of (graded) vector spaces whose morphisms will be denoted by the unadorned Hom. As usual, by limit we always mean projective or inverse limit and denote $\varprojlim$. Inductive or direct limit is often called colimit and is denoted by $\varinjlim$.

1.1 Simplicial categories

As usual, the *simplicial category* $\boldsymbol{\Delta}$ is the category whose objects are the ordered sets $[n] = \{0, \dots, n\}$, $n \geq 0$, and whose morphisms $\mathrm{Hom}_{\boldsymbol{\Delta}}([n],[m])$ are the non-decreasing maps. Any morphism can be written as a composition of the cofaces $\delta^i \colon [n-1] \to [n]$, with $i = 0, \dots, n$, $n \geq 1$, and codegeneracies $\sigma^i \colon [n+1] \to [n]$ with $i = 0, \dots, n$, $n \geq 0$, defined by

$$\delta^i(j) = \begin{cases} j, & \text{if} \quad j < i, \\ j+1, & \text{if} \quad j \geq i, \end{cases} \qquad \text{and} \qquad \sigma^i(j) = \begin{cases} j, & \text{if} \quad j \leq i, \\ j-1, & \text{if} \quad j > i. \end{cases}$$

More precisely, any morphism $f \colon [n] \to [m]$ of $\boldsymbol{\Delta}$ admits a unique factorization

$$f = \delta^{i_1} \cdots \delta^{i_r} \sigma^{j_1} \cdots \sigma^{j_s},$$

such that $m \geq i_1 \geq \cdots \geq i_r \geq 0$ and $0 \leq j_1 \leq \cdots \leq j_s \leq n-1$. The cofaces and codegeneracies satisfy the *cosimplicial identities:*

$$\begin{cases} \delta^j \delta^i = \delta^i \delta^{j-1}, & \text{if } i < j, \\ \sigma^j \delta^i = \begin{cases} \delta^i \sigma^{j-1}, & \text{if} \quad i < j, \\ \mathrm{id}, & \text{if} \quad i = j,\, j+1, \\ \delta^{i-1}\sigma^j, & \text{if} \quad i > j+1, \end{cases} \\ \sigma^j \sigma^i = \sigma^i \sigma^{j+1}, & \text{if } i \leq j. \end{cases} \tag{1.1}$$

A *simplicial object* in a category $\mathcal{C}$ is a contravariant functor from $\boldsymbol{\Delta}$ to $\mathcal{C}$. A *cosimplicial object* in $\mathcal{C}$ is a covariant functor from $\boldsymbol{\Delta}$ to $\mathcal{C}$.

A simplicial object in $\mathcal{C}$ is therefore a family of objects, $C = \{C_n\}_{n \geq 0}$, together with *face* operators $d_i \colon C_n \to C_{n-1}$, and *degeneracy* operators $s_j \colon C_n \to C_{n+1}$, satisfying the dual of the relations (1.1), called *simplicial identities:*

$$\begin{cases} d_i d_j = d_{j-1} d_i, & \text{if } i < j, \\ d_i s_j = \begin{cases} s_{j-1} d_i, & \text{if} \quad i < j, \\ \mathrm{id}, & \text{if} \quad i = j,\, j+1, \\ s_j d_{i-1}, & \text{if} \quad i > j+1, \end{cases} \\ s_i s_j = s_{j+1} s_i, & \text{if } i \leq j. \end{cases} \tag{1.2}$$

A similar description can be provided for a cosimplicial object in $\mathcal{C}$, that is, a family $\{C^n\}_{n\geq 0}$, with *coface* operators, $\delta^i\colon C^{n-1}\to C^n$, and *codegeneracy* operators, $\sigma^j\colon C^{n+1}\to C^n$, satisfying (1.1).

A *simplicial morphism* $f\colon C\to D$ between two simplicial objects in $\mathcal{C}$ is a sequence $\{f_n\colon C_n\to D_n\}_{n\geq 0}$ of morphisms in $\mathcal{C}$ commuting with faces and degeneracies. Simplicial objects and morphisms constitute a category.

1.1.1 Simplicial sets

Denote by **sset** the category of simplicial sets and simplicial maps. Given a simplicial set $X=\{X_n\}_{n\geq 0}$, we write $|x|=n$ if $x\in X_n$ and we say that x is an *n-simplex*. A simplex is *degenerate* if it is in the image of some degeneracy map. Otherwise, it is *non-degenerate*. A simplicial set is *finite* (respectively of *finite type*) if it has a finite number of non-degenerate simplices (respectively non-degenerate n-simplices for any n). Given a simplicial set X and a subset S of non-degenerate simplices of X, the *simplicial set generated by* S is the sub-simplicial set of X consisting of S, the faces of the elements in S and all their degeneracies.

For any $n\geq 0$ denote by $\underline{\Delta}^n=\{\underline{\Delta}^n_p\}_{p\geq 0}$ the simplicial set in which

$$\underline{\Delta}^n_p=\operatorname{Hom}_{\mathbf{\Delta}}([p],[n])=\{(j_0,\dots,j_p)\mid 0\leq j_0\leq\cdots\leq j_p\leq n\}.$$

The faces and degeneracies are given by

$$\begin{aligned}
d_i\colon \underline{\Delta}^n_p\to\underline{\Delta}^n_{p-1},\quad d_i(f)=f\circ\delta^i,\quad d_i(j_0,\dots,j_p)=(j_0,\dots,j_{i-1},j_{i+1},\dots,j_p),\\
s_i\colon \underline{\Delta}^n_p\to\underline{\Delta}^n_{p+1},\quad s_i(f)=f\circ\sigma^i,\quad s_i(j_0,\dots,j_p)=(j_0,\dots,j_i,j_i,\dots,j_p).
\end{aligned}$$

Observe that there is a unique non-degenerate n-simplex in $\underline{\Delta}^n_n$, namely $1_{[n]}=(0,\dots,n)$.

We denote by $\dot{\underline{\Delta}}^n$ the *boundary* of $\underline{\Delta}^n$, which is the sub-simplicial set generated by all non-degenerate simplices except $1_{[n]}=(0,\dots,n)$. On the other hand, the *ith horn* $\underline{\Lambda}^n_i$ of $\underline{\Delta}^n$, $0\leq i\leq n$, is the sub-simplicial set generated by all non-degenerate simplices except $(0,\dots,n)$ and $(0,\dots,\widehat{i},\dots,n)$.

Given a simplicial set X, there is a natural bijection

$$X_n\cong\operatorname{Hom}_{\mathbf{sset}}(\underline{\Delta}^n,X),\tag{1.3}$$

where each simplex $x\in X_n$ is naturally identified with the only simplicial map $x\in\operatorname{Hom}_{\mathbf{sset}}(\underline{\Delta}^n,X)$ which sends the only non-degenerate n-simplex $(0,\dots,n)$ to x. From this we obtain the classical formula

$$X=\varinjlim_{x\in X}\underline{\Delta}^{|x|}.\tag{1.4}$$

It is important to note that the family

$$\underline{\Delta}^\bullet=\{\underline{\Delta}^n\}_{n\geq 0}$$

is a cosimplicial object in the category of simplicial sets. The cofaces and codegeneracies are given by

$$\delta^i\colon \underline{\Delta}_p^{n-1} \to \underline{\Delta}_p^n \quad \text{and} \qquad \sigma^i\colon \underline{\Delta}_p^{n+1} \to \underline{\Delta}_p^n, \qquad p \geq 0,$$

where

$$\begin{aligned} &\delta^i(f) = \delta^i \circ f, \qquad \delta^i(i_0,\dots,i_p) = (j_0,\dots,j_p) \quad \text{with} \quad j_k = \begin{cases} i_k, & \text{if } i_k < i, \\ i_k+1, & \text{if } i_k \geq i, \end{cases} \\ &\sigma^i(f) = \sigma^i \circ f, \qquad \sigma^i(i_0,\dots,i_p) = (j_0,\dots,j_p) \quad \text{with} \quad j_k = \begin{cases} i_k, & \text{if } i_k \leq i, \\ i_k-1, & \text{if } i_k > i. \end{cases} \end{aligned} \tag{1.5}$$

The link between **sset** and the category **top** of topological spaces and continuous maps is built from the *standard topological n-simplex*. This is the topological space

$$\Delta^n = \{(t_0,\dots,t_n) \in \mathbb{R}^{n+1} \text{ such that } \sum_{i=0}^n t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i\}\,.$$

In other terms, Δ^n is the convex hull of its vertices $v_0,\dots,v_n$, where $v_i = (0, \dots, 1, \dots 0)$ with 1 in position i. The family $\Delta^\bullet = \{\Delta^n\}_{n\geq 0}$ form a cosimplicial topological space. For $i = 0,\dots,n$, the cofaces and codegeneracies,

$$\delta^i\colon \Delta^{n-1} \longrightarrow \Delta^n \quad \text{and} \qquad \sigma^i\colon \Delta^{n+1} \longrightarrow \Delta^n,$$

are given by

$$\delta^i(t_0,\dots,t_{n-1}) = (t_0,\dots,t_{i-1},0,t_i,\dots,t_{n-1}),$$

and

$$\sigma^j(t_0,\dots,t_{n+1}) = (t_0,\dots,t_j+t_{j+1},\dots,t_{n+1}).$$

Equivalently, these are the affine maps defined on the vertices $v_0,\dots,v_n$, by

$$\delta^i(v_j) = \begin{cases} v_j, & \text{if} \quad i < j, \\ v_{j+1}, & \text{if} \quad i \geq j \end{cases} \qquad \text{and} \qquad \sigma^i(v_j) = \begin{cases} v_j, & \text{if} \quad j \leq i, \\ v_{j-1}, & \text{if} \quad j > i. \end{cases}$$

The *singular simplicial set* $\operatorname{Sing} X$ of a topological space X is defined by

$$(\operatorname{Sing} X)_n = \operatorname{Hom}_{\mathbf{top}}(\Delta^n, X)\,.$$

The simplicial structure of $\operatorname{Sing} X$ is induced by the cosimplicial topological space $\Delta^\bullet$. This cosimplicial structure also gives a realization functor from **sset** to the category **top** of topological spaces. Given a simplicial set X, the *realization* $|X|$ of X is the quotient topological space

$$\bigsqcup_n (X_n \times \Delta^n)/\sim$$

where each X_n is equipped with the discrete topology, $\sqcup$ denotes the disjoint union and

$$(d_i x, u) \sim (x, \delta^i u) \quad \text{for} \quad (x,u) \in X_{n+1} \times \Delta^n,$$
$$(s_j x, u) \sim (x, \sigma^j u) \quad \text{for} \quad (x,u) \in X_{n-1} \times \Delta^n.$$

The realization $|X|$ is a CW-complex with one n-cell for each non-degenerate n-simplex.

A direct inspection shows that the realization functor $|\cdot|$ is left adjoint to the singular functor Sing,

$$\mathbf{sset} \underset{\mathrm{Sing}}{\overset{|\cdot|}{\rightleftarrows}} \mathbf{top}\,. \tag{1.6}$$

In particular, for any simplicial set X and any topological space Y, there is a bijection,

$$\mathrm{Hom}_{\mathbf{top}}(|X|, Y) \cong \mathrm{Hom}_{\mathbf{sset}}\big(X, \mathrm{Sing}(Y)\big).$$

Later on in the text, we will be using standard results about homotopy theory of simplicial sets, for which the reader may consult [63] or [102].

A final warning is in order: in what follows we will sometimes consider the particular subcategory $\mathbf{sset}_1$ of $\mathbf{sset}$ consisting of 2-reduced simplicial sets. These are simplicial sets with only one simplex in dimensions 0 and 1. In particular, they are simply connected and moreover, any simply connected simplicial set is weakly homotopy equivalent to a 2-reduced one. From now on, by an abuse of language, whenever we consider a simply connected simplicial set we will in fact be considering a 2-reduced replacement of it.

1.1.2 Simplicial complexes

A *simplicial complex* K consists of a family S of non-empty finite subsets of a given set V satisfying the following conditions:

(i) If $Y \subset X$, $X \in S$ and $Y \neq \emptyset$, then $Y \in S$.

(ii) For each $x \in V$, $\{x\} \in S$.

The elements of V are called vertices and the elements of S are *simplices*. The *dimension* of a simplex is its cardinality minus one.

The simplicial complex K is *ordered* if there is a partial order on V such that it induces a total order on the set of vertices of each simplex. Obviously, every simplicial complex can be ordered and from now on we will assume this is the case.

We denote also by Δ^n the simplicial complex formed by the non-empty subsets of $\{0,\dots,n\}$. The sub-complexes $\dot{\Delta}^n$ and Λ^n_i are the simplicial complexes containing the non-empty subsets of $\{0,\dots,n\}$ except $(0,\dots,n)$ and $(0,\dots,\widehat{i},\dots n)$, $(0,\dots,n)$, respectively.

Given an (ordered) simplicial complex K, there is an associated simplicial set $\underline{K}$, where $\underline{K}_n$ consists of the $(n+1)$-tuples of vertices $(v_0,\dots,v_n)$ such that $v_0 \le \cdots \le v_n$ and $\{v_0,\dots,v_n\}$ is a simplex of K. The face and degeneracy operators are given by

$$\begin{aligned} d_i\colon \underline{K}_n \to \underline{K}_{n-1}, \qquad & d_i(v_0,\dots,v_n) = (v_0,\dots,v_{i-1},v_{i+1},\dots,v_n),\\ s_i\colon \underline{K}_n \to \underline{K}_{n+1}, \qquad & s_j(v_0,\dots,v_n) = (v_0,\dots,v_j,v_j,\dots,v_n). \end{aligned}$$

Note that the non-degenerate simplices of $\underline{K}_n$ are precisely the n-simplices of K.

It trivially follows from the definition that the simplicial set associated to Δ^n is $\underline{\Delta}^n$ as previously defined, which confirms the compatibility with the chosen notation. The same applies to $\underline{\dot{\Delta}}^n$ and $\underline{\Lambda}^n_i$.

1.1.3 Simplicial chains

Given a simplicial set X, define the set of *simplicial chains on X* as

$$C_*(X) = \bigoplus_{n\ge 0} C_n(X),$$

where $C_n(X)$ is the vector space generated by the n-simplices. The face and degeneracy maps induce linear maps in the chains, denoted in the same way, which exhibit $C_*(X)$ as a simplicial vector space. Moreover, the linear map,

$$d\colon C_p(X) \longrightarrow C_{p-1}(X), \qquad dx = \sum_{i=0}^{p} (-1)^i d_i x,$$

given by the alternating sum of the faces, makes $(C_*(X), d)$ a chain complex whose homology $H(X)$ is called the *homology of X*.

Denote by $D_n(X) \subset C_n(X)$ the vector space generated by the degenerate simplices in X_n. Then, by the simplicial identities (1.2), $(D_*(X), d) \subset (C_*(X), d)$ is a sub-chain complex and we define the chain complex of *non-degenerate chains on X* as the quotient,

$$(N_*(X), d) = (C_*(X)/D_*(X), d).$$

Since $H(D_*(X), d) = 0$, killing the degenerate simplices has no effect on the homology, and therefore,

$$H\big(N_*(X)\big) = H(X).$$

On the other hand, given a simplicial complex K, the *simplicial chains* on K constitute the chain complex $(C_*(K), d)$, where $C_*(K) = \bigoplus_{p\ge 0} C_p(K)$, in which $C_p(K)$ is the vector space generated by the p-simplices, and $d\colon C_p(K) \to C_{p-1}(K)$ is given by,

$$d(v_0,\dots,v_p) = \sum_{i=0}^{p} (-1)^i (v_0,\dots,\widehat{v_i},\dots,v_p). \tag{1.7}$$

We denote by $H(K)$ the homology of this complex. Observe that, since p-simplices of K correspond to non-degenerate p-simplices of its associated simplicial set $\underline{K}$, we have

$$(N_*(\underline{K}), d) \cong (C_*(K), d). \tag{1.8}$$

Hence,

$$H(\underline{K}) = H(K)\,.$$

Applying the previous statements to the cosimplicial structure on $\underline{\Delta}^\bullet$ produces a cosimplicial object,

$$(C_*(\underline{\Delta}^\bullet), d)$$

in the category of chain complexes. The cofaces and codegeneracies

$$\delta^i \colon C_p(\underline{\Delta}^{n-1}) \to C_p(\underline{\Delta}^n) \quad \text{and} \qquad \sigma^i \colon C_p(\underline{\Delta}^{n+1}) \to C_p(\underline{\Delta}^n), \qquad p \geq 0,$$

are given exactly as in formula (1.5). By the cosimplicial identities (1.1), these maps preserve degenerate chains and therefore they induce a cosimplicial structure on the non-degenerate chains

$$(N_*(\underline{\Delta}^\bullet), d).$$

By (1.8),

$$(N_*(\underline{\Delta}^\bullet), d) \cong (C_*(\Delta^\bullet), d), \tag{1.9}$$

and thus, the cosimplicial structure in $(N_*(\underline{\Delta}^\bullet), d)$ induces also in $(C_*(\Delta^\bullet), d)$ a structure of cosimplicial chain complex. The cofaces and codegeneracies of $(C_*(\Delta^\bullet), d)$ are given as in formula (1.5) modulo degenerate chains. Their explicit expressions, which will be heavily used from Chapter 6 on, become

$$\begin{aligned}
&\delta^i \colon C_p(\Delta^{n-1}) \to C_p(\Delta^n), \quad \delta^i(i_0, \ldots, i_p) = (j_0, \ldots, j_p) \\
&\qquad\qquad\qquad\qquad\qquad \text{with} \;\; j_k = \begin{cases} i_k, & \text{if } i_k < i, \\ i_k + 1, & \text{if } i_k \geq i, \end{cases} \\
&\sigma^i \colon C_p(\Delta^{n+1}) \to C_p(\Delta^n), \quad \sigma^i(i_0, \ldots, i_p) = (j_0, \ldots, j_p) \\
&\qquad\qquad\qquad\qquad\qquad \text{with} \;\; j_k = \begin{cases} i_k, & \text{if } i_k \leq i, \\ i_k - 1, & \text{if } i_k > i, \end{cases}
\end{aligned} \tag{1.10}$$

if $(j_0, \ldots, j_p)$ is non-degenerate, i.e., if $j_0 < \cdots < j_p$. Otherwise, $\sigma^i(i_0, \ldots, i_p) = 0$. Note that the latter occurs if and only if $(i_0, \ldots, i_p)$ contains a pair $(i, i+1)$ of successive integers.

In other terms,

$$\sigma^i(i_0, \ldots, i_p) = \begin{cases} \big(\sigma^i(i_0), \ldots, \sigma^i(i_p)\big), & \text{if } \sigma^i(i_0) < \cdots < \sigma^i(i_p), \\ 0, & \text{otherwise.} \end{cases}$$

All of the above can be done mutatis mutandis replacing chains by cochains. Hence, for any simplicial set X, the *simplicial cochains*

$$(C^*(X), d), \quad \text{with} \quad C^*(X) = \bigoplus_{p\geq 0} C^p(X),$$

form the cochain complex where $C^p(X) = \mathrm{Hom}(C_p(X), \mathbb{Q})$, and the differential $d\colon C^p(X) \to C^{p+1}(X)$ is given by the alternating sum of the dual of the face operators. The cochain complex of the *non-degenerate cochains on* X, $(N^*(X), d)$ is defined accordingly.

The main advantage of dealing with simplicial cochains is the structure of differential graded algebra (see Section 1.2.1 for a precise definition) which then naturally arises in $(C^*(X), d)$: denote by

$$d_p^F\colon X_{p+q} \longrightarrow X_p, \;\; d_p^F = d_{p+1} \cdots d_{p+q}, \;\; \text{and} \;\; d_q^B\colon X_{p+q} \longrightarrow X_q, \;\; d_q^B = d_0 \overset{p}{\cdots} d_0,$$

the *front p-face* and the *back p-face* of X. Then, given $\alpha \in C^p(X)$ and $\beta \in C^q(X)$, define $\alpha \cup \beta \in C^{p+q}(X)$ by

$$\alpha \cup \beta(a) = \alpha(d_p^F a) \cdot \beta(d_q^B a), \quad \text{for} \quad a \in C_{p+q}(X).$$

Observe that the cochain in $C^0(X)$ which sends every 0-simplex of X to $1 \in \mathbb{Q}$, is an identity for this product. As the differential on $C^*(X)$ is a derivation for this product it follows that $(C^*(X), d)$ is a differential graded algebra. Moreover, by the simplicial identities, one checks that the degenerate simplicial cochains constitute a differential ideal and therefore the non-degenerate simplicial cochains $(N^*(X), d)$ inherit a structure of differential graded algebra.

As for any simplicial complex K, we define analogously the *simplicial cochains* $(C^*(K), d)$ and observe that,

$$(N^*(\underline{K}), d) \cong (C^*(K), d)$$

where $\underline{K}$ the associated simplicial set. In particular, for the cosimplicial object $\underline{\Delta}^\bullet$,

$$(N^*(\underline{\Delta}^\bullet), d) \cong (C^*(\Delta^\bullet), d), \tag{1.11}$$

and $(C^*(\Delta^\bullet), d)$ inherits a structure of simplicial differential graded algebra. Since we will need it in an explicit way, we specify here this structure:

For each $n \geq 0$ and each $p = 0, \ldots, n$ we denote by $\{\alpha_{i_0,\ldots,i_p}\}$ the basis of $C^p(\Delta^n) = \mathrm{Hom}(C_p(\Delta^n), \mathbb{Q})$ defined by,

$$\alpha_{i_0,\ldots,i_p}(j_0, \ldots, j_p) = \begin{cases} (-1)^{\frac{p(p-1)}{2}}, & \text{if } (j_0, \ldots, j_p) = (i_0, \ldots, i_p), \\ 0, & \text{otherwise.} \end{cases} \tag{1.12}$$

The differential $d\colon C^p(\Delta^n) \to C^p(\Delta^{n+1})$ is defined as usual by the formula $df(u) = -(-1)^{|f|} f(du)$, for $f \in C^*(\Delta^n)$ and $u \in C_*(\Delta^n)$. We thus deduce from (1.7) that

$$d(\alpha_{i_0,\dots,i_k}) = \sum_q \alpha_{q,i_0,\dots,i_k}, \tag{1.13}$$

with the following conventions: if some i_j is equal to q, then $\alpha_{qi_0,\dots,i_k} = 0$. Moreover, if σ is a permutation of the set $\{1,\dots,r\}$, then $\alpha_{i_{\sigma(0)},\dots,i_{\sigma(r)}} = \varepsilon(\sigma)\alpha_{i_0,\dots,i_r}$, with $\varepsilon(\sigma)$ denoting the sign of the permutation σ.

The face and degeneracy operators of $(C^*(\Delta^\bullet), d)$ are defined by

$$d_i\colon C^p(\Delta^n) \longrightarrow C^p(\Delta^{n-1}), \qquad s_i\colon C^p(\Delta^n) \longrightarrow C^p(\Delta^{n+1}), \quad \text{for} \quad p \geq 0, \tag{1.14}$$

where

$$d_i(\alpha_{i_0,\dots,i_p}) = \begin{cases} 0, & \text{if } i \in \{i_0,\dots,i_p\}, \\ \alpha_{j_0,\dots,j_q}, & \text{otherwise, with } \begin{cases} j_r = i_r, & \text{if } i_r < i, \\ j_r = i_{r-1}, & \text{if } i_r \geq i, \end{cases} \end{cases}$$

and

$$s_i(\alpha_{i_0,\dots,i_p}) = \begin{cases} \alpha_{i_0,\dots,i_r,i_{r+1}+1,\dots,i_p+1} + \alpha_{i_0,\dots,i_{r-1},i_r+1,\dots,i_p+1}, & \text{if } i = i_r, \\ \alpha_{i_0,\dots,i_r,i_{r+1}+1,\dots,i_p+1}, & \text{if } i_r < i < i_{r+1}. \end{cases}$$

Finally, the effect of the product of two cochains on a given simplex is, as defined before, the multiplication of the cochains applied to the front and back faces, respectively, of the given simplex.

1.2 Differential categories

We first set general notation.

Every algebraic object is considered to be $\mathbb{Z}$-graded unless explicitly stated otherwise. The usual convention given by the formula $V^n = V_{-n}$ for any $n \in \mathbb{Z}$ let us use both the "upper" or "lower" grading in what follows. For algebras and Lie coalgebras we usually use the upper grading, while for Lie algebras and coalgebras we often use the lower grading.

A *graded vector space* V, or simply a *vector space* when there is no possible ambiguity, is a family of vector spaces $V = \{V_n\}_{n\in\mathbb{Z}}$. If $v \in V_n$, we say that the *degree* of v is n and we write $|v| = n$. A *morphism* $f\colon V \to W$ of graded vector spaces is a collection of linear maps $\{f_n\colon V_n \to W_n\}_{n\in\mathbb{Z}}$. The category of graded vector spaces is denoted by **vect**.

We say that V is of, or has, *finite type* if $\dim V_n < \infty$ for all n.

For any $p \in \mathbb{Z}$, the *pth suspension* of V is the graded vector space s^pV defined by $(s^pV)_n = V_{n-p}$.

The *dual* of V is the graded vector space

$$V^\# = \{(V^\#)^n\}_{n\in\mathbb{Z}}, \quad \text{where} \quad (V^\#)^n = \operatorname{Hom}(V_n, \mathbb{Q}).$$

Remark that, for any $p \in \mathbb{Z}$,

$$(s^pV)^\# = s^{-p}V^\# \quad \text{and} \quad (s^{-p}V)^\# = s^pV^\#.$$

A *differential graded vector space* is simply a chain (or cochain) complex. In other terms, it is a graded vector space V endowed with a *differential*, that is, a collection of linear maps $d\colon V_n \to V_{n-1}$, $n \in \mathbb{Z}$, such that $d^2 = 0$.

A morphism $f\colon V \to W$ of differential graded vector spaces is a morphism of chain (or cochain) complexes. Such a morphism is said to be a *quasi-isomorphism*, and we write $f\colon V \xrightarrow{\simeq} W$, if $H(f)\colon H(V) \xrightarrow{\cong} H(W)$ is an isomorphism of graded vectors spaces. We will use the same nomenclature for any additional structure on the given differential graded vector space.

Denote by **dvect** the category of differential graded vector spaces. In this category we write either V or (V, d) to denote the same object. The latter is often reserved to avoid ambiguity or for situations in which we want to stress the existence of such a differential. The same applies henceforth to any other considered differential graded structure (differential algebra, coalgebra, Lie algebra, Lie coalgebra,...).

We will assume and use the basic facts concerning spectral sequences arising from filtrations on differential graded vector spaces, most of the times with some additional algebraic structures.

The *Koszul convention* is applied from this moment on: whenever two graded objects of degrees n and m are permuted in a formula, the sign $(-1)^{nm}$ appears.

1.2.1 Commutative differential graded algebras and the Sullivan model of a space

A *graded algebra*, or simply an algebra, is a graded vector space A endowed with an associative linear product,

$$A^p \otimes A^q \longrightarrow A^{p+q}, \qquad x \otimes y \longmapsto xy,$$

which has a unit $1 \in A^0$. A graded algebra is *commutative* if $xy = (-1)^{pq}yx$ for $x \in A^p$ and $y \in A^q$. The corresponding category is denoted by **cga**.

A *differential graded algebra*, dga henceforth, is a differential graded vector space A endowed with a graded algebra structure for which the differential d is a derivation:

$$d(xy) = (dx)y + (-1)^p x(dy), \quad \text{for} \quad x \in A^p \quad \text{and} \quad y \in A.$$

A morphism of dga's $f\colon A \to B$ is a morphism of differential vector spaces which preserves the unit and the product: $f(xy) = f(x)f(y)$. A dga A is *augmented* if there is a dga morphism $\varepsilon\colon A \to (\mathbb{Q}, 0)$. For such a dga the ideal $\overline{A} = \ker \varepsilon$ is called the *augmentation ideal*. If A is *connected*, that is, $A = A^{\geq 0}$ and $A^0 = \mathbb{Q}$, then A is automatically augmented and $\overline{A} = A^+ = A^{\geq 1}$. A morphism $f\colon A \to B$ of dga's augmented by ε_A and ε_B, respectively, is said to *preserve augmentations* if $\varepsilon_B \circ f = \varepsilon_A$.

A dga A is called *commutative*, cdga henceforth, if it is commutative as a graded algebra.

Unless otherwise mentioned, all dga's and cdga's will be augmented and all morphisms are assumed to preserve the augmentations. We denote by **dga** and **cdga** the corresponding categories. Of particular interest are the following subcategories of **cdga**:

On the one hand, $\mathbf{cdga_0}$ denotes the category of cdga's A non-negatively graded, $A = A^{\geq 0}$.

On the other hand, for $n > 0$, $\mathbf{cdga_n}$ denotes the category of n-connected cdga's. These are the cdga's A, satisfying $A^0 = \mathbb{Q}$ and $A^p = 0$ for $1 \leq p \leq n-1$. In other terms, $\overline{A} = \overline{A}^{\geq n}$.

The *tensor algebra* of a graded vector space V is the graded algebra

$$T(V) = \bigoplus_{n \geq 0} T^n(V), \quad \text{where} \quad T^0(V) = \mathbb{Q}, \quad T^n(V) = V \otimes \overset{n}{\cdots} \otimes V = V^{\otimes n},$$

and the product is given by juxtaposition. The *free commutative graded algebra on V* is the graded algebra defined as

$$\wedge V = T(V)/I$$

where I is the ideal generated by the elements $x \otimes y - (-1)^{|x|\,|y|} y \otimes x$, for any homogeneous elements $x, y \in V$. Any free commutative graded algebra $\wedge V$ is naturally augmented and satisfies the usual universality property: any linear map $V \to A$, in which A is a commutative graded algebra, extends uniquely to an algebra morphism $\wedge V \to A$.

One easily checks that for any given graded vector spaces V and W

$$\wedge(V \oplus W) = \wedge V \otimes \wedge W.$$

Indeed, the coproduct in **cdga** is the tensor product and the free functor preserves colimits.

In particular, writing $V = V^{\text{even}} \oplus V^{\text{odd}}$, it follows that $\wedge V$ is the tensor product of the symmetric algebra on the subspace of vectors of even degree with the exterior algebra on the subspace of vectors of odd degree,

$$\wedge V = S[V^{\text{even}}] \otimes E[V^{\text{odd}}].$$

A *Sullivan algebra* is a cdga of the form $(\wedge V, d)$ together with an increasing filtration of graded vector spaces on V, $V(0) \subset \cdots \subset V(n) \subset \cdots$, such that $V = \bigcup_n V(n)$, $dV(0) = 0$ and $dV(n) \subset \wedge V(n-1)$. A Sullivan algebra $(\wedge V, d)$ is called *minimal* if the differential is *decomposable*, i.e., $dV \subset \wedge^{\geq 2} V$.

For instance, the cdga $(\wedge(a,b,c), d)$ with a, b, c in degree 1 and $da = bc, db = ac$ and $dc = ab$ is not a Sullivan algebra, even though the differential is decomposable. Another example is given by the graded vector space $V = V^1$ with basis $\{a, x_n\}_{n\in\mathbb{Z}}$. The cdga $(\wedge V, d)$ defined by $da = 0$ and $dx_n = ax_{n-1}$, for any $n \in \mathbb{Z}$, is not a Sullivan algebra either, even though again the differential is decomposable.

Two cdga morphisms $f, g\colon (\wedge V, d) \to A$ from a Sullivan algebra to a given cdga are *homotopic*, and we write $f \sim g$, if there is a cdga morphism

$$\Phi\colon (\wedge V, d) \to A \otimes \wedge(t, dt)$$

such that $\mathrm{id}_A \otimes \varepsilon_0 \Phi = f$ and $\mathrm{id}_A \otimes \varepsilon_1 \Phi = g$. Here $\varepsilon_i\colon \wedge(t, dt) \to \mathbb{Q}$ are the augmentations defined by $\varepsilon_i(t) = i$, $i = 0, 1$. This is an equivalence relation among cdga morphisms from the Sullivan algebra $(\wedge V, d)$. We denote by $[(\wedge V, d), A]$ the set of homotopy classes of morphisms.

The commutative differential graded algebra Ω_n is the quotient

$$\Omega_n = (\wedge(t_0, \ldots, t_n, dt_0, \ldots, dt_n)/\mathcal{I},\ d),$$

where $|t_i| = 0$, $|dt_i| = 1$, $d(t_i) = dt_i$, and $\mathcal{I}$ is the ideal generated by $(\sum_{i=0}^n t_i) - 1$ and $\sum_{i=0}^n dt_i$.

The family

$$\Omega_\bullet = \{\Omega_n\}_{n\geq 0}$$

is a simplicial cdga. For a non-decreasing map $\varphi\colon [n] \to [m]$, the morphism of cdga's $\varphi^*\colon \Omega_m \to \Omega_n$ is determined by the formula

$$\varphi^*(t_i) = \sum_{j\in\varphi^{-1}(i)} t_j.$$

In particular, the faces and the degeneracies of $\Omega_\bullet$ are given by

$$\begin{aligned} &d_i\colon \Omega_n \to \Omega_{n-1}, \qquad d_i(t_j) = \begin{cases} t_j, & \text{if } j < i, \\ 0, & \text{if } j = i, \\ t_{j-1}, & \text{if } j > i, \end{cases} \\ &s_i\colon \Omega_n \to \Omega_{n+1}, \qquad s_i(t_j) = \begin{cases} t_j, & \text{if } j < i, \\ t_i + t_{i+1}, & \text{if } j = i, \\ t_{j+1}, & \text{if } j > i. \end{cases} \end{aligned} \tag{1.15}$$

Proposition 1.1 (Poincaré Lemma [34]). *The simplicial algebra $\Omega_\bullet$ is contractible, i.e., $\pi_q(\Omega_\bullet) = 0$ for $q \geq 0$.* □

Here $\pi_*(\Omega_\bullet)$ stands for the homotopy groups of $\Omega_\bullet$ considered just as a simplicial set.

Given a simplicial set X, the *algebra of* PL-*forms* on X is the cdga $A_{\mathrm{PL}}(X)$ in $\mathbf{cdga_0}$ defined by

$$A_{\mathrm{PL}}(X) = \mathrm{Hom}_{\mathbf{sset}}(X, \Omega_\bullet).$$

In particular, for any $n \geq 0$, we have an isomorphism of cdga's

$$A_{\mathrm{PL}}(\underline{\Delta}^n) \cong \Omega_n. \tag{1.16}$$

Moreover, the cosimplicial structure of $\underline{\Delta}^\bullet$ induces a simplicial structure on $A_{\mathrm{PL}}(\underline{\Delta}^\bullet)$ for which the above becomes an isomorphism

$$A_{\mathrm{PL}}(\underline{\Delta}^\bullet) \cong \Omega_\bullet$$

of simplicial cdga's. Indeed, with the notation of (1.5),

$$d_i = A_{PL}(\delta^i) \colon \Omega_n \longrightarrow \Omega_{n-1}, \quad \text{and} \quad s_i = A_{PL}(\sigma^i) \colon \Omega_n \longrightarrow \Omega_{n+1}.$$

On the other hand, the *Sullivan realization* of a cdga A is the simplicial set

$$\langle A \rangle^{\mathrm{S}} = \mathrm{Hom}_{\mathbf{cdga}}(A, \Omega_\bullet). \tag{1.17}$$

Theorem 1.2 ([12, §8.1], [51, §1.6], [128]). *The contravariant A_{PL} functor is left adjoint to the simplicial realization functor $\langle \cdot \rangle^{\mathrm{S}}$,*

$$\mathbf{cdga} \underset{\langle \cdot \rangle^{\mathrm{S}}}{\overset{A_{\mathrm{PL}}}{\leftrightarrows}} \mathbf{sset}\,.$$

In particular, for any simplicial set X and any cdga A, there is a natural bijection

$$\mathrm{Hom}_{\mathbf{cdga}}\big(A, A_{\mathrm{PL}}(X)\big) \cong \mathrm{Hom}_{\mathbf{sset}}(X, \langle A \rangle^{\mathrm{S}}). \qquad \square$$

If X is any topological space we abuse of notation and write $A_{\mathrm{PL}}(X)$ to denote $A_{\mathrm{PL}}(\mathrm{Sing}\, X)$.

For any cdga A whose cohomology satisfies $H(A) = H^{\geq 0}(A)$ and $H^0(A) \cong \mathbb{Q}$, there is a unique (up to cdga isomorphism) Sullivan minimal algebra $(\wedge V, d)$ and a quasi-isomorphism

$$\varphi \colon (\wedge V, d) \xrightarrow{\simeq} A.$$

The cdga $(\wedge V, d)$ is called the *Sullivan minimal model* of A. When $(\wedge V, d)$ is a Sullivan algebra (not necessarily minimal) we say accordingly that this is a *Sullivan model* of A. If $A = A_{\mathrm{PL}}(X)$ for a connected space or simplicial set X, then $(\wedge V, d)$ is called the *Sullivan minimal model* of X or simply a *Sullivan model* when it is not minimal. In any case, by construction,

$$H^*(X; \mathbb{Q}) \cong H^*(\wedge V, d)\,.$$

Moreover, the Sullivan minimal model of a connected simplicial set of finite type describes algebraically and faithfully the homotopy type of its $\mathbb{Q}$-completion in general, and its rationalization in the nilpotent case:

The *rationalization* of a nilpotent simplicial set X is another simplicial set $X_{\mathbb{Q}}$ equipped with a morphism $\rho\colon X \to X_{\mathbb{Q}}$ satisfying the following two properties:

(i) The reduced homology $\widetilde{H}_*(X_{\mathbb{Q}};\mathbb{Z})$ is a rational vector space.
(ii) The induced map $H_*(\rho;\mathbb{Q})$ is an isomorphism.

These two properties characterize the rationalization functor up to homotopy equivalences.

Concerning general, non-nilpotent simplicial sets, recall from [13] that the $\mathbb{Q}$-*completion functor*

$$\mathbb{Q}_\infty\colon \mathbf{sset} \longrightarrow \mathbf{sset}$$

assigns to a given simplicial set X the simplicial set $\mathbb{Q}_\infty X$ equipped with a natural morphism, $X \to \mathbb{Q}_\infty X$, governed by the following property:

A map $f\colon X \to Y$ induces an isomorphism in rational homology,

$$\widetilde{H}(f;\mathbb{Q})\colon \widetilde{H}_*(X;\mathbb{Q}) \xrightarrow{\cong} \widetilde{H}_*(Y;\mathbb{Q}),$$

if and only if the map

$$\mathbb{Q}_\infty f\colon \mathbb{Q}_\infty X \xrightarrow{\simeq} \mathbb{Q}_\infty Y$$

is a homotopy equivalence. The space $\mathbb{Q}_\infty X$ is called the $\mathbb{Q}$-*completion* of X, or the Bousfield–Kan $\mathbb{Q}$-completion of X.

Whenever X is a nilpotent simplicial set, the $\mathbb{Q}$-completion $\mathbb{Q}_\infty X$ and the rationalization $X_{\mathbb{Q}}$ have the same homotopy type [13, Chapter V, 4.3].

Then, given a connected simplicial set of finite type X, the natural map

$$\psi\colon X \longrightarrow \langle \wedge V, d\rangle^{\mathrm{S}},$$

the adjoint of

$$\varphi\colon (\wedge V, d) \xrightarrow{\simeq} A_{\mathrm{PL}}(X),$$

is the $\mathbb{Q}$-completion of X and is its rationalization in the nilpotent case. Moreover, the induced map $H_*(\psi;\mathbb{Q})$ is always injective.

In any case, for any $n \geq 1$, there are isomorphisms,

$$\mathrm{Hom}(V^n,\mathbb{Q}) \cong \pi_n\langle \wedge V, d\rangle^{\mathrm{S}},$$

which translates to isomorphisms,

$$\mathrm{Hom}(V^n,\mathbb{Q}) \cong \pi_n(X)\otimes\mathbb{Q}, \quad \text{for} \quad n > 1,$$

and

$$\operatorname{Hom}(V^1, \mathbb{Q}) \cong \mathbb{Q}_\infty \pi_1(X),$$

where the latter denotes the *Malcev* $\mathbb{Q}$*-completion* of the fundamental group of X. In general, a group G is *Malcev* $\mathbb{Q}$*-complete* if, given the central series of G,

$$G^1 \supset \cdots \supset G^n \supset \cdots \quad \text{where} \quad G^1 = G, \quad G^n = [G, G^{n-1}],$$

each G^n/G^{n+1} is a $\mathbb{Q}$-vector space and

$$G \cong \varprojlim_n G/G^n.$$

The *Malcev completion* of a group G is a morphism

$$G \longrightarrow \widehat{G}$$

where $\widehat{G}$ is Malcev $\mathbb{Q}$-complete and it induces isomorphisms,

$$G^n/G^{n+1} \otimes \mathbb{Q} \xrightarrow{\cong} \widehat{G}^n/\widehat{G}^{n+1}.$$

All of the above constitutes the basics of the Sullivan approach to rational homotopy theory which, in the past decades, has been the main tool for proving deep results in algebraic topology, differential geometry and mathematical physics. To name just one, we cite the dichotomy theorem for simply connected finite CW-complexes X: either $\pi_n(X)$ is a finite group for n large enough, or the sequence $\sum_{q \le n} \operatorname{rank} \pi_q(X)$ has an exponential growth. For further details on this topic the reader is referred to [51], and to [50] in the simply connected context.

1.2.2 Differential graded Lie algebras and the Quillen model of a space

A *graded Lie algebra*, or simply a Lie algebra, consists of a graded vector space L together with a linear product, called *the Lie bracket*,

$$[\ ,\ ]\colon L_p \otimes L_q \to L_{p+q}, \quad \text{for} \quad p, q \in \mathbb{Z},$$

that is antisymmetric,

$$[x, y] = -(-1)^{|x||y|}[y, x],$$

and satisfies the Jacobi identity,

$$(-1)^{|x||z|}\big[x, [y, z]\big] + (-1)^{|y||x|}\big[y, [z, x]\big] + (-1)^{|z||y|}\big[z, [x, y]\big] = 0.$$

A morphism of graded Lie algebras $f\colon L \to L'$ is a morphism of graded vector spaces preserving the Lie bracket: $f[x, y] = [f(x), f(y)]$.

The tensor algebra $T(V)$ on the graded vector space V is a graded Lie algebra in which the bracket is given by the commutator $[x,y] = x \otimes y - (-1)^{|x||y|} y \otimes x$. The *free graded Lie algebra* $\mathbb{L}(V)$ generated by V is the sub-Lie algebra of $T(V)$ generated by V. We denote by $\mathbb{L}^n(V)$ the linear span of the brackets of length n in V.

The free graded Lie algebra $\mathbb{L}(V)$ satisfies the following universality property: every linear map $f\colon V \to L$, where L is a graded Lie algebra, extends uniquely to a morphism of Lie algebras $\mathbb{L}(V) \to L$. In particular, as the free functor preserves colimits,

$$\mathbb{L}(V \oplus W) = \mathbb{L} \amalg \mathbb{L}(W),$$

where $\amalg$ denotes the coproduct.

The *universal enveloping algebra* UL of a graded Lie algebra L is defined as

$$UL = T(L)/I,$$

where I is the ideal generated by the elements of the form $x \otimes y - (-1)^{|x||y|} y \otimes x - [x,y]$. It readily follows that, given dgl's L and L',

$$U(L \times L') \cong UL \otimes UL'.$$

Moreover, for any free Lie algebra $\mathbb{L}(V)$,

$$U\mathbb{L}(V) \cong T(V).$$

Given a graded Lie algebra L, its *central series* is the sequence of ideals

$$L^1 \supset \cdots \supset L^n \supset L^{n+1} \supset \cdots$$

defined by $L^1 = L$ and $L^n = [L, L^{n-1}]$ for $n > 1$. A dgl is *nilpotent* if $L^n = 0$ for some n. Since dgl's are usually considered with the lower grading, there is no danger of confusion between the ideal L^n and the space L_n of elements of degree n. In a free Lie algebra $\mathbb{L}(V)$, the nth term of its central series is

$$\mathbb{L}(V)^n = \mathbb{L}^{\geq n}(V) = \bigoplus_{q \geq n} \mathbb{L}^q(V).$$

A *differential graded Lie algebra*, dgl for short, is a differential graded vector space L endowed with a graded Lie algebra structure for which the differential d is a derivation:

$$d[x,y] = [dx, y] + (-1)^{|x|}[x, dy].$$

A morphism $f\colon L \to L'$ of dgl's is both a morphism of differential graded vector spaces and of graded Lie algebras. We denote by **dgl** the category of differential graded Lie algebras, and by $\mathbf{dgl_n}$, for any given $n \in \mathbb{Z}$, the subcategory of **dgl** formed by the n-*connected* dgl's L, that is, $L = L_{\geq n}$. A dgl L is called *connected* if L is 0-connected, i.e., if $L = L_{\geq 0}$.

The tensor product $A \otimes L$ of a cdga A and a dgl L is a dgl with the bracket and the differential given by,

$$[a \otimes x, b \otimes y] = (-1)^{|b||x|} ab \otimes [x, y] \quad \text{and} \quad d(a \otimes x) = da \otimes x + (-1)^{|a|} a \otimes dx.$$

Here, A is considered with a differential of degree -1 and $A_n = A^{-n}$.

The set $\mathrm{Der}_n L$ of *derivations* of degree $n \in \mathbb{Z}$ of a dgl L consists of linear maps $\theta \colon L \to L$ of degree n such that

$$\theta[a, b] = [\theta(a), b] + (-1)^{|a|n}[a, \theta(b)], \quad \text{for} \quad a, b \in L.$$

The graded vector space

$$\mathrm{Der} L = \bigoplus_{n \in \mathbb{Z}} \mathrm{Der}_n L$$

is endowed with a dgl structure with the usual bracket and differential,

$$[\theta, \eta] = \theta \circ \eta - (-1)^{|\theta||\eta|} \eta \circ \theta,$$

$$(d\theta)(a) = d\theta(a) - (-1)^{|\theta|} \theta(da), \quad \text{for} \quad a \in L.$$

Observe that, given a derivation θ of L of even degree, and any $k \geq 1$,

$$\theta^k[a, b] = \sum_{i+j=k} \binom{k}{i} [\theta^i(a), \theta^j(b)], \quad \text{for} \quad a, b \in L. \tag{1.18}$$

By an abuse of language, a dgl L is called *free* if $L = (\mathbb{L}(V), d)$ is free as graded Lie algebra. A free dgl $(\mathbb{L}(V), d)$ is *minimal* if $dV \subset \mathbb{L}^{\geq 2}(V)$. For each dgl $L \in \mathbf{dgl}_1$, there is a unique (up to dgl isomorphism) minimal dgl $(\mathbb{L}(V), d)$ equipped with a quasi-isomorphism

$$\varphi \colon (\mathbb{L}(V), d) \xrightarrow{\simeq} L\,.$$

The dgl $(\mathbb{L}(V), d)$ is called *the Quillen minimal model* of L. If $(\mathbb{L}(V), d)$ is not minimal we accordingly say that this is just a *Quillen model.*

In his seminal paper [115], D. Quillen constructed a pair of functors

$$\mathbf{sset}_1 \underset{\langle\,\cdot\,\rangle^Q}{\overset{\lambda}{\rightleftarrows}} \mathbf{dgl}_1 \tag{1.19}$$

between the categories of 2-reduced simplicial sets and that of 1-connected dgl's, that is, differential graded Lie algebras which are positively graded. These functors are defined as the composition of the following pairs of adjoint functors in which the upper arrow denotes left adjoint,

$$\lambda \colon\ \mathbf{sset}_1 \underset{W}{\overset{G}{\rightleftarrows}} \mathbf{sgp_0} \underset{\mathcal{G}}{\overset{\widehat{\mathbb{Q}}}{\rightleftarrows}} \mathbf{sch_0} \underset{\mathcal{P}}{\overset{\widehat{U}}{\leftrightarrows}} \mathbf{sla_1} \underset{N}{\overset{N^*}{\leftrightarrows}} \mathbf{dgl}_1 : \langle\,\cdot\,\rangle^Q.$$

Here, $\mathbf{sgp_0}$, $\mathbf{sch_0}$ and $\mathbf{sla_1}$ denote respectively the categories of connected simplicial groups, connected simplicial complete Hopf algebras, and reduced simplicial Lie algebras. It turns out that each of these pairs induces equivalences on the corresponding homotopy categories when localizing on the family of rational homotopy equivalences in the simplicial categories $\mathbf{sset}_1$, $\mathbf{sgp_0}$, $\mathbf{sch_0}$, $\mathbf{sla_1}$ and on the family of quasi-isomorphisms in $\mathbf{dgl}_1$ [115, Theorem I]. One of the many interesting particularities of Quillen approach is the absence of finite type requirements in any of the categories involved.

By an abuse of notation, for any simply connected space X, we write $\lambda(X)$ to denote $\lambda(\operatorname{Sing} X)$. The *Quillen minimal model* of a simply connected space X is the Quillen minimal model of $\lambda(X)$. If $(\mathbb{L}(U), d)$ is the Quillen minimal model of X, there are isomorphisms, respectively of graded vector spaces and of graded Lie algebras,

$$U \cong s^{-1}\widetilde{H}_*(X;\mathbb{Q}) \quad \text{and} \quad H(\mathbb{L}(U), d) \cong \pi_*(\Omega X)\otimes\mathbb{Q},$$

where the Lie bracket in $\pi_*(\Omega X)\otimes\mathbb{Q}$, the *rational homotopy Lie algebra of* X, is given by the rationalization of the Samelson bracket.

The connection between the Sullivan minimal model $(\wedge V, d)$ of a simply connected CW-complex X of finite type and its Quillen minimal model $(\mathbb{L}(U), d)$ is provided by the work of M. Majewski [97]: the usual cochain algebra on $(\mathbb{L}(U), d)$ (see Definition 2.7) is quasi-isomorphic to the Sullivan minimal model of X. By this we mean that both cdga's are connected by a zigzag of quasi-isomorphisms.

Note here that the Lie algebra $\pi_*(\Omega X)\otimes\mathbb{Q}$ can also be deduced from the Sullivan minimal model $(\wedge V, d)$. Denote by $d_2\colon V\to\wedge^2 V$ the quadratic part of the differential d and define

$$L = s^{-1}\operatorname{Hom}(V,\mathbb{Q}). \tag{1.20}$$

Using the pairing $\langle\ ;\ \rangle\colon V\otimes sL\to\mathbb{Q}$, $\langle v; sx\rangle = (-1)^{|v|}sx(v)$, define a bilinear map $[\ ,\ ]\colon L\otimes L\to L$ by the formula

$$\langle v; s[x,y]\rangle = (-1)^{|y|+1}\langle d_2 v; sx, sy\rangle, \quad \text{for} \quad x,y\in L \quad \text{and} \quad v\in V. \tag{1.21}$$

The bracket $[\ ,\ ]$ makes L a graded Lie algebra, called the *rational homotopy Lie algebra of* $(\wedge V, d)$ and denoted by $\pi_{(\wedge V,d)}$. When X is simply connected and of finite type,

$$\pi_{(\wedge V,d)} \cong \pi_*(\Omega X)\otimes\mathbb{Q}.$$

1.2.3 Differential graded coalgebras

A *graded coalgebra*, or simply a coalgebra, is a graded vector space C equipped with a degree zero map $\Delta\colon C\to C\otimes C$ that is coassociative, i.e.,

$$(\Delta\otimes\mathrm{id}_C)\Delta = (\mathrm{id}_C\otimes\Delta)\Delta.$$

All graded coalgebras are assumed to have a *counit* and be coaugmented. That is, there is a degree-0 linear map $\varepsilon\colon C \to \mathbb{Q}$ and an element $u \in C_0$ such that

$$du = 0, \qquad \Delta(u) = u \otimes u, \qquad \varepsilon(u) = 1,$$

and for each $x \in C$,

$$\Delta x - (x \otimes u + u \otimes x) \in \ker \varepsilon \otimes \ker \varepsilon.$$

If we denote $\overline{C} = \ker \varepsilon$, the *reduced diagonal* $\overline{\Delta}\colon \overline{C} \to \overline{C} \otimes \overline{C}$ is defined by

$$\overline{\Delta}x = \Delta x - (u \otimes x + x \otimes u).$$

We set $\overline{\Delta}^1 = \overline{\Delta}$ and, for any $k \geq 2$, $\overline{\Delta}^k = (\overline{\Delta} \otimes \mathrm{id}_{\overline{C}}^{k-1}) \circ \overline{\Delta}^{k-1} : \overline{C} \to \overline{C}^{\otimes k+1}$. The subspace of *primitive elements of* C is defined as $\mathscr{P}(C) = \ker \overline{\Delta}$. A coalgebra C is *locally conilpotent* if $\overline{C} = \bigcup_{k \geq 1} \ker \overline{\Delta}^k$. Note that every coalgebra for which $\overline{C} = C_{\geq 1}$ is locally conilpotent.

A graded coalgebra C is (*co*)*commutative* if $\tau \circ \Delta = \Delta$, where $\tau\colon C \otimes C \to C \otimes C$ is the graded permutation of factors. We denote by **cgc** the category of commutative graded coalgebras.

The *tensor coalgebra* $T^{\mathrm{c}}(V)$ on the graded vector space V is the graded coalgebra on the vector space $T(V)$ whose reduced diagonal $\overline{\Delta}\colon T(V) \to T(V) \otimes T(V)$ is given by,

$$\overline{\Delta}[v_1|\cdots|v_n] = \sum_{i=1}^{n-1} [v_1|\cdots|v_i] \otimes [v_{i+1}|\cdots|v_n].$$

Here $[v_1|\cdots|v_n]$ denotes the element $v_1 \otimes \cdots \otimes v_n$.

On the other hand, the *cofree commutative coalgebra* generated by the graded vector space V is the graded coalgebra $\wedge V$ whose comultiplication $\Delta\colon \wedge V \to \wedge V \otimes \wedge V$ is the unique morphism of algebras such that

$$\Delta(v) = v \otimes 1 + 1 \otimes v\,.$$

For instance, $\overline{\Delta}(v_1 v_2) = v_1 \otimes v_2 + (-1)^{|v_1||v_2|} v_2 \otimes v_1$. This coalgebra is characterized by the following universality property: each linear map $C \to V$, from a locally conilpotent graded cocommutative coalgebra, extends uniquely to a morphism of coalgebras $C \to \wedge V$. For this reason, we will assume henceforth that any given coalgebra is locally conilpotent.

Note that the dual $C^{\#}$ of a (cocommutative) graded coalgebra C is a (commutative) graded algebra with the product,

$$(\alpha\beta)(c) = (\alpha \otimes \beta)(\Delta c), \quad \text{for} \quad \alpha, \beta \in C^{\#} \quad \text{and} \quad c \in C.$$

In particular, the dual $(\wedge V)^\#$ of the cofree cocommutative coalgebra is a commutative graded algebra which, whenever V is positively graded and of finite type, can be identified with the free commutative algebra $\wedge V^\#$ via the algebra isomorphism,

$$\wedge V^\# \xrightarrow{\cong} (\wedge V)^\# \tag{1.22}$$

induced by the inclusion $V^\# \hookrightarrow (\wedge V)^\#$. A detailed proof of this can be found in [50, Lemma 23.1].

A *differential graded coalgebra*, dgc from now on, or cdgc if it is commutative, is a differential graded vector space C with a graded coalgebra structure for which the differential d is a *coderivation*, that is,

$$\Delta d = (d \otimes \mathrm{id}_C + \mathrm{id}_C \otimes d)\Delta, \quad \text{with} \quad \varepsilon d = 0.$$

A *morphism* $f\colon C \to D$ of differential graded coalgebras is a map of chain complexes such that $(f \otimes f)\Delta = \Delta f$ and $\varepsilon_C = \varepsilon_D f$.

We denote by **dgc** and **cdgc** the categories of dgc's and cdgc's, respectively. Given $n > 0$, we denote also by $\mathbf{dgc}_n$ and $\mathbf{cdgc}_n$ the corresponding sub-categories consisting on dgc's or cdgc's C such that $\overline{C} = C_{\geq n}$.

The categories **dga** and **dgc** are related by the *bar* and *cobar* constructions which constitute a couple of adjoint functors,

$$\mathbf{dga} \underset{\Omega}{\overset{B}{\rightleftarrows}} \mathbf{dgc},$$

defined as follows [49], [83]:

Given $A \in \mathbf{dga}$, the usual (reduced) bar construction BA and the unreduced bar construction $B^{\mathrm{u}}A$ are the differential graded coalgebras defined by

$$BA = (T^{\mathrm{c}}(s\overline{A}), d_1 + d_2) \quad \text{and} \quad B^{\mathrm{u}}A = (T^{\mathrm{c}}(sA), d_1 + d_2),$$

with

$$d_1[sa] = -[sda] \quad \text{and} \quad d_2([sa_1|\cdots|sa_n]) = \sum_{i=2}^{n} (-1)^{n_i} [sa_1|\cdots|s(a_{i-1}a_i)|\cdots|sa_n]\,,$$

where $n_i = \sum_{j<i} |sa_j|$.

On the other hand, given $C \in \mathbf{dgc}$, the cobar construction ΩC is the dga given by

$$\Omega C = (T(s^{-1}\overline{C}), d_1 + d_2)$$

with

$$d_1(s^{-1}x) = -s^{-1}dx \quad \text{and} \quad d_2(s^{-1}x) = \sum_i (-1)^{|x_i|} s^{-1}x_i \otimes s^{-1}y_i,$$

where $\overline{\Delta}x = \sum x_i \otimes y_i$.

Theorem 1.3 ([49, Theorem 2.14]). *For any dga A and any dgc C the adjunction maps $\alpha_A\colon \Omega BA \xrightarrow{\simeq} A$ and $\beta_C\colon C \xrightarrow{\simeq} B\Omega C$ are quasi-isomorphisms.* □

A *differential graded Hopf algebra* is a graded vector space with structures of both dga and dgc for which the diagonal is a dga morphism. A Hopf algebra is *commutative* if it is so as a dga and as a dgc.

For instance, given a graded vector space V, consider in the tensor coalgebra $T^{\mathrm{c}}(V)$ the *shuffle product*

$$(v_1\otimes\cdots\otimes v_p)\cdot(v_{p+1}\otimes\cdots\otimes v_n)=\sum_{\sigma\in S(p,n-p)}\varepsilon_\sigma\; v_{\sigma^{-1}(1)}\otimes\cdots\otimes v_{\sigma^{-1}(n)},$$

where ε_σ is the associated Koszul sign and $S(p,n-p)$ denotes the set of $(p,n-p)$ shuffles, i.e., permutations σ such that $\sigma^{-1}(1)<\cdots<\sigma^{-1}(p)$ and $\sigma^{-1}(p+1)<\cdots<\sigma^{-1}(n)$. This product induces a structure of commutative algebra on $T^{\mathrm{c}}(V)$ which makes it a commutative Hopf algebra.

On the other hand, consider in the tensor algebra $T(V)$ the reduced diagonal,

$$\overline{\Delta}(v_1\otimes\cdots\otimes v_n)=\sum_{p=1}^{n-1}\sum_{\sigma\in S(p,n-p)}\varepsilon_\sigma(v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(p)})\otimes(v_{\sigma(p+1)}\otimes\cdots\otimes v_{\sigma(n)}).$$

This induces a structure of commutative graded coalgebra on $T(V)$ which makes it a cocommutative Hopf algebra.

In particular, given $A\in\mathbf{dga}$ and $C\in\mathbf{dgc}$, BA and ΩC are commutative and cocommutative differential graded Hopf algebras, respectively.

We finish by remarking that, for any Hopf algebra $\mathcal{H}$, the bracket given by the commutator $[x,y]=xy-(-1)^{|x||y|}yx$ induces on the space $\mathscr{P}(\mathcal{H})$ of primitives a structure of graded Lie algebra. Moreover, the *Milnor–Moore Theorem*, see [106, Theorem 5.18], asserts that whenever $\mathcal{H}$ is connected and of finite type, one can recover it from its primitives via the universal enveloping algebra:

$$U\mathscr{P}(\mathcal{H})\cong\mathcal{H}.$$

1.2.4 Differential graded Lie coalgebras

A *graded Lie coalgebra*, or simply a Lie coalgebra, is a graded vector space V with a comultiplication $\Delta\colon V\to V\otimes V$ such that its dual

$$V^\#\otimes V^\#\longrightarrow (V\otimes V)^\#\xrightarrow{\Delta^\#}V^\#$$

defines a Lie algebra structure on $V^\#$. This is equivalent to saying, see for instance [54, 104], that

$$(\mathrm{id}+\tau)\circ\Delta=0\quad\text{and}\quad(\mathrm{id}+\sigma+\sigma^2)\circ(\mathrm{id}\otimes\Delta)\circ\Delta=0,$$

where $\tau\colon V\otimes V\to V\otimes V$ is the graded permutation, $\sigma\colon V\otimes V\otimes V\to V\otimes V\otimes V$ is the graded cyclic permutation and id stands for the identity on V, $V\otimes V$ or $V\otimes V\otimes V$, respectively. Note, however, that in general the dual of a graded Lie algebra is not a Lie coalgebra unless it is of finite type.

As a first example observe that any graded coalgebra (C,Δ) admits a Lie coalgebra structure defined by $\Delta_{\mathrm{L}}=\Delta-\tau\circ\Delta$.

In particular, the tensor coalgebra $T^{\mathrm{c}}(V)$ on the graded vector space V is a Lie coalgebra with this structure. It turns out that the Lie coalgebra multiplication on $T^{\mathrm{c}}(V)$ induces also a Lie coalgebra structure on the indecomposables for the shuffle product. We denote this Lie coalgebra by

$$\mathbb{L}^{\mathrm{c}}(V)=T^{\mathrm{c}}(V)/T^{\mathrm{c}}(V)^{+}\cdot T^{\mathrm{c}}(V)^{+}$$

and call it the *free Lie coalgebra* on the graded vector space V.

The Lie coalgebra $\mathbb{L}^{\mathrm{c}}(V)$ satisfies the following universality property: If Γ is a graded Lie coalgebra and V a graded vector space, then every linear map $f\colon\Gamma\to V$ extends uniquely to a morphism of graded Lie coalgebras $\Gamma\to\mathbb{L}^{\mathrm{c}}(V)$ [54, §4.2.1].

Example 1.4. Denote by $p\colon T^{\mathrm{c}}(V)\to\mathbb{L}^{\mathrm{c}}(V)$ the projection, choose homogeneous elements $x,y\in V$, and set

$$[x,y]^{\mathrm{c}}=p(x\otimes y)\,.$$

Since $x\otimes y+(-1)^{|x||y|}y\otimes x$ is a shuffle product,

$$[x,y]^{\mathrm{c}}=p(x\otimes y)=-(-1)^{|x||y|}p(y\otimes x)=-(-1)^{|x||y|}[y,x]^{\mathrm{c}}\,.$$

As the Lie comultiplication Δ in $\mathbb{L}^{\mathrm{c}}$ is induced by the Lie comultiplication Δ_{L} in $T^{\mathrm{c}}(V)$, we have,

$$\Delta[x,y]^{\mathrm{c}}=p\Delta_{\mathrm{L}}(x\otimes y)=x\otimes y-(-1)^{|x||y|}y\otimes x\,.$$

Example 1.5. Let $(\wedge V,d)$ be a Sullivan minimal algebra, and let $d_2\colon V\to\wedge^2V$ be the quadratic part of the differential d. The *rational homotopy Lie coalgebra* of $(\wedge V,d)$ is (sV,Δ) where,

$$\Delta(sx)=-\sum_i(-1)^{|x_i|}\left(sx_i\otimes sx_i'-(-1)^{|sx_i||sx_i'|}sx_i'\otimes sx_i\right),$$

with $d_2x=\sum_i x_ix_i'$. The dual graded Lie algebra $(s^{-1}V^{\#},\Delta^{\#})$ is $\pi_{(\wedge V,d)}$, the rational homotopy Lie algebra of $(\wedge V,d)$ defined in Section 1.2.2.

A *differential graded Lie coalgebra*, dglc for short, is a graded Lie coalgebra C equipped with a differential d compatible with the comultiplication Δ:

$$\Delta\circ d=(d\otimes 1+1\otimes d)\circ\Delta.$$

We denote by **dglc** the category of dglc's. For any $n\in\mathbb{Z}$ we denote by $\mathbf{dglc}_n$ the subcategory of *n-connected* dglc's C which satisfy $\overline{C}=C^{\geq n}$.

1.2.5 A_∞-algebras

An A_∞-*algebra* is a graded vector space A, usually considered with upper grading, equipped with a sequence of linear maps of degree $2-n$,

$$m_n\colon A^{\otimes n} \longrightarrow A, \quad \text{for} \quad n \geq 1\,,$$

such that, for all $s \geq 1$ and $r, t \geq 0$,

$$\sum_{r+s+t=n} (-1)^{rs+t} m_{r+t+1}(\mathrm{id}^{\otimes r} \otimes m_s \otimes \mathrm{id}^{\otimes t}) = 0. \tag{1.23}$$

In particular, $m_1\colon A \to A$ is of degree 1 with $m_1^2 = 0$, $m_2\colon A \otimes A \to A$ has degree 0 and satisfies

$$m_1 \circ m_2 = m_2(m_1 \otimes \mathrm{id} + \mathrm{id} \otimes m_1)\,.$$

The map $m_3\colon A \otimes A \otimes A \to A$ has degree -1 and satisfies

$$m_1 m_3 - m_2(m_2 \otimes \mathrm{id}) + m_2(\mathrm{id} \otimes m_2) + m_3(m_1 \otimes \mathrm{id}^2 + \mathrm{id} \otimes m_1 \otimes \mathrm{id} + \mathrm{id}^2 \otimes m_1) = 0\,.$$

An A_∞-algebra A is *unital* if there is an element 1_A of degree 0 such that $m_1(1_A) = 0$, $m_2(1_A, a) = a = m_2(a, 1_A)$ for all $a \in A$ and such that, for all $i > 2$ and all $a_1, \ldots, a_i \in A$, the product $m_i(a_1, \ldots, a_i)$ vanishes if one of the a_i equals 1_A.

Note that a classical dga A is an A_∞-algebra with $m_i = 0$ for $i \geq 3$. In this case m_1 provides the differential and m_2 the product.

Remark that A_∞ structures on a graded vector space A are in one-to-one correspondence with differentials on the tensor coalgebra $T^{\mathrm{c}}(sA)$. Indeed, given an A_∞ structure on A, each map m_n produces a canonical map of degree 1, $d_n\colon (sA)^{\otimes n} \to sA$ (for recall $|sa| = |a| - 1$),

$$d_n = (-1)^n s \circ m_n \circ (s^{-1})^{\otimes n}\,, \tag{1.24}$$

that we extend to a coderivation on the coalgebra $T^{\mathrm{c}}(sA)$.

Since

$$\begin{aligned}
&d_{r+t+1}\,(\mathrm{id}^{\otimes r} \otimes d_k \otimes \mathrm{id}^{\otimes t}) \\
&\quad = (-1)^{r+t+1} s m_{r+t+1} (s^{-1})^{\otimes r+t+1}\,(\mathrm{id}^{\otimes r} \otimes d_k \otimes \mathrm{id}^{\otimes t}) \\
&\quad = (-1)^{r+t+1} s m_{r+t+1} (-1)^t \left((s^{-1})^{\otimes r} \otimes s^{-1} d_k \otimes (s^{-1})^{\otimes t}\right) \\
&\quad = (-1)^{r+t+k+1} (-1)^t s m_{r+t+1} \left((s^{-1})^{\otimes r} \otimes m_k (s^{-1})^{\otimes k} \otimes (s^{-1})^{\otimes t}\right) \\
&\quad = (-1)^{r+t+k+1} (-1)^{t+rk} s m_{r+t+1}\,(\mathrm{id}^{\otimes r} \otimes m_k \otimes \mathrm{id}^{\otimes t})\,(s^{-1})^{\otimes r+t+k},
\end{aligned}$$

in view of (1.23), we deduce that

$$\sum_{r+k+t=n} d_{r+t+1}(\mathrm{id}^{\otimes r} \otimes d_k \otimes \mathrm{id}^{\otimes t}) = 0,$$

and $(T^{\mathrm{c}}(sA), d)$ is a differential graded coalgebra. The converse also holds and thus $(A, \{m_i\})$ is an A_∞-algebra if and only if $(T^{\mathrm{c}}(sA), d)$ is a differential graded coalgebra.

We say that an A_∞-algebra A is *minimal* if $m_1 = 0$.

A *morphism of A_∞-algebras* $f\colon A \to A'$ consists of a sequence of linear maps $f_n\colon A^{\otimes n} \to B$ of degree $1-n$ such that the map, also denote by

$$f\colon (T^{\mathrm{c}}(sA), d) \longrightarrow (T^{\mathrm{c}}(sA'), d), \tag{1.25}$$

induced by the degree-0 linear maps,

$$(-1)^{\frac{n(n-1)}{2}} s \circ f_n \circ (s^{-1})^{\otimes k}\colon T^n(sA) \longrightarrow sA',$$

is a morphism of differential graded coalgebras. The morphism f is a *quasi-isomorphism* if $f_1\colon (A, m_1) \to (A', m_1)$ is a quasi-isomorphism of complexes.

A morphism between unital A_∞-algebras $f\colon A \to A'$ is *unital* if $f_1(1_A) = 1_{A'}$ and for $n > 1$ $f_n(a_1, \dots, a_n) = 0$ if one of the a_i equals 1_A.

An A_∞-algebra A is *augmented* if it is endowed with a unital morphism $\varepsilon\colon A \to \mathbb{Q}$ such that $\varepsilon(1_A) = 1$. The augmentation ideal $\overline{A} = \ker\varepsilon$ then inherits an A_∞-algebra structure. A *morphism of augmented A_∞-algebras* is a unital morphism $f\colon A \to A'$ such that $\varepsilon_{A'} \circ f = \varepsilon_A$. It induces a morphism between the augmentation ideals. We denote by $\mathbf{dga}_\infty$ the category of augmented A_∞-algebras.

The bar construction and its reduced version defined in Section 1.2.3 are extended to A_∞-algebras by the functors

$$B,\ B^u\colon\ \mathbf{dga}_\infty \longrightarrow \mathbf{dgc},$$

which assign to each A_∞-algebra A the corresponding dgc's,

$$BA = (T^{\mathrm{c}}(s\overline{A}), d) \quad \text{and} \quad B^{\mathrm{u}}A = (T^{\mathrm{c}}(sA), d). \tag{1.26}$$

Note that the unreduced version is defined in general for not necessarily augmented A_∞-algebras.

Proposition 1.6. *The functors B and B^{u} preserve quasi-isomorphisms.*

Proof. Let $f\colon A \xrightarrow{\simeq} A'$ be a quasi-isomorphism of augmented A_∞-algebras. The cdgc's BA and BA' are endowed with filtrations,

$$\{T^{\mathrm{c}\le p}(s\overline{A})\}_{p\ge 1} \quad \text{and} \quad \{T^{\mathrm{c}\le p}(s\overline{A'})\}_{p\ge 1},$$

that are preserved by Bf, which in view of (1.25), is denoted also by f. For each $p \ge 1$, let

$$f_p\colon T^{\mathrm{c}\le p}(s\overline{A}) \longrightarrow T^{\mathrm{c}\le p}(s\overline{A'})$$

be the induced map. It suffices to show that each f_p is a quasi-isomorphism. First, note that the induced map,

$$\begin{aligned} \overline{f}_p\colon (T^{\mathrm{c}})^p(s\overline{A}) &= T^{\mathrm{c}\le p}(s\overline{A}) \,/\, T^{\mathrm{c}\le p-1}(s\overline{A}) \\ &\xrightarrow{\simeq} T^{\mathrm{c}\le p}(s\overline{A'}) \,/\, T^{\mathrm{c}\le p-1}(s\overline{A'}) = (T^{\mathrm{c}})^p(s\overline{A'})\,, \end{aligned}$$

is a quasi-isomorphism for each p, since $f\colon A \xrightarrow{\simeq} A'$ is. Finally, recursively applying the five lemma to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T^{\mathrm{c}\le p-1}(s\overline{A}) & \longrightarrow & T^{\mathrm{c}\le p}(s\overline{A}) & \longrightarrow & (T^{\mathrm{c}})^p(s\overline{A}) & \longrightarrow & 0 \\ & & {\scriptstyle f_{p-1}}\big\downarrow & & {\scriptstyle f_{p-1}}\big\downarrow & & {\scriptstyle \overline{f}_p}\big\downarrow{\scriptstyle \simeq} & & \\ 0 & \longrightarrow & T^{\mathrm{c}\le p-1}(s\overline{A'}) & \longrightarrow & T^{\mathrm{c}\le p}(s\overline{A'}) & \longrightarrow & (T^{\mathrm{c}})^p(s\overline{A'}) & \longrightarrow & 0 \end{array}$$

we deduce that for each p the induced map f_p is a quasi-isomorphism as required. The same argument works for the unreduced version. □

An A_∞-algebra A is *commutative*, or it is a C_∞*-algebra* if, for each $k \ge 2$ and for each i, $1 \le i \le k-1$,

$$\sum_{\sigma \in S(i,k-i)} \varepsilon(\sigma)\, m_k \sigma(a_1 \otimes \cdots \otimes a_k) = 0,$$

where $\varepsilon(\sigma)$ is the signature of the permutation. For instance, any cdga is a C_∞-algebra with $m_i = 0$ for $i \ge 3$.

Any C_∞-algebra considered in this text is supposed to be augmented. We denote by $\mathbf{cdga}_\infty$ the category of C_∞-algebras.

When A is a C_∞-algebra, the differentials on $BA = (T^{\mathrm{c}}(s\overline{A}), d)$ and $B^{\mathrm{u}}A = (T^{\mathrm{c}}(sA), d)$ are derivations with respect to the shuffle products, see [88]. Hence, they induce differentials d on the quotients

$$\mathbb{L}^{\mathrm{c}}(s\overline{A}) = T^{\mathrm{c}}(s\overline{A})/T^{\mathrm{c}}(s\overline{A})^+ \cdot T^{\mathrm{c}}(s\overline{A})^+ \quad\text{and}\quad \mathbb{L}^{\mathrm{c}}(sA) = T^{\mathrm{c}}(sA)/T^{\mathrm{c}}(sA)^+ \cdot T^{\mathrm{c}}(sA)^+,$$

making

$$(\mathbb{L}^{\mathrm{c}}(s\overline{A}), d) \quad \text{and} \quad (\mathbb{L}^{\mathrm{c}}(sA), d) \tag{1.27}$$

differential graded Lie coalgebras.

Moreover, every morphism of C_∞-algebras $f\colon A \to A'$ preserves the shuffle products, and thus it induces natural morphisms of differential graded Lie coalgebras,

$$(\mathbb{L}^{\mathrm{c}}(s\overline{A}), d) \longrightarrow (\mathbb{L}^{\mathrm{c}}(s\overline{A'}), d') \qquad \text{and} \qquad (\mathbb{L}^{\mathrm{c}}(sA), d) \longrightarrow (\mathbb{L}^{\mathrm{c}}(sA'), d'). \tag{1.28}$$

This constitutes the definition of the functors $\mathscr{E}$ and $\mathscr{E}^{\mathrm{u}}$ on $\mathbf{cdga}_\infty$ (and in particular on $\mathbf{cdga}$) that will be studied in depth in the next chapter.

Next, we remark that any cdga is naturally quasi-isomorphic to a minimal C_∞-structure on its cohomology:

Theorem 1.7 ([87, 100]). *For any $A \in \mathbf{cdga}_0$ there exist a minimal C_∞-structure on its cohomology $H(A)$ and a natural quasi-isomorphism of C_∞-algebras*

$$H(A) \xrightarrow{\simeq} A. \qquad \square$$

This can be regarded from a general point of view with the aid of the *homotopy transfer theorem* [57, 90, 94, 103], a variation of the classical *homological perturbation lemma* [69, 70, 82, 86], which permits a transfer of any algebraic structure by a retraction. We give here a precise statement for cdga's as it is the only case we use.

A *transfer diagram*, also known as a *homotopy retraction*, is a diagram of the form

$$\phi \circlearrowright A \underset{i}{\overset{p}{\rightleftarrows}} V$$

where A and V are differential graded vector spaces, p and i are quasi-isomorphisms, $pi = \mathrm{id}_V$, and ϕ is a chain homotopy between id_A and ip, i.e., $\phi d + d\phi = \mathrm{id}_A - ip$, which satisfies $\phi i = p\phi = \phi^2 = 0$.

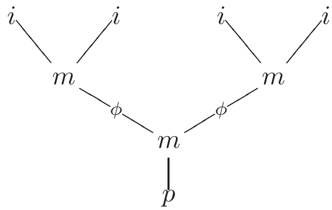
Theorem 1.8 ([36]). *Any transfer diagram in which A is a cdga induces a structure of C_∞-algebra on V with m_1 the differential on V, and quasi-isomorphisms of C_∞-algebras*

$$A \underset{I}{\overset{P}{\rightleftarrows}} (V, \{m_i\})$$

such that $PI = \mathrm{id}_{(V,\{m_i\})}$, $I_1 = i$ and $P_1 = p$. Moreover, if V is an augmented complex and p and i are augmentation preserving, then V becomes an augmented C_∞-algebra, and I, P are augmentation preserving. $\square$

The operations m_k's can be described explicitly. We give here such an expression in terms of trees, see for instance [90, §6.4]. A *planar tree* T is a directed, simply connected graph which can be embedded in the plane. The valence of a vertex is the number of edges having this vertex as the source. A leaf is a vertex of valence 0. A tree is *rooted* if there is a unique vertex (the root) with valence 1 which is the target of no edge. A rooted tree is *binary* if every vertex except the root and the leaves, has valence 2. For any $k \geq 2$, denote by $\mathcal{T}_k$ the set of isomorphism classes of planar rooted binary trees with k leaves.

To each $T \in \mathcal{T}_k$, we define a linear map $m_T \colon V^{\otimes k} \to V$ as follows: label the root by p, each internal edge by ϕ, each internal vertex by m, and each leaf by i. Then, m_T is defined as the composition of the different labels moving down from the leaves to the root. For instance the tree



produces $m_T = p \circ m \circ (\phi \circ m \otimes \phi \circ m) \circ i^{\otimes 4} \colon V^{\otimes 4} \to V$.

The transferred C_∞-algebra structure in V is given by $\{m_k\}_{k \geq 1}$, where $m_1 = d$ and, for $k \geq 2$,

$$m_k = \sum_{T \in \mathfrak{T}_k} m_T.$$

For instance,

$$\begin{aligned} m_2 &= p \circ m \circ (i \otimes i), \\ m_3 &= p \circ m \circ \big((\phi \circ m \circ (i \otimes i)) \otimes i\big) + p \circ m \circ \big(i \otimes (\phi \circ m \circ (i \otimes i))\big). \end{aligned} \tag{1.29}$$

We finish with three elementary results on transfer diagrams.

Proposition 1.9. *Let A be a differential graded vector space and let $H = (H(A), 0)$. Then, there exist a bijective correspondence between transfer diagrams of the form* $\phi \circlearrowright A \underset{i}{\overset{p}{\rightleftarrows}} H$ *and decompositions $A = B \oplus dB \oplus C$, where B is a complement of* $\ker d$ *(and thus $d \colon B \xrightarrow{\cong} dB$) and $C \xrightarrow{\cong} H$ is a given isomorphism. In particular, such diagrams always exist.*

Proof. Indeed, let $A = B \oplus dB \oplus C$ be such a decomposition. Define $i \colon H \xrightarrow{\cong} C \hookrightarrow A$, $p \colon A \twoheadrightarrow C \xrightarrow{\cong} H$ and $\phi(B) = \phi(C) = 0$, $\phi \colon dB \xrightarrow{\cong} B$ the inverse of d. Then $\phi \circlearrowright A \underset{i}{\overset{p}{\rightleftarrows}} H$ is a transfer diagram. Conversely, given such a diagram, notice that $d\phi d = d$. Then, $B = \phi dA$ is a complement of $\ker d$. Define $C = \operatorname{Im} i$; then an easy computation shows that $A = B \oplus dB \oplus C$. □

We now describe how to compose and tensor two given homotopy retractions. The proofs are mere checks.

Proposition 1.10. *Given homotopy retractions*

$$\phi \circlearrowright A \underset{i}{\overset{p}{\rightleftarrows}} V \quad \textit{and} \quad \psi \circlearrowright V \underset{j}{\overset{q}{\rightleftarrows}} W,$$

then $\phi + i\psi p \circlearrowright A \underset{ij}{\overset{qp}{\rightleftarrows}} W$ *is also a transfer diagram.* □

Proposition 1.11. *Given homotopy retractions*

$$\phi \circlearrowright \; A \underset{i}{\overset{p}{\rightleftarrows}} V \quad \text{and} \quad \psi \circlearrowright \; B \underset{j}{\overset{q}{\rightleftarrows}} W,$$

then $\gamma \circlearrowright \; A \otimes B \underset{i \otimes j}{\overset{p \otimes q}{\rightleftarrows}} V \otimes W$, *with* $\gamma(a \otimes b) = \phi(a) \otimes b + (-1)^{|a|} ip(a) \otimes \psi(b)$, *is also a transfer diagram.* □

1.3 Model categories

Closed model categories, or simply model categories in their current meaning, were first introduced by Quillen [114]. In this text, we follow the presentation given in [80] and [81], see also the introductory articles [46] and [64]. Recall that $\mathcal{C}$ is said to *have all small limits* (or *colimits*) if each functor $F\colon J \to \mathcal{C}$ from a small category has a limit (or a colimit).

A *model category* is a category $\mathcal{C}$ endowed with three distinguished classes of morphisms, called *fibrations*, *cofibrations* and *weak equivalences*, each of which is closed under composition, contains the isomorphisms, and is subject to the axioms below. A fibration that is also a weak equivalence is called a trivial fibration, and a cofibration that is also a weak equivalence is called a trivial cofibration.

CM 1. $\mathcal{C}$ is closed under small limits and colimits.

CM 2. If gf is defined and any two of f, g and gf are weak equivalences, then so is the third.

CM 3. A retract of a fibration, cofibration or weak equivalence is also a fibration, cofibration or weak equivalence respectively.

CM 4. Trivial cofibrations have the left lifting property with respect to fibrations, and cofibrations have the left lifting property with respect to trivial fibrations. Explicitly, if the following diagram commutes,

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y, \end{array}$$

where i is a cofibration and p is a fibration, and i or p is a weak equivalence, then there exists a map h making the diagram commutative.

CM 5. Every morphism f may be factored in two ways, $f = pi$ where p is a fibration, i a cofibration, and either p or i is a weak equivalence.

The axioms imply that any two of the three classes of maps determine the third one.

Also, the definition is self-dual: if $\mathcal{C}$ is a model category, then its opposite category $\mathcal{C}^{\mathrm{op}}$ also admits a model structure where weak equivalences correspond to their opposites, fibrations are the dual of cofibrations and cofibrations are the dual of fibrations.

From now on we often use $\twoheadrightarrow$, $\rightarrowtail$ and $\xrightarrow{\sim}$ to denote fibrations, cofibrations and weak equivalences, respectively.

A model category $\mathcal{C}$ has both a terminal object $*$ and an initial object $\emptyset$. An object $X \in \mathcal{C}$ is *cofibrant* if the only morphism $\emptyset \rightarrowtail X$ is a cofibration. Analogously, X is *fibrant* if the unique map $X \twoheadrightarrow *$ is a fibration. For any object X, applying the axiom CM5 to the map $\emptyset \to X$ gives a weak equivalence $Z \xrightarrow{\sim} X$ from a cofibrant object Z. Then Z is called a *cofibrant replacement* for X. Similarly, by starting from $X \to *$, we obtain a weak equivalence $X \xrightarrow{\sim} Z$ with Z fibrant; Z is called a *fibrant replacement* for X.

Under some restriction on the small category I, a model structure on a category $\mathcal{C}$ is inherited by the diagram category $\mathcal{C}^I$. In what follows I will denote a *direct category* [81, §5.1] or more generally, a *Reedy category* [80, §15.1]: identify first an ordinal λ with the category where there is a map $\alpha \to \beta$ if and only if $\alpha \leq \beta$. Then, a *direct category* is a small category $\mathcal{C}$ with a functor *degree* $d\colon \mathcal{C} \to \lambda$ such that the image of a non-identity map is a non-identity map. A *Reedy category* is a triple $\mathcal{C}, \mathcal{C}_+, \mathcal{C}_-$ consisting of a small category $\mathcal{C}$, two subcategories $\mathcal{C}_+$ and $\mathcal{C}_-$, both of which contain all the objects of $\mathcal{C}$, and a degree functor $d\colon \mathcal{C} \to \lambda$ for some ordinal λ such that:

(i) Every non-identity map in $\mathcal{C}_+$ raises the degree.

(ii) Every non-identity map in $\mathcal{C}_-$ lowers the degree.

(iii) Every map f can be factored uniquely as $f = gh$, where $h \in \mathcal{C}_-$ and $g \in \mathcal{C}_+$.

For instance, the category $\boldsymbol{\Delta}$ is a Reedy category in which $\boldsymbol{\Delta}_+$ contains the cofaces $\delta^i\colon [n-1] \to [n]$ and $\boldsymbol{\Delta}_-$ contains the codegeneracies $\sigma^i\colon [n+1] \to [n]$.

Theorem 1.12 ([80, Theorem 15.3.4], [81, Theorem 5.1.3])**.** *Given a direct or a Reedy category I and a model category $\mathcal{C}$, the category of diagrams $\mathcal{C}^I$ has a model structure for which a map $f = \{f_i\}_{i\in I}$ is a weak equivalence if each f_i is a weak equivalence. In the case of a direct category f is a fibration if each f_i is a fibration.* □

Let $\mathcal{C}$ be a model category and let $f, g\colon B \to X$ be maps in $\mathcal{C}$.

- A *cylinder object* for B is a factorization of the fold map $\nabla\colon B \amalg B \to B$ into a cofibration $i_0 \amalg i_1\colon B \amalg B \rightarrowtail \mathrm{Cyl}\,(B)$ followed by a weak equivalence $p\colon \mathrm{Cyl}\,(B) \xrightarrow{\sim} B$.
- A *path object* for X is a factorization of the diagonal map $\Delta\colon X \to X \times X$ into a weak equivalence $X \xrightarrow{\sim} X^I$ followed by a fibration $(p_0, p_1)\colon X^I \twoheadrightarrow X \times X$.
- A *left homotopy* from f to g is a map $H\colon \mathrm{Cyl}\,(B) \to X$ such that $Hi_0 = f$ and $Hi_1 = g$.

- A *right homotopy* from f to g is a map $H\colon B \to X^I$ such that $p_0 H = f$ and $p_1 H = g$.

Observe that cylinders and path objects always exist in a model category.

Proposition 1.13 ([81, Proposition 1.2.5]). *Let $f, g\colon B \to X$ be two maps in $\mathcal{C}$. If B is cofibrant and X is fibrant, then f and g are left homotopic if, and only if, they are right homotopic. We say that f is homotopic to g and write $f \sim g$; this equivalence relation does not depend on the choice of a cylinder or path object.* □

The *homotopy category* $\mathrm{Ho}\mathcal{C}$ of a model category $\mathcal{C}$ is the localization of $\mathcal{C}$ with respect to the class of weak equivalences.

The "fundamental theorem of model categories" states that $\mathrm{Ho}\mathcal{C}$ is equivalent to the category whose objects are the objects of $\mathcal{C}$ which are both fibrant and cofibrant, and whose morphisms are homotopy classes of maps, see for instance [81, Theorem 1.2.10].

A pair (F, G) of adjoint functors (F the left adjoint to G) between two model categories,

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D} \ ,$$

is a *Quillen pair* if any of the following equivalent conditions is satisfied:

(i) F preserves cofibrations and trivial cofibrations.

(ii) G preserves fibrations and trivial fibrations.

(iii) F preserves cofibrations and G preserves fibrations.

Given a Quillen pair (F, G), the functor F preserves weak equivalences between cofibrant objects and G preserves weak equivalences between fibrant objects. Moreover, they induce an adjunction

$$\mathrm{Ho}\,\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{Ho}\,\mathcal{D}$$

between the homotopy categories [81, Lemma 1.3.10]. In particular, for any fibrant object $D \in \mathcal{D}$ and any cofibrant object $C \in \mathcal{C}$ there is a natural bijection of homotopy classes of morphisms,

$$[F(C), D] \cong [C, G(D)]. \tag{1.30}$$

A *Quillen equivalence* is a Quillen pair (F, G) such that for each cofibrant object $C \in \mathcal{C}$ and each fibrant object $D \in \mathcal{D}$, a map $F(C) \xrightarrow{\sim} D$ is a weak equivalence in $\mathcal{D}$ if and only if its adjoint $C \xrightarrow{\sim} G(D)$ is a weak equivalence in $\mathcal{C}$.

Proposition 1.14 ([81, Proposition 1.3.13]). *A Quillen equivalence induces equivalences of categories between the associated homotopy categories.* □

Quillen pairs and Quillen equivalences are inherited by diagram categories. In what follows I is a direct or Reedy category.

Proposition 1.15 ([80, Proposition 15.4.1], [81, Corollary 5.1.6]). *Given a Quillen pair (F, G), the pair of induced functors on the diagram categories,*

$$\mathcal{C}^I \underset{G^I}{\overset{F^I}{\rightleftarrows}} \mathcal{D}^I,$$

is also a Quillen pair. In particular they induce adjoint functors in the homotopy categories,

$$\operatorname{Ho}\mathcal{C}^I \underset{G^I}{\overset{F^I}{\rightleftarrows}} \operatorname{Ho}\mathcal{D}^I\ .$$ □

Corollary 1.16. *In the adjunction induced by a Quillen pair,*

$$\operatorname{Ho}\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \operatorname{Ho}\mathcal{D},$$

the functor F preserves homotopy colimits and G preserves homotopy limits.

Proof. The homotopy colimit functor $\underleftarrow{\operatorname{hocolim}}\colon \operatorname{Ho}\mathcal{C}^I \to \operatorname{Ho}\mathcal{C}$ is the left adjoint to the "constant diagram" functor $K\colon \operatorname{Ho}\mathcal{C} \to \operatorname{Ho}\mathcal{C}^I$. Hence, for any objects $C^I \in \mathcal{C}^I$, $D \in \mathcal{D}$,

$$\begin{aligned} \operatorname{Hom}_{\operatorname{Ho}\mathcal{D}}(F\underleftarrow{\operatorname{hocolim}}\,C^I, D) &= \operatorname{Hom}_{\operatorname{Ho}\mathcal{C}}\big(\underleftarrow{\operatorname{hocolim}}\,C^I, GC\big) \\ = \operatorname{Hom}_{\operatorname{Ho}\mathcal{C}^I}(C^I, KGC) &= \operatorname{Hom}_{\operatorname{Ho}\mathcal{C}^I}\big(C^I, G^IK(C)\big) \\ = \operatorname{Hom}_{\operatorname{Ho}\mathcal{D}^I}(F^IC^I, KC) &= \operatorname{Hom}_{\operatorname{Ho}\mathcal{D}}(\underleftarrow{\operatorname{hocolim}}\,F^IC^I, C). \end{aligned}$$

Therefore, by the uniqueness of the adjoint, we have that

$$F\underleftarrow{\operatorname{hocolim}}\,C^I = \underleftarrow{\operatorname{hocolim}}\,F^IC^I.$$

The proof of the second assertion is completely analogous. □

We finish this introduction to model categories with two classical examples. On the one hand, **top** admits a standard model category structure with the usual (Serre) fibrations and weak homotopy equivalences as weak equivalences. The cofibrations are the maps that have the left lifting property with respect to the trivial Serre fibrations. Equivalently, they are the retracts of the relative cell complexes [81, Theorem 2.4.19]. All objects are fibrant in this structure.

On the other hand, **sset** admits a standard model category structure where fibrations are Kan fibrations, weak equivalences are precisely weak equivalences of simplicial sets, and cofibrations are monomorphisms of simplicial sets. In particular, all objects are cofibrant.

It turns out that the adjunction in (1.6),

$$\mathbf{sset} \underset{\mathrm{Sing}}{\overset{|\cdot|}{\rightleftarrows}} \mathbf{top},$$

is a Quillen equivalence. Therefore, the homotopy categories of simplicial sets and of topological spaces are equivalent. In other words, the category of topological spaces of the homotopy type of CW-complexes and homotopy classes of continuous maps is equivalent to the category of Kan complexes and homotopy classes of simplicial maps.

1.3.1 Differential model categories

There is a standard way, due to Hinich [77], to endow a differential category, i.e., the category **dvect** of (co)chain complexes enriched with some additional structure, with a model category structure.

Let $\mathcal{C}$ be a category admitting finite limits and arbitrary colimits and let

$$\mathbf{dvect} \underset{\mathcal{U}}{\overset{F}{\rightleftarrows}} \mathcal{C},$$

be a pair of adjoint functors where $\mathcal{U}$ commutes with filtered colimits and the following holds:

Let $(v, dv) \in \mathbf{dvect}$ be the differential graded vector space with one generator v in degree n and another generator dv in degree $n-1$. We assume that, for any $C \in \mathcal{C}$, the canonical map $C \to C \amalg F(v, dv)$ is taken to a weak equivalence,

$$\mathcal{U}(C) \xrightarrow{\sim} \mathcal{U}\big(C \amalg F(v, dv)\big).$$

Theorem 1.17 ([77, Theorem 2.2.1]). *The category $\mathcal{C}$ has a model category structure in which a morphism f is a weak equivalence if $\mathcal{U}(f)$ is a weak equivalence; f is a fibration if $\mathcal{U}(f)$ is surjective; and f is a cofibration if it has the left lifting property with respect to trivial fibrations.* □

We may apply this result to the categories **dga**, **cdga** or **dgl**, choosing as $\mathcal{U}$ the forgetful functor and as F the "free" functor which, on each of these cases is defined respectively by,

$$(T(V), d), \quad (\wedge V, d), \quad \text{and} \quad (\mathbb{L}(V), d), \quad \text{for} \quad (V, d) \in \mathbf{dvect}\,.$$

In particular, we have:

Theorem 1.18. *The categories* **dga**, **cdga** *and* **dgl** *admit a model category structure in which the weak equivalences are the quasi-isomorphisms, the fibrations are the surjective morphisms and the cofibrations are the morphisms satisfying the left lifting property with respect to trivial fibrations.* □

This generalizes previous classical bounded versions for connected dga's [12, Theorem 4.3], connected cdga's [84, Theorem 5] or reduced dgl's [115, Theorem II] (note the subtle difference in the latter category, where fibrations are dgl morphisms that are surjective in degrees greater than 1).

As for differential graded coalgebras, a similar result holds as long as we do not assume the existence of infinite limits. This restriction disappears if we restrict to bounded graded coalgebras and then the following generalizes [109, Proposition 5.2] for $\mathbf{cdgc}_1$.

Theorem 1.19 ([62])**.** *The category* **cdgc** *admits a model category structure* (*without infinite limits*) *in which the weak equivalences are the quasi-isomorphisms, the cofibrations are the injective morphisms and the fibrations are the morphisms satisfying the right lifting property with respect to trivial cofibrations.* □

In [77, §2.2.3] there is a description of the resulting cofibrations and cofibrant objects in the model category $\mathcal{C}$ when Theorem 1.17 is applied. In the particular differential categories we are interested in we obtain:

In **cdga** all objects are fibrant. Sullivan algebras are cofibrant objects and any cdga has a cofibrant replacement which is a Sullivan algebra. In $\mathbf{dgl}_0$ all objects are also fibrant. Free dgl's $(\mathbb{L}(V), d)$ are cofibrant and constitute cofibrant replacements of any given dgl.

On the other hand, in $\mathbf{cdgc}_1$ every object is cofibrant, while an object C is fibrant if and only if it is constructed by elementary extensions starting from the ground field $\mathbb{Q}$ ([109, Proposition 5.7 and 5.8]). Recall that an *elementary extension* of cdgc's is a sequence of cdgc's of the form

$$(\wedge V, d) \longrightarrow C \longrightarrow C'$$

which is obtained as a pullback diagram

$$\begin{array}{ccccc} (\wedge V, 0) & \longrightarrow & C & \longrightarrow & C' \\ \downarrow \cong & & \downarrow & & \downarrow f \\ (\wedge V, 0) & \longrightarrow & (\wedge(V \oplus E), d) & \longrightarrow & (\wedge E, 0) \end{array}$$

where $V = d(E)$. If we forget the differential, then $C \cong C' \otimes \wedge V$. The extension is called *non-primitive* if the map induced by f on the primitive elements is zero.

In particular, in $\mathbf{cdgc}_1$ the finite type cofibrant objects are cofree cdgc's of the form $(\wedge V, d)$ whose duals, via (1.22), are Sullivan algebras.

As for path and cylinder objects, a *path object* for a cdga B is defined by

$$B \longrightarrow B \otimes \wedge(t, dt) \overset{\varepsilon_1}{\underset{\varepsilon_0}{\rightrightarrows}} B\,,$$

where $|t| = 0$ and $\varepsilon_i(t) = i$. Therefore, for any cofibrant object A, two morphisms $f, g\colon A \to B$ are homotopic if there is a morphism $H\colon A \to B \otimes \wedge(t, dt)$ such that $\varepsilon_0 \circ H = f$ and $\varepsilon_1 \circ H = g$. Analogously, a *path object* for a dgl L is defined as,

$$L \longrightarrow L \otimes \wedge(t, dt) \overset{\varepsilon_1}{\underset{\varepsilon_0}{\rightrightarrows}} L\,,$$

where $|t| = 0$ and $\varepsilon_i(t) = i$.

On the other hand, a *cylinder object* for a dgl of the form $(\mathbb{L}(V), d)$ is the dgl

$$(\mathbb{L}(V \oplus V' \oplus sV), D),$$

where $(\mathbb{L}(V), D)$ and $(\mathbb{L}(V'), D)$ are copies of $(\mathbb{L}(V), d)$, and for $v \in V_n$, $D(sv) - (v - v') + sd_1 v \in \mathbb{L}^{\geq 2}(V_{<n} \oplus V'_{<n} + \oplus sV_{<n})$ (Here d_1 denotes the linear part of the differential, $d_1(V) \subset V$). The description of a cylinder object for a Sullivan algebra $(\wedge V, d)$ is analogous.

We finish with two examples of Quillen pairs involving these model differential categories that constitute the main results on which classical rational homotopy theory lies.

Theorem 1.20 ([12, Theorem 9.4], [128])**.** *The adjoint functors A_{PL} and $\langle \cdot \rangle^{\mathrm{S}}$ of Theorem 1.2 form a Quillen pair whose derived functors restrict to equivalences between the categories of connected minimal Sullivan algebras of finite type and rational nilpotent Kan complexes of finite $\mathbb{Q}$-type.* □

On the other hand, Quillen proved that all the pairs in (1.2.2) are Quillen equivalences with the single restriction being to the subcategory $\mathbf{sset}_1^{\mathbb{Q}}$ of simply connected rational simplicial sets. In particular:

Theorem 1.21 ([115, Theorem I])**.** *The functors λ and $\langle \cdot \rangle^{\mathrm{Q}}$ induce equivalences of categories*

$$\mathrm{Ho}\,\mathbf{sset}_1^{\mathbb{Q}} \underset{\langle \cdot \rangle^{\mathrm{Q}}}{\overset{\lambda}{\rightleftarrows}} \mathrm{Ho}\,\mathbf{dgl}_1$$

that are inverse of each other. □

1.3.2 Cofibrantly generated model categories

Let I be a collection of morphisms in a category $\mathcal{C}$. An object $A \in \mathcal{C}$ is *small relative to I* if for all sequences of maps in I, $X_0 \to \cdots \to X_\beta \to \ldots$, we have

$$\varinjlim_{\beta} \mathrm{Hom}_{\mathcal{C}}(A, X_\beta) = \mathrm{Hom}_{\mathcal{C}}(A, \varinjlim_{\beta} X_\beta)\,.$$

The set I is said to *permit the small object argument* if the domains of the maps in I are small relative to I.

A *relative I-cell complex* is a composition of morphisms $X_0 \to \cdots \to X_\beta \to \ldots$, each of which is obtained as a pushout,

$$\begin{array}{ccc} C_\beta & \longrightarrow & X_\beta \\ {\scriptstyle g_\beta}\downarrow & & \downarrow \\ D_\beta & \longrightarrow & X_{\beta+1}, \end{array}$$

with $g_\beta \in I$. An object $A \in \mathcal{C}$ is a *I-cell complex* if $0 \to A$ is a relative I-cell complex.

A model category $\mathcal{C}$ is *cofibrantly generated* if there are sets $\mathcal{I}$ and $\mathcal{J}$ of maps satisfying the following properties.

(i) The sets $\mathcal{I}$ and $\mathcal{J}$ permit the small object argument.

(ii) The fibrations are the maps that have the right lifting property with respect to the maps in $\mathcal{J}$.

(iii) The trivial fibrations are the maps that have the right lifting property with respect to the maps in $\mathcal{I}$.

We call $\mathcal{I}$ the set of *generating cofibrations* and $\mathcal{J}$ the set of *generating trivial cofibrations*. Indeed, directly from the definition it is not hard to see that cofibrations in $\mathcal{C}$ are precisely retracts of relative $\mathcal{I}$-complexes, and trivial cofibrations in $\mathcal{C}$ are retracts of $\mathcal{J}$-complexes.

The model category on **sset**, described at the beginning of this section, is cofibrantly generated by the sets of cofibrations $\mathcal{I}$ and trivial cofibrations $\mathcal{J}$ given by,

$$\mathcal{I} = \{\underline{\dot{\Delta}}^n \rightarrowtail \underline{\Delta}^n\}_{n\geq 0} \quad \text{and} \quad \mathcal{J} = \{\underline{\Lambda}_i^n \rightarrowtail \underline{\Delta}^n\}_{n\geq 0, o\leq i\leq n}.$$

A structure of cofibrantly generated model category may be transferred by the left adjoint functor of an adjunction, with the so-called *Transfer Principle* which we now detail.

Let $\mathcal{C}$ be a model category cofibrantly generated by the sets $\mathcal{I}$ and $\mathcal{J}$ of generating cofibrations and generating trivial cofibrations, respectively. Let $\mathcal{D}$ be a category with finite limits and small colimits, and let

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

be a pair of adjoint functors (upper arrow denotes left adjoint). A map f in $\mathcal{D}$ is called a weak equivalence or a fibration if $G(f)$ is a weak equivalence or a fibration respectively. Then, we have:

Theorem 1.22 ([80, Theorem 11.3.2])**.** *There is a model category in $\mathcal{D}$, cofibrantly generated by the families $F(\mathcal{I})$ and $F(\mathcal{J})$, and whose weak equivalences and fibrations are as above, provided the following two conditions:*

(i) *The sets $F(\mathcal{I})$ and $F(\mathcal{J})$ permit the small object argument.*

(ii) *The functor G takes relative $F(\mathcal{J})$-cell complexes to weak equivalences.* □

Remark 1.23. In some cases, property (ii) is often difficult to verify. However, as remarked in [7, §2.6], an argument of Quillen [114, II.4] can be applied to see that this condition is satisfied if the following holds:

(a) $\mathcal{D}$ has a fibrant replacement functor.

(b) $\mathcal{D}$ has functorial path objects for fibrant objects.

As the model on $\mathcal{D}$ inherited by the Transfer Principle deliberately preserve fibrations and weak equivalences, automatically it follows that:

Corollary 1.24. *The functors*

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

constitute a Quillen pair. □

Chapter 2

The Quillen Functors $\mathscr{L}$, $\mathscr{C}$ and their Duals $\mathscr{A}$, $\mathscr{E}$

The classical pair of adjoint functors

$$\mathbf{cdgc} \underset{\mathscr{C}}{\overset{\mathscr{L}}{\rightleftarrows}} \mathbf{dgl}$$

was first introduced by Quillen in [114, Appendix B] where their main homotopical properties were outlined and proved under certain connectivity assumptions. Namely, see [114, Theorem 7.5], as long as $L \in \mathbf{dgl}_1$ and $C \in \mathbf{cdgc}_2$, both functors preserve quasi-isomorphisms, and both adjunctions maps $\alpha_L\colon \mathscr{L}\mathscr{C}(L) \xrightarrow{\simeq} L$, $\beta_C\colon C \xrightarrow{\simeq} \mathscr{C}\mathscr{L}(C)$, are also quasi-isomorphisms.

Later on, Neisendorfer showed that these connectivity restrictions can be slightly relaxed: for any $L \in \mathbf{dgl}_0$ and any $C \in \mathbf{cdgc}_1$, both α_L and β_C are quasi-isomorpisms [109, Proposition 4.1]. Moreover, $\mathscr{C}$ preserves quasi-isomorphisms in $\mathbf{dgl}_0$ [109, Proposition 4.1(a)].

In this chapter we first present a detailed description of these functors. Moreover, see [78, Proposition 3.3.2], we extend once again these properties in a self contained presentation by showing that: α_L is a quasi-isomorphism for any $L \in \mathbf{dgl}$, $\mathscr{C}$ always preserves quasi-isomorphisms, and $\mathscr{L}$ preserves quasi-isomorphisms between finite type fibrant coalgebras in $\mathbf{cdgc}_1$. The need for this extension will be revealed in Chapters 3 and 9.

Also, as a key tool in some forthcoming fundamental results, we need functors that behave as "duals" of $\mathscr{L}$ and $\mathscr{C}$. Straight dualization defines a functor from **cdgc** to **cdga**, but the dual of a dgl is not a dglc unless finite type assumptions are imposed. Hence, inspired by the work by Sinha and Walter [122], we also introduce in this chapter another pair of adjoint functors,

$$\mathbf{cdga} \underset{\mathscr{E}}{\overset{\mathscr{A}}{\leftrightarrows}} \mathbf{dglc},$$

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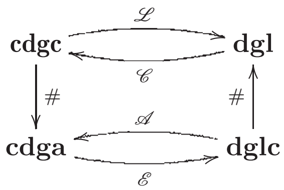
which solves this problem. Moreover, we extend the functor $\mathscr{E}$ and its "unreduced" version $\mathscr{E}^{\mathrm{u}}$ to the infinity category $\mathscr{E}, \mathscr{E}^{\mathrm{u}}\colon \mathbf{cdga}_\infty \to \mathbf{dglc}$. We then show the analog of the above homotopical properties whose almost complete lack of connectivity and finiteness assumptions were predicted in [122]: $\mathscr{E}$ and $\mathscr{E}^{\mathrm{u}}$ preserve quasi-isomorphisms, while $\mathscr{A}$ preserve quasi-isomorphisms when restricted to $\mathbf{dglc}_0$; the adjunction map $\alpha'_A\colon \mathscr{A}\mathscr{E}(A) \xrightarrow{\simeq} A$ is a quasi-isomorphism for any cdga A which is either of finite type or connected; and finally, the other adjunction $\beta'_E\colon E \xrightarrow{\simeq} \mathscr{E}\mathscr{A}(E)$ is a quasi-isomorphism for any "Sullivan dglc" E.

Again, while the dual of a dglc is always a dgl, this is not the case for a cdga unless finite type is assumed. Thus, as a general picture, these two pairs of adjoint functors fit schematically in the diagram



where the two squares that it contains are not commutative in general unless strong finiteness and connectivity assumptions are imposed (see Remark 2.17). Some of this restrictions can be avoided by considering the "completed categories" involved and replacing the dual functor by its topological dual version. However, except for the completion of dgl's of course, none of this will be needed and this material is therefore omitted.

2.1 The functors $\mathscr{L}$ and $\mathscr{C}$

We begin with their definitions. Given a cdgc C define

$$\mathscr{L}(C) = (\mathbb{L}(s^{-1}\overline{C}), d),$$

with $d = d_1 + d_2$, where

$$\begin{aligned} d_1(s^{-1}c) &= -s^{-1}dc, \\ d_2(s^{-1}c) &= \frac{1}{2}\sum_i (-1)^{|a_i|}[s^{-1}a_i, s^{-1}b_i] \quad \text{with} \quad \overline{\Delta}c = \sum_i a_i \otimes b_i. \end{aligned}$$

On the other hand, given a dgl L, define the *chain coalgebra*

$$\mathscr{C}(L) = (\wedge(sL), d),$$

with $d = d_1 + d_2$, where

$$d_1(sv_1 \wedge \cdots \wedge sv_n) = -\sum (-1)^{n_i} sv_1 \wedge \cdots \wedge s(dv_i) \wedge \cdots \wedge sv_n,$$
$$d_2(sv_1 \wedge \cdots \wedge sv_n) = \sum_{1\le i<j\le k} (-1)^{|sv_i|}\, \rho_{ij}\, s[v_i, v_j] \wedge sv_1 \wedge \cdots \wedge \widehat{sv_i} \wedge \cdots \wedge \widehat{sv_j} \wedge \cdots \wedge sv_n,$$

$n_i = \sum_{j<i} |sv_j|$ and ρ_{ij} is the Koszul sign of the permutation

$$sv_1 \wedge \cdots \wedge sv_n \longmapsto sv_i \wedge sv_j \wedge sv_1 \wedge \cdots \wedge \widehat{sv_i} \wedge \cdots \wedge \widehat{sv_j} \wedge \cdots \wedge sv_n.$$

In particular,

$$d_1(sv) = -sdv \quad \text{and} \quad d_2(sv \wedge sw) = (-1)^{|sv|} s[v, w]\,.$$

As usual, we denote by $\overline{\mathscr{C}}(L)$ the augmentation ideal of $\mathscr{C}(L)$.

We point out the classical relations of $\mathscr{L}$ and $\mathscr{C}$ with the universal enveloping algebra functor. On the one hand one easily checks that the universal enveloping algebra $U\mathscr{L}(C) = (T(s^{-1}\overline{C}), d)$ is precisely the differential Hopf algebra given by the cobar construction ΩC. In other words, as observed in Section 1.2.3, $\mathscr{L} = \mathscr{P}\Omega$, where $\mathscr{P}$ denotes the functor of primitive elements.

On the other hand, the cdgc $\mathscr{C}(L)$ is related to the bar construction on UL. More precisely, see [50, Theorem 22.7], there is a quasi-isomorphism,

$$\zeta\colon \mathscr{C}(L) \xrightarrow{\simeq} BUL\,, \tag{2.1}$$

defined by

$$\zeta(sx_1 \wedge \cdots \wedge sx_k) = \sum_{\sigma\in\Sigma_k} \varepsilon_\sigma [sx_{\sigma(1)}|\ldots|sx_{\sigma(k)}]\,.$$

Here, ε_σ is the Koszul sign of the permutation

$$sx_1 \wedge \cdots \wedge sx_k \longmapsto sx_{\sigma(1)} \wedge \cdots \wedge sx_{\sigma(k)}\,.$$

The following proposition is a direct consequence of the definition of $\mathscr{C}$ and $\mathscr{L}$ as such its proof is omitted.

Proposition 2.1. *The functor $\mathscr{L}$ is left adjoint to $\mathscr{C}$,*

$$\mathbf{cdgc} \underset{\mathscr{C}}{\overset{\mathscr{L}}{\rightleftarrows}} \mathbf{dgl}. \qquad \square$$

The adjunction maps

$$\alpha_L\colon \mathscr{L}\mathscr{C}(L) \longrightarrow L \quad \text{and} \quad \beta_C\colon C \longrightarrow \mathscr{C}\mathscr{L}(C)$$

are defined as follows: on the one hand, α_L is the unique dgl morphism

$$(\mathbb{L}(s^{-1}\wedge^+ sL), d) \to L$$

extending the projection

$$s^{-1}\wedge^{+}sL \longrightarrow s^{-1}\wedge^{+}sL/\left(s^{-1}\wedge^{\geq 2} sL\right) \cong L.$$

On the other hand, β_C is the unique cdgc morphism $C \to (\wedge s\mathscr{L}(C), d)$ lifting the inclusion

$$\overline{C} \cong ss^{-1}\overline{C} \subset s\mathbb{L}(s^{-1}\overline{C}).$$

To prove that, for any dgl and any cdgc in $\mathbf{cdgc}_1$, both adjunctions are quasi-isomorphisms, we need a very slight extension of [109, Proposition 4.2].

Note that, by definition, given a cofree cdgc of the form $(\wedge V, d)$, the chain complex $(s^{-1}V, s^{-1}d_1)$ is a sub-complex of $\mathscr{L}(\wedge V, d)$ whose underlying graded Lie algebra is $\mathbb{L}(s^{-1}\wedge^{+} V)$. In the same way, given a free dgl of the form $(\mathbb{L}(V), d)$, the chain complex (sV, sd_1) is a quotient complex of $\overline{\mathscr{C}}(\mathbb{L}(V), d)$ whose underlying graded coalgebra is $\wedge^{+}s\mathbb{L}(V)$. As always, d_1 denotes the linear part of d.

Lemma 2.2.

(i) *For any cdgc of the form $(\wedge V, d)$ the injection*

$$(s^{-1}V, s^{-1}d_1) \xhookrightarrow{\simeq} \mathscr{L}(\wedge V, d)$$

is a quasi-isomorphism of complexes.

(ii) *For any $L \in \mathbf{dgl}_0$ of the form $(\mathbb{L}(V), d)$, the projection*

$$\overline{\mathscr{C}}(L) \xrightarrow{\simeq} (sV, sd_1)$$

is a quasi-isomorphism of complexes.

Proof. (i) Suppose first that the differential is zero in $\wedge V$. Then let W be a graded vector space with $W = W_{\geq 2}$ equipped with a non-graded isomorphism $W \to V$ preserving the parity of the degree of elements. Hence, $\wedge V$ is isomorphic to $\wedge W$. Moreover, $\wedge W = \mathscr{C}(s^{-1}W)$ where $s^{-1}W$ is considered as an abelian Lie algebra in $\mathbf{dgl}_1$. In this reduced category, by [115, Theorem 7.5], we have a dgl quasi-isomorphism $\Phi\colon \mathscr{L}(\wedge W) \xrightarrow{\simeq} s^{-1}W$ such that the composition

$$s^{-1}W \hookrightarrow \mathscr{L}(\wedge W) \xrightarrow{\Phi} s^{-1}W$$

is the identity on $s^{-1}W$ and thus the inclusion $s^{-1}W \hookrightarrow \mathscr{L}(\wedge W)$ is a quasi-isomorphism of complexes. Therefore, by the commutativity of the diagram

$$\begin{array}{ccc} s^{-1}W & \xhookrightarrow{\quad\simeq\quad} & \mathscr{L}(\wedge W) \\ \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\ s^{-1}V & \xhookrightarrow{\qquad\quad} & \mathscr{L}(\wedge V), \end{array}$$

the lower injection is also a quasi-isomorphism.

For a general $C = (\wedge V, d)$ denote by Γ the quotient complex

$$\Gamma = \mathscr{L}(\wedge V, d)/(s^{-1}V, s^{-1}d).$$

The statement amounts to proving that $H(\Gamma) = 0$. To do this, observe that the bracket length in $\mathscr{L}(\wedge V, d) = (\mathbb{L}(s^{-1} \wedge^+ V), d)$ induces a grading on Γ for which we write $\Gamma = \bigoplus_{q\geq 1} \Gamma_q$. The differential of Γ is the sum $d = d_1 + d_2$, where d_1 preserves this grading and d_2 increases it by one. Next, note that $(\mathscr{L}(\wedge V), d_2) = \mathscr{L}(\wedge V, 0)$. Therefore, we can use the former special case to conclude that the inclusion $(s^{-1}V, 0) \xrightarrow{\simeq} \mathscr{L}(\wedge V, 0)$ is a quasi-isomorphism, which is equivalent to $H(\Gamma, d_2) = 0$.

Now, if $x \in \Gamma$ is a d-cycle, then for some $n \geq 1$, $x = x_1 + \cdots + x_n$ with $x_q \in \Gamma_q$ for $q = 1, \ldots, n$. Hence $d_2(x_n) = 0$ and thus $x_n = d_2(y)$ for some y. Then, replacing x by $x - dy$, we get a new cycle homologous to x in $\Gamma_{<n}$. By iteration, in the homology of Γ, $[x] = [z]$ with $z \in \Gamma_{<1} = 0$. Therefore, $H(\Gamma) = 0$ and the inclusion $(s^{-1}V, s^{-1}d_1) \xrightarrow{\simeq} \mathscr{L}(\wedge V, d)$ is a quasi-isomorphism.

(ii) We use an analogous argument. Consider first the case where L has zero differential. Then, we may modify the degrees and suppose that $L = L_{\geq 1}$. Then, $BUL = BT(V) = B\Omega Z$, where Z is the coalgebra $\mathbb{Q} \oplus sV$ with trivial comultiplication and differential. Write $B\Omega Z = T^c\big(sT(s^{-1}Z)\big)$.

In this particular case, the adjunction quasi-isomorphism $\beta_Z \colon Z \xrightarrow{\simeq} B\Omega Z$ of Theorem 1.3 has a retraction given by the projection $\gamma \colon B\Omega Z \to Z$ which is the identity on $s(s^{-1}Z)$ and maps $T^p\big(sT^q(s^{-1}Z)\big)$ to 0 when p or q are greater or equal than 2.

Finally, observe that the projection $\overline{\mathscr{C}}(L) \twoheadrightarrow (sV, sd_1)$ is the restriction to the respective augmentation ideals of the following sequence of quasi-isomorphisms

$$\mathscr{C}(L) \xrightarrow{\zeta} BUL \cong B\Omega Z \xrightarrow{\gamma} Z$$

and thus, it is a quasi-isomorphism. Here, ζ is the cdgc quasi-isomorphism of (2.1).

In the general case, denote by K the kernel of the projection $\overline{\mathscr{C}}(L) \twoheadrightarrow (sV, sd_1)$. The word length in $\mathscr{C}(L)$ induces a grading on K for which we write $K = \bigoplus_{q\geq 1} K_q$. Again, the differential in K can be written $d = d_1 + d_2$ with $d_1 \colon K_q \to K_q$, $d_2 \colon K_q \to K_{q-1}$. Reasoning exactly as in (i), taking into account the previous particular case, we obtain that $H(K, d_2) = 0$.

Now, let a by a cycle in K of usual degree n. As $L \in \mathbf{dgl}_0$, sL is concentrated in positive degrees $\overline{\mathscr{C}}(L) = \overline{\mathscr{C}}(L)_{\geq 1}$ and thus, we can write $a = \sum_{q=1}^n a_q$ with $a_q \in K_q$. Then, $d_2 a_1 = 0$, so $a_1 = d_2 b$ for some b. Therefore, replacing a by $a - db$ and iterating the process, the element a is homologous to an element in K_{n+1}. However, this element has to vanish as 0 is the only element in K_{n+1} of (usual) degree n. Hence $H(K) = 0$, and the considered projection is a quasi-isomorphism. □

Proposition 2.3. *For any $L \in \mathbf{dgl}$ and any $C \in \mathbf{cdgc}_1$, the adjunction maps*

$$\alpha_L \colon \mathscr{L}\mathscr{C}(L) \xrightarrow{\simeq} L \quad \text{and} \quad \beta_C \colon C \xrightarrow{\simeq} \mathscr{C}\mathscr{L}(C)$$

are quasi-isomorphisms.

Proof. For any $L \in \mathbf{dgl}$ write $\mathscr{C}(L) = (\wedge V, d)$, where $(s^{-1}V, s^{-1}d_1) = L$. It follows from Lemma 2.2(i) that the injection

$$L \xhookrightarrow{\simeq} \mathscr{L}\mathscr{C}L$$

is a quasi-isomorphism of complexes. But the composition of this injection with α_L,

$$L \xhookrightarrow{\simeq} \mathscr{L}\mathscr{C}L \xrightarrow{\alpha_L} L,$$

is the identity on L. Hence, α_L is also a quasi-isomorphism.

Similarly, for any $C \in \mathbf{cdgc}_1$, write $\mathscr{L}(C) = (\mathbb{L}(V), d)$, where $\overline{C} = (sV, sd_1)$. It follows from Lemma 2.2(ii) that the projection

$$\overline{\mathscr{C}}\mathscr{L}(C) \xrightarrow{\simeq} \overline{C}$$

is a quasi-isomorphism of complexes. Precomposing this with the restriction to the augmentation ideal of the adjunction map β_C

$$\overline{C} \xrightarrow{\beta_C} \overline{\mathscr{C}}\mathscr{L}(C) \xrightarrow{\simeq} \overline{C}$$

gives the identity on $\overline{C}$, and thus β_C is also a quasi-isomorphism. □

We now check that the functor $\mathscr{C}$ preserves quasi-isomorphisms and discuss under which conditions this is also the case for $\mathscr{L}$.

Proposition 2.4.

(1) *The functor $\mathscr{C}$ preserves quasi-isomorphisms in* $\mathbf{dgl}$.

(2) *The functor $\mathscr{L}$ preserves quasi-isomorphisms between finite type fibrant coalgebras in $\mathbf{cdgc}_1$ and also preserves all quasi-isomorphisms in $\mathbf{cdgc_2}$.*

Statement (1) and the second assertion of statement (2) are precisely [109, Proposition 4.4 and 6.4].

Proof. (1) Let $f \colon L \xrightarrow{\simeq} L'$ be a quasi-isomorphism of dgl's. The cdgc's $\mathscr{C}(L)$ and $\mathscr{C}(L')$ are naturally filtered by $\wedge^{\leq p} sL$ and $\wedge^{\leq p} sL'$ and the morphism $\mathscr{C}(f)$ preserves the filtration so that, for each p, it induces a map

$$f_p \colon \wedge^{\leq p} sL \longrightarrow \wedge^{\leq p} sL'.$$

It suffices to show that each f_p is a quasi-isomorphism. First, note that the induced map,

$$\overline{f}_p\colon \wedge^p sL = \wedge^{\le p} sL \,/\, \wedge^{\le p-1} sL \xrightarrow{\simeq} \wedge^{\le p} sL' \,/\, \wedge^{\le p-1} sL' = \wedge^p sL'\,,$$

is a quasi-isomorphism for each p, since f is. Finally, recursively applying the five lemma to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \wedge^{\le p-1} sL & \longrightarrow & \wedge^{\le p} sL & \longrightarrow & \wedge^p sL & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle f_{p-1}} & & \Big\downarrow{\scriptstyle f_{p-1}} & & \Big\downarrow{\scriptstyle \overline{f}_p\ \simeq} & & \\ 0 & \longrightarrow & \wedge^{\le p-1} sL' & \longrightarrow & \wedge^{\le p} sL' & \longrightarrow & \wedge^p sL' & \longrightarrow & 0 \end{array}$$

we deduce that for each p the induced map f_p is a quasi-isomorphism, as required.

(2) Let $f\colon C \xrightarrow{\simeq} C'$ be a quasi-isomorphism in $\mathbf{cdgc_2}$. The dgl's $\mathscr{L}(C) = \mathbb{L}(s^{-1}\overline{C})$ and $\mathscr{L}(C') = \mathbb{L}(s^{-1}\overline{C'})$ are naturally equipped with the decreasing filtrations $\mathbb{L}^{\ge p}(s^{-1}\overline{C})$ and $\mathbb{L}^{\ge p}(s^{-1}\overline{C'})$. Since $\mathscr{L}(C)$ and $\mathscr{L}(C')$ are in $\mathbf{dgl_1}$, the induced spectral sequences are first-quadrant spectral sequences and thus convergent. Therefore, since $E_1(f)$ is an isomorphism, by comparison, $\mathscr{L}(f)$ is a quasi-isomorphism.

Assume now that $f\colon C \xrightarrow{\simeq} C'$ is a quasi-isomorphism between fibrant cdgc's in $\mathbf{cdgc}_1$. Write $C = (\wedge V, d)$ and $C' = (\wedge V', d)$ and note, using (1.22) in Section 1.3.1, that the dual of a reduced, finite type fibrant cdgc is a Sullivan algebra. Hence, the dual of f can be written as the cdga quasi-isomorphism

$$f^\#\colon (\wedge V^\#, d^\#) \xrightarrow{\simeq} (\wedge V'^\#, d^\#).$$

It follows that the induced map on the indecomposable elements $f_1^\#$, and thus $f_1\colon (V, d_1) \xrightarrow{\simeq} (V', d_1)$ is a quasi-isomorphism of chain complexes. Now, by the naturality in Lemma 2.2(i), we have a commutative diagram of complexes,

$$\begin{array}{ccc} (s^{-1}V, s^{-1}d) & \xrightarrow[\simeq]{s^{-1}f_1} & (s^{-1}V', s^{-1}d) \\ \simeq\Big\downarrow & & \Big\downarrow\simeq \\ \mathscr{L}(C) & \xrightarrow{\mathscr{L}(f)} & \mathscr{L}(C'), \end{array}$$

in which both vertical arrows are injective quasi-isomorphisms. It follows that $\mathscr{L}(f)$ is a quasi-isomorphism. □

Example 2.5. Here we show that the functor $\mathscr{L}$ does not, in general, preserve quasi-isomorphisms in $\mathbf{cdgc}_1$ and the fibrant character of the cdgc is essential. Consider for instance the (non-fibrant!) cdgc

$$C = (\wedge(a, b, c), d)$$

with $d(ab) = c$, $d(bc) = a$, $d(ca) = b$, with $|a| = |b| = |c| = 1$. Then,

$$\mathscr{L}(C) = (\mathbb{L}(x, y, z, u, v, w, t), d)$$

with

$$x = -s^{-1}a,\ y = -s^{-1}b,\ z = -s^{-1}c,\ u = s^{-1}ab,$$
$$v = s^{-1}bc,\ w = s^{-1}ca,\ t = s^{-1}abc,$$

and

$$dx = dy = dz = 0,\ du = z - [x, y],\ dv = x - [y, z],$$
$$dw = y - [z, x],\ dt = [x, v] + [y, w] + [z, u].$$

On the other hand, consider the cofree cdgc $(\wedge e, 0)$ with e of degree 3 and the cdgc morphism

$$\varphi\colon C \xrightarrow{\simeq} (\wedge e, 0), \qquad \varphi(abc) = e.$$

Then it is an exercise to check that φ is a quasi-isomorphism, but $\mathscr{L}(\varphi)$ is not: $H_0\big(\mathscr{L}(\wedge e, 0)\big) = 0$, but $H_0\big(\mathscr{L}(C)\big)$ is the semi-simple Lie algebra $so(3)$, that is, the Lie algebra spanned by x, y and z with $[x, y] = z$, $[y, z] = x$ and $[z, x] = y$.

Recall that, under strong finiteness and connectivity restrictions, the pair of adjoint functors $\mathscr{C}$ and $\mathscr{L}$ becomes a Quillen equivalence with the usual model structures. Denote by $\mathbf{dgl^{hf}}_0$ the subcategory of $\mathbf{dgl}_0$ consisting of those dgl's whose homology is nilpotent and of finite type. On the other hand, let $\mathbf{cdgc^{cf}}_1$ be the subcategory of $\mathbf{cdgc}_1$ consisting of those cdgc's having a fibrant model of finite type. Then, J. Neisendorfer proved the following:

Theorem 2.6 ([109, Proposition 7.2]). *The adjoint functors*

$$\mathbf{cdgc^{cf}}_1 \underset{\mathscr{C}}{\overset{\mathscr{L}}{\rightleftarrows}} \mathbf{dgl^{hf}}_0$$

induce equivalences between the associated homotopy categories. In particular, a map $\mathscr{L}(C) \to L$ is a quasi-isomorphism if and only if the adjoint map $C \to \mathscr{C}(L)$ is a quasi-isomorphism. □

As a consequence, for fibrant connected cdgc's C_1 and C_2 of finite type, and for cofibrant connected dgl's L_1 and L_2 whose homology is nilpotent of finite type, the functors $\mathscr{L}$ and $\mathscr{C}$ induce bijections of homotopy classes of maps,

$$[C_1, C_2] \xrightarrow{\cong} [\mathscr{L}(C_1), \mathscr{L}(C_2)] \quad \text{and} \quad [L_1, L_2] \xrightarrow{\cong} [\mathscr{C}(L_1), \mathscr{C}(L_2)]\,.$$

The equivalence in Theorem 2.6 may be extended on the left to the homotopy category of $\mathbf{cdga^{hf}}_1$ which is the subcategory of $\mathbf{cdga}_1$ consisting of the cdga's that have a finite type Sullivan minimal model. This is done by considering the *cochain functor*, that is, the dual of the functor $\mathscr{C}$, which we now describe in detail as it will be of use later.

Definition 2.7. For any dgl L, the *cochain algebra on* L is the dual of the chain coalgebra $\mathscr{C}(L)$,

$$\mathscr{C}^*(L) = \mathrm{Hom}(\mathscr{C}(L), \mathbb{Q})\,.$$

The differential is given by $(df)(c) = -(-1)^{|f|} f(dc)$. In the particular case when L is connected and of finite type, this cdga has a well-known description: indeed, via the cga isomorphism (1.22),

$$\mathscr{C}^*(L) \cong (\wedge (sL)^\#, d) \tag{2.2}$$

in which $d = d_1 + d_2$, where

$$\begin{aligned} \langle d_1 v, sx\rangle &= (-1)^{|v|}\langle v, sdx\rangle, \\ \langle d_2 v, sx \wedge sy\rangle &= (-1)^{|y|+1}\langle v, s[x,y]\rangle, \end{aligned} \tag{2.3}$$

with $v \in (sL)^\#$ and $x, y \in L$. From this point on, the cochain functor will only be considered in this special case and thus, equation (2.2) can be considered as its definition.

Let $(\wedge V, d)$ be the minimal model of a 1-connected cdga with finite type homology. In that case, see for instance [130, Chapter II], $\mathscr{C}^*$ and the association $(\wedge V, d) \mapsto \mathscr{L}(\wedge V, d)^\#$ define equivalences between the homotopy categories of $\mathbf{cdga^{hf}}_1$ and $\mathbf{dgl^{hf}}_0$, which relate the minimal Sullivan and Quillen models of a simply connected space X of finite type:

If $(\wedge V, d)$ is the minimal Sullivan model of X, then $\mathscr{L}(\wedge V, d)^\#$ is a Quillen model of X.

2.2 The functors $\mathscr{A}$ and $\mathscr{E}$

The aim of this section is the study of the functors

$$\mathbf{cdga} \underset{\mathscr{E},\mathscr{E}^{\mathrm{u}}}{\overset{\mathscr{A}}{\rightleftarrows}} \mathbf{dglc}\,.$$

We begin with their definitions. Given a dglc (E, d), define

$$\mathscr{A}(E) = (\wedge s^{-1}E, d),$$

where

$$d(s^{-1}x) = \frac{1}{2}\sum_i (-1)^{|x_i|}\; s^{-1}x_i \wedge s^{-1}x_i' - s^{-1}dx,$$

with $\Delta x = \sum_i x_i \otimes x_i'$.

Remark 2.8. An easy and direct computation shows that, if E is a connected finite type dglc then,

$$\mathscr{A}(E) = \mathscr{C}^*(E^{\#}) = \# \circ \mathscr{C}(E^{\#}),$$

where $E^{\#}$ denotes the dgl dual to E. Equivalently, whenever L is a connected finite type dgl,

$$\mathscr{C}^*(L) = \mathscr{A}(L^{\#}), \qquad \text{that is,} \qquad \# \circ \mathscr{C}(L) = \mathscr{A} \circ \#(L).$$

On the other hand, for the construction of $\mathscr{E}$ choose a cdga A (which we recall is always assumed augmented). The differential on the bar construction $BA = T^{\rm c}(s\overline{A})$ preserves the ideal generated by the shuffle products (see Section 1.2.3), and therefore it induces a differential d on the indecomposables $B^+A/B^+A \cdot B^+A$, which is precisely (see Section 1.2.4) the free Lie coalgebra $\mathbb{L}^c(s\overline{A})$. This then defines the dglc,

$$\mathscr{E}(A) = (\mathbb{L}^c(s\overline{A}), d).$$

As a simple but illustrative example, consider a generic cdga A and homogeneous elements $a, b \in \overline{A}$. Recall that $[sa, sb]^{\rm c}$ denotes the projection of $[sa|sb] = sa \otimes sb \in BA$ on $\mathbb{L}^{\rm c}(s\overline{A})$. Then we have

$$d[sa, sb]^{\rm c} = (-1)^{|sa|} s(ab) - [sda, sb]^{\rm c} - (-1)^{|sa|}[sa, sdb]^{\rm c} .$$

Exactly the same construction can be performed on the unreduced bar construction $B^{\rm u}A$ to define

$$\mathscr{E}^{\rm u}(A) = (\mathbb{L}^{\rm c}(sA), d).$$

It is important to observe that the functors $\mathscr{E}$ and $\mathscr{E}^{\rm u}$ can be extended to the infinity category,

$$\mathscr{E}, \mathscr{E}^{\rm u} \colon \mathbf{cdga}_\infty \longrightarrow \mathbf{dglc}, \tag{2.4}$$

by means of the formulas (1.27) and (1.28) of Section 1.2.5, as we now recall: given a C_∞-algebra A, the differential on the bar construction $BA = (T^{\rm c}(s\overline{A}), d)$ preserves the ideal generated by the shuffle products and therefore it induces a differential on the quotient $B^+A/B^+A \cdot B^+A = \mathbb{L}^{\rm c}(s\overline{A})$. The same procedure is performed on the unreduced bar construction, and this defines the dglc's,

$$\mathscr{E}(A) = (\mathbb{L}^{\rm c}(s\overline{A}), d) \quad \text{and} \quad \mathscr{E}^u(A) = (\mathbb{L}^{\rm c}(sA), d).$$

Using the universality properties of the free Lie coalgebra and the free commutative algebra, and the explicit definitions of both functors, the following proposition is an easy exercise.

Proposition 2.9. *The functor $\mathscr{A}$ is left adjoint to $\mathscr{E}$.* □

The adjunction maps

$$\alpha'_A \colon \mathscr{A}\mathscr{E}(A) \longrightarrow A \quad \text{and} \quad \beta'_E \colon E \longrightarrow \mathscr{E}\mathscr{A}(E)$$

are defined as follows:

On the one hand, α'_A is the unique cdga morphism $(\wedge s^{-1}\mathscr{E}(A), d) \to A$ extending the projection

$$s^{-1}\mathscr{E}(A) \longrightarrow s^{-1}s\overline{A} = \overline{A}.$$

On the other hand, β'_E is the unique dglc morphism $E \to (\mathbb{L}^{\mathrm{c}}\big(s\wedge^+ s^{-1}E), d)$ induced by the inclusion

$$E = ss^{-1}E \hookrightarrow s\wedge^+ s^{-1}E.$$

Preservation of quasi-isomorphisms is proven by similar arguments to Proposition 2.4.

Proposition 2.10.

(1) *The functors $\mathscr{E}$ and $\mathscr{E}^{\mathrm{u}}$ preserve quasi-isomorphisms in both* **cdga** *and* $\mathbf{cdga}_\infty$.
(2) *The functor $\mathscr{A}$ preserves quasi-isomorphisms in* $\mathbf{dglc}_0$.

Proof. (1) The proof is analogous to that of Proposition 1.6: for any cdga or C_∞-algebra A, the dglc $\mathscr{E}(A) = (\mathbb{L}^{\mathrm{c}}(s\overline{A}), d)$ is filtered by the sub-dglc's $\mathbb{L}^{\mathrm{c}\leq p}(s\overline{A})$, each of which is the image of $T^{\mathrm{c}\leq p}(s\overline{A})$ of the projection $BA \to \mathscr{E}(A)$.

Let $f\colon A \xrightarrow{\simeq} A'$ be a quasi-isomorphism of cdga's or C_∞-algebras. Then, $\mathscr{E}(f)$ preserve the filtration and we get maps

$$f_p\colon \mathbb{L}^{\mathrm{c}\leq p}(s\overline{A}) \longrightarrow \mathbb{L}^{\mathrm{c}\leq p}(s\overline{A'}).$$

It is sufficient to show that each f_p is a quasi-isomorphism. Note that the induced map

$$\begin{aligned}\overline{f}_p\colon (\mathbb{L}^{\mathrm{c}})^p(s\overline{A}) &= \mathbb{L}^{\mathrm{c}\leq p}(s\overline{A}) \,/\, \mathbb{L}^{\mathrm{c}\leq p-1}(s\overline{A})\\ &\xrightarrow{\simeq} (\mathbb{L}^{\mathrm{c}})^p(s\overline{A'}) = \mathbb{L}^{\mathrm{c}\leq p}(s\overline{A'}) \,/\, \mathbb{L}^{\mathrm{c}\leq p-1}(s\overline{A'}) = (\mathbb{L}^{\mathrm{c}})^p(s\overline{A'})\,,\end{aligned}$$

is a quasi-isomorphism for each p, since f is. Recursively applying the five lemma as in Proposition 1.6 we obtain the result. The unreduced version is proved in the same way.

(2) Given any $E \in \mathbf{dglc}_0$, filter $\mathscr{A}(E) = (\wedge s^{-1}E, D)$ by the word-length ideals $\wedge^{\geq p}s^{-1}E$. Since $s^{-1}E = (s^{-1}E)^{\geq 1}$, the induced spectral sequence E_* converges. Now, given a quasi-isomorphism $f\colon E \xrightarrow{\simeq} E'$ in $\mathbf{dglc}_0$ it follows that $E_1\big(\mathscr{A}(f)\big)$ is already an isomorphism since f is. We conclude that $\mathscr{A}(f)$ is also a quasi-isomorphism. □

Example 2.11. The functor $\mathscr{A}$ does not preserve quasi-isomorphisms of non-connected dglc's. As an example, consider the free dglc $L = \mathbb{L}(x)$ on one generator x of degree -1, equipped with the differential $d[x,x]^{\mathrm{c}} = x, dx = 0$. Then the injection $f\colon 0 \xrightarrow{\simeq} L$ is a quasi-isomorphism. On the other hand, $\mathscr{A}(L) = \wedge(u,v)$ with $|u| = 0$, $|v| = -1$, $du = 0$ and $dv = u - u^2$. Therefore, $H\big(\mathscr{A}(L)\big)$ is the direct sum $\mathbb{Q}1 \oplus \mathbb{Q}[u]$, where $[u]$ is an idempotent. In particular $\mathscr{A}(f)$ is not a quasi-isomorphism.

To prove that the adjunction maps are quasi-isomorphisms we need some preparation.

We remark that, given a cofree dglc $(\mathbb{L}^c(V),d)$, the chain complex $\mathbb{Q}\oplus(s^{-1}V,s^{-1}d_1)$ is a sub-complex of $\mathscr{A}(\mathbb{L}^c(V),d)$ whose underlying graded algebra is $\wedge s^{-1}\mathbb{L}^c(V)$. On the other hand, given a Sullivan algebra $(\wedge V,d)$, the chain complex (sV,sd_1) is a quotient complex of $\mathscr{E}(A)$ whose underlying graded Lie coalgebra is $\mathbb{L}^c(s\wedge^+V)$. The analogue of Lemma 2.2, whose proof also uses similar arguments, reads as follows:

Lemma 2.12.

(i) *For any cofree dglc $(\mathbb{L}^c(V),d)$ which can be written as the union $\bigcup_i(\mathbb{L}^c(V_i),d)$ with each V_i of finite type, the injection*

$$\mathbb{Q}\oplus(s^{-1}V,s^{-1}d_1)\xhookrightarrow{\simeq}\mathscr{A}(\mathbb{L}^c(V),d)$$

is a quasi-isomorphism of complexes.

(ii) *For any Sullivan algebra $(\wedge V,d)$, the projection*

$$\mathscr{E}(\wedge V,d)\xrightarrow{\simeq}(sV,sd_1)$$

is a quasi-isomorphism of complexes.

Proof. (i) Since $\mathscr{A}$ is left adjoint, it commutes with direct limits and thus,

$$\mathscr{A}(\mathbb{L}^c(V),d)=\varinjlim_i\mathscr{A}(\mathbb{L}^c(V_i),d).$$

Denote by $\gamma\colon\mathbb{Q}\oplus(s^{-1}V,s^{-1}d_1)\hookrightarrow\mathscr{A}(\mathbb{L}^c(V),d)$ the inclusion, which is also the direct limit

$$\gamma=\varinjlim_i\gamma_i,\quad\text{where}\quad\gamma_i\colon\mathbb{Q}\oplus(s^{-1}V_i,s^{-1}d_1)\to\mathscr{A}(\mathbb{L}^c(V_i,d),$$

is the corresponding injection. Since an inductive limit of a quasi-isomorphism is again a quasi-isomorphism, we can assume without loss of generality that V is of finite type.

First, if the differential on $\mathbb{L}^c(V)$ is zero we can rearrange the degrees and suppose that $V=V^{\geq1}$, in which case $\mathbb{L}^c(V)=\left(\mathbb{L}(V^\#)\right)^\#$, and from Remark 2.8 it follows that

$$\mathscr{A}(\mathbb{L}^c(V),0)=\left(\mathscr{C}(\mathbb{L}(V^\#),0)\right)^\#.$$

Hence, $\gamma=\mathrm{id}_\mathbb{Q}\oplus\rho^\#$ with

$$\rho\colon\overline{\mathscr{C}}(\mathbb{L}(V^\#),0)\xrightarrow{\simeq}(V^\#,0)$$

the quasi-isomorphic projection of Lemma 2.2(ii). Thus, γ is a quasi-isomorphism.

In the general case, denote by

$$\Gamma = \mathscr{A}(\mathbb{L}^{\mathrm{c}}(V), d)/(s^{-1}V, s^{-1}d_1)$$

the cokernel of γ. We finish by proving that $H(\Gamma) = 0$.

For this, observe that for any dglc E the differential in $\mathscr{A}(E)$ can be written as $d = d_1 + d_2$, in which d_1 preserves the word length and d_2 increases it by 1. Moreover,

$$(\mathscr{A}(E), d_2) = \mathscr{A}(E, 0),$$

and in particular $(\mathscr{A}(\mathbb{L}^{\mathrm{c}}(V), d), d_2) = \mathscr{A}(\mathbb{L}^{\mathrm{c}}(V), 0)$. Now, by the preceding special case,

$$\mathbb{Q} \oplus (s^{-1}V, 0) \xhookrightarrow{\simeq} \mathscr{A}(\mathbb{L}^{\mathrm{c}}(V), 0)$$

and therefore, $H(\Gamma, d_2) = 0$. Note also that the word length in $\mathscr{A}\big(\mathbb{L}^{\mathrm{c}}(V)\big)$ induces a grading on $\Gamma = \bigoplus_{q\geq 1} \Gamma_q$. Now, if $x \in \Gamma$ is a d-cycle, then for some $n \geq 1$, $x = x_1 + \cdots + x_n$ with $x_q \in \Gamma_q$. Then, $d_2(x_n) = 0$ and thus $x_n = d_2(y)$ for some y. Replacing x by $x - dy$, we get a cycle in $\Gamma_{<n}$ homologous to x. By iteration $[x] = [z]$ with $z \in \Gamma_{<1} = 0$. Therefore, $H(\Gamma, d) = 0$ and γ is a quasi-isomorphism.

(ii) Denote by $\rho\colon \mathscr{E}(\wedge V, d) \xrightarrow{\simeq} (sV, sd_1)$ the projection and observe that any Sullivan algebra $(\wedge V, d)$ can be written as the increasing union $\bigcup_{i\geq 0}(\wedge V_i, d)$ of Sullivan algebras

$$(\wedge V_0, d_0 \subset) \cdots \subset (\wedge V_i, d_i) \subset \cdots,$$

where each V_i is of finite global dimension. Even though $\mathscr{E}$ may not commute in general with direct limits, it does in this particular case and

$$\mathscr{E}(\wedge V, d) = \varinjlim_i \mathscr{E}(\wedge V_i, d) \quad \text{and} \quad \rho = \varinjlim_i \rho_i\colon \mathscr{E}(\wedge V_i, d) \to (sV_i, sd_1).$$

Hence, as the direct limit of quasi-isomorphisms is a quasi-isomorphism, we may assume again that V is of finite dimension.

When $d = 0$ in $(\wedge V, d)$, we can modify the degrees and suppose that $V = V^{\geq 2}$. Then,

$$\mathscr{E}(\wedge V, 0)^{\#} \cong \big(\mathbb{L}^{\mathrm{c}}(s \wedge^+ V)\big)^{\#} \cong \mathbb{L}(s \wedge^+ V^{\#}) = \mathscr{L}(\wedge V^{\#}, 0)$$

and $\rho = \gamma^{\#}$, with

$$\gamma\colon (s^{-1}V^{\#}, 0) \xhookrightarrow{\simeq} \mathscr{L}(\wedge V^{\#}, 0)$$

the injective quasi-isomorphism of Lemma 2.2(i). Hence, ρ is also a quasi-isomorphism.

In the general case, if we denote $K = \ker\rho$ the statement will follow by proving that $H(K) = 0$.

Note that, for any cdga A, the differential on $\mathscr{E}(A)$ can be written as $d = d_1 + d_2$, in which d_1 preserves the cobracket length and d_2 decreases it by 1. Moreover,

$$(\mathscr{E}(A), d_2) = \mathscr{E}(A, 0).$$

Since, by the previous particular case, the projection

$$\mathscr{E}(\wedge V, 0) \xrightarrow{\simeq} (sV, 0)$$

is a quasi-isomorphism, we obtain that $H(K, d_2) = 0$.

Now, the cobracket length on $\mathscr{E}(\wedge V, d)$ induces also a grading on $K = \bigoplus_{q\geq 1} K^q$ for which d_1 preserves this grading and d_2 decreases it by 1.

Choose a cocycle $x \in K$ and write $x = \sum_{q=1}^n x_q$ with $x_q \in K^q$. Then $d_2 x_1 = 0$, so $x_1 = d_2 y$ for some y. Therefore, replacing x by $x - dy$ and iterating the process, x is homologous to a cycle x' in K^n. Now construct a sequence z_i, $i \geq 1$, with $d_2 z_1 = x$ and for $i \geq 2$, $d_2 z_i = d_1 z_{i-1}$ Suppose that $z_1, \dots, z_q$ have been constructed; then $d_1 z_q$ is a d_2-cycle and so is of the form $d_2 z_{q+1}$. If we assure that $z_q = 0$ from some q on, then $x = d(z_1 + \cdots + z_{q-1})$ and the result would follow.

To check that this is the case we set a multigrading in $\mathscr{E}(\wedge V, d)$, and thus in K, as follows:

Since V is of finite dimension write $V = \bigoplus_{q=1}^n V_{[q]}$ with $dV_{[0]} = 0$ and $dV_{[q]} \subset \wedge\left(\bigoplus_{i=1}^{q-1} V_{[i]}\right)$. We fix a basis for each $V_{[q]}$. The product of those elements forms a basis $\mathcal{B}$ for $\wedge V$. Now to each element $\omega \in \mathcal{B}$ we associate a multidegree $\ell(\omega) \in \mathbb{Z}^n$ as follows:

If $v \in V_{[q]}$, $\ell(v)$ is the n-tuple $(a_1, \dots, a_n)$ with $a_i = 0$ for $i \neq n - q$ and $a_{n-q} = 1$. If $\omega = v_1 \wedge \cdots \wedge v_r \in \mathcal{B}$, we set $\ell(\omega) = \sum_{i=1}^r \ell(v_i)$.

The bar construction $B(\wedge V, d)$ admits then a basis $\mathcal{D}$ formed by elements of the form $a = [s\omega_1|s\omega_2|\cdots|s\omega_t]$ with $\omega_j \in \mathcal{B}$, and we extend ℓ to this basis by setting $\ell(a) = \sum_{i=1}^t \ell(\omega_i)$.

Next, we select in the graded vector space $\mathscr{E}(\wedge V, d)$ a basis $\mathcal{F}$ extracted from the elements of $\mathcal{D}$ along the projection $B(A, d) \to \mathscr{E}(A, d)$. In $\mathcal{F}$ we keep the multidegree ℓ and consider on it the lexicographic order. Observe that, for each $V_{[q]}$ and any $a \in \mathcal{F}$ corresponding to an element of a basis of $V_{[q]}$, $d_1(a)$ is a linear combination $\sum \lambda_j a_j$ with $\lambda_j \in \mathbb{Q}$, $a_j \in \mathcal{F}$ and $\ell(a_j) < \ell(a)$.

Returning to the proof of $z_q = 0$ for q big enough, observe that, by the definition of d_2, if α is a linear combination of elements of multidegree $\leq r$ and α is a d_2-boundary, then there is a linear combination β of elements of multidegree $\leq r$ with $d_2\beta = \alpha$. Hence, for any q, if $d_1 z_q$ is a linear combination of elements of multidegree $< r$, then the same is true for z_{q+1}. Since d_1 decreases strictly the multidegree, $z_q = 0$ after some number of steps. $\square$

Lemma 2.12(ii) can be sharpened whenever A is a minimal Sullivan algebra.

Proposition 2.13. *Let $(\wedge V, d)$ be a minimal Sullivan algebra. Then*

$$H(\rho)\colon H(\mathscr{E}(\wedge V, d)) \xrightarrow{\cong} sV$$

is an isomorphism of graded Lie coalgebras.

Here, the Lie coalgebra structure of sV is the one given in Example 1.5.

Proof. Let $v \in V$ with $d_2 v = \sum_i v_i v_i'$. Then using Lemma 2.12(ii), there is a cycle

$$\widetilde{v} = sv - \sum_i (-1)^{|v_i|}[sv_i, sv_i']^{\mathrm{c}} + \omega$$

with $\rho(\widetilde{v}) = sv$ and $\omega \in \sum_{p+q\geq 3} \mathbb{L}^q(\wedge^p V)$. Since $(p \otimes p)\Delta\omega = 0$,

$$(p\otimes p)\Delta(\widetilde{v}) = -\sum_i (-1)^{|v_i|}\left(sv_i \otimes sv_i' - (-1)^{|sv_i||sv_i'|} sv_i' \otimes sv_i\right) = \Delta(sv)\,. \quad \square$$

Definition 2.14. A *Sullivan differential graded Lie coalgebra* is a dglc E which is the union of an increasing sequence of dglc's $0 = E_0 \subset \cdots \subset E_{i-1} \subset E_i \subset \cdots$ such that, for each $i \geq 1$, the differential and comultiplication on each E_i satisfy

$$dE_i \subset E_{i-1} \quad \text{and} \quad \Delta E_i \subset E_{i-1} \otimes E_{i-1}.$$

Observe that for each Sullivan dglc E, the cdga $\mathscr{A}(E)$ is a Sullivan algebra.

For instance, any $E \in \mathbf{dglc}_1$ is a Sullivan dglc as $\mathscr{A}(E)$ is a 1-connected free cdga and hence, a Sullivan cdga.

Proposition 2.15. *For any cdga A which is either of finite type or in $\mathbf{cdga}_1$, and any Sullivan dglc E, the adjunction maps*

$$\alpha_A'\colon \mathscr{A}\mathscr{E}(A) \xrightarrow{\simeq} A \quad \textit{and} \quad \beta_E'\colon E \xrightarrow{\simeq} \mathscr{E}\mathscr{A}(E)$$

are quasi-isomorphisms.

Proof. We first suppose that A is of finite type and we prove that α_A' is a quasi-isomorphism. As $\mathscr{E}(A) = (\mathbb{L}^{\mathrm{c}}(s^{-1}\overline{A}), d)$, Lemma 2.12(i) guarantees that the injection

$$(A, d) \overset{\simeq}{\hookrightarrow} \mathscr{A}\mathscr{E}(A)$$

is a quasi-isomorphism of complexes. Composing this inclusion with α_A' yields the identity on A and therefore, α_A' is a quasi-isomorphism.

We now assume that $A \in \mathbf{cdga}_1$. In this case, every finitely generated sub-cdga $R \subset A$ is of finite type and we denote by $\alpha_R'\colon \mathscr{A}\mathscr{E}(R) \xrightarrow{\simeq} R$ the corresponding quasi-isomorphism. If x is a cycle in A, there is a finitely generated sub-cdga R

such that $x \in R$, and since α'_R is a quasi-isomorphism, there is a cycle $y \in \mathscr{A}\mathscr{E}(R)$ with $\alpha'_R(y) = x$. Then $y \in \mathscr{A}\mathscr{E}(A)$ and $\alpha'_A(y) = x$. That is, $H(\alpha'_A)$ is surjective. For the injectivity, let z be a cycle in $\mathscr{A}\mathscr{E}(A)$ such that $\alpha'_A(z) = dy$. Then there is a finitely sub-cdga $R \subset A$ such that $y \in R$ and $z \in \mathscr{A}\mathscr{E}(R)$. Since α'_R is a quasi-isomorphism, z is a boundary and so $H(\alpha'_A)$ is injective.

We see now that β'_E is a quasi-isomorphism for any Sullivan dglc E. In this case, $\mathscr{A}(E)$ is a Sullivan algebra and by Lemma 2.12(ii), the projection $\rho\colon \mathscr{E}\mathscr{A}(E) \xrightarrow{\simeq} ss^{-1}E = E$ is a quasi-isomorphism. Since $\rho \circ \beta'_E$ is the identity, β'_E is a quasi-isomorphism. □

Example 2.16. Let us show that the Sullivan character of the dglc E in Proposition 2.15(ii) is necessary. For this (see Example 2.5), consider the dgl $so(3)$ concentrated in degree 0 and with trivial differential. It is spanned as vector space by x, y and z with $[x, y] = z$, $[y, z] = x$ and $[z, x] = y$. Write $E = so(3)^\#$ and observe that, by Remark 2.8,

$$\mathscr{A}(E) = \mathscr{C}^*\big(so(3)\big).$$

By definition,

$$\mathscr{C}^*\big(so(3)\big) = (\wedge(a, b, c), d),$$

where

$$|a| = |b| = |c| = 1 \quad \text{and} \quad da = bc,\ db = ca,\ dc = ab.$$

This is not a Sullivan algebra and thus, E is not a Sullivan dglc. Consider the cdga quasi-isomorphism

$$(\wedge e, 0) \xrightarrow{\simeq} (\wedge(a, b, c), d), \quad e \longmapsto abc,$$

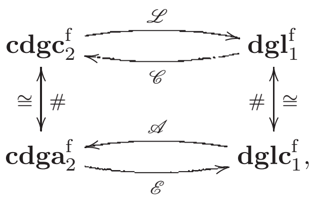
and apply Proposition 2.10(1) to obtain that

$$\mathscr{E}\mathscr{A}(E) \cong \mathscr{E}(\wedge e, 0) = (\mathbb{L}^c(se), 0).$$

Therefore, $\beta'_E\colon E \to \mathscr{E}\mathscr{A}(E)$ is obviously not a quasi-isomorphism.

We finish the section with the following observation.

Remark 2.17. Under extra connectivity assumptions, Remark 2.8 has a stronger version: restrict the domain and codomain of the pair of functors $\mathscr{C}$, $\mathscr{L}$ and $\mathscr{A}$, $\mathscr{E}$ as indicated in this diagram,



where the superscript f denotes the subcategory of finite type objects of the corresponding category. Then, an easy inspection shows that the usual dual functors are self inverse equivalences and all squares inside the diagram are commutative. That is:

$$\mathscr{L} \circ \# = \# \circ \mathscr{E} \quad \text{and} \quad \mathscr{C} \circ \# = \# \circ \mathscr{A}. \tag{2.5}$$

In particular, given $A \in \mathbf{cdga}_2^{\mathrm{f}}$, there is a cdgc $C \in \mathbf{cdgc}_2^{\mathrm{f}}$ with $A = C^{\#}$ and $\mathscr{L}(C) = \big(\mathscr{E}(A)\big)^{\#}$.

In this special setting the preservation of quasi-isomorphisms by $\mathscr{L}$ (respectively by $\mathscr{C}$) is then equivalent to the preservation of quasi-isomorphisms by $\mathscr{E}$ (respectively $\mathscr{A}$).

Chapter 3

Complete Differential Graded Lie Algebras

In this chapter we introduce and carefully study the category **cdgl** of *complete differential graded Lie algebras*, which is the main algebraic category in this text. This category, together with the completion functor, which can be analogously built on any of the considered algebraic categories in Section 1.2, is defined in the usual, filtered way:

Let L be a dgl provided with a filtration $L = F^1 \supset F^2 \supset \cdots$ of Lie ideals compatible with the bracket. The completion of L is defined as the dgl

$$\widehat{L} = \varprojlim_n L/F^n.$$

On the other hand, a filtered dgl L is complete if the natural morphism

$$L \xrightarrow{\cong} \varprojlim_n L/F^n$$

is already a dgl isomorphism.

In most situations the considered filtration in L is given by its central series $\{L^n\}_{n\geq 1}$. In particular, if L is nilpotent, then L is automatically complete. Moreover, by definition, any complete dgl is an inverse (or projective) limit of nilpotent differential graded Lie algebras. Thus, the category of complete differential graded Lie algebras (cdgl's) should be thought of as the right generalization of the category of nilpotent dgl's to extend Quillen rational homotopy theory to non-simply connected, even to non-nilpotent finite type spaces.

After setting the main properties of the completion functor and the category **cdgl** in Section 3.1, we analyze in detail the completion of dgl's that are free as Lie algebras. These constitute a particularly important class of cdgl's which turn out to be essential for several reasons. First, they are specially well adapted to computations as we will show in Section 3.2. Also, a special subclass of these cdgl's become cofibrant replacements in the model structure defined in Chapter 6.

U. Buijs et al., *Lie Models in Topology*, Progress in Mathematics 335,
https://doi.org/10.1007/978-3-030-54430-0_3

We finish Section 3.2 by considering the completion $\widehat{\mathscr{L}}$ of the Quillen functor $\mathscr{L}$ defined in Chapter 2, which also takes values in this particular class of free cdgl's. In this completed context we are able to extend some of the results in Section 2.1.

We close this chapter by warning the reader that the completion of a dgl should not be confused with its "profinite completion", which is the inverse limit of all nilpotent, finite type quotients of the given dgl. We highlight the differences between these functors and give the restrictive, but necessary, finiteness requirements for them to coincide.

3.1 Complete differential graded Lie algebras

Unless specifically stated otherwise, a *filtration* in a dgl L will always denote a decreasing filtration

$$L = F^1 \supset \cdots \supset F^n \supset F^{n+1} \supset \cdots$$

of differential Lie ideals compatible with the Lie bracket, which means that

$$[F^p, F^q] \subset F^{p+q} \quad \text{for} \quad p, q \geq 1.$$

In particular $L^n \subset F^n$ for all n.

Definition 3.1. A *complete differential graded Lie algebra* , cdgl henceforth, is a dgl L equipped with a filtration $\{F^n\}_{n\geq 1}$ for which the natural map

$$L \xrightarrow{\cong} \varprojlim_n L/F^n$$

is a dgl isomorphism. A *morphism* $f\colon L \to L'$ between cdgl's, associated to filtrations $\{F^n\}_{n\geq 1}$ and $\{G^n\}_{n\geq 1}$, respectively, is a dgl morphism which preserves the filtrations, that is, $f(F^n) \subset G^n$ for each $n \geq 1$.

We denote by **cdgl** the corresponding category.

By forgetting differentials we analogously define the category **cgl** of *complete graded Lie algebras.* All that follows in this section remains valid replacing **cdgl** by **cgl**.

Observe that, by definition, any element a in the cdgl $L \cong \varprojlim_n L/F^n$ can be written as a sequence,

$$a = (a_n)_{n\geq 1}, \quad \text{where} \quad a_n \in L/F^n \quad \text{and} \quad \rho_{n+1}(a_{n+1}) = a_n, \tag{3.1}$$

with $\rho_{n+1}\colon L/F^{n+1} \to L/F^n$. More specifically, a can be represented by a formal series,

$$a = \textstyle\sum_{n\geq 1} x_n \quad \text{with} \quad x_n \in F^n, \tag{3.2}$$

where two such series, $\sum x_n$ and $\sum y_n$, represent the same element a if $\sum_{n=1}^q (x_n - y_n) \in F^{q+1}$ for all $q \geq 1$. To see this, it suffices to choose the term a_n of the sequence in (3.1) represented by $x_1 + \cdots + x_{n-1}$ with $x_p \in F^p$, for $1 \leq p \leq n-1$. Then, since cdgl morphisms preserve filtrations, the following is immediate:

Lemma 3.2. *Let $f\colon L \to L'$ be a cdgl morphism. Then, for every $a = (a_n) \in L$, $f(a) = \big(f(a_n)\big)$. In particular, for every series $\sum_{n\geq 1} x_n \in L$,*

$$f\left(\sum_{n\geq 1} x_n\right) = \sum_{n\geq 1} f(x_n).$$

□

On the other hand, we may complete any filtered dgl as follows:

Definition 3.3. Let L be a dgl filtered by $\{F^n\}_{n\geq 1}$. Its *completion* $\widehat{L}$ is defined as the dgl

$$\widehat{L} = \varprojlim_n L/F^n.$$

The natural map $L \to \widehat{L}$ is called the *completion morphism.*

We now check that the completion of a filtered dgl is always complete with respect to a certain natural filtration. Let L be a dgl endowed with the filtration $\{F^n\}_{n\geq 1}$. For each $n \geq 1$, consider the commutative diagram

$$\begin{array}{ccc} L & & \\ {\scriptstyle i}\downarrow & \searrow^{p_n} & \\ \widehat{L} & \xrightarrow{q_n} & L/F^n \end{array} \tag{3.3}$$

where p_n is the projection and i is the natural induced map. Consider in $\widehat{L}$ the filtration $\{\widehat{F}^n\}_{n\geq 1}$ given by

$$\widehat{F}^n = \ker q_n \colon \widehat{L} \to L/F^n. \tag{3.4}$$

It is readily seen that i induces isomorphisms

$$L/F^n \cong \widehat{L}/\widehat{F}^n, \quad \text{for} \quad n \geq 1. \tag{3.5}$$

Then, with respect to the filtration $\{\widehat{F}^n\}_{n\geq 1}$, we have:

Proposition 3.4. *$\widehat{L}$ is a cdgl.*

Proof. By the identity (3.5), $\widehat{L} = \varprojlim_n L/F^n \cong \varprojlim_n \widehat{L}/\widehat{F}^n$. □

Proposition 3.5. *The category* **cdgl** *is complete and cocomplete.*

Proof. It is enough to check that **cdgl** has equalizers, coequalizers, small products and small coproducts.

Let $f, g\colon L \to L'$ be cdgl morphisms, write $L = \varprojlim_n L/F^n$, and consider $E = \ker(f-g)$. Then, E is a sub-dgl of L for which it is clear that $E = \varprojlim_n E/(E\cap F^n)$. That is, E is a complete cdgl which is trivially the equalizer of f and g. A dual argument shows that **cdgl** has coequalizers.

On the other hand, given cdgl's L and L' filtered by $\{F^n\}$ and $\{G^n\}$, respectively, their usual dgl product $L \times L'$ is complete with respect to the filtration $\{M^n\}$, where $M^n = F^n \times G^n$.

Finally we describe the coproduct

$$L \mathbin{\widehat{\amalg}} L'$$

in the category **cdgl**. Let $L \amalg L'$ be the usual coproduct in **dgl** and consider the filtration $\{R^n\}_{n\geq 1}$ of $L \amalg L'$ given by

$$R^n = \sum_{p_1+q_1+\cdots+p_r+q_r=n} \big[F^{p_1}, [G^{q_1}, [\dots [F^{p_r}, G^{q_r}]] \dots \big],$$

where, if an index is zero, the corresponding term does not appear. Now, consider the completion of $L \amalg L'$ with respect to this filtration,

$$\widehat{L \amalg L'} = \varprojlim_n (L \amalg L')/R^n, \tag{3.6}$$

and observe that, since $L = \varprojlim_n L/F^n$ and $L' = \varprojlim_n L'/G^n$, there are natural injections

$$L \hookrightarrow \widehat{L \amalg L'} \hookleftarrow L'.$$

It is a straightforward exercise to check that this is in fact the coproduct of L and L' in **cdgl**. That is,

$$L \mathbin{\widehat{\amalg}} L' = \widehat{L \amalg L'}. \qquad \square$$

Observe that, denoting by $\mathcal{F}$-**dgl** the category of filtered dgl's and filtration preserving morphisms, the completion procedure defines a functor

$$\widehat{\cdot} : \mathcal{F}\text{-}\mathbf{dgl} \longrightarrow \mathbf{cdgl}$$

which has a right adjoint:

Proposition 3.6. *The completion functor is left adjoint to the forgetful functor $\mathcal{K}$:*

$$\mathcal{F}\text{-}\mathbf{dgl} \underset{\mathcal{K}}{\overset{\widehat{\cdot}}{\rightleftarrows}} \mathbf{cdgl}.$$

Proof. Let L be a dgl filtered by $\{F^n\}_{n\geq 1}$ and let L' be a cdgl associated to the filtration $\{G^n\}$. We define the map

$$\varphi\colon \mathrm{Hom}_{\mathbf{cdgl}}(\widehat{L}, L') \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{F}\text{-}\mathbf{dgl}}(L, \mathcal{K}L'), \quad \varphi(f) = f \circ i,$$

with i as in diagram (3.3), and check that it is a bijection. If $g\colon L \to \mathcal{K}L'$ is a filtered dgl morphism, it induces morphisms $g_n\colon L/F^n \to L'/G^n$ for each $n \geq 1$. Define $\varphi^{-1}(g) = \varprojlim_n g_n \colon \widehat{L} \to L'$.

On the one hand, consider $\varphi(f) = f \circ i$ and observe that, for each $n \geq 1$ the induced morphism

$$(f \circ i)_n = f_n \circ i_n \colon L/F^n \xrightarrow{\cong} \widehat{L}/\widehat{F^n} \xrightarrow{f_n} L'/G^n$$

can be identified with f_n, since i_n is an isomorphism, see identity (3.5). Hence,

$$\varphi^{-1}\varphi(f) = \varprojlim_n (f \circ i)_n = \varprojlim_n f_n = f,$$

since f is a morphism between complete dgl's.

On the other hand it is obvious that $\varprojlim_n g_n \circ i = g$, which proves that $\varphi\varphi^{-1}(g) = g$. □

We finish the section with two clarifying observations.

Remark 3.7. Any dgl L is always equipped with the filtration given by the central series (or bracket length) $\{L^n\}_{n\geq 1}$. Recall that $L^1 = L$ and $L^n = [L^{n-1}, L]$ for $n > 1$. From now on, and unless explicitly stated otherwise, this is the filtration we consider by default to complete any given dgl. As any dgl morphism preserves this filtration, this choice defines a full embedding

$$\mathbf{dgl} \hookrightarrow \mathcal{F}\text{-}\,\mathbf{dgl}\,.$$

Observe that any dgl L which is either nilpotent or in $\mathbf{dgl_1}$ is automatically complete with respect to this filtration, as $L \cong \varprojlim_n L/L^n$.

Remark 3.8. Let L be a dgl filtered by $\{F^n\}_{n\geq 1}$. By declaring this family a basis of open neighbourhoods at 0, this endows L with a structure of *topological dgl*, that is, a differential graded Lie algebra and a topological space for which addition, bracket and differential are continuous maps. Note that, by translation, the family $\{x + F^n\}_{n\geq 1}$ is a basis of open neighbourhoods at any point $x \in L$.

Now, observe that any filtration preserving dgl morphism is automatically continuous, but the converse is not true in general. For instance, in any abelian dgl L consider the filtrations $\mathcal{F}_1 = \{L, L, 0, \dots\}$ and $\mathcal{F}_2 = \{L, 0, \dots\}$. Then, the identity $(L, \mathcal{F}_1) \to (L, \mathcal{F}_2)$ is obviously continuous, but not filtration preserving.

Nevertheless, we could define a cdgl morphism to be a continuous dgl morphism between cdgl's and the resulting category contains, faithfully but not fully, the category **cdgl**. Most of the forthcoming results remain valid in this more general context, but the arguments needed are substantially more technical.

For the reader acquainted with pro-categories, the above remark has the following translation to the context of *pro-objects* in the category of dgl's.

Remark 3.9. Recall that a *pronilpotent* dgl L is, by definition, a projective limit

$$L = \varprojlim_n L_{(n)}$$

of a *tower*

$$\dots \longrightarrow L_{(p)} \longrightarrow L_{(p-1)} \longrightarrow \dots \longrightarrow L_{(2)} \longrightarrow L_{(1)}$$

of nilpotent dgl's.

Observe that a dgl L is complete if and only if is pronilpotent: obviously, every complete dgl $L = \varprojlim_n L/F^n$ is pronilpotent, as each L/F^n is nilpotent. Conversely, let $L = \varprojlim_n L_{(n)}$ be a pronilpotent dgl. Then, $L = \varprojlim_n L/F^n$ with

$$F^n = \ker(L \to \varprojlim_p L_{(p)}/L^n_{(p)}).$$

Indeed, the natural map

$$L \longrightarrow \varprojlim_n L/F^n$$

is easily seen to be injective because its kernel $\bigcap_n F^n = 0$, since each $L_{(n)}$ is nilpotent. On the other hand, by definition, $L/F^n \to \varprojlim_p L_{(p)}/L^n_{(p)}$ is injective and, since inverse limits preserves injections, we have an injective map

$$\varprojlim_n L/F^n \longrightarrow \varprojlim_n \varprojlim_p L_{(p)}/L^n_{(p)} = \varprojlim_p \varprojlim_n L_{(p)}/L^n_{(p)} = \varprojlim_p L_{(p)} = L.$$

Finally the composition of the above two injections is the identity on L and the assertion follows.

However, the set of pro-morphisms between two pronilpotent dgl's (defined as in any other pro-category) strictly contains the set of cdgl morphisms with respect to the corresponding filtrations. In fact, one can see that pronilpotent morphisms correspond to continuous morphisms arising from these filtrations.

In other terms, if we denote by **pro-dgl** the category of pronilpotent dgl's and morphisms, the obvious functor

$$\textbf{cdgl} \longrightarrow \textbf{pro-dgl}$$

is bijective on objects, but not full on morphisms.

Again, mainly to simplify the computations, especially when dealing with morphisms, we have chosen to work in **cdgl** instead of **pro-dgl**.

3.2 The completion of free Lie algebras

The completion of dgl's which are free as Lie algebras plays an essential role in what follows and deserves to be studied in depth.

Given a free Lie algebra $\mathbb{L}(V)$ we denote its completion with respect to the bracket length (see Remark 3.7) by

$$\widehat{\mathbb{L}}(V) = \varprojlim_n \mathbb{L}(V)/\mathbb{L}^{\geq n}(V).$$

Observe that, as a graded vector space,

$$\widehat{\mathbb{L}}(V) \cong \prod_{n\geq 1} \mathbb{L}^n(V),$$

and thus, any element $a \in \widehat{\mathbb{L}}(V)$ can be uniquely written as a series

$$a = \sum_{n\geq 1} a_n, \quad \text{where} \quad a_n \in \mathbb{L}^n(V).$$

Note also that the induced filtration on $\widehat{\mathbb{L}}(V)$, see (3.4), is given by the ideals,

$$\widehat{\mathbb{L}}^{\geq n}(V) = \prod_{q\geq n} \mathbb{L}^q(V), \quad \text{for} \quad n \geq 1.$$

In particular, in view of (3.5),

$$\widehat{\mathbb{L}}(V)/\widehat{\mathbb{L}}^{\geq n}(V) = \mathbb{L}(V)/\mathbb{L}^{\geq n}(V). \tag{3.7}$$

It is convenient to keep in mind that if $V = V_{\geq 1}$, then $\mathbb{L}(V) = \widehat{\mathbb{L}}(V)$. It is also important to note that $\widehat{\mathbb{L}}(V)$ is the *free complete Lie algebra* generated by V:

Proposition 3.10. *Any linear map $V \to L$ to a complete Lie algebra extends uniquely to a cgl morphism $\widehat{\mathbb{L}}(V) \to L$. In particular, the functor $\widehat{\mathbb{L}}$:* **vect** $\to$ **cgl**, *$V \mapsto \widehat{\mathbb{L}}(V)$, is left adjoint to the forgetful functor.*

Proof. By the universal character of $\mathbb{L}(V)$, the map $V \to L$ extends uniquely to a Lie algebra morphism $\mathbb{L}(V) \to L$ which is filtration preserving, and thus it induces the sought-for cgl morphism. From this, the second assertion is obvious. □

From the general property of left adjoint functors we deduce:

Corollary 3.11. *The functor $\widehat{\mathbb{L}}$ preserves colimits and, in particular,*

$$\widehat{\mathbb{L}}(V) \,\widehat{\amalg}\, \widehat{\mathbb{L}}(W) \cong \widehat{\mathbb{L}}(V \oplus W).$$ □

By an abuse of language a cdgl of the form $(\widehat{\mathbb{L}}(V), d)$ will often be called a *free cdgl*, although it is clear that this is not a free object in the category **cdgl**. Nevertheless, for two such objects we still have

$$(\widehat{\mathbb{L}}(V), d) \,\widehat{\amalg}\, (\widehat{\mathbb{L}}(W), d) \cong (\widehat{\mathbb{L}}(V \oplus W), d).$$

Let $f\colon (\widehat{\mathbb{L}}(V), d) \to (\widehat{\mathbb{L}}(W), d)$ be a cdgl morphism. We write

$$f = \sum_{i\geq 1} f_i, \quad \text{where} \quad f_i(V) \subset \mathbb{L}^i(W).$$

Observe that d induces in V a differential d_1 for which $f_1\colon (V, d_1) \to (W, d_1)$ is a morphism of chain complexes.

Proposition 3.12. *If f_1 is a quasi-isomorphism (respectively, an isomorphism), then f is a quasi-isomorphism (respectively, an isomorphism).*

Remark 3.13. Observe that this statement fails if we work with non-complete free Lie algebras. Consider, for instance, the morphism $f\colon (\mathbb{L}(a,b),0) \to (\mathbb{L}(u,v),0)$, defined by $f(a) = u$ and $f(b) = v + [u,v]$, $|u| = |a| = 0$, $|b| = |v| = n \in \mathbb{Z}$. Then, f_1 is an isomorphism but not f, because v is not in the image of f. However, the completion $\widehat{f}\colon \widehat{\mathbb{L}}(a,b) \to \widehat{\mathbb{L}}(u,v)$ of f is an isomorphism. In particular,

$$v = \widehat{f}\left(\sum_{i=0}^{\infty}(-1)^n \ \mathrm{ad}_a^n(b)\right).$$

In fact, Proposition 3.12 is an immediate consequence of a more general fact on *complete differential graded vector spaces.* As the reader may easily guess, such a space is simply a differential graded vector space V endowed with a decreasing filtration $V = F^1 \supset F^2 \supset \cdots$, which is preserved by the differential, and such that the induced map

$$V \xrightarrow{\cong} \varprojlim_n V/F^n$$

is an isomorphism. Note in particular that the injective character of this isomorphism implies that $\bigcap_n F^n = 0$. A morphism of complete differential vector spaces is one which preserves the corresponding filtrations.

Lemma 3.14. *Let $f\colon V \to W$ a morphism of complete differential graded vector spaces filtered by $\{F^n\}_{n\geq 1}$ and $\{G^n\}_{n\geq 1}$, respectively. If for each $n \geq 1$ the induced map $f^n\colon F^n/F^{n+1} \to G^n/G^{n+1}$ is an isomorphism (respectively, a quasi-isomorphism), then f is an isomorphism (respectively, a quasi-isomorphism).*

Proof. Suppose first that each f^n is an isomorphism, and let $a \in \ker f$. If $a \neq 0$, there is a maximal q such that $a \in F^q$. Then $f^q(a) = 0$ and $a \in F^{q+1}$ by hypothesis. This is in contradiction with the maximality of q. For the surjectivity of f, let $0 \neq b \in W$ and let q be the maximal integer such that $b \in G^q$. Then $0 \neq [b] \in G^q/G^{q+1}$, and there exists $a_q \in F^q$ with $f^q[a_q] = [b]$. Then $b - f(a_q) \in G^{q+1}$ and we take $a_{q+1} \in F^{q+1}$ such that $f^{q+1}[a_{q+1}] = [b - f(a_q)]$. In this way we construct a sequence of elements $a_n \in F^n$, with $n \geq q$, such that $b - f(a_q + a_{q+1} + \cdots + a_n) \in F^{n+1}W$. Then, see (3.2), the series $\sum_{n\geq q} a_n$ represents an element in $\varprojlim_n V/F^n \cong V$ such that $b = f(\sum_{n\geq q} a_n)$.

Now we suppose that each f^n is a quasi-isomorphism and show that f is also a quasi-isomorphism. To this end consider the spectral sequences $E(V)$ and $E(W)$ on V and W induced by the respective filtrations, and let $E(f)\colon E(V) \to E(W)$ the induced morphism. Observe that at the E_0-term,

$$E_0(f) = \bigoplus_{n\geq 1} f^n\colon \bigoplus_{n\geq 1} F^n/F^{n+1} \xrightarrow{\simeq} \bigoplus_{n\geq 1} G^n/G^{n+1},$$

and thus $E_1(f)$ is an isomorphism. As $\bigcap_n F^n = \bigcap_n G^n = 0$, the spectral sequences converge and therefore, by comparison, $H(f)$ is an isomorphism. ∎

Proof of Proposition 3.12. Simply apply Lemma 3.14 to $f\colon (\widehat{\mathbb{L}}(V),d)\to(\widehat{\mathbb{L}}(W),d)$. In fact, recall from (3.7) that $\widehat{\mathbb{L}}(V)/\widehat{\mathbb{L}}^{\geq n}(V) = \mathbb{L}(V)/\mathbb{L}^{\geq n}(V)$ and therefore,

$$\widehat{\mathbb{L}}^{\geq n}(V)/\widehat{\mathbb{L}}^{\geq n+1}(V) = \mathbb{L}^{\geq n}(V)/\mathbb{L}^{\geq n+1}(V) = \mathbb{L}^n(V).$$

Hence, in this particular case, and for each $n \geq 1$, the morphism f^n of Lemma 3.14 is of the form

$$f^n\colon (\mathbb{L}^n(V),\bar{d}) \xrightarrow{\simeq} (\mathbb{L}^n(W),\bar{d}),$$

which is trivially a quasi-isomorphism since $f_1\colon (V,d_1) \xrightarrow{\simeq} (W,d_1)$ is. Indeed,

$$H(\mathbb{L}^n(V),\bar{d}) \cong \mathbb{L}^n\big(H(V,d_1)\big)$$

and, under this identification, $H(f^n)$ becomes $\mathbb{L}^n\big(H(f_1)\big)$. □

As in the classical case we now introduce the notion of minimality of free cdgl's and check that, even in the general, non-reduced case, every complete cdgl has a minimal model.

Definition 3.15. A *minimal cdgl* is a free cdgl $(\widehat{\mathbb{L}}(V),d)$ such that $d(V) \subset \widehat{\mathbb{L}}^{\geq 2}(V)$. A *minimal Lie model* of a cdgl L is such a minimal cdgl together with a quasi-isomorphism

$$\varphi\colon (\widehat{\mathbb{L}}(V),d) \xrightarrow{\simeq} L\,.$$

Proposition 3.16. *Every $L \in \mathbf{cdgl}_0$ admits a minimal Lie model.*

Proof. We will construct by induction on n a minimal cdgl $(\widehat{\mathbb{L}}(V_{\leq n}),d)$ and a morphism

$$f_n\colon (\widehat{\mathbb{L}}(V_{\leq n}),d) \to L$$

such that $H_q(f_n)$ is an isomorphism for $q < n$ and is surjective in degree n. To begin, let $(a_i), i \in I$ be elements in L_0 whose classes form a basis of $H_0(L)$. We define V_0 to be the vector space spanned by the variables x_i, $i \in I$, and we define $dx_i = 0$ and $f_0(x_i) = a_i$.

Now we suppose to have constructed $V_{\leq n}$ and f_n with the above properties. We denote by $(a_i)_{i\in I}$ a family of cycles in L_{n+1} whose classes form a basis of $\operatorname{coker} H_{n+1}(f_n)$ and we define W_{n+1} to be the vector space generated by elements $x_i, i \in I$. As above, we define $dx_i = 0$ and $f_{n+1}(x_i) = a_i$. Now we select cycles b_j in $\widehat{\mathbb{L}}(V_{\leq n})$ that form a basis of $\ker H_n(f_n)$. Then, we introduce a vector space R_{n+1} generated by corresponding variables y_j and we set $d(y_j) = b_j$. Moreover, since there are elements v_j with $f_n(b_j) = d(v_j)$, we define f_{n+1} extending f_n by $f_{n+1}(y_j) = v_j$. Finally, we set $V_{n+1} = W_{n+1} \oplus R_{n+1}$. □

The next results also extend classical results involving free dgl's to the completed context.

Definition 3.17. A *contractible* cdgl is a cdgl of the form $\widehat{\mathbb{L}}(R\oplus dR)$. Observe that, by Lemma 3.14, any contractible cdgl has indeed trivial homology.

Proposition 3.18. *Every free cdgl* $(\widehat{\mathbb{L}}(V),d)$ *in* $\mathbf{dgl}_0$ *is the coproduct*

$$(\widehat{\mathbb{L}}(V),d)\cong(\widehat{\mathbb{L}}(Z),d)\ \widehat{\amalg}\ \widehat{\mathbb{L}}(R\oplus dR),$$

of a minimal cdgl and a contractible cdgl.

Proof. Decompose the differential d in the form $d=\sum_{i\geq 1}d_i$ with $d_q(V)\subset\mathbb{L}^q(V)$, and write $V=Z\oplus R\oplus d_1(R)$, with $d_1Z=0$. It follows that the projection

$$p\colon(\widehat{\mathbb{L}}(V),d_1)\longrightarrow(\widehat{\mathbb{L}}(Z),0)$$

defined by $p(R)=p(d_1R)=0$ is a quasi-isomorphism. We denote its kernel by I. Thus, $H(I,d_1)=0$. Write $I^q=I\cap\mathbb{L}^q(V)$, and let $\{z_i\}$ be a graded basis for Z.

For each i we now construct by induction a sequence of elements $z_i(n)$, for $n\geq 1$, such that

$$z_i(1)=z_i,\quad z_i(n+1)-z_i(n)\in I^{n+1}\quad\text{and}\quad d\big(Z(n)\big)\subset I^{>n}\oplus\widehat{\mathbb{L}}\big(Z(n)\big),$$

where $Z(n)$ is the vector space generated by the elements $z_i(n)$.

Suppose that $z_i(q)$ has been defined with the above properties for $q\leq n$ and all i. For simplicity of notation, fix some index i and set $z=z_i(n)$. We write then

$$d_{n+1}(z)=\alpha(z)+\beta(z),$$

with $\alpha(z)\in\mathbb{L}^{n+1}(Z(n))$ and $\beta(z)\in I^{n+1}$. Since $d^2=0$, by the induction hypothesis,

$$d_1d_{n+1}z=-\sum_{i=2}^{n}d_id_{n+2-i}z\in\mathbb{L}^{n+1}\big(Z(n)\big).$$

It follows that $d_1\beta(z)=0$ and $\beta(z)=d_1\mu(z)$ for some $\mu(z)\in I^{n+1}$. To conclude, define $z_i(n+1)=z-\mu(z)$.

Now, consider the elements

$$z_i'=\sum_{n\geq 1}z_i(n)\in\widehat{\mathbb{L}}(V)$$

and let Z' the vector space generated by the z_i'. Observe that, by construction, both $(\mathbb{L}(Z'),d)$ and $\widehat{\mathbb{L}}(R\oplus dR)$ are sub-cdgl's of $(\widehat{\mathbb{L}}(V),d)$.

To finish, observe that the linear part j_1 of the induced injection

$$j\colon(\widehat{\mathbb{L}}(Z'),d)\ \widehat{\amalg}\ \widehat{\mathbb{L}}(R\oplus dR)\longrightarrow(\widehat{\mathbb{L}}(V),d)$$

is an isomorphism. Hence, by Lemma 3.14, j is an isomorphism. □

Theorem 3.19. *Every quasi-isomorphism* $f\colon(\widehat{\mathbb{L}}(V),d)\xrightarrow{\simeq}(\widehat{\mathbb{L}}(W),d)$ *between minimal cdgl's in* $\mathbf{dgl}_0$ *is an isomorphism. In particular, the minimal Lie model of a cdgl in* $\mathbf{cdgl}_0$ *is unique up to isomorphism.*

Proof. By Proposition 3.12, it is enough to prove that $f_1\colon V \to W$ is an isomorphism.

Let $w \in W_0$. Then $dw = 0$ and since f is a quasi-isomorphism, there is a cycle $a = \sum_{n\geq 1} a_n \in \widehat{\mathbb{L}}(V)$ such that $f(a) - w$ is a boundary. Since $(\widehat{\mathbb{L}}(W), d)$ is minimal, it follows that $f(a) = w$. Thus, $f_1(a_1) = w$, and $f_1\colon V_0 \to W_0$ is surjective.

Now, let $v \in V_0$ with $f_1(v) = 0$, which amounts to saying that $f(v) \in \widehat{\mathbb{L}}^{\geq 2}(W_0)$. By the surjectivity of f_1, there is an element $a_2 \in \mathbb{L}^2(V_0)$ such that $f(v - a_2) \in \widehat{\mathbb{L}}^{\geq 3}(W_0)$. The same argument applies to find $a_3 \in \mathbb{L}^3(V_0)$ with $f(v - a_2 - a_3) \in \widehat{\mathbb{L}}^{\geq 4}(W_0)$. In this way, we construct an element $a = \sum_{n\geq 2} a_n \in \mathbb{L}^{\geq 2}(V_0)$ such that $f(v-a) = 0$. As f is a quasi-isomorphism, $v-a$ is necessarily a boundary, but since $(\widehat{\mathbb{L}}(V), d)$ is minimal it follows that $v = 0$ and thus $f_1\colon V_0 \stackrel{\cong}{\to} W_0$ is an isomorphism.

We suppose by induction that $f_1\colon V_{<n} \to W_{<n}$ is an isomorphism, which implies in particular that

$$f\colon \widehat{\mathbb{L}}(V_{<n}) \stackrel{\cong}{\longrightarrow} \widehat{\mathbb{L}}(W_{<n}) \tag{3.8}$$

is an isomorphism.

Let $w \in W_n$ and note that dw is a boundary in $\widehat{\mathbb{L}}(W_{<n})$. Taking into account that f is a quasi-isomorphism and the isomorphism in (3.8), it follows that there is an element $u \in \widehat{\mathbb{L}}(V)_n$ with $f(du) = dw$. Then, the element $w - f(u) \in \widehat{\mathbb{L}}(W)$ is a cycle, and again since f is a quasi-isomorphism, there is a cycle v in $\widehat{\mathbb{L}}(V)$ such that $f(v) - \big(w - f(u)\big)$ is a boundary.

In particular, $f_1(u_1 + v_1) = w$, where u_1 and v_1 denotes the linear part of u and v. Hence, the morphism $f_1\colon V_n \to W_n$ is surjective.

Now suppose $u \in V_n$ with $f_1(u) = 0$. By the surjectivity of f_1, and using the same argument as for the injectivity of $f_1|_{V_0}$, there is an element $v \in \widehat{\mathbb{L}}^{\geq 2}(V)_n$ such that $f(u+v) = 0$. The element $u+v$ is a cycle because f is an isomorphism in degrees $< n$ and $df(u+v) = 0$. Therefore, as f is a quasi-isomorphism, $u+v$ is a boundary. But, since $d_1 = 0$, this implies that $u = 0$. Hence $f_1\colon V_n \to W_n$ is injective and the proposition follows. □

Proposition 3.20. *Let $f\colon (\widehat{\mathbb{L}}(V), d) \to (\widehat{\mathbb{L}}(W), d)$ be a morphism of cdgl's.*

1. *If f is an isomorphism, then f_1 is an isomorphism.*
2. *If both $V_{\geq 0}$ and $W = W_{\geq 0}$, and f is a quasi-isomorphism, then f_1 is a quasi-isomorphism.*

Remark 3.21. The restriction to $\mathbf{cdgl_0}$ in the second part of the statement is necessary. For instance the inclusion $0 \hookrightarrow (\mathbb{L}(x), d)$ where $|x| = -1$ and $dx = [x, x]$ is a quasi-isomorphism, but its linear part obviously is not.

Proof. (1) Since f is an isomorphism, for each element $w \in W$ there is an element $a = \sum_{n\geq 1} a_n$ in $\widehat{\mathbb{L}}(V)$ with $a_n \in \mathbb{L}^n(V)$ and $f(a) = w$. Then $f_1(a_1) = w$, and so f_1 is surjective.

Now suppose $v \in V$ with $f_1(v) = 0$. By the surjectivity of f_1, and using the same argument as in the proof of Theorem 3.19 we can find elements $a_n \in \mathbb{L}^n(V)$ for $n \geq 2$, such that $f(v + \sum_{n\geq 2} a_q) = 0$. The injectivity of f implies that $v = 0$.

(2) By Proposition 3.18, there are minimal cdgl's $(\widehat{\mathbb{L}}(Z), d)$, $(\widehat{\mathbb{L}}(Z'), d)$ and quasi-isomorphisms

$$g\colon (\widehat{\mathbb{L}}(Z), d) \xrightarrow{\simeq} (\widehat{\mathbb{L}}(V), d) \quad \text{and} \quad h\colon (\widehat{\mathbb{L}}(W), d) \xrightarrow{\simeq} (\widehat{\mathbb{L}}(Z'), d),$$

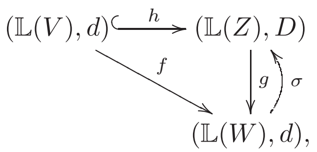
such that g_1 and h_1 are also quasi-isomorphisms. In fact g is the injection of the non-contractible minimal factor and h is the projection onto it. Then, hfg is a quasi-isomorphism between minimal cdgl's and thus, it is an isomorphism by Theorem 3.19. By (1), it follows that $h_1 f_1 g_1$ is an isomorphism and therefore, f_1 is also a quasi-isomorphism. □

Next, we show that the completion functor preserves quasi-isomorphisms between free cdgl's in $\mathbf{dgl}_0$.

Proposition 3.22. *Let $f\colon (\mathbb{L}(V), d) \xrightarrow{\simeq} (\mathbb{L}(W), d)$ be a quasi-isomorphism in $\mathbf{dgl}_0$. Then, the completion $\widehat{f}\colon (\widehat{\mathbb{L}}(V), d) \xrightarrow{\simeq} (\widehat{\mathbb{L}}(W), d)$ is also a quasi-isomorphism.*

Proof. It is enough to show that f_1 is a quasi-isomorphism and then apply Proposition 3.12 to conclude.

To begin, we first extend f into a surjective quasi-isomorphism g,



by setting

$$Z = V \oplus W \oplus s^{-1}W, \quad Dv = dv, \; Dw = s^{-1}w,$$
$$g(w) = w \quad \text{and} \quad g(s^{-1}w) = dw.$$

Now, since g is a surjective quasi-isomorphism, it admits a section σ. We give here a detailed proof of this standard fact to preserve the self-contained character of the text.

First of all, each $a \in W_0$ is a cycle and thus, there is a cycle $b \in \mathbb{L}(Z)$ with $g(b) = a + dc$ for some c. By the surjectivity of g, there is a $c' \in \mathbb{L}(Z)$ with $g(c') = c$. We define $\sigma(a) = b - dc'$ and so we have $g\sigma(a) = a$.

Suppose σ is defined on $W_{<n}$ and let $a \in W_n$. Then, $\sigma(da)$ is a cycle and $g\sigma(da)$ is a boundary. Hence, $\sigma(da) = db$ for some b and therefore $a - g(b)$ is a cycle. As g is a quasi-isomorphism, it follows that

$$a - g(b) = g(b') + dc$$

for some cycle $b' \in \mathbb{L}(Z)$ and $c \in \mathbb{L}(W)$. Once again, by the surjectivity of g, $c = g(c')$ and we define $\sigma(a) = b + b' + dc'$. Clearly, $g\sigma(a) = a$, $\sigma(da) = d\sigma(a)$ and the section of g has been constructed.

Since $g\sigma = \mathrm{id}$, $H(g_1)H(\sigma_1) = \mathrm{id}$ and thus, $H(g_1)$ is surjective.

Also, σ is a quasi-isomorphism since g is. Hence, we can repeat the same process, replacing σ by a surjective quasi-isomorphism with a section, to assert that $H(\sigma_1)$ is also surjective.

Next, we prove that $H(g_1)$ is injective: let $z \in Z$ be a D_1-cycle such that $H(g_1)[z] = 0$. By the surjectivity of $H(\sigma_1)$, $[z] = H(\sigma_1)[a]$ for some d_1-cycle $a \in W$. It follows that $0 = H(g_1)H(\sigma_1)[a] = [a]$ and therefore, $[z] = 0$. Hence g_1 is a quasi-isomorphism.

On the other hand, an easy inspection shows that h and h_1 are also quasi-isomorphisms. Simply note that since $W = W_{\geq 0}$, then $dW_n \subset \mathbb{L}(W_{<n})$ for each $n \geq 0$.

Thus, we conclude that $f_1 = g_1 h_1$ is also a quasi-isomorphism. □

The following result shows that free cgl's appear naturally as the duals of cofree graded Lie coalgebras.

Proposition 3.23. *Let V be a finite type graded vector space. Then, $\mathbb{L}^{\mathrm{c}}(V)^{\#} = \widehat{\mathbb{L}}(V^{\#})$.*

Proof. Recall from Section 1.2.4 that any cofree graded Lie coalgebra $\mathbb{L}^{\mathrm{c}}(V)$ is the union of sub-Lie coalgebras $\mathbb{L}^{\mathrm{c}\leq n}(V)$, $n \geq 1$, defined as the image of $T^{\leq n}(V)$ by the projection $T(V) \twoheadrightarrow \mathbb{L}^{\mathrm{c}}(V)$. That is,

$$\mathbb{L}^{\mathrm{c}}(V) = \varinjlim_n \mathbb{L}^{\mathrm{c}\leq n}(V). \tag{3.9}$$

Note also that, since V is of finite type, $\mathbb{L}^{\mathrm{c}\leq n}(V)$ is finite-dimensional and

$$\mathbb{L}^{\mathrm{c}\leq n}(V)^{\#} = \mathbb{L}(V^{\#})/\mathbb{L}^{>n}(V^{\#}). \tag{3.10}$$

Then,

$$\begin{aligned}\mathbb{L}^{\mathrm{c}}(V)^{\#} &= \mathrm{Hom}(\varinjlim_n \mathbb{L}^{\mathrm{c}\leq n}(V), \mathbb{Q}) \\ &= \varprojlim_n \mathrm{Hom}\big(\mathbb{L}^{\mathrm{c}\leq n}(V), \mathbb{Q}\big) \\ &= \varprojlim_n \mathbb{L}(V^{\#})/\mathbb{L}^{>n}(V^{\#}) = \widehat{\mathbb{L}}(V^{\#}).\end{aligned}$$

□

Remark 3.24. The converse is not true, that is, $\mathbb{L}^c(V^\#)$ is not $\widehat{\mathbb{L}}(V)^\#$, as this may not be even a Lie coalgebra. Nevertheless, by identity (3.10), $\mathbb{L}^{c\leq n}(V^\#) = \big(\mathbb{L}(V)/\mathbb{L}^{>n}(V)\big)^\#$ as long as V is of finite type. And thus, by formula (3.9),

$$\mathbb{L}^c(V^\#) = \varinjlim_n \big(\mathbb{L}(V)/\mathbb{L}^{>n}(V)\big)^\#.$$

We finish the section with a first application which highlights the strength of the extension of the classical theory of free dgl's to the completed context.

Observe that the completion $\widehat{\mathscr{L}}$ of the Quillen functor $\mathscr{L}$ takes values precisely in the category of free cdgl's. We extend and complement some of the results in Chapter 2 whenever completion is considered. First, we prove that the restriction to fibrant objects in (2) of Proposition 2.4 disappears when completing $\mathscr{L}$.

Proposition 3.25. *Let $f\colon C \xrightarrow{\simeq} C'$ be a quasi-isomorphism of cdgc's. Then,*

$$\widehat{\mathscr{L}}(f)\colon \widehat{\mathscr{L}}(C) \xrightarrow{\simeq} \widehat{\mathscr{L}}(C')$$

is a quasi-isomorphism.

Proof. It follows directly from Proposition 3.12 as $\widehat{\mathscr{L}}(f)_1 = s^{-1}f$ is a quasi-isomorphism. □

Example 3.26. Consider the quasi-isomorphism of differential graded coalgebras $\varphi\colon C \xrightarrow{\simeq} (\wedge e, 0)$, described in Example 2.5 for which $\mathscr{L}(\varphi)$ fails to be a quasi-isomorphism. Nonetheless, by Proposition 3.25, $\widehat{\mathscr{L}}(C)$ is a quasi-isomorphism. Indeed, using the same notation as in Example 2.5, we have seen that

$$H_0(\mathscr{L}(\wedge e, 0)) = 0,$$

in contrast with

$$H_0(\mathscr{L}(C)) = \langle \overline{x}, \overline{y}, \overline{z}\rangle, \quad \text{where} \quad [\overline{x},\overline{y}] = \overline{z},\ [\overline{y},\overline{z}] = \overline{x} \quad \text{and} \quad [\overline{z},\overline{x}] = \overline{y}.$$

Despite z being a cycle in $\mathscr{L}(C)$ defining a non-zero class in $H_0(\mathscr{L}(C))$, it is a boundary in $\widehat{\mathscr{L}}(C)$: write

$$\begin{aligned} z &= du + [x,y] = du + d[v,y] + [[y,z],y] \\ &= du + d[v,y] + d[[w,z],y] + [[[z,x],z],y] = \cdots \end{aligned}$$

and proceed inductively in this way to show that $z = dv$, where $v = \sum_{i\geq 0} v_i$ with $v_0 = u + [v,y] + [[w,z],y]$ and $v_{i+1} = -\mathrm{ad}_y \mathrm{ad}_z \mathrm{ad}_x(v_i)$.

A similar computation shows that the cycles x and y are boundaries in $\widehat{\mathscr{L}}(C)$.

Proposition 3.27. *Let $(\wedge V, d)$ be a Sullivan algebra where V is a finite type graded vector space. Then,*

(i) $\widehat{\mathscr{L}}(\wedge V, d)^\# \cong \big(\mathscr{E}(\wedge V, d)\big)^\#$.

(ii) *The completion map $\mathscr{L}(\wedge V, d)^\# \xrightarrow{\simeq} \widehat{\mathscr{L}}(\wedge V, d)^\#$ is a quasi-isomorphism.*

Proof. (i) By definition and using Proposition 3.23, it follows that

$$\big(\mathscr{E}(\wedge V,d)\big)^{\#} = \big(\mathbb{L}^c(s\wedge^+ V),d\big)^{\#} \cong (\widehat{\mathbb{L}}(s\wedge^+ V)^{\#},d^{\#}) \cong \widehat{\mathscr{L}}\big((\wedge V,d)^{\#}\big).$$

(ii) By Lemma 2.12(2), the projection $\mathscr{E}(\wedge V,d) \xrightarrow{\simeq} s^{-1}V$ is a quasi-isomorphism and therefore, the injection

$$s^{-1}V^{\#} \xrightarrow{\simeq} \big(\mathscr{E}(\wedge V,d)\big)^{\#}$$

is also a quasi-isomorphism. On the other hand, by Lemma 2.2, the injection

$$s^{-1}V^{\#} \xrightarrow{\simeq} \mathscr{L}(\wedge V,d)^{\#}$$

is also a quasi-isomorphism. The statement follows from the commutativity of the diagram

$$\begin{array}{ccc} & s^{-1}V^{\#} & \\ {}^{\simeq}\swarrow & & \searrow^{\simeq} \\ \mathscr{L}(\wedge V,d)^{\#} & \longrightarrow & \widehat{\mathscr{L}}(\wedge V,d)^{\#}. \end{array}$$

□

The topological translation of this result is the following.

Definition 3.28. Let $(\wedge V,d)$ be the minimal Sullivan model of a nilpotent simplicial set of finite type. Since V has finite type, its dual $(\wedge V,d)^{\#}$ is a well-defined cocommutative differential graded coalgebra and we can consider (see Section 2.1) the dgl

$$\mathscr{L}(\wedge V,d)^{\#}.$$

This is the *Neisendorfer model of X* as introduced in [109].

Then, Proposition 3.27(ii) reads:

Proposition 3.29. *The Neisendorfer model of any nilpotent simplicial set of finite type is quasi-isomorphic to its completion.* □

Assertion (ii) of Proposition 3.27 suggests the possibility of comparing, in general, the homology of a free dgl with that of its completion. We finish the section with two results in this direction.

Proposition 3.30. *Let $(\mathbb{L}(V),d)$ be a dgl in $\mathbf{dgl}_0$ with V a finite type graded vector space, whose homology is a finite type nilpotent Lie algebra. Then, the completion morphism*

$$(\mathbb{L}(V),d) \xrightarrow{\simeq} (\widehat{\mathbb{L}}(V),d)$$

is a quasi-isomorphism.

Proof. By the Neisendorfer equivalence in Theorem 2.6, there is a Sullivan minimal algebra $(\wedge Z, d)$ with Z a finite type graded vector space such that $(\mathbb{L}(V), d)$ is quasi-isomorphic to $\mathscr{L}(\wedge Z, d)^{\#}$. By Proposition 3.22 it follows then that their completions are also quasi-isomorphic

$$(\widehat{\mathbb{L}}(V), d) \xrightarrow{\simeq} \widehat{\mathscr{L}}(\wedge Z, d)^{\#}\,.$$

Now, by Proposition 3.27, the completion morphism $\mathscr{L}(\wedge Z, d)^{\#} \xrightarrow{\simeq} \widehat{\mathscr{L}}(\wedge Z, d)^{\#}$ is a quasi-isomorphism. The result is then a consequence of the commutativity of the following diagram

$$\begin{array}{ccc} (\mathbb{L}(V), d) & \longrightarrow & (\widehat{\mathbb{L}}(V), d) \\ \simeq\big\downarrow & & \big\downarrow\simeq \\ \mathscr{L}(\wedge Z, d)^{\#} & \xrightarrow[\simeq]{} & \widehat{\mathscr{L}}(\wedge Z, d)^{\#}. \end{array}$$ □

Remark 3.31. The nilpotency hypothesis on the homology is necessary. For instance, when $\dim V_0 \geq 2$ and the differential is 0, the inclusion $(\mathbb{L}(V), 0) \to (\widehat{\mathbb{L}}(V), 0)$ is clearly not a quasi-isomorphism.

Concerning this general question, we also show that the homology of a free cdgl generated by a finite type vector space is always a limit of nilpotent Lie algebras:

Proposition 3.32. *Let $(\widehat{\mathbb{L}}(V), d)$ be a free cdgl in which V has finite type. Then,*

$$H\big(\widehat{\mathbb{L}}(V)\big) \cong \varprojlim_n H\big(\mathbb{L}(V)/\mathbb{L}^{>n}(V)\big).$$

Proof. In general, under no finiteness assumptions, a classical result on the homology of inverse limits, see for instance [131, Theorem 3.5.8], provides a short exact sequence

$$0 \longrightarrow \varprojlim_n{}^1 H_{q+1}\big(\mathbb{L}(V)/\mathbb{L}^{>n}(V)\big) \longrightarrow H_q\big(\widehat{\mathbb{L}}(V)\big) \longrightarrow \varprojlim_n H_q\big(\mathbb{L}(V)/\mathbb{L}^{>n}(V)\big) \longrightarrow 0.$$

Moreover, if V is a finite type graded vector space, then each $H_q\big(\mathbb{L}(V)/\mathbb{L}^{>n}(V)\big)$ is also a finite type graded vector space and the tower is Mittag-Leffler. Therefore, $\varprojlim_n{}^1$ vanishes on it and the result follows. □

3.3 Completion vs profinite completion

This section, which can be regarded simply as an appendix to this chapter, is devoted to stressing the difference between the completion and profinite completion procedures and to giving some particular instances in which they coincide or are related.

Definition 3.33. A *profinite vector space* is an inverse limit

$$V = \varprojlim_{\alpha \in I} V_\alpha,$$

endowed with the inverse limit topology, considering on each V_α the discrete topology. As 0 is open in each V_α, one easily sees that the kernels of the maps $V \to V_\alpha$ form a system of open neighborhoods of $0 \in V$.

The profinite vector spaces form a category **pvect** in which the morphisms are the continuous linear maps.

Proposition 3.34. *The category* **pvect** *is isomorphic to the opposite category* $\mathbf{vect^{op}}$ *of graded vector spaces.*

Proof. Consider the functors

$$\mathbf{vect^{op}} \underset{\#_c}{\overset{\#}{\rightleftarrows}} \mathbf{pvect},$$

where $\#$ is the usual dual and $\#_c$ denotes the *topological dual.* That is, given V a profinite vector space, $V^{\#_c}$ consists of continuous maps $V \to \mathbb{Q}$. We will show that these functors are inverses of each other.

For this, consider $V = \varprojlim V_\alpha \in \mathbf{pvect}$. Then the continuous dual of each projection $p_\alpha \colon V \to V_\alpha$ produces a map $V_\alpha^\# \to V^{\#_c}$ (observe that, since each V_α is of finite type, its topological dual is just the usual dual). These maps induce a morphism,

$$\varinjlim_\alpha V_\alpha^\# \longrightarrow V^{\#_c}. \tag{3.11}$$

We verify that it is an isomorphism. This amounts to saying that it is surjective, i.e., that each continuous map $f \colon V \to \mathbb{Q}$ in $V^{\#_c}$ factors through some V_α.

For such a map, $W = f^{-1}(0)$ is a closed vector subspace of codimension 1 and we denote by $\overline{f} \colon V/W \to \mathbb{Q}$ the induced isomorphism.

Now, since the topology of V is given by the inverse limit topology where each V_α is discrete, every closed subset of V is the intersection of closed subsets of the form $p_\alpha^{-1}(F_\alpha)$, with $p_\alpha \colon V \to V_\alpha$ the corresponding projection and $F_\alpha \subset V_\alpha$. Therefore, there is a subset $Z_\alpha \subsetneq V_\alpha$ with $W \subset p_\alpha^{-1}(Z_\alpha)$. It follows that $p_\alpha(W) \subset Z_\alpha$, and $p_\alpha(W) \neq V_\alpha$. In particular, the induced map

$$\overline{p}_\alpha \colon V/W \xrightarrow{\cong} V_\alpha / p_\alpha(W)$$

is an isomorphism. Define $f_\alpha \colon V_\alpha \to \mathbb{Q}$ as the composition

$$V_\alpha \longrightarrow V_\alpha/p_\alpha(W) \xrightarrow{\overline{p}_\alpha^{\,-1}} V/W \xrightarrow{\overline{f}} \mathbb{Q}\,.$$

Then $f = f_\alpha \circ p_\alpha$ and the map (3.11) is in fact an isomorphism. In other words,

$$\varinjlim_\alpha V_\alpha^\# \cong (\varprojlim_\alpha V_\alpha)^{\#_c}. \tag{3.12}$$

From this, the proposition easily follows:

On the one hand, let V a graded vector space. Then, V is the inductive limit $V = \varinjlim_\alpha V_\alpha$ of its finite type subspaces. Then, $V^\# = \varprojlim V_\alpha^\#$ and, using (3.12) together with the finite type character of each V_α, we get:

$$\#_{\rm c} \circ \#(V) = (\varprojlim V_\alpha^\#)^{\#_{\rm c}} \cong \varinjlim_\alpha (V_\alpha^\#)^\# \cong \varinjlim_\alpha V_\alpha = V.$$

Analogously, given $V = \varprojlim_\alpha V_\alpha \in \mathbf{pvect}$,

$$\# \circ \#_{\rm c}(V) \cong (\varinjlim_\alpha V_\alpha^\#)^\# \cong \varprojlim_\alpha (V_\alpha^\#)^\# \cong V. \qquad \square$$

We now extend the notion of profiniteness to differential graded Lie algebras.

Definition 3.35. A *profinite dgl* L is an inverse limit

$$L = \varprojlim L_\alpha$$

of finite type nilpotent dgl's, endowed with the inverse limit topology, considering on each L_α the discrete topology. Profinite dgl's form the category **pdgl** in which the morphisms are the continuous dgl morphisms.

The isomorphism in Proposition 3.34 restricts to an isomorphism between profinite Lie algebras and the opposite category of conilpotent Lie coalgebras, which we now introduce:

Definition 3.36. A *conilpotent differential graded Lie coalgebra* is an inductive limit

$$E = \varinjlim_\alpha E_\alpha$$

of finite type dglc's such that each $E_\alpha^\#$ is a nilpotent dgl.

Then we have, cf. [10, A.15]:

Proposition 3.37. *The category* **pdgl** *is isomorphic to the opposite category of conilpotent dglc's.*

Proof. By definition, the linear dual of a conilpotent Lie coalgebra $E = \varinjlim E_\alpha$ is a profinite dgl,

$$E^\# = \varprojlim E_\alpha^\# \,.$$

On the other hand, let $L = \varprojlim_\alpha L_\alpha$ be a profinite dgl. Then, in view of (3.12), its topological dual,

$$L^{\#_{\rm c}} \cong \varinjlim L_\alpha^\#,$$

is an inductive limit of dglc's whose duals are nilpotent finite type dgl's. Hence, the isomorphism in Proposition 3.34 restricts to the sought-for isomorphism. $\square$

Definition 3.38. The *profinite completion* $\widehat{L}^{\rm f}$ of a dgl L is the profinite dgl

$$\widehat{L}^{\rm f} = \varprojlim_{\alpha} L/F_\alpha \,,$$

where F_α runs over all ideals such that L/F_α is of finite type and nilpotent. The natural map $L \to \widehat{L}^{\rm f}$ is the *profinite completion morphism.*

As in [10, Remark 7.2] we prove:

Proposition 3.39. *Every profinite dgl is a cdgl.*

Proof. Let $L = \varprojlim_\alpha L_\alpha$ a profinite dgl and consider, for each α, the short exact sequence

$$0 \longrightarrow L_\alpha^n \longrightarrow L_\alpha \longrightarrow L_\alpha/L_\alpha^n \longrightarrow 0.$$

Since the projective limit is an exact functor on finite type spaces, we get the short exact sequence

$$0 \longrightarrow \varprojlim_{\alpha} L_\alpha^n \longrightarrow L \longrightarrow \varprojlim_{\alpha} L_\alpha/L_\alpha^n \longrightarrow 0.$$

But a simple inspection shows that $\varprojlim_\alpha L_\alpha^n = L^n$ for each $n \geq 1$ and therefore $L/L^n = \varprojlim_\alpha L_\alpha/L_\alpha^n$. On the other hand, since each L_α is of finite type, $L_\alpha \cong \varprojlim_n L_\alpha/L_\alpha^n$. Therefore,

$$\varprojlim_{n} L/L^n \cong \varprojlim_{n} \varprojlim_{\alpha} L_\alpha/L_\alpha^n \cong \varprojlim_{\alpha} \varprojlim_{n} L_\alpha/L_\alpha^n \cong \varprojlim_{\alpha} L_\alpha = L,$$

that is, L is complete (with respect to the bracket length filtration). □

The converse however is not true.

Example 3.40.

(1) Let L be an infinite-dimensional abelian Lie algebra with basis $\{x_\alpha\}$. Then, $\widehat{L} = L$, while $\widehat{L}^{\rm f} = \prod_\alpha \mathbb{Q}x_\alpha$.

(2) Denote by $L = L_0$ the Lie algebra $\mathbb{L}(a_i, b_i, i \geq 1)/I$, where I is the ideal generated by the brackets $[a_i, a_j], [b_i, b_j]$ and $[a_i, b_j]$ for $i \neq j$, and the elements $[a_i, b_i] - [a_1, b_1]$ for $i \geq 2$. Denote by $q\colon L \to E$ a morphism from L to a finite-dimensional nilpotent Lie algebra E and let $a_n + \sum_{i<n} \lambda_i a_i$ be an element in $\ker q$. Then,

$$0 = q[a_n + \sum_{i<n} \lambda_i a_i, b_n] = q[a_n, b_n] + \sum_{i<n} \lambda_i \, q[a_i, b_n] = q[a_n, b_n].$$

Therefore, $[a_1, b_1]$ belongs to the kernel of the natural morphism $L \to \widehat{L}^{\rm f}$. In this case $\widehat{L}^{\rm f}$ is abelian, but $\widehat{L} = L$.

Let us show that if the space of "indecomposables" of the considered dgl is of finite type, then the completion coincides with the profinite completion.

Proposition 3.41. *Let L be a dgl for which $L/[L,L]$ is of finite type. Then, $\widehat{L} \cong \widehat{L}^{\mathrm{f}}$. In particular, for any free Lie algebra $\mathbb{L}(V)$ in which V is of finite type, $\widehat{\mathbb{L}}(V) \cong \widehat{\mathbb{L}}^{\mathrm{f}}(V)$.*

Here, $\widehat{\mathbb{L}}^{\mathrm{f}}(V)$ denotes the profinite completion of $\mathbb{L}(V)$.

Proof. If $L/[L,L]$ is of finite type, then each L/L^n is also of finite type. To see this, observe that if the classes of the elements x_i generate $L/[L,L]$, then the family $\big[[\dots[x_{i_1},x_{i_2}],x_{i_3}]\dots],x_{i_n}\big]$ generates L^n/L^{n+1} and therefore, all of these are finite type vector spaces. The result is then obtained by induction on the short exact sequence $0 \to L^n/L^{n+1} \to L/L^{n+1} \to L/L^n \to 0$.

Write any $x \in \widehat{L} = \varprojlim_n L/L^n$ as a sequence (x_n) with $n \geq 1$ and $x_n \in L/L^n$. In the same way write any $y \in \widehat{L}^f$ as the family (y_α) of its projections in the quotients L/F_α.

Define an explicit isomorphism

$$\widehat{L} \xrightarrow{\cong} \widehat{L}^{\mathrm{f}}$$

as follows: since each L/L^n is finite-dimensional and nilpotent, to a family $y = (y_\alpha) \in \widehat{L}^f$, we associate the sub-family $(x_n) \in \widehat{L}$.

Conversely, let $x = (x_n) \in \widehat{L}$ and let L/F_α be a finite-dimensional nilpotent quotient. Then, the projection $p\colon L \to L/F_\alpha$ factors through some $q_{n,\alpha}\colon L/L^n \to L/L_\alpha$. We define $x_\alpha = q_{n,\alpha}(x_n)$. This construction does not depend on the choice of the integer n and defines a map $\widehat{\mathbb{L}} \to \widehat{\mathbb{L}}^{\mathrm{f}}$ which is clearly the inverse of the morphism above. □

Corollary 3.42. *Let V be a finite type graded vector space. Then, $\mathbb{L}^{\mathrm{c}}(V)^\# \cong \widehat{\mathbb{L}}^{\mathrm{f}}(V^\#)$.*

Proof. Apply Propositions 3.23 and 3.41. □

We finish with the following:

Proposition 3.43. *Let $(\widehat{\mathbb{L}}(V),d)$ be a cdgl where $V = V_{\geq 0}$ and $H(V,d_1)$ is a finite type graded vector space. Then, the profinite completion morphism*

$$\alpha_V\colon (\widehat{\mathbb{L}}(V),d) \xrightarrow{\simeq} (\widehat{\mathbb{L}}(V)^{\mathrm{f}},d)$$

is a quasi-isomorphism.

Here, $\widehat{\mathbb{L}}(V)^{\mathrm{f}}$ denotes the profinite completion of $\widehat{\mathbb{L}}(V)$ and is not to be confused with $\widehat{\mathbb{L}}^{\mathrm{f}}(V)$, the profinite completion of $\mathbb{L}(V)$.

Proof. By Proposition 3.18, we have a quasi-isomorphism

$$\varphi\colon (\widehat{\mathbb{L}}(W),d) \xrightarrow{\simeq} (\widehat{\mathbb{L}}(V),d),$$

where W is a finite type graded vector space. The naturality of the profinite completion yields then a commutative diagram

$$\begin{array}{ccc} (\widehat{\mathbb{L}}(W), d) & \xrightarrow[\simeq]{\varphi} & (\widehat{\mathbb{L}}(V), d) \\ {\scriptstyle \alpha_W}\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \alpha_V} \\ (\widehat{\mathbb{L}}(W)^{\mathrm{f}}, d) & \xrightarrow[\simeq]{\varphi^{\mathrm{f}}} & (\widehat{\mathbb{L}}(V)^{\mathrm{f}}, d)\,. \end{array}$$

By Proposition 3.20, φ_1 is a quasi-isomorphism and a similar proof to that of Proposition 3.12 shows that φ^{f} is a quasi-isomorphism. Finally, by the first assertion of Proposition 3.41, α_W is an isomorphism and therefore, α_V is a quasi-isomorphism. □

Chapter 4

Maurer–Cartan Elements and the Deligne Groupoid

In a dgl L, a Maurer–Cartan element is an element z of degree -1 that satisfies the so-called *Maurer–Cartan equation*,

$$dz = -\frac{1}{2}[z, z].$$

Whenever the dgl is complete, the set of these elements is endowed with a particular equivalence relation given by the *gauge action*, a particular action of L_0 in the set of Maurer–Cartan elements. This plays a fundamental role in the geometrical interpretation of complete differential graded Lie algebras.

For the time being, the reader may heuristically think of Maurer–Cartan elements of a cdgl L as "points" in the topological space represented by L. Moreover, the path component of this space containing the point represented by the Maurer–Cartan element a can be identified with the cdgl L with a new differential d_a obtained by perturbing the original differential d by a,

$$d_a = d + \mathrm{ad}_a\,.$$

In the same way, the fact that two Maurer–Cartan elements are gauge related should be thought as a "path" in L joining these two points. This path is represented by an element of L_0 acting as a *gauge transformation*, sending a Maurer–Cartan element into the corresponding gauge related one. We prove then that the group of gauge transformations can be regarded as the vector space L_0 with the group structure given by the *Baker–Campbell–Hausdorff* formula.

With this point of view, the reader may then think of the *Deligne groupoid* of a given cdgl L as the category of points and paths in L. With this analogy, the orbit space of the gauge action of L_0 on Maurer–Cartan elements, intuitively for now, represents the set of "path components" of L, or equivalently, of its Deligne groupoid.

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In this language, the *Goldman–Millson Theorem* establishes that a cdgl morphism, which induces a quasi-isomorphism at each term of the corresponding filtrations, also induces a bijection between the sets of Maurer–Cartan elements modulo the gauge action. That is, this cdgl morphism "is" a homotopy equivalence between the spaces represented by the given cdgl's.

All of this is carefully presented in this chapter, from a purely algebraic and self-contained point of view. Nevertheless, in subsequent chapters, specially when the realization of a cdgl and its homotopical behavior are finally presented, the intuition gives way to a formal homotopy theory of cdgl's in which all of the above makes perfect sense. In particular, the gauge relation between Maurer–Cartan elements would precisely become a simple homotopy between two points, in the geometrical realization of the considered cdgl.

From this point on we advise the reader to pay attention to the following: as we have seen, elements in cdgl's appear naturally as series and, as such, their manipulation is often easier when they are expressed by their corresponding generating functions. This chapter contains many computations of this kind in which the advantage of using such generating operators will be revealed.

4.1 Maurer–Cartan elements

Given L a dgl, a *Maurer–Cartan element* or MC element, is an element $a \in L_{-1}$ satisfying the *Maurer–Cartan equation*

$$da + \frac{1}{2}[a,a] = 0\,.$$

For instance, the element 0 is always an MC element and is the only one if $L = L_{\geq 0}$. On the other hand, each central cycle in L_{-1} is also an MC element.

The set of Maurer–Cartan elements of L is denoted by $\mathrm{MC}(L)$. Since $\mathrm{MC}(L)$ is preserved by dgl morphisms, MC can be regarded as a functor

$$\mathrm{MC}\colon \mathbf{cdgl} \longrightarrow \mathbf{set}^*,$$

where $\mathbf{set}^*$ is the category of pointed sets and $0 \in \mathrm{MC}(L)$ is the base point for any L.

Lemma 4.1. *The functor* MC *commutes with small inverse limits.*

Proof. This follows directly from the fact that MC commutes with products and equalizers. □

Example 4.2. Let $(L,d) = (\mathbb{L}(a,b)/[a,b],d)$ with $|a| = |b| = -1$, $da = -\frac{1}{2}[a,a]$ and $db = -\frac{1}{2}[b,b]$. Then, $\mathrm{MC}(L) = \{0, a, b, a+b\}$, and therefore, nonzero different Maurer–Cartan elements are not necessarily linearly independent.

Example 4.3. Let A be the cdga, with trivial differential, generated by the elements $e_1,\dots,e_n$ in degree 0 with $e_i^2 = e_i$ for all i and $e_ie_j = 0$ for $i \neq j$. That is, $\{e_1,\dots,e_n\}$ is a family of *orthogonal idempotents*. Then, by definition (see Section 2.2),

$$\mathscr{E}(A) = (\mathbb{L}^{\mathrm{c}}(se_1,\dots,se_n), d),$$

with $|se_i| = -1$ and $d([se_i, se_i]^{\mathrm{c}}) = -se_i$. By Proposition 3.23,

$$\mathscr{E}(A)^{\#} = (\widehat{\mathbb{L}}(x_1,\dots,x_n), d),$$

where $\{x_1,\dots,x_n\}$ is a dual basis of the vector space generated by the family $\{e_1,\dots,e_n\}$ and the Lie bracket is precisely the dual of the comultiplication of $\mathscr{E}(A)$. In particular, if we denote by $\langle\, ,\, \rangle$ the pairing between a vector space and its dual, we have $\langle x_i, se_j\rangle = \delta_{ij}$ and

$$\langle [x_i, x_i], [e_i, e_i]^{\mathrm{c}}\rangle = \langle x_i \otimes x_i, \Delta([e_i, e_i]^{\mathrm{c}})\rangle = 2.$$

It follows that

$$dx_i = -\frac{1}{2}[x_i, x_i],$$

and therefore,

$$\mathrm{MC}(\mathscr{E}(A)^{\#}) = \mathrm{MC}(\widehat{\mathbb{L}}(x_1,\dots,x_n), d) = \{0, x_1,\dots,x_n\}.$$

Observe that A is isomorphic to the cdga $A_{\mathrm{PL}}(X)$ of PL-forms on X, the discrete topological space of n points. As we will see later in Chapter 7, the cdgl $(\widehat{\mathbb{L}}(x_1,\dots,x_n), d)$ becomes a Lie model of X.

Definition 4.4. Given $a \in \mathrm{MC}(L)$, the derivation, $d_a = d + \mathrm{ad}_a$ is again a differential on L, called the *perturbed differential by a* .

Here, ad_a denotes the usual adjoint operator $\mathrm{ad}_a\, b = [a, b]$, which is clearly a derivation of L of degree $|a|$.

A simple computation shows the following.

Proposition 4.5. *Let (L, d) be a dgl. Then $b \in \mathrm{MC}(L, d)$ if and only if $b - a \in \mathrm{MC}(L, d_a)$. In particular, the map $b \mapsto b - a$ defines a bijection $\mathrm{MC}(L, d) \xrightarrow{\cong} \mathrm{MC}(L, d_a)$.* □

The "truncation" of a dgl with respect to a given MC element constitutes an important object in our theory:

Definition 4.6. Let L be a dgl and $a \in \mathrm{MC}(L)$. *The component of L at a* is the non-negatively graded cdgl L^a in which,

$$L^a_p = \begin{cases} \ker d_a, & \text{if } p = 0,\\ L_p, & \text{if } p > 0, \end{cases}$$

with the perturbed differential d_a.

Observe that L^a is not a quotient, neither an ideal, but just a sub-dgl of (L, d_a) for which the injection $(L^a, d_a) \hookrightarrow (L, d_a)$ induces an isomorphism in homology in degrees ≥ 0.

We finish with the following trivial, yet useful observation in which we use the notation in (3.1).

Lemma 4.7. *Let $L = \varprojlim_n L/F^n$ be a cdgl and $a = (a_n)_{n\geq 1} \in L_{-1}$ such that each a_n is a Maurer–Cartan element in L/F^n. Then $a \in \mathrm{MC}(L)$.* □

4.2 Exponential automorphisms and the Baker–Campbell–Hausdorff product

Given a cgl L, the sub-Lie algebra L_0 can be equipped with the group structure given by the *Baker–Campbell–Hausdorff product*. This is an important object in our theory which is carefully presented in this section. Specific details of what follows can be found in [51, §2.4].

Let UL be the universal enveloping algebra of a graded Lie algebra L. Let I be the ideal of UL_0 generated by L_0 and consider the filtration in the graded algebra UL_0 given by

$$I^0 \supset I^1 \cdots \supset I^n \supset \cdots,$$

where $I^0 = UL_0$, $I^1 = I$ and $I^n = I^{n-1}I$, $n \geq 2$. We now complete the graded algebra UL_0 and the ideal I with respect to this filtration, to obtain

$$\widehat{UL_0} = \varprojlim_{n\geq 0} UL_0/I^n \quad \text{and} \quad \widehat{I} = \varprojlim_{n\geq 1} I/I^n. \tag{4.1}$$

It is well known that the maps

$$\widehat{I} \underset{\log}{\overset{\exp}{\rightleftarrows}} 1 + \widehat{I} \quad (\cong)$$

are bijections and inverses of each other. Here,

$$\exp(x) = e^x = \sum_{n\geq 0} \frac{x^n}{n!} \quad \text{and} \quad \log(1+x) = \sum_{n\geq 1} (-1)^{n+1}\frac{x^n}{n}.$$

On the other hand, given two graded vector spaces V and W, filtered respectively by

$$F^0 = V \supset F^1 \cdots \supset F^n \supset \cdots \quad \text{and} \quad G^0 = V \supset G^1 \cdots \supset G^n \supset \cdots,$$

their tensor product $V \otimes W$ is naturally filtered by $\{J^n\}_{n\geq 0}$ where

$$J^n = \oplus_{i+j=n} F^i \otimes G^j.$$

Define the *complete tensor product* of V and W as the completion of $V \otimes W$ with respect to this filtration,

$$V \widehat{\otimes} W = \varprojlim_n (V \otimes W)/J^n.$$

Given $x \in V$ and $y \in W$, denote by $x \widehat{\otimes} y$ the image of $x \otimes y$ under the canonical map $V \otimes W \to V \widehat{\otimes} W$.

Now consider the *diagonal map* on the Lie algebra L_0,

$$\Delta\colon L_0 \longrightarrow L_0 \times L_0, \quad \Delta(x) = (x, x),$$

and its extension to $UL_0 \to U(L_0 \times L_0) = UL_0 \otimes UL_0$. As this map preserves the filtration $\{I^n\}_{n \geq 0}$ in UL_0, and the induced one in $UL_0 \otimes UL_0$, by completing, we obtain an algebra morphism, which we denote in the same way,

$$\Delta\colon \widehat{UL_0} \longrightarrow \widehat{UL_0} \widehat{\otimes} \widehat{UL_0}.$$

Let

$$G = \{x \in \widehat{UL_0} \mid \Delta(x) = x \widehat{\otimes} x\}$$

be the group of *grouplike elements* in $\widehat{UL_0}$, and denote by

$$P = \{x \in \widehat{I} \mid \Delta(x) = x \widehat{\otimes} 1 + 1 \widehat{\otimes} x\}$$

the set of *primitive elements* in $\widehat{UL_0}$. Whenever $L_0 = \varprojlim_n L_0/L_0^n$ is complete, see for instance [51, Proposition 2.3], the injection $L_0 \hookrightarrow \widehat{I}_0$ restricts to an isomorphism $L_0 \xrightarrow{\cong} P$ and the exponential and logarithm restrict to bijections

$$L_0 \underset{\log}{\overset{\exp}{\rightleftarrows}} G. \tag{4.2}$$

In particular, the group structure on G induces a multiplication law in L_0, called the *Baker–Campbell–Hausdorff product,* BCH product henceforth, defined by

$$a * b = \log(e^a \cdot e^b).$$

As the multiplication in G is associative, the BCH product is associative. An explicit form of the product is given by the classical Baker–Campbell–Hausdorff formula,

$$a * b = a + b + \frac{1}{2}[a, b] + \frac{1}{12}\big[a, [a, b]\big] - \frac{1}{12}\big[b, [a, b]\big] + \cdots$$

Note that $a * (-a) = 0$ and therefore, $-a$ is the inverse of a for the BCH product. We also use the notation $-a = a^{-1}$.

A particularly useful class of automorphisms of a given cdgl consists of exponential maps of certain derivations, which often are adjoint operators. The rest of the section is devoted to a detailed analysis of this class.

Definition 4.8. Let L be a cdgl associated to the filtration $\{F^n\}_{n\geq 1}$. We denote by $\mathcal{D}er L$ the sub-dgl of $\operatorname{Der} L$ consisting on derivations θ which increase the filtration degree, that is,

$$\theta(F^n) \subset F^{n+1}, \quad \text{for any } n \geq 1.$$

Note that $\mathcal{D}er L$ is also complete with respect to the filtration

$$\mathcal{F}^n = \{\theta \in \mathcal{D}er L,\ \theta(F^i) \subset F^{i+n},\ i \geq 1\}, \quad \text{for} \quad n \geq 1.$$

Observe also that $\operatorname{ad}_x \in \mathcal{D}er L$ for any $x \in L_0$.

Definition 4.9. Let L be a cdgl and let $\theta \in \mathcal{D}er_0 L$. Define the *exponential map of* θ as

$$e^\theta = \sum_{i\geq 0} \frac{\theta^i}{i!}.$$

Proposition 4.10. *For any $\theta \in \mathcal{D}er_0 L$, the exponential map e^θ is a cgl automorphism of L. Moreover, if θ is a cycle, then e^θ is a cdgl automorphism and as such, it preserves Maurer–Cartan elements.*

In particular, for any $x \in L_0$, e^{ad_x} is always a cgl automorphism and it commutes with the differential only when x is a cycle.

Proof. Note first that the exponential map is well defined, since L is complete and θ increases the filtration degree. In view of the general formula (1.18), given $x, y \in L$ and $i \geq 0$,

$$\frac{\theta^i[x,y]}{i!} = \sum_{j+k=i} \frac{[\theta^j(x), \theta^k(y)]}{j!\,k!}.$$

Therefore,

$$e^\theta[x,y] = \sum_{i\geq 0}\sum_{j+k=i} \frac{[\theta^j(x), \theta^k(y)]}{j!\,k!} = \sum_{j\geq 0}\sum_{k\geq 0} \frac{[\theta^j(x), \theta^k(y)]}{j!\,k!} = [e^\theta(x), e^\theta(y)],$$

and thus e^θ is a cgl morphism, which trivially commutes with the differential whenever θ is a cycle.

On the other hand, one can easily prove that

$$e^\theta e^\gamma = e^{\theta * \gamma}, \quad \text{for} \quad \theta, \gamma \in \mathcal{D}er_0 L, \tag{4.3}$$

where $*$ denotes, as usual, the BCH product in $\mathcal{D}er_0 L$. Hence, e^θ is clearly bijective with inverse $e^{-\theta}$, as

$$e^\theta e^{-\theta} = e^{\theta * (-\theta)} = \operatorname{id}_L.$$

For completeness, we include now a more conceptual proof of this result: observe that the injection $\mathcal{D}er_0 L \hookrightarrow \operatorname{Hom}_{\mathbf{cgl}}(L, L)$ extends to an algebra morphism

$$\widehat{U\mathcal{D}er_0 L} \longrightarrow \operatorname{Hom}_{\mathbf{cgl}}(L, L),$$

where the product on the endomorphisms of L is given by composition. By restricting to the grouplike elements G of $\widehat{U\mathcal{D}er_0 L}$ we obtain a group morphism

$$\varphi\colon G \longrightarrow \operatorname{Aut} L,$$

where $\operatorname{Aut} L$ denotes the cgl automorphisms of L. Now, as in (4.2), we have isomorphisms

$$\mathcal{D}er_0 L \underset{\log}{\overset{\exp}{\rightleftarrows}} G,$$

and $e\colon \mathcal{D}er_0 L \to \operatorname{Aut} L$ is simply the composition $\varphi \circ \exp$. This also readily implies formula (4.3). □

Lemma 4.11. *Let L be a cdgl and let $x, y \in L_0$. Then,*

$$\operatorname{ad}_{x*y} = \operatorname{ad}_x * \operatorname{ad}_y \ .$$

Proof. From the Jacobi identity we immediately deduce that, in $\operatorname{Der} L$,

$$\operatorname{ad}_{[x,y]} = [\operatorname{ad}_x, \operatorname{ad}_y].$$

Therefore, by the BCH formula, we get $\operatorname{ad}_{x*y} = \operatorname{ad}_x * \operatorname{ad}_y$. □

The following is an immediate consequence of Lemma 4.11 and formula (4.3).

Corollary 4.12. *Let L be a cdgl and let $x, y \in L_0$. Then,*

$$e^{\operatorname{ad}_{x*y}} = e^{\operatorname{ad}_x} \circ e^{\operatorname{ad}_y} \ .$$ □

Next, we show that, when applied to elements of degree 0, the exponential of an adjoint operator can be explicitly given in terms of the BCH product:

Proposition 4.13. *Let L be a cdgl and let $x, y \in L_0$. Then,*

$$x * y * (-x) = e^{\operatorname{ad}_x}(y).$$

With the previous convention, this formula also reads

$$x * y * x^{-1} = e^{\operatorname{ad}_x}(y).$$

Proof. We first show that the extension of e^{ad_x} to the ideal $\widehat{I}$ of $\widehat{UL_0}$, see (4.1), satisfies

$$e^{\operatorname{ad}_x}(y) = e^x y e^{-x}. \tag{4.4}$$

Indeed,

$$\begin{aligned} e^{\operatorname{ad}_x}(y) &= \sum_{n=0}^{\infty} \frac{\operatorname{ad}_x^n(y)}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i=0}^{n} (-1)^i \binom{n}{i} x^{n-i} y x^i \\ &= \sum_{i=0}^{\infty} \sum_{n=i}^{\infty} \frac{x^{n-i}}{(n-i)!}\, y\, \frac{(-x)^i}{i!} = e^x y e^{-x}. \end{aligned}$$

Replacing y by e^y, we see that

$$e^{\mathrm{ad}_x}(e^y) = e^x e^y e^{-x}. \tag{4.5}$$

In a second step, considering also the extension of e^{ad_x} to $\widehat{I}$, we prove the equality

$$(e^{\mathrm{ad}_x})(y^n) = \left(e^{\mathrm{ad}_x}(y)\right)^n. \tag{4.6}$$

On the left-hand side, the term in x of length k, with $k \geq 0$, equals

$$\frac{\mathrm{ad}_x^k(y^n)}{k!}.$$

On the right-hand side, this term is

$$\sum_{k_1+\cdots+k_n=k} \frac{\mathrm{ad}_x^{k_1}(y)}{k_1!} \cdots \frac{\mathrm{ad}_x^{k_n}(y)}{k_n!}.$$

As ad_x is a derivation of $\widehat{I}$, the two expressions coincide and the equality (4.6) is proved. Therefore,

$$e^{\mathrm{ad}_x}(e^y) = \sum_{n\geq 0} \frac{(e^{\mathrm{ad}_x})(y^n)}{n!} = \sum_{n\geq 0} \frac{\left(e^{\mathrm{ad}_x}(y)\right)^n}{n!} = e^{e^{\mathrm{ad}_x} y}. \tag{4.7}$$

Finally, the proposition follows from the chain of equalities

$$\begin{aligned} x * y * (-x) = \log(e^x e^y e^{-x}) &=_{(4.5)} \log\left(e^{\mathrm{ad}_x}(e^y)\right) \\ &=_{(4.7)} \log(e^{e^{\mathrm{ad}_x}(y)}) = e^{\mathrm{ad}_x}(y). \end{aligned}$$ □

Corollary 4.14. *Let L be a cdgl, $x, y \in L_0$ and $\lambda \in \mathbb{Q}$. Then,*

$$x * (\lambda y) * (-x) = \lambda\,(x * y * (-x)).$$

Proof. $x * (\lambda y) * (-x) = e^{\mathrm{ad}_x}(\lambda y) = \lambda e^{\mathrm{ad}_x}(y)$. □

4.3 The gauge action and the Deligne groupoid

In a cdgl L, the *gauge action* of an element $x \in L_0$ on a Maurer–Cartan element $b \in \mathrm{MC}(L)$ is defined by

$$x \,\mathcal{G}\, b = \sum_{i\geq 0} \frac{\mathrm{ad}_x^i(b)}{i!} - \sum_{i\geq 0} \frac{\mathrm{ad}_x^i(dx)}{(i+1)!}.$$

Observe that the first series is by definition $e^{\mathrm{ad}_x}(b)$. The second one can also be expressed as an operator considering its "generating function": in view of the equality,

$$\sum_{n\geq 0} \frac{t^n}{(n+1)!} = \frac{e^t-1}{t},$$

we write

$$\sum_{i\geq 0} \frac{\mathrm{ad}_x^i(dx)}{(i+1)!} = \frac{e^{\mathrm{ad}_x}-1}{\mathrm{ad}_x}(dx),$$

where $1=\mathrm{id}_L$. Hence, the gauge action takes the classical form,

$$x \mathcal{G} b = e^{\mathrm{ad}_x}(b) - \frac{e^{\mathrm{ad}_x}-1}{\mathrm{ad}_x}(dx). \tag{4.8}$$

By means of the perturbed differential d_b the gauge action adopts a particularly simple form:

$$\begin{aligned} x \mathcal{G} b &= e^{\mathrm{ad}_x}(b) - \frac{e^{\mathrm{ad}_x}-1}{\mathrm{ad}_x}(dx) \\ &= b + \frac{e^{\mathrm{ad}_x}-1}{\mathrm{ad}_x}([x,b]-dx) \\ &= b - \frac{e^{\mathrm{ad}_x}-1}{\mathrm{ad}_x}(d_b x). \end{aligned} \tag{4.9}$$

The fact that this action is preserved by cdgl morphisms is a simple observation.

Proposition 4.15. *Let $f\colon L\to L'$ be a cdgl morphism, $x\in L_0$ and $b\in \mathrm{MC}(L)$. Then, $f(x\mathcal{G} b) = f(x)\mathcal{G} f(b)$.* □

Theorem 4.16. *Let L be a cdgl. The gauge action is an action of the group L_0, equipped with the* BCH *product, on* $\mathrm{MC}(L)$.

The proof is the result of the next two lemmas.

Lemma 4.17. *Let L be a cdgl, $b\in\mathrm{MC}(L)$ and $x\in L_0$. Then, $x\mathcal{G} b\in \mathrm{MC}(L)$.*

Proof. Write

$$a = e^{\mathrm{ad}_x}(b) - \frac{e^{\mathrm{ad}_x}-1}{\mathrm{ad}_x}(u),$$

where u denotes dx to avoid excessive notation. In view of the general formula for the gauge action (4.8), the lemma reduces to showing that a is a Maurer–Cartan element. We proceed by direct computation.

On the one hand,

$$[a,a] = [e^{\mathrm{ad}_x}(b), e^{\mathrm{ad}_x}(b)] - 2\left[e^{\mathrm{ad}_x}(b), \sum_{i\geq 0}\frac{\mathrm{ad}_x^i(u)}{(i+1)!}\right] + \left[\sum_{i\geq 0}\frac{\mathrm{ad}_x^i(u)}{(i+1)!}, \sum_{j\geq 0}\frac{\mathrm{ad}_x^j(u)}{(j+1)!}\right].$$

On the other hand,

$$da = e^{\mathrm{ad}_x}(db) + \sum_{i\geq 0}\sum_{j<i} \frac{1}{i!}\,\mathrm{ad}_x^j\,\mathrm{ad}_u\,\mathrm{ad}_x^{i-j-1}(b) - \sum_{i\geq 0}\sum_{j<i} \frac{1}{(i+1)!}\,\mathrm{ad}_x^j\,\mathrm{ad}_u\,\mathrm{ad}_x^{i-j-1}(u). \tag{4.10}$$

Next, we show that in this expression, each of the three summands can be written as follows:

$$\begin{aligned} e^{\mathrm{ad}_x}(db) &= -\frac{1}{2}[e^{\mathrm{ad}_x}b, e^{\mathrm{ad}_x}b],\\ \sum_{i\geq 0}\sum_{j<i} \frac{1}{i!}\,\mathrm{ad}_x^j\,\mathrm{ad}_u\,\mathrm{ad}_x^{i-j-1}(b) &= \left[e^{\mathrm{ad}_x}(b), \sum_{i\geq 0}\frac{\mathrm{ad}_x^i(u)}{(i+1)!}\right],\\ \sum_{i\geq 0}\sum_{j<i} \frac{1}{(i+1)!}\,\mathrm{ad}_x^j\,\mathrm{ad}_u\,\mathrm{ad}_x^{i-j-1}(u) &= \frac{1}{2}\left[\sum_{i\geq 0}\frac{\mathrm{ad}_x^i(u)}{(i+1)!}, \sum_{j\geq 0}\frac{\mathrm{ad}_x^j(u)}{(j+1)!}\right]. \end{aligned} \tag{4.11}$$

The first identity is trivial as b is Maurer–Cartan and e^{ad_x} is a cgl morphism in view of Proposition 4.10.

For the second identity in (4.11), recall from (1.18) that, for any $\alpha, \beta \in L$,

$$\mathrm{ad}_x^j[\alpha, \beta] = \sum_{k=0}^{j}\binom{j}{k}[\mathrm{ad}_x^k\,\alpha, \mathrm{ad}_x^{j-k}\,\beta].$$

Hence,

$$\sum_{i\geq 0}\sum_{j<i} \frac{1}{i!}\,\mathrm{ad}_x^j\,\mathrm{ad}_u\,\mathrm{ad}_x^{i-j-1}(b) = \sum_{i\geq 0}\sum_{j<i}\sum_{k=0}^{j} \frac{1}{i!}\binom{j}{k}[\mathrm{ad}_x^k\,u, \mathrm{ad}_x^{i-k-1}\,b]. \tag{4.12}$$

By setting $r = i - k - 1$, and taking into account that

$$\sum_{j=k}^{r+k}\binom{j}{k} = \binom{r+k+1}{k+1},$$

equation (4.12) becomes

$$\sum_{r,k}\left(\sum_{j=k}^{r+k}\binom{j}{k}\frac{1}{(r+k+1)!}\right)[\mathrm{ad}_x^k\,u, \mathrm{ad}_x^r\,b] = \sum_{r,k}\left[\frac{\mathrm{ad}_x^k\,u}{(k+1)!}, \frac{\mathrm{ad}_x^r\,b}{r!}\right],$$

in which the right-hand side term is precisely

$$\left[e^{\mathrm{ad}_x}(b), \sum_{i\geq 0}\frac{\mathrm{ad}_x^i(u)}{(i+1)!}\right].$$

For the third identity in (4.11), we begin as above and write

$$\sum_{i\geq 0}\sum_{j<i}\frac{1}{(i+1)!}\operatorname{ad}_x^j\operatorname{ad}_u\operatorname{ad}_x^{i-j-1}(u)=\sum_{i\geq 0}\sum_{j<i}\sum_{k=0}^{j}\frac{1}{(i+1)!}\binom{j}{k}\left[\operatorname{ad}_x^k u,\operatorname{ad}_x^{i-k-1}u\right]$$
$$=\sum_{r,k}\frac{1}{(r+k+2)!}\binom{r+k+1}{k+1}[\operatorname{ad}_x^k u,\operatorname{ad}_x^r u].$$

Since $[\operatorname{ad}_x^k u,\operatorname{ad}_x^r u]=[\operatorname{ad}_x^r u,\operatorname{ad}_x^k u]$, this equals

$$\frac{1}{2}\sum_{r,k}\frac{1}{(r+k+2)!}\left(\binom{r+k+1}{k+1}+\binom{r+k+1}{k+1}\right)[\operatorname{ad}_x^k u,\operatorname{ad}_x^r u]$$
$$=\frac{1}{2}\sum_{r,k}\frac{1}{(r+k+2)!}\binom{r+k+2}{k+1}[\operatorname{ad}_x^k u,\operatorname{ad}_x^r u]=\frac{1}{2}\left[\sum_k\frac{\operatorname{ad}_x^k u}{(k+1)!},\sum_r\frac{\operatorname{ad}_x^r u}{(r+1)!}\right].$$

Finally, equations (4.10) and (4.11) readily imply that

$$da=-\frac{1}{2}[a,a].$$ □

Lemma 4.18. *For any $a\in\mathrm{MC}(L)$ and any $x,y\in L_0$,*

$$(x*y)\,\mathcal{G}\,a=x\,\mathcal{G}\,(y\,\mathcal{G}\,a)\,.$$

Proof. Denote $L\oplus\mathbb{Q}c$ the cdgl where $|c|=-1$, $dc=0$ and the bracket, extending that on L, is defined by $[x,c]=-(-1)^{|x|}dx$ for any $x\in L$, and $[c,c]=0$. Then, for $x\in L_0$, a simple computation shows that

$$e^{\operatorname{ad}_x}(b+c)=x\,\mathcal{G}\,b+c\,.$$

Using Corollary 4.12, we thus have

$$(x*y)\,\mathcal{G}\,b+c=e^{\operatorname{ad}_{x*y}}(b+c)=e^{\operatorname{ad}_x}e^{\operatorname{ad}_y}(b+c)=e^{\operatorname{ad}_x}\,(y\,\mathcal{G}\,b+c)=x\,\mathcal{G}\,(y\,\mathcal{G}\,b)+c,$$

and therefore $(x*y)\,\mathcal{G}\,a=x\,\mathcal{G}\,(y\,\mathcal{G}\,a)$. □

Corollary 4.19. *$x\,\mathcal{G}\,b=a$ if and only if $-x\,\mathcal{G}\,a=b$.*

Proof. Indeed, if $x\,\mathcal{G}\,b=a$, then

$$(-x)\,\mathcal{G}\,a=(-x)\,\mathcal{G}\,(x\,\mathcal{G}\,b)=((-x)*x)\,\mathcal{G}\,b=b\,.$$ □

Next, we show that the isotropy group of a given Maurer–Cartan element consists of the cycles of the corresponding perturbed differential.

Proposition 4.20. *Let L be a cdgl, $a\in\mathrm{MC}(L)$, $x\in L_0$. Then $x\,\mathcal{G}\,a=a$ if and only if $d_a x=0$. In particular, if $x\,\mathcal{G}\,a=a$, then for each $\lambda\in\mathbb{Q}$, $(\lambda x)\,\mathcal{G}\,a=a$.*

Proof. Note that for each $x \in L_0$,

$$\frac{e^{\mathrm{ad}_x} - 1}{\mathrm{ad}_x}$$

is a cgl automorphism of L with inverse

$$\frac{\mathrm{ad}_x}{e^{\mathrm{ad}_x} - 1}.$$

Hence, in view of formula (4.9), $x \mathcal{G} a = a$ if and only if $d_a x = 0$. □

Let (L, d) be a cdgl and $a \in \mathrm{MC}(L)$. We denote by $\mathcal{G}_a$ the gauge action in the perturbed cdgl (L, d_a).

Proposition 4.21. *With the above notation, for any $b \in \mathrm{MC}(L)$,*

$$x \,\mathcal{G}_a\, (b - a) = (x \,\mathcal{G}\, b) - a\,.$$

Proof. Recall from Proposition 4.5 that $b - a$ is indeed in $\mathrm{MC}(L, d_a)$. Then,

$$\begin{aligned} x \,\mathcal{G}_a\, (b - a) &= e^{\mathrm{ad}_x}(b - a) - \frac{e^{\mathrm{ad}_x} - 1}{\mathrm{ad}_x}(dx + [a, x]) \\ &= e^{\mathrm{ad}_x}(b) - \frac{e^{\mathrm{ad}_x} - 1}{\mathrm{ad}_x}(dx) - \left(e^{\mathrm{ad}_x}(a) - \frac{e^{\mathrm{ad}_x} - 1}{\mathrm{ad}_x}[x, a]\right) \\ &= (x \,\mathcal{G}\, b) - a. \end{aligned}$$

□

We now introduce the Deligne groupoid as the groupoid associated to the gauge action.

Definition 4.22. The *Deligne groupoid* of L is the groupoid that has $\mathrm{MC}(L)$ as objects, and the elements $x \in L_0$ as arrows from $x\mathcal{G}a$ to a.

The next proposition shows that, if a and b are in the same path component of the Deligne groupoid, then the cdgl's (L, d_a) and (L, d_b) are isomorphic and thus, the corresponding components L^a and L^b are also isomorphic. The first step in the proof is the computation of $d(e^{\mathrm{ad}_x} v)$ for any element $v \in L$. This already appears in [91, Lemma 1].

Lemma 4.23. *Let L be a cdgl, $x \in L_0$ and $v \in L$. Then,*

$$d(e^{-\mathrm{ad}_x} v) = e^{-\mathrm{ad}_x}(dv) + (-1)^{|v|} e^{-\mathrm{ad}_x}\, \mathrm{ad}_v \frac{e^{\mathrm{ad}_x} - 1}{\mathrm{ad}_x}(dx)\,.$$

Proof. By definition,

$$d(e^{-\mathrm{ad}_x} v) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\; d(\mathrm{ad}_x^n\, v)\,. \tag{4.13}$$

In each summand, we apply the Leibniz rule to get

$$d(\mathrm{ad}_x^n v) = \sum_{m=0}^{n-1} \mathrm{ad}_x^{n-m-1} \mathrm{ad}_{dx} \mathrm{ad}_x^m(v) + \mathrm{ad}_x^n(dv). \tag{4.14}$$

Now $\mathrm{ad}_{dx} \mathrm{ad}_x^m(v) = (-1)^{|v|+1}[\mathrm{ad}_x^m(v), dx]$ and an easy inductive argument using the Jacobi identity shows that

$$[\mathrm{ad}_x^m(v), dx] = \sum_{k=0}^{m}(-1)^k \binom{m}{k} \mathrm{ad}_x^{m-k} \mathrm{ad}_v \mathrm{ad}_x^k(dx).$$

Then, using the identity $\sum_{m=k}^{n-1} \binom{m}{k} = \binom{n}{k+1}$ in the last equality of the following formula, (4.14) becomes

$$\begin{aligned} d(\mathrm{ad}_x^n v) &= (-1)^{|v|+1} \left(\sum_{m=0}^{n-1} \sum_{k=0}^{m} (-1)^k \binom{m}{k} \mathrm{ad}_x^{n-k-1} \mathrm{ad}_v \mathrm{ad}_x^k(dx) \right) + \mathrm{ad}_x^n(dv) \\ &= (-1)^{|v|+1} \left(\sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} \mathrm{ad}_x^{n-k-1} \mathrm{ad}_v \mathrm{ad}_x^k \right)(dx) + \mathrm{ad}_x^n(dv)\,. \end{aligned}$$

To finish, insert this expression in (4.13); the result can be easily rewritten as

$$d(e^{-\mathrm{ad}_x} v) = (-1)^{|v|} e^{-\mathrm{ad}_x} \mathrm{ad}_v \frac{e^{\mathrm{ad}_x}-1}{\mathrm{ad}_x}(dx) + e^{-\mathrm{ad}_x}(dv)\,. \qquad \square$$

Proposition 4.24. *Let L be a cdgl, $x \in L_0$ and $a, b \in \mathrm{MC}(L)$ with $x \mathcal{G} b = a$. Then, for any $v \in L$,*

$$d_a\, e^{\mathrm{ad}_x}(v) = e^{\mathrm{ad}_x}(d_b v).$$

In other words, the map $e^{\mathrm{ad}_x}\colon (L, d_b) \xrightarrow{\cong} (L, d_a)$ is cdgl isomorphism which restricts to an isomorphism $e^{\mathrm{ad}_x}\colon L^b \xrightarrow{\cong} L^a$ between the corresponding components.

Proof. Apply first Lemma 4.23 for the cdgl (L, d_a), choosing $-x \in L_0$ and $v \in L$, to obtain,

$$d_a\, e^{\mathrm{ad}_x}(v) = e^{\mathrm{ad}_x}(d_a v) - (-1)^{|v|} e^{\mathrm{ad}_x} \mathrm{ad}_v \frac{e^{-\mathrm{ad}_x}-1}{-\mathrm{ad}_x}(d_a x) \tag{4.15}$$

In the second summand of this expression note that

$$\begin{aligned} \frac{e^{-\mathrm{ad}_x}-1}{-\mathrm{ad}_x}(d_a x) &= \frac{e^{-\mathrm{ad}_x}-1}{-\mathrm{ad}_x}(dx + [a,x]) = \frac{e^{-\mathrm{ad}_x}-1}{-\mathrm{ad}_x}(dx) + \frac{e^{-\mathrm{ad}_x}-1}{-\mathrm{ad}_x}(-\mathrm{ad}_x a) \\ &= \frac{e^{-\mathrm{ad}_x}-1}{-\mathrm{ad}_x}(dx) + e^{-\mathrm{ad}_x}(a) - a. \end{aligned} \tag{4.16}$$

Next, by Corollary 4.19, $-x \mathcal{G} a = b$, that is,

$$b = e^{-\mathrm{ad}_x}(a) + \frac{e^{-\mathrm{ad}_x}-1}{-\mathrm{ad}_x}(dx)\,,$$

or equivalently,

$$\frac{e^{-\operatorname{ad}_x}-1}{-\operatorname{ad}_x}(dx) = -e^{-\operatorname{ad}_x}(a) + b\,.$$

Inserting this in the last equality of (4.16) we deduce that,

$$\frac{e^{-\operatorname{ad}_x}-1}{-\operatorname{ad}_x}(d_a x) = b - a.$$

Replacing this in (4.15) we finally obtain,

$$\begin{aligned} d_a\, e^{\operatorname{ad}_x}(v) &= e^{\operatorname{ad}_x}(d_a v) - (-1)^{|v|} e^{\operatorname{ad}_x} \operatorname{ad}_v(b-a) \\ &= e^{\operatorname{ad}_x}\big(d_a v - (-1)^{|v|} \operatorname{ad}_v(b-a)\big) \\ &= e^{\operatorname{ad}_x}(d_b v)\,. \end{aligned}$$

□

From a geometrical point of view the orbit set of the gauge action will become a crucial object.

Definition 4.25. We denote by $\widetilde{\mathrm{MC}}(L) = \mathrm{MC}(L)/\mathcal{G}$ the orbit set of the gauge action, that is, the set of equivalence classes of Maurer–Cartan elements modulo the gauge action. For a pair a, b of MC elements we write $a \sim b$ if they define the same class, that is, if there exists $x \in L_0$ such that $x\,\mathcal{G}\,a = b$.

For the rest of the text, to simplify notation, we will often denote in the same way an element in $\mathrm{MC}(L)$ and its corresponding class in $\widetilde{\mathrm{MC}}(L)$. Nevertheless, it will always be clear in which set such an element is considered.

Note that, since cdgl morphisms preserve the gauge action, the functor MC induces a functor

$$\widetilde{\mathrm{MC}}\colon \mathbf{cdgl} \longrightarrow \mathbf{set}^*\,.$$

Example 4.26. Let L be the cdgl defined in Example 4.2, $L = \mathbb{L}(a,b)/[a,b]$ where a and b are MC elements. Then, all the MC elements are gauge non-equivalent. Therefore,

$$\widetilde{\mathrm{MC}}(L) = \{a, b, 0, a+b\}.$$

In particular, classes of non-zero Maurer–Cartan elements are not necessarily linearly independent.

Example 4.27. Let L be an abelian Lie algebra. Then, the Maurer–Cartan elements are the cycles of L_{-1}. Moreover, two MC elements are equivalent if and only if they represent the same homology class, i.e.,

$$\widetilde{\mathrm{MC}}(L) = H_{-1}(L).$$

For instance, in the abelian dgl $(L, 0)$ where $L = \mathbb{Q}a$ with $|a| = -1$, each element λa, $\lambda \in \mathbb{Q}$ represents a different class in $\widetilde{\mathrm{MC}}(L)$.

From Propositions 4.5 and 4.21 we immediately deduce:

Proposition 4.28. *Let (L, d) be a cdgl and $a \in \mathrm{MC}(L, d)$. If $\widetilde{\mathrm{MC}}(L, d) = \{[a_i], i \in I\}$, then $\widetilde{\mathrm{MC}}(L, d_a) = \{[a_i - a], i \in I\}$.* □

4.4 Applications to deformation theory

In this section we apply the main results of previous sections to the deformation theory of cdgl's, in the sense of homological algebra. More specifically, we classify perturbations of the differential of a given cdgl by means of gauge related elements in the corresponding cdgl of derivations. We begin with the natural generalization of Definition 4.4.

Definition 4.29. Let L be a cdgl. A *perturbation* of the differential d of L is another differential in L of the form $d+\psi$, where $\psi \in \mathcal{D}er L$.

Given a cdgl L, note that a Maurer–Cartan element in $\mathcal{D}er L$ is a derivation ψ of degree -1 satisfying

$$d\psi + \psi d = -\frac{1}{2}[\psi,\psi] = -\psi^2.$$

Moreover, the following is a simple observation

Lemma 4.30. *Let (L,d) be a cdgl. Then, $d+\psi$ is a perturbation of d if and only if $\psi \in \mathrm{MC}(\mathcal{D}er L)$.* ☐

That is, perturbations of d are in bijective correspondence with Maurer–Cartan elements of $\mathcal{D}er L$. We show next that gauge related MC derivations correspond to particular isomorphisms between the perturbed cdgl's.

Theorem 4.31. *Let (L,d) be a cdgl for which $d \in \mathcal{D}er L$, that is, d increases the filtration degree, and let $d+\varphi$, $d+\psi$ be perturbations of d. Then, $\varphi \sim \psi$ if and only if there exists an isomorphism*

$$f\colon (L, d+\varphi) \xrightarrow{\cong} (L, d+\psi)$$

such that $f - \mathrm{id}_L$ increases the filtration degree.

Proof. Assume $\varphi \sim \psi$. That is, there exists $\theta \in \mathcal{D}er_0 L$ such that $\theta \,\mathcal{G}\, \varphi = \psi$. Then

$$\psi = e^{\mathrm{ad}_\theta}(\varphi) - \frac{e^{\mathrm{ad}_\theta}-1}{\mathrm{ad}_\theta}(d\theta).$$

Recall that, in $\mathcal{D}er L$, $d\theta = [d,\theta]$. Hence, the second summand in the right-hand side is

$$\frac{e^{\mathrm{ad}_\theta}-1}{\mathrm{ad}_\theta}(d\theta) = \sum_{i\geq 0} \frac{\mathrm{ad}_\theta^i}{(i+1)!}[d,\theta] = \sum_{i\geq 1} \frac{\mathrm{ad}_\theta^i}{i!} d.$$

Therefore,

$$d+\psi = d + e^{\mathrm{ad}_\theta}(\varphi) - \sum_{i\geq 0} \frac{\mathrm{ad}_\theta^i}{i!}(d) = e^{\mathrm{ad}_\theta}(d+\varphi).$$

But, taking into account that $d+\varphi \in \mathcal{D}er_0 L$, the general formula (4.4) yields,

$$e^{\mathrm{ad}_\theta}(d+\varphi) = e^\theta (d+\varphi) e^{-\theta}.$$

That is,

$$d+\psi = e^\theta (d+\varphi) e^{-\theta}.$$

By Proposition 4.10, e^θ is a cgl automorphism with inverse equal to $e^{-\theta}$. Hence, the above equation becomes

$$(d+\psi)e^\theta = e^\theta (d+\varphi)$$

and therefore,

$$e^\theta \colon (L, d+\varphi) \xrightarrow{\cong} (L, d+\psi),$$

is the sought-for isomorphism.

On the other hand, given $f\colon (L, d+\varphi) \stackrel{\cong}{\to} (L, d+\psi)$, write $f = e^\theta$ with $\theta = \log(f - \mathrm{id}_L)$. Note that θ is a derivation in $\mathcal{D}er_0 L$. Finally, since f commutes with differentials, $(d+\psi)e^\theta = e^\theta(d+\varphi)$, and the argument above shows that $\theta \,\mathcal{G}\, \varphi = \psi$. □

Remark 4.32. We briefly consider here an important application of Theorem 4.31.

First, note that there is no obstruction to state and prove this result replacing the cdgl $\mathcal{D}er L$ by the cdgl $\mathcal{D}er A$ whenever A is a complete cdga whose differential d increases the filtration degree, that is, $d \in \mathcal{D}er_{-1} A$. Again, perturbations of d correspond to Maurer–Cartan elements in $\mathcal{D}er A$, and two such elements $\varphi, \psi \in \mathrm{MC}(\mathcal{D}er A)$ are gauge equivalent if and only if there is an isomorphism $f\colon (A, d+\varphi) \stackrel{\cong}{\to} (A, d+\psi)$ such that $f - \mathrm{id}_A$ increases the filtration degree.

Now, fix a 1-connected cga H and construct the minimal model $(\wedge V, d) \xrightarrow{\simeq} (H, 0)$ in the standard bigraded way, see [74, §3], which induces a particular filtration on $(\wedge V, d)$. Then, given a simply connected CW-complex X with rational cohomology algebra H, there exist a perturbation $d+\varphi_X$ of d and a quasi-isomorphism $(\wedge V, d+\varphi_X) \stackrel{\simeq}{\to} A_{PL}(X)$, see [74, Theorem 4.4]. An application of the above version of Theorem 4.31 allows us to conclude that two simply connected complexes X and Y, sharing the same rational cohomology algebra H, have the same rational homotopy type if and only if the Maurer–Cartan elements φ_X and φ_Y are gauge related.

In other words, denoting by $\mathcal{CW}_H$ the class of rational homotopy types of simply connected complexes with rational cohomology algebra H, we have:

$$\mathcal{CW}_H \cong \widetilde{\mathrm{MC}}\big(\mathcal{D}er(\wedge V, d)\big).$$

The reader can find in [125] detailed explanations of every assertion in this remark.

4.5 The Goldman–Millson Theorem

In general, cdgl quasi-isomorphisms are not preserved by perturbation of differentials. That is, given a cdgl quasi-isomorphism $f\colon L \xrightarrow{\simeq} L'$ and $a \in \mathrm{MC}(L)$, the morphism $f\colon (L, d_a) \to (L' d_{f(a)})$ is not necessarily a quasi-isomorphism. Moreover, the set map $\mathrm{MC}(f)$, or even $\widetilde{\mathrm{MC}}(f)$, is not in general a bijection. An elementary example consists on the injection $0 \to (\mathbb{L}(a), d)$, where a is an MC element.

Nevertheless, the following result, whose earliest version was proved by Schlessinger and Stasheff in [125, Theorem 5.3], shows that any cdgl morphism whose associated graded cdgl morphism is a quasi-isomorphism, induces a bijection on the set of equivalence classes of Maurer–Cartan elements. Moreover, it is also a quasi-isomorphism for every perturbed differential.

Theorem 4.33. *Let $f\colon L \to L'$ be a morphism of cdgl's, filtered by $\{F^n\}_{n\geq 1}$ and $\{G^n\}_{n\geq 1}$, respectively, such that the induced map $F^n/F^{n+1} \xrightarrow{\simeq} G^n/G^{n+1}$ is a quasi-isomorphism for any $n \geq 1$. Then, $\widetilde{\mathrm{MC}}(f)$ is a bijection and, for each $a \in \mathrm{MC}(L)$, $f\colon (L, d_a) \xrightarrow{\simeq} (L', d_{f(a)})$ is a quasi-isomorphism.*

Remark 4.34. The first (and main) assertion of this result, that is, the bijective character of $\widetilde{\mathrm{MC}}(L)$, is commonly known as the *Goldman–Millson Theorem*. However, and even acknowledged by these authors in the introduction of [65], it was first stated by Deligne in [38] and proved by Schlessinger and Stasheff in op. cit., under slightly different hypotheses on L. An exact analogue of our statement can be found in [41, Theorem C.2], while the original Goldman–Millson version [65, Theorem 2.4] is precisely Proposition 4.40 below.

The relevance of Theorem 4.33 will be revealed later in the text: in Chapter 8 we introduce a model structure on **cdgl** where a morphism $f\colon L \to L'$ is a weak equivalence if and only if $\widetilde{\mathrm{MC}}(f)$ is a bijection and for every $a \in \widetilde{\mathrm{MC}}(L)$, $f^a\colon (L, d_a) \xrightarrow{\simeq} (L', d_{f(a)})$ is a quasi-isomorphism (see Theorem 8.1). Moreover, via the realization functor from **cdgl** to the category of simplicial sets, to be introduced in Chapter 7, we will see that the realization $\langle f \rangle$ of a weak equivalence is in fact a weak homotopy equivalence of simplicial sets. Hence, in this language, the theorem above asserts that $\langle f \rangle$ is a weak homotopy equivalence.

On the other hand, in Theorem 11.13 of Chapter 11, we will show that the realization $\langle L \rangle$ of a given cdgl has the same homotopy type as the simplicial set $\mathrm{MC}_\bullet(L)$ given by the Deligne–Getzler–Hinich groupoid. Hence, Theorem 4.33 amounts to saying that, under the given hypotheses, $\mathrm{MC}_\bullet(f)$ is a weak homotopy equivalence. This is precisely the cdgl version of [42, Theorem 1.1], proved by Dolgushev and Rogers in the wider context of complete L_∞-algebras.

Proof of Theorem 4.33. We first show that, for each $a \in \mathrm{MC}(L)$, $f\colon (L, d_a) \xrightarrow{\simeq} (L', d_{f(a)})$ is a quasi-isomorphism. Observe that, with the perturbed differentials, f also preserves filtrations and is a cdgl morphism. Moreover, the differentials induced by d_a and $d_{f(a)}$ on F^p/F^{p+1} and G^p/G^{p+1}, respectively, coincide with the differentials induced by d, since $d - d_a$ and $d - d_{f(a)}$ increase the filtration degree. To finish, simply apply Lemma 3.14.

For the first assertion consider the following commutative diagram where the vertical arrows are all induced by f:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^n/F^{n+1} & \xrightarrow{j} & L/F^{n+1} & \xrightarrow{p} & L/F^n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle k} & & \\ 0 & \longrightarrow & G^n/G^{n+1} & \xrightarrow{j} & L'/G^{n+1} & \xrightarrow{q} & L'/G^n & \longrightarrow & 0 \end{array}$$

In what follows and as in (3.1), any element of L (or L') will be written as a sequence,

$$a = (a_n), \quad \text{where} \quad a_n \in L/F^n \quad \text{and} \quad p(a_{n+1}) = a_n.$$

We first show that $\widetilde{\mathrm{MC}}(f)$ is surjective. To see this, given $b = (b_n) \in \mathrm{MC}(L')$ we construct by induction on $n \geq 1$ a sequence of elements a_n in $\mathrm{MC}(L/F^n)$ and a sequence of elements $x_n \in (L'/G^n)_0$, such that

$$q(x_{n+1}) = x_n, \qquad p(a_{n+1}) = a_n \qquad \text{and} \qquad k(a_n) = x_n \Game b_n.$$

This will prove the assertion. Indeed, by Lemma 4.7, the sequence (a_n) then defines a Maurer–Cartan element a in $L = \varprojlim L/F^n$. On the other hand, the sequence (x_n) defines an element x in L'_0 such that $f(a) = x \Game b$. In other words, $\widetilde{\mathrm{MC}}(f)(a) = b$.

Suppose elements a_r and x_r for $r \leq n$ with the above properties are already constructed. Since p is surjective, there is an element $z \in F^n/F^{n+1}$ with $p(z) = a_n$. Let $x' \in (L'/G^{n+1})_0$ be a lifting of x_n: $q(x') = x_n$. Then,

$$q(x' \Game b_{n+1} - g(z)) = x_n \Game b_n - kp(z) = x_n \Game b_n - k(a_n) = 0,$$

which implies that the element $x' \Game b_{n+1} - g(z)$ belongs to the abelian Lie algebra G^n/G^{n+1}. In particular,

$$[x' \Game b_{n+1} - g(z), x' \Game b_{n+1}] = 0 = [x' \Game b_{n+1} - g(z), g(z)].$$

Therefore,

$$[g(z), g(z)] = [x' \Game b_{n+1}, g(z)] = [x' \Game b_{n+1}, x' \Game b_{n+1}]. \tag{4.17}$$

Write $u = dz + \frac{1}{2}[z, z] \in (L/F^{n+1})_{-2}$. Since $p(z) = a_n$ is an MC element, $p(u) = 0$, so that $u \in F^n/F^{n+1}$. As F^n/F^{n+1} is abelian,

$$du = [dz, z] = [u, z] - \frac{1}{2}\big[[z, z], z\big] = 0.$$

Since b_{n+1} belongs to $\mathrm{MC}(G^n/G^{n+1})$, it follows that $x' \mathcal{G}\, b_{n+1} \in \mathrm{MC}(G^n/G^{n+1})$:

$$d(x' \mathcal{G}\, b_{n+1}) = -\frac{1}{2}[x' \mathcal{G}\, b_{n+1}, x' \mathcal{G}\, b_{n+1}].$$

Now, in view of (4.17),

$$\begin{aligned} jh(u) &= dg(z) + \frac{1}{2}[g(z), g(z)] = dg(z) + \frac{1}{2}[x' \mathcal{G}\, b_{n+1}, x' \mathcal{G}\, b_{n+1}] \\ &= d(g(z) - [x' \mathcal{G}\, b_{n+1}, x' \mathcal{G}\, b_{n+1}]). \end{aligned}$$

Since h is a quasi-isomorphism and $h(u)$ is a boundary, there are elements $t \in (F^n/F^{n+1})_0$ and $s \in (G^n/G^{n+1})_1$ such that

$$dt = u \qquad \text{and} \qquad h(t) = (g(z) - x' \mathcal{G}\, b_{n+1}) - ds.$$

Since t is central, $[z, z] = [z - t, z - t]$. Also,

$$d(z - t) = dz - u = -\frac{1}{2}[z, z] = -\frac{1}{2}[z - t, z - t],$$

that is, $z - t$ is an MC element. On the other hand, since s is central,

$$\begin{aligned} (x' - n) \mathcal{G}\, b_{n+1} &= \sum_{i \geq 0} \frac{\mathrm{ad}^i_{x'-s}(b_{n+1})}{i!} - \sum_{i \geq 0} \frac{\mathrm{ad}^i_{x'-s}(d(x' - s))}{(i+1)!} \\ &= \sum_{i \geq 0} \frac{\mathrm{ad}^i_{x'}(b_{n+1})}{i!} + ds - \sum_{i \geq 0} \frac{\mathrm{ad}^i_{x'}\, d(x')}{(i+1)!} \\ &= x' \mathcal{G}\, b_{n+1} + ds. \end{aligned}$$

We set $x_{n+1} = x' - s$ and $a_{n+1} = z - t$. Clearly,

$$q(x_{n+1}) = x_n, \qquad p(a_{n+1}) = a_n,$$

and

$$g(a_{n+1}) = g(z) - h(t) = x' \mathcal{G}\, b_{n+1} + ds = (x' - s) \mathcal{G}\, b_{n+1} = x_{n+1} \mathcal{G}\, b_{n+1}.$$

As previously observed, $\widetilde{\mathrm{MC}}(f)$ is then surjective.

Next, we show that $\widetilde{\mathrm{MC}}(f)$ is injective. Let $a, b \in \mathrm{MC}(L)$ such that

$$\widetilde{\mathrm{MC}}(f)(a) = \widetilde{\mathrm{MC}}(f)(b),$$

that is, $f(a) = y \mathcal{G} f(b)$ for some $y \in L'_0$. We construct by induction on $n \geq 1$ a sequence of elements $x_n \in (L/F^n)_0$ such that

$$p(x_{n+1}) = x_n \quad \text{and} \quad x_n \mathcal{G}\, b_n = a_n.$$

This would define an element $x = (x_n) \in L_0$ such that $x \,\mathcal{G}\, b = a$ which would prove the assertion. To do so, we will need to find at the same time, and also inductively, a sequence of elements $y[n] \in L'_0$, $n \geq 1$, such that $y[n] \,\mathcal{G}\, f(b) = f(a)$ and $k(x_n) = y[n]_n$.

Suppose we have constructed x_m and $y[m]$, for $m \leq n$, with the above properties. Let $x' \in L/F^{n+1}$ be a lifting of x_n and denote $u = x' \,\mathcal{G}\, b_{n+1} \in L/F^{n+1}$. Then, since $p(a_{n+1} - u) = a_n - p(u) = a_n - x_n \,\mathcal{G}\, b_{n+1} = 0$, $a_{n+1} - u$ is a central element in F^n/F^{n+1}. Now, as a_{n+1} and u are Maurer–Cartan elements,

$$d(a_{n+1} - u) = -\frac{1}{2}[a_{n+1}, a_{n+1}] + \frac{1}{2}[u, u] = -\frac{1}{2}[a_{n+1} + u, a_{n+1} - u] = 0.$$

On the other hand, in G^n/G^{n+1}, $v = y[n]_{n+1} * g(x')^{-1}$ is in the kernel of q and is a central element, because $q(v) = y[n] * k(x_n)^{-1} = 0$. Here $*$ denotes the BCH product on the elements of degree zero.

On the one hand, by Theorem 4.16, it follows that in L'/G^{n+1}:

$$\begin{aligned} v \,\mathcal{G}\, g(u) &= \big(y[n]_{n+1} * g(x')^{-1} \big) \,\mathcal{G}\, g(u) = y[n]_{n+1} \,\mathcal{G}\, g(b_{n+1}) \\ &= (y[n] \,\mathcal{G}\, f(b))_{n+1} = g(a_{n+1}). \end{aligned}$$

On the other hand, since v is central,

$$v \,\mathcal{G}\, g(u) = g(u) - dv.$$

As a result of these two equations,

$$h(u - a_{n+1}) = dv.$$

Since h is a quasi-isomorphism, there exist elements $w \in F^n/F^{n+1}$ and $r \in G^n/G^{n+1}$ such that $dw = u - a_{p+1}$ and $h(w) = v + dr$.

Since w is central, $a_{n+1} = u - dw = w \,\mathcal{G}\, u$. Set $x_{n+1} = w * x'$ and observe that

$$x_{n+1} \,\mathcal{G}\, b_{n+1} = w \,\mathcal{G}\, (x' \,\mathcal{G}\, b_{n+1}) = w \,\mathcal{G}\, u = a_{n+1}.$$

We have $p(x_{n+1}) = p(x') = x_n$ and, since dr is central,

$$g(x_{n+1}) = h(w) * g(x') = \big(y[n]_{n+1} * g(x')^{-1} + dr\big) * g(x') = y[n]_{n+1} + dr.$$

Finally, let $t \in L'_1$ with $t_{n+1} = r$, set $s = d_{f(a)}t$ and $y[n+1] = s * y[n]$. Since $ds = [s, f(a)]$, by Proposition 4.20, $s \,\mathcal{G}\, f(a) = f(a)$ and so $y[n+1] \,\mathcal{G}\, f(b) = f(a)$. On the other hand, $y[n+1]_{n+1} = dr + y[n]_{n+1} = g(x_{n+1})$. □

We will now apply Theorem 4.33 in various situations to detect cdgl quasi-isomorphisms which, in addition, induce bijections on the corresponding MC sets. All these consequences will be very important later on.

In what follows, given a cdgl morphism $f\colon L \to L'$, we will call the induced map

$$f^n\colon F^n/F^{n+1} \longrightarrow G^n/G^{n+1}$$

the (nth) *associated graded* morphism.

Proposition 4.35. *Let* $f\colon (\widehat{\mathbb{L}}(V), d) \to (\widehat{\mathbb{L}}(W), d)$ *be a morphism of cdgl's whose linear part* $f_1\colon (V, d_1) \xrightarrow{\simeq} (W, d_1)$ *is a quasi-isomorphism. Then,* $\widetilde{\mathrm{MC}}(f)$ *is a bijection.*

Proof. In the proof of Proposition 3.12 we checked that for the usual filtration $F^n = \widehat{\mathbb{L}}^{\geq n}(V)$ in $(\widehat{\mathbb{L}}(V), d)$, and the analogous one in $(\widehat{\mathbb{L}}(W), d)$,

$$F^n/F^{n+1} = (\mathbb{L}^n(V), \bar{d})$$

and that the associated graded morphism

$$(\mathbb{L}^n(V), \bar{d}) \xrightarrow{\simeq} (\mathbb{L}^n(W), \bar{d})$$

is a quasi-isomorphism since f_1 is. To finish, apply Theorem 4.33. □

Given a graded vector space U, consider the contractible cdgl $(\widehat{\mathbb{L}}(U \oplus sU), d)$ (see Definition 3.17). In other words, sU is the suspension of U and the differential d is defined by $dsu = u$ and $du = 0$.

Let L be any cdgl and consider the coproduct

$$B = L \,\widehat{\amalg}\, (\widehat{\mathbb{L}}(U \oplus sU), d)$$

together with the natural cdgl morphisms,

$$\iota\colon L \hookrightarrow B \qquad \text{and} \qquad p\colon B \to L,$$

given by the inclusion into the first factor and the projection onto it.

Proposition 4.36. *Given any cdgl* L*, both* $\widetilde{\mathrm{MC}}(\iota)$ *and* $\widetilde{\mathrm{MC}}(p)$ *are bijections. Moreover, for any* $a \in \mathrm{MC}(L)$,

$$\iota\colon (L, d_a) \xrightarrow{\simeq} (B, d_a) \qquad \textit{and} \qquad p\colon (B, d_a) \xrightarrow{\simeq} (L, d_a)$$

are quasi-isomorphisms.

Proof. Let $\{F^n\}_{n\geq 1}$ and $\{G^n\}_{n\geq 1}$ be the filtrations on L and $\widehat{\mathbb{L}}(U \oplus sU)$, respectively. Recall that

$$G^n = \widehat{\mathbb{L}}^{\geq n}(U \oplus sU).$$

Recall also from (3.6) in Proposition 3.5 that

$$B = \varprojlim_n \big(L \amalg \widehat{\mathbb{L}}(U \oplus sU)\big)/R^n,$$

where

$$R^n = \sum_{p_1+q_1+\cdots+p_r+q_r=n} \big[F^{p_1}, [G^{q_1}, [\dots [F^{p_r}, G^{q_r}]] \dots \big].$$

Therefore, in view of (3.5), for each $n \geq 1$ the associated graded morphism of p is

$$p^n \colon R^n/R^{n+1} \longrightarrow F^n/F^{n+1}.$$

We prove that this is a quasi-isomorphism by showing, more generally, that

$$p \colon R^n \xrightarrow{\simeq} F^n$$

is also a quasi-isomorphism. As $p\iota = \mathrm{id}_L$ it suffices to show that $H(p)$ is injective.

To this end, observe first that

$$R^n = F^n \oplus \Gamma$$

where every bracket in Γ contains at least one non-zero element of $U \oplus sU$. Thus, $p(\Gamma) = 0$ and is the identity on F^n.

We define a derivation s of degree $+1$ in B by $s(L) = 0$, $s(u) = su$ and $s(su) = 0$. The derivation $\theta = sd + ds$ is equal to the identity on $U \oplus sU$ and therefore is simply multiplication by i on each bracket of Γ containing exactly i non-zero elements of $U \oplus sU$. In particular, if z is such an element, which is also a cycle, then z is also a boundary:

$$z = \frac{1}{i}\theta(z) = (dsz + sdz) = d\big(\frac{1}{i}sz\big).$$

Therefore, since $sz \in \Gamma$, this is an acyclic graded vector space.

Now let $x = y + z \in R^n$, with $y \in F^n$ and $z \in \Gamma$ represent a class in $\ker H(p)$. This amounts to saying that both y and z are cycles and there exists $a \in F^n$ such that $da = y$. As Γ is acyclic, $z = dz'$ and $x = d(a + z')$. Hence, $H(p)$ is injective and thus, it is an isomorphism.

Again, as $p\iota$ is the identity on L, and the associated graded morphism of p is a quasi-isomorphism, the corresponding associated graded morphism

$$\iota^n \colon F^n/F^{n+1} \xrightarrow{\simeq} R^n/R^{n+1}$$

is also a quasi-isomorphism. To finish, apply Theorem 4.33. □

Definition 4.37. Let A be a cdga and L be a cdgl filtered by $\{F^n\}_{n\geq 1}$. Define the cdgl,

$$A\widehat{\otimes}L = \varprojlim_n A \otimes (L/F^n),$$

where the differential and the bracket in $A \otimes (L/F^n)$ are defined as usual by

$$d(a \otimes x) = da \otimes x + (-1)^{|a|} a \otimes dx \quad \text{and} \quad [a \otimes x, a' \otimes x'] = (-1)^{|a'||x|} aa' \otimes [x, x'] .$$

Observe that this is indeed a complete dgl with respect to the usual filtration, see (3.4), given by the kernels of the natural maps $A\widehat{\otimes}L \to A \otimes L/F^n$.

Proposition 4.38. *Let $\varphi\colon A \xrightarrow{\simeq} B$ be a cdga quasi-isomorphism and let*

$$\varphi\widehat{\otimes}\,\mathrm{id} : A\widehat{\otimes}L \xrightarrow{\simeq} B\widehat{\otimes}L$$

be the induced quasi-isomorphism. Then, $\widetilde{\mathrm{MC}}(\varphi\widehat{\otimes}\,\mathrm{id})$ is a bijection.

Proof. First of all, note the following general fact: let C, D and E be chain complexes and let $f\colon C \xrightarrow{\simeq} D$ be a quasi-isomorphism. Then, there are a quasi-isomorphism $g\colon D \xrightarrow{\simeq} C$ and homotopies h and k with $dh + hd = gf - \mathrm{id}_C$ and $dk+kd = fg-\mathrm{id}_D$. Denote $F = \mathrm{id}_E \otimes f$ and $G = \mathrm{id}_E \otimes g$ and observe that $\mathrm{id}_E \otimes h$ and $\mathrm{id}_E \otimes k$ are respectively homotopies between GF and $\mathrm{id}_{E\otimes C}$, and between FG and $\mathrm{id}_{E\otimes D}$. Therefore, F is a quasi-isomorphism.

The morphism $\varphi\widehat{\otimes}\,\mathrm{id}$ preserves the filtrations and, by the comment above, the associated graded morphism

$$A \otimes F^n/F^{n+1} \xrightarrow{\simeq} B \otimes F^n/F^{n+1}$$

is a quasi-isomorphism for every $n \geq 1$. Theorem 4.33 shows that $\widetilde{\mathrm{MC}}(\varphi\widehat{\otimes}\,\mathrm{id})$ is a bijection. □

A completely analogous argument proves the following:

Proposition 4.39. *Let A be a cdga and $f\colon L \to L'$ a cdgl morphism such that the associated graded morphism $f^n\colon F^n/F^{n+1} \xrightarrow{\simeq} G^n/G^{n+1}$ is a quasi-isomorphism for $n \geq 1$. Then,*

$$\widetilde{\mathrm{MC}}(\mathrm{id}_A \widehat{\otimes} f)\colon \widetilde{\mathrm{MC}}(A\widehat{\otimes}L) \xrightarrow{\cong} \widetilde{\mathrm{MC}}(A\widehat{\otimes}L')$$

is a bijection. □

For completeness, we now state and prove the original Goldman–Millson Theorem as appears in [65, Theorem 2.4]:

Proposition 4.40. *Let $\mathcal{R}$ be an Artinian local ring with maximal ideal $\mathfrak{m}$ and let $f\colon L \xrightarrow{\simeq} L'$ be a quasi-isomorphism of dgl's. Then,*

$$\widetilde{\mathrm{MC}}(\mathrm{id}\otimes f)\colon \widetilde{\mathrm{MC}}(\mathfrak{m}\otimes L) \xrightarrow{\cong} \widetilde{\mathrm{MC}}(\mathfrak{m}\otimes L')$$

is a bijection.

Proof. Filter $\mathfrak{m}\otimes L$ and $\mathfrak{m}\otimes L'$ by $\{\mathfrak{m}^n \otimes L\}_{n\geq 1}$ and $\{\mathfrak{m}^n \otimes L'\}_{n\geq 1}$, respectively. As $\mathcal{R}$ is Artinian, the dgl's $\mathfrak{m}\otimes L$ and $\mathfrak{m}\otimes L'$ are complete. Clearly, the associated graded morphism of $\mathrm{id}_{\mathfrak{m}} \otimes f$ is a quasi-isomorphism and thus, the result follows from Theorem 4.33. □

Chapter 5

The Lawrence–Sullivan Interval

Let X be a simply connected CW-complex. Recall from Section 1.2.2 that a Quillen model of X is a free dgl $(\mathbb{L}(V), d)$ positively graded and not necessarily minimal, together with a quasi-isomorphism $(\mathbb{L}(V), d) \xrightarrow{\simeq} \lambda(X)$.

Fix a cellular structure on X with a single 0-cell and no 1-cell, and denote by $\mathrm{Cell}_*(X)$ the cellular rational chain complex of X. It is well known that one can obtain a Quillen model of X of the form $(\mathbb{L}(V), d)$, where $V = s^{-1}\,\mathrm{Cell}_{\geq 2}$, and in which $d(s^{-1}e)$ encodes the attaching map of the cell e. In particular, if we write as usual $d = \sum_{i\geq 1} d_i$ with each d_i increasing the bracket length by $i-1$, then d_1 corresponds to the desuspension of the chain differential on $\mathrm{Cell}_*(X)$ and

$$d_2 \colon s^{-1}\,\mathrm{Cell}_*(X) \longrightarrow \mathbb{L}^2(V) \subset s^{-1}\,\mathrm{Cell}_*(X) \otimes s^{-1}\,\mathrm{Cell}_*(X)$$

is the desuspension of an approximation to the diagonal.

Our intention is to extend this procedure to any simplicial set X, not necessarily simply connected, so that we allow the existence of simplices in all dimensions. To do this, let $N^*(X)$ be the non-degenerate rational simplicial chains on X and denote by

$$V = s^{-1}N_*(X)$$

its desuspension. Our goal is to construct a differential $d = \sum_{i\geq 1} d_i$ on the complete free graded Lie algebra $\widehat{\mathbb{L}}(V)$ which faithfully reflects the simplicial structure of X. Again we may choose d_1 to be the chain differential on $N_*(X)$ and

$$d_2 \colon V = s^{-1}N_*(X) \longrightarrow \mathbb{L}^2(V) \subset s^{-1}N_*(X) \otimes s^{-1}N_*(X)$$

to be exactly the desuspension of an approximation of the diagonal of $N_*(X)$. Observe that any vertex σ of X, or equivalently, any generator of $N_0(X)$, corresponds to a degree -1 element $a = s^{-1}\sigma \in V_{-1}$. Thus, if we choose the diagonal approximation so that σ is sent to $-\sigma \otimes \sigma$, then

$$d_2 a = -a \otimes a = -\frac{1}{2}[a, a].$$

U. Buijs et al., *Lie Models in Topology*, Progress in Mathematics 335,
https://doi.org/10.1007/978-3-030-54430-0_5

That is, any generator of V_{-1} must be a Maurer–Cartan element. The question is then whether it is possible to define $d_3, d_4, \dots$ so that their sum becomes a differential on $\widehat{\mathbb{L}}(V)$.

An affirmative answer to this question was given by R. Lawrence and D. Sullivan in [91] for the simplest possible non-trivial simplicial set, namely the closed interval $\underline{\Delta}^1$. In this case $V = s^{-1}N_*(\underline{\Delta}^1)$ is generated by two elements a, b of degree -1, corresponding to vertices, and an element x of degree zero, corresponding to the only non-degenerate 1-simplex. By the above argument, for the differential d to be set on $\widehat{\mathbb{L}}(V)$, a and b must be Maurer–Cartan elements,

$$da = -\frac{1}{2}[a,a], \qquad db = -\frac{1}{2}[b,b],$$

and the linear part of d on x is simply $b - a$. The fundamental result in op. cit. asserts that, under these constraints, there is a unique expression for the differential dx so that

$$(\widehat{\mathbb{L}}(a,b,x), d)$$

becomes a cdgl. Namely,

$$dx = \mathrm{ad}_x\, b + \sum_{n=0}^{\infty} \frac{B_n}{n!}\ \mathrm{ad}_x^n(b-a),$$

where B_n denotes the nth Bernoulli number. A detailed analysis of this particular cdgl constitutes the core of this chapter and the departure point for the development of a consistent homotopy theory in **cdgl**.

5.1 Introducing the Lawrence–Sullivan interval

We first recall some elementary facts concerning the *Bernoulli numbers* $\{B_n\}_{n\geq 1}$ as they play an important role in what follows. Among other ways, these are defined by the series

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n. \tag{5.1}$$

Taking into account that

$$\frac{e^x - 1}{x} = \sum_{n=0}^{\infty} \frac{1}{n+1!} x^n,$$

we calculate the first Bernoulli numbers from the equation $\dfrac{x}{e^x-1} \cdot \dfrac{e^x-1}{x} = 1$:

$$B_0 = 1\,, \quad B_1 = -\frac{1}{2}\,, \quad B_2 = \frac{1}{6}\,, \quad B_3 = 0\,, \quad B_4 = -\frac{1}{30}\,, \quad B_5 = 0, \quad B_6 = \frac{1}{42}, \dots$$

On the other hand, the formula

$$\left(\frac{-x}{e^{-x}-1}\right) = x + \left(\frac{x}{e^{x}-1}\right) \tag{5.2}$$

implies that

$$\sum \frac{B_n}{n!}(-x)^n - \sum \frac{B_n}{n!}x^n = x,$$

and in particular $B_n = 0$ if n is odd and $n \geq 3$.

To begin, recall from Section 1.1.3 that $N^*(\underline{\Delta}^1) = C_*(\Delta^1)$. The desuspension of this chain complex is thus generated by cycles a, b of degree -1 and an element x of degree 0 whose chain boundary is $b - a$.

Definition 5.1. The *Lawrence–Sullivan interval*, LS interval for short, is the cdgl

$$\mathfrak{L}_1 = (\widehat{\mathbb{L}}(a, b, x), d),$$

in which a and b are Maurer–Cartan elements, x is of degree 0 and

$$dx = \operatorname{ad}_x b + \sum_{n=0}^{\infty} \frac{B_n}{n!} \operatorname{ad}_x^n (b-a).$$

From (5.1) we may write dx in terms of operators as

$$dx = \operatorname{ad}_x b + \frac{\operatorname{ad}_x}{e^{\operatorname{ad}_x}-1}(b-a). \tag{5.3}$$

Remark 5.2. There are several other useful ways of expressing dx. For instance, using the identity (5.2), we write

$$\frac{\operatorname{ad}_x}{e^{\operatorname{ad}_x}-1}(b-a) = \left(-\operatorname{ad}_x + \frac{-\operatorname{ad}_x}{e^{-\operatorname{ad}_x}-1}\right)(b-a)\,.$$

Therefore, dx can also be written as

$$dx = \operatorname{ad}_x a + \frac{-\operatorname{ad}_x}{e^{-\operatorname{ad}_x}-1}(b-a). \tag{5.4}$$

From this and (5.3) we obtain formulas for the perturbed differentials:

$$d_b x = \frac{\operatorname{ad}_x}{e^{\operatorname{ad}_x}-1}(b-a) \quad \text{and} \quad d_a x = \frac{-\operatorname{ad}_x}{e^{-\operatorname{ad}_x}-1}(b-a)\,. \tag{5.5}$$

On the other hand, starting from (5.3), we have

$$dx = \operatorname{ad}_x b + \frac{\operatorname{ad}_x}{e^{-\operatorname{ad}_x}-1}(b-a) = \frac{\operatorname{ad}_x}{1-e^{\operatorname{ad}_x}}(a) + \operatorname{ad}_x b + \frac{\operatorname{ad}_x}{e^{-\operatorname{ad}_x}-1}(b).$$

Then, applying (5.2) to the last term we get a more symmetric expression of dx:

$$dx = \frac{\operatorname{ad}_x}{1-e^{\operatorname{ad}_x}}(a) + \frac{\operatorname{ad}_x}{1-e^{-\operatorname{ad}_x}}(b)\,.$$

We postpone the proof of the fact that $d^2 = 0$ for the next sections and present now some important properties of $\mathfrak{L}_1$.

First, a direct computation shows that the MC elements a, b are gauge related through x:

Proposition 5.3. *In the* LS *interval,* $x \mathcal{G} b = a$.

Proof. Indeed,

$$\begin{aligned} x \mathcal{G} b &= e^{\operatorname{ad}_x}(b) - \frac{e^{\operatorname{ad}_x} - 1}{\operatorname{ad}_x}(dx) \\ &= e^{\operatorname{ad}_x}(b) - \frac{e^{\operatorname{ad}_x} - 1}{\operatorname{ad}_x}\Big(\operatorname{ad}_x b + \frac{\operatorname{ad}_x}{e^{\operatorname{ad}_x} - 1}(b-a)\Big) \\ &= e^{\operatorname{ad}_x}(b) - \frac{e^{\operatorname{ad}_x} - 1}{\operatorname{ad}_x}(\operatorname{ad}_x b) - (b-a) = b - (b-a) = a. \end{aligned}$$ □

Corollary 5.4. *Let L be a cdgl, $u, v \in \operatorname{MC}(L)$, $y \in L_0$. Then, $y \mathcal{G} v = u$ if and only if there exists a cdgl morphism $\varphi\colon \mathfrak{L}_1 \to L$ with $\varphi(a) = u$, $\varphi(b) = v$ and $\varphi(x) = y$.*

Proof. By Proposition 5.3, if such φ exists, then $y \mathcal{G} v$ obviously equals u, as gauge related elements are preserved by cdgl morphisms (see Proposition 4.15).

On the other hand, isolating dy from the identity $y \mathcal{G} v = u$, that is, from

$$u = e^{\operatorname{ad}_x}(v) - \frac{e^{\operatorname{ad}_x} - 1}{\operatorname{ad}_x}(dy),$$

we see that

$$dy = \operatorname{ad}_y u + \frac{\operatorname{ad}_y}{e^{\operatorname{ad}_y} - 1}(v-u)\,.$$

This means precisely that $\varphi(dx) = d\varphi(x)$, that is, φ is a cdgl morphism. □

We also show that the differential in $\mathfrak{L}_1$ is completely and uniquely determined by its linear part provided that a and b are Maurer–Cartan elements.

Theorem 5.5. *Let $(\widehat{\mathbb{L}}(a,b,x), d')$ be a cdgl in which a, b are Maurer–Cartan elements and the linear part of the differential satisfies $d'_1 x = b - a$. Then*

$$(\widehat{\mathbb{L}}(a,b,x), d') = \mathfrak{L}_1.$$

Proof. Let d be the differential in $\mathfrak{L}_1$. Write $d = \sum_{n\ge1} d_n$ and $d' = \sum_{n\ge1} d'_n$, where d_n and d'_n increase the bracket length by $n-1$. We show that $d_n x = d'_n x$ for any $n \ge 1$. For $n = 1$ this is by definition.

For $n = 2$, since a, b are MC elements for both differentials and $d_1 x = d' x = b - a$, it follows that $0 = d'_1 d'_2 + d'_2 d'_1 = d_1 d'_2 + d_2 d_1$. Then, $d_1 d_2 x = d_1 d'_2 x$ and therefore, $d_2 x - d'_2 x$ is d_1-cycle of bracket length 2 and degree -1. But $H(\widehat{\mathbb{L}}(a,b,x), d_1) = \mathbb{L}(a)$ and thus, any such cycle must be a boundary: $d_2 x - d'_2 x = d_1 \alpha$ with $\alpha \in \mathbb{L}^2(a,b,x)_0 = 0$. Hence, $d_2 x = d'_2 x$.

Similarly, if $d_m x = d'_m x$ for $1 \leq m < n$, then from $d^2 = d'^2 = 0$ and the induction hypotheses we deduce that $d_n x - d'_n x$ is a d_1-cycle of bracket length n and degree -1. Again, there are no non-trivial such d_1-cycles and thus, $d_n x = d'_n x$. $\square$

Next, we show that the derivation d in $\mathfrak{L}_1$ certainly defines a differential. We will do this by using two completely different approaches: classical rational homotopy theory and the flow generated by a particular differential equation.

5.2 The LS interval as a cylinder

Here, we identify the LS interval $\mathfrak{L}_1$ with the classical *Tanré cylinder* [130, §II. 5] on the free dgl generated by a Maurer–Cartan element. This procedure will be generalized in Section 8.3.

Definition 5.6. The *cylinder construction on the free Lie algebra generated by an* MC *element* a is the cdgl $(\widehat{\mathbb{L}}(a, u, v), d)$, where $|u| = -1$, $|v| = 0$, $du = 0$ and $dv = u$.

Let i be the derivation of degree $+1$ on the cylinder defined by $i(a) = v$, $i(v) = i(u) = 0$. Then,

$$\theta = di + id$$

is also a degree-0 derivation of $(\widehat{\mathbb{L}}(a, u, v), d)$ commuting with d and satisfying $\theta(u) = \theta(v) = 0$.

Let $\widehat{\mathbb{L}}(a, b, x)$ be the complete free Lie algebra on the three elements a, b and x with $|a| = |b| = -1$, $|x| = 0$. We define a morphism of graded Lie algebras

$$\begin{gathered} \psi\colon \widehat{\mathbb{L}}(a, b, x) \longrightarrow \widehat{\mathbb{L}}(b, u, v), \\ \psi(a) = a,\ \psi(b) = e^{\theta}(a),\ \psi(x) = v. \end{gathered} \tag{5.6}$$

Theorem 5.7. *The map ψ is a cgl isomorphism, and the induced derivation $\psi^{-1} d\psi$ on $\widehat{\mathbb{L}}(a, b, x)$ is precisely the original derivation in the Lawrence–Sullivan interval. In particular, it is a differential for which ψ is a cdgl isomorphism.*

Proof. Note that the linear part of ψ is trivially an isomorphism. By Proposition 3.12, ψ is also an isomorphism.

Next, observe that the elements a and b are Maurer–Cartan elements for the differential $\psi^{-1} d\psi$:

$$\begin{aligned} da = \psi^{-1} d\psi a &= -\frac{1}{2}[\psi^{-1} a, \psi^{-1} a] = -\frac{1}{2}[a, a], \\ db = \psi^{-1} d\psi b = \psi^{-1} d e^{\theta}(a) &= -\frac{1}{2}\psi^{-1}[e^{\theta}(a), e^{\theta}(a)] = -\frac{1}{2}[b, b]. \end{aligned}$$

Note that in the last equation we have used that e^{θ} is a cdgl automorphism (see Proposition 4.10).

Now, an inductive argument establishes the identity

$$\theta^{k+1}(a) = (-1)^k \mathrm{ad}_v^k(u) + (-1)^{k+1}\mathrm{ad}_v^{k+1}(a), \quad \text{for} \quad k \geq 0,$$

which in terms of operators translates to

$$e^\theta(a) = \frac{e^{-\mathrm{ad}_v} - 1}{-\mathrm{ad}_v}(u) + e^{-\mathrm{ad}_v}(a).$$

We isolate u from this expression and $e^\theta(a)$ by $\psi(b)$ to see that,

$$u = \frac{-\mathrm{ad}_v}{e^{-\mathrm{ad}_v} - 1}(\psi(b) - a) + \mathrm{ad}_v(a),$$

and finally obtain

$$dx = \psi^{-1} d\psi x = \psi^{-1} dv = \psi^{-1} u = \frac{-\mathrm{ad}_x}{e^{-\mathrm{ad}_x} - 1}(b - a) + \mathrm{ad}_x a,$$

which is the form in (5.4) of the differential in the LS interval. □

5.3 The flow of a differential equation, the gauge action and the LS interval

Consider the following generic situation: the differential equation

$$u'(t) = y + \mathrm{ad}_x\, u(t),$$

defined on any manifold M endowed with a smooth Lie bracket, is invariant under translation of the variable t and therefore its solution defines a flow in M, that is, an action of $(\mathbb{R}, +)$ on M, given by $t \cdot b = u(t)$ with $u(0) = b$. Note the similarity of this equation with others on certain principal bundles whose flows define the so-called *gauge transformations*.

In our context, replace M by any given graded Lie algebra L, choose $\mathbb{Q}$ instead of $\mathbb{R}$, and define a curve in L as an element $u(t)$ in the cgl

$$L \widetilde{\otimes} \wedge t,$$

defined by completing the dgl $L \otimes (\wedge t, 0)$ in which $|t| = 0$, with respect to the filtration $\{L \otimes \wedge^{\geq n} t\}_{n \geq 0}$. That is,

$$L \widetilde{\otimes} \wedge t = \varprojlim_n (L \otimes \wedge t)/(L \otimes \wedge^{\geq n} t),$$

whose elements can be written as series of the form,

$$\sum_{n \geq 0} a_n t^n, \quad \text{with} \quad a_n \in L.$$

The derivative operator is defined by

$$\Big(\sum_{n\geq 0} a_n t^n\Big)' = \sum_{n\geq 1} n a_n t^{n-1}.$$

Then, we have:

Lemma 5.8. *Given $x \in L_0$ and $y \in L_r$, with $r \in \mathbb{Z}$, the formal differential equation in $L\widetilde{\otimes} \wedge t$,*

$$\begin{cases} u'(t) = y + \mathrm{ad}_x u(t), \\ u(0) = c \in L_r, \end{cases} \tag{5.7}$$

has a unique solution given by

$$u(t) = \frac{e^{t\,\mathrm{ad}_x} - 1}{-\,\mathrm{ad}_x}(y) + e^{t\,\mathrm{ad}_x}(c) \in (L\widetilde{\otimes} \wedge t)_r.$$

Proof. If we write $u(t) = \sum_{n\geq 0} a_n t^n$, $a_n \in L$, the equation (5.7) can be written as the following recurrence relation:

$$\begin{aligned} a_0 &= c, \\ a_1 &= y + \mathrm{ad}_x(a_0), \\ a_n &= \frac{\mathrm{ad}_x}{n}(a_{n-1}),\ \ n > 1. \end{aligned}$$

Then,

$$a_n = \frac{(\mathrm{ad}_x)^{n-1}}{n!}(a_1) = \frac{(\mathrm{ad}_x)^{n-1}}{n!}(y) + \frac{(\mathrm{ad}_x)^{n}}{n!}(c), \quad n > 1,$$

and the unique solution is

$$\begin{aligned} u(t) = \sum_{n\geq 0} a_n t^n &= \sum_{n\geq 1} \frac{(t^n)(\mathrm{ad}_x)^{n-1}}{n!}(y) + \sum_{n\geq 0} \frac{(t\,\mathrm{ad}_x)^n}{n!}(c) \\ &= \frac{e^{t\,\mathrm{ad}_x} - 1}{\mathrm{ad}_x}(y) + e^{t\,\mathrm{ad}_x}(c). \end{aligned}$$

□

Remark 5.9. As we observed above, whenever y is fixed, this produces a flow, i.e., an action of $(\mathbb{Q}, +)$ on L_r given by

$$t \cdot c = u(t), \quad \text{with} \quad u(0) = c.$$

Consider now the cgl $\widehat{\mathbb{L}}(a, b, x)$ with $|a| = |b| = -1$ and $|x| = 0$. In this cgl we define a derivation d by declaring a and b to be MC elements and

$$dx = \mathrm{ad}_x(b) + \frac{\mathrm{ad}_x}{e^{\mathrm{ad}_x} - 1}(b - a).$$

Theorem 5.10. *$(\widehat{\mathbb{L}}(a,b,x),d)$ is a cdgl, i.e., $d^2x=0$.*

Proof. In $\widehat{\mathbb{L}}(a,b,x)$ we consider the differential equation (5.7) in which we choose $y=dx$ and the initial condition $c=b$:

$$\begin{cases} u'(t)=dx+\mathrm{ad}_x u(t), \\ u(0)=b. \end{cases} \tag{5.8}$$

By Lemma 5.8, the corresponding unique solution is

$$u(t)=\frac{e^{t\,\mathrm{ad}_x}-1}{-\mathrm{ad}_x}(dx)+e^{t\,\mathrm{ad}_x}(b), \tag{5.9}$$

whose evaluation at 1 gives

$$\begin{aligned} u(1)&=\frac{e^{\mathrm{ad}_x}-1}{-\mathrm{ad}_x}(dx)+e^{\mathrm{ad}_x}(b)=\frac{e^{\mathrm{ad}_x}-1}{-\mathrm{ad}_x}\Big(ad_x(b)+\frac{\mathrm{ad}_x}{e^{\mathrm{ad}_x}-1}(b-a)\Big)+e^{\mathrm{ad}_x}(b) \\ &=\frac{e^{\mathrm{ad}_x}-1}{-\mathrm{ad}_x}\big(ad_x(b)\big)+e^{\mathrm{ad}_x}(b)-(b-a)=b-(b-a)=a. \end{aligned}$$

Now, consider the element of degree -2

$$f(t)=du(t)+\frac{1}{2}[u(t),u(t)],$$

which satisfies

$$f(0)=f(1)=0,$$

because a and b are MC elements, and whose derivative is

$$\begin{aligned} f'(t)&=du'(t)+[u(t),u'(t)]=d\big(dx+\mathrm{ad}_x\,u(t)\big)+[u(t),dx+\mathrm{ad}_x\,u(t)] \\ &=d^2x+[x,du(t)]+\frac{1}{2}\big[x,[u(t),u(t)]\big]=d^2x+\mathrm{ad}_x\,f(t). \end{aligned}$$

Therefore, the curvature function is a solution of the differential equation

$$\begin{cases} f'(t)= & d^2x+\mathrm{ad}_x f(t), \\ f(0)= & 0, \end{cases}$$

which, again by Lemma 5.8, has a unique solution of the form

$$f(t)=\frac{e^{t\,\mathrm{ad}_x}-1}{-\mathrm{ad}_x}(d^2x)+e^{t\,\mathrm{ad}_x}(0)=\frac{e^{t\mathrm{ad}_x}-1}{-\mathrm{ad}_x}(d^2x),$$

Evaluating at $t=1$ we have

$$0=f(1)=\frac{e^{\mathrm{ad}_x}-1}{-\mathrm{ad}_x}(d^2x)=0.$$

As $\frac{e^{\mathrm{ad}_x}-1}{-\mathrm{ad}_x}$ is a cgl isomorphism, $d^2x=0$. □

Now let L be a cdgl. In the differential equation (5.7), choose any $x \in L_0$ and $y = dx$. Then, for the flow generated by this equation (see Remark 5.9), we have:

Corollary 5.11. *For any $t \in \mathbb{Q}$ and any $b \in \mathrm{MC}(L)$, $t \cdot b = (tx) \mathcal{G} b$.*

Proof. Simply observe that, by Lemma 5.8, the unique solution of solution of (5.7) is:

$$u(t) = \frac{e^{t\,\mathrm{ad}_x} - 1}{-\mathrm{ad}_x}(dx) + e^{t\,\mathrm{ad}_x}(b) = (tx) \mathcal{G} b. \qquad \square$$

Remark 5.12. In particular, $u(1) = x \mathcal{G} b$. In other words, given a cdgl L, the flow generated by $u(t)$ with $y = dx$, takes the MC element b to the MC element $x \mathcal{G} b$ in time $t = 1$. Hence, interpreting Maurer–Cartan elements as points, this can be thought of as a path joining a to $x \mathcal{G} a$. In particular, the differential in $\mathfrak{L}_1$, providing a and b are MC elements, is the only one for which $x \mathcal{G} b = a$.

5.4 Subdivision of the LS interval and a model of the triangle

We first proceed to "subdivide" the LS interval as in [91, Theorem 2]. Consider two LS intervals $(\widehat{\mathbb{L}}(a, u, y), d)$ and $(\widehat{\mathbb{L}}(u, b, z), d)$ and glue them together to obtain the cdgl

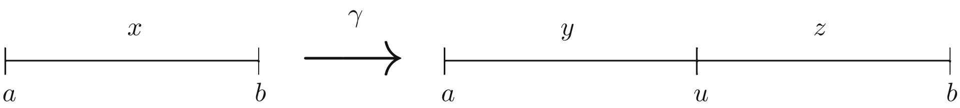
$$(\widehat{\mathbb{L}}(a, u, v, y, z), d)$$

with the obvious differential. Note that this is simply the coproduct of the given LS intervals in which both u's have been identified. Then, we can state

Theorem 5.13. *The map defined by*

$$\gamma\colon \mathfrak{L}_1 \longrightarrow (\widehat{\mathbb{L}}(a, u, b, y, z), d), \qquad \gamma(a) = a, \quad \gamma(b) = b, \quad \gamma(x) = y * z,$$

is a cdgl morphism.



Proof. Theorem 4.16 and Proposition 5.3 imply that in $(\widehat{\mathbb{L}}(a, u, b, y, z), d)$,

$$(y * z) \mathcal{G} b = y \mathcal{G} (z \mathcal{G} b) = y \mathcal{G} u = a.$$

Hence, by Corollary 5.4, the map γ is a cdgl morphism. $\square$

Now, recall that the LS interval $\mathfrak{L}_1$ is the only solution to the question with which we started this chapter for the simplicial set $\underline{\Delta}^1$. Suppose we want to go a step further and try to solve this problem for $\underline{\Delta}^2$, whose non-degenerate chains are

$$N_*(\underline{\Delta}^2) = C_*(\Delta^2).$$

We are then aiming to find a cdgl

$$\mathfrak{L}_2 = (\widehat{\mathbb{L}}(s^{-1}C_*(\Delta^2)), d)$$

satisfying the following conditions: the generators given by the vertices are MC elements, the sub-cdgl given by each edge is an LS interval, and the linear part of the differential is the simplicial chain differential of $s^{-1}C_*(\Delta^2)$. If we denote by

$$\{a_0, a_1, a_2, a_{01}, a_{02}, a_{12}, a_{012}\}$$

the generators of $s^{-1}C_*(\Delta^2)$, the solution is given in the following:

Proposition 5.14 (Model of the triangle)**.** *There is a cdgl*

$$\mathfrak{L}_2 = (\widehat{\mathbb{L}}(a_0, a_1, a_2, a_{01}, a_{12}, a_{02}, a_{012}), d),$$

in which $(\widehat{\mathbb{L}}(a_0, a_1, a_{01}), d)$, $(\widehat{\mathbb{L}}(a_1, a_2, a_{12}), d)$ *and* $(\widehat{\mathbb{L}}(a_0, a_2, a_{02}), d)$ *are* LS *intervals and*

$$da_{012} = a_{01} * a_{12} * a_{02}^{-1} - [a_0, a_{012}], \quad \textit{or equivalently,} \quad d_{a_0}a_{012} = a_{01} * a_{12} * a_{02}^{-1}.$$

Proof. Note first that the linear part of

$$da_{012} = a_{01} * a_{12} * a_{02}^{-1} - [a_0, a_{012}]$$

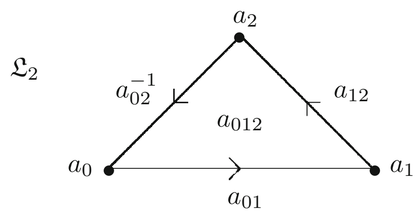
is $a_{01} - a_{12} + a_{02}$. Hence, we only need to check that $d^2 a_{012} = 0$ as, by definition, this is the case for all the simplices of lower degrees. Equivalently, we prove that $d_{a_0}^2 a_{012} = 0$. The composition of two morphisms as in Theorem 5.13 yields the cdgl morphism

$$\psi\colon \mathfrak{L}_1 \longrightarrow (\widehat{\mathbb{L}}(a_0, a_1, a_2, a_{01}, a_{12}, a_{02}), d),$$

with $\psi(a) = \psi(b) = a_0$ and $\psi(x) = a_{01} * a_{12} * a_{02}^{-1}$. Hence,

$$d(a_{01} * a_{12} * a_{02}^{-1}) = d\psi(x) = \psi(dx) = \psi\Big(\mathrm{ad}_x(b) + \sum_{k\geq 0} \frac{B_k}{k!}\,\mathrm{ad}_x^k(b-a)\Big) = \psi[x,b]$$
$$= \mathrm{ad}_{a_{01}*a_{12}*a_{02}^{-1}}(a_0) = -ad_{a_0}(a_{01} * a_{12} * a_{02}^{-1}).$$

It follows that $a_{01} * a_{12} * a_{02}^{-1}$ is a d_{a_0}-cycle, and thus $d_{a_0}^2(a_{012}) = 0$. □



In geometrical terms, the differential of the 2-face a_{012} draws the border of Δ^2 starting from the base point a_0 and connecting each edge with the following one by their BCH product. If we start drawing the border of the triangle from the vertex a_1, the same process produces a new cdgl

$$\mathfrak{L}_2' = (\widehat{\mathbb{L}}(a_0, a_1, a_2, a_{01}, a_{12}, a_{02}, a_{012}), d'),$$

where

$$d' a_{012} = a_{12} * a_{02}^{-1} * a_{01} - [a_1, a_{012}].$$

Fortunately, we have:

Proposition 5.15. *The cdgl's $\mathfrak{L}_2$ and $\mathfrak{L}_2'$ are isomorphic. An explicit isomorphism is given by*

$$\mathfrak{L}_2 \xrightarrow{\cong} \mathfrak{L}_2', \qquad \varphi(a_{012}) = e^{\mathrm{ad}_{a_{01}}}(a_{012}),$$

and is the identity on all generators of smaller dimension.

Proof. The following computation checks that φ is in fact a cdgl morphism (the second and fourth equalities follow from Propositions 4.24 and 4.13, respectively):

$$\begin{aligned} d'\varphi(a_{012}) &= d'_{a_0} e^{\mathrm{ad}_{a_{01}}} a_{012} - [a_0, e^{\mathrm{ad}_{a_{01}}} a_{012}] \\ &= e^{\mathrm{ad}_{a_{01}}} d'_{a_1}(a_{012}) - [a_0, e^{\mathrm{ad}_{a_{01}}}(a_{012})] \\ &= e^{\mathrm{ad}_{a_{01}}}(a_{12} * a_{02}^{-1} * a_{01}) - [a_0, e^{\mathrm{ad}_{a_{01}}}(a_{012})] \\ &= a_{01} * a_{12} * a_{02}^{-1} - [a_0, e^{\mathrm{ad}_{a_{01}}}(a_{012})] = \varphi(da_{012})\,. \end{aligned}$$

The result is then a consequence of Proposition 3.12. □

We prove next that the differential d on $\mathfrak{L}_2$ is completely determined provided the edges are LS intervals and the perturbed differential of the top simplex is in the cdgl generated by the 1-skeleton.

Proposition 5.16. *Let D be a differential in $\widehat{\mathbb{L}}(s^{-1}C_*\Delta^2)$ such that $D = d$ on $\widehat{\mathbb{L}}(s^{-1}C_*\dot{\Delta}^2)$, $D_1 = d_1$, and $D_{a_0}(a_{012}) \in \widehat{\mathbb{L}}(s^{-1}C_*\dot{\Delta}^2)$. Then $D = d$.*

Proof. Write $e = a_{012}$ and

$$D = D_1 + D_2 + \cdots, \qquad d = d_1 + d_2 + \cdots,$$

where D_n and d_n increase the bracket length by $n-1$. We show inductively that $D_n = d_n$ for $n \geq 1$. By definition, $D_1 = d_1$. Suppose $D_i = d_i$ for $i < n$. Since $D^2 = 0$, by induction we have

$$d_1 D_n = -\sum_{i=2}^{n} D_i D_{n+1-i} = -\sum_{i=2}^{n} d_i d_{n+1-i} = d_1 d_n.$$

Therefore, $d_1(D_n - d_n)(e) = 0$. Since $H(\widehat{\mathbb{L}}(s^{-1}C_*\dot{\Delta}^2), d_1)$ is generated by the class of a_0 in degree -1 and that of $d_1 e$ in degree 0, it follows that every decomposable d_1-cycle in degree 0 must be a d_1-boundary. In particular, the difference $(D_n - d_n)(e)$ is a d_1-boundary. But, since $\widehat{\mathbb{L}}(s^{-1}\dot{\Delta}^2) = \widehat{\mathbb{L}}(s^{-1}\dot{\Delta}^2)_{\leq 0}$, we conclude that $D_n e = d_n e$. □

5.5 Paths in a cdgl

The geometrical interpretation of the LS interval given in Remark 5.12, together with Corollary 5.4, motivates the following:

Definition 5.17. Given a cdgl L and $u, v \in \mathrm{MC}(L)$, a *path from u to v* is a morphism $\varphi\colon \mathfrak{L}_1 \to L$ with $\varphi(a) = u$ and $\varphi(b) = v$. By abuse of language we often identify the path φ with the element $\varphi(x) \in L_0$ as long as the endpoints of the path are unambiguously determined.

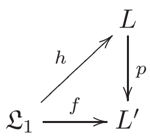
Equivalently, in view of Corollary 5.4, a path in L joining u with v is an element $y \in L_0$ such that $y \,\mathcal{G}\, v = u$.

With this terminology notice that, in view of Proposition 5.13, if y is a path in L from u to v and z is another path in L from v to w, then $y * z$ is a path from u to w.

The results in this section might be deduced from general facts if we endow, as we do in Chapter 8, the category **cdgl** with a closed model structure reflecting the geometry of **sset**. Nevertheless, we collect them here to illustrate that the LS interval $\mathfrak{L}_1$ and a cdgl morphism $\mathfrak{L}_1 \to L$ behave exactly like the closed interval and a path, respectively.

The first result asserts that any surjective cdgl morphism p has the path lifting property. If moreover, $H_0(p)$ is an isomorphism, the second result states that any path can be lifted with prescribed endpoints.

Proposition 5.18 (Path lifting lemma)**.** *Let $p\colon L \to L'$ be a surjective cdgl morphism, let $c \in \mathrm{MC}(L)$ and let $f\colon \mathfrak{L}_1 \to L'$ be a path with $f(a) = p(c)$. Then, there is a path $h\colon \mathfrak{L}_1 \to L$ with $h(a) = c$ and $p \circ h = f$,*



Proof. Since p is surjective, there is an element $y \in L$ with $p(y) = f(x)$. Let $(\widehat{\mathbb{L}}(a, u, v), d)$ be the cylinder construction on the MC element a (see Definition 5.6) and let

$$\psi\colon \mathfrak{L}_1 \xrightarrow{\cong} (\widehat{\mathbb{L}}(a, u, v), d)$$

be the isomorphism of Theorem 5.7. We define a map $\rho\colon (\widehat{\mathbb{L}}(a,u,v),d) \to L$ by $\rho(a) = c$, $\rho(v) = y$ and $\rho(u) = dy$. Clearly $p \circ \rho = f \circ \psi^{-1}$. We set $h = \rho \circ \psi\colon \mathfrak{L}_1 \to L$ and get $p \circ h = f$. $\square$

Corollary 5.19. *Let $p\colon L \to L'$ be a surjective morphism such that $\widetilde{\mathrm{MC}}(p)$ is surjective. Then, $\mathrm{MC}(p)$ is also surjective.*

Proof. Let $v' \in \mathrm{MC}(L')$. By hypothesis, there exist $u \in \mathrm{MC}(L)$ and a path x in L' from $p(u)$ to v'. The path lifting lemma above gives a path in L from u to an MC element v with $p(v) = v'$. $\square$

Example 5.20. In Corollary 5.19, the surjection hypothesis on $\widetilde{\mathrm{MC}}(p)$ is necessary. Consider for instance the surjective morphism

$$f\colon (\widehat{\mathbb{L}}(a,b),d) \longrightarrow (\mathbb{L}(a),d'), \qquad f(a) = a, \quad f(b) = -\frac{1}{2}[a,a].$$

Here, $|a| = -1$, $da = b$, $db = 0$ and $d'(a) = -\frac{1}{2}[a,a]$. Observe that

$$\widetilde{\mathrm{MC}}(\widehat{\mathbb{L}}(a,b),d) = \{0\} \quad \text{and} \quad \widetilde{\mathrm{MC}}(\mathbb{L}(a),d') = \{0,a\}.$$

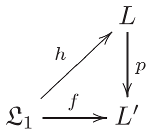
Thus, $\mathrm{MC}(f)$ is obviously not surjective.

In the proof of the next proposition we use the following basic fact whose proof is completely straightforward.

Lemma 5.21. *Let $p\colon L \to L'$ be a surjective morphism such that $H_n(p)$ is an isomorphism for some n. Then:*

1) *For any cycle $z' \in L'_n$, there exists a cycle $z \in L_n$ such that $p(z) = z'$.*
2) *For any cycle $z \in L_n$ such that $p(z) = dx$, there exists $y \in L_n$ such that $z = dy$ and $p(y) = x$.* $\square$

Proposition 5.22. *Let $p\colon L \to L'$ be a surjective cdgl morphism for which $H_0(p)$ is an isomorphism, and let $u \sim v \in \mathrm{MC}(L)$, $u' \sim v' \in \mathrm{MC}(L')$ with $p(u) = u'$, $p(v) = v'$. Then, given a path f from u' to v', there exists a path h from u to v such that $p \circ h = f$,*



Proof. Choose a path g from u to v. In view of Theorem 5.13, $p(g(x))^{-1} * f(x)$ is a $d_{v'}$-cycle. By Lemma 5.21, there is a d_v-cycle y with $p(y) = p(g(x))^{-1} * f(x)$. Observe that y is a loop on v and therefore $g(x) * y$ is a path $h\colon \mathfrak{L}_1 \to L$ from u to v. As $p(g(x) * y) = f(x)$, it follows that $p \circ h = f$. $\square$

Bibliographical notes

The fundamental work of Lawrence and Sullivan has been the cornerstone to develop ideas concerning rational models of non-connected spaces, see [32, 33].

On the other hand, the central problem stated in the introduction of this chapter was attacked "cellularly" (instead of simplicially) by D. Sullivan in an appendix to [129]: let X be a CW-complex in which the closure of each cell has the rational homology of a point. Consider $V = s^{-1}\,\mathrm{Cell}_*$, the desuspension of the rational cellular complex. Then, see [129, Theorem A.1], there is a complete differential Lie algebra $(\widehat{\mathbb{L}}(V), d)$, such that the linear part of d is the boundary operator of the cells and the quadratic part comes from a cellular approximation of the diagonal. Applied to the interval I, with two 0-cells and one 1-cell, one obtains a complete free Lie algebra $(\widehat{\mathbb{L}}(a,b,x), d)$ with $d_1x = b - a$. This was conjectured to agree with the LS interval in [91] and [129], and it was proved in [112].

It is also worth mentioning the implications of the analysis of the LS interval in number theory. Recall that an *Euler-type identity* is a convolution equation involving Bernoulli numbers of the form

$$\sum_{k=0}^{n} \lambda_k B_k B_{n-k} = 0,$$

where both k and n are even and the λ_k's are rational numbers depending on k and n. Classical identities of this type are the *Euler equation*,

$$-(n+1)B_n = \sum_{\substack{k=2 \\ k \text{ even}}}^{n-2} \binom{n}{k} B_k B_{n-k}, \quad n \text{ even with } n \geq 4,$$

and the *Miki identity* [105],

$$2H_nB_n = \sum_{\substack{k=2 \\ k \text{ even}}}^{n-2} \frac{n}{k(n-k)} \Big(1 - \binom{n}{k}\Big) B_k B_{n-k}, \quad n \text{ even with } n \geq 4,$$

where H_n is the nth harmonic number. In [19] the authors deduce a very general Euler type identity, which includes both the Euler equation and the Miki identity, from the fact that $d^2 = 0$ on the LS interval. In the same context, new relations among Bernoulli numbers are also obtained in [112]. Direct proofs of the Miki identity can be found in [37] and [59].

Chapter 6

The Cosimplicial cdgl $\mathfrak{L}_\bullet$

In this chapter we further develop the ideas and machinery developed in Chapter 5, specially those involving the construction of the LS interval and the "model of the triangle" in Section 5.4, to find compatible models of the simplicial sets $\underline{\Delta}^n$, for any $n \geq 0$.

More specifically, our goal is to find a family of cdgl's

$$\mathfrak{L}_\bullet = \{\mathfrak{L}_n\}_{n\geq 0}$$

such that, for each $n \geq 0$, the following holds:

- $\mathfrak{L}_n = \widehat{\mathbb{L}}(s^{-1}C_*(\Delta^n), d)$ is the free cdgl generated by the desuspension $s^{-1}N_*(\underline{\Delta}^n) = s^{-1}C_*(\Delta^n)$ of the non-degenerate simplicial chains on the simplicial set $\underline{\Delta}^n$.

- The differential on each vertex is a Maurer–Cartan element, i.e., for each $a = s^{-1}x$, with x a generator of $C_0(\Delta^n)$,

$$da = -\frac{1}{2}[a, a].$$

- The linear part $d_1\colon s^{-1}C_*(\Delta^n) \to s^{-1}C_*(\Delta^n)$ of d is precisely the simplicial chain boundary.

- For each $i = 0, \ldots, n$, the maps

$$\delta^i\colon \mathfrak{L}_{n-1} \longrightarrow \mathfrak{L}_n \qquad \text{and} \qquad \sigma^i\colon \mathfrak{L}_{n+1} \longrightarrow \mathfrak{L}_n$$

induced by the cofaces and codegeneracies,

$$\delta^i\colon C_*(\Delta^{n-1}) \longrightarrow C_*(\Delta^n) \qquad \text{and} \qquad \sigma^i\colon C_*(\Delta^{n+1}) \longrightarrow C_*(\Delta^n),$$

are cdgl morphisms.

U. Buijs et al., *Lie Models in Topology*, Progress in Mathematics 335,
https://doi.org/10.1007/978-3-030-54430-0_6

As a result, $\mathfrak{L}_\bullet$ becomes a cosimplicial cdgl, which is unique up to cdgl isomorphism, and constitutes the central object around which we will develop the geometrical realization of any cdgl and the cdgl model of any simplicial set.

The construction of $\mathfrak{L}_\bullet$ is obtained inductively by choosing $\mathfrak{L}_0$ to be the free cdgl generated by an MC element, $\mathfrak{L}_1$ as the LS interval and $\mathfrak{L}_2$ to be the "model of the triangle" of Proposition 5.14. Along the way we will prove many interesting properties of this cosimplicial cdgl and give an explicit formula for the differential in the model $\mathfrak{L}_3$ of the tetrahedron.

We finish the chapter by noticing that MC elements are ubiquitous, not only in $\mathfrak{L}_n$ but more generally, in the "model" of any finite simplicial complex. Moreover, we see that these MC elements can be chosen to be invariant under actions of any given subgroup of automorphisms of the 1-skeleton of the given simplicial complex.

6.1 The main result

In this short section, we give a precise and detailed statement of our main objective. A very helpful preliminary consists in setting and simplifying the notation used:

Recall from Section 1.1.3 the cosimplicial object

$$C_*(\Delta^\bullet) = \{C_*(\Delta^n)\}_{n\geq 0}$$

in the category of chain complexes, whose cofaces and codegeneracies,

$$\delta^i\colon C_p(\Delta^{n-1}) \longrightarrow C_p(\Delta^n) \quad \text{and} \quad \sigma^i\colon C_p(\Delta^{n+1}) \longrightarrow C_p(\Delta^n),$$

for $i = 0, \dots, n$, are explicitly given in formula (1.10). The differential d on each $C_*(\Delta^n)$ is the usual one, given in formula (1.7).

Consider the desuspension $s^{-1}C_*(\Delta^\bullet)$, which is trivially again a cosimplicial chain complex whose cofaces and codegeneracies,

$$s^{-1}\circ\delta^i\circ s\colon s^{-1}C_p(\Delta^{n-1}) \longrightarrow s^{-1}C_p(\Delta^n)$$

and

$$s^{-1}\circ\sigma^i\circ s\colon s^{-1}C_p(\Delta^{n+1}) \longrightarrow s^{-1}C_p(\Delta^n),$$

are simply the ones induced by those of $C_*(\Delta^\bullet)$. From now on, and to simplify notation we write

$$s^{-1}\Delta^\bullet = s^{-1}C_*(\Delta^\bullet), \quad \delta^i = s^{-1}\delta^i s, \quad \sigma^i = s^{-1}\sigma^i s,$$

and denote by

$$a_{i_0\dots i_p} \in s^{-1}\Delta^n$$

the generator of degree $p-1$ corresponding to the p-simplex $(i_0,\dots,i_p)\in\Delta^n_p$.

For each $n \geq 0$, we denote by d_1 the differential $s^{-1} \circ d \circ s$ on the chain complex $s^{-1}\Delta^n$ which is defined on generators by:

$$d_1 a_{i_0 \dots i_p} = \sum_{j=0}^{p} (-1)^j a_{i_0 \dots \widehat{i_j} \dots i_p}.$$

It is convenient to keep in mind that, in view of (1.9),

$$s^{-1}\Delta^\bullet = s^{-1}N(\underline{\Delta}^\bullet),$$

where the latter is the desuspension of the non-degenerate simplicial chains on the $\underline{\Delta}^\bullet$.

We then consider the family

$$\{(\widehat{\mathbb{L}}(s^{-1}\Delta^n), d_1)\}_{n\geq 0} \tag{6.1}$$

of free cdgl's in which the differential on each of them is simply induced by d_1, and thus, it only has a linear part. For each $i = 0, \dots, n$, the coface δ^i and the codegeneracy σ^i can be extended "bracket-wise" to obtain morphisms

$$\delta^i \colon \widehat{\mathbb{L}}(s^{-1}\Delta^{n-1}) \longrightarrow \widehat{\mathbb{L}}(s^{-1}\Delta^n), \qquad \sigma^i \colon \widehat{\mathbb{L}}(s^{-1}\Delta^{n+1}) \longrightarrow \widehat{\mathbb{L}}(s^{-1}\Delta^n),$$

which are given, once again as in (1.10), by

$$\delta^i(a_{i_0 \dots i_p}) = a_{j_0 \dots j_p} \quad \text{with} \quad j_k = \begin{cases} i_k, & \text{if } i_k < i, \\ i_k + 1, & \text{if } i_k \geq i, \end{cases} \tag{6.2}$$

$$\sigma^i(a_{i_0 \dots i_p}) = 0 \quad \text{if} \quad (i, i+1) \subset \{i_0, \dots, i_p\}, \qquad \text{and otherwise,}$$

$$\sigma^i(a_{i_0, \dots, i_p}) = a_{j_0 \dots j_p} \quad \text{with} \quad j_k = \begin{cases} i_k, & \text{if } i_k \leq i, \\ i_k - 1, & \text{if } i_k > i. \end{cases} \tag{6.3}$$

With the above notation, this chapter is mainly devoted to the proof of the following:

Theorem 6.1. *For each $n \geq 0$, there is a differential d on $\widehat{\mathbb{L}}(s^{-1}\Delta^n)$ such that:*

(1) *The generators $a_0, \dots, a_n$ are* MC *elements.*
(2) *The linear part of d is precisely d_1.*
(3) *Every coface $\delta^i \colon (\widehat{\mathbb{L}}(s^{-1}\Delta^{n-1}), d) \to (\widehat{\mathbb{L}}(s^{-1}\Delta^n), d)$ is a cdgl morphism.*
(4) *Every codegeneracy $\sigma^i \colon (\widehat{\mathbb{L}}(s^{-1}\Delta^{n+1}), d) \to (\widehat{\mathbb{L}}(s^{-1}\Delta^n), d)$ is a cdgl morphism.*

In particular, the family

$$\mathfrak{L}_\bullet = \{(\widehat{\mathbb{L}}(s^{-1}\Delta^n), d)\}_{n\geq 0},$$

is a cosimplicial cdgl.

Moreover, if $= \{(\widehat{\mathbb{L}}(s^{-1}\Delta^n), d)\}_{n\geq 0}$ *is a family of cdgl's satisfying properties* (1), (2), *and* (3), *then each* $(\widehat{\mathbb{L}}(s^{-1}\Delta^n), d)$ *is unique up to cdgl isomorphism.*

We outline the strategy for its proof. First, we inductively prove the existence and uniqueness of such a family of cdgl's satisfying (1), (2), and (3) of the statement. Then, we show that, among this class of isomorphic families there is one for which the codegeneracies are also cdgl morphisms. For this, a particular equivariant behaviour of the differential is needed.

6.2 Inductive sequences of models of the standard simplices

We begin by considering particular families of cdgl's satisfying (1), (2), and (3) of Theorem 6.1, plus an inductive property on their differentials.

Definition 6.2. A *sequence of models* (of the standard simplices) is a family of cdgl's of the form

$$\{(\widehat{\mathbb{L}}(s^{-1}\Delta^n), d)\}_{n\geq 0}$$

satisfying:

(1) For each cdgl in this family the generators $a_0, \dots, a_n$ are MC elements.

(2) For each cdgl in this family the linear part of d is d_1.

(3) For each $i = 0, \dots, n$, the coface $\delta^i\colon (\widehat{\mathbb{L}}(s^{-1}\Delta^{n-1}), d) \hookrightarrow (\widehat{\mathbb{L}}(s^{-1}\Delta^n), d)$ is a cdgl morphism.

A sequence of models is called *inductive* if the following additional property is satisfied:

(4) For each $n \geq 2$,

$$da_{0\dots n} = \operatorname{ad}_{a_0} a_{0\dots n} + \Phi, \quad \text{with} \quad \Phi \in \widehat{\mathbb{L}}(s^{-1}\dot{\Delta}^n).$$

In other words, for the perturbed differential,

$$d_{a_0} a_{0\dots n} \in \widehat{\mathbb{L}}(s^{-1}\dot{\Delta}^n).$$

For this definition, and in what follows, the following observation is necessary and should be taken into account.

Remark 6.3. Recall from Section 1.1.1 that $\dot{\Delta}^n$ and Λ^n_i, for $i = 0, \dots, n$, are the sub-simplicial complexes of Δ^n consisting of its boundary and the ith horn. Hence, $s^{-1}C_*(\dot{\Delta}^n)$ and $s^{-1}C_*(\dot{\Delta}^n)$ are sub-chain complexes of $s^{-1}\Delta^n$ which, by consistency on the notation, are denoted henceforth by $s^{-1}\dot{\Delta}^n$ and $s^{-1}\Lambda^n_i$, respectively. In particular $\widehat{\mathbb{L}}(s^{-1}\dot{\Delta}^n)$ and $\widehat{\mathbb{L}}(s^{-1}\Lambda^n_i)$ are free sub-cgl's of $\widehat{\mathbb{L}}(s^{-1}\Delta^n)$.

Moreover, for any element $(\widehat{\mathbb{L}}(s^{-1}\Delta^n), d)$ of a sequence of models, the differential restricts to $\widehat{\mathbb{L}}(s^{-1}\dot{\Delta}^n)$ and $\widehat{\mathbb{L}}(s^{-1}\Lambda^n_i)$ and thus the cdgl's

$$(\widehat{\mathbb{L}}(s^{-1}\dot{\Delta}^n), d) \quad \text{and} \quad (\widehat{\mathbb{L}}(s^{-1}\Lambda^n_i, d)$$

are well defined. Indeed, note that $s^{-1}\dot{\Delta}^n$ is the sub-chain complex of $s^{-1}\Delta^n$ generated by the images of all cofaces $\delta^k\colon s^{-1}\Delta^{n-1} \hookrightarrow s^{-1}\Delta^n$, $k = 0, \dots, n$. Therefore, in view of Definition 6.2(3), given a generator v of $s^{-1}\dot{\Delta}^n$,

$$dv = d\delta^k(v') = \delta^k(dv') \in \mathbb{L}(s^{-1}\dot{\Delta}^n).$$

The same applies to $s^{-1}\Lambda^n_i$, as this is the sub-chain complex of $s^{-1}\Delta^n$ generated by the images of all cofaces δ^k, for $k \neq i$.

More generally, let $\{(\widehat{\mathbb{L}}(s^{-1}\Delta^n), d)\}_{n\geq 0}$ be a sequence of models and let K be a finite simplicial complex. Then K is a sub-simplicial complex of Δ^n for some n and we denote by $s^{-1}K = s^{-1}C_*(K)$ the corresponding sub-simplicial chain complex of $s^{-1}\Delta^n$. Then, the same argument implies that d preserves $\widehat{\mathbb{L}}(s^{-1}K)$ and defines a sub-cdgl

$$(\widehat{\mathbb{L}}(s^{-1}K), d) \subset (\widehat{\mathbb{L}}(s^{-1}\Delta^n), d).$$

All of the above also trivially applies when we replace d by a perturbed differential d_a with a a Maurer–Cartan element of any of the chosen sub-cdgl's.

Before proving the existence and uniqueness of an (inductive) sequence of models, we need the following preliminary results which are of particular importance.

Proposition 6.4. *Let $\{(\widehat{\mathbb{L}}(s^{-1}\Delta^n), d)\}_{n\geq 0}$ be a sequence of models. Then, for any $n \geq 2$, any $i = 0, \dots, n$, and any Maurer–Cartan element a of $(\widehat{\mathbb{L}}(s^{-1}\Lambda^n_i), d)$:*

(i) $H(\widehat{\mathbb{L}}(s^{-1}\Delta^n), d_a) = H(\widehat{\mathbb{L}}(s^{-1}\Lambda^n_i), d_a) = 0.$

(ii) *There is a cdgl isomorphism*

$$(\widehat{\mathbb{L}}(s^{-1}\Delta^n), d_a) \cong (\widehat{\mathbb{L}}(s^{-1}\Lambda^n_i), d_a) \,\widehat{\amalg}\, \widehat{\mathbb{L}}(u, du) \quad \textit{with} \quad |u| = n-1.$$

Notice that, in particular, the above applies also to the original differential d choosing $a = 0$.

Proof. (i) Observe that the linear part of d_a obviously coincides with d_1 independently of the chosen a, and consider the inclusions,

$$(\widehat{\mathbb{L}}(s^{-1}\Delta^n), d_a) \overset{j}{\hookleftarrow} (\mathbb{L}(a), d_a) \overset{k}{\hookrightarrow} (\widehat{\mathbb{L}}(s^{-1}\Lambda^n_i), d_a),$$

whose linear parts,

$$(s^{-1}\Delta^n, d_1) \xleftarrow{j_1} (\mathbb{Q}a, 0) \xhookrightarrow{k_1} (s^{-1}\Lambda^n_i, d_1)$$

are clearly quasi-isomorphisms. Then, by Proposition 3.12, j and k are also quasi-isomorphisms. But $H(\mathbb{L}(a), d_a) = 0$ and the statement follows.

(ii) Extend the inclusion

$$(\widehat{\mathbb{L}}(s^{-1}\Lambda^n_i), d_a) \hookrightarrow (\widehat{\mathbb{L}}(s^{-1}\Delta^n), d_a)$$

to a cdgl morphism

$$f\colon (\widehat{\mathbb{L}}(s^{-1}\Lambda^n_i), d_a)\,\widehat{\amalg}\,\widehat{\mathbb{L}}(u, du) \longrightarrow (\widehat{\mathbb{L}}(s^{-1}\Delta^n), d_a) \tag{6.4}$$

by setting $f(u) = a_{0,\dots,n}$ and thus $f(du) = d_a a_{0,\dots,n}$. Since the linear part f_1 is again an isomorphism, by Proposition 3.12, f is an isomorphism. □

From this result we obtain two immediate but important consequences:

Corollary 6.5. *For any cdgl L, any $i = 0, \dots, n$, and any* MC *element of $(\widehat{\mathbb{L}}(\Lambda^n_i), d)$, the map*

$$\operatorname{Hom}_{\mathbf{cdgl}}\big((\widehat{\mathbb{L}}(s^{-1}\Delta^n), d), L\big) \xrightarrow{\cong} \operatorname{Hom}_{\mathbf{cdgl}}\big((\widehat{\mathbb{L}}(\Lambda^n_i), d_a), L\big) \times L_n,$$
$$f \longmapsto \big(f|_{\widehat{\mathbb{L}}(\Lambda^n_i)}, f(a_{o,\dots,n})\big),$$

is a bijection. In particular, given $x \in L_n$, any cdgl morphism $f\colon \mathfrak{L}_{\Lambda^n_i} \to L$ extends to a cdgl morphism $f\colon \mathfrak{L}_{\Delta^n} \to L$ such that $f(a_{0\dots n}) = x$.

Proof. Indeed, using 6.4(ii), we have:

$$\begin{aligned}
&\operatorname{Hom}_{\mathbf{cdgl}}\big((\widehat{\mathbb{L}}(s^{-1}\Delta^n), d), L\big)\\
&\quad\cong \operatorname{Hom}_{\mathbf{cdgl}}\big((\widehat{\mathbb{L}}(s^{-1}\Lambda^n_i), d_a)\,\widehat{\amalg}\,\widehat{\mathbb{L}}(u, du), L\big)\\
&\quad\cong \operatorname{Hom}_{\mathbf{cdgl}}\big((\widehat{\mathbb{L}}(s^{-1}\Lambda^n_i), d_a), L\big) \times \operatorname{Hom}_{\mathbf{cdgl}}(\widehat{\mathbb{L}}(u, du), L).
\end{aligned}$$

But, trivially, the map,

$$\operatorname{Hom}_{\mathbf{cdgl}}(\widehat{\mathbb{L}}(u, du), L) \xrightarrow{\cong} L_n, \qquad f \longmapsto x,$$

is bijective. The composition of these identifications is the bijective map of the statement. □

The second consequence of Proposition 6.4 reads as follows.

Corollary 6.6. *Let $\{(\widehat{\mathbb{L}}(s^{-1}\Delta^n), d)\}_{n\geq 0}$ be an inductive sequence of models. Then, for any $n \geq 2$,*

$$H(\widehat{\mathbb{L}}(s^{-1}\dot{\Delta}^n), d_{a_0}) = \mathbb{L}(\Omega), \quad \textit{with} \quad \Omega = d_{a_0} a_{0\dots n}.$$

Proof. Observe that the isomorphism f in (6.4) restricts to an isomorphism,

$$f\colon (\widehat{\mathbb{L}}(s^{-1}\Lambda^n_i), d_{a_0})\,\widehat{\amalg}\,(\widehat{\mathbb{L}}(du), 0) \xrightarrow{\cong} (\widehat{\mathbb{L}}(s^{-1}\dot{\Delta}^n), d_{a_0}).$$

By Proposition 6.4(ii), $H(\widehat{\mathbb{L}}(s^{-1}\Lambda^n_i), d_{a_0}) = 0$ and therefore,

$$H(f)\colon \mathbb{L}(du) \xrightarrow{\cong} H(\widehat{\mathbb{L}}(s^{-1}\dot{\Delta}^n), d_{a_0})$$

is the isomorphism which maps du to $[d_{a_0} a_{0,\dots,n}]$. □

We now show the existence and uniqueness of sequences of models.

Theorem 6.7.

(i) *There exist inductive sequences of models.*

(ii) *Let* $\{(\widehat{\mathbb{L}}(s^{-1}\Delta^n), d)\}_{n\geq 0}$ *and* $\{(\widehat{\mathbb{L}}(s^{-1}\Delta^n), d')\}_{n\geq 0}$ *be two sequences of models. Then, there are cdgl isomorphisms,*

$$\varphi_n\colon (\widehat{\mathbb{L}}(s^{-1}\Delta^n), d) \xrightarrow{\cong} (\widehat{\mathbb{L}}(s^{-1}\Delta^n), d'), \quad \textit{for} \quad n \geq 0,$$

which commute with the coface morphisms δ^i, *for* $i = 0, \dots, n$,

$$\begin{array}{ccc} (\widehat{\mathbb{L}}(s^{-1}\Delta^n), d) & \xrightarrow[\cong]{\varphi_n} & (\widehat{\mathbb{L}}(s^{-1}\Delta^n), d') \\ \delta^i \big\uparrow & & \big\uparrow \delta^i \\ (\widehat{\mathbb{L}}(s^{-1}\Delta^{n-1}), d) & \xrightarrow[\cong]{\varphi_{n-1}} & (\widehat{\mathbb{L}}(s^{-1}\Delta^{n-1}), d') \end{array} \tag{6.5}$$

and such that $\operatorname{Im}(\varphi_n - \mathrm{id}) \subset \widehat{\mathbb{L}}^{\geq 2}(s^{-1}\Delta^n)$.

Proof. (i) We construct by induction on n a inductive sequence of models

$$\{\mathfrak{L}_n\}_{n\geq 0} = \{(\widehat{\mathbb{L}}(s^{-1}\Delta^n), d)\}_{n\geq 0}.$$

For $n = 0$, let

$$\mathfrak{L}_0 = (\mathbb{L}(a_0), d),$$

where a_0 is a Maurer–Cartan element.

For $n = 1$, let

$$\mathfrak{L}_1 = (\widehat{\mathbb{L}}(a_0, a_1, a_{01}), d)$$

be the LS interval. Note that a_0, a_1 are MC elements, $d_1 a_{01} = a_1 - a_0$, and trivially $\delta_0, \delta_1 \colon \mathfrak{L}_0 \to \mathfrak{L}_1$ are cdgl morphisms. Hence (1), (2) and (3) of Definition 6.2 are satisfied.

For $n = 2$, let

$$\mathfrak{L}_2 = (\widehat{\mathbb{L}}(a_0, a_1, a_2, a_{01}, a_{12}, a_{02}, a_{012}), d)$$

be the cdgl of Proposition 5.14 in which $(\widehat{\mathbb{L}}(a_0,a_1,a_{01}),d)$, $(\widehat{\mathbb{L}}(a_1,a_2,a_{12}),d)$ and $(\widehat{\mathbb{L}}(a_0,a_2,a_{02}),d)$ are LS intervals and

$$da_{012} = a_{01} * a_{12} * a_{02}^{-1} - [a_0, a_{012}], \quad \text{or equivalently,} \quad d_{a_0}a_{012} = a_{01} * a_{12} * a_{02}^{-1}.$$

By definition, this cdgl verifies (1), (2), (3) and (4) of Definition 6.2.

Let $n > 3$ and suppose we have a sequence of cdgl's $\{(\widehat{\mathbb{L}}(s^{-1}\Delta^m),d)\}_{m<n}$ such that each of them satisfies all conditions of Definition 6.2. We will define a differential d on $\widehat{\mathbb{L}}(s^{-1}\Delta^n)$ so that $(\widehat{\mathbb{L}}(s^{-1}\Delta^n),d)$ satisfies the same properties. Note that condition (3) of Definition 6.2 determines the differential d on each generator of $s^{-1}\dot{\Delta}^n$. Indeed, as this chain complex is generated by the images of the cofaces, given $v \in s^{-1}\dot{\Delta}^n$ write it as $v = \delta^i(v')$ for some i, so that

$$dv = d\delta^i(v') = \delta^i(dv')$$

is already defined. It suffices then to define $da_{0\dots n}$ so that it squares to 0 and satisfies properties (2) and (4).

For this, note that, by condition (4) applied to $(\widehat{\mathbb{L}}(s^{-1}\Delta^{n-1}),d)$,

$$d_{a_0}a_{0\dots n-1} \in \widehat{\mathbb{L}}(s^{-1}\dot{\Delta}^{n-1}).$$

Also, observe that $\widehat{\mathbb{L}}(s^{-1}\dot{\Delta}^{n-1})$ is mapped injectively to $\widehat{\mathbb{L}}(s^{-1}\Lambda^n_n)$ by δ^n. Moreover, by (3),

$$\delta^n(d_{a_0}a_{0\dots n-1}) = d_{a_0}\delta^n(a_{0\dots n-1}) = d_{a_0}a_{0\dots n-1}.$$

Hence, as $H(\widehat{\mathbb{L}}(s^{-1}\Lambda^n_n),d_{a_0}) = 0$ by Proposition 6.4(i), there exists $\Gamma \in \widehat{\mathbb{L}}(s^{-1}\Lambda^n_n)$, of degree $n-2$, such that

$$d_{a_0}a_{0\dots n-1} = d_{a_0}\Gamma.$$

We set

$$d_{a_0}a_{0\dots n} = (-1)^n\left(a_{0\dots n-1} - \Gamma\right). \tag{6.6}$$

By definition, $d_{a_0}a_{0\dots n}$ is a cycle satisfying property (4) of Definition 6.2. To prove (2) denote by Γ_1 the linear part of Γ and let ω be the difference

$$\omega = (-1)^{n-1}\Gamma_1 - \sum_{i=0}^{n-1}(-1)^i a_{0\dots\widehat{i}\dots n}.$$

Since $d_1(\sum_{i=0}^n(-1)^i a_{0\dots\widehat{i}\dots n}) = d_1^2(a_{0\dots n}) = 0$ and $d_1\Gamma_1 = d_1(a_{0\dots n-1})$, it follows that,

$$d_1\omega = (-1)^{n-1}d_1(a_{0\dots n-1}) + (-1)^n d_1(a_{0\dots n-1}) = 0.$$

Hence, ω is a d_1-cycle of degree $n-2$ in $\widehat{\mathbb{L}}(s^{-1}\Lambda^n_n)$.

Note however that $H(\widehat{\mathbb{L}}(s^{-1}\Lambda^n_n),d_1) \cong \mathbb{L}(a_0)$ and thus, as $n-2>0$, ω must be a d_1-boundary. That is, there is a linear element γ of degree $n-1$ in $\widehat{\mathbb{L}}(s^{-1}\Lambda^n_n)$

such that $d_1\gamma = \omega$. But observe that $\widehat{\mathbb{L}}(s^{-1}\Lambda^n_n)$ is generated by elements of degree $\leq n-2$, and therefore $\gamma = 0$. Hence, $\omega = 0$ and

$$d_1 a_{0\ldots n} = (-1)^n a_{0\ldots n-1} + \sum_{i=0}^{n-1} (-1)^i a_{0\ldots\widehat{i}\ldots n-1}$$

as required.

(ii) We now prove the uniqueness up to isomorphism of sequences of models (not necessarily inductive). Obviously, there is only one choice for $(\widehat{\mathbb{L}}(s^{-1}\Delta^0), d)$ while $(\widehat{\mathbb{L}}(s^{-1}\Delta^1), d)$ is also unique by Theorem 5.5.

Let $\{(\widehat{\mathbb{L}}(s^{-1}\Delta^n), d)\}_{n\geq 0}$ and $\{(\widehat{\mathbb{L}}(s^{-1}\Delta^n), d')\}_{n\geq 0}$ be two sequences of models, fix $n \geq 2$ and suppose that, for $m < n$, we have isomorphisms,

$$\varphi_m\colon (\widehat{\mathbb{L}}(s^{-1}\Delta^m), d) \xrightarrow{\cong} (\widehat{\mathbb{L}}(s^{-1}\Delta^m), d')$$

making the diagram (6.5) commutative. We construct

$$\varphi_n\colon (\widehat{\mathbb{L}}(s^{-1}\Delta^n), d) \xrightarrow{\cong} (\widehat{\mathbb{L}}(s^{-1}\Delta^n), d')$$

with the same property. By condition (3) of Definition 6.2, φ_n is already determined on every generator of $\widehat{\mathbb{L}}(s^{-1}\Delta^n)$, except on $x = a_{0\ldots n}$, and its restriction to $(\widehat{\mathbb{L}}(s^{-1}\dot{\Delta}^n)$ is already an isomorphism. Note that there is no loss of generality in assuming that the sequence $\{(\widehat{\mathbb{L}}(s^{-1}\Delta^n), d)\}_{n\geq 0}$ is inductive, as we will show then that any other sequence of models is isomorphic to it.

Hence, as $d_{a_0}x \in \widehat{\mathbb{L}}(s^{-1}\dot{\Delta}^n)$, the induction hypothesis

$$\mathrm{Im}\,(\varphi_i - \mathrm{id}) \subset \widehat{\mathbb{L}}^{\geq 2}(s^{-1}\Delta^i) \quad \text{for } i < n,$$

implies that

$$\varphi_n d_{a_0}x - d_{a_0}x$$

is a decomposable element. Now since $d_1 = d_1'$ the element

$$u = \varphi_n d_{a_0}x - d'_{a_0}x$$

is also decomposable. Write

$$u = \sum_{p\geq 2} u_p, \quad \text{with} \quad u_p \in \mathbb{L}^p(s^{-1}\Delta^n).$$

As $d'_{a_0}u = 0$, it follows that $d_1'u_2 = 0$. Now, since $|u_r| \geq 0$ and $H(\widehat{\mathbb{L}}(s^{-1}\Delta^n), d_1) \cong \mathbb{L}(a_0)$, there is an element $v_2 \in \mathbb{L}^2(s^{-1}\Delta^n)$ such that $d_1v_2 = u_2$. Then,

$$u - d'_{a_0}v_2 \in \widehat{\mathbb{L}}^{>2}(s^{-1}\Delta^n).$$

The same argument is then applied to find $v_3 \in \mathbb{L}^3(s^{-1}\Delta^n)$ such that $u - d'_{a_0}v_2 - d'_{a_0}v_3 \in \widehat{\mathbb{L}}^{>3}(s^{-1}\Delta^n)$. In this way we construct a sequence of elements $v_q \in \mathbb{L}^q(s^{-1}\Delta^n)$, $q \geq 2$, such that

$$u = d'_{a_0}\Big(\sum_{q\geq 2} v_q\Big).$$

We define

$$\varphi_n(x) = x + \sum_{q\geq 2} v_q.$$

Then,

$$\varphi_n d_{a_0} x = d'_{a_0} x + u = d'_{a_0}\varphi_n x,$$

and hence, φ_n is trivially an isomorphism for which the condition $\operatorname{Im}(\varphi_n - \mathrm{id}) \subset \widehat{\mathbb{L}}^{\geq 2}(s^{-1}\Delta^n)$ is fulfilled. □

Property (4) of Definition 6.2 guarantees that for any element $(\widehat{\mathbb{L}}(s^{-1}\Delta^n), d)$, with $n \geq 2$, of a given inductive sequence of models,

$$d_{a_0}a_{0\ldots n} \in \widehat{\mathbb{L}}(s^{-1}\dot{\Delta}^n).$$

Moreover, by formula (6.6) in the proof of Theorem 6.7(i), we have been more specific and have shown that for $n \geq 3$ we may choose

$$d_{a_0}a_{0\ldots n} = (-1)^n(a_{0\ldots n-1} - \Gamma), \quad \text{with} \quad \Gamma \in \widehat{\mathbb{L}}(s^{-1}\Lambda^n_n).$$

We can go a step further and assume that the expression of $d_{a_0}a_{0\ldots n}$ involves none of the vertices $a_0, \ldots, a_n$. To simplify the notation, in what follows write

$$s^{-1}\widetilde{\Delta}^n = (s^{-1}\dot{\Delta}^n)_{\geq 0} \quad \text{and} \quad s^{-1}\widetilde{\Lambda}^n_n = (s^{-1}\Lambda^n_n)_{\geq 0}.$$

Proposition 6.8. *There exist inductive sequences of models* $= \{(\widehat{\mathbb{L}}(s^{-1}\Delta^n), d)\}_{n\geq 0}$ *such that, for* $n \geq 2$,

$$d_{a_0}(a_{0\ldots n}) \in \mathbb{L}(s^{-1}\widetilde{\Delta}^n).$$

Moreover, for $n \geq 3$ *we may choose*

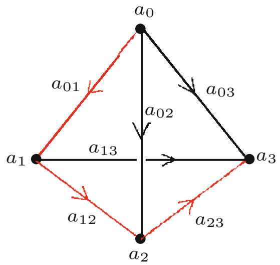
$$d_{a_0}a_{0\ldots n} = (-1)^n(a_{0\ldots n-1} - \Gamma), \quad \textit{with} \quad \Gamma \in \widehat{\mathbb{L}}(s^{-1}\widetilde{\Lambda}^n_n).$$

Proof. We require the following notation:

For each $n \geq 2$ and each $i = 0, \ldots, n$, define elements c_i, of degree 0 in $\widehat{\mathbb{L}}(s^{-1}\Delta^n)$ by

$$c_0 = 0 \quad \text{and} \quad c_i = a_{01} * a_{12} * \cdots * a_{i-1,i}.$$

That is, for $i \geq 1$, c_i is the path (see Section 5.5) joining a_0 and a_i by elements of the form $a_{r,r+1}$. For instance, in the picture below the red line represents c_3 in $\widehat{\mathbb{L}}(s^{-1}\Delta^3)$:



Also, for $0 \leq i < j \leq n$ define degree-0 elements,

$$b_{ij} = c_i * a_{ij} * c_j^{-1},$$

which are clearly loops at a_0. Observe that, by Proposition 4.13,

$$b_{ij} = e^{\mathrm{ad}_{c_i}}(a_{ij}),$$

and therefore, this definition can be extended to any generator $a_{i_0 \dots i_r}$ of $s^{-1}\widetilde{\Delta}^n$, $0 < r \leq n$, by

$$b_{i_0 \dots i_r} = e^{\mathrm{ad}_{c_{i_0}}}(a_{i_0 \dots i_r}).$$

Note however that, by construction, $b_{ij} = 0$ when $j = i + 1$. With this notation, and for any $n \geq 2$ define

$$E_n \subset \widehat{\mathbb{L}}(s^{-1}\widetilde{\Delta^n})$$

as the subspace generated by

$$b_{i_0 \dots, i_r} \quad \text{with} \quad 0 < r \leq n.$$

On the other hand,

$$F_n \subset \widehat{\mathbb{L}}(s^{-1}\widetilde{\Lambda}^n_n),$$

will denote the subspace generated by

$$b_{i_0 \dots, i_r} \quad \text{with} \quad 1 < r < n \quad \text{and} \quad \{i_0, \dots, i_r\} \neq \{0, 1, \dots, n-1\}.$$

The proposition will follow once we prove, by induction on $n \geq 2$, that

$$d_{a_0} a_{0 \dots n} \in \widehat{\mathbb{L}}(E_{n-1}). \tag{6.7}$$

By doing so we check that, for $n \geq 3$, the element $da_{0 \dots n}$ may also be chosen with the required special property.

For $n = 2$ this is immediate having chosen the model of the triangle $\mathfrak{L}_2$ in Proposition 5.14:

$$d_{a_0} a_{012} = a_{01} * a_{12} * a_{02}^{-1} = -(a_{02} * a_{12}^{-1} * a_{01}^{-1}) = -b_{02}.$$

As induction hypothesis, assume that

$$d_{a_0} a_{0\dots m} \in \widehat{\mathbb{L}}(E_{m-1}), \quad \text{for} \quad 2 \le m \le n-1.$$

With this assumption, we first see that F_n is d_{a_0}-stable, that is,

$$d_{a_0}(b_{i_0\dots i_r}) \in \widehat{\mathbb{L}}(F_n), \quad \text{for} \quad b_{i_0\dots i_r} \in F_n. \tag{6.8}$$

We use induction on r. For $r=2$:

$$\begin{aligned} d_{a_0} b_{ijk} &= e^{\mathrm{ad}_{c_i}} d_{a_i}(a_{ijk}) && \text{(Proposition 4.24)}\\ &= e^{\mathrm{ad}_{c_i}}(a_{ij} * a_{jk} * a_{ik}^{-1}) && \text{(Proposition 5.14)}\\ &= c_i * a_{ij} * a_{jk} * a_{ik}^{-1} * c_i^{-1} && \text{(Proposition 4.13)}\\ &= (c_i * a_{ij} * c_j^{-1}) * (c_j * a_{jk} * c_k^{-1}) * (c_k * a_{ik}^{-1} * c_i^{-1})\\ &= b_{ij} * b_{jk} * b_{ik}^{-1}. \end{aligned}$$

Next, given $a_{i_0\dots i_r} \in s^{-1}\widetilde{\Lambda}^n_n$, with $1<r$, and a path c from a_0 to a_i $\widehat{\mathbb{L}}(s^{-1}\Lambda^n_n)$, we check that

$$e^{\mathrm{ad}_c}(a_{i_0\dots i_r}) \in \widehat{\mathbb{L}}(F_n). \tag{6.9}$$

Indeed, write $c = c * c_{i_0}^{-1} * c_{i_0}$ and observe that, by Corollary 4.12,

$$e^{\mathrm{ad}_c}(a_{i_0\dots i_r}) = e^{\mathrm{ad}_{c*c_{i_0}^{-1}}} e^{\mathrm{ad}_{c_{i_0}}}(a_{i_0\dots i_r}) = e^{\mathrm{ad}_{c*c_{i_0}^{-1}}}(b_{i_0\dots i_r}). \tag{6.10}$$

But notice that $c * c_{i_0}^{-1}$ is a loop at a_0 and that every loop in $\widehat{\mathbb{L}}(s^{-1}\Lambda^n_n)$ based at a_0 is a Baker–Campbell–Hausdorff product of loops of the form b_{ij}. The assertion follows from expressing $c * c_{i_0}^{-1}$ as such a product, replacing it in (6.10), and repeatedly applying Corollary 4.12.

Assume the assertion (6.8) holds for $2<s<r$ with $r<n$ and let us prove it for r. For this, let

$$b_{i_0\dots i_r} = e^{\mathrm{ad}_{c_{i_0}}}(a_{i_0\dots i_r})$$

be a generator of F_n. By Proposition 4.24,

$$d_{a_0}(b_{i_0\dots i_r}) = e^{\mathrm{ad}_{c_{i_0}}} d_{a_{i_0}}(a_{i_0\dots i_r}).$$

Now, by the inductive hypothesis (6.7),

$$d_{a_0}(a_{0\dots r}) \in \widehat{\mathbb{L}}(E_r) \subset \widehat{\mathbb{L}}(F_n).$$

On the other hand, the element $a_{i_0\dots i_r}$ is the image of $a_{0\dots r}$ under a composition of coface operators δ^i. Since the δ^i are morphisms of cdgl's, we conclude that $d_{a_{i_0}}(a_{i_0\dots i_r})$ belongs to the complete free Lie algebra generated by two types of elements: degree-0 loops in $\widehat{\mathbb{L}}(s^{-1}\Lambda^n_n)$ based in a_{i_0}, and elements of the form $e^{\mathrm{ad}_c}(a_{j_0\dots j_s})$, where c is a path joining a_{i_0} to a_{j_0}.

It follows that $d_{a_0}(b_{i_0\dots i_r})$ is in the complete free Lie algebra generated by loops in $\widehat{\mathbb{L}}(s^{-1}\Lambda^n_n)$ based at a_0 (which are obviously in $\widehat{\mathbb{L}}(F_n)$), and elements of the form $e^{\mathrm{ad}_t}(a_{j_0\dots j_s})$, where t is a path joining a_0 to a_{j_0} (which are also in $\widehat{\mathbb{L}}(F_n)$ in view of (6.9)). Therefore, $d_{a_0}(b_{i_0\dots i_r}) \in \widehat{\mathbb{L}}(F_n)$. We have thus shown the d_{a_0}-stability of $\widehat{\mathbb{L}}(F_n)$.

Once this is assured we close the induction process by proving that

$$d_{a_0}a_{0\dots n} \in \widehat{\mathbb{L}}(E_{n-1}).$$

Recall that, for $n \geq 3$, we showed in (6.6) of the proof of Theorem 6.7(i) that there is a $\Gamma \in \widehat{\mathbb{L}}(s^{-1}\Lambda^n_n)$ with

$$d_{a_0}\Gamma = d_{a_0}(a_{0\dots n-1}) \quad \text{and} \quad d_{a_0}(a_{0\dots n}) = (-1)^n(a_{0\dots n-1} - \Gamma).$$

We will slightly modify the differential so that Γ can be chosen in $\widehat{\mathbb{L}}(F_n)$, which would finish the proof, because $F_n \subset \widehat{\mathbb{L}}(s^{-1}\widetilde{\Lambda}^n_n)$. By the induction hypothesis we first observe that

$$\alpha = d_{a_0}(a_{0\dots n-1}) \in \widehat{\mathbb{L}}(F_n).$$

Then write

$$\alpha = \sum_{p\geq 0} \alpha_p, \quad \text{with} \quad \alpha_p \in \mathbb{L}^p(F_n).$$

Then, α_1 is a d_1-cycle. But observe that, since $H_*(\widehat{\mathbb{L}}(s^{-1}\Lambda^n_n), d_1) = \mathbb{L}(a_0)$, we have $H(\widehat{\mathbb{L}}(F_n), d_1) = 0$. Hence, there is a $u_1 \in F_n$ with $d_1u_1 = \alpha_1$. In particular $\alpha - d_1u_1 \in \widehat{\mathbb{L}}^{>1}(F_n)$. Suppose, by induction, that we have constructed $u_1, \dots, u_p \in F_n$ such that

$$v = \alpha - d_{a_0}(u_1 + \dots + u_p) \in \widehat{\mathbb{L}}^{>r}(F_n).$$

Then, the component v_{r+1} in bracket length $r+1$ of v is a d_1-cycle and so a d_1-boundary, $v_{r+1} = d_1u_{r+1}$. Thus,

$$\alpha - d_{a_0}(u_1 + \dots + u_{r+1}) \in \widehat{\mathbb{L}}^{>r+1}(F_n).$$

We have just built an element

$$\Gamma = \sum_{p\geq 1} u_p \in \widehat{\mathbb{L}}(F_n)$$

so that $d_{a_0}\alpha = d_{a_0}\Gamma$, that is,

$$d_{a_0}(a_{0\dots n-1}) = d_{a_0}\Gamma.$$

To finish, set

$$d_{a_0}(a_{0\dots n}) = (-1)^n(a_{0\dots n-1} - \Gamma),$$

which is clearly in $\widehat{\mathbb{L}}(E_n)$. □

6.3 Sequences of equivariant models of the standard simplices

Note that, up to this point, Theorem 6.7 guarantees the existence and uniqueness of an inductive sequence of models $\{(\widehat{\mathbb{L}}(s^{-1}\Delta^n), d)\}_{n\geq 0}$ satisfying properties (1), (2), and (3) of Theorem 6.1. However, for such a sequence, the codegeneracies $\sigma^i\colon \widehat{\mathbb{L}}(s^{-1}\Delta^{n+1}) \to \widehat{\mathbb{L}}(s^{-1}\Delta^n)$ defined by formula (6.3) may not be cdgl morphisms, as they do not commute with d in general. We will now slightly modify the differential on each model of the sequence so that it satisfies a particular equivariance property which fixes the problem.

To do this, given a generator $a_{i_0\dots i_p}$ of $\widehat{\mathbb{L}}(s^{-1}\Delta^n)$, with $0\leq i_0<\cdots<i_p\leq n$, and a permutation $\sigma\in\Sigma_{p+1}$ we denote

$$a_{i_{\sigma(0)}\dots i_{\sigma(p)}} = \varepsilon_\sigma\, a_{i_0\dots i_p}, \tag{6.11}$$

where ε_σ is the signature of σ.

Definition 6.9. For any $n\geq 0$, consider the action of the symmetric group Σ_{n+1} on $s^{-1}\Delta^n$ defined on generators by

$$\sigma\, a_{i_0\dots i_p} = a_{\sigma(i_0)\dots\sigma(i_p)},$$

and then extended linearly to $s^{-1}\Delta^n$. Note that, if $\sigma(i_0)<\cdots<\sigma(i_p)$, then the element $a_{\sigma(i_0)\dots\sigma(i_p)}$ is well defined. Otherwise, use the convention in (6.11). In particular, this action extends the natural permutation action on the vertices, and for the top generator,

$$\sigma a_{0\dots n} = \varepsilon_\sigma a_{0\dots n}.$$

Finally, extend this action bracket-wise to a Σ_{n+1}-action on $\widehat{\mathbb{L}}(s^{-1}\Delta^n)$.

Definition 6.10. A sequence of models $\{(\widehat{\mathbb{L}}(s^{-1}\Delta^n), d)\}_{n\geq 0}$ is *equivariant* if, for each $n\geq 0$, $(\widehat{\mathbb{L}}(s^{-1}\Delta^n), d)$ is a Σ_{n+1}-cdgl, that is, if the differential d is equivariant for this action: $\sigma d = d\sigma$ for any $\sigma\in\Sigma_{n+1}$.

Theorem 6.11. *There exist equivariant sequences of models.*

In this proof we will use an auxiliary result. Notice that the linear differential d_1 on $\widehat{\mathbb{L}}(s^{-1}\Delta^n)$ is equivariant for the Σ_{n+1}-action, that is, $d_1\sigma=\sigma d_1$ for any permutation σ, and thus makes $(\widehat{\mathbb{L}}(s^{-1}\Delta^n), d_1)$ a Σ_{n+1}-cdgl.

Lemma 6.12. *Let $(V,d_1)\subset(\widehat{\mathbb{L}}(s^{-1}\Delta^n), d_1)$ be the chain complex of "ε-invariants":*

$$V=\{u\in\widehat{\mathbb{L}}(s^{-1}\Delta^n),\ \sigma u=\varepsilon_\sigma u,\ \sigma\in\Sigma_{n+1}\}.$$

Then,

$$H_q(V,d_1)=0 \quad \textit{for} \quad q\neq -1,-2.$$

Proof. Given $u \in V$,

$$\sigma d_1 u = d_1 \sigma u = d_1 \varepsilon_\sigma u = \varepsilon_\sigma d_1 u,$$

and thus (V, d_1) is in fact a sub-chain complex of $(\widehat{\mathbb{L}}(s^{-1}\Delta^n), d_1)$.

On the other hand, if $u \in \widehat{\mathbb{L}}(s^{-1}\Delta^n)$, then $\sum_{\sigma\in\Sigma_{n+1}} \varepsilon_\sigma \sigma u \in V$. Indeed, given $\tau \in \Sigma_{n+1}$,

$$\tau \sum_{\sigma\in\Sigma_{n+1}} \varepsilon_\sigma \sigma u = \sum_{\sigma\in\Sigma_{n+1}} \varepsilon_\sigma \tau\sigma u = \varepsilon_\tau \sum_{\sigma\in\Sigma_{n+1}} \varepsilon_{\tau\sigma} \tau\sigma u.$$

Hence, the map

$$(\widehat{\mathbb{L}}(s^{-1}\Delta^n), d_1) \longrightarrow (V, d_1), \qquad u \longmapsto \frac{1}{(n+1)!} \sum_{\sigma\in\Sigma_{n+1}} \varepsilon_\sigma \sigma u$$

is a retraction of the inclusion $(V,d_1) \hookrightarrow (\widehat{\mathbb{L}}(s^{-1}\Delta^n), d_1)$. This implies that $H(V, d_1)$ is a subspace of $H(\widehat{\mathbb{L}}(s^{-1}\Delta^n), d_1) \cong \mathbb{L}(a_0)$. □

Proof of Theorem 6.11. We inductively modify the differential on each element of a given sequence of models $\{(\widehat{\mathbb{L}}(s^{-1}\Delta^n), d)\}_{n\geq 0}$ to make it equivariant.

The model of Δ^0 is trivially equivariant and there is nothing to do.

For $n = 1$, the LS interval $\mathfrak{L}_1 = (\widehat{\mathbb{L}}(a_0, a_1, a_{01}), d)$ is also equivariant: let σ be the generator of Σ_2. By definition,

$$\sigma a_0 = a_1, \qquad \sigma a_1 = a_0 \qquad \text{and} \qquad \sigma a_{01} = -a_{01}.$$

Then,

$$d\sigma a_0 = da_1 = -\frac{1}{2}[a_1, a_1] = \sigma d a_0.$$

Analogously, $d\sigma a_1 = \sigma d a_1$. Finally,

$$\begin{aligned}
\sigma d a_{01} &= \sigma\left(\operatorname{ad}_{a_{01}}(a_1) + \frac{\operatorname{ad}_{a_{01}}}{e^{\operatorname{ad}_{a_{01}}} - 1}(a_1 - a_0)\right) \\
&= -\operatorname{ad}_{a_{01}}(a_0) + \frac{\operatorname{ad}_{-a_{01}}}{e^{\operatorname{ad}_{-a_{01}}} - 1}(a_0 - a_1) \\
&= -\left(\operatorname{ad}_{a_{01}}(a_0) + \frac{\operatorname{ad}_{-a_{01}}}{e^{\operatorname{ad}_{-a_{01}}} - 1}(a_1 - a_0)\right) \\
&\overset{(*)}{=} -da_{01} = d\sigma a_{01},
\end{aligned}$$

where the equality $(*)$ is given by formula (5.4).

As induction hypothesis, assume that the model $(\widehat{\mathbb{L}}(s^{-1}\Delta^m), d)$ of the sequence is equivariant for $m < n$ with $n \geq 2$.

We first check that the restriction of the differential in $(\widehat{\mathbb{L}}(s^{-1}\Delta^n), d)$ to $\widehat{\mathbb{L}}(s^{-1}\dot{\Delta}^n)$ is equivariant, that is,

$$d\sigma u = \sigma d u, \quad \text{for} \quad \sigma \in \Sigma_{n+1} \quad \text{and} \quad u \in s^{-1}\dot{\Delta}^n.$$

For this, it suffices to choose

$$u = \delta^i(v), \quad v \in s^{-1}\Delta^{n-1}, \quad i = 0, \dots, n,$$

and

$$\sigma = (q, q+1),$$

a transposition of adjacent elements. Indeed, every permutation is a composition of such transpositions and $s^{-1}\dot{\Delta}^n$ is the sub-chain complex of $s^{-1}\Delta^n$ generated by the images of all cofaces $\delta^i\colon s^{-1}\Delta^{n-1} \hookrightarrow s^{-1}\Delta^n$, $i = 0, \dots, n$.

Notice first that, if $i = q$, then $\sigma\delta^i = \delta^{i+1}$. Hence, using property (3) of Definition 6.2,

$$d\sigma u = d\sigma\delta^i(v) = d\delta^{i+1}(v) = \delta^{i+1}(dv) = \sigma\delta^i(dv)\sigma d\delta^i(v) = \sigma du.$$

If $i = q+1$, then $\sigma\delta^i = \delta^{i-1}$ and an analogous argument applies.

Assume then that $i \notin \{q, q+1\}$. Then, one easily checks that there is a permutation $\sigma' \in \Sigma_n$ such that $\sigma \circ \delta^i = \delta^i \circ \sigma'$. Hence, since $(\widehat{\mathbb{L}}(s^{-1}\Delta^{n-1}), d)$ is an equivariant model,

$$d\sigma u = d\sigma\delta^i(v) = d\delta^i(\sigma' v) = \delta^i(d\sigma' v) = \delta^i(\sigma' dv) = \sigma\delta^i(dv) = \sigma d\delta^i(v) = \sigma du.$$

Therefore, we only need to modify $d(a_{0\dots n})$ in an equivariant way. Write $x = a_{0\dots n}$ and

$$dx = \sum_{p\geq 1} d_p x, \quad \text{with} \quad d_p x \in \mathbb{L}^p(s^{-1}\Delta^n).$$

We suppose by induction on q that, for $1 \leq p < q$, the elements $d_q x$ have been defined, are equivariant and satisfy

$$d\left(\sum_{p=1}^{q-1} d_p x\right) \in \widehat{\mathbb{L}}^{\geq q}(\Delta^n).$$

To simplify in the notation, write $u = \sum_{p=1}^{q-1} d_p x$. Since $\sigma x = \varepsilon_\sigma x$ for any permutation σ, it follows that $\sigma du = \varepsilon_\sigma du$. In particular, if $(du)_q$ denotes the component of du in $\mathbb{L}^q(\Delta^n)$, it also satisfies

$$\sigma(du)_q = \varepsilon_\sigma(du)_q.$$

Hence, $(du)_q$ is in the space V of Lemma 6.12 and is trivially a d_1-cycle. By Lemma 6.12, there exists an element $\omega_q \in V$ such that

$$(du)_q = -d_1\omega_q.$$

We set $d_q x = \omega_q$, so that $d(\sum_{p=1}^{q} d_p x) \in \mathbb{L}^{\geq q+1}(s^{-1}\Delta^n)$. By construction, $dx = \sum_{p\geq 1} d_p x$ is equivariant and $d^2 x = 0$. $\square$

Remark that the model of the triangle given in Proposition 5.14 is not equivariant. Nevertheless, we will provide such an equivariant model in Proposition 6.21.

6.4 The cosimplicial cdgl $\mathfrak{L}_\bullet$

Let

$$\mathfrak{L}_\bullet = \{\mathfrak{L}_n\}_{n\geq 0} = \{(\widehat{\mathbb{L}}(s^{-1}\Delta^n), d)\}_{n\geq 0}$$

be a sequence of equivariant models. We finish the proof of Theorem 6.1 by showing property (4). That is, for any $n \geq 0$ and any $i = 0, \dots, n$, the codegeneracy

$$\sigma^i\colon (\widehat{\mathbb{L}}(s^{-1}\Delta^{n+1}), d) \longrightarrow (\widehat{\mathbb{L}}(s^{-1}\Delta^n), d)$$

defined by formula (6.3) is a cdgl morphism.

Let $a_{i_0\dots i_p}$ be a generator of $\mathfrak{L}_n$ and assume first that $\sigma^i(a_{i_0}) < \cdots < \sigma^i(a_{i_p})$. Then, $a_{i_0\dots i_p} \in \operatorname{Im}\delta^i$, i.e., $a_{i_0\dots i_p} = \delta^i(x)$. The identity $\sigma^i\delta^i = \mathrm{id}_{s^{-1}\Delta^n}$ yields the equalities

$$d\sigma^i a_{i_0\dots i_p} = d\sigma^i\delta^i(x) = dx = \sigma^i\delta^i(dx) = \sigma^i d\delta^i(x) = \sigma^i da_{i_0\dots i_p}.$$

Otherwise, see formula (6.3), the sequence $i_0, \dots, i_p$ contains consecutive elements i and $i+1$ and therefore, by definition,

$$\sigma^i(a_{i_0\dots i_p}) = 0.$$

Denote by $\tau = (i, i+1)$ the transposition of those adjacent elements and observe that $\sigma^i\tau = \sigma^i$. Thus,

$$\sigma^i da_{i_0\dots i_p} = \sigma^i\tau da_{i_0\dots i_p} = \sigma^i d\tau a_{i_0\dots i_p} = -\sigma^i d(a_{i_0\dots i_p}),$$

whence,

$$\sigma^i d(a_{i_0\dots i_p}) = 0 = d\sigma^i(a_{i_0\dots i_p}).$$

This completes the proof of Theorem 6.1. □

We finish by observing that, if $\mathfrak{L}_\bullet = \{\mathfrak{L}_n\}_{n\geq 0}$ is a (not necessarily equivariant) inductive sequence of models, we still have a cosimplicial cdgl structure on it, with the same cofaces but with slightly different codegeneracies.

Theorem 6.13. *Any inductive sequence $\{\mathfrak{L}_n\}_{n\geq 0}$ of models admits a cosimplicial cdgl structure for which the cofaces are the maps δ^i's and each codegeneracy*

$$\widetilde{\sigma}^i\colon \mathfrak{L}_{n+1} \longrightarrow \mathfrak{L}_n$$

satisfies

$$\operatorname{Im}(\widetilde{\sigma}^i - \sigma^i) \subset \widehat{\mathbb{L}}^{\geq 2}(s^{-1}\Delta^n).$$

Proof. Let $\{\mathfrak{L}'_n\}_{n\geq 0}$ be an equivariant sequence of models whose existence is guaranteed by Theorem 6.11. By the uniqueness property of Theorem 6.7, we have isomorphisms

$$\varphi_n\colon \mathfrak{L}_n \xrightarrow{\cong} \mathfrak{L}'_n,$$

which commute with the coface morphisms δ^i, for $i = 0, \dots, n$,

$$\begin{array}{ccc} \mathfrak{L}_n & \xrightarrow[\cong]{\varphi_n} & \mathfrak{L}'_n \\ {\scriptstyle \delta^i}\uparrow & & \uparrow{\scriptstyle \delta^i} \\ \mathfrak{L}_{n-1} & \xrightarrow[\cong]{\varphi_{n-1}} & \mathfrak{L}'_{n-1} \end{array}$$

We then define the codegeneracies by

$$\widetilde{\sigma}^i \colon \mathfrak{L}_{n+1} \longrightarrow \mathfrak{L}_n, \qquad \widetilde{\sigma}^i = \varphi_n^{-1} \sigma^i \varphi_{n+1},$$

with σ_i as in (6.3). It is an easy exercise to check that the cosimplicial identities are satisfied. Finally, as $\operatorname{Im}(\varphi_n - \mathrm{id}) \in \widehat{\mathbb{L}}^{\geq 2}(s^{-1}\Delta^n)$ for each $n \geq 0$, it follows that $\operatorname{Im}(\widetilde{\sigma}^i - \sigma^i) \subset \widehat{\mathbb{L}}^{\geq 2}(s^{-1}\Delta^n)$. □

6.5 An explicit model for the tetrahedron

In this section we build an explicit model

$$\mathfrak{L}_3 = (\widehat{\mathbb{L}}(s^{-1}\Delta^3), d)$$

of the tetrahedron. This example highlights its geometrical approach and the combinatorial issues that one encounters when producing a closed formula for the differential in a sequence of inductive models.

We remark here that the existence of explicit formulae for the differential in the model of Δ^n remains an open problem for $n \geq 4$. Although the differentials in $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are unambiguously determined by Theorem 5.5 and Proposition 5.16, no such uniqueness results hold for $s^{-1}\Delta^n$ when $n \geq 3$.

Given L a cdgl, we first introduce an operation $x \maltese y$ on the elements of L_1, defined via the property

$$d(x \maltese y) = dx * dy\,.$$

Definition 6.14. Let L be a cdgl and $e_1, \dots, e_n \in L_1$. Consider the contractible cdgl

$$L' = \widehat{\mathbb{L}}(u_i, du_i)_{1 \leq i \leq n}, \quad \text{with} \quad |u_i| = 1,$$

and the cdgl morphism defined by

$$\gamma \colon L' \longrightarrow L, \qquad \gamma(u_i) = e_i, \quad \gamma(du_i) = de_i.$$

Since $H(L') = 0$, there is an element ω with $d\omega = du_1 * \cdots * du_n$. Define,

$$e_1 \maltese e_2 \maltese \cdots \maltese e_n = \gamma(\omega).$$

Then,

$$d(e_1 \maltese e_2 \maltese \cdots \maltese e_n) = de_1 * de_2 * \cdots * de_n. \tag{6.12}$$

Remark 6.15. Note that the linear part of $e_1 \maltese \cdots \maltese e_n$ is precisely $\sum_{i=1}^n e_i$.

Observe also that $\maltese$ is uniquely defined by property (6.12), up to a boundary. That is, if

$$d(e_1 \maltese e_2 \maltese \cdots \maltese e_n) = d(e_1 \maltese' e_2 \maltese' \cdots \maltese' e_n) = de_1 * de_2 * \cdots * de_n,$$

then,

$$e_1 \maltese e_2 \maltese \cdots \maltese e_n - e_1 \maltese' e_2 \maltese' \cdots \maltese' e_n$$

is always a boundary.

In the same way $\maltese$ is also associative up to a boundary: for any elements e_1, e_2, e_3 in L_1, the difference

$$e_1 \maltese (e_2 \maltese e_3) - (e_1 \maltese e_2) \maltese e_3$$

is always a boundary.

By the above observation, there are many choices for the element $e_1 \maltese e_2 \maltese \cdots \cdots \maltese e_n$. The most heuristic one is to simply consider the element $de_1 * de_2 * \cdots * de_n$ and replace one and only one of the de_i's appearing in the bracketing of each summand by the corresponding e_i.

A canonical construction can be done as follows. With L' as in Definition 6.14, consider the adjoint action of UL' on L', denoted simply by juxtaposition. Let E be the UL'_0-module in L' generated by the elements

$$x_{ij} = [u_i, du_j] - [u_j, du_i], \quad \text{with} \quad 1 \leq i < j \leq n.$$

Then

$$L'_1 = dL'_2 \oplus E$$

and therefore, there is a unique choice for $e_1 \maltese e_2 \maltese \cdots \maltese e_n$ as a sum $\sum_{i<j} \omega_{ij} x_{ij} \in E$, with $\omega_{ij} \in UL'_0$.

A natural and more explicit construction of the operation $\maltese$ goes as follows. Recall from [116, Corollary 3.24] that, given a cdgl L and elements $a, b \in L_0$, the component of $a * b$ containing only one a on the bracketing of each of its summands is

$$z = \frac{\mathrm{ad}_b}{e^{\mathrm{ad}_b} - 1}(a)\,.$$

Next, given $z \in \widehat{\mathbb{L}}(a,b)$, let $z\frac{\partial}{\partial b}$ denote the derivation of $\widehat{\mathbb{L}}(a,b)$ that is zero on a and maps b to z. Then, following [116, Corollary 3.25], we have

$$a * b = \exp\left(z\frac{\partial}{\partial b}\right)(b) = b + z + \sum_{n \geq 2} \frac{1}{n!}\left(z\frac{\partial}{\partial b}\right)^n(b)\,.$$

Then, given $x, y \in L_1$, write $dx = a$, $dy = b$ and define,

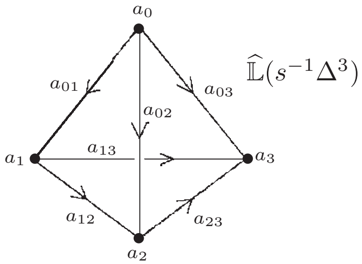
$$x \maltese y = y + \sum_{n\geq 1} \frac{1}{n!}\left(z\frac{\partial}{\partial b}\right)^{n-1}(z'), \quad \text{with} \quad z' = \frac{\operatorname{ad}_b}{e^{\operatorname{ad}_b} - 1}(x)\,.$$

By the preceding discussion, $d(x \maltese y) = dx * dy$.

The reader may choose his/her favourite form of the operation $\maltese$ for the following.

Proposition 6.16. *A model $\mathfrak{L}_3 = (\widehat{\mathbb{L}}(s^{-1}\Delta^3), d)$ for the tetrahedron is given by:*

$$d_{a_0}(a_{0123}) = e^{\operatorname{ad}_{a_{01}}} a_{123} - (a_{012} \maltese a_{023} \maltese a_{013}^{-1}).$$



Observe that giving the differential of the top simplex a_{0123} completely determines the model. In fact, in view of the compatibility condition (3) of Definition 6.2, the differential is already defined in any face of smaller dimension.

Proof. The expressions of differential on the 2-faces $d_{a_0}a_{012}$, $d_{a_0}a_{023}$ and $d_{a_0}a_{013}$ are already explicitly given in Proposition 5.14 as the BCH product of the edges bounding the corresponding face. With this, the arithmetic properties of the BCH product, and the properties of the $\maltese$ product, the following is a simple computation:

$$\begin{aligned} d_{a_0}(a_{012} \maltese a_{023} \maltese a_{013}^{-1}) &= d_{a_0}a_{012} * d_{a_0}a_{023} * d_{a_0}a_{013}^{-1} \\ &= a_{01} * a_{12} * a_{23} * a_{13}^{-1} * a_{01}^{-1}. \end{aligned}$$

On the other hand, a direct computation, using first Proposition 4.24 and then Proposition 4.13, gives

$$\begin{aligned} d_{a_0}\left(e^{\operatorname{ad}_{a_{01}}} a_{123}\right) = e^{\operatorname{ad}_{a_{01}}} d_{a_1}(a_{123}) &= e^{\operatorname{ad}_{a_{01}}}(a_{12} * a_{23} * a_{13}^{-1}) \\ &= a_{01} * a_{12} * a_{23} * a_{13}^{-1} * a_{01}^{-1}. \end{aligned}$$

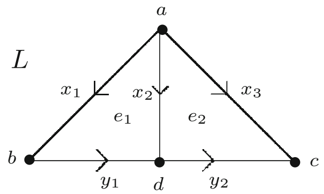
Hence, $e^{\operatorname{ad}_{a_{01}}} a_{123} - (a_{012} \maltese a_{023} \maltese a_{013}^{-1})$ is a cycle and we set,

$$d_{a_0}a_{0123} = e^{\operatorname{ad}_{a_{01}}} a_{123} - (a_{012} \maltese a_{023} \maltese a_{013}^{-1}).$$

Finally, by Remark 6.15, $da_{0123} = d_{a_0}a_{0123} - [a_0, a_{0123}]$ has the right linear part and, by definition, $\mathfrak{L}_3$ is inductive. $\square$

The reader may take advantage of the geometrical process leading to this model which we now explain.

First, we subdivide the triangle as follows: let L be the cdgl given by two glued models of Δ^2 as in Proposition 5.14, and whose generators are denoted as in the following picture.



Proposition 6.17. *The map* $\varphi\colon (\widehat{\mathbb{L}}(s^{-1}\Delta^2), d) \longrightarrow L$ *given by,*

$$\begin{aligned} &\varphi(a_0) = a, \quad \varphi(a_1) = b, \quad \varphi(a_2) = c,\\ &\varphi(a_{01}) = x_1,\ \varphi(a_{02}) = x_3,\ \varphi(a_{12}) = y_1 * y_2,\\ &\varphi(a_{012}) = e_1 \maltese e_2, \end{aligned}$$

is a cdgl morphism.

Proof. Trivially, the differential commutes with φ in any vertex and in the 1-simplices a_{01}, a_{02}. In view of Theorem 5.13, d commutes with φ also in a_{12}. For the top face a_{012} this is also straightforward:

$$\begin{aligned} d_a\varphi(a_{012}) &= d_a(e_1 \maltese e_2) = d_a e_1 * d_a e_2 = (x_1 * y_1 * x_2^{-1}) * (x_2 * y_2 * x_3^{-1})\\ &= x_1 * y_1 * y_2 * x_3^{-1} = \varphi(d_{a_0} a_{012}). \end{aligned}$$ □

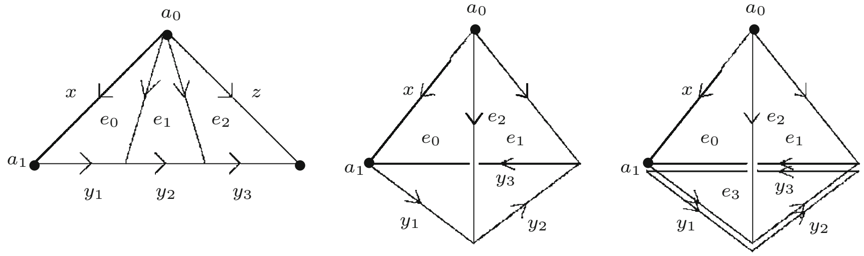
Once we know how to subdivide the triangle, do it twice to obtain the cdgl on the left in the picture below, in which

$$d_{a_0}(e_0 \maltese e_1 \maltese e_2) = x * y_1 * y_2 * y_3 * z^{-1},$$

that is, it shapes the outer triangle. Then, identify $x = -z$ to obtain the model of the horn $(\widehat{\mathbb{L}}(s^{-1}\Lambda^3_3), d)$ as the middle cdgl in the picture. In this cdgl,

$$d_{a_0}(e_0 \maltese e_1 \maltese e_2) = x * y_1 * y_2 * y_3 * x^{-1},$$

which equals $e^{\operatorname{ad}_x}(y_1 * y_2 * y_3)$ by Proposition 4.13.



Finally, build $(\mathbb{L}(s^{-1}\Delta^3), d)$ by attaching the bottom triangle e_3 as in the right cdgl in the picture, for which $d_{a_1}e_3 = y_1 * y_2 * y_3$, or equivalently, by Proposition 4.24, $d_{a_0}e^{\mathrm{ad}_x}(e_3) = e^{\mathrm{ad}_x}(y_1 * y_2 * y_3)$. In this way the d_{a_0}-cycle

$$e_0 \maltese e_1 \maltese e_2 - e^{\mathrm{ad}_x}(e_3)$$

of Proposition 6.16 arises naturally.

Other approaches can be also followed, for instance gluing appropriately two subdivided triangles, to find different forms of the differential of the maximal face a_{0123}. However, by the uniqueness property in Theorem 6.7, the obtained models will all be isomorphic.

6.6 Symmetric MC elements of simplicial complexes

Recall that any finite simplicial complex K can be considered as a subcomplex of Δ^n for some n. Hence, recall also from Remark 6.3, the differential on an inductive model $\mathfrak{L}_n = (\widehat{\mathbb{L}}(s^{-1}\Delta^n), d)$ preserves $\widehat{\mathbb{L}}(s^{-1}K)$ and defines a sub-cdgl

$$(\widehat{\mathbb{L}}(s^{-1}K), d) \subset (\widehat{\mathbb{L}}(s^{-1}\Delta^n), d).$$

We denote it by $\mathfrak{L}_K$ and call it, for the time being, a *model of* K. This terminology will be made precise in the next chapter.

In this final section we show that in the model of a given finite simplicial complex, Maurer–Cartan elements are ubiquitous and are not restricted to the vertices. Moreover, we will see that they can be chosen to be invariant under the action of automorphisms of the 1-skeleton of the given simplicial complex.

Let Γ be a connected, simple, non-oriented, finite graph. In particular it is a finite simplicial complex and, as previously remarked, we can consider the cdgl

$$\mathfrak{L}_\Gamma = (\widehat{\mathbb{L}}(s^{-1}\Gamma), d)$$

which is always a sub-cdgl of some $(\widehat{\mathbb{L}}(s^{-1}\Delta^n), d)$.

Fix a subgroup $G \subset \mathrm{aut}(\Gamma)$ of the group of automorphisms of Γ. Extend the action of G on Γ linearly first to the chain complex $s^{-1}\Gamma$, and then bracket-wise to $\mathfrak{L}_\Gamma$. Finally, fix a vertex in Γ which corresponds to an MC element a in $\mathfrak{L}_\Gamma$.

Theorem 6.18. *The cdgl* $\mathfrak{L}_\Gamma$ *contains an* MC *element that is G-invariant, and whose linear part is*

$$\frac{1}{|G|}\sum_{g\in G} ga.$$

We warn the reader that in what follows the lower grading corresponds to the bracket length grading and not to the usual homological degree.

We start by an auxiliary result of a general nature:

Lemma 6.19. *Let* $L = (\widehat{\mathbb{L}}(V), d)$ *be a free cdgl, let* $b \in \mathrm{MC}(L)$ *and let* $x \in L_0$ *be such that for some* $n \geq 2$,

$$(x \mathcal{G} b)_{<n} = b_{<n}.$$

Then,

$$(dx)_{<n} = [x, b]_{<n} \qquad \text{and} \qquad (x \mathcal{G} b)_n = b_n + [x, b]_n - (dx)_n.$$

Proof. It is convenient to keep in mind the original definition of the gauge action,

$$x \mathcal{G} b = e^{\mathrm{ad}_x}(b) - \frac{e^{\mathrm{ad}_x} - 1}{\mathrm{ad}_x}(dx).$$

We prove the first identity by showing inductively that

$$(dx)_r = [x, b]_r, \quad \text{for} \quad r < n. \tag{6.13}$$

For $r = 1$ this means that $(dx)_1 = 0$, which is obvious since, on the one hand

$$(x \mathcal{G} b)_1 = b_1 - (dx)_1,$$

while, by hypothesis, $(x \mathcal{G} b)_1 = b_1$.

Suppose (6.13) holds for $s < r$, and write

$$(dx)_r = [x, b]_r + \alpha_r.$$

Hence, we have

$$(x \mathcal{G} b)_r = \big(e^{\mathrm{ad}_x}(b)\big)_r - \left(\frac{e^{\mathrm{ad}_x} - 1}{\mathrm{ad}_x}[x, b]\right)_r - \left(\frac{e^{\mathrm{ad}_x} - 1}{\mathrm{ad}_x}(\alpha_r)\right)_r = b_r - \alpha_r.$$

But again, in view of the hypothesis, $(x \mathcal{G} b)_r = b_r$. Hence, $\alpha_r = 0$ and

$$(dx)_r - [x, b]_r.$$

Finally, the same computation yields

$$(x \mathcal{G} b)_n = b_n - \alpha_n = b_n + [x, b]_n - (dx)_n\,. \qquad \square$$

Proof of Theorem 6.18. Recall from Definition 5.17 that a path in a cdgl L joining two Maurer–Cartan elements u and v is an element $y \in L_0$ such that $y \mathcal{G} v = u$.

Now, given $g \in G$, ga is (the desuspension of) another vertex of Γ and therefore, it is an MC element of $\mathfrak{L}_\Gamma$. As Γ is connected, we can join a with ga by a path: take a sequence of adjacent edges of Γ going from a to ga which are LS intervals

$$\widehat{\mathbb{L}}(a, a_1, x_1),\, \widehat{\mathbb{L}}(a_1, a_2, x_2), \dots, \widehat{\mathbb{L}}(a_{k-1}, ga, x_k)$$

inside $\mathfrak{L}_\Gamma$. Then (see Section 5.5), $x_1 * \cdots * x_k$ is a path joining a with ga.

For each $g \in G$ fix such a path v_g from a to ga, and let

$$y = -\frac{1}{|G|} \sum_{g \neq 1} v_g.$$

Then,

$$(y \mathcal{G} a)_1 = a + \frac{1}{|G|} \sum_{g \neq 1} (ga - a) = \frac{1}{|G|} \sum_{g \in G} ga,$$

which is obviously invariant under G.

Defining $b(1) = (y \mathcal{G} a)$, we construct a sequence of MC elements $b(1), \dots, b(n), \dots$ of $\mathfrak{L}_\Gamma$ satisfying:

$$b(n)_{<n} = b(n-1)_{<n} \quad \text{and} \quad \big(gb(n)\big)_{\leq n} = b(n)_{\leq n}, \quad \text{for} \quad g \in G.$$

Suppose by induction we have constructed

$$b = b(n-1)$$

such that $(gb)_{<n} = b_{<n}$ for all $g \in G$. Then for each g we denote by x_g a path from b to gb. By Lemma 6.19,

$$(dx_g)_{<n} = [x_g, b]_{<n} \qquad \text{and} \qquad (x_g \mathcal{G} b)_n = b_n + [x_g, b]_n - (dx_g)_n \,.$$

We define

$$x = -\frac{1}{|G|} \sum_{g \neq 1} x_g \qquad \text{and} \qquad b(n) = x \mathcal{G} b \,.$$

Then, $(dx)_1 = 0$ and $(dx)_r = [x, b]_r$ for $r < n$. Hence,

$$(x \mathcal{G} b)_r = b_r + \sum_{i=1}^{r-1} \frac{1}{i!} (\mathrm{ad}_x^i(b))_r - \sum_{i=0}^{r-1} \frac{1}{(i+1)!} (\mathrm{ad}_x^i(dx))_r = b_r.$$

This shows that for $r < n$, $b(n)_r = b_r$, and $(x \mathcal{G} b)_r = b_r$ is G-invariant. It remains to verify that the component of $b(n)$ in Lie bracket of length n is G-invariant. For this write

$$(dx)_n = [x, b]_n + \alpha_n,$$

and using formula (5.4) note also that,

$$(dx_g)_n = (gb)_n - b_n + [x_g, b]_n.$$

Then,

$$\begin{aligned}
b(n) &= (x \mathcal{G} b)_n \\
&= b_n + \sum_{i=1}^{n-1} \frac{1}{i!}(\mathrm{ad}_x^i\, b)_n - \sum_{i=0}^{n-2} \frac{1}{(i+1)!}(\mathrm{ad}_x^i(dx))_n \\
&= b_n - \alpha_n = b_n + [x,b]_n - (dx)_n \\
&= b_n - \frac{1}{|G|}\sum_{g\neq 1}([x_g,b]_n - (dx_g)_n) \\
&= b_n + \frac{1}{|G|}\sum_{g\neq 1}((gb)_n - b_n) = \frac{1}{|G|}\sum_{g\in G}(gb)_n\,,
\end{aligned}$$

which is trivially G-invariant.

Finally, by construction,

$$\sum_{n\geq 1} b(n)_n \in \mathfrak{L}_\Gamma$$

is an invariant MC element that has the prescribed linear part. □

Corollary 6.20. *For each $n \geq 1$, the cdgl $\mathfrak{L}_n$ has an* MC *element invariant under the action of the symmetric group Σ_{n+1}. In particular, its linear part is the barycentre of Δ^n.*

Proof. As usual, $a_0,\dots,a_n$ denote the generators of $\mathfrak{L}_n$ of degree -1. Observe that Σ_{n+1} acts by automorphisms on the graph given by the 1-skeleton of Δ^n. By Theorem 6.18, there is an MC element b whose linear part is

$$\frac{1}{|\Sigma_{n+1}|}\sum_{\sigma\in\Sigma_{n+1}} \sigma a_0.$$

Denote by E_r the set of permutations $\sigma\in\Sigma_{n+1}$ such that $\sigma a_0 = a_r$. Then E_0 is the stabilizer of a_0. For each r choose an element σ_r such that $\sigma_r a_0 = a_r$. This choice induces a bijection $\varphi_r\colon E_0 \to E_r$ given by $\varphi_r(h) = \sigma_r h$. Then

$$\frac{1}{|\Sigma_{n+1}|}\sum_{\sigma}\sigma a_0 = \frac{1}{|\Sigma_{n+1}|}\sum_{r=0}^{n}\Big(\sum_{\sigma\in E_r}\sigma a_0\Big) = \frac{|E_0|}{|\Sigma_{n+1}|}\sum_{r=0}^{n} a_r = \frac{1}{n+1}\sum_{r=0}^{n} a_r,$$

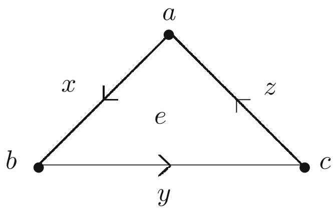
which is the barycenter of Δ^n. □

As an application, we deduce a model for the triangle which is symmetric by the action of Σ^3. Recall from Proposition 5.14 that an inductive model of Δ^2 is

$$\mathfrak{L}_2 = \widehat{\mathbb{L}}(a,b,c,x,y,z,e),$$

where a, b, c are MC elements, $\widehat{\mathbb{L}}(a, b, x)$, $\widehat{\mathbb{L}}(b, c, y)$ and $\widehat{\mathbb{L}}(c, a, z)$ are LS intervals and

$$d_a e = x * y * z.$$



Observe that, in this case, the graph given by the 1-skeleton of the triangle is precisely $\dot{\Delta}^2$. The group Σ_3 is generated by $\tau = (a, b, c)$ and $\sigma = (a, b)$. The action of Σ_3 on $\dot{\Delta}^2$ is defined by

$$\tau(x) = y, \qquad \tau(y) = z, \qquad \tau(z) = x,$$

and

$$\sigma(x) = -x, \quad \sigma(y) = -z, \quad \sigma(z) = -y.$$

By Theorem 6.18, choose an MC element $u \in \mathfrak{L}_{\dot{\Delta}^2}$ invariant under the natural action of Σ_3, and denote by ω a path in $\mathfrak{L}_\Gamma$ joining a to u. That is, $\omega \,\mathcal{G}\, u = a$. Then,

$$(x * \tau(\omega) * (-\omega)) \,\mathcal{G}\, a = a \qquad \text{and} \qquad (x * \sigma\omega * (-\omega)) \,\mathcal{G}\, a = a.$$

By Proposition 4.20,

$$d_a(x * \tau(\omega) * (-\omega)) = 0 = d_a(x * \sigma\omega * (-\omega)).$$

However, in view of Corollary 6.6, $H(\mathfrak{L}_{\dot{\Delta}^2}, d_a) = \mathbb{L}(x * y * z)$. Hence, there are integers λ and μ such that

$$x * \tau(\omega) * (-\omega) = \lambda(x * y * z) \qquad \text{and} \qquad x * \sigma(\omega) * (-\omega) = \mu(x * y * z).$$

Denote

$$e' = e^{\mathrm{ad}_{-\omega}}(e).$$

Proposition 6.21. *With the above notation,*

$$(\widehat{\mathbb{L}}(a, b, c, x, y, z, e'), d)$$

is a symmetric model for the triangle.

Proof. We have to verify that

$$\tau(e') = e' \qquad \text{and} \qquad \sigma(e') = -e'.$$

By Proposition 4.24, $d(e') = e^{\mathrm{ad}_{-\omega}}(d_a e)$. Then by Proposition 4.13, this can be written as

$$d(e') = (-\omega) * x * y * z * \omega.$$

Now a simple computation shows that the extensions of τ and σ to e' commute with the differentials. $\square$

Remark 6.22. Notice that the invariant MC element produced by Theorem 6.18 is not unique. For instance, consider again the 1-skeleton $\dot{\Delta}^2$ of the triangle and let u be a Σ_3-invariant MC element in $\mathfrak{L}_{\dot{\Delta}^2}$. Consider the degree 0-element

$$\gamma = [[x, y] + [y, z] + [z, x], x + y + z],$$

which is trivially Σ_3-invariant. Therefore, $\gamma \mathcal{G} u$ is also an MC-invariant element.

Chapter 7

The Model and Realization Functors

The cosimplicial cdgl $\mathfrak{L}_\bullet$ leads naturally to the definition of cdgl models for any simplicial set and to a geometrical realization for any given cdgl. Indeed, the *global model* $\mathfrak{L}_X$ of a simplicial set X is the cdgl obtained as the colimit,

$$\mathfrak{L}_X = \varinjlim_{\sigma \in X} \mathfrak{L}_{|\sigma|}.$$

On the other hand, the realization of a given cdgl L is the simplicial set,

$$\langle L \rangle = \operatorname{Hom}_{\mathbf{cdgl}}(\mathfrak{L}_\bullet, L).$$

These constructions constitute a pair of adjoint functors

$$\mathbf{sset} \underset{\langle \cdot \rangle}{\overset{\mathfrak{L}}{\rightleftarrows}} \mathbf{cdgl}\ ,$$

which are crucial objects in our theory.

The first thing to notice is that this pair provides the precise Eckmann–Hilton dual of the classical Sullivan approach to rational homotopy theory. Indeed, the role of $\mathfrak{L}_\bullet$ in this context is the Eckmann–Hilton analogue of the simplicial cdga $\Omega_\bullet$ (see Section 1.2.1), which is identified with the piecewise linear forms on the standard simplices. Recall that the Sullivan realization of a given cdga A is

$$\langle A \rangle^{\mathrm{S}} = \operatorname{Hom}_{\mathbf{cdga}}(A, \Omega_\bullet).$$

After presenting in detail the global model and realization functors, we prove some of their first features, arising immediately from adjointness. In particular we notice that

$$\mathfrak{L}_X = (\widehat{\mathbb{L}}(s^{-1}X), d)$$

U. Buijs et al., *Lie Models in Topology*, Progress in Mathematics 335,
https://doi.org/10.1007/978-3-030-54430-0_7

is a free cdgl generated by the desuspension of the non-degenerate simplicial chains of X. On the other hand, $\langle L\rangle$ is always a Kan complex.

Then, we describe in detail the path components and the homotopy groups of the realization of a given cdgl L. As for $\pi_0\langle L\rangle$, we see that the number of path components of $\langle L\rangle$ is given by (the cardinality of) $\widetilde{\mathrm{MC}}(L)$. Moreover, for any $z\in\widetilde{\mathrm{MC}}(L)$, the corresponding path component has the homotopy type of the realization $\langle L^z\rangle$ of the component of L at z (see Definition 4.6), which is already a connected cdgl. In particular,

$$\langle L\rangle \simeq \amalg_{z\in\widetilde{\mathrm{MC}}(L)} \langle L^z\rangle.$$

On the other hand, we find the first strong evidence that the realization functor extends the classical Quillen realization: for any connected cdgl L and any $n\geq 1$,

$$\pi_n\langle L\rangle = H_{n-1}(L),$$

where, for $n=1$, $H_0(L)$ is considered with the group structure given by the Baker–Campbell–Hausdorff product.

Dually, we describe the homological behaviour of the global model $\mathfrak{L}_X$ of a given simplicial set X. The first thing we notice is its acyclicity,

$$H(\mathfrak{L}_X)=0.$$

The reader may be either confused or surprised by this fact, since $\mathfrak{L}_X$ is supposed to contain all the rational information about the homotopy type of X. However, to recover this information for any of the path components of X, we have to look at the component of the global model at a given Maurer–Cartan element.

In fact, the homology of the component $\mathfrak{L}_X^a$ of $\mathfrak{L}_X$ at a non-trivial MC element a drastically changes and is far from being trivial. Recall that $\mathfrak{L}_X^a$ is a connected sub-cdgl of the perturbed cdgl $(\mathfrak{L}_X,d_a)$. We see that the inclusion $\mathfrak{L}_X^a \stackrel{\simeq}{\hookrightarrow} (L,d_a)$ is a quasi-isomorphism, and thus

$$H(\mathfrak{L}_X,d_a)=H(\mathfrak{L}_X^a).$$

Under this point of view, the fact that $H(\mathfrak{L}_X)=0$ amounts to saying that the realization of the component of $\mathfrak{L}_X$ at the MC element 0 has the homotopy type of a point.

We support the geometrical flavour of all these results and ideas by showing also that the cardinality of the $\widetilde{\mathrm{MC}}$ set of $\mathfrak{L}_X$ coincides with the number of path connected components of X plus one.

7.1 Introducing the global model and realization functors. Adjointness

Having constructed the cosimplicial cdgl $\mathfrak{L}_\bullet$, one is tempted to define right away, for each simplicial set X, its global Lie model $\mathfrak{L}_X$ as

$$\mathfrak{L}_X = \varinjlim_{\sigma \in X} \mathfrak{L}_{|\sigma|}.$$

However, to make the recipe functorial, we need to be more precise. For this, recall that given functors

$$F\colon \mathcal{A} \longrightarrow \mathcal{C} \qquad \text{and} \qquad G\colon \mathcal{A} \longrightarrow \mathcal{B},$$

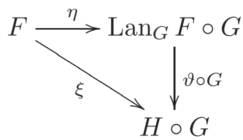
the (left) *Kan extension of F along G* consists of a functor, usually denoted by

$$\mathrm{Lan}_G F\colon \mathcal{B} \longrightarrow \mathcal{C},$$

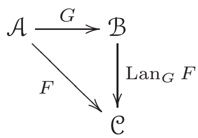
and a natural transformation

$$\eta\colon F \longrightarrow \mathrm{Lan}_G F \circ G$$

which is universal in the following sense: given any other functor $H\colon \mathcal{B} \to \mathcal{C}$ and any other natural transformation $\xi\colon F \to H \circ G$, there exists a unique natural transformation $\vartheta\colon \mathrm{Lan}_G F \to H$ such that the following diagram of natural transformations commutes:



Even though F is different in general from $\mathrm{Lan}_G F \circ G$, by abuse of notation we write



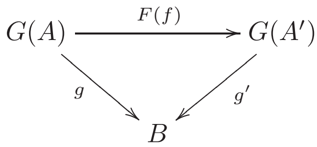
If $\mathcal{A}$ is small and $\mathcal{C}$ is cocomplete, then the left Kan extension of F along G exists. Moreover, for any object $B \in \mathcal{B}$, $\mathrm{Lan}_G F(B)$ is the colimit of the functor

$$G \downarrow B \longrightarrow \mathcal{C} \tag{7.1}$$

which assigns to each $(A, g) \in G \downarrow B$ the object $F(A)$. That is,

$$\mathrm{Lan}_G F(B) = \varinjlim_{(A,g) \in G\downarrow B} F(A).$$

As usual, $G \downarrow B$ denotes the *comma category* whose objects are pairs (A, g) where $A \in \mathcal{A}$ and $g\colon G(A) \to B$. On the other hand, $\mathrm{Hom}_{G\downarrow B}\big((A,g),(A',g')\big)$ is the subset of $\mathrm{Hom}_{\mathcal{A}}(A, A')$ consisting of morphisms $f\colon A \to A'$ such that the following diagram commutes:



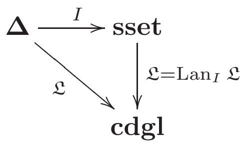
Next, let $\mathfrak{L}_\bullet$ be the cosimplicial cdgl introduced in Section 6.4. The cosimplicial character of this object amounts to saying that

$$\mathfrak{L}\colon \mathbf{\Delta} \longrightarrow \mathbf{cdgl}, \quad \text{defined by} \quad \mathfrak{L}[n] = \mathfrak{L}_n,$$

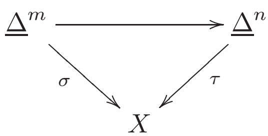
is a functor. Consider also the functor

$$I\colon \mathbf{\Delta} \longrightarrow \mathbf{sset}, \quad \text{defined by} \quad I[n] = \underline{\Delta}^n.$$

Definition 7.1. The *global model* functor $\mathfrak{L}\colon \mathbf{sset} \to \mathbf{cdgl}$ is defined as the left Kan extension of $\mathfrak{L}$ along I,



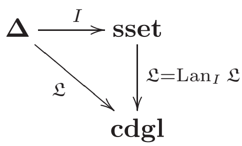
As $\mathbf{\Delta}$ is small and $\mathbf{cdgl}$ is cocomplete, this functor exists. Moreover, given a simplicial set X, any object $([n], \sigma) \in I \downarrow X$ corresponds to a simplicial map $\sigma\colon \underline{\Delta}^n \to X$, which in turn defines an n-simplex $\sigma \in X_n$. In this way, a morphism $([m], \sigma) \to ([n], \tau)$ in $I \downarrow X$ corresponds to a commutative diagram of simplicial maps



It follows that

$$\mathfrak{L}_X = \varinjlim_{\sigma \in X} \mathfrak{L}_{|\sigma|}. \tag{7.2}$$

Proposition 7.2. *The diagram*



is commutative. In other terms, for each $n \geq 0$,

$$\mathfrak{L}_{\underline{\Delta}^n} = \mathfrak{L}_n.$$

Proof. Observe that the comma category $I \downarrow \underline{\Delta}^n$ has as final object $([n], \mathrm{id}_{\underline{\Delta}^n})$. Therefore, the limit of the functor (7.1) in this particular case is attained by its image on the final object. That is, $\mathfrak{L}_{\underline{\Delta}^n} = \mathfrak{L}_n$. □

On the other hand, the right adjoint to the global model functor is given by the following:

Definition 7.3. The *realization functor* $\langle \cdot \rangle \colon \mathbf{cdgl} \to \mathbf{sset}$ assigns to each cdgl L the simplicial set

$$\langle L \rangle = \mathrm{Hom}_{\mathbf{cdgl}}(\mathfrak{L}_\bullet, L),$$

where the faces and degeneracies are given respectively by,

$$\begin{aligned} d_i \colon \langle L \rangle_{n+1} &\longrightarrow \langle L \rangle_n, \qquad d_i = \mathrm{Hom}_{\mathbf{cdgl}}(\delta^i, L), \\ s_j \colon \langle L \rangle_n &\longrightarrow \langle L \rangle_{n+1}, \quad s_j = \mathrm{Hom}_{\mathbf{cdgl}}(\sigma^j, L), \end{aligned}$$

for $n \geq 0$, $i = 0, \ldots, n+1$ and $j = 0, \ldots, n$.

Theorem 7.4. *The global model functor is left adjoint to the realization functor,*

$$\mathbf{sset} \underset{\langle \cdot \rangle}{\overset{\mathfrak{L}}{\rightleftarrows}} \mathbf{cdgl}\,.$$

That is, for any simplicial set X *and any cdgl* L*, there is a bijection,*

$$\mathrm{Hom}_{\mathbf{sset}}(X, \langle L \rangle) \cong \mathrm{Hom}_{\mathbf{cdgl}}(\mathfrak{L}_X, L).$$

Proof. The result follows from formulas (1.3), (1.4), and (7.2), together with classical properties of commutation of limits and colimits with Hom functors:

$$\begin{aligned} \mathrm{Hom}_{\mathbf{cdgl}}(\mathfrak{L}_X, L) &= \mathrm{Hom}_{\mathbf{cdgl}}(\varinjlim_{\sigma \in X} \mathfrak{L}_{|\sigma|}, L) = \varprojlim_{\sigma \in X} \mathrm{Hom}_{\mathbf{cdgl}}(\mathfrak{L}_{|\sigma|}, L) \\ &= \varprojlim_{\sigma \in X} \langle L \rangle_{|\sigma|} = \varprojlim_{\sigma \in X} \mathrm{Hom}_{\mathbf{sset}}(\underline{\Delta}^{|\sigma|}, \langle L \rangle) \\ &= \mathrm{Hom}_{\mathbf{sset}}(\varinjlim_{\sigma \in X} \underline{\Delta}^{|\sigma|}, \langle L \rangle) = \mathrm{Hom}_{\mathbf{sset}}(X, \langle L \rangle). \end{aligned}$$

□

7.2 First features of the global model and realization functors

Essentially as corollaries of Theorem 7.4, we deduce in this section the first properties satisfied by the global model and realization functors.

We start with the following, trivially satisfied by any other pair of adjoint functors:

Proposition 7.5. *The functors $\mathfrak{L}$ and $\langle\cdot\rangle$ preserve inductive and projective limits respectively.* □

In particular:

Corollary 7.6. *Let $\{X_i\}_{i\in I}$ be the path components of the simplicial set X. Then,*

$$\mathfrak{L}_X = \widehat{\amalg}_{i\in I}\mathfrak{L}_{X_i}.$$ □

Another consequence is the following: recall, for instance from [81, Lemma 3.1.4], the refinement of formula (1.4) by which any simplicial set X can be recovered from its non-degenerate simplices as

$$X = \varinjlim_{\sigma\in X'} \underline{\Delta}^{|\sigma|}.$$

Here, X' denotes the category that has the non-degenerate simplices of X as objects and the composition of faces as morphisms. Hence, applying directly Propositions 7.2 and 7.5 we get:

Corollary 7.7. *For any simplicial set X,*

$$\mathfrak{L}_X = \varinjlim_{\sigma\in X'} \mathfrak{L}_{|\sigma|}.$$ □

Based on this corollary we now give an explicit description of the cdgl structure of the global model of any simplicial set X as a free cdgl and characterize its differential. As usual, let

$$N_*(X) = C_*(X)/D_*(X)$$

denote the chain complex of non-degenerate simplicial chains on X obtained by taking the quotient of the simplicial chains by the degenerate ones. To avoid excessive notation, we often simply denote the desuspension of this complex by

$$s^{-1}X = s^{-1}N_*(X).$$

Note that this is in accordance with the notation previously used in Chapter 6 where, for any $n \geq 0$, we denoted

$$s^{-1}\Delta^n = s^{-1}C_*(\Delta^n) = s^{-1}N_*(\underline{\Delta}^n).$$

Proposition 7.8. *For any simplicial set X,*

$$\mathfrak{L}_X = \widehat{\mathbb{L}}(s^{-1}X)$$

as a graded Lie algebra. Moreover, the differential d on $\mathfrak{L}_X$ is completely determined by the following:

- *The* 0*-simplices are Maurer–Cartan elements.*
- *The linear part* ∂_1 *of* ∂ *is the desuspension of the differential in* $N_*(X)$*.*
- *If* $j\colon Y \subset X$ *is a sub-simplicial set, then* $\mathfrak{L}_j = \widehat{\mathbb{L}}\big(s^{-1}N_*(j)\big)$*.*

Proof. We begin by recalling the *Dold–Kan correspondence*, see for instance [63, Corollary 2.3], which establishes inverse equivalences,

$$\mathbf{sAbGrp} \underset{\mathcal{N}}{\overset{\Gamma}{\rightleftarrows}} \mathbf{dvect_0} \tag{7.3}$$

between the category of abelian simplicial groups and that of connected differential graded vector spaces, i.e., non-negatively graded chain complexes. All we need from the Dold–Kan correspondence is an explicit description of the functor $\mathcal{N}$: given $A \in \mathbf{sAbGrp}$ we consider the chain complex (A, d) where

$$d\colon A_n \longrightarrow A_{n-1}, \qquad d = \sum_{i=0}^{n} (-1)^i d_i m,$$

is given by the alternating sum of the faces of A. We define

$$\mathcal{N}A = A/D,$$

where D is the subgroup of degenerate simplices. By the simplicial identities, d preserves D and makes of $(\mathcal{N}A, d)$ a chain complex.

On the other hand, consider the pair of adjoint functors,

$$\mathbf{sset} \underset{U}{\overset{F}{\rightleftarrows}} \mathbf{sAbGrp}, \tag{7.4}$$

where U is the forgetful functor and F assigns to each simplicial set X the free abelian simplicial group generated by X. Composing (7.3) and (7.4) we obtain a pair of adjoint functors

$$\mathbf{sset} \underset{U\Gamma}{\overset{\mathcal{N}F}{\rightleftarrows}} \mathbf{dvect_0}\,.$$

Notice that $\mathcal{N}F$ is precisely the functor N of non-degenerate simplicial chains and thus this functor preserves inductive limits.

On the other hand, recall from Corollary 3.11 that the functor $\widehat{\mathbb{L}}\colon \mathbf{vect} \to \mathbf{cgl}$ also preserves inductive limits.

Hence, by all of the above, and starting with Corollary 7.7, we have:

$$\begin{aligned}\mathfrak{L}_X = \varinjlim_{\sigma\in X'} \mathfrak{L}_{\Delta^{|\sigma|}} &= \widehat{\mathbb{L}}\big(\varinjlim_{\sigma\in X'} s^{-1}\Delta^{|\sigma|}\big) = \widehat{\mathbb{L}}\big(\varinjlim_{\sigma\in X'} s^{-1}N_*(\underline{\Delta}^{|\sigma|})\big)\\ &= \widehat{\mathbb{L}}\big(s^{-1}N_*(\varinjlim_{\sigma\in X'} \underline{\Delta}^{|\sigma|})\big) = \widehat{\mathbb{L}}\big(s^{-1}N_*(X)\big).\end{aligned}$$

To finish the proof simply take into account the behaviour of the differential on each $\mathfrak{L}_n$ given in detail in the past chapter. □

A direct consequence of this result is:

Corollary 7.9. *Let* $\mathfrak{L}_X = (\widehat{\mathbb{L}}(V), d)$ *be the global model of a simplicial set* X*. Then, for any* $q \geq -1$,

$$H_q(V, d_1) \cong H_{q+1}(X; \mathbb{Q}). \qquad \square$$

Another consequence of this description of the global model is the following: let $X \subset Y$ be a simplicial inclusion. Note that $s^{-1}X \subset s^{-1}Y$ and, by Proposition 7.8, $\mathfrak{L}_X = (\widehat{\mathbb{L}}(s^{-1}X), d)$ is a sub-cdgl of $\mathfrak{L}_Y = (\widehat{\mathbb{L}}(s^{-1}Y), d)$. Then:

Proposition 7.10. $\mathfrak{L}_{Y/X} \cong \mathfrak{L}_Y/I$*, where* I *is the ideal generated by* $(s^{-1}X)_{\geq 0}$ *and by* $\{a - a_0\}_{a \in (s^{-1}X)_{-1}}$*, where* a_0 *is a fixed* 0*-simplex.*

Proof. Notice that the simplicial set Y/X is the colimit of

$$\langle x_0 \rangle \twoheadleftarrow X \hookrightarrow Y$$

where $\langle x_0 \rangle$ denotes the simplicial set generated by a 0-simplex x_0 of X. Then, by Proposition 7.5, $\mathfrak{L}_{Y/X}$ is the pushout

$$\begin{array}{ccc} \mathfrak{L}_X & \longrightarrow & \mathfrak{L}_Y \\ \downarrow & & \downarrow \\ \mathfrak{L}_{\langle x_0 \rangle} & \longrightarrow & \mathfrak{L}_{Y/X} \end{array}$$

of the induced cdgl morphisms: on the one hand, the upper horizontal arrow is just the inclusion. On the other hand, by Proposition 7.8,

$$\mathfrak{L}_{\langle x_0 \rangle} = \mathfrak{L}_0 = (\mathbb{L}(a_0,), d),$$

where a_0 is an MC element. Thus, the left vertical arrow is just the projection which sends $(s^{-1}X)_{\geq 0}$ to 0 and any Maurer–Cartan generator a_i of $(s^{-1}X)_{-1}$ to a_0. From this the result follows trivially. $\square$

In particular,

Corollary 7.11. *Let* $X \subset Y$ *be a sub-simplicial set and let* $a \in X_0$*. Then*

$$\mathfrak{L}^a_{Y/X} \cong \mathfrak{L}^a_Y / \mathfrak{L}^a_X. \qquad \square$$

It is important to note that this quotient – how could it be otherwise – is not taken over the cdgl $\mathfrak{L}^a_X$, but over the ideal generated by this cdgl.

The next example is a generalization of Proposition 7.2:

Example 7.12 (Simplicial complexes)**.** Let K be a simplicial complex and consider the corresponding simplicial set $\underline{K}$ (see Section 1.1.1). Recall from (1.8) that the chain complex $N_*(\underline{K})$ of non-degenerate chains on $\underline{K}$ is precisely the chain complex $C_*(K)$ of simplicial chains on K. Hence, by Proposition 7.8,

$$\mathfrak{L}_{\underline{K}} = (\widehat{\mathbb{L}}(s^{-1}K), d)$$

is generated by the desuspension of the simplicial chains on K and the linear part d_1 of d is the desuspension of the chain map on K.

In the special case of K being a finite simplicial complex, observe that, by definition, $\mathfrak{L}_{\underline{K}}$ is precisely the cdgl $\mathfrak{L}_K$ defined in Remark 6.3, as a sub-cdgl of $\mathfrak{L}_n$ with n big enough so that $K \subset \Delta^n$. For this reason, we often write $\mathfrak{L}_K$ to denote $\mathfrak{L}_{\underline{K}}$.

Concerning the realization functor, we prove here how Theorem 7.4 readily implies that this functor takes values in the category of Kan complexes.

Proposition 7.13. *The realization $\langle L \rangle$ of a cdgl L is a Kan complex.*

Proof. Recall that $\langle L \rangle$ is a Kan complex if for any $n \geq 1$, any $i = 0, \dots, n$ and any pair of solid arrows

$$\begin{array}{ccc} \underline{\Lambda}_i^n & \longrightarrow & \langle L \rangle \\ \downarrow & \nearrow & \\ \underline{\Delta}^n & & \end{array}$$

there exists an extension making this diagram commutative. By adjunction, this is equivalent to the existence of the dotted lifting in the corresponding cdgl diagram

$$\begin{array}{ccc} \mathfrak{L}_{\Lambda_i^n} & \longrightarrow & L \\ \downarrow & \nearrow & \\ \mathfrak{L}_n & & \end{array}$$

To construct this extension, recall from Proposition 6.4(ii) the isomorphism

$$\mathfrak{L}_n \cong \mathfrak{L}_{\Lambda_i^n} \,\widehat{\amalg}\, \widehat{\mathbb{L}}(u, du).$$

Now trivially extend the morphism $\mathfrak{L}_{\Lambda_i^n} \to L$ to $\mathfrak{L}_n$ by sending u to any degree n element in L and du to its boundary. $\square$

7.3 The path components and homotopy groups of $\langle L \rangle$

We first recall that a *path* in a simplicial set X joining two 0-simplices $y_0, y_1 \in X_0$ is a 1-simplex $x \in X_1$ such that $d_0 x = y_0$ and $d_1 x = y_1$. If X is a Kan complex this is an equivalence relation whose set $\pi_0(X)$ of equivalence classes consists of the *path components* of X.

Moreover, given a Kan complex X and a 0-simplex $y \in X_0$, the path component of X containing y is homotopy equivalent (i.e., it is weakly equivalent) to the reduced simplicial set X^y with only one 0-simplex, resulting by the identification of all the 0-simplices of this path component to y, and all the simplices of X which are connected by faces to y. That is,

$$X_n^y = \begin{cases} \{y\}, & \text{if } n = 0, \\ x \in X_n \text{ such that } d_{i_1} \cdots d_{i_n} x = y \text{ for some } i_1, \dots, i_n, & \text{if } n > 0. \end{cases}$$

The simplicial structure is induced by that on X, with the mentioned identification on the set of 0-simplices.

In this section, and for any cdgl L, we explicitly describe the path components of $\langle L \rangle$ and the homotopy groups of each of them.

Proposition 7.14. *Given a cdgl L, there are natural bijections*

$$\langle L \rangle_0 \cong \mathrm{MC}(L) \qquad \textit{and} \qquad \pi_0 \langle L \rangle \cong \widetilde{\mathrm{MC}}(L).$$

Proof. Note that the 0-simplices of $\langle L \rangle$ are the cdgl morphisms $\psi \colon \mathfrak{L}_0 \to L$. Since $\mathfrak{L}_0 = \mathbb{L}(a_0)$, where a_0 is an MC element, such a morphism is identified with the MC element $\psi(a_0)$, and the first bijection is thus trivial.

By virtue of Proposition 7.13, $\langle L \rangle$ is a Kan complex and thus, two 0-simplices $z_0, z_1 \in \langle L \rangle = \mathrm{MC}(L)$ are in the same path component if there is a 1-simplex,

$$\varphi \in \langle L \rangle_1 = \mathrm{Hom}_{\mathbf{cdgl}}(\mathfrak{L}_1, L) \quad \text{with} \quad d_1(\varphi) = z_0,\ d_0(\varphi) = z_1.$$

By definition, this amounts to saying that $\varphi(a_0) = z_0$ and $\varphi(a_1) = z_1$. In other terms, invoking Corollary 5.4, $z_0 \,\mathcal{G}\, z_1$, and so both represent the same element in $\widetilde{\mathrm{MC}}(L)$. □

Remark 7.15. Notice that, in view of Definition 5.17, paths in $\langle L \rangle$ correspond to paths in L. Hence, in what follows we often use the same notation for both.

Next we identify the homotopy type of each path component of $\langle L \rangle$.

Theorem 7.16. *Given a cdgl L and $z \in \mathrm{MC}(L)$, the path component of $\langle L \rangle$ containing z has the same homotopy type as the realization $\langle L^z \rangle$ of the component of L at z. More specifically, there is a simplicial isomorphism,*

$$\langle L \rangle^z \cong \langle L^z \rangle.$$

Proof. Notice first that the reduced simplicial set $\langle L \rangle^z$, homotopy equivalent to the path component of $\langle L \rangle$ containing z, takes the form,

$$\langle L \rangle_n^z = \{ f \colon \mathfrak{L}_n \to L,\ f(a_i) = z \text{ for any vertex } a_i,\ i = 0, \dots, n \}.$$

We will prove that the map

$$\varphi \colon \langle L \rangle^z \xrightarrow{\cong} \langle L^z \rangle$$

that assigns to each n-simplex $f\colon \mathfrak{L}_n \to L$ of $\langle L\rangle^z$ the n-simplex of $\langle L^z\rangle$ given by,

$$\varphi(f)(a_i)=0,\quad i=0,\dots,n,\quad \text{and}\quad \varphi(f)(a_{i_0\dots i_r})=f(a_{i_0\dots i_r}),\quad \text{if } r>0\,,$$

is an isomorphism of simplicial sets. To begin, recall from Definition 4.6 that L^z is the connected sub-cdgl of the perturbed (L,d_z) where,

$$(L^z)_n=\begin{cases}\ker d_z, & \text{if } n=0,\\ L_n, & \text{if } n>0.\end{cases}$$

Next, it is easy to see that φ is a bijection between the corresponding morphisms of Lie algebras. We now check that $\varphi(f)$ is a cdgl morphism, that is,

$$\varphi(f)\circ d=d_z\circ\varphi(f).$$

For generators of degree -1 and 0 this is an easy computation:

$$\varphi(f)\circ d(a_i)=-\frac{1}{2}\varphi(f)[a_i,a_i]=0=d_z\varphi(f)(a_i).$$

On the other hand, since $f(a_i)=z$ for all i, it follows that $df(a_{ij})=fd(a_{ij})=[-z,f(a_{ij})]$. Hence,

$$d_z\varphi(f)(a_{ij})=d_zf(a_{ij})=df(a_{ij})+[z,f(a_{ij})]=0=\varphi(f)d(a_{ij}).$$

Now, let $a_{i_0\dots i_r}$ be a generator of $\mathfrak{L}_n$ with $r>1$. By Proposition 6.8, we may assume that $d_{a_{i_0}}(a_{i_0\dots i_r})$ does not contain any generator a_i and therefore $d_{a_{i_0}}f$ and $\varphi(d_{a_{i_0}}f)$ are equal. Hence,

$$\begin{aligned}d_z\varphi(f)(a_{i_0\dots i_r})&=d_zf(a_{i_0\dots i_r})=df(a_{i_0\dots i_r})+[z,f(a_{i_0\dots i_r})]\\&=fd(a_{i_0\dots i_r})+[f(a_{i_0}),f(a_{i_0\dots i_r})]=fd_{a_{i_0}}(a_{i_0\dots i_r})=\varphi(f)d_{a_{i_0}}(a_{i_0\dots i_r})\\&=\varphi(f)\left(d(a_{i_0\dots i_r})+[a_0,a_{i_0\dots i_r}]\right)=\varphi(f)d(a_{i_0\dots i_r}).\end{aligned}$$

Finally, the compatibility of φ with the faces and degeneracies follows easily from the cosimplicial structure of $\mathfrak{L}_n$. □

As an immediate consequence we obtain:

Corollary 7.17. *Let L be a cdgl. Then,*

$$\langle L\rangle\simeq \amalg_{z\in\widetilde{\mathrm{MC}}(L)}\,\langle L^z\rangle.$$ □

The next result computes the homotopy groups of $\langle L\rangle$.

Theorem 7.18. *Let L be a connected cdgl. Then, $\langle L\rangle$ is a connected simplicial set and the natural morphism*

$$\rho_n\colon \pi_n\langle L\rangle \xrightarrow{\cong} H_{n-1}(L), \qquad \rho_n(f)=[f(a_{0\dots n})], \qquad n\geq 1,$$

is a group isomorphism. When $n=1$, the group $H_0(L)$ is considered with the BCH *product.*

Proof. By Proposition 7.14, $\langle L\rangle$ is connected. Moreover, it is reduced as 0 is the only MC element which is taken the 0-simplex to compute $\pi_*\langle L\rangle$. Even though the face operators $d_i\colon \langle L\rangle_n\to\langle L\rangle_{n-1}$ are not linear, we write

$$\ker d_i=\{f\colon \mathfrak{L}_n\to L\mid f\circ\delta^i=0\}$$

and recall that

$$\pi_n\langle L\rangle=\bigcap_{i=0}^n \ker d_i/\sim$$

where $f\sim g$ if there is an $h\in\langle L\rangle_{n+1}$ such that $d_nh=f$, $d_{n+1}h=g$ and $d_ih=0$ for $i<n$. Observe that,

$$\bigcap_{i=0}^n \ker d_i=\{f\colon \mathfrak{L}_n\to L,\ f(a_{i_0\dots i_q})=0 \quad\text{for}\quad (i_0,\dots,i_q)\neq(0,\dots,n)\}.$$

In particular, $f(a_{0\dots n})$ is a cycle and therefore,

$$\rho_n\colon \bigcap_{i=0}^n \ker d_i\longrightarrow H_{n-1}(L)\,, \qquad \rho_n(f)=[f(a_{0\dots n})],$$

defines a surjective map.

We first check that, for $n>1$, ρ_n induces a group isomorphism on the quotient $\pi_n\langle L\rangle$. If $h\in\langle L\rangle_{n+1}$ is a homotopy between f and g, then $d_ih=0$ for $i<n$ is equivalent to the vanishing of $h\colon\mathfrak{L}_{n+1}\to L$ on all the generators, except $a_{0\dots n}$, $a_{0\dots\widehat{n}n+1}$ and $a_{0\dots n+1}$. On the other hand, $d_nh=f$ and $d_{n+1}h=g$ translate to

$$h(a_{0\dots\widehat{n}n+1})=f(a_{0\dots n}) \quad\text{and}\quad h(a_{0\dots n})=g(a_{0\dots n}).$$

Now, for degree reasons,

$$dh(a_{0\dots n+1})=(-1)^n\big(f(a_{0\dots n})-g(a_{0\dots n})\big),$$

and therefore ρ_n induces a quotient map

$$\rho_n\colon \pi_n\langle L\rangle\longrightarrow H_{n-1}(L),$$

which is obviously surjective. For the injectivity suppose $\rho_n(f) = \rho_n(g)$, that is, there exists an element $u \in L_{n+1}$ with

$$du = f(a_{0\dots n}) - g(a_{0\dots n}).$$

Then, define a homotopy $h\colon \mathfrak{L}_{n+1} \to L$ between f and g by

$$h(a_{0\dots n+1}) = (-1)^n u, \qquad h(a_{0\dots n}) = g(a_{0\dots n}) \quad \text{and} \quad h(a_{0\dots\widehat{n}n+1}) = f(a_{0\dots n}).$$

Finally, we see that ρ is a group morphism: given $f, g \in \bigcap_{i=0}^n \ker d_i$, the sum $[f] + [g] \in \pi_n\langle L\rangle$ is by definition $[d_n(h)]$, where $h\colon \mathfrak{L}_{n+1} \to L$ belongs to $\bigcap_{i=0}^{n-1} \ker d_i$, $d_{n+1}h = f$ and $d_{n-1}h = g$. Define such a morphism by

$$h(a_{0\dots n+1}) = 0, \quad h(a_{0\dots\widehat{n}n+1}) = f(a_{0\dots n}) + g(a_{0\dots n}).$$

Then, we obviously have $\rho_n([f] + [g]) = \rho_n[f] + \rho_n[g]$.

Next, we prove that, for $n = 1$, ρ_1 also induces a group isomorphism. Since $L = L_{\geq 0}$, a map $f\colon \mathfrak{L}_1 \to L$ corresponds to the cycle $f(a_{01}) \in L$. Suppose $f \sim g$, which means that there is a map

$$h\colon \mathfrak{L}_2 \longrightarrow L \quad \text{such that} \quad h \circ \delta^2 = g, \quad h \circ \delta^1 = f \quad \text{and} \quad h \circ \delta^0 = 0.$$

Recall that, in $\mathfrak{L}_2$,

$$d_{a_0}(a_{012}) = a_{01} * a_{12} * a_{02}^{-1}.$$

Applying h to this expression we get

$$f(a_{01}) * g(a_{01})^{-1},$$

whose homology class vanishes being a boundary. Therefore, $\rho_1(f) = \rho_1(g)$ and thus ρ_1 also induces a map $\rho_1\colon \pi_1\langle L\rangle \to H_0(L)$ which is also bijective.

We show that ρ_1 is also a group morphism. Again by definition, given

$$f, g\colon \mathfrak{L}_1 \to L,$$

the product $[f] \cdot [g]$ is defined as $[d_1 h]$, where $h\colon \mathfrak{L}_2 \to L$ is a cdgl morphism such that $d_0 h = f$ and $d_2 h = g$. Choose such a morphism by setting,

$$h(a_{012}) = 0 \quad \text{and} \quad h(a_{02}) = f(a_{01}) * g(a_{01}).$$

This shows that $\rho_1([f] \cdot [g]) = f(a_{01}) * g(a_{01}) = \rho_1[f] * \rho_1[g]$. □

Example 7.19. Let $L = (L, 0)$ be a cdgl concentrated in degree 0, with zero differential. It follows that $H(L) = L$, and so $\langle L\rangle$ is an Eilenberg–MacLane space whose fundamental group is the vector space L equipped with the Baker–Campbell–Hausdorff product.

7.4 Homological behaviour of $\mathfrak{L}_X$

From now on, and as we did for the standard simplices in the preceding chapter, we will often not distinguish a 0-simplex x of X from the MC element $a = s^{-1}x$ in $\mathfrak{L}_X$.

We begin this section by proving the acyclicity of the global model functor.

Theorem 7.20. *Let X be a simplicial set. Then, $H(\mathfrak{L}_X) = 0$.*

Proof. As homology commutes with inductive limits, the result trivially follows from Proposition 6.4(i) and formula (7.2):

$$H(\mathfrak{L}_X) = \varinjlim_{\sigma\in X} H(\mathfrak{L}_{|\sigma|}) = 0. \qquad \square$$

As we remarked in the introduction to this chapter, the reader should not be surprised by this fact. As we will see shortly, this only means that the path component of $\langle \mathfrak{L}_X \rangle$ containing the MC element 0, considered as a 0-simplex, has the homotopy type of a point.

The situation, however, changes drastically when we perturb the differential of the global model $\mathfrak{L}_X$ of a given simplicial set X by an MC element given by one of its 0-simplices. As a simple example consider, for any $n \geq 2$, the simplicial set $\underline{\dot{\Delta}}^n$ and let a be any of its vertices. Then, by Corollary 6.6,

$$H(\mathfrak{L}_{\dot{\Delta}^n}, d_a) \cong \mathbb{Q}$$

while, by Theorem 7.20,

$$H(\mathfrak{L}_{\dot{\Delta}^n}) = 0.$$

In Theorems 7.23 and 7.27 below we prove that the homology $H(\mathfrak{L}_X, d_a)$ of the global model perturbed by an MC element a corresponding to a 0-simplex of X coincides with the homology $H(\mathfrak{L}_X^a)$ of the component of $\mathfrak{L}_X$ at a. In other words, by Corollary 7.17 and Theorem 7.18, $H(\mathfrak{L}_X, d_a)$ provides the homotopy groups of the path component of $\langle \mathfrak{L}_X \rangle$ containing the vertex a.

We begin with a general useful decomposition of the perturbed global model of a given simplicial set X. Recall from Proposition 7.8 that

$$\mathfrak{L}_X = (\widehat{\mathbb{L}}(V), d),$$

where $V = s^{-1}X$ is the desuspension of non-degenerate chains on X. Consider as usual the set $\{a_i\}$ of 0-simplices of X as Maurer–Cartan elements in V_{-1} and fix one of them a. Define the graded vector space $Z = Z_{\geq -1}$ as follows:

$$Z_p = \begin{cases} \operatorname{Span}\{a_i - a\}, & \text{if } p = -1, \\ V_p, & \text{if } p \geq 0. \end{cases}$$

Proposition 7.21. *The differential d in $\mathfrak{L}_X$ can be chosen so that $(\widehat{\mathbb{L}}(Z), d_a)$ is a sub-cdgl of $(\mathfrak{L}_X, d_a)$. In particular,*

$$(\mathfrak{L}_X, d_a) = (\mathbb{L}(a), d_a) \,\widehat{\amalg}\, (\widehat{\mathbb{L}}(Z), d_a).$$

In the unperturbed global model $\mathfrak{L}_X$,

$$dz = d_a z - [a, z], \quad \text{for} \quad z \in Z.$$

Proof. Note that, as cgl's,

$$\mathfrak{L}_X = \mathbb{L}(a) \,\widehat{\amalg}\, \widehat{\mathbb{L}}(Z)$$

so we only have to prove the first assertion. To this end, and by the description of $\mathfrak{L}_X$ in Proposition 7.8, it is enough to prove it for $X = \underline{\Delta}^n$, i.e., for $\mathfrak{L}_n$. Fix a a vertex of Δ^n. We show that for any $p = -1, \dots, n$ we can choose d in $\mathfrak{L}_n$ so that $d_a(Z_p) \subset \widehat{\mathbb{L}}(Z_p)$. For $p = -1$ this is obvious, since

$$d_a(a_i - a) = -\frac{1}{2}[a_i - a, a_i - a].$$

The case $p = 0$ is also clear in view of the form of the differential in $\mathfrak{L}_1$,

$$d_a x = [x, b - a] + \sum_{i=0}^{\infty} \frac{B_i}{i!} \operatorname{ad}_x^i (b - a).$$

In higher degrees apply Proposition 6.8 to find a form of d so that d_a of any generator of degree greater than 0 does not contain any vertex. In particular, $d_a(Z_p) \subset \widehat{\mathbb{L}}(Z_p)$. $\square$

From this, we extract the following important consequence, where (a) denotes the ideal of $\mathfrak{L}_X$ generated by a.

Corollary 7.22. *$(\mathfrak{L}_X/(a), \overline{d}_a)$ is a free cdgl and a quasi-isomorphic retract of the cdgl $(\mathfrak{L}_X, d_a)$. Moreover,*

$$(\mathfrak{L}_X/(a), \overline{d}_a) = \mathfrak{L}_X/(a)$$

and thus the projection,

$$(\mathfrak{L}_X, d_a) \xrightarrow{\simeq} \mathfrak{L}_X/(a)$$

is a quasi-isomorphism.

Proof. As $H(\widehat{\mathbb{L}}(a), d_a) = 0$, both the inclusion

$$(\widehat{\mathbb{L}}(Z), d_a) \overset{\simeq}{\hookrightarrow} (\mathbb{L}(a), d_a) \,\widehat{\amalg}\, (\widehat{\mathbb{L}}(Z), d_a)$$

and the projection

$$(\mathbb{L}(a), d_a) \,\widehat{\amalg}\, (\widehat{\mathbb{L}}(Z), d_a) \overset{\simeq}{\twoheadrightarrow} (\widehat{\mathbb{L}}(Z), d_a)$$

are quasi-isomorphisms. But, since $(\mathfrak{L}_X, d_a) = (\mathbb{L}(a), d_a) \,\widehat{\amalg}\, (\widehat{\mathbb{L}}(Z), d_a)$,

$$(\widehat{\mathbb{L}}(Z), d_a) = (\mathfrak{L}_X/(a), \overline{d}_a),$$

and the first assertion follows. For the identity, note that $d_a = d + \mathrm{ad}_a$ and thus $\overline{d}_a$ coincides with the differential induced by d. □

Theorem 7.23. *Let X be a connected simplicial set and let a be one of its 0-simplices. Then, the injection*

$$\mathfrak{L}^a_X \xhookrightarrow{\simeq} (\mathfrak{L}_X, d_a)$$

is a quasi-isomorphism, and

$$H(\mathfrak{L}^a_X) \cong H(\mathfrak{L}_X, d_a).$$

Proof. We first see that $H(\mathfrak{L}_X, d_a)$ is a connected Lie algebra, that is,

$$H(\mathfrak{L}_X, d_a) = H_{\geq 0}(\mathfrak{L}_X, d_a).$$

Assume first that X is a finite simplicial complex and $X \subset \Delta^n$. Let $\{x_i\}_{i\in\mathcal{I}}$ be the 0-simplices of X and let $\{x_{jk}\}$, $(j,k) \in \mathcal{J}$, be the set of edges of a *spanning tree* of X, i.e., a maximal tree in the 1-skeleton of X which contains all its vertices. In view of Example 7.12 and Remark 6.3,

$$\mathfrak{L}_X \subset \mathfrak{L}_n.$$

Fix a 0-simplex x and let a be the corresponding MC element of $\mathfrak{L}_X = (\widehat{\mathbb{L}}(s^{-1}X, d))$.

With the notation of Proposition 7.21, consider the vector subspace of Z given by

$$E = \mathrm{Span}\{a_i - a,\ a_{jk},\ \text{with } i \in \mathcal{I} \text{ and } (j,k) \in \mathcal{J}\}.$$

Then, $(\widehat{\mathbb{L}}(E), d_a)$ is a sub-cdgl of $(\widehat{\mathbb{L}}(Z), d_a) = (\mathfrak{L}_X/(a), \overline{d}_a)$ and, if we denote by L the quotient cdgl, we have a short exact sequence,

$$(\widehat{\mathbb{L}}(E), d_a) \hookrightarrow (\mathfrak{L}_X/(a), \overline{d}_a) \twoheadrightarrow L.$$

Since $H(E, d_1) = 0$, we can apply Proposition 3.12 to deduce that $(\widehat{\mathbb{L}}(E), d_a)$ is acyclic and thus the projection

$$(\mathfrak{L}_X/(a), \overline{d}_a) \xrightarrow{\simeq} L \tag{7.5}$$

is a quasi-isomorphism. However, observe that, by construction, $L = L_{\geq 0}$ is connected and therefore $H(\mathfrak{L}_X/(a), \overline{d}_a)$ is also non-negatively graded. But, by Corollary 7.22, $H(\mathfrak{L}_X/(a), \overline{d}_a) = H(\mathfrak{L}_X, d_a)$ and the result follows.

For a general connected simplicial set X write it as an increasing union of finite simplicial sets containing the 0-simplex a and apply the usual limit argument taking into account that both $\mathfrak{L}$ and H preserve inductive limits.

To finish, notice that the injection $\mathfrak{L}^a_X \hookrightarrow (\mathfrak{L}_X, d_a)$ is trivially an isomorphism in homology in non-negative degrees. But, as $H(\mathfrak{L}_X, d_a)$ is non-negatively graded, this injection is a quasi-isomorphism. □

Combining Theorems 7.18 and 7.23 yields

Corollary 7.24. *Let X be a connected simplicial set and a be a 0-simplex. Then, for any $n \geq 0$,*

$$H_n(\mathfrak{L}_X, d_a) \cong \pi_{n+1}\langle \mathfrak{L}_X^a \rangle. \qquad \square$$

However, the trivial element $0 \in \mathfrak{L}_X$ is a Maurer–Cartan element which does not correspond to a 0-simplex of X. For it we have

Corollary 7.25. *Let X be a simplicial set. Then, $\langle \mathfrak{L}_X^0 \rangle \simeq *$.*

Proof. By Theorems 7.20 and 7.23, $H(\mathfrak{L}_X^0) = H(\mathfrak{L}_X) = 0$ which, by Theorem 7.18, implies that $\pi_i\langle \mathfrak{L}_X^0 \rangle = 0$ for all i. □

The following is also an immediate but essential consequence in our theory:

Corollary 7.26. *For any connected simplicial set X and any 0-simplex a, the composition*

$$\mathfrak{L}_X^a \xrightarrow{\simeq} (\mathfrak{L}_X, d_a) \xrightarrow{\simeq} \mathfrak{L}_X/(a)$$

is an injective quasi-isomorphism.

Proof. The first map is the injective quasi-isomorphism of Theorem 7.23. The second is the surjective quasi-isomorphism of Corollary 7.22. Their composition is trivially injective. □

We next see that the homology of the global model of a simplicial set perturbed by a 0-simplex only depends on the path component containing the given simplex.

Theorem 7.27. *Let Y be a connected component of a simplicial set X and let a be a 0-simplex in Y. Then, the injection $(\mathfrak{L}_Y, d_a) \overset{\simeq}{\hookrightarrow} (\mathfrak{L}_X, d_a)$ is a quasi-isomorphism.*

Combining this result with Theorem 7.23 we immediately obtain a refinement of the latter.

Corollary 7.28. *Let Y be a path connected component of the simplicial set X and let a be a 0-simplex of Y. Then, the injection*

$$\mathfrak{L}_Y^a \overset{\simeq}{\hookrightarrow} (\mathfrak{L}_X, d_a)$$

is a quasi-isomorphism,

$$H(\mathfrak{L}_Y^a) \cong H(\mathfrak{L}_X, d_a). \qquad \square$$

The rest of the section is devoted to the proof of Theorem 7.27. We begin by a technical but interesting ancillary result.

Lemma 7.29. *Let $\widehat{\mathbb{L}}(\mathbb{Q}a \oplus Z)$ be the complete graded Lie algebra in which a is of degree -1 and let Z be a graded vector space such that $Z_p = 0$ if p does not belong to some interval $[m, M]$. Then, the ideal I generated by Z in $\widehat{\mathbb{L}}(\mathbb{Q}a \oplus Z)$ is the free complete Lie algebra on the vector space W generated by the elements $\mathrm{ad}_a^n v_i$, $n \geq 0$, where v_i is a graded basis of Z.*

Proof. Since $W \in \mathbb{L}^{\geq 1}(\mathbb{Q}a \oplus Z)$, we have that $\mathbb{L}^n(W) \subset \mathbb{L}^{\geq n}(\mathbb{Q}a \oplus Z)$ for any $n \geq 1$, and thus

$$\widehat{\mathbb{L}}(W) \subset I\,.$$

On the other hand, since $Z_p = 0$ for $p < m$ and $p > M$,

$$\left(I \cap \mathbb{L}^n(\mathbb{Q}a \oplus Z) \right)_q \subset \left(\mathbb{L}^{\geq \frac{q+n}{M+1}}(W) \right)_q,$$

and we deduce that $I \subset \widehat{\mathbb{L}}(W)$. □

Remark 7.30. The bounding hypothesis on Z is necessary as shown by the following example. Let Z be the vector space generated by the elements v_i, $i \geq 0$ with $|v_i| = i$. Then, $\sum_{i\geq 0} \mathrm{ad}_a^i v_i$ is a degree-0 element of I, but does not belong to $\widehat{\mathbb{L}}(W)$. The inclusion $\widehat{\mathbb{L}}(W) \subset I$ is thus strict in this case.

Proof of Theorem 7.27. Assume first that X is a finite simplicial set. As in Proposition 7.21, write

$$\mathfrak{L}_Y = \widehat{\mathbb{L}}(a) \,\widehat{\amalg}\, \widehat{\mathbb{L}}(Z) \qquad \text{and} \qquad \mathfrak{L}_X = \widehat{\mathbb{L}}(a) \,\widehat{\amalg}\, \widehat{\mathbb{L}}(Z \oplus V).$$

Denote by I the ideal of $\mathfrak{L}_Y$ generated by Z, and by J the ideal of $\mathfrak{L}_X$ generated by $Z \oplus V$. Consider the commutative diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (I, d_a) & \longrightarrow & (\mathfrak{L}_Y, d_a) & \longrightarrow & (\mathfrak{L}_Y/I, \overline{d}_a) & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & (J, d_a) & \longrightarrow & (\mathfrak{L}_X, d_a) & \longrightarrow & (\mathfrak{L}_X/J, \overline{d}_a) & \longrightarrow & 0 \end{array}$$

Since $\mathfrak{L}_X/J = \mathfrak{L}_Y/I = \mathbb{L}(a)$, the right vertical arrow is the identity and thus $(\mathfrak{L}_Y, d_a) \hookrightarrow (\mathfrak{L}_X, d_a)$ is a quasi-isomorphism if $(I, d_a) \hookrightarrow (J, d_a)$ is. By Lemma 7.29,

$$I = \widehat{\mathbb{L}}(U) \quad \text{and} \quad J = \widehat{\mathbb{L}}(W \oplus U),$$

where

$$W = \{\mathrm{ad}_a^n(z_i), n \geq 0\,,\ \text{with } z_i \text{ a basis of } Z\}$$

and

$$U = \{\mathrm{ad}_a^n(v_j), n \geq 0\,,\ \text{with } v_j \text{ a basis of } V\}\,.$$

Hence, the inclusion $(I, d_a) \hookrightarrow (J, d_a)$ has the form

$$(\widehat{\mathbb{L}}(W), d_a) \hookrightarrow (\widehat{\mathbb{L}}(W \oplus U), d_a)$$

and its quotient is

$$(\widehat{\mathbb{L}}(U), \overline{d}_a).$$

We finish by showing that this cdgl is acyclic.

Denote by δ the linear part of d_a in $(\widehat{\mathbb{L}}(W \oplus U), d_a)$. We show inductively that for any element $\mathrm{ad}_a^q(v_j)$ of the basis of U, with q even,

$$\delta(\mathrm{ad}_a^q(v_j)) = \mathrm{ad}_a^{q+1}(v_j).$$

This is true for $q = 0$. Suppose it holds for $q - 2$ and write

$$\mathrm{ad}_a^q(v_j) = \frac{1}{2}\left[[a,a], \mathrm{ad}_a^{q-2}(v_j)\right].$$

Then,

$$\delta(\mathrm{ad}_a^q(v_j)) = \frac{1}{2}\delta\left[[a,a], \mathrm{ad}_a^{q-2}(v_j)\right] = \frac{1}{2}\left[[a,a], \mathrm{ad}_a^{q-1}(v_j)\right] = \mathrm{ad}_a^{q+1}(v_j).$$

This amounts to saying that $(\widehat{\mathbb{L}}(U), \overline{d}_a)$ is a contractible cdgl and the assertion follows.

For a general, not necessarily finite, simplicial set X, write it as an increasing union of finite simplicial sets and apply a standard limit argument. □

7.5 The Deligne groupoid of the global model

Here we show that, given a simplicial set X, the cardinality of $\widetilde{\mathrm{MC}}(\mathfrak{L}_X)$ is the number of path components of X plus one. We first consider an example which illustrates this assertion:

Let X be a disjoint union of points $x_i, i \in I$. Then $\mathfrak{L}_X = (\mathbb{L}(V), d)$, where $V = V_{-1}$ is generated by MC elements $a_i, i \in I$. Since each element in degree -1 is a linear combination of the a_i, a simple computation shows that

$$\widetilde{\mathrm{MC}}(\mathfrak{L}_X) = \{a_i\}_{i \in I} \cup \{0\}.$$

A natural generalization of this fact constitutes our main theorem in this section. Let $X = \amalg_{i \in I} X_i$ be the decomposition of X into path connected components. For each $i \in I$, we choose a 0-simplex in X_i, and denote by a_i the corresponding Maurer–Cartan element.

Theorem 7.31. *With the above notation,*

$$\widetilde{\mathrm{MC}}(\mathfrak{L}_X) = \{a_i\}_{i\in I} \cup \{0\}\,.$$

In particular, there is a bijection

$$\pi_0(X^+) \cong \widetilde{\mathrm{MC}}(\mathfrak{L}_X),$$

where $X^+ = X \amalg \{*\}$.

Corollary 7.32. *For any connected simplicial set* X *and any vertex* $a \in X_0$, $\widetilde{\mathrm{MC}}(\mathfrak{L}_X) = \{0, a\}$ *and* $\widetilde{\mathrm{MC}}(\mathfrak{L}_X/(a)) = \{0\}$.

Proof. The first assertion is immediate. For the second assume first that X is a connected simplicial complex. Then, apply Proposition 4.35 to the quasi-isomorphic projection $\mathfrak{L}_X/(a) \xrightarrow{\simeq} L$ of (7.5) to conclude that

$$\widetilde{\mathrm{MC}}(\mathfrak{L}_X/(a)) = \widetilde{\mathrm{MC}}(L) = \{0\},$$

as L is non-negatively graded. Taking into account that $\widetilde{\mathrm{MC}}$ preserves inductive limits, this extends trivially to a general connected simplicial set. □

Remark 7.33. By Corollary 7.17, the bijection in Theorem 7.31 can now be restated as follows: for any simplicial set,

$$\pi_0(X^+) = \pi_0\langle\mathfrak{L}_X\rangle.$$

The rest of the section is devoted to the proof of Theorem 7.31.

Definition 7.34. Let $n \geq 1$. Two Maurer–Cartan elements u, v in a cdgl $(\widehat{\mathbb{L}}(V), d)$ are said to be *n-equivalent* if there is a morphism

$$\varphi\colon \mathfrak{L}_1 = (\widehat{\mathbb{L}}(a,b,x), d) \longrightarrow (\widehat{\mathbb{L}}(V), d)$$

with $\varphi(x) \in \mathbb{L}^{\geq n}(V)$, $\varphi(a) = u$ and $\varphi(b) = v$. We denote this relation by $u \sim_n v$.

Lemma 7.35. *Let* α *be a Maurer–Cartan element in* $(\widehat{\mathbb{L}}(V), d)$. *Suppose that* $\alpha = \beta + w$ *with* $w \in \mathbb{L}^{\geq n}(V)$, *and there is* $z \in \mathbb{L}^{\geq n}(V)$ *with* $dz = w + t$ *and* $t \in \mathbb{L}^{\geq n+1}(V)$. *Then,* $\alpha \sim_n \beta + w'$ *with* $w' \in \mathbb{L}^{\geq n+1}(V)$.

Proof. Recall the isomorphism ψ defined in Theorem 5.7,

$$\psi\colon (\widehat{\mathbb{L}}(a,b,x), d) \xrightarrow{\cong} (\widehat{\mathbb{L}}(a,u,v), d), \qquad dv = u, du = 0,$$

with $\psi(a) = a$, $\psi(x) = v$ and

$$\psi(b) = e^{\mathrm{ad}_{-v}}(a) + \frac{e^{\mathrm{ad}_{-v}} - 1}{\mathrm{ad}_{-v}}(u)\,.$$

Let

$$f\colon (\widehat{\mathbb{L}}(a,u,v),d) \longrightarrow (\widehat{\mathbb{L}}(V),d)$$

be the cdgl morphism defined by $f(a)=\alpha$, $f(v)=-z$ and $f(u)=-dz$. Then, $f\circ\psi$ is a path in $(\widehat{\mathbb{L}}(V),d)$ which starts at $f\psi(a)=\alpha$ and $f\psi(x)=-z$. To locate the MC element $f\psi(b)$, we observe that $\psi(b)-a+u\in\mathbb{L}^{\geq r+1}(V)$. Therefore,

$$f\psi(b)-f(a)-f(u)=f\psi(b)-\alpha+dz\in\mathbb{L}^{\geq r+1}(V).$$

Hence, $\alpha\sim_r f\psi(b)$ and $f\psi(b)-\beta\in\mathbb{L}^{\geq r+1}(V)$. □

Lemma 7.36. *Let $(u_r)_{r\geq n_0}$ be a sequence of Maurer–Cartan elements in $(\widehat{\mathbb{L}}(V),d)$ of the form $u_r=z+v_r$, with $v_r\in\widehat{\mathbb{L}}^{\geq r}(V)$, and such that $u_r\sim_r u_{r+1}$ for each $r\geq n_0$. Then, $u_{n_0}\sim_{n_0} z$.*

Proof. By hypothesis, for $r\geq n_0$ there is a morphism

$$\varphi_r\colon (\widehat{\mathbb{L}}(a,b,x),d) \longrightarrow (\widehat{\mathbb{L}}(V),d)$$

with $\varphi_r(a)=u_r$, $\varphi_r(b)=u_{r+1}$ and $\varphi_r(x)\in\mathbb{L}^{\geq r}(V)$. For each $r>n_0$, we define w_r to be the Baker–Campbell–Hausdorff product

$$w_r=\varphi_{n_0}(x)*\varphi_{n_0+1}(x)*\cdots*\varphi_{r-1}(x).$$

Theorem 5.13 implies that the element w_r is a path from u_{n_0} to u_r. We form the infinite product

$$w=\varphi_{n_0}(x)*\varphi_{n_0+1}(x)*\cdots,$$

which is well defined in $\widehat{\mathbb{L}}(V)$. We claim that w is a path of order n_0 from u_{n_0} to z, i.e., $u_{n_0}\sim_{n_0} z$, or equivalently,

$$dw=[w,z]-\sum_{n\geq 0}\frac{B_n}{n!}\,\mathrm{ad}_w^n(z-u_{n_0}).$$

Consider the element

$$y=dw-[w,z]-\sum_{n\geq 0}\frac{B_n}{n!}\,\mathrm{ad}_w^n(z-u_{n_0}),$$

and observe that y has the same image in $\widehat{\mathbb{L}}(V)/\widehat{\mathbb{L}}^{\geq r}(V)$ as

$$dw_r-[w_r,u_r]-\sum_{n\geq 0}\frac{B_n}{n!}\,\mathrm{ad}_{w_r}^n(u_r-u_{n_0}).$$

This last expression is equal to 0 because w_r is a path from u_{n_0} to u_r. This implies $y=0$ and proves the result. □

Proof of Theorem 7.31. Assume first that X is a finite and connected simplicial set. Let

$$\mathfrak{L}_X = (\widehat{\mathbb{L}}(Z), d),$$

and let $a \in Z_{-1}$ be a 0-simplex of X. Denote by W_{-1} the vector space generated by the difference $b-a$ where b runs over the MC elements of Z_{-1}. Then, Z_0 contains a subvector space W_0 such that, if $W = W_{-1} \oplus W_0$, we have $d(W) \subset \widehat{\mathbb{L}}(W)$ and $H(W, d_1) = 0$. If we denote by I the ideal of $\mathfrak{L}_X$ generated by W, it follows that the projection

$$\mathfrak{L}_X \xrightarrow{\simeq} \mathfrak{L}_X/I$$

is a quasi-isomorphism. Hence, by Proposition 4.35, it induces a bijection

$$\widetilde{\mathrm{MC}}(\mathfrak{L}_X) \cong \widetilde{\mathrm{MC}}(\mathfrak{L}_X/I).$$

By construction,

$$\mathfrak{L}_X/I = (\widehat{\mathbb{L}}(V), d)$$

where

$$V = \mathbb{Q}a \oplus V_{\geq 0} \qquad \text{and} \qquad dx - [a,x] \in \mathbb{L}(V_{\geq 0}) \quad \text{for} \quad x \in V.$$

Consider the ideal of $(\widehat{\mathbb{L}}(V), d)$ generated by $V_{\geq 0}$. Since V is finite-dimensional, by Lemma 7.29, this ideal has the form $(\widehat{\mathbb{L}}(U), d)$, where U is generated by the elements $\{\mathrm{ad}_a^r(v_k)\}_{r,k}$, where $r \geq 0$ and $\{v_k\}$ is a graded basis of $V_{\geq 0}$.

We denote by E_r the subvector space of $\mathfrak{L}_X$ generated by the Lie words containing exactly r elements of $V_{\geq 0}$. The differential d can be written as a series $d = \sum_{i\geq 1} d_i$ with $d_i(V) \subset E_i$. By hypothesis, $d_1(v) = -[a,v]$ if $v \in V_{\geq 0}$. A simple computation gives:

$$d_1 \, \mathrm{ad}_a^r(v) = \begin{cases} \mathrm{ad}_a^{r+1}(v), & \text{if } r \text{ is even,} \\ 0, & \text{if } r \text{ is odd.} \end{cases}$$

On the other hand, the derivation defined by $\theta = -\mathrm{ad}_a - d_1$ verifies

$$\theta\big(\mathrm{ad}_a^r(v)\big) = \begin{cases} 0, & \text{if } r \text{ is even,} \\ -\mathrm{ad}_a^{r+1}(v), & \text{if } r \text{ is odd.} \end{cases}$$

In particular, $\theta^2 = 0$.

Observe that $H_{-1}(E_{\geq 1}, \theta) = 0$.

Now, we see that any MC element is gauge related either to a or to 0. Consider a general MC element of $(\widehat{\mathbb{L}}(V), d)$, which is necessarily of the form

$$u = \lambda a + \xi, \quad \text{with} \quad \lambda = 0, 1, \quad \text{and} \quad \xi \in \widehat{\mathbb{L}}^{\geq 2}(V).$$

Assume first that $\lambda = 1$, i.e., $u = a+\xi$. We will build a sequence of Maurer–Cartan elements $\{u_n\}_{n\geq 1}$ such that $u_1 = u$, $u_n - a \in E_{\geq n}$ and $u_n \sim_n u_{n+1}$. Suppose that for some $n \geq 1$ the MC element u_n has been constructed. Then we can write it as

$$u_n = a + \omega_n + \gamma, \quad \text{with} \quad \omega_n \in E_n \quad \text{and} \quad \gamma \in E_{>n}.$$

Since u_n is a Maurer–Cartan element, we have $d_1\omega_n = -[a, \omega_n]$ and $\theta(\omega_n) = 0$. From $H_{-1}(E_{\geq 1}, \theta) = 0$, we deduce the existence of $t \in E_n$ such that $\omega_n = \theta(t)$. This implies that

$$\omega_n = -[a,t] - d_1 t.$$

Recall the cylinder isomorphism in Theorem 5.7,

$$\psi\colon (\widehat{\mathbb{L}}(a,b,x),d) \longrightarrow (\widehat{\mathbb{L}}(a,e,c),d),$$

and construct a morphism $\mu\colon (\widehat{\mathbb{L}}(a,e,c),d) \to (\widehat{\mathbb{L}}(V),d)$, by $\mu(a) = u_n$, $\mu(e) = t$ and $\mu(c) = d$t. A short computation shows that

$$\mu\circ\psi(b) = a + \gamma', \text{ with } \gamma' \in E_{>n}.$$

The path $\mu\circ\psi$ defines u_{n+1} such that $u_n \sim_n u_{n+1}$. By Lemma 7.36, $a \sim u$.

Suppose now that in our generic MC element u, $\lambda = 0$, i.e., u is decomposable. Then,

$$u = \sum_{i\geq 1} \omega_i, \quad \text{where} \quad \omega_i \in E_i \quad \text{for} \quad i \geq 1.$$

Since u is a Maurer–Cartan element, $d\omega_1 = 0$. Since $H_{-1}(\widehat{\mathbb{L}}(V),d) = 0$, we deduce the existence of an ω_1' such that $\omega_1 = d\omega_1'$ and Lemma 7.35 implies that $u \sim_1 u_2$ with $u_2 \in E_{\geq 2}$. By the same process we obtain a sequence of Maurer–Cartan elements $u_n \in E_{\geq n}$ such that $u_n \sim_n u_{n+1}$. Finally, by Lemma 7.36, $u \sim 0$.

All of the above shows that $\widetilde{\mathrm{MC}}(\mathfrak{L}_X) = \{a, 0\}$ and proves the theorem when X is finite and connected.

For any connected (non-finite) X,

$$\widetilde{\mathrm{MC}}(\mathfrak{L}_X) = \widetilde{\mathrm{MC}}\Big(\varinjlim_{\substack{Y\subset X,\\ a\in Y \text{ finite}}} \mathfrak{L}_Y\Big) \cong \varinjlim_{\substack{Y\subset X,\\ a\in Y \text{ finite}}} \widetilde{\mathrm{MC}}(\mathfrak{L}_Y) = \{a,0\}.$$

For a general simplicial set $X = \amalg_{i\in I} X_i$,

$$\begin{aligned}\widetilde{\mathrm{MC}}(\mathfrak{L}_X) &= \widetilde{\mathrm{MC}}(\amalg_{i\in I}\mathfrak{L}_{X_i}) \cong \textstyle\bigcup_{i\in I}\widetilde{\mathrm{MC}}(\mathfrak{L}_{X_i})\\ &= \textstyle\bigcup_{i\in I}\{0,a_i\} = \{a_i\}_{i\in I}\cup\{0\}.\end{aligned}$$ □

Chapter 8

A Model Category for cdgl

As we recalled in Section 1.3.1, practically all categories of chain complexes enriched with some additional structure, in particular **cdgl**, also possess a model category structure in which fibrations and weak equivalences are surjections and quasi-isomorphisms, respectively.

However, this model category structure does not, in general, reflect the homotopical properties inherited in **cdgl** from **sset** via the realization and model functors,

$$\mathbf{sset} \underset{\langle\,\cdot\,\rangle}{\overset{\mathfrak{L}}{\rightleftarrows}} \mathbf{cdgl}.$$

For instance, take the free cdgl $(\mathbb{L}(a), d)$ generated by a Maurer–Cartan element and consider the quasi-isomorphism

$$0 \xrightarrow{\simeq} (\mathbb{L}(a), d).$$

Notice that the realization of this cdgl morphism is not a homotopy equivalence of simplicial sets, since $\langle(\mathbb{L}(a), d)\rangle$ has two path connected components, while $\langle 0\rangle$ has only one (see Corollary 7.17). On the other hand, any map $f\colon X \to Y$ between simplicial sets is trivially modeled by a quasi-isomorphism

$$\mathfrak{L}_X \xrightarrow{\simeq} \mathfrak{L}_Y$$

since, in view of Theorem 7.20, $H(\mathfrak{L}_X) = H(\mathfrak{L}_Y) = 0$.

To overcome this handicap we use the *Transfer Principle* of Section 1.3.2 to endow **cdgl** with a model category structure which perfectly matches the classical one on **sset** through the above pair of adjoint functors. In this new category structure the fibrations turn out to be cdgl morphisms which are surjective only in non-negative degrees. On the other hand, weak equivalences are cdgl morphisms $f\colon L \to L'$ such that $\widetilde{\mathrm{MC}}(f)$ is a bijection and for each $z \in \widetilde{\mathrm{MC}}(L)$ the component

$$f^z\colon L^z \xrightarrow{\simeq} L'^{f(z)}$$

of f at z is a quasi-isomorphism.

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Automatically, the global model and realization functors become a Quillen pair and we extract some important consequences from this fact. In particular, these functors preserve, respectively, the homotopy type of Kan complexes and that of cofibrant cdgl's. We also give explicit path and cylinder objects in this new model structure which enable us to describe in detail homotopies between cdgl morphisms.

For computational purposes, cofibrant replacements of a given object are important in any given model category. In our setting we describe a special class of cofibrations and, in particular, of cofibrant replacements of any cdgl.

As a crucial special case we define the minimal model of a connected simplicial set X as the minimal model of $\mathfrak{L}_X^a$,

$$m_X \xrightarrow{\simeq} \mathfrak{L}_X^a,$$

where a is any 0-simplex of X. This object is an invariant of the homotopy type of X and, as we will see in following chapters, it contains all its rational information. Nevertheless, the tools which we have up to now enable us to show that, if $m_X = (\widehat{\mathbb{L}}(V), d)$, and for each $q \geq 1$,

$$\pi_q\langle \mathfrak{L}_X^a \rangle \cong H_{-1}(m_X) \qquad \text{and} \qquad H_q(X;\mathbb{Q}) \cong V_{q-1}.$$

8.1 The model category

In this section we show that **cdgl** is endowed with the particular model structure proposed in the introduction of this chapter. The first important consequences are also listed and proved. For the particularities of cofibrantly generated model categories we refer the reader to Section 1.3.2 for a brief compendium.

Theorem 8.1. *There is a cofibrantly generated model category structure on* **cdgl** *for which:*

- *A morphism $f\colon L \to M$ is a fibration if it is surjective in non-negative degrees,*
$$f\colon L_{\geq 0} \twoheadrightarrow M_{\geq 0}.$$
- *A morphism $f\colon A \to B$ is a weak equivalence if*
$$\widetilde{\mathrm{MC}}(f)\colon \widetilde{\mathrm{MC}}(L) \xrightarrow{\cong} \widetilde{\mathrm{MC}}(M)$$
is a bijection and
$$f^a\colon L^a \xrightarrow{\simeq} M^{f(a)}$$
is a quasi-isomorphism for each $a \in \widetilde{\mathrm{MC}}(L)$.
- *A morphism is a cofibration if it has the left lifting property with respect to trivial fibrations.*

Moreover, the cdgl morphisms

$$\{\mathfrak{L}_{\dot{\Delta}^n} \hookrightarrow \mathfrak{L}_{\Delta^n}\}_{n\geq 0} \quad \textit{and} \quad \{\mathfrak{L}_{\Lambda^n_i} \overset{\sim}{\hookrightarrow} \mathfrak{L}_{\Delta^n}\}_{n\geq 0,\, i=0,\dots,n},$$

are generating sets of cofibrations and trivial cofibrations, respectively.

Proof. We first prove that a morphism

$$f\colon L \longrightarrow M$$

is a fibration or a weak equivalence, with the notation of the statement, if and only if its realization

$$\langle f\rangle\colon \langle L\rangle \longrightarrow \langle M\rangle$$

is a fibration or a weak equivalence respectively of simplicial sets which, by Proposition 7.13, are necessarily Kan complexes.

By definition, the realization

$$\langle f\rangle\colon \langle L\rangle \xrightarrow{\simeq} \langle M\rangle$$

of a morphism $f\colon L \to M$ is a weak equivalence of simplicial sets if $\pi_0\langle f\rangle$ is bijective and, for the restriction of $\langle f\rangle$ to each path component, $\pi_n(f)$ is an isomorphism for any $n \geq 1$.

On the one hand, by Proposition 7.14, $\pi_0\langle f\rangle$ is identified with

$$\widetilde{\mathrm{MC}}(f)\colon \widetilde{\mathrm{MC}}(L) \longrightarrow \widetilde{\mathrm{MC}}(M).$$

Hence, $\pi_0\langle f\rangle$ is bijective if and only if $\widetilde{\mathrm{MC}}(f)$ is.

On the other hand, for $n \geq 1$, recall also from Theorem 7.16 that the component of $\langle L\rangle$ containing the 0-simplex $a \in \widetilde{\mathrm{MC}}(L)$ has the homotopy type of the realization $\langle L^a\rangle$ of the component of L at a. Hence, the restriction of $\langle f\rangle$ to this path component has the homotopy type of the map

$$\langle f^a\rangle\colon \langle L^a\rangle \longrightarrow \langle M^{f(a)}\rangle.$$

Finally, from Theorem 7.18, $\pi_n\langle f^a\rangle$ is an isomorphism if and only if $H_n(f^a)$ is.

Summarizing, f is a weak equivalence if and only if $\langle f\rangle$ is.

Now, the realization $\langle f\rangle\colon \langle L\rangle \to \langle M\rangle$ is by definition a fibration, i.e., a Kan fibration, if there exists a lifting in any square commutative diagram of the form

$$\begin{array}{ccc} \underline{\Lambda}^n_i & \longrightarrow & \langle A^a\rangle \\ \big\downarrow & \nearrow & \big\downarrow {\scriptstyle \langle f\rangle} \\ \underline{\Delta}^n & \longrightarrow & \langle B\rangle \end{array}$$

By the adjunction between the global model and realization functors, this is equivalent to the existence of the lifting in the corresponding cdgl diagram

$$\begin{array}{ccc} \mathfrak{L}_{\Lambda^n_i} & \longrightarrow & A^a \\ \big\downarrow & \nearrow & \big\downarrow f \\ \mathfrak{L}_{\Delta^n} & \longrightarrow & B \end{array}$$

We will see that this lifting exists if and only if f is surjective in non-negative degrees.

Assume first that this is the case. By Corollary 6.5, the morphism $\mathfrak{L}_{\Delta^n} \to B$ is uniquely determined by the image $x \in B_n$ of $a_{0\dots n}$, as it is

$$\mathfrak{L}_{\Lambda^n_i} \longrightarrow A \xrightarrow{f} B$$

on $\mathfrak{L}_{\Lambda^n_i}$. Since f is surjective in non-negative degrees and $n \geq 0$, there exists $y \in A$ such that $f(y) = x$. Using again Corollary 6.5 define the lifting $\phi\colon \mathfrak{L}_{\Delta^n} \to A^a$ as the only morphism extending $\mathfrak{L}_{\Lambda^n_i} \to A$ for which $\phi(a_{0\dots n}) = y$.

Conversely, assume the lifting exists for any such commutative square and let $x \in B_n$ with $n \geq 0$. By Corollary 6.5, there exists a unique cdgl morphism $\mathfrak{L}_{\Delta^n} \to B$ which is zero on $\mathfrak{L}_{\Lambda^n_i}$ and sends $a_{0\dots n}$ to x. Hence, its lifting $\mathfrak{L}_{\Delta^n} \to A$ sends $a_{0\dots n}$ to y with $f(y) = x$.

Hence, f is a fibration if and only if $\langle f \rangle$ is.

Once the classes of fibrations and weak equivalences of cdgl's have been characterized in geometrical terms, we plan to transfer the usual model structure on **sset** (see Section 1.3) along the adjunction

$$\mathbf{sset} \underset{\langle\,\cdot\,\rangle}{\overset{\mathfrak{L}}{\rightleftarrows}} \mathbf{cdgl},$$

by applying the Transfer Principle described in Theorem 1.22. It is enough then to check that all the hypotheses of this theorem are satisfied in our case.

Recall first that **sset** is cofibrantly generated by the sets

$$\mathcal{I} = \{\dot{\Delta}^n \hookrightarrow \Delta^n\}_{n\geq 0} \quad \text{and} \quad \mathcal{J} = \{\Lambda^n_i \overset{\sim}{\hookrightarrow} \Delta^n\}_{n\geq 0,\, i=0,\dots,n}$$

of cofibrations and trivial cofibrations, respectively. Moreover (see Proposition 3.5), the category **cdgl** has arbitrary limits and colimits. Hence, it only remains to verify the following two conditions:

(1) The sets $\{\mathfrak{L}(\mathcal{I})\}$ and $\{\mathfrak{L}(\mathcal{J})\}$ permit the small object argument.

(2) The realization functor takes relative $\mathfrak{L}(\mathcal{J})$-cell complexes to weak equivalences.

Concerning condition (1), since $\mathfrak{L}_{\dot{\Delta}^n}$ and $\mathfrak{L}_{\Lambda^n_i}$ are complete free Lie algebras on a finite number of generators, it follows by a simple inspection that the sets of morphisms,

$$\mathfrak{L}(\mathcal{I}) = \{\mathfrak{L}_{\dot{\Delta}^n} \hookrightarrow \mathfrak{L}_{\Delta^n}\}_{n\geq 0} \quad \text{and} \quad \mathfrak{L}(\mathcal{J}) = \{\mathfrak{L}_{\Lambda^n_i} \hookrightarrow \mathfrak{L}_{\Delta^n}\}_{n\geq 0,\, i=0,\dots,n},$$

permit the small object argument.

For condition (2), observe that each of the morphisms $\mathfrak{L}_{\Lambda^n_i} \overset{\simeq}{\hookrightarrow} \mathfrak{L}_{\Delta^n}$ in $\mathfrak{L}(\mathcal{J})$ is identified with the natural inclusion

$$\mathfrak{L}_{\Lambda^n_i} \overset{\simeq}{\hookrightarrow} \mathfrak{L}_{\Lambda^n_i} \,\widehat{\amalg}\, \widehat{\mathbb{L}}(u \oplus du), \quad \text{with} \quad |u| = n-1,$$

via the cdgl isomorphism

$$\mathfrak{L}_{\Delta^n} \cong \mathfrak{L}_{\Lambda^n_i} \,\widehat{\amalg}\, \widehat{\mathbb{L}}(u \oplus du), \quad \text{with} \quad |u| = n-1,$$

of Proposition 6.4(ii). Therefore, the pushout of $\mathfrak{L}_{\Delta^n} \overset{\simeq}{\hookleftarrow} \mathfrak{L}_{\Lambda^n_i} \to L$ is necessarily of the form

$$\begin{array}{ccc}
\mathfrak{L}_{\Lambda^n_i} & \longrightarrow & L \\
\simeq \big\downarrow & & \big\downarrow \simeq \\
\mathfrak{L}_{\Lambda^n_i} \,\widehat{\amalg}\, \widehat{\mathbb{L}}(u \oplus du) & \longrightarrow & L \,\widehat{\amalg}\, \widehat{\mathbb{L}}(u, du)
\end{array}$$

for any morphism $\mathfrak{L}_{\Lambda^n_i} \to L$. Hence, any $\mathfrak{L}(\mathcal{J})$-cell complex f is an injective quasi-isomorphism of the sort

$$f\colon L \overset{\simeq}{\hookrightarrow} L \,\widehat{\amalg}\, \widehat{\mathbb{L}}(V \oplus dV),$$

for some cdgl L and some graded vector space V. By Proposition 4.36,

$$\widetilde{\mathrm{MC}}(f)\colon \widetilde{\mathrm{MC}}(L) \xrightarrow{\cong} \widetilde{\mathrm{MC}}\big(L \,\widehat{\amalg}\, \widehat{\mathbb{L}}(V \oplus dV)\big)$$

is an isomorphism and, for each $a \in \mathrm{MC}(L)$,

$$f^a\colon (L, d_a) \overset{\simeq}{\hookrightarrow} (L \,\widehat{\amalg}\, \widehat{\mathbb{L}}(V \oplus dV), d_a)$$

is a quasi-isomorphism. In particular, the restriction of f^a to

$$f^a\colon (L^a, d_a) \overset{\simeq}{\hookrightarrow} \big((L \,\widehat{\amalg}\, \widehat{\mathbb{L}}(V \oplus dV))^a, d_a\big)$$

is also a quasi-isomorphism. Hence, f is a weak equivalence and, as previously proved, $\langle f \rangle$ is a weak equivalence of simplicial sets, which proves condition (2).

To finish we apply the Transfer Principle in Theorem 1.22 and obtain a model category in cdgl for which fibrations and weak equivalences are as stated. $\square$

The first consequences are summarized in the following.

Corollary 8.2. *The realization and model functors,*

$$\mathbf{sset} \underset{\langle\,\cdot\,\rangle}{\overset{\mathfrak{L}}{\rightleftarrows}} \mathbf{cdgl},$$

form a Quillen pair and therefore, they induce adjoint functors in the homotopy categories,

$$\operatorname{Ho}\mathbf{sset} \underset{\langle\,\cdot\,\rangle}{\overset{\mathfrak{L}}{\rightleftarrows}} \operatorname{Ho}\mathbf{cdgl}\,.$$

In particular,

(i) *The realization functor $\langle\,\cdot\,\rangle$ preserves homotopies between morphisms of cofibrants cdgl's.*

(ii) *The global model functor $\mathfrak{L}$ preserves homotopies between maps of Kan complexes.*

(iii) *The realization functor $\langle\,\cdot\,\rangle$ preserves weak equivalences of cdgl's and the global model functor preserves weak equivalences of simplicial sets.*

(iv) *For any cdgl L and any simplicial set X, we have a natural bijection between homotopy classes of maps*

$$[\mathfrak{L}_X, L] \cong [X, \langle L\rangle].$$

Proof. The main assertion follows at once from Corollary 1.24. Recall from Section 1.3 that both functors of a Quillen pair preserve homotopies of maps between fibrant and cofibrant objects. This is precisely (i) and (ii), taking into account that every cdgl is fibrant, every simplicial set is cofibrant and Kan complexes are the fibrant simplicial sets. Concerning (iii), recall that the left adjoint (respectively, right adjoint) of a Quillen pair preserves weak equivalences between cofibrant (respectively, fibrant) objects. Finally, (iv) is the general fact summarized in (1.30). □

Another important feature of these functors is the following direct consequence of Corollary 1.16. Here, limits and colimits are considered over a direct or Reedy category.

Proposition 8.3. *The realization functor $\langle\,\cdot\,\rangle$ preserves homotopy limits, while the model functor $\mathfrak{L}$ preserves homotopy colimits.* □

Corollary 8.4. *Let $p\colon L \to L'$ be a cdgl morphism which is surjective in non-negative degrees. Then, the simplicial set $\langle\,\ker p\,\rangle$ is the homotopy fibre of*

$$\langle p\rangle\colon \langle L\rangle \to \langle L'\rangle.$$

Proof. Since p is a fibration in **cdgl**, the homotopy limit of $L \xrightarrow{p} L' \leftarrow 0$ is its ordinary limit $\ker p$. To finish, apply Proposition 8.3. □

Remark 8.5. The isomorphism $\rho_n\colon \pi_n\langle L\rangle \cong H_{n-1}(L)$ described for any connected cdgl in Theorem 7.18 is a direct consequence of a special version of Theorem 8.2(iv). In fact, the naturality of the adjunction induces a bijection

$$\Phi\colon \operatorname{Hom}_{\mathbf{cdgl}}\big((\mathfrak{L}_{\Delta^n}, \mathfrak{L}_{\dot\Delta^n}), (L,0)\big) \xrightarrow{\cong} \operatorname{Hom}_{\mathbf{set}}\big((\Delta^n, \dot\Delta^n), (\langle L\rangle, a)\big).$$

Note then that ρ_n is the composition

$$[(S^n,*),(\langle L\rangle,a)] \xrightarrow{\cong} [(\Delta^n,\dot\Delta^n),(\langle L\rangle,a)]$$
$$\xrightarrow{\Phi} [(\mathfrak{L}_{\Delta^n}, \mathfrak{L}_{\dot\Delta^n}),(L,0)] \xrightarrow[\cong]{\text{ev}} H_{n-1}(L),$$

where $\operatorname{ev}(g) = g(a_{0\dots n})$.

Remark 8.6. From now on we write

$$L \simeq L'$$

to denote that L and L' are weakly equivalent cdgl's. We also often use the classical terminology by which L and L' have the same homotopy type. Whenever L and L' are connected, $L \simeq L'$ if and only if they are related by a zigzag of quasi-isomorphisms. In this case, by abuse of language we say that L and L' are "quasi-isomorphic".

To finish, it is worth noting that, if we restrict this model structure on **cdgl** to the subcategory $\mathbf{dgl_1}$ of 1-connected dgl's, we get the classical one in [115, Theorem 5.1] in which fibrations are surjections in degrees greater than 1 and weak equivalences are quasi-isomorphisms.

8.2 Weak equivalences and free extensions

In this section we analyze in detail some weak equivalences and cofibrations in this new model structure on **cdgl**. In particular, we explicitly describe convenient cofibrant replacements of any given cdgl.

We begin with the following important observation. Let X be a connected simplicial set and let a be any of its 0-simplices. Then, the non-trivial component $\mathfrak{L}_X^a$ may not be easy to handle, while $\mathfrak{L}_X/(a)$ is free as cgl, as guaranteed by Corollary 7.22. Nevertheless, we have:

Proposition 8.7. *The injection* $j\colon \mathfrak{L}_X^a \overset{\simeq}{\hookrightarrow} \mathfrak{L}_X/(a)$ *is a weak equivalence.*

Proof. By Corollary 7.32, 0 is the only element in $\widetilde{\mathrm{MC}}\big(\mathfrak{L}_X/(a)\big)$, while this is trivially the case for $\mathfrak{L}_X^a$. Hence, $\widetilde{\mathrm{MC}}(j)$ is a bijection. On the other hand, Corollary 7.26 asserts that $j = j^0$ is a quasi-isomorphism. □

Remark 8.8. Despite the preceding result, it is convenient to keep in mind that quasi-isomorphisms are not always weak equivalences. For instance, the inclusion

$$0 \xhookrightarrow{\simeq} (\mathbb{L}(a), d),$$

where a is a Maurer–Cartan element, is a quasi-isomorphism, but it is not an equivalence because $\widetilde{\mathrm{MC}}(\mathbb{L}(a), d) = \{0, a\}$.

Another illustrative set of examples is the following: recall from Corollary 7.26 that the injection j of Proposition 8.7 is the composition of the quasi-isomorphisms

$$\mathfrak{L}_X^a \xhookrightarrow{\simeq} (\mathfrak{L}_X, d_a) \xrightarrow{\simeq} \mathfrak{L}_X/(a).$$

However, by Theorem 7.31 and Proposition 4.28,

$$\widetilde{\mathrm{MC}}(\mathfrak{L}_X, d_a) = \{0, -a\}.$$

This shows that none of these quasi-isomorphisms is a weak equivalence.

Nevertheless, certain quasi-isomorphisms are always weak equivalences. The next result is simply the translation of Theorem 4.33 to this model category vocabulary:

Theorem 8.9. *Let $f\colon L \to L'$ be a morphism of cdgl's, filtered respectively by $\{F^n\}_{n\geq 1}$ and $\{G^n\}_{n\geq 1}$, such that the induced map*

$$F^n/F^{n+1} \xrightarrow{\simeq} G^n/G^{n+1}$$

is a quasi-isomorphism for any $n \geq 1$. Then, f is a weak equivalence. □

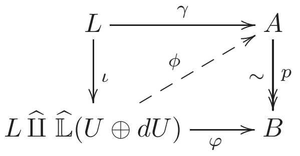
Another set of special weak equivalences is given by the following:

Proposition 8.10. *Let L be a cdgl and let U be a graded vector space. Then:*

(i) *The injection $\iota\colon L \xhookrightarrow{\sim} L \,\widehat{\amalg}\, \widehat{\mathbb{L}}(U \oplus dU)$ is a weak equivalence. Moreover, it is a cofibration if and only if $U = U_{\geq 0}$ is non-negatively graded.*

(ii) *The projection $p\colon L \,\widehat{\amalg}\, \widehat{\mathbb{L}}(U \oplus dU) \overset{\sim}{\twoheadrightarrow} L$ is a trivial fibration.*

Proof. (i) The fact that ι is a weak equivalence follows from Proposition 4.36. It is also easy to check that ι is a cofibration whenever $U = U_{\geq 0}$: consider a commutative square



in which p is a trivial fibration. As p is surjective in non-negative degrees, for any generator $u \in U$ choose $a \in A$ such that $p(a) = \varphi(u)$. Define the morphism ϕ as being γ on L, $\phi(u) = a$, and $\phi(du) = da$.

However, if U contains an element u of negative degree, then ι is no longer a cofibration: let $p\colon (\mathbb{L}(du), 0) \to \widehat{\mathbb{L}}(u \oplus du)$ be the inclusion. It is obviously a trivial fibration, as it is surjective at non-negative degrees and the component of p at the only MC element 0 is the zero map. However, the identity $\mathrm{id}_{\widehat{\mathbb{L}}(u\oplus du)}$ does not lift to p. The same example extends to the general inclusion i as long as there is a generator of U of negative degree.

(ii) As it is surjective, p is trivially a fibration. On the other hand, again by Proposition 4.36, p is also a weak equivalence. □

Next, we present a large and useful class of cofibrations.

Definition 8.11. A *free extension* of a cdgl L is an inclusion

$$L \hookrightarrow (L \,\widehat{\amalg}\, \widehat{\mathbb{L}}(V), d)$$

such that the following properties are satisfied:

- $V = V_{\geq -1}$ and V_{-1} is generated by Maurer–Cartan elements.
- $V_0 = V_0' \oplus V_0''$, where $dV_0' = 0$ and V_0'' is generated by paths between MC elements in $L \,\widehat{\amalg}\, \widehat{\mathbb{L}}(V_{-1})$.
- For $x \in V_n$, with $n \geq 1$, there is a Maurer–Cartan element a such that $d_a x \in L \,\widehat{\amalg}\, \widehat{\mathbb{L}}(V_{<n})$.

Theorem 8.12. *Every free extension is a cofibration.*

Proof. We need to prove that for every commutative square

$$\begin{array}{ccc} L & \longrightarrow & A \\ \big\downarrow & \overset{\phi}{\nearrow} & \big\downarrow{\scriptstyle p}\ {\scriptstyle\sim} \\ (L \,\widehat{\amalg}\, \widehat{\mathbb{L}}(V), d) & \xrightarrow[\varphi]{} & B \end{array} \tag{8.1}$$

in which p is a trivial fibration, there exists a morphism ϕ making commutative both triangles. We first observe that the injection

$$L \hookrightarrow (L \,\widehat{\amalg}\, \widehat{\mathbb{L}}(V_{-1} \oplus V_0''), d)$$

is a cofibration. Indeed, this morphism is obtained by successive pushouts of diagrams of these two kinds:

$$L \longleftarrow 0 \longrightarrow \mathfrak{L}_{\Delta^0},$$

one for each generator of V_{-1}, and

$$L \,\widehat{\amalg}\, \widehat{\mathbb{L}}(V_{-1}) \longleftarrow \mathfrak{L}_{\dot{\Delta}^1} \longrightarrow \mathfrak{L}_{\Delta_1},$$

one for each generator of V_0''. By definition, this is an $\mathfrak{L}(\mathfrak{J})$-cell complex, and therefore it is a cofibration. Thus, there exists a cdgl morphism

$$\phi\colon (L \,\widehat{\amalg}\, \widehat{\mathbb{L}}(V_{-1} \oplus V_0''), d) \longrightarrow A$$

as in diagram (8.1).

On the other hand, since $dV_0' = 0$, p is surjective in non-negative degrees, and $p^0\colon A^0 \to B^0$ is a quasi-isomorphism, it follows that ϕ is easily extended to

$$\phi\colon (L\,\widehat{\amalg}\,\widehat{\mathbb{L}}(V_{-1} \oplus V_0), d) \longrightarrow A.$$

We finish by defining ϕ inductively on $V_{\geq 1}$. Assume ϕ is defined on $V_{<n}$ with $n \geq 1$ and let $x \in V_n$. We denote by a the Maurer–Cartan element for which $d_a x \in L\,\widehat{\amalg}\,\widehat{\mathbb{L}}(V_{<n})$. In the restriction to the component of a,

$$\begin{array}{ccc} & & A^{\phi(a)} \\ & & \simeq \big\downarrow p \\ \big(L\,\widehat{\amalg}\,\widehat{\mathbb{L}}(V_{<n})\big)^a & \xrightarrow{\varphi} & B^{\varphi(a)} \end{array}$$

the element $\phi d_a(x)$ is a cycle in $A^{\phi(a)}$ with $p(\phi d_a x) = d_{\varphi(a)}\varphi(x)$. Therefore, since p is surjective in non-negative degrees and the restriction $p\colon A^{\phi(a)} \to B^{\varphi(a)}$ is a quasi-isomorphism, there exists $y \in A$ with $\phi d_a x = d_{\phi(a)} y$ and $p(y) = \varphi(x)$. We define $\phi(x) = y$ and observe that

$$\phi(dx) = \phi(d_a x) - \phi([a,x]) = d_{\phi(a)} y - [\phi(a), y] = dy = d\phi(x). \qquad \square$$

Corollary 8.13. *The model $\mathfrak{L}_X$ of any simplicial set is a cofibrant cdgl.*

Proof. Simply observe that $0 \to \mathfrak{L}_X$ is in fact a free extension in view of Proposition 7.8. □

Proposition 8.14. *Every cdgl morphism $f\colon L \to L'$ can be factored as*

$$\begin{array}{ccc} L & \xrightarrow{f} & L' \\ & {\scriptstyle i}\searrow & \sim \big\uparrow \varphi \\ & & (L\,\widehat{\amalg}\,\widehat{\mathbb{L}}(V), d) \end{array}$$

where i is a free extension and φ is a weak equivalence.

Proof. For each $z \in \widetilde{\mathrm{MC}}(L')$ not in $\operatorname{Im}\widetilde{\mathrm{MC}}(f)$ define a Maurer–Cartan element $v \in V_{-1}$ and set $\varphi(v) = z$. On the other hand, if $\widetilde{\mathrm{MC}}(f)(a) = \widetilde{\mathrm{MC}}(f)(b)$ define $v \in V_0''$ as a path from a to b and set $\varphi(v) = x \in L_0'$, which is a path from $\mathrm{MC}(f)(a)$ to $\mathrm{MC}(f)(b)$. This produces a cdgl morphism

$$\varphi\colon (L\,\widehat{\amalg}\,\widehat{\mathbb{L}}(V_{-1} \oplus V_0''), d) \longrightarrow L'$$

for which $\widetilde{\mathrm{MC}}(\varphi)$ is a bijection.

Next, to avoid excessive notation, write

$$M = (L\,\widehat{\amalg}\,\widehat{\mathbb{L}}(V_{-1} \oplus V_0'), d).$$

Then, for every $a \in \widetilde{\mathrm{MC}}(M)$, the usual inductive argument enables us to construct a quasi-isomorphism

$$(M^a \,\widehat{\amalg}\, \widehat{\mathbb{L}}(W^a), d) \xrightarrow{\simeq} L'^{\varphi(a)}$$

extending φ and in which:

- W^a is a non-negatively graded vector space and in particular $dW_0^a = 0$;
- $dw \in M^a \,\widehat{\amalg}\, \widehat{\mathbb{L}}(W^a_{<n})$ for every $w \in W^a_n$.

Then, setting

$$V = V_{-1} \oplus V_0' \oplus (\oplus_{a \in \widetilde{\mathrm{MC}}(M)} W^a),$$

assembling these quasi-isomorphisms produces a weak equivalence

$$\varphi\colon (L \,\widehat{\amalg}\, \widehat{\mathbb{L}}(V), d) \xrightarrow{\sim} L'$$

extending f and in which $(L \,\widehat{\amalg}\, \widehat{\mathbb{L}}(V), d)$ is a free extension. □

Corollary 8.15. *Every cdgl L has a cofibrant replacement*

$$(\widehat{\mathbb{L}}(V), d) \xrightarrow{\sim} L,$$

in which $V = V_{\geq -1}$, V_{-1} is generated by MC elements and $dV_0 = 0$.

Proof. Simply apply Proposition 8.14 to $0 \to L$. □

Note that the minimal models of connected cdgl's (see Definition 3.15) are examples of such cofibrant replacements.

8.3 A path object, a cylinder object and homotopy of morphisms

Staying with the analysis of the model structure in **cdgl**, we construct in this section a functorial path object for any cdgl and a cylinder object for any free cdgl. This in particular lets us describe the notion of homotopy between cdgl morphisms. For a brief introduction to these concepts we refer the reader to Section 1.3.

Definition 8.16. Let L be a cdgl associated to the filtration $\{F^n\}_{n\geq 1}$, that is,

$$L = \varprojlim_n L/F^n.$$

Define the *path object* of L as the cdgl,

$$L^I = L \widehat{\otimes} \wedge (t, dt) = \varprojlim_n \big(L/F^n \otimes \wedge (t, dt) \big)$$

with $|t| = 0$, $|dt| = -1$. The inclusions $L/F^n \hookrightarrow L/F^n \otimes \wedge(t, dt)$, $n \geq 1$, induce an injective morphism

$$j\colon L \hookrightarrow L^I.$$

On the other hand, the projections $L/F^n \otimes \wedge(t, dt) \to L/F^n$, sending t to 0 and 1, respectively, induce surjective morphisms

$$\varepsilon_0, \varepsilon_1 \colon L^I \longrightarrow L.$$

Then:

Proposition 8.17. *For any cdgl L, the sequence*

$$L \xrightarrow[\sim]{j} L^I \xrightarrow{(\varepsilon_0,\varepsilon_1)} L \times L$$

is in fact a functorial path object.

Proof. Obviously, the path construction is functorial on L.

Also, $(\varepsilon_0, \varepsilon_1)$ is a fibration, as it is surjective, and $(\varepsilon_0, \varepsilon_1)j$ is the diagonal. It remains to show that j is a weak equivalence. For this, recall from (3.4) and (3.5), that L^I is naturally filtered by $\{G^n\}_{n\geq 0}$, where

$$G^n = \ker\big(L^I \to L/F^n \otimes \wedge(t, dt)\big)$$

and

$$G_n/G_{n+1} = \big(L/F^n \otimes \wedge(t, dt)\big)/\big(L/F^{n+1} \otimes \wedge(t, dt)\big) = F^n/F^{n+1} \otimes \wedge(t, dt).$$

Therefore, for each $n \geq 1$, the induced map

$$j^n \colon F^n/F^{n+1} \xrightarrow{\simeq} F^n/F^{n+1} \otimes \wedge(t, dt)$$

is a quasi-isomorphism. Now apply Theorem 8.9 to conclude that j is then a weak equivalence. □

Definition 8.18. We say that two cdgl morphisms $f, g \colon L \to L'$ are *right homotopic*, and write $f \sim_r g$, if there is a cdgl morphism $\Phi \colon L \to L'^I$ such that $\varepsilon_0\Phi = f$ and $\varepsilon_1\Phi = f$. The morphism Φ is called a *right homotopy* between f and g.

Since every cdgl is fibrant, as in any model category, the right homotopy is an equivalence relation among cdgl morphisms.

Example 8.19. Consider the coproduct

$$L \,\widehat{\amalg}\, \widehat{\mathbb{L}}(U \oplus dU)$$

of a given cdgl with a contractible one, and denote by

$$\iota \colon L \longrightarrow L \,\widehat{\amalg}\, \widehat{\mathbb{L}}(U \oplus dU) \quad \text{and} \quad p \colon L \,\widehat{\amalg}\, \widehat{\mathbb{L}}(U \oplus dU) \longrightarrow L$$

the inclusion and projection, respectively. Then,

$$\iota p \sim_{\mathrm{r}} \mathrm{id}_{L \,\widehat{\amalg}\, \widehat{\mathbb{L}}(U \oplus dU)}$$

via the homotopy

$$\Phi\colon L\,\widehat{\amalg}\,\widehat{\mathbb{L}}(U\oplus dU)\longrightarrow \big(L\,\widehat{\amalg}\,\widehat{\mathbb{L}}(U\oplus dU)\big)^I$$

defined by

$$\Phi(x)=x\otimes 1,\quad x\in L,\qquad \Phi(u)=u\otimes t,\quad \Phi(du)=du\otimes t+(-1)^{|u|}u\otimes dt,\quad u\in U.$$

Next, we construct a cylinder object for certain free cdgl's along the same line as the original Tanré cylinder defined for ordinary dgl's in [130, II.5].

Let $(\widehat{\mathbb{L}}(V),d)$ be a free cdgl and let

$$V\cong U$$

be an isomorphism which maps the graded basis $\{v_i\}$ of V to the basis $\{u_i\}$. Construct the cdgl

$$(\widehat{\mathbb{L}}(V\oplus U\oplus sU),d),\quad \text{where}\quad d|_V=d,\quad dsu=u\quad\text{and}\quad du=0.$$

A derivation i of degree $+1$ is defined on $(\widehat{\mathbb{L}}(V\oplus U\oplus sU),d)$ by setting

$$i(v)=su\quad\text{and}\quad i(u)=i(su)=0.$$

Then $\theta=i\circ d+d\circ i$ is a derivation commuting with d and thus, by Proposition 4.10,

$$e^\theta\colon (\widehat{\mathbb{L}}(V\oplus U\oplus sU),d)\xrightarrow{\cong}(\widehat{\mathbb{L}}(V\oplus U\oplus sU),d)$$

is a cdgl automorphism. We introduce graded vector spaces V' and $\overline{V}$ isomorphic to V and sV, respectively, and we define an isomorphism

$$\psi\colon \widehat{\mathbb{L}}(V\oplus V'\oplus\overline{V})\xrightarrow{\cong}\widehat{\mathbb{L}}(V\oplus U\oplus sU)$$

of graded Lie algebras by $\psi(v)=v$, $\psi(v')=e^\theta(v)$ and $\psi(\overline{v})=su$. This induces a differential

$$D=\psi^{-1}d\psi$$

on $\widehat{\mathbb{L}}(V\oplus V'\oplus\overline{V})$ which makes ψ a cdgl isomorphism. Since e^θ is an automorphism commuting with d, we have:

$$Dv'=\psi^{-1}d\psi(v')=\psi^{-1}e^\theta dv\in\widehat{\mathbb{L}}(V').$$

Therefore, $(\widehat{\mathbb{L}}(V'),D)$ is a sub-cdgl of $(\widehat{\mathbb{L}}(V\oplus V'\oplus\overline{V},D)$ isomorphic to $(\widehat{\mathbb{L}}(V),d)$.

Definition 8.20. The *cylinder construction* on $L=(\widehat{\mathbb{L}}(V),d)$ is the sequence of cdgl's

$$L\,\widehat{\amalg}\,L\xrightarrow{\iota_0\,\widehat{\amalg}\,\iota_1}\text{Cyl }L\xrightarrow{p}L,$$

where

$$\text{Cyl }L=(\widehat{\mathbb{L}}(V\oplus V'\oplus\overline{V}),D),$$

$$\iota_0(v)=v,\quad \iota_1(v)=v',\quad p(v)=p(v')=v\quad\text{and}\quad p(\overline{v})=0.$$

Proposition 8.21. *For any free cdgl $L = (\widehat{\mathbb{L}}(V), d)$ in which $V = V_{\geq -1}$, the above construction is a cylinder object for L.*

Proof. Since $\overline{V}$ is non-negatively graded, and in view of the cdgl isomorphism ψ, Proposition 8.10(i) guarantees that both ι_0 and ι_1 are trivial cofibrations. Thus, $\iota_0 \,\widehat{\amalg}\, \iota_1$ is also a trivial cofibration. On the other hand, since $p \circ \iota_0$ is the identity, p is also a weak equivalence. □

Consider the LS interval $\mathfrak{L}_1 = (\widehat{\mathbb{L}}(a, b, x), d)$ and recall from Theorem 5.7 the cdgl isomorphism

$$\psi \colon \mathfrak{L}_1 \xrightarrow{\cong} (\widehat{\mathbb{L}}(a, u, su), d), \quad \text{where} \quad dsu = u,$$

defined by

$$\psi(a) = a, \quad \psi(x) = su \quad \text{and} \quad \psi(b) = e^{\theta}(a) = e^{\mathrm{ad}_{-su}}(a) + \frac{e^{\mathrm{ad}_{-su}} - 1}{\mathrm{ad}_{-su}}(u).$$

This immediately translates to:

Corollary 8.22. $\mathfrak{L}_1 \cong \mathrm{Cyl}\, \mathfrak{L}_0$. □

Remark 8.23. Observe that, in Proposition 8.21, for ι_0 and ι_1 to be cofibrations, it is necessary to assume that $V = V_{\geq -1}$ (see (i) of Proposition 8.10). We will assume that for the remaining of the section whenever a free cdgl $(\widehat{\mathbb{L}}(V), d)$ is considered. This assumption does not entail any restriction as, by Corollary 8.15, every cdgl L admits a cofibrant replacement of this type. On the topological side, the global model $\mathfrak{L}_X$ of any simplicial set X is also of this kind.

Definition 8.24. We say that two cdgl morphisms $f, g \colon (\widehat{\mathbb{L}}(V), d) \to L$ are *left homotopic*, and write $f \sim_l g$, if there is a cdgl morphism $\Psi \colon \mathrm{Cyl}\,(\widehat{\mathbb{L}}(V), d) \to L$ such that $f = \Psi \circ \iota_0$ and $g = \Psi \circ \iota_1$. The morphism Ψ is called a *left homotopy* between f and g. As in any model category, the left homotopy is an equivalence relation among cdgl morphisms with cofibrant domain.

The following is a useful application:

Proposition 8.25. *Let $f, g \colon (\widehat{\mathbb{L}}(V), d) \to (\widehat{\mathbb{L}}(W), d)$ be left homotopic morphisms. Then, the induced maps*

$$f_1, g_1 \colon (V, d_1) \longrightarrow (W, d_1)$$

are homotopic morphisms of chain complexes.

Proof. Let $H \colon \mathrm{Cyl}\,(\widehat{\mathbb{L}}(V), d) \to (\widehat{\mathbb{L}}(W), d)$ be a homotopy between f and g. Observe that in the cylinder,

$$D_1(\overline{v}) = v' - v - \overline{d_1 v}.$$

Hence,

$$g_1v - f_1v = H_1\overline{d_1v} + d_1H_1\overline{v}\,.$$

Then, the morphism $h\colon V \to W$ defined by $h(v) = H_1\overline{v}$ is a chain homotopy between f_1 and g_1. □

The following is simply Proposition 1.13 in our context:

Proposition 8.26. *Let $f, g\colon (\widehat{\mathbb{L}}(V), d) \to L$ be cdgl morphisms. Then, $f \sim_{\mathrm{r}} g$ if and only if $f \sim_{\mathrm{l}} g$.* □

In what follows, for the class of cdgl morphisms of cofibrant domain, we simply denote by $\sim$ either the right or left homotopy relation.

In particular, regarding a Maurer–Cartan element $a \in \mathrm{MC}(L)$ as a morphism $\mathfrak{L}_0 \to L$, we expand the characterization in Corollary 5.4 of gauge related Maurer–Cartan elements. For the following just take into account Corollary 8.22 and Proposition 8.26:

Corollary 8.27. *Let L be a cdgl and $a, b \in \mathrm{MC}(L)$. Then, the following assertions are equivalent:*

(1) *$a \sim b$, that is, there exists $x \in L_0$ such that $x \,\mathcal{G}\, a = b$.*

(2) *There exists $\Phi \in \mathrm{MC}(L^I)$ such that $\varepsilon_0(\Phi) = a$ and $\varepsilon_1(\Phi) = b$.*

(3) *There exists a morphism $\varphi\colon \mathfrak{L}_1 \to L$ such that $\varphi(a) = a$ and $\varphi(b) = b$.* □

Proposition 8.28. *Let $f, g\colon L \to L'$ be homotopic morphisms with cofibrant domain. Then, $\widetilde{\mathrm{MC}}(f) = \widetilde{\mathrm{MC}}(g)$.*

Proof. Let $a \in \mathrm{MC}(L)$ and consider the morphism $i_a\colon \mathfrak{L}_0 \to L$, $i_a(a_0) = a$. Then, $fi_a \sim gi_a$. By the above corollary, this amounts to saying that $f(a) \sim g(a)$. In other terms, $\widetilde{\mathrm{MC}}(f)(a) = \widetilde{\mathrm{MC}}(g)(a)$. □

We finish the section introducing, by means of the cylinder object, the cone and suspension of the model of any simplicial set. These are special cases of "homotopy cofibres" in the category **cdgl** which, as observed in Proposition 8.3 for general homotopy colimits, are preserved by the model functor. As a particular instance, we give another inductive process for building the models of $\underline{\Delta}^n$.

Consider the projection $X \to \{*\}$ of any given simplicial set X to a 0-simplex which is modeled by $\mathfrak{L}_X \to \mathfrak{L}_0$.

Definition 8.29. The *cone of* $\mathfrak{L}_X$ is the cdgl $\mathrm{Cone}\,\mathfrak{L}_X$ obtained as the pushout

$$\begin{array}{ccc}
\mathfrak{L}_X & \longrightarrow & \mathfrak{L}_0 \\
{\scriptstyle \iota_1}\big\downarrow & & \big\downarrow \\
\mathrm{Cyl}\,\mathfrak{L}_X & \longrightarrow & \mathrm{Cone}\,\mathfrak{L}_X
\end{array}$$

If we write $\mathfrak{L}_X = (\widehat{\mathbb{L}}(V), d)$ and $\mathfrak{L}_0 = (\widehat{\mathbb{L}}(a_0), d)$, where a_0 is an MC element, a simple inspection shows that

$$\text{Cone } \mathfrak{L}_X \cong (\widehat{\mathbb{L}}(V \oplus \mathbb{Q}a_0 \oplus \overline{V}), D),$$

where the D is induced in this pushout by the differential of the cylinder.

Definition 8.30. The *suspension of* $\mathfrak{L}_X$ is the cdgl $\Sigma\mathfrak{L}_X$ obtained as the pushout

$$\begin{array}{ccc} \mathfrak{L}_X & \longrightarrow & \mathfrak{L}_0 \\ {\scriptstyle \iota_0}\downarrow & & \downarrow \\ \text{Cone } \mathfrak{L}_X & \longrightarrow & \Sigma\mathfrak{L}_X \end{array}$$

Again, if $\mathfrak{L}_X = (\widehat{\mathbb{L}}(V), d)$ and, to avoid confusion, we set $\mathfrak{L}_0 = (\widehat{\mathbb{L}}(b_0), d)$, where b_0 is an MC element, we have:

$$\Sigma\mathfrak{L}_X \cong (\widehat{\mathbb{L}}(\mathbb{Q}b_0 \oplus \mathbb{Q}a_0 \oplus \overline{V}), D).$$

Theorem 8.31. *For any $n \geq 1$, the cdgl $\mathfrak{L}_n$ is isomorphic to the cone of $\mathfrak{L}_{n-1}$:*

$$\mathfrak{L}_n \cong \text{Cone } \mathfrak{L}_{n-1}.$$

Proof. Let

$$\{a_{i_0\dots i_p}\}_{0\leq i_0<\dots<i_p\leq n-1}$$

denote, as usual, the generators of $\mathfrak{L}_{n-1} = (\widehat{\mathbb{L}}(s^{-1}\Delta^{n-1}), d)$ and consider

$$\iota_1 \colon \mathfrak{L}_{n-1} \hookrightarrow \text{Cyl } \mathfrak{L}_{n-1}, \qquad \iota_1(a_{i_0\dots i_p}) = a'_{i_0\dots i_p},$$

the injection of Definition 8.29. Denote by J the ideal generated by the elements $a'_i - a'_0$ and the elements $a'_{i_0\dots i_p}$ for $p > 0$. Thus,

$$\text{Cone}\,\mathfrak{L}_{n-1} = \text{Cyl } \mathfrak{L}_{n-1}/J \cong (\widehat{\mathbb{L}}(\{a_{i_0\dots i_p}\} \oplus \mathbb{Q}a'_o \oplus \{\overline{a}_{i_0\dots i_p}\}), D).$$

Finally, it is easy to check that the map

$$\Psi\colon \text{Cone}\,\mathfrak{L}_{n-1} = \text{Cyl } \mathfrak{L}_{n-1}/J \xrightarrow{\cong} \mathfrak{L}_n,$$

defined by

$$\Psi(a_{i_0\dots i_q}) = a_{i_0\dots i_q}, \qquad \Psi\overline{a}_{i_0\dots i_p} = (-1)^p a_{i_0\dots i_p n} \qquad \text{and} \qquad \Psi(a'_0) = a_n,$$

is an isomorphism. □

8.4 Minimal models of simplicial sets

For this section it is convenient to keep in mind the main properties satisfied by the minimal model of a connected cdgl (see Definition 3.15 and subsequent results).

Let X be a connected simplicial set and let a be any 0-simplex in X. As usual, we also denote by a the corresponding Maurer–Cartan element in $\mathfrak{L}_X$.

Definition 8.32. The *minimal Lie model of* X is the minimal Lie model of $\mathfrak{L}_X^a$. This is a minimal cdgl $m_X = (\widehat{\mathbb{L}}(V), d)$, equipped with a quasi-isomorphism

$$\varphi\colon m_X \xrightarrow{\simeq} \mathfrak{L}_X^a.$$

As for maps, let $f\colon X \to Y$ be a map of simplicial sets and let $m_X \xrightarrow{\simeq} \mathfrak{L}_X^a$, $m_Y \xrightarrow{\simeq} \mathfrak{L}_Y^{f(a)}$ be the corresponding minimal models. By classical lifting arguments in model category theory, there is a cdgl morphism $m_f\colon m_X \to m_Y$, unique up to homotopy, such that the diagram

$$\begin{array}{ccc} \mathfrak{L}_X^a & \xrightarrow{\mathfrak{L}_f} & \mathfrak{L}_Y^{f(a)} \\ \uparrow{\simeq} & & \simeq\uparrow \\ m_X & \xrightarrow[m_f]{} & m_Y \end{array}$$

commutes up to homotopy. The *minimal Lie model of* f is the cdgl morphism m_f.

By Proposition 3.16 and Theorem 3.19, the minimal model of X exists and is unique up to cdgl isomorphisms. Also, as observed at the end of Section 8.2, m_X is a cofibrant object and φ is a weak equivalence. Moreover, it is easy to see that the minimal model is independent of the chosen 0-simplex and is an invariant of the homotopy type of the given simplicial set:

Proposition 8.33. *The minimal model of X does not depend on the chosen* 0-*simplex.*

Proof. Since X is connected, given 0-simplices $a, b \in X_0$ there is a path in X from a to b. Then, by construction, there is also a path in $\mathfrak{L}_X$ from a to b. Hence, by Proposition 4.24, $\mathfrak{L}_X^a \cong \mathfrak{L}_X^b$ and thus m_X is independent of the chosen 0-simplex. □

Proposition 8.34. *Homotopy equivalent simplicial sets have isomorphic minimal models.*

Proof. Let $f\colon X \xrightarrow{\sim} Y$ be a homotopy equivalence between simplicial sets. By Corollary 8.2(iii),

$$\mathfrak{L}_f\colon \mathfrak{L}_X \xrightarrow{\sim} \mathfrak{L}_Y$$

is a weak equivalence. In particular, for any given 0-simplex $a \in X_0$,

$$\mathfrak{L}^a_f \colon \mathfrak{L}^a_X \xrightarrow{\simeq} \mathfrak{L}^{f(a)}_Y$$

is a quasi-isomorphism. Hence,

$$m_f \colon m_X \xrightarrow{\cong} m_Y$$

is a quasi-isomorphism between minimal models which, by Theorem 3.19, is necessarily an isomorphism. □

Next, we show how to extract basic homotopy invariants from the minimal model.

Proposition 8.35. *Let $(\widehat{\mathbb{L}}(V), d)$ be the minimal Lie model of a connected simplicial set X. Then, there are natural isomorphisms,*

$$V \cong s^{-1}\widetilde{H}_*(X;\mathbb{Q}) \qquad \text{and} \qquad H_*(\widehat{\mathbb{L}}(V), d) \cong s^{-1}\pi_*\langle \mathfrak{L}^a_X \rangle.$$

Proof. By Corollary 7.26, we get a sequence of quasi-isomorphisms

$$(\widehat{\mathbb{L}}(V), d) \xrightarrow{\simeq} (\mathfrak{L}^a_X, d_a) \xrightarrow{\simeq} (\mathfrak{L}_X, d_a) \xrightarrow{\simeq} (\mathfrak{L}_X/(a), d),$$

which in turn induces quasi-isomorphisms on the spaces of indecomposable elements. From Corollary 7.9 it follows that $V \cong s^{-1}\widetilde{H}_*(X;\mathbb{Q})$. For the second identity simply apply Theorem 7.18. □

The following is just a simple but illustrative consequence.

Proposition 8.36. *Let Y be a connected simplicial set and let $i \colon X \hookrightarrow Y$ be the injection of a connected sub-simplicial set such that $H_*(i;\mathbb{Q})$ is injective. Then, the minimal Lie model $m_i \colon m_X \hookrightarrow m_Y$ is an injection and m_Y/m_X is the minimal Lie model of Y/X.*

Proof. Since $H(i;\mathbb{Q}))$ is injective, the first isomorphism in Proposition 8.35 guarantees that $m_i \colon m_X \hookrightarrow m_Y$ induces an injective morphism at the indecomposables. In particular, m_i is injective and m_Y/m_X is a minimal cdgl. On the other hand, by Corollary 7.11,

$$\mathfrak{L}^a_{Y/X} = \mathfrak{L}^a_Y/\mathfrak{L}^a_X.$$

Hence, by the five lemma, the right vertical map in the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & m_X & \xrightarrow{m_i} & m_Y & \longrightarrow & m_Y/m_X & \longrightarrow & 0 \\
 & & \downarrow{\simeq} & & \downarrow{\simeq} & & \downarrow{\simeq} & & \\
0 & \longrightarrow & \mathfrak{L}^a_X & \longrightarrow & \mathfrak{L}^a_Y & \longrightarrow & \mathfrak{L}^a_{Y/X} & \longrightarrow & 0
\end{array}$$

is a quasi-isomorphism. In other words, m_Y/m_X is the minimal Lie model of Y/X. □

We finish this section by deriving a special property of the minimal models of a simplicial complex.

Proposition 8.37. *Let X be a connected simplicial complex, a one of its 0-simplices, and m_X its minimal model. Then, there is an isomorphism of cdgl's*

$$\mathfrak{L}_X/(a) \cong m_X \,\widehat{\amalg}\, \widehat{\mathbb{L}}(R \oplus dR).$$

Here, $\widehat{\mathbb{L}}(R \oplus dR)$ simply denotes a contractible cdgl (see Definition 3.17).

Proof. Denote by Γ a maximal tree in the 1-skeleton of X. Then,

$$\mathfrak{L}_\Gamma/(a) \subset \mathfrak{L}_X/(a)$$

is an inclusion and both are free cdgls (see Proposition 7.22). We write, as cgl's,

$$\mathfrak{L}_X/(a) = \mathfrak{L}_\Gamma/(a) \,\widehat{\amalg}\, \widehat{\mathbb{L}}(V).$$

Note that, since Γ contains all the vertices, $V = V_{\geq 0}$. On the other hand, since Γ is a tree, $\mathfrak{L}_\Gamma/(a)$ is contractible.

We now replace the set of generators of $\widehat{\mathbb{L}}(V)$ so that the differential d of $\mathfrak{L}_X/(a)$ preserve $\widehat{\mathbb{L}}(V)$. We do it inductively so that $dV_n \subset \widehat{\mathbb{L}}(V_{\leq n})$ and, by an abuse of notation, still use V to denote the new generators.

Assume this is the case up to n. This induction hypothesis, plus the fact that $\mathfrak{L}_\Gamma$ is acyclic, imply that the projection

$$\mathfrak{L}_\Gamma/(a) \,\widehat{\amalg}\, \widehat{\mathbb{L}}(V_{\leq n}) \xrightarrow{\simeq} \widehat{\mathbb{L}}(V_{\leq n})$$

is a quasi-isomorphism. Let I be its kernel and let $v \in V_{n+1}$. Write

$$dv = \alpha + \beta, \quad \text{with} \quad \alpha \in \widehat{\mathbb{L}}(V_{\leq n}) \quad \text{and} \quad \beta \in I.$$

Then, $d\beta = 0$ and thus $\beta = d\gamma$ for some $\gamma \in I$. Replace then v by $v - \gamma$ and the assertion follows.

Now, by Proposition 3.18, we have a decomposition

$$(\widehat{\mathbb{L}}(V), d) \cong m_X \,\widehat{\amalg}\, \widehat{\mathbb{L}}(S \oplus dS).$$

Moreover, observe that $\mathfrak{L}_\Gamma/(a)$ can be written as

$$\mathfrak{L}_\Gamma/(a) = \widehat{\mathbb{L}}(T \oplus dT),$$

where T is generated by a set of 1-simplices of Γ. Hence, this time as cdgl's,

$$\mathfrak{L}_X/(a) = \mathfrak{L}_\Gamma/(a) \,\widehat{\amalg}\, \widehat{\mathbb{L}}(V) \cong m_X \,\widehat{\amalg}\, \widehat{\mathbb{L}}(R \oplus dR)$$

with $R = S \oplus T$. □

From the above result the following is immediate in light of Example 8.19:

Corollary 8.38. *Let X be a connected simplicial complex and a be one of its vertices. Then, there are quasi-isomorphisms*

$$m_X \underset{\psi}{\overset{\varphi}{\underset{\simeq}{\rightleftarrows}}} \mathfrak{L}_X/(a)$$

such that $\psi\varphi = \mathrm{id}_{m_X}$ and $\varphi\psi \sim \mathrm{id}_{\mathfrak{L}_X/(a)}$. □

Bibliographical notes

In [115] D. Quillen endowed the category $\mathbf{dgl}_1$ of simply connected dgl's with a model category in which fibrations and weak equivalences are surjections in degrees greater than 1 and quasi-isomorphisms, respectively. This was extended later to the unbounded category **dgl** by V. Hinich in [77, Theorem 2.2.1]. The general results in this reference could also be used to set up the same model structure in **cdgl**: the pair of adjoint functors

$$\mathbf{dvect} \underset{\mathcal{U}}{\overset{F}{\rightleftarrows}} \mathbf{cdgl}\,,$$

in which $\mathcal{U}$ is the forgetful functor and $F(V,d) = \widehat{\mathbb{L}}(V,d)$, satisfies the hypotheses of Theorem 1.17, and therefore there is a model structure on **cdgl** for which fibrations and weak equivalences are surjective morphisms and quasi-isomorphisms, respectively.

All fibrations in this structure are also fibrations in the model structure presented in this chapter. However, the zero map $0 \to \mathbb{L}(a)$, where a is a Maurer–Cartan element, is not surjective, but it is a fibration in our model structure. Scattered in this chapter there are many examples of quasi-isomorphisms that are not weak equivalences. However, the zero map $(L,0) \to 0$ with $L = L_{\le -2}$, is a weak equivalence in our structure, but not a quasi-isomorphism. Finally, $0 \to \widehat{\mathbb{L}}(u,du)$ is always a cofibration in the Hinich model structure, but in ours it is a cofibration only when u is of non-negative degree (see (i) of Proposition 8.10).

On the other hand, in [93, §9], A. Lazarev and M. Markl consider the adjoint functors

$$\mathbf{cdga} \underset{\mathscr{A}^f}{\overset{\mathscr{E}^f}{\rightleftarrows}} \mathbf{pdgl}$$

given by $\mathscr{A}^f(L) = \mathscr{A}(L^\#)$, $\mathscr{E}^f(A) = ((\widehat{\mathscr{E}(A))^\#}^f, d)$ which endow a model category structure on the category **pdgl** of profinite dgls (see Definition 3.35) by transferring the usual model structure on **cdga**. In this model structure, see [93, Theorem 9.16], fibrations are surjections and weak equivalences are morphisms f such that $\mathscr{A}^f$ are cdga quasi-isomorphism. With this structure on **pdgl** and the usual model structure on **cdga**, the pair of functors $\mathscr{A}^f$ and $\mathscr{E}^f$ form a Quillen equivalence, and therefore they induce equivalences between the associated homotopy categories.

Chapter 9

The Global Model Functor via Homotopy Transfer

In subsequent chapters we will be testing our global model functor by comparing it with other known Lie and cdga models of simplicial sets, mainly designed to algebraically mirror their rational homotopy types. On the geometrical side, we will also be comparing our realization functor with other known realization constructions of differential graded Lie and commutative algebras.

In this, the functor

$$A_{\mathrm{PL}}\colon \mathbf{sset} \longrightarrow \mathbf{cdga}$$

of PL-forms (see Section 1.2.1) will play an essential role, as it serves as common nexus in all these proposed comparisons. Our first task is then to obtain the global model functor, which up to now has been meticulously and intentionally developed in a self-contained manner, by means of the A_{PL} functor. This is attained by a particular procedure which constitutes the core of this chapter.

The starting point is considering the so-called *Dupont contraction*

$$A_{\mathrm{PL}}(\underline{\Delta}^\bullet) \underset{\iota_\bullet}{\overset{p_\bullet}{\rightleftarrows}} C^*(\Delta^\bullet).$$

This is given by way of $i_\bullet$ to regard the simplicial cochains on the standard simplices inside their PL-forms and a projection $p_\bullet$ from them to the cochains in a simplicial way and such that

$$p_\bullet \iota_\bullet = \mathrm{id}_{C^*(\Delta^\bullet)} \qquad \text{and} \qquad \iota_\bullet p_\bullet \sim \mathrm{id}_{A_{\mathrm{PL}}(\underline{\Delta}^\bullet)}\,.$$

Then, the *homotopy transfer* Theorem 1.8 readily produces a simplicial C_∞-structure on $C^*(\Delta^\bullet)$. Next we apply the functor $\mathscr{E}$ to this C_∞-algebra to obtain a differential graded Lie coalgebra. Then, by dualizing, we obtain a cosimplicial cdgl which is precisely $\mathfrak{L}_\bullet$.

A usual inductive limit procedure will let us obtain $\mathfrak{L}_X$ in the same manner for any simplicial set X of finite type.

U. Buijs et al., *Lie Models in Topology*, Progress in Mathematics 335,
https://doi.org/10.1007/978-3-030-54430-0_9

For all of the above, we advise the reader to review the main homotopical properties of the functors

$$\mathscr{E}, \mathscr{E}^u \colon \mathbf{cdga}_\infty \longrightarrow \mathbf{dglc}$$

listed in Section 2.2.

9.1 The Dupont calculus on $A_{\mathrm{PL}}(\underline{\Delta}^\bullet)$

Recall from formula (1.11) in Section 1.1.3 that

$$(N^*(\underline{\Delta}^\bullet), d) \cong (C^*(\Delta^\bullet), d)$$

and thus $(C^*(\Delta^\bullet), d)$ is a simplicial object in the category of cochain complexes.

The *Dupont calculus* [44, 45, 132] establishes a simplicial transfer diagram of the form

$$\kappa_\bullet \circlearrowright A_{\mathrm{PL}}(\underline{\Delta}^\bullet) \underset{\iota_\bullet}{\overset{p_\bullet}{\rightleftarrows}} C^*(\Delta^\bullet).$$

Except for its existence and the particular trivial case for Δ^0, the general expression of the simplicial chain homotopy $\kappa_\bullet$ will not be needed. However, the explicit descriptions of $\iota_\bullet$ and $p_\bullet$ are necessary to attain our goals and thus we outline their construction here. For further details on this well-known subject we refer to [45].

We begin by describing the simplicial inclusion $i_\bullet$.

Definition 9.1. The *Whitney elementary form* $\omega_{i_0\dots i_k} \in A^k_{PL}(\underline{\Delta}^n)$ is defined by

$$\omega_{i_0\dots i_k} = k! \sum_{j=0}^{k} (-1)^j t_{i_j} dt_{i_0} \cdots \widehat{dt_{ij}} \cdots dt_{i_k}.$$

In particular, $\omega_{i_0} = t_{i_0}$ and

$$\omega_{0\dots n} = n!\, dt_1 \cdots dt_n\,.$$

We denote by C_n the subspace of $A_{\mathrm{PL}}(\underline{\Delta}^n)$ generated by the Whitney elementary forms. It is a simple computation to check that the face and degeneracy operators of $A_{PL}(\underline{\Delta}^\bullet)$ given in (1.15) restrict to $C_\bullet$. In fact one obtains precisely the expression in (1.14) replacing the α's by the ω's. Hence,

$$C_\bullet \subset A_{PL}(\underline{\Delta}^\bullet)$$

becomes a sub-simplicial cochain complex.

Definition 9.2. For each $n \geq 0$ consider the basis,

$$\{\alpha_{i_0 \dots i_k}\}$$

of $C^*(\Delta^n)$ given in formula (1.12) of Section 1.1.3, and define,

$$\iota_n \colon C^*(\Delta^n) \longrightarrow C_n, \qquad \iota_n(\alpha_{i_0 \dots i_k}) = \omega_{i_0 \dots i_k}.$$

Proposition 9.3. *For each $n \geq 0$,*

$$\iota_\bullet \colon C^*(\Delta^\bullet) \xrightarrow{\cong} C_\bullet$$

is an isomorphism of simplicial cochain complexes.

Proof. By definition, ι_n is a vector space isomorphism. Also, the faces and degeneracies of $C^*(\Delta^\bullet)$, given in formula (1.14), commute with $\iota_\bullet$ by the above observation. Hence, we only have to check that each ι_n commutes with the differential. Recall from (1.13) of Section 1.1.3 that

$$d(\alpha_{i_0 \dots i_k}) = \sum_q \alpha_{q i_0 \dots i_k},$$

subject to the conventions mentioned in the referred formula. Hence, we have to show that,

$$d(\omega_{i_0 \dots i_k}) = \sum_q \omega_{q i_0 \dots i_k}\,.$$

On the one hand,

$$d\omega_{i_0 \dots i_k} = (k+1)! dt_{i_0} \cdots dt_{i_k}.$$

On the other hand,

$$\omega_{q i_0 \dots i_k} = (k+1)! \left(t_q dt_{i_0} \cdots dt_{i_k} + dt_q \left(\sum_{j=0}^{k} (-1)^{j+1} t_{i_j} dt_{i_0} \cdots \widehat{dt_{i_j}} \cdots dt_{i_k} \right) \right).$$

It is an easy exercise to check that the two expressions coincide, based on the identities $\sum t_q = 1$ and $\sum dt_q = 0$. □

Next, we construct the simplicial projection $p_\bullet$ as originally done by Whitney in [132].

Definition 9.4. For each $n \geq 0$ define

$$p_n \colon A_{\mathrm{PL}}(\underline{\Delta}^n) \longrightarrow C^*(\Delta^n)$$

by

$$p_n(\omega) = \sum_{k=0}^{n} \sum_{i_0 < \cdots < i_k} J_{i_0 \dots i_k}(\omega)\, \alpha_{i_0 \dots i_k}$$

where

$$J_{i_0\dots i_k}(t_{i_1}^{a_1}\cdots t_{i_k}^{a_k}dt_{i_1}\cdots dt_{i_k}) = \frac{a_1!\cdots a_k!}{(a_1+\cdots+a_k+k)!}$$

and

$$J_{i_0\dots i_k}(t_{j_1}^{a_1}\cdots t_{j_k}^{a_k}dt_{\ell_1}\cdots dt_{\ell_k}) = 0 \quad \text{if one of the } j_r \text{ or } \ell_r \notin \{i_0,\dots,i_k\}\,.$$

Observe that

$$J_{i_0\dots i_k}\colon A_{\mathrm{PL}}(\underline{\Delta}^n) \longrightarrow \mathbb{Q}$$

is precisely the integral of the given form over the k-simplex generated by the vertices $i_0,\dots,i_k$.

It is easily checked that

$$J_{i_0\dots i_k}(\omega_{j_0\dots j_k}) = \begin{cases} 0, & \text{if } \{i_0\dots i_k\}\neq\{j_0\dots j_k\},\\ 1, & \text{otherwise,}\end{cases}$$

and thus, $p_n\iota_n = \mathrm{id}_{C^*(\Delta^n)}$ for each $n\geq 0$. In particular, each p_n is a projection.

In this context, it is proven in [44, 45, 60] that

$$p_\bullet\colon A_{\mathrm{PL}}(\underline{\Delta}^\bullet) \longrightarrow C^*(\Delta^\bullet)$$

is a morphism of simplicial cochain complexes. Moreover, there exists a simplicial cochain map $\kappa_\bullet$ of degree 1 on $A_{\mathrm{PL}}(\underline{\Delta}^\bullet)$ for which the following holds:

Theorem 9.5. [44, 45, 60] *The diagram*

$$\kappa_\bullet \circlearrowright \; A_{\mathrm{PL}}(\underline{\Delta}^\bullet) \underset{\iota_\bullet}{\overset{p_\bullet}{\rightleftarrows}} C^*(\Delta^\bullet)$$

is a simplicial homotopy retraction. □

This means that $p_\bullet$, $\iota_\bullet$ and $\kappa_\bullet$ are simplicial maps and

$$p_\bullet\iota_\bullet = \mathrm{id}_{C^*(\Delta^\bullet)},\quad d\kappa_\bullet+\kappa_\bullet d = \mathrm{id}_{A_{PL}(\underline{\Delta}^\bullet)} - \iota_\bullet p_\bullet \quad\text{and}\quad \kappa_\bullet^2 = \kappa_\bullet\iota_\bullet = p_\bullet\kappa_\bullet = 0.$$

Example 9.6. Consider the particular case $n=1$. Then (see 1.2.1),

$$A_{\mathrm{PL}}(\underline{\Delta}^1)\cong \wedge(t_1,dt_1),$$

and simply using the definitions, we have:

$$\omega_0 = t_0 = 1-t_1, \qquad \omega_1 = t_1 \qquad\text{and}\qquad \omega_{01} = t_0dt_1 - t_1dt_0 = dt_1.$$

Moreover,

$$p_1(t_1^n) = \begin{cases} \omega_0+\omega_1, & \text{if } n=0,\\ \omega_1, & \text{if } n>0,\end{cases}$$

and

$$p_1(t_1^n dt_1) = \frac{\omega_{01}}{n+1}, \quad \text{for} \quad n \geq 0.$$

In this case there is only one choice for the operator κ_1, namely,

$$\kappa_1(t_1^n) = 0 \qquad \text{and} \qquad \kappa_1(t_1^n dt_1) = \frac{t_1^{n+1} - t_1}{n+1} \qquad \text{for} \qquad n \geq 0.$$

Theorem 9.5 has the following important generalization to any simplicial set.

Corollary 9.7. *Let X be a simplicial set. Then, there is a transfer diagram of the form,*

$$\kappa \circlearrowright \; A_{\mathrm{PL}}(X) \underset{\iota}{\overset{p}{\rightleftarrows}} N^*(X).$$

Proof. Write the given simplicial set as

$$X = \varinjlim_{\sigma \in X} \underline{\Delta}^{|\sigma|}.$$

Since the contravariant functor A_{PL} is left adjoint of the realization functor (see Theorem 1.2), it sends colimits to limits. The same applies to the functor N^* of non-degenerate cochains. See for instance the proof of Proposition 7.8 where a right adjoint of the functor N of non-degenerate chains is explicitly described. Therefore,

$$A_{\mathrm{PL}}(X) \cong \varprojlim_{\sigma \in X} A_{PL}(\underline{\Delta}^{|\sigma|}) \qquad \text{and} \qquad N^*(X) \cong \varprojlim_{\sigma \in X} N^*(\underline{\Delta}^{|\sigma|}) = \varprojlim_{\sigma \in X} C^*(\Delta^{|\sigma|}).$$

Hence, taking limits as σ runs through the simplices of X in the homotopy retraction of Theorem 9.5 the assertion follows. □

Remark 9.8. It is important to note the existence of an "augmented" version of Corollary 9.7. Let X be a simplicial set and let $a \in X_0$ be a 0-simplex. The inclusion $* \hookrightarrow X$ mapping the trivial simplicial set to a defines augmentations

$$\varepsilon \colon A_{\mathrm{PL}}(X) \longrightarrow \mathbb{Q} \qquad \text{and} \qquad \varepsilon \colon N^*(X) \longrightarrow \mathbb{Q},$$

whose kernels are denoted respectively by

$$\overline{A}_{\mathrm{PL}}(X) \qquad \text{and} \qquad \overline{N}^*(X).$$

Observe that, for $X = \underline{\Delta}^n$ and choosing the 0-simplex a_0, the augmentations in $C^*(\Delta^n)$ and $A_{\mathrm{PL}}(\Delta^n)$ are given by,

$$\varepsilon(\alpha_i) = \begin{cases} 1, & \text{if } i = 0, \\ 0, & \text{if } i \neq 0, \end{cases} \qquad \text{and} \qquad \varepsilon(t_i) = \begin{cases} 1, & \text{if } i = 0, \\ 0, & \text{if } i \neq 0. \end{cases}$$

Hence, the maps $\iota_\bullet$ and $p_\bullet$ in the transfer diagram of Theorem 9.5 obviously preserve augmentations. Thus, by construction, ι and p of the transfer diagram in Corollary 9.7 also preserve augmentations for any simplicial set X with a fixed 0-simplex. In particular, there is a transfer diagram of the form

$$\kappa \circlearrowright \overline{A}_{\rm PL}(X) \underset{\iota}{\overset{p}{\rightleftarrows}} \overline{N}^*(X).$$

9.2 Obtaining $\mathfrak{L}_\bullet$ and $\mathfrak{L}_X$ by transfer

The Dupont calculus and the functors

$$\mathscr{E}, \mathscr{E}^u\colon \mathbf{cdga}_\infty \longrightarrow \mathbf{cdgl}$$

defined in (2.4) of Section 2.2 are the basic tools to obtain the cosimplicial cdgl $\mathfrak{L}_\bullet$ by means of the $A_{\rm PL}$ functor, as we explain in this section.

First, apply Theorem 1.8 to the transfer diagram of Theorem 9.5 to obtain a simplicial C_∞ structure on $C^*(\Delta^\bullet)$ and a quasi-isomorphism of simplicial C_∞-algebras

$$C^*(\Delta^\bullet) \xrightarrow{\simeq} A_{\rm PL}(\underline{\Delta}^\bullet).$$

whose linear part is $\iota_\bullet$.

Next, apply the functor $\mathscr{E}^{\rm u}$ to this map to obtain a morphism of simplicial dglc's

$$\mathscr{E}^{\rm u}\big(C^*(\Delta^\bullet)\big) \xrightarrow{\simeq} \mathscr{E}^{\rm u}\big(A_{\rm PL}(\underline{\Delta}^\bullet)\big)$$

which is necessarily a quasi-isomorphism by Proposition 2.10(1). From now on, we denote

$$\mathfrak{L}^{\rm c}_\bullet = \mathscr{E}^{\rm u}\big(C^*(\Delta^\bullet)\big)$$

which, by definition, has the form

$$\mathfrak{L}^{\rm c}_\bullet = \big(\mathbb{L}^{\rm c}\big(sC^*(\Delta^\bullet)\big), d\big), \tag{9.1}$$

and whose dual is clearly a cosimplicial cdgl.

Finally, we have:

Theorem 9.9. *As cosimplicial cdgl's,*

$$\mathfrak{L}_\bullet \cong (\mathfrak{L}^{\rm c}_\bullet)^\#.$$

Proof. First, notice that for each $n \geq 0$,

$$\big(sC^*(\Delta^n)\big)^\# = s^{-1}C_*(\Delta^n) = s^{-1}\Delta^n.$$

Hence, by Proposition 3.23, and using the notation in (9.1), we get an isomorphism of cgl's:

$$(\mathfrak{L}^{\rm c}_n)^\# \cong \widehat{\mathbb{L}}(s^{-1}\Delta^n) = \mathfrak{L}_n.$$

Moreover, by construction, the linear parts of the differentials in both sides coincide.

On the other hand, the face operators in $(\mathfrak{L}^c_\bullet)$ are defined as follows: first, extend "tensor-wise" to $T^c(sC^*\Delta^\bullet)$ the usual face maps of $C^*(\Delta^\bullet)$. Then quotient out the resulting maps by the indecomposables of the shuffle product to obtain the faces of $\mathbb{L}^c\big(sC^*(\Delta^\bullet)\big)$. By dualizing, we then get precisely the cofaces of $\mathfrak{L}_\bullet$ given in (6.2).

Next, we prove that for $n \geq 0$ and $i = 0, \ldots, n$, any generator

$$a_i \in (\mathbb{L}^c(sC^*\Delta^n))^\#$$

of degree -1 is a Maurer–Cartan element. By naturality, it is enough to check it for $n = 0$. In this case, the inclusion of the transfer diagram in Theorem 9.5 is given by

$$\iota_0 \colon C^*(\Delta^0) \longrightarrow A_{\rm PL}(\Delta^0), \qquad \iota(\alpha_0) = \omega_0,$$

with α_0 the only generator of $C^*(\Delta^0)$ which is of degree 0. But notice that $\omega_0 = t_0 = 1$ is a unit,

$$\omega_0^2 = \omega_0.$$

Hence, see (1.29), for the multiplication of the C_∞ structure in $C^*(\Delta^0)$ induced by Theorem 9.5, we have:

$$\alpha_0^2 = \alpha_0.$$

Thus, in the corresponding bar construction $T^c\big(sC^*(\Delta^0)\big)$, check formula (1.24),

$$d(s\alpha_0 \otimes s\alpha_0) = -s\alpha_0.$$

Notice however (see Section 1.2.4 and particularly Example 1.4), that for the Lie coalgebra structure on $\mathbb{L}^c\big(sC^*(\Delta^0)\big)$,

$$\Delta_L(s\alpha_0 \otimes s\alpha_0) = 2[s\alpha_0, s\alpha_0]^c,$$

where $[\ ,\ ]^c$ denotes the Lie cobracket. Hence, in the dual Lie algebra

$$(\mathfrak{L}^c_0)^\# = (\widehat{\mathbb{L}}(s^{-1}\Delta^0), d)$$

this produces

$$da_0 = -\frac{1}{2}[a_0, a_0],$$

where a_0 is dual to α_0.

Summarizing, the cosimplicial cdgl $(\mathfrak{L}^c_\bullet)^\#$ shares with $\mathfrak{L}_\bullet$ the underlying cgl structure, the linear differential and the coface operators. Moreover, all the generators of degree -1 are Maurer–Cartan elements. Hence, by the uniqueness property of Theorem 6.7(2),

$$\mathfrak{L}_\bullet \cong (\mathfrak{L}^c_\bullet)^\#. \qquad \square$$

The same procedure can be followed, replacing $\underline{\Delta}^\bullet$ by any simplicial set X: start by applying Theorem 1.8, this time to the transfer diagram of Corollary 9.7, to obtain a C_∞ structure on $N^*(X)$ and a quasi-isomorphism of C_∞-algebras

$$N^*(X) \xrightarrow{\simeq} A_{\mathrm{PL}}(X). \tag{9.2}$$

Applying the functor $\mathscr{E}^{\mathrm{u}}$ we obtain then a quasi-isomorphism of dglc's

$$\mathscr{E}^{\mathrm{u}}\big(N^*(X)\big) \xrightarrow{\simeq} \mathscr{E}^{\mathrm{u}}\big(A_{\mathrm{PL}}(X)\big),$$

which again is a quasi-isomorphism by Proposition 2.10(1). We denote

$$\mathfrak{L}^{\mathrm{c}}_X = \mathscr{E}^{\mathrm{u}}\big(N^*(X)\big)$$

which is of the form

$$\mathfrak{L}^{\mathrm{c}}_X = \big(\mathbb{L}^{\mathrm{c}}\big(sN^*(X)\big), d\big).$$

Then, the analogue of the previous theorem reads:

Theorem 9.10. *Let X be a finite type simplicial set. Then, there is an isomorphism of cdgl's*

$$\mathfrak{L}_X \cong (\mathfrak{L}^{\mathrm{c}}_X)^{\#}.$$

Proof. The proof is completely analogous to the previous one. Since X is of finite type, $sN^*(X)$ is a finite type cochain complex whose dual is precisely $s^{-1}X$, with the notation of Section 7.2. Hence, by Proposition 3.23, and as cgl's

$$(\mathfrak{L}^{\mathrm{c}}_X)^{\#} \cong \widehat{\mathbb{L}}(s^{-1}X) = \mathfrak{L}_X.$$

Moreover, the linear parts of the differentials in both sides coincide. The same argument used in the proof of Theorem 9.9 shows that every generator of $(\mathfrak{L}^{\mathrm{c}}_X)^{\#}$ of degree -1 corresponding to a 0 simplex of X is a Maurer–Cartan element. To finish, apply Proposition 7.8. $\square$

An "augmented" version of this result is easily obtained, again using the same process: Let X be a simplicial set and let $a \in X_0$ be any of its 0-simplices. in view of Remark 9.8, the C_∞ quasi-isomorphism (9.2) respects the augmentations. Hence, we may apply the "reduced" functor $\mathscr{E}$ to obtain, via Proposition 2.10(1), a quasi-isomorphism of cdgl's:

$$\mathscr{E}\big(N^*(X)\big) \xrightarrow{\simeq} \mathscr{E}\big(A_{PL}(X)\big). \tag{9.3}$$

Denote the Lie coalgebra

$$\overline{\mathfrak{L}}^{\mathrm{c}}_X = \mathscr{E}\big(N^*(X)\big) \tag{9.4}$$

which is of the form

$$\overline{\mathfrak{L}}^{\mathrm{c}}_X = \big(\mathbb{L}^{\mathrm{c}}\big(s\overline{N}^*(X)\big), d\big).$$

where $\overline{N}^*(X)$ denotes the augmentation ideal of $N^*(X)$. Then, the same argument in the proof of the two previous theorems shows:

Theorem 9.11. *Let X be a finite type simplicial set X and let a be a 0-simplex. Then, there is an isomorphism of cdgl's*

$$\mathfrak{L}_X/(a) \cong (\overline{\mathfrak{L}}^{\mathrm{c}}_X)^{\#}. \qquad \square$$

In particular,

Corollary 9.12. *There is an isomorphism of cosimplicial cdgl's,*

$$\mathfrak{L}_\bullet/(a_0) \cong (\overline{\mathfrak{L}}^{\mathrm{c}}_\bullet)^{\#}. \qquad \square$$

Remark 9.13. In the augmented case it will be more convenient later on to give now a more precise reformulation of (9.3) whenever $X = \underline{\Delta}^n$ and for any $n \geq 0$. Indeed, in that case, and as we proceeded above, first apply Theorem 1.8 to the simplicial diagram of Theorem 9.5, and then the functor $\mathscr{E}$ to obtain simplicial dglc quasi-isomorphisms

$$\overline{\mathfrak{L}}^{\mathrm{c}}_\bullet \underset{q_\bullet}{\overset{j_n}{\underset{\longleftarrow}{\overset{\longrightarrow}{\simeq}}}} \mathscr{E}\big(A_{\mathrm{PL}}(\underline{\Delta}^\bullet)\big)$$

such that

$$q_\bullet i_\bullet = \mathrm{id}_{\overline{\mathfrak{L}}^{\mathrm{c}}_n} \qquad \text{and} \qquad i_\bullet q_\bullet \sim \mathrm{id}_{\mathscr{E}\left(A_{\mathrm{PL}}(\underline{\Delta}^\bullet)\right)}.$$

In particular, this exhibits $\overline{\mathfrak{L}}^{\mathrm{c}}_\bullet$ as a simplicial deformation retract of $\mathscr{E}\big(A_{\mathrm{PL}}(\underline{\Delta}^\bullet)\big)$.

Bibliographical notes

The idea of using the homotopy transfer theorem to construct a cdgl associated to a given simplicial set has also been developed by R. Bandiera [2], and by D. Robert-Nicoud within an operadic framework [117].

The appearance of the Lawrence–Sullivan interval from an C_∞-model for the interval has been noted by several authors, for instance by Fiorenza and Manetti [53], Cheng and Getzler [36, Proposition 19], Bandiera [2, Example 5.2.25], and Robert-Nicoud [118, Proposition 5.4].

Chapter 10

Extracting the Sullivan, Quillen and Neisendorfer Models from the Global Model

One of the main goals in this chapter is to show that our global model functor

$$\mathfrak{L}\colon \ \mathbf{sset} \longrightarrow \mathbf{cdgl}$$

effectively provides the (homotopy type of the) Quillen functor

$$\lambda\colon \ \mathbf{sset}_1 \longrightarrow \mathbf{dgl}_1$$

when restricted to 1-reduced simplicial sets of finite type. Let X be such a simplicial set and let $a \in X_0$ be a 0-simplex. Then,

$$\mathfrak{L}_X^a \simeq \lambda(X).$$

More generally, if X is nilpotent, $\mathfrak{L}_X^a$ has the homotopy type of the Neisendorfer model of X (see Definition 3.28).

The key ingredient to attain this result is the existence of a quasi-isomorphism

$$\mathscr{E}\big(A_{\mathrm{PL}}(X)\big)^{\#} \xrightarrow{\simeq} \mathfrak{L}_X/(a),$$

easily deduced from the machinery developed in the past chapter, which relates, by means of the functor $\mathscr{E}$, the component of the global model of a connected simplicial set X at a given 0-simplex with $A_{\mathrm{PL}}(X)$.

On the other hand, this weak equivalence also constitutes the starting point for the construction of a bridge linking Sullivan models with Lie models. In fact, we provide explicit algorithms to obtain a Sullivan model from the minimal Lie model of a given connected simplicial set, and vice versa.

As a result of all of the above we can show, for instance, that the homotopy Lie algebra $\pi_{(\wedge,V,d)}$ associated to the minimal model $(\wedge V, d)$ of a connected simplicial

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set X of finite type is isomorphic to the graded Lie algebra $H(\mathfrak{L}_X^a)$. In particular $H_0(\mathfrak{L}_X^a)$, with the group structure given by the BCH product, is isomorphic to the Malcev $\mathbb{Q}$-completion of $\pi_1(X)$.

The explicit link between the Sullivan and the Lie model also enables us to introduce the notion of coformality by means of the following result: for a given connected simplicial set of finite type, the differential on its Sullivan minimal model is purely quadratic if and only if its minimal Lie model is quasi-isomorphic to a cdgl with zero differential.

10.1 Connecting the global model with the Sullivan, Quillen and Neisendorfer models

In this section and as usual, a will denote a 0-simplex of the simplicial set X or a degree -1 generator of $\mathfrak{L}_X$.

We also note that, in what follows, given a cdga A, the nomenclature $\mathscr{E}A^\#$ is not ambiguous as it can only denote the cdgl dual of the dglc $\mathscr{E}A$. In the same way, if A is of finite type, $\mathscr{L}A^\#$ does not lead to confusion as can only denote the Quillen functor $\mathscr{L}$ applied to the cdgc $A^\#$.

Recall also (see Corollary 7.26) the injective quasi-isomorphism

$$\mathfrak{L}_X^a \overset{\simeq}{\hookrightarrow} \mathfrak{L}_X/(a)$$

which is also a weak equivalence, as observed in Proposition 8.7. Thus, in the forthcoming results any of these cdgl's can be chosen if one is just interested on its homotopy type. The following crucial observation precisely relates these cdgl's with the PL-forms and therefore, with a given Sullivan model of X.

Theorem 10.1. *Let X be a connected simplicial set of finite type and let $(\wedge V, d)$ be a Sullivan model of X. Then, there are cdgl quasi-isomorphisms*

$$\mathscr{E}(\wedge V, d)^\# \overset{\simeq}{\longleftarrow} \mathscr{E}\big(A_{\mathrm{PL}}(X)\big)^\# \overset{\simeq}{\longrightarrow} \mathfrak{L}_X/(a).$$

That is, in terms of the model category in **cdgl**, the cdgl's $\mathscr{E}(\wedge V, d)^\#$ and $\mathfrak{L}_X/(a)$ have the same homotopy type.

Proof. For the first quasi-isomorphism choose a cdga quasi-isomorphism,

$$(\wedge V, d) \overset{\simeq}{\longrightarrow} A_{\mathrm{PL}}(X),$$

apply the functor $\mathscr{E}$ and Proposition 2.10(1) to obtain a dglc quasi-isomorphism

$$\mathscr{E}(\wedge V, d) \overset{\simeq}{\longrightarrow} \mathscr{E}\big(A_{\mathrm{PL}}(X)\big),$$

and finally, take its dual.

For the second quasi-isomorphism consider the dglc quasi-isomorphism in formula (9.3),

$$\overline{\mathfrak{L}}^c_X \xrightarrow{\simeq} \mathscr{E}\big(A_{\mathrm{PL}}(X)\big).$$

By taking duals and applying Theorem 9.11 we get the desired map

$$\mathscr{E}\big(A_{\mathrm{PL}}(X)\big)^{\#} \xrightarrow{\simeq} \mathfrak{L}_X/(a). \qquad \square$$

We see now that the previous result is all we need to show that, whenever X is a nilpotent simplicial set of finite type, then $\mathfrak{L}^a_X$ has the same homotopy type as the Neisendorfer model of X. In particular, if X is simply connected, we also recover $\lambda(X)$, the Quillen functor on X.

Recall from Definition 3.28 that, given the minimal Sullivan model $(\wedge V, d)$ of a nilpotent simplicial set of finite type, the Neisendorfer model of X is the dgl

$$\mathscr{L}(\wedge V, d)^{\#}.$$

Recall also that, when X is simply connected, a well-known theorem of M. Majewski in [97] connects $\lambda(X)$ with the Neisendorfer model of X by a sequence of quasi-isomorphisms, and thus, these two dgl's have the same homotopy type.

Theorem 10.2. *Let X be a nilpotent, finite type simplicial set. Then, $\mathfrak{L}^a_X$ is quasi-isomorphic to the Neisendorfer model of X. In particular, if X is simply connected,*

$$\mathfrak{L}^a_X \simeq \lambda(X).$$

Proof. By Proposition 3.27, together with Theorem 10.1, we have the following chain of weak equivalences:

$$\mathfrak{L}^a_X \simeq \mathscr{E}(\wedge V, d)^{\#} \cong \widehat{\mathscr{L}}(\wedge V, d)^{\#} \simeq \mathscr{L}(\wedge V, d)^{\#}. \qquad \square$$

Remark 10.3. Observe that this result partially justifies working with complete cdgl's, or more generally (see Remark 3.9) with pronilpotent dgl's, instead of profinite dgl's. For instance, if X is an infinite wedge of spheres of dimension 2, then $\lambda(X)$ is the free graded Lie algebra $\mathbb{L}(V)$ on an infinite-dimensional vector space V concentrated in degree 1. As for any other simply connected dgl, $\lambda(X)$ is equal to its completion, and weakly equivalent to $\mathfrak{L}^a_X$ by the previous result. On the other hand however, the profinite completion $\lambda(X)^{\mathrm{f}}$ is much bigger than $\lambda(X)$.

Next, we give another important geometrical consequence of Theorem 10.1. Given the Sullivan minimal model $(\wedge V, d)$ of a connected simplicial set of finite type, we denote by $\pi_{(\wedge V,d)}$ its homotopy Lie algebra. See Section 1.2.2 and in particular formula (1.21) for an explicit description of this graded Lie algebra.

Theorem 10.4. *Let X be a connected simplicial set of finite type and let $(\wedge V, d)$ be its Sullivan minimal model. Then, as graded Lie algebras,*

$$\pi_{(\wedge V,d)} \cong H(\mathfrak{L}^a_X).$$

Proof. Recall from Lemma 2.13 that

$$H(\mathscr{E}(\wedge V, d)) \cong sV \tag{10.1}$$

as graded Lie coalgebras, where the Lie coalgebra structure in sV is given in Example 1.5. In the same example it is also shown that the dual graded Lie algebra is precisely the homotopy Lie algebra of $(\wedge V, d)$. Hence, taking duals in equation (10.1) and applying Theorem 10.1 we obtain the following chain of isomorphisms of graded Lie algebras:

$$H(\mathfrak{L}_X^a) \cong H(\mathscr{E}(\wedge V, d))^\# \cong \pi_{(\wedge V, d)}. \qquad \square$$

As a corollary, we obtain:

Theorem 10.5. *Let X be a connected simplicial set of finite type. Then, $H_0(\mathfrak{L}_X^a)$ is the Malcev completion of the fundamental group $\pi_1(X)$.*

Proof. Let $(\wedge V, d)$ be the Sullivan minimal model of X. Then, by the previous theorem, $H_0(\mathfrak{L}_X^a)$ is isomorphic to the Lie algebra $V_1^\#$, which is the Malcev completion of $\pi_1(X)$ by [51, Theorem 7.5]. $\square$

Recall from Theorem 10.1 that, for any connected simplicial set of finite type X,

$$\mathscr{E}(A_{\mathrm{PL}}(X))^\# \simeq \mathfrak{L}_X/(a). \tag{10.2}$$

We finish the section by detecting a dglc inside $\mathscr{E}(A_{\mathrm{PL}}(X))^\#$ whose dual is precisely the minimal Lie model of X:

Theorem 10.6. *Let X be a connected simplicial set. Then, there is a free dglc*

$$m_X^{\mathrm{c}} = \big(\mathbb{L}^{\mathrm{c}}(s\widetilde{H}^*(X)), d\big),$$

in which the linear part of the differential is zero, which is quasi-isomorphic to $\mathscr{E}(A_{\mathrm{PL}}(X))$.

Moreover, whenever X is of finite type, the dual of m_X^{c} is the minimal Lie model m_X of X.

Proof. As in Remark 9.8 fix a 0-simplex $a \in X$ and its corresponding augmentation in $N^*(X)$. This induces an augmentation in $H^*(X)$ whose augmentation ideal is precisely the reduced homology $\widetilde{H}^*(X)$. Choose an augmentation transfer diagram of the form

$$\circlearrowleft\; N^*(X) \rightleftarrows (H^*(X), 0),$$

whose existence is guaranteed by Proposition 1.9. Also, consider the transfer diagram of Corollary 9.7,

$$\kappa \circlearrowleft\; A_{\mathrm{PL}}(X) \underset{\iota}{\overset{p}{\rightleftarrows}} N^*(X),$$

in which again we assume that all maps preserve the corresponding augmentations. Composing both homotopy retractions via Proposition 1.10 produces a new transfer diagram of augmented maps,

$$\circlearrowright \; A_{\mathrm{PL}}(X) \rightleftarrows (H^*(X), 0).$$

Then, by Theorem 1.8, there is a quasi-isomorphism of C_∞-algebras

$$(H^*(X), 0) \xrightarrow{\simeq} A_{\mathrm{PL}}(X),$$

to which we apply the functor $\mathscr{E}$ obtaining in this way a dglc quasi-isomorphism:

$$\big(\mathbb{L}^c\big(s\widetilde{H}^*(X)\big), d\big) \xrightarrow{\simeq} \mathscr{E}\big(A_{\mathrm{PL}}(X)\big).$$

The linear part of the differential d is 0, by construction. By Proposition 3.23, if X is of finite type, the dual dgl,

$$\big(\mathbb{L}^{\mathrm{c}}\big(s\widetilde{H}^*(X)\big), d\big)^\# \cong \big(\widehat{\mathbb{L}}\big(s^{-1}\widetilde{H}_*(X)\big), d\big),$$

is a minimal cdgl which is then quasi-isomorphic to $\mathscr{E}\big(A_{\mathrm{PL}}(X)\big)^\#$. In view of (10.2), it follows that

$$\big(\widehat{\mathbb{L}}\big(s^{-1}\widetilde{H}_*(X)\big), d\big) \simeq \mathfrak{L}^a_X,$$

and thus, by the uniqueness of the minimal model of X,

$$\big(\widehat{\mathbb{L}}\big(s^{-1}\widetilde{H}_*(X)\big), d\big) \cong m_X. \qquad \square$$

Definition 10.7. Given a connected simplicial set X, the dglc m^{c}_X obtained in the preceding theorem is called the *minimal dglc model of* X.

10.2 From the Lie minimal model to the Sullivan model and vice versa

Let X be a simplicial set of finite type. It is quite easy to construct a Sullivan model of X from its minimal Lie model.

Theorem 10.8. *Let m_X be the minimal Lie model of the connected, finite type simplicial set X. Then, the cdga*

$$\varinjlim_n \mathscr{C}^*(m_X/m_X^n)$$

is a Sullivan model of X.

Proof. Let $m_X = (\widehat{\mathbb{L}}(V), d)$ and $m_X^{\rm c} = (\mathbb{L}^c(W), {\rm d})$ be the minimal Lie model and minimal dglc model of X, respectively. Hence, by Theorem 10.6,

$$\mathscr{E}\big(A_{\rm PL}(X)\big) \simeq m_X^{\rm c} \qquad \text{and} \qquad (m_X^{\rm c})^\# \cong m_X.$$

First apply the functor $\mathscr{A}$ to the above quasi-isomorphism so that, in view of Propositions 2.10(2) and 2.15, we obtain,

$$A_{\rm PL}(X) \simeq \mathscr{A}\mathscr{E}\big(A_{\rm PL}(X)\big) \simeq \mathscr{A}(m_X^{\rm c}). \tag{10.3}$$

Note that

$$m_X^{\rm c} = \varinjlim_n m_n^{\rm c}, \qquad \text{with} \qquad m_n^{\rm c} = \big((\mathbb{L}^{\rm c})^{\leq n}(W), d\big). \tag{10.4}$$

Since $\mathscr{A}$ is a left adjoint functor, it commutes with inductive limits, and therefore

$$A_{\rm PL}(X) \simeq \mathscr{A}(\varinjlim_n m_n^{\rm c}) \simeq \varinjlim_n \mathscr{A}(m_n^{\rm c}).$$

Now, since X is of finite type, each $m_n^{\rm c}$ is also of finite type and its dual cdgl is

$$m_n^{{\rm c}\,\#} = m_X/m_X^n. \tag{10.5}$$

Hence, as observed in Remark 2.8,

$$\mathscr{A}(m_n^{\rm c}) = \mathscr{C}^*(m_n^{{\rm c}\,\#}) = \mathscr{C}^*(m_X/m_X^n),$$

and therefore,

$$A_{\rm PL}(X) \simeq \varinjlim_n \mathscr{C}^*(m_X/m_X^n).$$

Finally, observe that the right-hand side of this equation is a Sullivan algebra which, being quasi-isomorphic to $A_{\rm PL}(X)$, is a Sullivan model of X. □

Recovering the minimal Lie model of a simplicial set from a given Sullivan model is more complicated. We first need the following.

Lemma 10.9. *Any cdga A whose cohomology is of finite type with $H^0(A) = \mathbb{Q}$ has a Sullivan model $(\land W, d)$ in which the differential has only linear and quadratic terms and each $W(n)$ is of finite type.*

Proof. Consider first $H(A)$ with trivial differential and also, with trivial multiplication, except on $H^0(A)$. We then construct, in the standard and classical bigraded way (see, for instance, Example 7 of [50, §12(d)]), the minimal model of $H(A)$,

$$\rho\colon (\land W, d_2) \xrightarrow{\simeq} H(A).$$

By construction, d_2 is quadratic and W inherits a second lower gradation

$$W = \bigoplus_{q\geq 0} W_q \qquad \text{such that} \qquad d_2 W_q \subset (\land^2 W)_{q-1}.$$

Moreover, for each $n \geq 0$,

$$W(n) = \bigoplus_{q \leq n} W_q$$

is a finite type graded vector space. Starting from ρ we now construct a cdga morphism

$$\varphi \colon (\wedge W, d) \longrightarrow A,$$

in which $d = d_1 + d_2$ with $d_1 W \subset W$ and d_2 exactly as before. For this, we define d_0 and $\varphi(n)$ inductively on $\wedge W(n)$ so that

$$\varphi(n) \colon (\wedge W(n), d_1 + d_2) \longrightarrow A$$

is a cdga morphism. For $n = 0$, set

$$\varphi(0) \colon \wedge W(0) \longrightarrow A$$

by declaring $\varphi(0)(w)$ to be a cycle representing $\rho(w)$. Suppose we have constructed

$$\varphi(n-1) \colon (\wedge W(n-1), d_1 + d_2) \longrightarrow (A, d),$$

and let w be an element of a given basis of W_n. Since $d_2^2 = 0$ in W and $d^2 = (d_1 + d_2)^2 = 0$ in $W(n-1)$, we have $d_2 d_1 d_2 w = -d_1 d_2^2 w = 0$. Therefore, $d_1 d_2 w$ is a d_2-cycle in $\wedge^2 W$. Since ρ is a quasi-isomorphism, this is a d_2-boundary:

$$d_1 d_2 w = d_2 z, \quad \text{with} \quad z \in W_1 \oplus \cdots \oplus W_{n-1}.$$

Moreover, $d_2 d_1 z = -d_1 d_2 z = -d_1^2 d_2 w = 0$. Therefore, $d_1 z$ is a d_2-cycle in $W_1 \oplus \cdots \oplus W_{n-1}$, so that $d_1 z = 0$.

It follows that $d_2 w - z$ is a $(d_1 + d_2)$-cycle in $\wedge W(n-1)$ and there are an element u in W_0 and an $x \in A$ such that

$$\varphi(n-1)(d_2 w - z - u) = dx.$$

We define

$$d_1 w = -z - u \qquad \text{and} \qquad \varphi(n)(w) = x.$$

This defines a differential $d = d_1 + d_2$ on $\wedge W$ and the cdga morphism $\varphi \colon (\wedge W, d) \to A$. Finally, Since ρ is a quasi-isomorphism, a spectral sequence argument shows that φ is also a quasi-isomorphism. Hence, $(\wedge W, d)$ is the Sullivan model with the desired properties. □

Observe that, for each $n \geq 0$, since $W(n)$ is of finite type and the differential in $(\wedge W, d)$ only has linear and quadratic terms, then

$$(\wedge W(n), d) \cong \mathscr{C}^*(L_n)$$

for a finite-dimensional nilpotent dgl L_n. Indeed, if $V = W_0^{\#}$, one easily checks that

$$L_n = (\mathbb{L}(V)/\mathbb{L}^{>n}(V), d).$$

Consider the cdgl

$$L = \varprojlim_n L_n = (\widehat{\mathbb{L}}(V), d).$$

With the above notation, let X be a connected simplicial set of finite type and let A be its Sullivan minimal model. Then:

Proposition 10.10. *L is the minimal Lie model of X.*

Proof. By construction,

$$L = (\mathbb{L}^{\mathrm{c}}(V), d)^{\#},$$

where $(\mathbb{L}^{\mathrm{c}}(V), d)$ is a Sullivan dglc (see Definition 2.14). Therefore, by Proposition 2.15, we have a quasi-isomorphism

$$(\mathbb{L}^{\mathrm{c}}(V), d) \xrightarrow{\simeq} \mathscr{E}\mathscr{A}(\mathbb{L}^{\mathrm{c}}(V), d). \tag{10.6}$$

The functor $\mathscr{A}$ is a left adjoint functor and thus commutes with inductive limits. Take also into account Remark 2.8, by which $\mathscr{C}^*(M) = \mathscr{A}(M^{\#})$ whenever M is finite type and connected. Finally, recall from Theorem 10.1 that

$$\mathscr{E}(\wedge W, d)^{\#} \simeq \mathfrak{L}_X/(a).$$

With all this in mind, plus equation (10.6), we obtain the following sequence of quasi-isomorphisms:

$$\begin{aligned} L &= (\mathbb{L}^{\mathrm{c}}(V), d)^{\#} \simeq (\mathscr{E}\mathscr{A}(\mathbb{L}^{\mathrm{c}}(V), d))^{\#} \\ &= \big(\mathscr{E}\mathscr{A}\varinjlim_n(\mathbb{L}^{\mathrm{c}\leq n}(V), d)\big)^{\#} = \big(\mathscr{E}\varinjlim_n \mathscr{A}(\mathbb{L}^{\mathrm{c}\leq n}(V), d)\big)^{\#} \\ &\cong \big(\mathscr{E}\varinjlim_n \mathscr{C}^*(L/L^{>n})\big)^{\#} \cong \big(\mathscr{E}\varinjlim(\wedge W(n), d)\big)^{\#} \\ &= \mathscr{E}(\wedge W, d)^{\#} \simeq \mathfrak{L}_X^a. \end{aligned}$$

Thus, L is necessarily the minimal Lie model of X. □

10.3 Coformal spaces

Let X be a connected simplicial set of finite type, with minimal Lie model m_X and minimal Sullivan model $(\wedge V, d)$.

Proposition 10.11. *With the above notations, the following assertions are equivalent:*

(1) *The differential d in $(\wedge V, d)$ is quadratic, $d\colon V \to \wedge^2 V$.*

(2) *m_X is quasi-isomorphic to $(H(m_X), 0)$.*

Proof. First assume that the differential in the Sullivan minimal model $(\wedge V, d)$ is quadratic and, as in (1.20), denote by

$$L = \pi_{(\wedge V,d)}$$

the rational homotopy Lie algebra of $(\wedge V, d)$, which is considered henceforth as cdgl with zero differential. Since X is of finite type, L is isomorphic to $H(\mathfrak{L}^a_X)$ (see Theorem 10.4) and in particular, each L/L^n is also of finite type. In view of the explicit form of the bracket in L given in (1.21), one readily sees that,

$$(\wedge V, d) = \varinjlim_n \mathscr{C}^*(L/L^n).$$

Note also that in view of Remark 2.8, for each $n \geq 1$,

$$\mathscr{C}^*(L/L^n) = \mathscr{A}\big((L/L^n)^\#\big).$$

Let m^{c}_X be the minimal dglc model of X (see Definition 10.7) and recall that, as noted in (10.3),

$$A_{\mathrm{PL}}(X) \simeq \mathscr{A}(m^{\mathrm{c}}_X).$$

Finally, take into account that $\mathscr{A}$ preserves inductive limits as it is left adjoint.

By all of the above, we have the identities

$$\begin{aligned}\mathscr{A}(m^{\mathrm{c}}_X) \simeq A_{\mathrm{PL}}(X) \simeq (\wedge V, d) = \varinjlim_n \mathscr{C}^*(L/L^n) &= \varinjlim_n \mathscr{A}\big((L/L^n)^\#\big)\\ &= \mathscr{A}\big(\varinjlim_n (L/L^n)^\#\big).\end{aligned}$$

In particular,

$$\mathscr{A}(m^{\mathrm{c}}_X) \simeq \mathscr{A}\big(\varinjlim_n (L/L^n)^\#\big).$$

Now we apply the functor $\mathscr{E}$ to both sides and take into account Proposition 2.10(1) together with Proposition 2.15 to obtain:

$$m^{\mathrm{c}}_X \simeq \mathscr{E}\mathscr{A}(m^{\mathrm{c}}_X) \simeq \mathscr{E}\mathscr{A}\big(\varinjlim_n (L/L^n)^\#\big) \simeq \big(\varinjlim_n (L/L^n)^\#\big).$$

By taking duals, it follows that

$$m_X \simeq (\varprojlim L/L^n, 0) = (L, 0) \tag{10.7}$$

and thus, m_X is quasi-isomorphic to $(H(m_x), 0)$.

Conversely, assume that there is a quasi-isomorphism

$$m_X \xrightarrow{\simeq} (L, 0), \quad \text{with} \quad L = H(m_X).$$

Consider, for each $n \geq 1$, the induced morphism,

$$m_X/m_X^n \xrightarrow{\simeq} L/L^n$$

and the corresponding dual

$$(L/L^n)^{\#} \xrightarrow{\simeq} (m_X/m_X^n)^{\#}.$$

Take inductive limits to obtain a quasi-isomorphism,

$$\varinjlim_n (L/L^n)^{\#} \xrightarrow{\simeq} \varinjlim_n (m_X/m_X^n)^{\#}.$$

In view of the identities (10.4) and (10.5), the codomain of this map is

$$\varinjlim (m_X/m_X^n)^{\#} = m_X^{\mathrm{c}},$$

that is, the minimal dglc model of X.

Hence, if we denote by L^{c} the dglc $\varinjlim_n (L/L^n)^{\#}$ with zero differential, we have a quasi-isomorphism

$$L^{\mathrm{c}} \xrightarrow{\simeq} m_X^{\mathrm{c}}.$$

By Proposition 2.10(2), the functor $\mathscr{A}$ preserves quasi-isomorphisms in $\mathbf{dglc}_0$ and, as observed above, $\mathscr{A}(m_X^{\mathrm{c}}) \simeq A_{\mathrm{PL}}(X)$. Hence,

$$\mathscr{A}(L^{\mathrm{c}}) \simeq \mathscr{A}(m_X^{\mathrm{c}}) \simeq A_{\mathrm{PL}}(X) \simeq (\wedge V, d).$$

But, since the differential in the dglc L^{c} is 0, we have that $\mathscr{A}(L^c)$ is a minimal cdga and its differential is purely quadratic. Hence, $\mathscr{A}$ must be isomorphic to $(\wedge V, d)$. □

Definition 10.12. A connected simplicial set of finite type satisfying any of the equivalent conditions of the above proposition is called *coformal*.

The following is an immediate consequence of the identity (10.7) in the proof of Proposition 10.11.

Corollary 10.13. *Let X be coformal space. Then, the minimal Lie model of X is the minimal Lie model of $(L, 0)$, where L is the rational homotopy Lie algebra of the minimal Sullilvan model of X.* □

Chapter 11

The Deligne–Getzler–Hinich Functor $\mathrm{MC}_\bullet$ and Equivalence of Realizations

Recall from Section 1.2.1 that the Sullivan geometrical realization of a given cdga A is given by the simplicial set

$$\langle A\rangle^{\mathrm{S}} = \mathrm{Hom}_{\mathbf{cdga}}\big(A, A_{\mathrm{PL}}(\underline{\Delta}^\bullet)\big).$$

In the case where the cdga is of the particular form $A = \mathscr{C}^*(L)$ for a connected finite type dgl L, we start by showing that

$$\langle \mathscr{C}^*(L)\rangle^{\mathrm{S}} \cong \mathrm{MC}\big(A_{\mathrm{PL}}(\underline{\Delta}^\bullet)\otimes L\big).$$

The latter simplicial set however makes sense for any cdgl, considering the complete tensor product,

$$\mathrm{MC}_\bullet(L) = \mathrm{MC}\big(A_{\mathrm{PL}}(\underline{\Delta}^\bullet)\widehat{\otimes} L\big)$$

and it is known as the *Deligne–Getzler–Hinich* ∞*-groupoid* of L. This is a well-studied realization functor of cdgl's whose main features are collected here in detail.

After that we present one of the main results in this text: for any connected cdgl L whose indecomposables $L/[L,L]$ are of finite type,

$$\langle L\rangle \simeq \mathrm{MC}_\bullet(L).$$

Then, we show the following, which constitutes another highlight of this book. Let X be a connected simplicial set of finite type, let a be one of its 0-simplices, and let $(\wedge V, d)$ be a Sullivan model of X. Then, there is a sequence of homotopy equivalences of simplicial sets,

$$\mathbb{Q}_\infty X \simeq \langle \wedge V, d\rangle \simeq \mathrm{MC}_\bullet(\mathfrak{L}_X^a) \simeq \langle \mathfrak{L}_X^a\rangle,$$

where $\mathbb{Q}_\infty X$ denotes the Bousfield–Kan $\mathbb{Q}$-completion of X. In particular,

$$\langle \mathfrak{L}_X\rangle \simeq \mathbb{Q}_\infty X^+,$$

where $\mathbb{Q}_\infty X^+$ denotes the disjoint union of $\mathbb{Q}_\infty X$ with an external point.

U. Buijs et al., *Lie Models in Topology*, Progress in Mathematics 335,
https://doi.org/10.1007/978-3-030-54430-0_11

An immediate consequence is that, in the simply connected finite type context, the realization functor coincides with the classical Quillen realization: if L is a simply connected dgl of finite type, then

$$\langle L\rangle \simeq \langle L\rangle^Q.$$

11.1 The set of Maurer–Cartan elements as a set of morphisms

Let L be a connected dgl and let A be a cdga. We assume that either L is finite-dimensional, or that it has finite type and $A^q = 0$ unless $0 \le q \le N$ for some N. Let

$$\mathscr{C}^*(L) = (\wedge (sL)^\#, d)$$

be the cochain algebra on L (see Definition 2.7), and choose a graded basis $\{x_i\}$ for L which provides a basis $\{\overline{x_i}\}$ for $(sL)^\#$ via the pairing

$$\langle \overline{x_i}, sx_j\rangle = -\delta_{ij}.$$

Finally, consider the linear isomorphism

$$\operatorname{Hom}_0((sL)^\#, A) = \operatorname{Hom}_{\mathbf{cga}}(\mathscr{C}^*(L), A) \stackrel{\varphi}{\longrightarrow} (A\otimes L)_{-1},$$

defined by

$$\varphi(f) = \sum_i (-1)^{|\overline{x_i}|} f(\overline{x_i})\otimes x_i.$$

Proposition 11.1. *Under the above hypotheses on L and A,*

(i) *The morphism φ restricts to a natural bijection*

$$\operatorname{Hom}_{\mathbf{cdga}}(\mathscr{C}^*(L), A) \cong \operatorname{MC}(A\otimes L)\,.$$

(ii) *Moreover, φ sends homotopic maps to gauge equivalent* MC *elements and induces a bijection*

$$[\mathscr{C}^*(L), A] \cong \widetilde{\operatorname{MC}}(A\otimes L)\,.$$

Proof. (i) We use the structure coefficients α_i^j and c_{ij}^k, defined by

$$dx_i = \sum_j \alpha_i^j x_j \quad \text{and} \quad [x_i, x_j] = \sum_k c_{ij}^k\, x_k\,.$$

Recall from (2.3) the description of the differential on the cochain functor. From the expression of the linear part $\langle d_1 g, c\rangle = -(-1)^{|g|}\langle g, dc\rangle$ we deduce that

$$d_1\overline{x_i} = (-1)^{|\overline{x_i}|}\sum_r \alpha_r^i\,\overline{x_r}\,.$$

On the other hand, the form of the quadratic part $\langle d_2\overline{x_i}, sx_j \wedge sx_k\rangle = (-1)^{|\overline{x_k}|}\langle \overline{x_i}, s[x_j, x_k]\rangle$ shows that

$$d_2\overline{x_i} = -\sum_{j\le k}(-1)^{|\overline{x_k}|}c^i_{jk}\overline{x_k}\,\overline{x_j}\,.$$

A straightforward computation yields that

$$\varphi(f) = \sum(-1)^{|\overline{x_i}|}f(\overline{x_i})\otimes x_i$$

is an MC element.

Conversely, if $\sum a_i\otimes x_i$ is an MC element, consider the cdga morphism

$$f\colon \mathscr{C}^*(L)\longrightarrow A,\quad \text{defined by}\quad f(\overline{x_i}) = a_i.$$

In fact, another straightforward computation, similar to the one above, shows that f commutes with the differentials.

(ii) Let

$$\Phi\colon \mathscr{C}^*(L)\longrightarrow A\otimes\wedge(t,dt)$$

be a homotopy between the morphisms $f, g\colon \mathscr{C}^*(L)\to A$. Then, by (i),

$$\varphi(\Phi)\in \mathrm{MC}(A\otimes\wedge(t,dt)\otimes L)$$

and satisfies

$$\varepsilon_0\varphi(\Phi) = \varphi(f)\qquad\text{and}\qquad \varepsilon_1\varphi(\Phi) = \varphi(f).$$

Since $A\otimes\wedge(t,dt)\otimes L$ is the path object of $A\otimes L$ (Proposition 8.17), by Corollary 8.27 this implies that $\varphi(f)$ and $\varphi(g)$ are gauge equivalent.

For the converse, if u and v are gauge equivalent Maurer–Cartan elements in $A\otimes L$, then by Corollary 8.27, there is a Maurer–Cartan element

$$z\in\mathrm{MC}(A\otimes\wedge(t,dt)\otimes L)\quad\text{such that}\quad \varepsilon_0(z) = u,\quad \varepsilon_1(z) = v.$$

By (i), this yields a cdga morphism

$$\Phi\colon \mathscr{C}^*(L)\longrightarrow A\otimes\wedge(t,dt)\quad\text{such that}\quad \varphi(\Phi) = z.$$

Then $u = \varphi(\varepsilon_0\circ\Phi)$ and $v = \varphi(\varepsilon_1\circ\Phi)$. Therefore $\varphi^{-1}u$ and $\varphi^{-1}v$ are homotopic morphisms. □

If we want to extend this result to **cdgl** we need to be precise on the boundedness and/or finiteness assumptions. Let

$$L = \varprojlim_n L/F^n$$

be a connected cdgl and let A be a cdga. We assume that either each L/F^n is finite-dimensional, or each L/F^n has finite type and $A^q = 0$ unless $0\le q\le N$ for some N. Note, for instance, that each L/F^n is finite-dimensional or has finite type if that is the case for $L/[L,L]$.

Proposition 11.2. *Under the above hypotheses on L and A there is a natural bijection*

$$\varphi\colon \operatorname{Hom}_{\mathbf{cdga}}(\varinjlim_n \mathscr{C}^*(L/F^n), A) \xrightarrow{\cong} \mathrm{MC}(A\widehat{\otimes}L)\,.$$

Moreover, φ sends homotopic maps to gauge equivalent MC *elements and induces a bijection*

$$[\varinjlim_n \mathscr{C}^*(L/F^n), A] \cong \widetilde{\mathrm{MC}}(A\widehat{\otimes}L)\,.$$

Proof. For each $n \geq 1$ consider the bijection given by Proposition 11.1,

$$\varphi_n\colon \operatorname{Hom}_{\mathbf{cdga}}(\mathscr{C}^*(L/F^n), A) \xrightarrow{\cong} \mathrm{MC}(A\otimes L/F^n)$$

whose hypotheses are clearly fulfilled. Take the inverse limit of these bijections to obtain in this case another bijection

$$\varphi\colon \varprojlim_n \operatorname{Hom}_{\mathbf{cdga}}(\mathscr{C}^*(L/F^n), A) \xrightarrow{\cong} \varprojlim_n \mathrm{MC}(A\otimes L/F^n).$$

But,

$$\varprojlim_n \operatorname{Hom}_{\mathbf{cdga}}(\mathscr{C}^*(L/F^n), A) \cong \operatorname{Hom}_{\mathbf{cdga}}(\varinjlim_n \mathscr{C}^*(L/F^n), A)$$

and, since MC commutes with inverse limits,

$$\varprojlim_n \mathrm{MC}(A\otimes L/F^n) \cong \mathrm{MC}(\varprojlim_n A\otimes L/F^n) = \mathrm{MC}(A\widehat{\otimes}L).$$

Thus, φ is the asserted bijection.

The second part of the statement follows directly as in Proposition 11.1. □

It is convenient to note that, if we only care about the homotopy classes of morphisms or the set $\widetilde{\mathrm{MC}}$, we can relax the hypotheses in the two previous results.

Let first $L \simeq L'$ be connected, quasi-isomorphic cdgl's and let $A \simeq B$ be quasi-isomorphic cdga's. Assume that L and A satisfy the conditions of Proposition 11.1. Then,

Corollary 11.3. $[\mathscr{C}^*(L'), B] \cong \widetilde{\mathrm{MC}}(B\widehat{\otimes}L')$.

Proof. This follows directly from Proposition 11.1 combined with Proposition 4.38 and the fact that $\mathscr{C}^*$ preserves quasi-isomorphisms (see Section 2.1). □

On the other hand, let $A \simeq B$ be quasi-isomorphic cdga's and let $f\colon L \to L'$ be a cdgl morphism such that the induced morphism $f^n\colon F^n/F^{n+1} \xrightarrow{\simeq} G^n/G^{n+1}$ is a quasi-isomorphism for $n \geq 1$. Assume that L and A satisfy the conditions of Proposition 11.2. Then, the same argument applies to yield:

Corollary 11.4. $[\varinjlim_n \mathscr{C}^*(L'/G^n), B] \cong \widetilde{\mathrm{MC}}(B\widehat{\otimes}L')$. □

Next, we analyze the "dual" situation. Given a cdgc C and a dgl L, we consider the classical *convolution* dgl structure on $\operatorname{Hom}(C,L)$ in which the differential and the bracket are given by

$$(df)(c) = d(fc) - (-1)^{|f|} f(dc),$$

$$[f,g](c) = \sum_i (-1)^{|g|\,|c_i|}[f(c_i), g(c_i')] \quad \text{with} \quad \Delta c = \sum_i c_i \otimes c_i'\,.$$

We define the linear map

$$\theta\colon \operatorname{Hom}_{\mathbf{cgc}}\big(C, \mathscr{C}(L)\big) \longrightarrow \operatorname{Hom}_{-1}(C,L)$$

where $\theta(g)$ is the composition

$$C \xrightarrow{g} \mathscr{C}(L) \xrightarrow{\omega} L,$$

in which $\omega(sx) = -x$, $\omega(1) = \omega(\wedge^{\geq 2} sL) = 0$.

Proposition 11.5. *Let C be a cdgc and L a cdgl. Then, with the above notation,*

(i) *θ induces a natural bijection*

$$\theta\colon \operatorname{Hom}_{\mathbf{cdgc}}\big(C, \mathscr{C}(L)\big) \cong \operatorname{MC}\big(\operatorname{Hom}(C,L)\big).$$

(ii) *If moreover L and $C^\#$ satisfy the condition of Proposition* 11.2*, we have a commutative diagram*

$$\begin{array}{ccc} \operatorname{Hom}_{\mathbf{cdgc}}\big(C, \mathscr{C}(L)\big) & \xrightarrow[\cong]{\theta} & \operatorname{MC}\big(\operatorname{Hom}(C,L)\big) \\ \cong \big\downarrow \rho_1 & & \cong \big\downarrow \rho_2 \\ \operatorname{Hom}_{\mathbf{cdga}}(\varinjlim_n \mathscr{C}^*(L/F^n), C^\#) & \xrightarrow[\cong]{\varphi} & \operatorname{MC}(C^\# \widehat{\otimes} L) \end{array}$$

where $\{F^n\}$ is the filtration associated to L and ρ_1 and ρ_2 are induced by dualization and φ is the isomorphism of Proposition 11.2.

Proof. Part (i) follows from an easy computation. For (ii) note first that, with the notation in the beginning of the section, we have

$$\omega = \sum_i \overline{x}_i\, x_i,$$

considering each $\overline{x}_i$ as a linear map in $\operatorname{Hom}(\mathscr{C}(L), \mathbb{Q})$. Then,

$$\rho_2\theta(g) = \sum_i \overline{x_i} \circ g \otimes x_i = \sum_i \rho_1(g)(\overline{x}_i) \otimes x_i = \varphi\rho_1(g). \qquad \square$$

11.2 Simplicial contractions of $A_{\mathrm{PL}}(\underline{\Delta}^\bullet)$

Recall from Proposition 1.1 that $A_{\mathrm{PL}}(\underline{\Delta}^\bullet)$ is a contractible simplicial cdga. However, in the next section we need to be much more precise and we will make use of a particular contraction which is compatible with faces and degeneracies. The purpose here is to present this contraction in detail along the lines of [45]. To this end we first make some arrangements concerning notation:

Recall from the identity (1.16) in Section 1.2.1 that

$$A_{\mathrm{PL}}(\underline{\Delta}^n) \cong \Omega_n = (\wedge(t_0,\dots,t_n,dt_0,\dots,dt_n)/\mathcal{I},\, d),$$

where $|t_i|=0$, $|dt_i|=1$, $d(t_i)=dt_i$, and $\mathcal{I}$ is the ideal generated by $(\sum_{i=0}^n t_i)-1$ and $\sum_{i=0}^n dt_i$.

In particular

$$A_{\mathrm{PL}}(\underline{\Delta}^1) = (\wedge(t_0,t_1,dt_0,dt_1)/\mathcal{I}) \cong \wedge(t,dt).$$

On the other hand,

$$A_{\mathrm{PL}}(\underline{\Delta}^n\times\underline{\Delta}^1) \cong A_{\mathrm{PL}}(\underline{\Delta}^n)\otimes A_{\mathrm{PL}}(\underline{\Delta}^1),$$

whose elements can therefore be written as

$$\omega = \omega_1(t)+\omega_2(t)dt, \quad \text{with} \quad \omega_1(t),\omega_2(t)\in A_{\mathrm{PL}}(\underline{\Delta}^n)\otimes\wedge t. \tag{11.1}$$

We will rename this cdga as

$$A_{\mathrm{PL}}(\underline{\Delta}^n\times I).$$

Note that

$$A_{\mathrm{PL}}(\underline{\Delta}^\bullet\times I)$$

is a simplicial cdga with the faces and degeneracies induced by those of $A_{\mathrm{PL}}(\underline{\Delta}^\bullet)$.

Also, for any $n\ge 0$ and any $i=0,\dots,n$ we define the *evaluation at the vertex* i as the augmentation

$$\mathrm{ev}_i\colon\ A_{\mathrm{PL}}(\underline{\Delta}^n)\longrightarrow\mathbb{Q}, \qquad \mathrm{ev}_i(t_j)=\begin{cases}1, & \text{if } i=j,\\ 0, & \text{otherwise.}\end{cases}$$

Finally, and for simplicity of notation, we write f^* to denote the morphism $A_{\mathrm{PL}}(f)$ for any given simplicial map f.

The special simplicial contraction we need is given by the following:

Proposition 11.6. *There are linear maps $h_i\colon A_{\mathrm{PL}}(\underline{\Delta}^n)\to A_{\mathrm{PL}}(\underline{\Delta}^n)$ of degree 1, with $i=0,\dots,n$, such that*

$$dh_i+h_id=\mathrm{id}-\mathrm{ev}_i, \qquad \mathrm{ev}_i\,h_i=h_i^2=0,$$

and the following identities are satisfied:

$$d_jh_i=\begin{cases}h_id_j, & \text{if } i<j,\\ h_{i+1}d_j, & \text{if } i\ge j,\end{cases} \qquad h_is_j=\begin{cases}s_jh_i, & \text{if } i\le j,\\ s_jh_{i-1}, & \text{if } i>j.\end{cases} \tag{11.2}$$

Here we always consider $\mathbb{Q}=\mathrm{Im}\,\mathrm{ev}_i$ as a sub-cdga of $A_{\mathrm{PL}}(\underline{\Delta}^n)$.

We begin by defining *integration along the fibre* as the following linear map of degree 1:

$$\mu\colon A_{PL}(\underline{\Delta}^n \times I) \longrightarrow A_{PL}(\underline{\Delta}^n), \qquad \mu(\omega) = (-1)^{p-1}\int_0^1 \omega_2(t)dt\,,$$

with $\omega = \omega_1(t) + \omega_2(t)dt$ as in (11.1).

Next, consider the topological injections $i_0, i_1\colon \Delta^n \to \Delta^n \times I$, $i_k(x) = (x,k)$, and denote in the same way the corresponding induced injections

$$i_0, i_1\colon \underline{\Delta}^n \longrightarrow \underline{\Delta}^n \times \underline{\Delta}^1.$$

Then we have:

Lemma 11.7. *Integration along the fibre is a simplicial map which satisfies*

$$d\mu + \mu d = i_1^* - i_0^*.$$

Proof. First, an easy computation shows that the map

$$i_o^* - i_1^*\colon\ A_{\mathrm{PL}}(\underline{\Delta}^n \times I) \longrightarrow A_{\mathrm{PL}}(\underline{\Delta}^n)$$

is given by

$$(i_1^* - i_0^*)(\omega) = \omega_1(1) - \omega_1(0).$$

Now, for $\omega \in A_{\mathrm{PL}}(\underline{\Delta}^n)$,

$$d\omega = d\omega_1(t) + (-1)^p \frac{\partial\omega_1(t)}{\partial t}\,dt + d\omega_2(t)\,dt.$$

Therefore,

$$\mu d\omega = \int_0^1 \frac{\partial\omega_1(t)}{\partial t}dt + (-1)^p\int_0^1 d\omega_2(t)dt\,.$$

On the other hand,

$$d\mu(\omega) = (-1)^{p-1}\int_0^1 d_\Delta\omega_2(t)dt.$$

Hence,

$$(d\mu + \mu d)(\omega) = \int_0^1 \frac{\partial\omega_1(t)}{\partial t}dt = \omega_1(1) - \omega_1(0) = (i_1^* - i_0^*)(\omega).$$

Finally, one easily checks the commutativity of these diagrams, which proves the simplicial character of μ:

$$\begin{array}{ccc} A_{\mathrm{PL}}(\underline{\Delta}^n \times I) & \xrightarrow{\ \mu\ } & A_{\mathrm{PL}}(\underline{\Delta}^n) \\ {\scriptstyle A_{\mathrm{PL}}(\delta^i\times\mathrm{id})}\big\downarrow & & \big\downarrow{\scriptstyle d_i} \\ A_{\mathrm{PL}}(\underline{\Delta}^{n-1} \times I) & \xrightarrow{\ \mu\ } & A_{\mathrm{PL}}(\underline{\Delta}^{n-1})\,, \end{array} \qquad \begin{array}{ccc} A_{\mathrm{PL}}(\underline{\Delta}^n \times I) & \xrightarrow{\ \mu\ } & A_{\mathrm{PL}}(\underline{\Delta}^n) \\ {\scriptstyle A_{\mathrm{PL}}(\sigma^i\times\mathrm{id})}\big\downarrow & & \big\downarrow{\scriptstyle s_i} \\ A_{\mathrm{PL}}(\underline{\Delta}^{n+1} \times I) & \xrightarrow{\ \mu\ } & A_{\mathrm{PL}}(\underline{\Delta}^{n+1})\,. \end{array}$$

□

Proof of Proposition 11.6. For each $i = 0, \ldots, n$, consider the (topological) map

$$b_i \colon \Delta^n \times I \longrightarrow \Delta^n, \qquad b_i(x,t) = tx + (1-t)e_i\,,$$

where e_i is the ith vertex of Δ^n and, as before, denote in the same way the induced simplicial map $b_i \colon \underline{\Delta}^n \times \underline{\Delta}^1 \to \underline{\Delta}^n$.

We remark that, as topological maps $\Delta^n \to \Delta^n$, the composition $b_i \circ i_0$ is the constant map at the vertex e_i and $b_i \circ i_1$ is the identity on Δ^n. Therefore,

$$dh_i + h_i d = d\mu b_i^* + \mu b_i^* d = (i_1^* - i_0^*)b_i^* = \mathrm{id} - \mathrm{ev}_i\ .$$

The identities in (11.2) are direct consequences of the definition of h_i and the simplicial character of μ.

It remains to prove that $\mathrm{ev}_i\, h_i = 0 = h_i^2$. For the first identity, note that the map

$$I \xrightarrow{e_i \times \mathrm{id}} \Delta^n \times I \xrightarrow{b_i} \Delta^n$$

is the constant map at the vertex e_i. Hence, $(\mathrm{ev}_i \otimes \mathrm{id})b_i^* = 0$ on $A_{\mathrm{PL}}^{\geq 1}(\underline{\Delta}^n)$, and so

$$\mathrm{ev}_i\, \mu\, b_i^* = \mu(\mathrm{ev}_i \otimes \mathrm{id})b_i^* = 0.$$

On the other hand, consider the map

$$\varphi = b_i \circ (b_i \times \mathrm{id})\colon \Delta^n \times I \times I \longrightarrow \Delta^n, \qquad \varphi(x,s,t) = tsx + (1-ts)e_i,$$

and denote in the same way the corresponding simplicial map. For $\omega \in A_{\mathrm{PL}}^p(\underline{\Delta}^n)$, write

$$\varphi^*(\omega) = \omega_1 + \omega_2 dt + \omega_3 ds + \omega_4 ds dt,$$

with

$$\omega_k \in A_{\mathrm{PL}}(\underline{\Delta}^n) \otimes \wedge(t,s) \quad \text{for} \quad k = 1,2,3,4.$$

Then,

$$h_i^2(\omega) = \iint_{I\times I} \omega_4 ds\, dt.$$

But observe that φ factorizes as the composition

$$\Delta^n \times I \times I \xrightarrow{\psi} \Delta^n \times I \xrightarrow{b_i} \Delta^n,$$

with $\psi(x,s,t) = (x,st)$. Hence, the double integral is equal to zero. □

Example 11.8. An easy computation shows that

$$b_i^*(t_j) = \begin{cases} tt_j, & \text{if } j \neq i, \\ tt_i + (1-t), & \text{if } j = i. \end{cases}$$

Hence, we have:

$$h_0(dt_1 \cdots dt_q) = \frac{1}{q}\sum_{i=1}^{q} (-1)^{i-1} t_i dt_1 \cdots \widehat{dt_i} \cdots dt_q = \frac{1}{q!}\omega_{i_1 \ldots i_q}\,.$$

11.3 The Deligne–Getzler–Hinich ∞-groupoid

Given a cdgl L and any $n \geq 0$, consider the cdgl

$$A_{\mathrm{PL}}(\underline{\Delta}^n)\widehat{\otimes}L.$$

The morphisms $d_i \otimes \mathrm{id}_L$ and $s_j \otimes \mathrm{id}_L$ induce cdgl morphisms, which we also denote by

$$d_i\colon\ A_{\mathrm{PL}}(\underline{\Delta}^n)\widehat{\otimes}L \longrightarrow A_{\mathrm{PL}}(\underline{\Delta}^{n-1})\widehat{\otimes}L$$

and

$$s_j\colon\ A_{\mathrm{PL}}(\underline{\Delta}^n)\widehat{\otimes}L \longrightarrow A_{\mathrm{PL}}(\underline{\Delta}^{n+1})\widehat{\otimes}L,$$

which makes of

$$A_{\mathrm{PL}}(\underline{\Delta}^\bullet)\widehat{\otimes}L$$

a simplicial cdgl.

Definition 11.9. The *Deligne–Getzler–Hinich ∞-groupoid* is the functor

$$\mathrm{MC}_\bullet\colon\ \mathbf{cdgl} \longrightarrow \mathbf{sset}$$

which associates to each cdgl L the simplicial set

$$\mathrm{MC}_\bullet(L) = \mathrm{MC}(A_{\mathrm{PL}}(\underline{\Delta}^\bullet)\widehat{\otimes}L).$$

This functor was introduced in [76] and then studied in depth in [60]. As we will see later, this is in fact a generalization of the Deligne groupoid of L. Here, we only collect two important properties of this functor that will be essential in the next section.

The first one, which already appears in [76, Proposition 2.2.3] and then in [60, Proposition 4.7] for L_∞-algebras, asserts that $\mathrm{MC}_\bullet$ takes surjective cdgl morphisms to Kan fibrations. In particular, it preserves fibrations of connected cdgl's.

Proposition 11.10. *If $f\colon L \to L'$ is a cdgl surjective morphism, then*

$$\mathrm{MC}_\bullet(f)\colon \mathrm{MC}_\bullet(L) \longrightarrow \mathrm{MC}_\bullet(L')$$

is a Kan fibration with fibre $\mathrm{MC}_\bullet(K)$*, where* $K = \ker f$*.*

For the proof we need some preliminaries. Given any cdgl L, we extend the degree -1 linear maps h_i's defined in Proposition 11.6 to

$$h_i\colon A_{\mathrm{PL}}(\underline{\Delta}^n)\widehat{\otimes}L \longrightarrow A_{\mathrm{PL}}(\underline{\Delta}^n)\widehat{\otimes}L$$

by tensoring with the identity on L. Then, by the same result, we have

$$dh_i + h_i d = \mathrm{id} - \mathrm{ev}_i\,,$$

where

$$\mathrm{ev}_i\colon A_{\mathrm{PL}}(\underline{\Delta}^n)\widehat{\otimes}L \longrightarrow L, \qquad \mathrm{ev}_i(\omega\otimes\gamma)=\mathrm{ev}_i(\omega)\gamma.$$

Denote

$$K_i=\ker\mathrm{ev}_i\,.$$

A first version of the following lemma can be found in [78, Section 8.2], while a general statement for L_∞-algebras is in [60, Lemma 4.6].

Lemma 11.11. *For any $i=0,\dots,n$, the map*

$$\Phi\colon \mathrm{MC}_n(L)\xrightarrow{\cong}\mathrm{MC}(L)\times d(K_i)_0, \qquad \Phi(\alpha)=\big(\mathrm{ev}_i(\alpha),dh_i(\alpha)\big),$$

is a bijection.

Proof. Obviously, ev_i preserves MC elements, as it is a cdgl morphism. Also, since $\mathrm{ev}_i\,h_i=0$ (see Proposition 11.6), $dh_i(\alpha)\in d(K_i)_0$ for any $\alpha\in\mathrm{MC}_n(L)$.

We first prove the surjectivity of Φ. Let $\mu\in\mathrm{MC}(L)$ and $\nu\in d(K_i)_0$. For each $k\geq 1$ we inductively construct degree -1 elements $\alpha_k\in A_{PL}(\underline{\Delta}^n)\widehat{\otimes}L$, not necessarily Maurer–Cartan, such that

$$\mathrm{ev}_i(\alpha_k)=\mu \qquad\text{and}\qquad dh_i(\alpha_k)=\nu.$$

For $k=0$, set

$$\alpha_1=\mu+\nu.$$

As ev_i commutes with differentials and $\nu\in dK_i$,

$$\mathrm{ev}_i(\alpha_1)=\mathrm{ev}_i(\mu)=\mu.$$

On the other hand, write $\nu=d\gamma$ with $\gamma\in K_i$ to obtain,

$$dh_i(\alpha_1)=dh_i(\mu+\nu)=dh_i(\nu)=d(\gamma-\mathrm{ev}_i\,\gamma-dh_i\gamma)=d\gamma=\nu\,.$$

Suppose α_k has been constructed and define

$$\alpha_{k+1}=\alpha_1-\frac{1}{2}h_i[\alpha_k,\alpha_k].$$

By Proposition 11.6,

$$\mathrm{ev}_i\,h_i=h_i^2=0,$$

and so the required properties are trivially satisfied.

Now, since

$$\alpha_{n+1}-\alpha_n=-\frac{1}{2}h_i[\alpha_n-\alpha_{n-1},\alpha_n+\alpha_{n-1}],$$

the element $\alpha_{n+1}-\alpha_n$ belongs to G^{n+1} being $\{G^n\}_{n\geq 1}$ the filtration associated to the cdgl $A_{\mathrm{PL}}(\underline{\Delta}^n)\widehat{\otimes}L$ (see Definition 4.37). Indeed, note that each h_i preserves the filtration. Hence, recalling the expression in (3.1), we may consider the element

$$\alpha=(\overline{\alpha}_n)_{n\geq 1}\in A_{\mathrm{PL}}(\underline{\Delta}^n)\widehat{\otimes}L,$$

where each $\overline{\alpha}_n$ denotes the class of α_n in $(A_{\mathrm{PL}}(\underline{\Delta}^n)\widehat{\otimes}L)/G^n$. This element trivially satisfies

$$\mathrm{ev}_i(\alpha)=\mu \qquad \text{and} \qquad dh_i(\alpha)=\nu.$$

Moreover,

$$\alpha=\alpha_0-\frac{1}{2}h_i[\alpha,\alpha]\,.$$

We finish by showing that $\alpha\in\mathrm{MC}_n(L)$. For this, set

$$F(\alpha)=d\alpha+\frac{1}{2}[\alpha,\alpha].$$

Then,

$$[\alpha,F\alpha]=[\alpha,d\alpha]=-\frac{1}{2}d[\alpha,\alpha].$$

Next, since $\mathrm{ev}_i(\alpha)=\mu$, we have

$$\begin{aligned}F(\alpha)&=d\alpha+\frac{1}{2}[\alpha,\alpha]=d\alpha_0-\frac{1}{2}dh_i[\alpha,\alpha]+\frac{1}{2}[\alpha,\alpha]\\&=d\mu+\frac{1}{2}\left(h_id[\alpha,\alpha]+\mathrm{ev}_i[\alpha,\alpha]-[\alpha,\alpha]\right)+\frac{1}{2}[\alpha,\alpha]\\&=d\mu+\frac{1}{2}h_id[\alpha,\alpha]+\frac{1}{2}[\mu,\mu]=\frac{1}{2}h_id[\alpha,\alpha]=-h_i[\alpha,F(\alpha)]\,.\end{aligned}$$

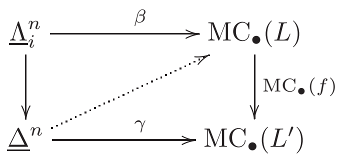
Therefore, $F(\alpha)\in\bigcap_n G^n=0$ and thus α is an MC element.

We now prove that Φ is injective. Let α and β in $\mathrm{MC}_n(L)$ be such that $\mathrm{ev}_i(\alpha)=\mathrm{ev}_i(\beta)$ and $dh_i\alpha=dh_i\beta$. Then,

$$\alpha-\beta=h_id(\alpha-\beta)=-\frac{1}{2}h_i([\alpha,\alpha]-[\beta,\beta])=-\frac{1}{2}h_i[\alpha-\beta,\alpha+\beta]\,.$$

Thus $\alpha-\beta\in\bigcap_n G^n=0$ and $\alpha=\beta$. $\square$

Proof of Proposition 11.10. Let $n\geq 0$ and $i=0,\dots,n$. We have to find a lifting for any given commutative square



In other words, if we denote

$$e=\gamma(0\dots n)\quad\text{and}\quad \beta_j=\beta(0\dots\widehat{j}\dots n)\quad\text{for}\quad j\neq i,$$

and we assume that $\mathrm{MC}_\bullet(f)(\beta_j)=f(\beta_j)=d_je$, we have to find $\alpha\in\mathrm{MC}_n(L)$ such that

$$d_j\alpha=\beta_j\quad\text{for}\quad j\neq i\qquad\text{and}\qquad \mathrm{MC}_n(f)(\alpha)=e.$$

Since $\mathrm{id}\,\widehat{\otimes}f\colon A_{\mathrm{PL}}(\underline{\Delta}^\bullet)\widehat{\otimes}L\to A_{\mathrm{PL}}(\underline{\Delta}^\bullet)\widehat{\otimes}L'$ is trivially a fibration, there is an element $\rho\in A_{\mathrm{PL}}(\underline{\Delta}^n)\widehat{\otimes}L$ such that

$$d_j\rho=\beta_j\quad\text{for}\quad j\neq i\qquad\text{and}\qquad f(\rho)=e.$$

To shorten the argument, we regard each face map d_j as the morphism

$$d_j\colon\ A_{\mathrm{PL}}(\underline{\Delta}^n)\widehat{\otimes}L\longrightarrow A_{\mathrm{PL}}(\underline{\Delta}^n)\widehat{\otimes}L,\qquad d_j(t_k\otimes x)=\begin{cases}t_k\otimes x, & \text{if } k\neq j,\\ 0, & \text{if } k=j.\end{cases}$$

With this convention notice that for $j\neq i$,

$$\mathrm{ev}_i\,d_j=\mathrm{ev}_i\qquad\text{and}\qquad h_id_j=d_jh_i.$$

Indeed, the first identity is obvious and the second follows from the first identity in (11.2) of Proposition 11.6.

In particular $\mathrm{ev}_i(\rho)=\mathrm{ev}_i(\beta_j)$ and, since β_j is an MC element, $\mathrm{ev}_i(\rho)\in\mathrm{MC}(L)$ is also Maurer–Cartan. On the other hand, again by Proposition 11.6, $h_i(\rho)\in K_i$. Hence, we use Lemma 11.11 to obtain an element $\alpha\in\mathrm{MC}_n(L)$ such that

$$\mathrm{ev}_i(\alpha)=\mathrm{ev}_i(\rho)\qquad\text{and}\qquad dh_i(\alpha)=dh_i(\rho).$$

Thus, for $j\neq i$,

$$dh_i(d_j\alpha)=d_jdh_i(\alpha)=d_jdh_i(\rho)=dh_i(d_j\rho)=dh_i(\beta_j)\,.$$

On the other hand,

$$\mathrm{ev}_i(d_j\alpha)=\mathrm{ev}_i(\alpha)=\mathrm{ev}_i(\rho)=\mathrm{ev}_i(d_j\rho)=\mathrm{ev}_i(\beta_j).$$

That is, $d_j\alpha$ and β_j are MC elements for which $\Phi(d_j\alpha)=\Phi(\beta_j)$. Therefore, by Proposition 11.11, $d_j\alpha=\beta_j$ for $j\neq i$.

Similar calculations also show that

$$\mathrm{ev}_i\big(f(\alpha)\big)=\mathrm{ev}_i(e)\qquad\text{and}\qquad dh_i\big(f(\alpha)\big)=dh_i(e),$$

that is, $\Phi\big(f(\alpha)\big)=\Phi(e)$. Hence, once again in view of Proposition 11.11, $f(\alpha)=e$, and we have shown that $\mathrm{MC}_\bullet(f)$ is a fibration.

Finally, we identify the fibre as asserted. Let

$$\omega=\sum\omega_i\otimes\gamma_i\in\ker\mathrm{MC}_\bullet(f),$$

where the ω_i's are linearly independent. Since $\mathrm{MC}_\bullet(f)(\omega)=\sum\omega_i\otimes f(\gamma_i)$, it follows that, for all i, $f(\gamma_i)=0$ and $\omega\in\mathrm{MC}_\bullet(K)$. □

Along the lines of [60, Theorem 4.8], we also show that the Deligne–Getzler–Hinich ∞-groupoid also preserves weak equivalences between connected cdgl's.

Proposition 11.12. *Let $f\colon L \xrightarrow{\simeq} L'$ be a quasi-isomorphism of connected cdgl's. Then*

$$\mathrm{MC}_\bullet(f)\colon \mathrm{MC}_\bullet(L) \xrightarrow{\simeq} \mathrm{MC}_\bullet(L')$$

is a weak equivalence of simplicial sets.

Proof. Denote by Z_n (respectively, B_n) the vector space of cycles (respectively, of boundaries) in L_n. We define a decreasing sequence F^k of sub-cdgl's of L, for $k \geq 0$, as follows.

$$F^{2k}_i = \begin{cases} 0, & \text{if } i < k,\\ Z_k, & \text{if } i = k,\\ L_i, & \text{if } i > k \end{cases} \quad \text{and} \quad F^{2k+1}_i = \begin{cases} 0, & \text{if } i < k,\\ B_k, & \text{if } i = k,\\ L_i, & \text{if } i > k. \end{cases}$$

In particular, $L = F^0$. Therefore,

$$(F^{2k}/F^{2k+1})_i = \begin{cases} 0, & \text{if } i \neq k,\\ H_k(L), & \text{if } i = k \end{cases}$$

and

$$(F^{2k+1}/F^{2k+2})_i = \begin{cases} 0, & \text{if } i \neq k, k+1,\\ B_k, & \text{if } i = k,\\ S_i, & \text{if } i = k+1, \end{cases}$$

where S_i is a complement of Z_i in L_i.

It follows then that

$$\mathrm{MC}_\bullet(F^{2k}/F^{2k+1}) = \mathrm{MC}(A_{PL}(\underline{\Delta}^\bullet) \otimes H_k(L)).$$

Analogous constructions are performed in L' to obtain the sequence G^k of sub-cdgl's of L', for $k \geq 0$.

By the above discussion, since f is a quasi-isomorphism, the induced map,

$$\mathrm{MC}_\bullet(F^{2k}/F^{2k+1}) \xrightarrow{\cong} \mathrm{MC}_\bullet(G^{2k}/G^{2k+1})$$

is an isomorphism.

On the other hand, for $k \geq 1$, notice that $\mathrm{MC}_\bullet(F^{2k+1}/F^{2k+2})$ is the vector space of cycles in $A_{PL}(\underline{\Delta}^\bullet) \otimes (F^{2k+1}/F^{2k+2})$. For $k \geq 2$, this is for degree reasons. For $k = 1$, this follows from the fact that $[B_1, B_1] \subset Z_2$. Thus, we have an isomorphism of simplicial sets,

$$\theta\colon A_{\mathrm{PL}}(\underline{\Delta}^\bullet) \otimes B_k(L) \xrightarrow{\cong} \mathrm{MC}_\bullet(F^{2k+1}/F^{2k+2}),$$

defined by

$$\theta(\omega \otimes du) = \omega \otimes du + (-1)^{|\omega|} d\omega \otimes u.$$

It follows from the Poincaré Lemma in Proposition 1.1 that both

$$\mathrm{MC}_\bullet(F^{2k+1}/F^{2k+2}) \qquad \text{and} \qquad \mathrm{MC}_\bullet(G^{2k+1}/G^{2k+2}),$$

are contractible. In particular, the morphism

$$\mathrm{MC}_\bullet(F^{2k+1}/F^{2k+2}) \xrightarrow{\simeq} \mathrm{MC}_\bullet(G^{2k+1}/G^{2k+2})$$

induced by f is trivially a homotopy equivalence.

Now, for each $k \geq 0$, applying $\mathrm{MC}_\bullet$ to the diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^k/F^{k+1} & \longrightarrow & L/F^{k+1} & \longrightarrow & L/F^k & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & G^k/G^{k+1} & \longrightarrow & L'/G^{k+1} & \longrightarrow & L'/G^k & \longrightarrow & 0 \end{array}$$

we get a commutative diagram of simplicial sets,

$$\begin{array}{ccccc} MC_\bullet(F^k/F^{k+1}) & \longrightarrow & MC_\bullet(L/F^{k+1}) & \longrightarrow & MC_\bullet(L/F^k) \\ \downarrow \simeq & & \downarrow & & \downarrow \\ MC_\bullet(G^k/G^{k+1}) & \longrightarrow & MC_\bullet(L'/G^{k+1}) & \longrightarrow & MC_\bullet(L'/G^k) \end{array}$$

where, by Proposition 11.10, both horizontal lines are fibration sequences and, by all of the above, the left vertical arrow is a homotopy equivalence. Hence, since for $k = 0$ the left vertical map is the identity on a point, an inductive argument shows that

$$\mathrm{MC}_\bullet(L/F^k) \xrightarrow{\simeq} \mathrm{MC}_\bullet(L'/G^k)$$

is a homotopy equivalence for all $k \geq 0$.

Finally, since MC preserves inverse limits,

$$\mathrm{MC}_\bullet(L) = \varprojlim_n \mathrm{MC}_\bullet(L/F^n)$$

and therefore, by [13, Chapter IX, Theorem 3.1], we have a commutative diagram of short exact sequences,

$$\begin{array}{ccccccccc} 1 \to & {\varprojlim_r}^1 \pi_{n+1}\mathrm{MC}_\bullet(L/F^r(L)) & \longrightarrow & \pi_n(\mathrm{MC}_\bullet(L)) & \longrightarrow & \varprojlim_r \pi_n(\mathrm{MC}_\bullet(L/F^r(L))) & \to 1 \\ & \downarrow \cong & & \downarrow \pi_n(\mathrm{MC}_\bullet(f)) & & \downarrow \cong & \\ 1 \to & {\varprojlim_r}^1 \pi_{n+1}\mathrm{MC}_\bullet(L'/F^r(L')) & \longrightarrow & \pi_n(\mathrm{MC}_\bullet(L')) & \longrightarrow & \varprojlim_r \pi_n(\mathrm{MC}_\bullet(L'/F^r(L'))) & \to 1 \end{array}$$

Hence, $\mathrm{MC}_\bullet(f)$ is a homotopy equivalence. □

11.4 Equivalence of realizations and Bousfield–Kan completion

We first prove that the Deligne–Getzler–Hinich ∞-groupoid functor coincides, up to homotopy, with the realization functor.

Theorem 11.13. *Let L be a connected cdgl such that L/L^2 is a finite type vector space. Then the simplicial sets $\mathrm{MC}_\bullet(L)$ and $\langle\, L\,\rangle$ are homotopy equivalent.*

Proof. Since L/L^2 is of finite type, each L/L^n is also of finite type (see the argument in the proof of 3.41). Hence, if $\{F^n\}_{n\geq 1}$ is the filtration associated to L, each L/F^n is also of finite type and thus

$$L = \varprojlim_n L/F^n$$

is a profinite dgl (see Definition 3.35). Then, By Proposition 3.37,

$$L = E^{\#},$$

where E is a conilpotent dglc.

Note that, since L is connected, for any $n \geq 0$,

$$\mathrm{Hom}_{\mathbf{cdgl}}(\mathfrak{L}_n, L) = \mathrm{Hom}_{\mathbf{cdgl}}(\mathfrak{L}_n/(a_0), L).$$

On the other hand, simply by taking duals and applying Corollary 9.12, we obtain that

$$\mathrm{Hom}_{\mathbf{cdgl}}(\mathfrak{L}_n/(a_0), L) = \mathrm{Hom}_{\mathbf{dglc}}(E, \overline{\mathfrak{L}}^{\mathrm{c}}_n).$$

Therefore, we have an isomorphism of simplicial sets

$$\langle L\rangle \cong \mathrm{Hom}_{\mathbf{dglc}}(E, \overline{\mathfrak{L}}^{\mathrm{c}}_\bullet). \tag{11.3}$$

Next, by composing with the simplicial homotopy equivalence given in Remark 9.13,

$$\overline{\mathfrak{L}}^{\mathrm{c}}_\bullet \xrightarrow{\simeq} \mathscr{E}\big(A_{\mathrm{PL}}(\underline{\Delta}^\bullet)\big),$$

we obtain a homotopy equivalence,

$$\mathrm{Hom}_{\mathbf{dglc}}(E, \overline{\mathfrak{L}}^{\mathrm{c}}_\bullet) \xrightarrow{\simeq} \mathrm{Hom}_{\mathbf{dglc}}\Big(E, \mathscr{E}\big(A_{\mathrm{PL}}(\underline{\Delta}^\bullet)\big)\Big). \tag{11.4}$$

Now recall from Proposition 2.9 that $\mathscr{E}$ is right adjoint to the functor $\mathscr{A}$. Hence, we have a simplicial isomorphism,

$$\mathrm{Hom}_{\mathbf{dglc}}\Big(E, \mathscr{E}\big(A_{\mathrm{PL}}(\underline{\Delta}^\bullet)\big)\Big) \cong \mathrm{Hom}_{\mathbf{cdga}}\big(\mathscr{A}(E), A_{\mathrm{PL}}(\underline{\Delta}^\bullet)\big). \tag{11.5}$$

Note that, by Proposition 3.37, the conilpotent dglc E can be chosen to be the increasing union of finite type dglc's E_n, with $n \geq 1$, such that

$$E_n^{\#} \cong L/F^n, \quad \text{for} \quad n \geq 1.$$

Hence, since $\mathscr{A}$ is left adjoint, it preserves direct limits and

$$\mathscr{A}(E) \cong \varprojlim_n \mathscr{A}(E_n). \tag{11.6}$$

But observe from Remark 2.8 that, since each E_n is connected and of finite type,

$$\mathscr{A}(E_n) = \mathscr{C}^*(E_n^{\#}) = \mathscr{C}(L/F^n). \tag{11.7}$$

Finally, taking into account the identities (11.6) and (11.7) together with Proposition 11.2, we have a sequence of isomorphisms of simplicial sets

$$\begin{aligned}\operatorname{Hom}_{\mathbf{cdga}}\big(\mathscr{A}(E), A_{\mathrm{PL}}(\underline{\Delta}^\bullet)\big) &\cong \operatorname{Hom}_{\mathbf{cdga}}\big(\varinjlim_n \mathscr{C}^*(L,F^n), A_{\mathrm{PL}}(\underline{\Delta}^\bullet)\big)\\ &\cong \mathrm{MC}(A_{\mathrm{PL}}(\underline{\Delta}^\bullet)\widehat{\otimes} L)\\ &= \mathrm{MC}_\bullet(L).\end{aligned}$$

This, in view of (11.3), (11.4) and (11.5), produces a homotopy equivalence of simplicial sets

$$\langle L\rangle \simeq \mathrm{MC}_\bullet(L). \qquad \square$$

With the above result we can easily show that, for any connected simplicial set X of finite type, all known geometrical realizations of $\mathfrak{L}_X^a$ have the homotopy type of the Bousfield–Kan $\mathbb{Q}$-completion of X (see Section 1.2.1).

Theorem 11.14. *Let X be a connected simplicial set of finite type, a be a 0-simplex and $(\wedge V, d)$ be a Sullivan model of X. Then, there are homotopy equivalences*

$$\mathbb{Q}_\infty X \simeq \langle \wedge V, d\rangle^S \simeq \mathrm{MC}_\bullet(\mathfrak{L}_X^a) \simeq \langle \mathfrak{L}_X^a\rangle.$$

Proof. The equivalence between the Bousfield–Kan completion of X and the Sullivan realization of $(\wedge V, d)$ is a classical result proved by Bousfield and Gugenheim in [12, Theorem 12.2]. The third equivalence follows immediately from Theorem 11.13.

It remains to prove the second equivalence. Let m_X be the minimal Lie model of $\mathfrak{L}_X^a$. By Theorem 10.8, the cdga

$$\varinjlim_n \mathscr{C}^*(m_X/m_X^n)$$

is a Sullivan model of X, which we denote by $(\wedge W, d)$. Therefore, we have a homotopy equivalence of the Sullivan realizations,

$$\langle \wedge W, d\rangle^{\mathrm{S}} \simeq \langle \wedge V, d\rangle^{\mathrm{S}}.$$

Now, Proposition 11.2 provides a sequence of bijections,

$$\langle \wedge W, d\rangle_n^{\mathrm{S}} = \operatorname{Hom}_{\mathbf{cdga}}\big((\wedge W, d), A_{\mathrm{PL}}(\underline{\Delta}^n)\big) \xrightarrow{\cong} \mathrm{MC}\big(A_{\mathrm{PL}}(\underline{\Delta}^n)\widehat{\otimes} m_X\big) = \mathrm{MC}_n(m_X)$$

which are compatible with the simplicial structures, and therefore

$$\langle \wedge W, d\rangle^S \cong \mathrm{MC}_\bullet(m_X).$$

Finally, by Proposition 11.12,

$$\mathrm{MC}_\bullet(m_X) \simeq \mathrm{MC}_\bullet(\mathfrak{L}_X^a). \qquad \square$$

The first consequence of Theorem 11.14 is immediate in view of Theorem 7.18:

Corollary 11.15. *Let X be a connected simplicial set of finite type and let a be a 0-simplex. Then, for any $n \geq 1$,*

$$\pi_n(\mathbb{Q}_\infty X) = H_n(\mathfrak{L}_X^a). \qquad \square$$

Remark 11.16. In particular we recover Theorem 10.5, since the fundamental group of the Bousfield–Kan completion of a finite type simplicial set X is the Malcev completion of the fundamental group of X, see [51, Theorem 7.5].

Another important consequence of Theorem 11.14 shows that in the simply connected finite type case, all known realizations of dgl's coincide up to homotopy.

Corollary 11.17. *Let L be a simply connected dgl of finite type. Then,*

$$\langle L\rangle \simeq \langle L\rangle^{\mathrm{Q}}$$

where the latter denotes the classical Quillen realization functor.

Proof. In view of the equivalence of homotopy categories given in Theorem 1.21, there is a simply connected simplicial complex X of finite type such that

$$L \simeq \lambda(X).$$

Using the aforementioned equivalence, together with Theorems 10.2 and 11.14, we have:

$$\langle L\rangle \simeq \langle \lambda(X)\rangle \simeq \langle \mathfrak{L}_X^a\rangle \simeq X_{\mathbb{Q}} \simeq \langle \lambda(X)\rangle^{\mathrm{Q}} \simeq \langle L\rangle^{\mathrm{Q}}.$$

Recall from Section 1.2.1 that in the simply connected case the Bousfield–Kan completion coincides with the rationalization. $\square$

Another fundamental consequence of Theorem 11.14 is the following.

Corollary 11.18. *Let X be a connected, finite type simplicial set. Then,*

$$\langle \mathfrak{L}_X\rangle \simeq \mathbb{Q}_\infty X^+,$$

that is, $\langle \mathfrak{L}_X \rangle$ has the homotopy type of the disjoint union of the $\mathbb{Q}$-completion of X with a point. Moreover, given a simplicial map $f\colon X \to Y$ there is a homotopy commutative square of the form

$$\begin{array}{ccc} \mathbb{Q}_\infty X^+ & \xrightarrow{\mathbb{Q}_\infty f^+} & \mathbb{Q}_\infty Y^+ \\ \simeq \uparrow & & \uparrow \simeq \\ \langle \mathfrak{L}_X \rangle & \xrightarrow{\langle \mathfrak{L}_f \rangle} & \langle \mathfrak{L}_Y \rangle \end{array}$$

where the vertical arrows are homotopy equivalences, and $\mathbb{Q}_\infty f^+$ is defined as $\mathbb{Q}_\infty f$ on $\mathbb{Q}_\infty X$ and preserves the external point.

Proof. Recall from Remark 7.33 that, for any connected simplicial set X and any 0-simplex a,

$$\langle \mathfrak{L}_X \rangle = \langle \mathfrak{L}_X^0 \rangle \amalg \langle \mathfrak{L}_X^a \rangle.$$

On the one hand, by Corollary 7.25,

$$\langle \mathfrak{L}_X^0 \rangle \simeq *.$$

On the other hand, by Theorem 11.14,

$$\langle \mathfrak{L}_X^a \rangle \simeq \mathbb{Q}_\infty X.$$

The second assertion follows from the naturality up to homotopy of all the involved constructions. □

Remark 11.19. This is the right place to stress that our theory models the *free* homotopy category of $\mathbb{Q}$-complete spaces, but inside the *pointed* category:

The category **sset** of *free* simplicial set is fully embedded in the *pointed* category $\mathbf{sset}^*$ by means of the functor

$$\iota\colon \mathbf{sset} \hookrightarrow \mathbf{sset}^*,$$

defined as follows: $\iota(X) = X^+$ and ι sends the map $f\colon X \to Y$ to the pointed map $f^+\colon X^+ \to Y^+$, which preserves the external point and is f on X. Hence, what the functors $\mathfrak{L}$ and $\langle \cdot \rangle$ faithfully model in view of Corollary 11.18 is the rational homotopy category of $\operatorname{Im} \iota$.

Remark 11.20. Recall Theorem 8.9, which in turn is a reformulation of Theorem 4.33. Note that, in view of Theorem 11.13, any of these results is equivalent to [42, Theorem 1.1] by which every cdgl morphism $L \to L'$ which induces quasi-isomorphisms $L/F^n \xrightarrow{\simeq} L'/G^n$, for each $n \geq 1$, also induces a homotopy equivalence $\mathrm{MC}_\bullet(L) \xrightarrow{\simeq} \mathrm{MC}_\bullet(L')$.

Finally, we show the necessity of the finite type hypothesis in one of the homotopy equivalences in Theorem 11.14.

Proposition 11.21. *Let* $(\land V, d)$ *be the Sullivan minimal model of a simply connected simplicial set* X *whose Betti numbers are countable and at least one of them is infinite. Then,* $\langle \land V, d\rangle^S \neq \langle \mathfrak{L}_X^a\rangle$.

Proof. Let p be the smallest degree i such that $H^i(X;\mathbb{Q})$ is infinite-dimensional. We decompose

$$(\land V^{\leq p}, d) = (\land V^{<p} \otimes \land V^p, d).$$

As $\dim H(\land V^{<p}, d) < \infty$ and $\dim H^p(\land V, d) = \infty$, we have $\dim V^p = \infty$. This implies that

$$\dim \pi_p\langle \land V, d\rangle^{\mathrm{S}} = \dim\big(\mathrm{Hom}(V^p, \mathbb{Q})\big)$$

is not countable. Therefore, the dimension of $H_p\langle \land V, d\rangle^{\mathrm{S}}$ is not countable. On the other hand,

$$\pi_p\langle \mathfrak{L}_X^a\rangle = H_{p+1}(\mathfrak{L}_X^a)$$

is countable by hypothesis. □

Bibliographical notes

A first sketch of the Deligne–Getzler–Hinich ∞-groupoid $\mathrm{MC}_\bullet(L)$, as a generalization of the Deligne groupoid of L, already appears in a letter of P. Deligne to L. Breen [39]. In this letter, Deligne already poses the question of whether this simplicial set coincides with the Quillen realization in the reduced case. Theorem 11.13 together with Corollary 11.17 constitute an answer to this question.

Later on, the Deligne–Getzler–Hinich ∞-groupoid was extended to L_∞-algebras, which can be roughly described as Lie algebras up to homotopy and have the advantage of being stable by homotopy transfer. This extension was first done by E. Getzler for nilpotent L_∞-algebras in [60], where the *nerve of* L, a manageable deformation retract $\gamma_\bullet(L)$ of $\mathrm{MC}_\bullet(L)$, is also built for any nilpotent L_∞-algebra. Explicitly,

$$\gamma_\bullet(L) = \ker \kappa_\bullet \widehat{\otimes} L,$$

where $\kappa_\bullet$ is the simplicial chain homotopy of Theorem 9.5. Then, in his thesis [2, Chapter 5], R. Bandiera introduced $\mathrm{MC}_\bullet(L)$ for any complete L_∞-algebra L through a clear and detailed presentation.

To keep the self-contained feature of this text, the homotopy equivalence between $\langle L\rangle$ and $\mathrm{MC}_\bullet(L)$ in Theorem 11.13 was stated for connected cdgl's whose decomposables $L/[L,L]$ have finite type. However, the same result was proved for any connected cdgl L independently by D. Robert-Nicoud in [118, Theorems 3.2 and 5.2] and by the authors in [27, Theorem 0.1]. The common core of the proof is the following: tensor the simplicial diagram in Theorem 9.5 with a given connected cdgl to obtain a simplicial transfer diagram

$$\kappa_\bullet \circlearrowright \quad A_{PL}(\underline{\Delta}^\bullet)\widehat{\otimes} L \underset{\iota_\bullet \widehat{\otimes} L}{\overset{p_\bullet \widehat{\otimes} L}{\rightleftarrows}} C^*(\Delta^\bullet)\widehat{\otimes} L.$$

Then, by a Lie version of Theorem 1.8, the simplicial cdgl structure in $A_{\mathrm{PL}}(\underline{\Delta}^\bullet)\widehat{\otimes}L$ is transferred through this diagram to a simplicial L_∞-algebra structure on $C^*(\Delta^\bullet)\otimes L$ for which, as simplicial sets,

$$\mathrm{MC}_\bullet(L)\simeq \mathrm{MC}(C^*(\Delta^\bullet)\otimes L).$$

Finally, the existence of a simplicial isomorphism

$$\mathrm{MC}(C^*(\Delta^\bullet)\otimes L)\cong\langle L\rangle$$

closes the argument. Moreover, it is proved in [27, Theorem 0.2] that, as simplicial sets, $\langle L\rangle$ is isomorphic to the nerve $\gamma_\bullet(L)$.

Before that, A. Berglund had already proven in [8, Theorem 1.1] the existence of a surprisingly manageable and explicit isomorphism,

$$B_n\colon H_{n-1}(L)\xrightarrow{\cong},\mathrm{MC}_\bullet(L),\quad \text{for}\quad n\geq 1,$$

for any connected nilpotent dgl, or more generally, any connected nilpotent L_∞-algebra. This is defined by

$$B_n[\alpha]=\omega_{0\dots n}\otimes\alpha,$$

where $\omega_{0\dots n}\in {A_{\mathrm{PL}}}^n(\underline{\Delta}^n)$ denotes the only Whitney elementary form of maximum degree n (see Definition 9.1).

As a final remark, the fact arising from Proposition 11.1 that, for any connected dgl L of finite type,

$$\mathrm{MC}_\bullet(L)\cong \mathrm{Hom}_{\mathbf{cdga}}\big(\mathscr{C}^*(L),A_{\mathrm{PL}}(\underline{\Delta}^\bullet)\big),$$

is a classical result, see for instance [72], based on the concept of twisting cochains [15]. Indeed, the image of the inclusion

$$\mathrm{Hom}_{\mathbf{cdga}}\big(\mathscr{C}^*(L),A_{\mathrm{PL}}(\underline{\Delta}^\bullet)\big)\hookrightarrow \mathrm{Hom}_{-1}\big(L,A_{\mathrm{PL}}(\underline{\Delta}^\bullet)\big)\cong\big(A_{\mathrm{PL}}(\underline{\Delta}^\bullet)\otimes L\big)_{-1}$$

is precisely the set of twisting cochains, i.e., the set of Maurer–Cartan elements.

Chapter 12

Examples

In this chapter we use all the material collected up to this point to present a considerable number of selected examples.

Since the rational homotopy theory of simply connected spaces is classical and well understood, we have focused on examples that can be applied to the non-simply connected, non-nilpotent setting. Nevertheless, as our theory extends the classical Quillen approach, we refer the reader to standard references like [50] or [130], where a large number of useful applications in the simply connected context can be found.

We begin by constructing a Lie model of any 2-dimensional CW-complex as follows: let X be obtained by attaching a family of 2-cells $\{e_j\}_{j\in J}$ to a wedge of circles $\bigvee_{i\in I} S^1_i$ along the maps

$$\omega_j = y_{j_1}^{r_{j_1}} \cdots y_{j_{q_j}}^{r_{q_j}}, \quad \text{for} \quad j \in J.$$

Here, for $i \in I$, each y_i denotes a generator of $\pi_1(S^1_i)$. Then, X has a Lie model of the form

$$(\widehat{\mathbb{L}}(y_i, e_j), d), \quad \text{where each } y_i \text{ is a 0-cycle and} \quad de_j = y_{j_1}^{r_{j_1}} * \cdots * y_{j_{q_j}}^{r_{q_j}}, \quad \text{for} \quad j \in J.$$

As a particular instance, we construct a Lie model of any surface.

This approach also allows us to give an explicit description of the Malcev completion of a finitely presented group as follows: let

$$G = \langle\, a_1, \dots, a_p \mid b_1, \dots, b_k \,\rangle, \quad \text{with} \quad b_j = a_{j_1}^{r_{j_1}} \cdots a_{j_{q_j}}^{r_{q_j}}, \quad \text{for} \quad j = 1, \dots, k,$$

be a finitely presented group. The Malcev completion of G is the group

$$\mathbb{Q}_\infty G = \widehat{\mathbb{L}}(a_1, \dots, a_p)/(b_1, \dots, b_k)$$

where each a_i is of degree zero and

$$b_k = a_{i_1}^{r_1} * \cdots * a_{i_q}^{r_q}, \quad \text{for} \quad j = 1, \dots, k.$$

U. Buijs et al., *Lie Models in Topology*, Progress in Mathematics 335,
https://doi.org/10.1007/978-3-030-54430-0_12

From all of the above we will also deduce that for a finite 2-dimensional complex X, its $\mathbb{Q}_\infty$ completion is an Eilenberg–MacLane space. More precisely,

$$\mathbb{Q}_\infty X \simeq K(\mathbb{Q}_\infty \pi_1(X), 1).$$

As another particularly interesting example we also compute the Malcev completion of any right-angled Artin group, starting with the minimal Lie model of a torus.

Then, we will see that, up to homotopy, the model functor commutes with products. Namely, given simplicial sets X and Y,

$$\mathfrak{L}^{(a,b)}_{X\times Y} \simeq \mathfrak{L}^a_X \times \mathfrak{L}^b_Y,$$

for any choice of 0-simplices a and b. Moreover, given connected Lie models $(\widehat{\mathbb{L}}(V), d)$ and $(\widehat{\mathbb{L}}(W), d)$ of X and Y, we give an explicit model of $X \times Y$ of the form $(\widehat{\mathbb{L}}(V \oplus W \oplus s(V \otimes W), D)$. Various particular examples are also presented.

We then devote our attention to constructing Lie models of mapping spaces. The general picture is the following: choose a cdgl model A of the connected simplicial set X and a cdgl model L of another connected simplicial set Y. Then, under mild finiteness conditions, $A\widehat{\otimes}L$ is a Lie model of the simplicial mapping space $\operatorname{Map}(X, \mathbb{Q}_\infty Y)$, that is, there is a homotopy equivalence of simplicial sets,

$$\operatorname{Map}(X, \mathbb{Q}_\infty Y) \simeq \langle A\widehat{\otimes}L\rangle.$$

For pointed mappings we also provide a homotopy equivalence

$$\operatorname{Map}^*(X, \mathbb{Q}_\infty Y) \simeq \langle A^+\widehat{\otimes}L\rangle.$$

In particular, the model of the component $\operatorname{Map}_f(X, \mathbb{Q}_\infty Y)$ of a given map $f\colon X \to \mathbb{Q}_\infty Y$ is simply $(A\widehat{\otimes}L, d_z)$, where z is the MC element corresponding to f. As a result, the homotopy groups of this component can be computed as

$$\pi_n \operatorname{Map}_f(X, \mathbb{Q}_\infty Y) \cong H_{n-1}(A\widehat{\otimes}L, d_z), \quad \text{for} \quad n \geq 1.$$

In the pointed setting we also give a procedure to obtain the homotopy groups of a given component in terms of derivations. Namely, given $\ell\colon L \to L'$, a model of $f\colon X \to Y$ in which L is a free cdgl, then,

$$\pi_n \operatorname{Map}^*_f(X, \mathbb{Q}_\infty Y) \cong H_n \operatorname{Der}^\ell(L, L'), \quad \text{for} \quad n \geq 1,$$

where Der^ℓ denotes the chain complex of ℓ-derivations.

Based on these general results we present a list of interesting applications including, for instance, the minimal model of free loop spaces on connected simplicial sets. We finish the study of the rational homotopy type of mapping spaces by simplicially enriching the category **cdgl**. For the enriched category $\mathbf{cdgl}^{\boldsymbol{\Delta}}$ we show that, given connected simplicial sets X and Y with Lie models L and L', respectively, there is a homotopy equivalence of simplicial sets,

$$\operatorname{Map}(X, \mathbb{Q}_\infty Y) \simeq \operatorname{Hom}_{\mathbf{cdgl}^\Delta}(L, L').$$

Notice that up to this point we have collected examples and explicit procedures to algebraically model various geometrical objects. After that we explore the opposite point of view and describe how to read some homotopy invariants in the realization $\langle L\rangle$ of a given cdgl. For instance, we show that given a connected cdgl L, the map

$$H_0(L) \times H(L) \longrightarrow H(L), \qquad (\alpha, \beta) \longmapsto e^{\operatorname{ad}_\alpha}(\beta),$$

is precisely the natural action

$$\pi_1\langle L\rangle \times \pi_*\langle L\rangle \longrightarrow \pi_*\langle L\rangle.$$

Moreover, there is an isomorphism of Lie algebras

$$H(L) = \pi_*\Omega\langle L\rangle.$$

We finish with an easy description of the Postnikov decomposition of $\langle L\rangle$ for any given connected cdgl.

12.1 Lie models of 2-dimensional complexes. Surfaces

Recall that if Y is a sub-simplicial set of the simplicial set X, the canonical map

$$|X|/|Y| \xrightarrow{\cong} |X/Y|$$

is always a homeomorphism. Hence, in the following, a quotient of simplicial complexes will always denote the quotient of the associated simplicial sets.

We first construct the minimal model of a finite wedge of circles. For $n \geq 3$, we denote by A_n the boundary of the n-gon. This is the 1-dimensional simplicial complex having

$$\{a_0, \dots, a_{n-1}\}$$

as vertices and whose 1-simplices are

$$\{x_j\}_{j=1}^n, \quad \text{where} \quad x_j = (a_{j-1}, a_j), \quad j = 1, \dots, n-1, \quad \text{and} \quad x_n = (a_{n-1}, a_0).$$

By Proposition 7.8, its global model is then

$$\mathfrak{L}_{A_n} = (\widehat{\mathbb{L}}(a_0, \dots, a_{n-1}, x_1, \dots, x_n), d)$$

where a_i is a Maurer–Cartan element for $i = 0, \dots, n-1$, x_j is a path from a_{j-1} to a_j for $1 \leq j < n$ and x_n is a path from a_{n-1} to a_0.

Let Y_n be the 0-dimensional subcomplex of A_n given by its vertices, whose global model is the sub-cdgl of $\mathfrak{L}_{A_n}$,

$$\mathfrak{L}_{Y_n} = (\widehat{\mathbb{L}}(a_0, \dots, a_{n-1}), d)$$

From now on, to simplify the notation, we set $a = a_0$. Also, we will write $\widehat{\mathbb{L}}(a_i)$ or $\widehat{\mathbb{L}}(x_j)$ to denote $\widehat{\mathbb{L}}(a_0, \dots, a_{n-1})$ or $\widehat{\mathbb{L}}(x_1, \dots, x_n)$.

Consider the quotient simplicial set

$$W_n = A_n/Y_n, \tag{12.1}$$

whose realization is homeomorphic to a wedge of n circles.

Lemma 12.1. *The minimal model of W_n is*

$$(\widehat{\mathbb{L}}(x_1, \dots x_n), 0), \quad \textit{with} \quad |x_i| = 0, \quad i = 1, \dots, n.$$

Proof. By Corollary 7.11 applied to $Y_n \subset A_n$ we obtain that

$$\mathfrak{L}^a_{W_n} = \mathfrak{L}^a_{A_n}/\mathfrak{L}^a_{Y_n} = (\widehat{\mathbb{L}}(x_j), 0),$$

which is already a minimal cdgl and thus, it is the minimal model of W_n. □

On the other hand, let K_n be the n-gon, that is, A_n with a 2-cell attached along the perimeter. For $n = 3$, this is the triangle. For $n > 3$, we consider the simplicial complex structure in K_n given by drawing all possible diagonals (in fact $n - 3$ diagonals) in A_n starting from the vertex $a = a_0$ (see the picture below). As a result, K_n has the same set of vertices as A_n, which we denote in the same way: $a = a_0, a_1, \dots, a_{n-1}$. The edges $x_1, \dots, x_n$ of A_n are also edges of K_n, which contains $n - 3$ extra 1-simplices given by the diagonals $v_1, \dots, v_{n-3}$. Finally, K_n has $n - 2$ 2-simplices $e_1, \dots, e_{n-2}$, where $e_k = (a_0, a_k, a_{k+1})$.

In the next result we keep the above notation and write

$$(\mathfrak{L}_{A_n}, d_a) = (\widehat{\mathbb{L}}(a_i, x_j), d_a).$$

Lemma 12.2. *There is a quasi-isomorphism*

$$(\widehat{\mathbb{L}}(a_i, x_j, e), d_a) \xrightarrow{\simeq} (\mathfrak{L}_{K_n}, d_a)$$

with $d_a e = x_1 * \cdots * x_n$

Proof. First note that the argument of Proposition 5.14 shows that $x_1 * \cdots * x_n$ is a d_a-cycle, so that $(\widehat{\mathbb{L}}(a_i, x_j, e), d_a)$ is in fact a cdgl. Next, consider the map

$$\varphi\colon (\widehat{\mathbb{L}}(a_i, x_j, e), d_a) \xrightarrow{\simeq} (\mathfrak{L}_{K_n}, d_a), \tag{12.2}$$

defined by

$$\varphi(a_i) = a_i, \quad \varphi(x_j) = x_j \quad \text{and} \quad \varphi(e) = e_1 \maltese \cdots \maltese e_{n-2}.$$

This map can be thought as the "subdivision of the n-gon" into $n - 2$ triangles.

By construction (see Definition 6.14),

$$d_a(e_1 \maltese \cdots \maltese e_{n-2}) = d_a e_1 * \cdots * d_a e_{n-2}.$$

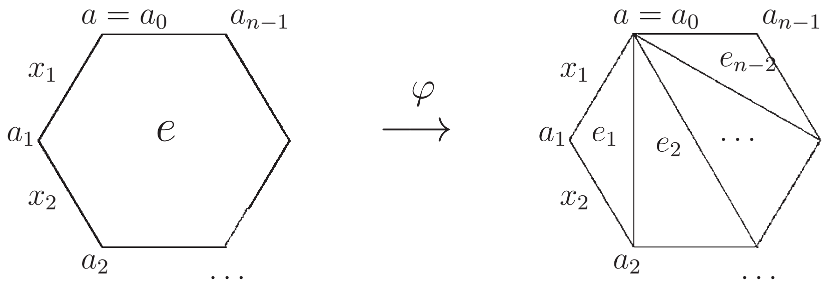
On the other hand, each e_k is a triangle inside $\mathfrak{L}_{K_n}$ and thus (see Proposition 5.14),

$$d_a e_1 = x_1 * x_2 * v_1^{-1}, \quad d_a e_2 = v_1 * x_2 * v_2^{-1}, \dots, \quad d_a e_{n-2} = v_{n-3} * x_{n-1} * x_n.$$

Therefore,

$$d_a \varphi(e) = d_a e_1 * \cdots * d_a e_{n-2} = x_1 * \cdots * x_n = \varphi(d_a e),$$

and thus φ is a cdgl morphism. An easy computation shows that the linear part φ_1 is a quasi-isomorphism. Hence, by Proposition 3.12, φ is also a quasi-isomorphism. □



Finally, consider the quotient

$$Z_n = K_n / Y_n,$$

whose realization is homeomorphic to a wedge of n circles with a 2-cell attached along the product of the circles. Again, we keep the notation for $\mathfrak{L}_{K_n}$.

Lemma 12.3. *Z_n has a Lie model of the form*

$$(\widehat{\mathbb{L}}(x_j, e), d)$$

where each x_j is a 0-cycle, $j = 1, \dots, n$, and

$$de = x_1 * \cdots * x_n.$$

Proof. Restricting to the component of a both sides of the quasi-isomorphism given in Lemma 12.2 we get another quasi-isomorphism,

$$(\widehat{\mathbb{L}}(x_j, e), d) \xrightarrow{\simeq} \mathfrak{L}^a_{K_n}.$$

But, by Corollary 7.11,

$$\mathfrak{L}^a_{Z_n} \cong \mathfrak{L}^a_{K_n} / \mathfrak{L}^a_{Y_n} = \mathfrak{L}^a_{K_n},$$

and the lemma holds. □

We will follow an analogous procedure to obtain a Lie model of the simplicial set X whose realization is the CW-complex

$$(S^1 \vee \overset{p}{\cdots} \vee S^1) \cup_w e$$

obtained by attaching a 2-cell e to a wedge of p circles $y_1, \dots, y_p$ along the map

$$\omega = y_{i_1}^{r_1} \cdots y_{i_q}^{r_q}, \quad \text{where} \quad i_\ell \in \{1, \dots, p\} \quad \text{and} \quad r_\ell \in \mathbb{Z}.$$

Here, for $i = 1, \dots, p$, each y_i denotes a generator of the fundamental group of the corresponding circle.

Theorem 12.4.

(i) *X has a Lie model of the form*

$$(\widehat{\mathbb{L}}(y_1, \dots, y_p, e), d),$$

where each y_k is a 0-cycle and

$$de = y_{i_1}^{r_1} * \cdots * y_{i_q}^{r_q}.$$

In particular, this is the minimal Lie model of X if and only if $r_1 y_{i_1} + \cdots + r_q y_{i_q} = 0$.

(ii) *If $de \neq 0$, then*

$$H_{>1}(\widehat{\mathbb{L}}(y_1, \dots, y_p, e), d) = 0.$$

Here, powers of degree 0 generators are also taken with respect to the BCH product.

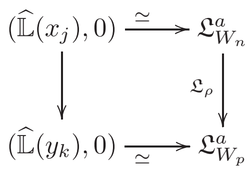
Proof. (i) Let $n = \sum_{j=i}^q |r_j|$ and observe that X is the pushout,

$$\begin{array}{ccc} W_n & \hookrightarrow & Z_n \\ {\scriptstyle \rho}\downarrow & & \downarrow \\ W_p & \longrightarrow & X \end{array}$$

where ρ denotes the map drawing the word ω. As $\mathfrak{L}$ preserves inductive limits, this produces a cdgl pushout

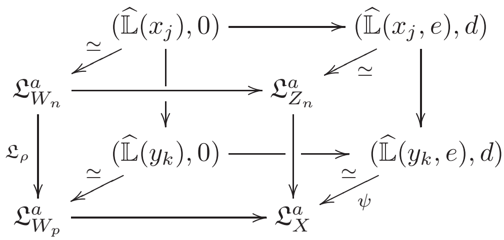
$$\begin{array}{ccc} \mathfrak{L}^a_{W_n} & \hookrightarrow & \mathfrak{L}^a_{Z_n} \\ {\scriptstyle \mathfrak{L}_\rho}\downarrow & & \downarrow \\ \mathfrak{L}^a_{W_p} & \longrightarrow & \mathfrak{L}^a_X \end{array}$$

Note also that, in view of Lemma 12.2, a Lie model of ρ is given by



where the vertical map sends $x_1 * \cdots * x_n$ to $y_{i_1}^{r_1} * \cdots * y_{i_q}^{r_q}$.

To finish, consider the commutative diagram,



where: $j = 1, \dots, n$; $k = 1, \dots, p$; the front face is the former pushout; the back face is also a pushout whose vertical maps send $x_1 * \cdots * x_n$ to $y_{i_1}^{r_1} * \cdots * y_{i_q}^{r_q}$; the top right quasi-isomorphism is given by Lemma 12.3; and ψ is the morphism induced by the pushout universality property, which is necessarily a quasi-isomorphism; Note that in $(\widehat{\mathbb{L}}(y_k, e), d)$ the differential is then as stated,

$$de = y_{i_1}^{r_1} * \cdots * y_{i_q}^{r_q}.$$

(ii) Write

$$de = \sum_{m \geq s} \alpha_m, \quad \text{with} \quad \alpha_m \in \mathbb{L}^m(y_k) \quad \text{and} \quad \alpha_s \neq 0.$$

We then consider the differential graded Lie algebra

$$(\mathbb{L}(y_k, e), \delta), \quad \text{where} \quad \delta y_k = 0 \quad \text{and} \quad \delta e = \alpha_s,$$

and claim that

$$H_{>0}(\mathbb{L}(y_k, e), \delta) = 0. \tag{12.3}$$

To see this, we modify the degrees, setting $|y_k| = 2$ and $|e| = 2s + 1$. We can now use [73, Theorem 3.12] to get the isomorphism

$$H_*(\mathbb{L}(y_k, e), \delta) \cong \mathbb{L}(y_k)/(\delta e).$$

This proves (12.3) as, for the original degrees $|y_k| = 0$, $|e| = 1$, the right-hand side Lie algebra in this identity is concentrated in degree 0.

Now, let β be a cycle of degree $r \geq 1$ in $(\widehat{\mathbb{L}}(y_k, e), d)$ and write

$$\beta = \sum_{m \geq \ell} \beta_m, \quad \text{with} \quad \beta_m \in \mathbb{L}^m(y_k, e) \quad \text{and} \quad \beta_\ell \neq 0.$$

Observe that as β is of degree r, each β_m must contain r times the element e in each bracket. Note also that each β_ℓ is a δ-cycle. By (12.3), there exists $\mu_\ell \in \mathbb{L}^{\ell-s+1}(y_k, e)$ such that

$$\beta_\ell = \delta \mu_\ell$$

and therefore,

$$\beta - d\mu_\ell \in \widehat{\mathbb{L}}^{>\ell}(y_k, e).$$

We inductively construct in this way a sequence $\mu_m \in \widehat{\mathbb{L}}^{m-s+1}(y_k, e)$, $m \geq \ell$, such that

$$\beta - d(\sum_{m=\ell}^{t} \mu_m) \in \widehat{\mathbb{L}}^{>t}(y_k, e).$$

Hence, the element

$$\mu = \sum_{m \geq \ell} \mu_m$$

satisfies $d\mu = \beta$. □

The extension of Theorem 12.4 to any 2-dimensional complex is straightforward. Let X be the 2-dimensional CW-complex obtained by attaching a family, not necessarily finite, of 2-cells $\{e_j\}_{j\in J}$, to a wedge, not necessarily finite, of circles $\bigvee_{i\in I} S^1_i$, along the elements $\omega_j \in \pi_1(\bigvee_{i\in I} S^1)$, with $j \in J$.

As before, for each $i \in I$, denote by y_i a generator of $\pi_1(S^1_i)$ and write the attaching map of each 2-cell as

$$\omega_j = y_{j_1}^{r_{j_1}} \cdots y_{j_{q_j}}^{r_{q_j}}, \quad \text{for} \quad j \in J.$$

Then, we have:

Theorem 12.5.

(i) *X has a Lie model of the form*

$$(\widehat{\mathbb{L}}(y_i, e_j), d),$$

where each y_i is a 0-cycle and

$$de_j = y_{j_1}^{r_{j_1}} * \cdots * y_{j_{q_j}}^{r_{q_j}}, \quad \text{for} \quad j \in J.$$

(ii) *If $de_j \neq 0$ for every $j \in J$, then*

$$H_{\geq 1}(\widehat{\mathbb{L}}(y_i, e_j), d) = 0.$$

As before, by an abuse of notation, we denote by $\widehat{\mathbb{L}}(y_i, e_j)$ the complete free Lie algebra generated by $\{y_i, e_j\}_{i\in I, j\in J}$. Again, powers of degree-0 generators are considered here with respect to the BCH product.

Proof. (i) First, note that $\bigvee_{i\in I} S^1_i$ is the inductive limit of finite wedges. In others words, it is the realization of the inductive limit of finite simplicial sets as in (12.1). Therefore, as the global model functor preserves colimits, apply Lemma 12.1 to conclude that a Lie model of $\bigvee_{i\in I} S^1_i$ is

$$(\widehat{\mathbb{L}}(y_i), 0), \quad \text{with} \quad |y_i| = 0 \quad \text{for} \quad i \in I.$$

Next, exactly the same proof of Theorem 12.4(i) permits us assert that, if we attach a 2-cell e to $\bigvee_{i\in I} S^1_i$ along the map $\omega = y_{i_1}^{r_1} \cdots y_{i_q}^{r_q}$, the 2-dimensional complex $(\bigvee_{i\in I} S^1_i) \cup_\omega e$ has a Lie model of the form

$$(\widehat{\mathbb{L}}(y_i, e), d), \quad \text{with } |y_i| = 0 \quad \text{and} \quad de = y_{i_1}^{r_1} * \cdots * y_q^{r_q}.$$

Finally, since

$$X = \varinjlim_{j\in J} (\bigvee_{i\in I} S^1_i) \cup_{\omega_j} e_j,$$

apply again that the model functor preserve colimits to obtain the desired Lie model of X.

(ii) If $de_j \neq 0$ for any $j \in J$, apply Theorem 12.4(ii) to conclude that, for any fixed $j_0 \in J$,

$$H_{\geq 1}(\widehat{\mathbb{L}}(y_i, e_{j_0}), d) = 0.$$

The result follows by observing that $(\widehat{\mathbb{L}}(y_i, e_j), d)$ is the inductive limit of

$$(\widehat{\mathbb{L}}(y_i, e_{j_0}), d),$$

for $j_0 \in J$, and that homology commutes with inductive limits. □

This result has important implications. The first one is algebraic:

Corollary 12.6. *Let*

$$G = \langle\, a_1, \ldots, a_p \mid b_1, \ldots, b_k \,\rangle$$

be a finitely presented group, where

$$b_j = a_{j_1}^{r_{j_1}} \cdots a_{j_{q_j}}^{r_{q_j}}, \quad \textit{for} \quad j = 1, \ldots, k.$$

Then, the Malcev completion of G is the group

$$\mathbb{Q}_\infty G = \widehat{\mathbb{L}}(a_1, \ldots, a_p)/(b_1, \ldots, b_k),$$

where each a_i is of degree zero,

$$b_k = a_{i_1}^{r_1} * \cdots * a_{i_q}^{r_q}, \quad \textit{for} \quad j = 1, \ldots, k,$$

and the group law is given by the Baker–Campbell–Hausdorff product.

Proof. Let X be the 2-dimensional complex obtained by adding 2-cells $e_1, \dots, e_k$ to a wedge of p circles along the words $b_1, \dots, b_k$. Clearly, $\pi_1(X) = G$.

On the one hand, by Theorem 12.5(i), and for any fixed 0-simplex a of X,

$$H_0(\mathfrak{L}_X^a) = H_0(\widehat{\mathbb{L}}(a_1, \dots, a_p, e_1, \dots, e_k), d) = \widehat{\mathbb{L}}(a_1, \dots, a_p)/(b_1, \dots, b_k).$$

On the other hand, by Theorem 10.5, $H_0(\mathfrak{L}_X^a)$ is the Malcev completion of $\pi_1(X)$. □

The second consequence of Theorem 12.5 is purely topological.

Corollary 12.7. *Let X be obtained by attaching a non trivial 2-cell to a finite wedge of circles. Then $\mathbb{Q}_\infty X$ is an Eilenberg–MacLane space. More, precisely,*

$$\mathbb{Q}_\infty X \simeq K(\mathbb{Q}_\infty \pi_1(X), 1).$$

Proof. Recall from Corollary 11.15 that

$$H_{n-1}(\mathfrak{L}_X^a) \cong \pi_n(\mathbb{Q}_\infty X), \qquad n \geq 1.$$

The statement follows by the two assertions in Theorem 12.4 and Corollary 12.6. □

This corollary was first proven in [47] and it generalizes the following result of Lyndon [96]: If X is obtained by attaching a non-trivial 2-cell to a finite wedge of circles and $\pi_1(X)$ has no torsion, then X is an Eilenberg–MacLane space.

As a final and immediate consequence of Theorem 12.4 we exhibit a Lie model of any surface, which is minimal in the orientable case.

Corollary 12.8. *A Lie model of the compact connected surface*

$$S = T^{\# m} \# (\mathbb{R}P^2)^{\# n}, \quad \text{for} \quad m, n \geq 0,$$

obtained as the connected sum of m copies of the torus T and n copies of $\mathbb{R}P^2$, is given by

$$(\widehat{\mathbb{L}}(x_1, \dots, x_m, y_1, \dots, y_m, z_1, \dots, z_n, e), d),$$

where every generator except e is a 0-cycle and

$$de = x_1 * y_1 * x_1^{-1} * y_1^{-1} * \cdots * x_m * y_m * x_m^{-1} * y_m^{-1} * z_1^2 * \cdots * z_n^2.$$

Moreover, this model is minimal if and only if $n = 0$, that is, if and only if S is orientable. □

Example 12.9. A Lie model of the Klein bottle

$$K = \mathbb{R}P^2 \# \mathbb{R}P^2$$

is given by

$$(\widehat{\mathbb{L}}(z_1, z_2, e), d), \quad \text{where} \quad dz_1 = dz_2 = 0 \quad \text{and} \quad de = z_1^2 * z_2^2.$$

However, the Klein bottle can be obtained from different attachments of the 2-cell. These variations provide different Lie models and thus, different presentations of the Malcev completion of its fundamental group. For instance,

$$(\widehat{\mathbb{L}}(u, v, e), d), \quad \text{where} \quad du = dv = 0 \quad \text{and} \quad de = u * v * u * v^{-1},$$

is such an alternative model.

12.2 Lie models of tori and classifying spaces of right-angled Artin groups

An r-dimensional torus

$$T = S_1^1 \times \cdots \times S_r^1$$

is a nilpotent space whose minimal Sullivan model is

$$(\wedge V, d) = (\wedge(x_1, \ldots, x_r), 0), \quad \text{with} \quad |x_i| = 1 \quad \text{for} \quad i = 1, \ldots, r.$$

By Theorem 10.2, its minimal Lie model is

$$\widehat{\mathscr{L}}(\wedge V, d)^\#.$$

Using the definition of the functor $\mathscr{L}$ (see Section 2.1), and denoting

$$x_{i_1 \ldots i_s} = s^{-1}(x_{i_1} \ldots x_{i_s})^\#, \quad \text{with} \quad 1 \leq i_1 < \cdots < i_s \leq r,$$

this minimal Lie model can be written as,

$$(\widehat{\mathbb{L}}(x_{i_1 \ldots i_s})_{1 \leq i_1 < \cdots < i_s \leq r}, d), \tag{12.4}$$

where the differential is quadratic and is given as follows: fix integers $1 \leq i_1 < \cdots < i_s \leq r$ and let E be the set of decompositions of $\{i_1, \ldots, i_s\}$ into two disjoint tuples, $\{j_1, \ldots, j_p\}$ and $\{k_1, \ldots, k_q\}$, with $j_1 < \cdots < j_p$ and $k_1 < \cdots < k_q$. Then,

$$d(x_{i_1 \ldots i_s}) = \frac{1}{2} \sum_E \varepsilon_E [x_{j_1 \ldots j_p}, x_{k_1 \ldots k_q}],$$

where ε_E denotes the sign of the permutation

$$i_1, \ldots, i_s \longmapsto j_1, \ldots, j_p, k_1, \ldots, k_q.$$

Recall that a *right-angled Artin group* A is a group with a presentation of the form

$$A = \langle x_1, \dots, x_n \,|\, x_i x_j = x_j x_i \text{ for a subset } \mathcal{S} \text{ of pairs } (i,j)\rangle.$$

These groups include the finitely generated free abelian groups and finitely generated free groups.

Every right-angled Artin group A acts freely on a particularly interesting finite CW-complex called its *Salvetti complex* K_A. Important consequences in geometric group theory arise from the study of this complex, see for instance [9].

The Salvetti complex K_A is defined as follows: denote by P_r the subset of r-tuples $(x_{i_1}, \dots, x_{i_r})$ of generators of A that commute with each other. Then,

$$K_A = \bigcup_r \bigcup_{(x_{i_1},\dots,x_{i_r})\in P_r} S^1_{i_1} \times S^1_{i_1} \times \cdots \times S^1_{i_r}.$$

That is, K_A is a union of tori $\{T_\gamma\}_{\gamma\in\Gamma}$ of different dimensions. If we denote by $(\widehat{\mathbb{L}}(V_\gamma), d)$ the minimal model of each such torus as in (12.4), then since the global model commutes with inductive limits, the minimal models of K_A is of the form,

$$L_{K_A} = (\widehat{\mathbb{L}}(\bigcup_{\gamma\in\Gamma} V_\gamma), d).$$

We use this to prove:

Proposition 12.10. *The Malcev completion of the right-angled Artin group A is*

$$\mathbb{Q}_\infty A = \widehat{\mathbb{L}}(x_1, \dots, x_n)/([x_i, x_j],\ (i,j) \in \mathcal{S}),$$

where the law group is given by the BCH *product.*

Proof. It is known, see [35], that K_A is a classifying space for A. In particular, it is an Eilenberg–MacLane space, $K_A \simeq K(A,1)$. Moreover, Papadima and Suciu proved in [111] that the minimal Sullivan model of K_A has the form $(\wedge V, d)$ with $V = V^1$, see also [48]. In particular, the Sullivan realization $\langle \wedge V, d\rangle^{\mathrm{S}}$ is the Eilenberg–MacLane space $K(\mathbb{Q}_\infty A, 1)$.

Now, by Proposition 11.14, $\langle \wedge V, d\rangle^{\mathrm{S}} = \langle L_{K_A}\rangle$, and therefore,

$$H(L) = H_0(L_{K_A}) = \mathbb{Q}_\infty A.$$

But, in view of the general model of the torus and that of L_{K_A} one immediately concludes that

$$H_0(L_{K_A}) = \widehat{\mathbb{L}}(x_1, \dots, x_n)/([x_i, x_j],\ (i,j) \in \mathcal{S}). \qquad \square$$

12.3 Lie model of a product

In this section we see that the model of a product of simplicial sets has the homotopy type of the product of their models. Let X and Y be connected simplicial sets, fix 0-simplices a and b in X and Y, and let $e = (a,b) \in X \times Y$. Then,

Theorem 12.11. *There is a natural quasi-isomorphism*

$$\mathfrak{L}_{X\times Y}/(e) \xrightarrow{\simeq} \mathfrak{L}_X/(a) \times \mathfrak{L}_Y/(b)\,.$$

In particular, the minimal Lie model of $X \times Y$ is quasi-isomorphic to the product of the minimal Lie models of X and Y.

Proof. We suppose first that X and Y are finite type simplicial sets. Consider the Dupont transfer diagrams of Corollary 9.7 associated to X and Y,

$$A_{PL}(X) \underset{\iota_X}{\overset{p_X}{\rightleftarrows}} N^*(X) \qquad \text{and} \qquad A_{PL}(Y) \underset{\iota_Y}{\overset{p_Y}{\rightleftarrows}} N^*(Y)\,, \tag{12.5}$$

respectively. As observed in Remark 9.8, all the objects in these diagrams have augmentations induced by the 0-simplices $a \in X$ and $b \in Y$, and all the maps preserve these augmentations. As usual, we denote by

$$\overline{N}^*(X) \subset N^*(X) \qquad \text{and} \qquad \overline{N}^*(Y) \subset N^*(Y)$$

the augmentation ideals. Then, as in (9.4), consider then the dglc's resulting from applying the functor $\mathscr{E}$,

$$\overline{\mathfrak{L}}^{\mathrm{c}}_X = \big(\mathbb{L}^{\mathrm{c}}\big(s\overline{N}^*(X)\big), d\big) \quad \text{and} \quad \overline{\mathfrak{L}}^{\mathrm{c}}_Y = \big(\mathbb{L}^{\mathrm{c}}\big(s\overline{N}^*(Y)\big), d\big),$$

whose respective duals are cdgl's of the form

$$(\widehat{\mathbb{L}}(V), d) \quad \text{and} \quad (\widehat{\mathbb{L}}(W), d),$$

with

$$V = s^{-1}\overline{N}^*(X)^{\#} \quad \text{and} \quad W = s^{-1}\overline{N}^*(Y)^{\#}.$$

By Theorem 9.11, these cdgl's are isomorphic to

$$\mathfrak{L}_X/(a) \quad \text{and} \quad \mathfrak{L}_Y/(b),$$

respectively.

On the other hand, tensor both diagrams in (12.5) via Proposition 1.11 to obtain another transfer diagram of augmented maps,

$$A_{\mathrm{PL}}(X) \otimes A_{\mathrm{PL}}(Y) \underset{\iota}{\overset{p}{\rightleftarrows}} N^*(X) \otimes N^*(Y), \tag{12.6}$$

where $p = p_X \otimes p_Y$ and $\iota = \iota_X \otimes \iota_Y$. Consider also the transfer diagram

$$A_{\mathrm{PL}}(X\times Y) \underset{\iota_{X\times Y}}{\overset{p_{X\times Y}}{\rightleftarrows}} N^*(X\times Y) \tag{12.7}$$

given by Corollary 9.7, applied this time to $X \times Y$. Note that the maps involved also preserve the augmentations induced by the 0-simplex $e = (a, b) \in X \times Y$.

Join (12.6) and (12.7) via the cdga isomorphism

$$A_{\mathrm{PL}}(X) \otimes A_{\mathrm{PL}}(Y) \cong A_{\mathrm{PL}}(X \times Y)$$

to obtain the following diagram of augmented maps:

$$\begin{array}{ccc} A_{\mathrm{PL}}(X)\otimes A_{\mathrm{PL}}(Y) & \underset{i}{\overset{p}{\rightleftarrows}} & N^*(X)\otimes N^*(Y) \\ \cong\updownarrow & & \\ A_{\mathrm{PL}}(X\times Y) & \underset{\iota_{X\times Y}}{\overset{p_{X\times Y}}{\rightleftarrows}} & N^*(X\times Y). \end{array} \tag{12.8}$$

Via Theorem 1.8 we obtain augmented structures of C_∞-algebras on $N^*(X) \otimes N^*(Y)$ and $N^*(X \times Y)$ so that all the maps in (12.8) become augmented C_∞ quasi-isomorphisms.

In particular, we have an augmented C_∞ quasi-isomorphism,

$$N^*(X) \otimes N^*(Y) \xrightarrow{\simeq} N^*(X \times Y).$$

Apply $\mathscr{E}$ to this morphism to obtain, in view of (9.4) and via Proposition 2.10(1), a dglc quasi-isomorphism,

$$\mathscr{E}\big(N^*(X) \otimes N^*(Y)\big) \xrightarrow{\simeq} \overline{\mathfrak{L}}^{\mathrm{c}}_{X\times Y}.$$

Dualizing this morphism produces, by Theorem 9.11, a cdgl quasi-isomorphism of the form

$$\mathfrak{L}_{X\times Y}/(e) \xrightarrow{\simeq} \big(\widehat{\mathbb{L}}(s^{-1}\overline{N^*(X)\otimes N^*(Y)}^{\#}), d\big). \tag{12.9}$$

Observe that the augmentation ideal of $N^*(X) \otimes N^*(Y)$ is

$$\overline{N^*(X)\otimes N^*(Y)} = \big(\overline{N}^*(X)\otimes \mathbb{Q}\beta\big) \oplus \big(\mathbb{Q}\alpha \otimes \overline{N}^*(Y)\big) \oplus \big(\overline{N}^*(X)\otimes \overline{N}^*(Y)\big),$$

where α and β are the 0-cochains dual to a and b, respectively, and thus they are unit elements in $N^*(X)$ and $N^*(Y)$ respectively. Hence,

$$s^{-1}\overline{N^*(X)\otimes N^*(Y)}^{\#} \cong V \oplus W \oplus s(V\otimes W),$$

because

$$V \cong s^{-1}\big(\overline{N}^*(X)\otimes \mathbb{Q}\beta\big)^{\#}, \quad W \cong s^{-1}\big(\mathbb{Q}\alpha\otimes \overline{N}^*(Y)\big)^{\#}$$

and

$$s(V\otimes W)\cong s^{-1}\big(\overline{N}^*(X)\otimes\overline{N}^*(Y)\big)^{\#}.$$

Then, the quasi-isomorphism (12.9) takes the simpler form

$$\mathfrak{L}_{X\times Y}/(e)\xrightarrow{\simeq}\big(\widehat{\mathbb{L}}\big(V\oplus W\oplus s(V\otimes W)\big),d\big). \tag{12.10}$$

On the other hand, consider the diagrams of augmented maps

$$\begin{array}{ccc} A_{\rm PL}(X)\otimes A_{\rm PL}(Y) & \overset{p}{\underset{i}{\rightleftarrows}} & N^*(X)\otimes N^*(Y)\\ \uparrow & & \\ A_{\rm PL}(X) & \overset{p_X}{\underset{\iota_X}{\rightleftarrows}} & N^*(X), \end{array}\qquad \begin{array}{ccc} A_{\rm PL}(X)\otimes A_{\rm PL}(Y) & \overset{p}{\underset{i}{\rightleftarrows}} & N^*(X)\otimes N^*(Y)\\ \uparrow & & \\ A_{\rm PL}(Y) & \overset{p_Y}{\underset{\iota_Y}{\rightleftarrows}} & N^*(Y), \end{array}$$

in which the vertical morphisms are the inclusions (along the units in $A_{\rm PL}(X)$ and $A_{\rm PL}(Y)$ induced by the 0-simplices a and b, respectively). Again use Theorem 1.8 to make them diagrams of augmented C_∞-algebras. In particular, we have morphisms of augmented C_∞-algebras

$$N^*(X)\longrightarrow N^*(X)\otimes N^*(Y),\qquad N^*(Y)\longrightarrow N^*(X)\otimes N^*(Y).$$

Applying again first the functor $\mathscr{E}$, and then dualizing, we obtain cdgl morphisms of the form

$$\big(\widehat{\mathbb{L}}\big(V\oplus W\oplus s(V\otimes W)\big),d\big)\longrightarrow(\widehat{\mathbb{L}}(V),d),\quad \big(\widehat{\mathbb{L}}\big(V\oplus W\oplus s(V\otimes W)\big),d\big)\longrightarrow(\widehat{\mathbb{L}}(W),d).$$

which induce another cdgl morphism,

$$\varphi\colon\big(\widehat{\mathbb{L}}\big(V\oplus W\oplus s(V\otimes W)\big),d\big)\xrightarrow{\simeq}(\widehat{\mathbb{L}}(V),d)\times(\widehat{\mathbb{L}}(W),d). \tag{12.11}$$

Note that, by construction, this morphism is the identity on V and W. We now show that φ is a quasi-isomorphism. For this, define a new grading on these cdgl's by letting V and W be of degree 1, $s(V\otimes W)$ of degree 2, and extending the grading bracket-wise on the Lie algebras. With respect to this grading write the differentials in both sides and the morphism φ as

$$d=\sum_{i\ge1}d_i,\qquad \varphi=\sum_{i\ge1}\varphi_i,$$

where each d_i and φ_i increases the degree by $i-1$. Note that in the C_∞ structure on $N^*(X)\otimes N^*(Y)$,

$$m_2\big((x\otimes1)\otimes(1\otimes y)\big)=x\otimes y,\quad\text{for}\quad x\in N^*(X)\quad\text{and}\quad y\in N^*(Y).$$

Thus, by construction, the linear part d_1 in $\widehat{\mathbb{L}}\big(V\oplus W\oplus s(V\otimes W)\big)$, which depends only on m_2, satisfies:

$$d_1\big(s(v\otimes w)\big)=-s(d_1v\otimes w)-(-1)^{|v|}s(v\otimes d_1w)\pm[v,w],\quad\text{with}\quad v\in V\quad\text{and}\quad w\in W.$$

Therefore, we may proceed as in the classical "reduced" case in [130, VII.1] to show that

$$\varphi_1\colon \big(\widehat{\mathbb{L}}\big(V\oplus W\oplus s(V\otimes W)\big),d_1\big)\xrightarrow{\simeq}(\widehat{\mathbb{L}}(V),d_1)\times(\widehat{\mathbb{L}}(W),d_1)$$

is a quasi-isomorphism. Then, a straightforward spectral sequence argument implies that φ is also a quasi-isomorphism.

Hence, the composition of the quasi-isomorphisms in (12.10) and (12.11) yields the desired quasi-isomorphism

$$\mathfrak{L}_{X\times Y}/(e)\xrightarrow{\simeq}\mathfrak{L}_X/(a)\times\mathfrak{L}_Y/(b).$$

Finally, for generic X and Y, not necessarily of finite type, write them, together with their product, as the colimit of their finite type sub-simplicial sets and take the colimit of the corresponding quasi-isomorphisms as above. □

Observe that from the quasi-isomorphisms (12.10) and (12.11) in the preceding proof one immediately deduces:

Corollary 12.12. *Let $(\widehat{\mathbb{L}}(V),d)$ and $(\widehat{\mathbb{L}}(W),d)$ be connected Lie models of the connected simplicial sets X and Y. Then, $X\times Y$ has a connected model of the form*

$$\big(\widehat{\mathbb{L}}\big(V\oplus W\oplus s(V\otimes W)\big),d\big),$$

whose differential extends the one on $(\widehat{\mathbb{L}}(V),d)$ and $(\widehat{\mathbb{L}}(W),d)$. □

As an application, we construct the minimal Lie model of $X\times S^n$, for any connected simplicial set X and any $n\geq 1$. Let $(\widehat{\mathbb{L}}(V),d)$ be the minimal Lie model of X. We form the cgl

$$L=\widehat{\mathbb{L}}(V\oplus\mathbb{Q}y\oplus s^nV),\quad\text{where}\quad |y|=n-1,$$

and we define on it the derivations i of degree n and D of degree -1 by

$$i(v)=s^nv,\quad i(y)=i(s^nv)=0,$$
$$Dy=0,\quad Dv=dv\quad\text{and}\quad Ds^nv=(-1)^n idv+[y,v].$$

In the simply connected case, the following is [95, Theorem 3.3].

Proposition 12.13. *With the above notations, (L,D) is the minimal Lie model of $X\times S^n$. Moreover, the injection*

$$(\widehat{\mathbb{L}}(V),d)\,\widehat{\amalg}\,(\mathbb{L}(y),0)\longrightarrow(L,D)$$

is a Lie model for the injection $X\vee S^n\to X\times S^n$.

Proof. We first show that $D^2 = 0$. This is obviously the case for y and V. On the other hand, by definition, $[D, i] = \mathrm{ad}_y$ on V. Then,

$$\begin{aligned}(-1)^{n-1}[y, dv] &= D[D, i](v) = D^2 i(v) - (-1)^n Di(Dv)\\ &= D^2 iv - (-1)^n\big((-1)^n iDDv + \mathrm{ad}_y\, Dv\big) = D^2 iv + (-1)^{n-1}[y, dv].\end{aligned}$$

Therefore, $D^2 iv = D^2 s^n v = 0$, and thus D is indeed a differential. Notice also that D is decomposable, so (L, D) is a minimal free cdgl.

On the other hand, consider the quasi-isomorphism

$$(\widehat{\mathbb{L}}(V), d) \times (\mathbb{L}(y), 0) \xrightarrow{\simeq} \mathfrak{L}^a_X \times \mathfrak{L}^b_{S^n}$$

given by choosing the minimal models of both factors. By Theorem 12.11, the first assertion is thus proved once we show that the cdgl morphism,

$$\varphi\colon (L, D) \xrightarrow{\simeq} (\widehat{\mathbb{L}}(V), d) \times (\mathbb{L}(y), 0), \tag{12.12}$$

defined by

$$\varphi(v) = v, \quad \varphi(y) = y \quad \text{and} \quad \varphi(s^n v) = 0,$$

is in fact a quasi-isomorphism.

We define decreasing filtrations in L and in $(\widehat{\mathbb{L}}(V), d) \times \mathbb{L}(y)$ by defining in both cases F^p to be the linear span of the Lie brackets the sum of whose entries in V and $s^n V$ is $\geq p$. Then $D(F^p) \subset F^p$ and, in the induced spectral sequence, $d_0(s^n v) = [y, v]$. The morphism φ is compatible with the filtrations and, since $E_0(\varphi)$ is an isomorphism, φ is a quasi-isomorphism.

The last assertion follows directly from Proposition 8.36. □

Again, let $(\widehat{\mathbb{L}}(V), d)$ be the minimal Lie model of the connected simplicial set X. Consider the cdgl

$$L = (\widehat{\mathbb{L}}(V \oplus \mathbb{Q}y_1 \oplus \mathbb{Q}y_2 \oplus V' \oplus V''), D),$$

in which $V' \cong V'' \cong s^n V$ and D is defined as follows: let i' and i'' be degree-n derivations given by $i'(v) = v'$, $i''(v) = v''$, and are zero on the rest of generators. Define,

$$Dv = dv, \quad Dy_1 = Dy_2 = 0, \quad D(v') = (-1)^n i'(dv) + [y_1, v]$$

and

$$D(v'') = (-1)^n i''(dv) + [y_2, v].$$

As a consequence of the above proposition we obtain:

Corollary 12.14. *For any $n \geq 1$, L is the minimal Lie model of $X \times (S^n \vee S^n)$.*

Proof. Note that

$$X \times (S^n \vee S^n)$$

is the colimit of

$$X \times S^n \hookleftarrow X \times * \hookrightarrow X \times S^n$$

which, in view of Proposition 12.13, is modeled by

$$(\widehat{\mathbb{L}}(V \oplus \mathbb{Q}y_1 \oplus s^n V), D) \hookleftarrow (\widehat{\mathbb{L}}(V), s) \hookrightarrow (\widehat{\mathbb{L}}(V \oplus \mathbb{Q}y_2 \oplus s^n V), D).$$

To complete the proof, recall that $\mathfrak{L}$ preserves colimits and observe that the colimit of this diagram is precisely L. □

We finish with another illustrative example in the same direction for which we need the following observation.

Proposition 12.15. *For $n \geq 1$, the minimal model of the pinching map*

$$\nabla \colon S^n \longrightarrow S^n \vee S^n$$

is given by

$$\nu \colon (\mathbb{L}(y), 0) \longrightarrow (\mathbb{L}(y_1, y_2), 0), \quad |y| = |y_1| = |y_2| = n - 1,$$
$$\nu(y) = \begin{cases} y_1 + y_2, & \text{if } n \geq 2, \\ y_1 * y_2, & \text{if } n = 1. \end{cases}$$

Proof. Let $f_1, f_2 \colon S^n \vee S^n \to S^n$ be the maps which are the identity on one factor and trivial on the other. Any minimal model of ∇ is necessarily of the form $\nu \colon (\mathbb{L}(y), 0) \xrightarrow{\simeq} (\mathbb{L}(y_1, y_2), 0)$. Choose the generators y_1 and y_2 so that the composition of ν with the projections on the first and the second generator are minimal Lie models for $f_1 \circ \nabla$ and $f_2 \circ \nabla$, respectively. Since the linear part ν_1 of ν is the desuspension of the map induced in homology, we get $\nu_1(y) = y_1 + y_2$. Now, if $n \geq 2$ then, for degree reasons, $\nu = \nu_1$ and the result holds.

In the case $n = 1$ observe that the cdgl morphism given in Theorem 5.13, for which we use the same notation,

$$\gamma \colon \mathfrak{L}_1 \longrightarrow (\widehat{\mathbb{L}}(a, u, b, y, z), d), \qquad \gamma(a) = a, \quad \gamma(b) = b, \quad \gamma(x) = y * z,$$

is precisely the Lie model of the subdivision of the interval,

$$\mathfrak{L}_{\Delta^1} \longrightarrow \mathfrak{L}_{\Delta^1 \vee \Delta^1}.$$

Observe that the pinching map $\nabla \colon S^1 \to S^1 \vee S^1$ is obtained from the subdivision of the interval by collapsing in both side the sub-simplicial complexes consisting of the vertices. By Corollary 7.11, a model of ∇,

$$\mathfrak{L}^a_{S^1} \longrightarrow \mathfrak{L}^a_{S^1 \vee S^1},$$

which turns out to be minimal, is the morphism induced by γ on the quotients by all the MC elements,

$$\overline{\gamma}\colon (\widehat{\mathbb{L}}(x),0) \longrightarrow (\widehat{\mathbb{L}}(y,z),0), \quad \overline{\gamma}(x) = y * z. \qquad \square$$

Let again X be a connected simplicial set with minimal model $(\widehat{\mathbb{L}}(V),d)$. With the notation of Proposition 12.13 and Corollary 12.14, we have:

Proposition 12.16. *For $n \geq 2$, the minimal Lie model of the map*

$$\mathrm{id}_X \times \nabla\colon X \times S^n \longrightarrow X \times (S^n \vee S^n)$$

is given by the cdgl morphism

$$\psi\colon (\widehat{\mathbb{L}}(V \oplus \mathbb{Q}y \oplus s^nV),D) \longrightarrow (\widehat{\mathbb{L}}(V \oplus \mathbb{Q}y_1 \oplus \mathbb{Q}y_2 \oplus V' \oplus V''),D),$$

defined by

$$\psi(v) = v, \quad \psi(y) = y_1 + y_2 \quad \text{and} \quad \psi(s^nv) = v' + v''.$$

Proof. Consider the morphism,

$$\varphi\colon (\widehat{\mathbb{L}}(V \oplus \mathbb{Q}y_1 \oplus \mathbb{Q}y_2 \oplus V' \oplus V''),D) \xrightarrow{\simeq} (\widehat{\mathbb{L}}(V),d) \times (\mathbb{L}(y_1,y_2),d),$$

defined by

$$\varphi(v) = v, \quad \varphi(y_1) = y_1, \quad \varphi(y_2) = y_2 \quad \text{and} \quad \varphi(v') = \varphi(v'') = 0,$$

which is the analogue of that in (12.12). The same argument as in the proof of Proposition 12.13 shows that φ is a quasi-isomorphism. On the other hand, the following diagram is trivially commutative:

$$\begin{array}{ccc}
(\widehat{\mathbb{L}}(V \oplus \mathbb{Q}y \oplus s^nV),D) & \xrightarrow{\psi} & (\widehat{\mathbb{L}}(V \oplus \mathbb{Q}y_1 \oplus \mathbb{Q}y_2 \oplus V' \oplus V''),D) \\
\varphi \downarrow \simeq & & \varphi \downarrow \simeq \\
(\widehat{\mathbb{L}}(V),d) \times (\mathbb{L}(y),0) & \xrightarrow[\mathrm{id} \times \nu]{} & (\widehat{\mathbb{L}}(V),d) \times (\mathbb{L}(y_1,y_2),0).
\end{array}$$

Here, the left vertical quasi-isomorphism is that in (12.12) and ν is the model of ∇ of Proposition 12.15.

As the bottom morphism is trivially a Lie model of $\mathrm{id}_X \times \nabla$, we only have to check that ψ is indeed a cdgl morphism, that is, it commutes with differentials. For that write on both sides $D = \sum_{i \geq 2} D_i$, where D_i increases the bracket length by $i-1$, and check that $D_i\psi(s^nv) = D_i(v' + v'') = \psi(D_is^nv)$ for any $s^nv \in s^nV$. $\square$

12.4 Mapping spaces

In this section we give Lie models of mapping spaces arising from models of the involved simplicial sets. We first fix some notation, make some general assumptions, and prove an essential result.

Henceforth, we denote by $\mathrm{Map}(X,Y)$ the simplicial mapping space,

$$\mathrm{Map}_n(X,Y) = \mathrm{Hom}_{\mathbf{sset}}(X \times \underline{\Delta}^n, Y).$$

Also, in what follows, Theorem 11.14 and Proposition 11.2 will be key. Hence, we will assume that any considered cdgl L is such that $L/[L,L]$ is of finite type. Moreover, we will assume that X is a connected finite simplicial set or else that $L/[L,L]$ is finite-dimensional. These assumptions remain in force throughout this section. Given connected simplicial set X and Y, by a cdga model of X we mean, as usual, a cdga of the same homotopy type of $A_{\mathrm{PL}}(X)$. In the same way, a cdgl model of Y stands for any cdgl of the homotopy type of $\mathfrak{L}^a_Y$.

Theorem 12.17. *Let X and Y be connected simplicial sets and let L be a cdgl model of Y. Then, there is a homotopy equivalence of Kan complexes*

$$\mathrm{MC}_\bullet\big(A_{\mathrm{PL}}(X)\widehat{\otimes}L\big) \xrightarrow{\simeq} \mathrm{Map}\big(X, \mathrm{MC}_\bullet(L)\big)\,.$$

Proof. First of all, by definition,

$$\mathrm{MC}_n\big(A_{\mathrm{PL}}(X)\widehat{\otimes}L\big) = \mathrm{MC}\big((A_{\mathrm{PL}}(X)\widehat{\otimes}A_{\mathrm{PL}}(\underline{\Delta}^n))\,\widehat{\otimes}L\big) = \mathrm{MC}\big(A_{\mathrm{PL}}(X\times\underline{\Delta}^n)\widehat{\otimes}L\big).$$

Next, Let $m \xrightarrow{\simeq} L$ be the minimal model of L. Then, by Theorem 10.8,

$$\varinjlim_n \mathscr{C}^*(m/m^n)$$

is a Sullivan model of $\langle L\rangle$ which we denote by $(\wedge V,d)$. Thus, by Proposition 11.2, we have a natural bijection,

$$\mathrm{Hom}_{\mathbf{cdga}}\big((\wedge V,d), A_{\mathrm{PL}}(X\times\underline{\Delta}^n)\big) \xrightarrow{\cong} \mathrm{MC}\big(A_{\mathrm{PL}}(X\times\underline{\Delta}^n)\widehat{\otimes}L\big).$$

On the other hand, the adjunction of Theorem 1.2 gives a natural bijection

$$\mathrm{Hom}_{\mathbf{cdga}}\big((\wedge V,d), A_{\mathrm{PL}}(X\times\underline{\Delta}^n)\big) \xrightarrow{\cong} \mathrm{Hom}_{\mathbf{sset}}(X\times\underline{\Delta}^n, \langle\wedge V,d\rangle^{\mathrm{S}}).$$

Finally, notice that, by Theorem 11.14,

$$\langle\wedge V,d\rangle^{\mathrm{S}} \simeq \mathrm{MC}_\bullet(L).$$

The combination of this homotopy equivalence with the above bijections gives a homotopy equivalence of simplicial sets

$$\mathrm{MC}_\bullet\big(A_{\mathrm{PL}}(X)\widehat{\otimes}L\big) \xrightarrow{\simeq} \mathrm{Map}(X, \mathrm{MC}_\bullet(L)\big),$$

as stated. □

12.4.1 Lie models of mapping spaces

Our main result in this section reads:

Theorem 12.18. *Let X and Y be connected simplicial sets. Let A be a cdga model of X and let L be a connected cdgl model of Y. Then, there are natural homotopy equivalences,*

$$\langle A\widehat{\otimes}L\rangle \simeq \operatorname{Map}(X,\langle L\rangle) \simeq \operatorname{Map}(X,\mathbb{Q}_\infty Y).$$

Proof. The second equivalence follows immediately from Theorem 11.14. For the first equivalence, use this result and Theorem 12.17 to obtain,

$$\operatorname{Map}(X,\langle L\rangle) \simeq \operatorname{Map}\big(X,\operatorname{MC}_\bullet(L)\big) \cong \operatorname{MC}_\bullet(A_{\mathrm{PL}}(X)\widehat{\otimes}L) \simeq \langle A_{\mathrm{PL}}(X)\widehat{\otimes}L\rangle\,. \quad (12.13)$$

Now, since A and $A_{\mathrm{PL}}(X)$ are connected by a sequence of quasi-isomorphisms, Proposition 4.38 shows that

$$A\widehat{\otimes}L \simeq A_{\mathrm{PL}}(X)\widehat{\otimes}L.$$

Again, since the realization functor preserves weak equivalences,

$$\langle A\widehat{\otimes}L\rangle \simeq \langle A_{\mathrm{PL}}(X)\widehat{\otimes}L\rangle.$$

With this and (12.13), the result follows. □

An interesting particular situation is the following: recall that a connected simplicial set is *formal* if $A_{\mathrm{PL}}(X)$ has the same homotopy type as $H^*(X;\mathbb{Q})$. On the other hand (see Definition 10.12), recall that a connected simplicial set of finite type Y is coformal if the differential on its Sullivan minimal model $(\wedge V,d)$ is quadratic, or, its minimal Lie model m_X is quasi-isomorphic to a connected cdgl L with zero differential. In particular, L can be chosen to be $H(m_X)$ or, as observed in Corollary 10.13, the homotopy Lie algebra of $(\wedge V,d)$. By Theorem 12.18, the following is an immediate consequence:

Corollary 12.19. *Let X be a formal simplicial set and let Y be a coformal simplicial set of finite type. Write $H = H^*(X;\mathbb{Q})$ and $L=(L,0)$ a cdgl model of Y. Then,*

$$\operatorname{Map}(X,\mathbb{Q}_\infty Y) \simeq \langle H\widehat{\otimes}L\rangle. \qquad \square$$

We now see how to extract from Theorem 12.18 Lie models for each path component of a given mapping space. Again, Let A and L be, respectively, a cdga model and a cdgl model of the connected simplicial sets X and Y. We begin by noticing that, in view of Theorem 12.18 and Proposition 7.14, we have bijections

$$\pi_0\operatorname{Map}(X,\mathbb{Q}_\infty Y) \cong \pi_0\langle A\widehat{\otimes}L\rangle \cong \widetilde{\operatorname{MC}}(A\widehat{\otimes}L).$$

Thus, a simplicial map $f\colon X\to\mathbb{Q}_\infty Y$, regarded as a 0-simplex

$$f\in\operatorname{Map}(X,\mathbb{Q}_\infty Y)_0,$$

corresponds to a Maurer–Cartan element

$$z \in \mathrm{MC}(A\widehat{\otimes}L).$$

With this notation, Theorems 7.16 and 7.18 immediately lead to

Corollary 12.20. *Let X be a connected simplicial set with cdga model A and let Y be a connected simplicial set with cdga model L. Then, there is a homotopy equivalence of simplicial sets,*

$$\langle (A\widehat{\otimes}L)^z\rangle \simeq \mathrm{Map}_f(X, \mathbb{Q}_\infty Y),$$

where $\mathrm{Map}_f(X,\mathbb{Q}_\infty)$ *denotes the path component of* $\mathrm{Map}(X,\mathbb{Q}_\infty Y)$ *containing* f*. In particular,*

$$\pi_n \,\mathrm{Map}_f(X,\mathbb{Q}_\infty) \cong H_{n-1}(A\widehat{\otimes}L, d_z). \qquad \square$$

As an immediate consequence we obtain:

Corollary 12.21. *Let $c\colon X\to \mathbb{Q}_\infty Y$ denote the constant map. Then, for any $n\geq 1$,*

$$\pi_n \,\mathrm{Map}_c(X,\mathbb{Q}_\infty Y) \cong H_{n-1}(A\widehat{\otimes}L).$$

Moreover, if X is a finite simplicial set or Y is nilpotent,

$$\pi_n \,\mathrm{Map}_c(X,\mathbb{Q}_\infty Y) \cong \bigoplus_{p+q=n-1} H^p(X;\mathbb{Q})\otimes \pi_q(\mathbb{Q}_\infty Y).$$

Proof. The first assertion is immediate since the Maurer–Cartan element in $A\widehat{\otimes}L$ associated to c is 0. For the second, choose either a finite-dimensional cdga model A of X if it is finite, or else, a nilpotent dgl model of Y if it is nilpotent. In this case,

$$A\widehat{\otimes}L = \varprojlim_n (A\otimes L/F^n) \cong A\otimes \varprojlim_n L/F^n = A\otimes L.$$

Therefore, in view of Corollary 11.15,

$$H(A\widehat{\otimes}L)\cong H(A\otimes L)\cong H^*(X;\mathbb{Q})\otimes \pi_*(\mathbb{Q}_\infty Y). \qquad \square$$

To illustrate the above results we now present a list of interesting examples.

Example 12.22. Let X be a connected simplicial set which has a connected Lie model L of finite type. Then,

$$\mathscr{C}^*(L) = (\land V, d)$$

is a Sullivan model of X and, by Theorem 12.18, $(\land V,d)\widehat{\otimes}L$ is a Lie model of $\mathrm{Map}(X,\mathbb{Q}_\infty X)$, that is,

$$\mathrm{Map}(X,\mathbb{Q}_\infty X)\simeq \langle \land V\widehat{\otimes}L\rangle.$$

Assume now that L is finite-dimensional. Note that in this case $\wedge V\widehat{\otimes}L = \wedge V\otimes L$. As at the beginning of Section 11.1, choose a graded basis $\{x_i\}$ for L and the corresponding basis $\{\overline{x_i}\}$ of $(sL)^{\#}$ via the usual pairing $\langle\overline{x_i}, sx_j\rangle = -\delta_{ij}$. In this particular case, the bijection in Proposition 11.1,

$$\operatorname{Hom}_{\mathbf{cdga}}\big((\wedge V,d),(\wedge V,d)\big)\xrightarrow{\cong}\operatorname{MC}\big(\wedge V\otimes L\big),$$

sends the identity to $z=\sum_i \overline{x}_i\otimes x_i$. Then, by Corollary 12.20,

$$\operatorname{Map}_\iota(X,\mathbb{Q}_\infty X)\simeq\langle(\wedge V\otimes L)^z\rangle$$

where $\iota\colon X\to\mathbb{Q}_\infty X$ is the completion map. Moreover, by Corollary 12.21,

$$\pi_n\operatorname{Map}_\iota(X,\mathbb{Q}_\infty X)\cong H_{n-1}(\wedge V\otimes L,d_z),\quad\text{for}\quad n\geq 1.$$

In particular, if X is a finite nilpotent simplicial set and, as usual, $\operatorname{aut}X$ denotes the self homotopy equivalences of X, then

$$\pi_n(\operatorname{aut}X,\mathrm{id}_X)_{\mathbb{Q}}\cong H_{n-1}(\wedge V\otimes L,d_z),\quad\text{for}\quad n\geq 1.$$

Indeed, if X is nilpotent, its $\mathbb{Q}$-completion $\mathbb{Q}_\infty X$ coincides with its rationalization $X_{\mathbb{Q}}$. Also, since X is finite, the canonical map $\operatorname{Map}_f(X,X)_{\mathbb{Q}}\xrightarrow{\simeq}\operatorname{Map}_{\iota f}(X,X_{\mathbb{Q}})$ is a homotopy equivalence for any map f [75, Theorem 3.11].

Example 12.23. Here we describe $\operatorname{Map}(T_g,\mathbb{Q}_\infty T_g)$, where T_g denotes an orientable surface of genus g. It is well known that

$$H=H^*(T_q;\mathbb{Q})\cong\wedge(a_i,b_j)_{1\leq i,j\leq g}/I,\quad\text{with}\quad |a_i|=|b_i|=1,$$

where I is the ideal generated by the elements a_ia_j, b_ib_j, $a_ib_i-a_1b_1$ and the elements a_ib_j for $i\neq j$. On the other hand, see Corollary 12.8,

$$\pi_1(\mathbb{Q}_\infty T_g)=\widehat{\mathbb{L}}(\alpha_i,\beta_j)_{1\leq i,j\leq g}/\sum\alpha_i*\beta_i*\alpha_i^{-1}*\beta_i^{-1},\quad\text{where}\quad|\alpha_i|=|\beta_j|=0.$$

Denote this Lie algebra by L. As T_g is formal and coformal, via Corollary 12.19 the space $\operatorname{Map}(T_g,\mathbb{Q}_\infty T_g)$ is homotopy equivalent to the realization of the cdgl $H\otimes L$. A straightforward computation shows that an element

$$\omega=\sum_i a_i\otimes\omega_i+\sum_j b_j\otimes\omega'_j\in(H\otimes L)_{-1}$$

is an MC element if and only if

$$\sum_i[\omega_i,\omega'_i]=0.$$

The gauge action of an element $1\otimes\rho\in(H\otimes L)_0$ is given by

$$(1\otimes\rho)\,\mathcal{G}\,\omega=\sum_i a_i\otimes e^{\operatorname{ad}_\rho}\omega_i+\sum_j b_j\otimes e^{\operatorname{ad}_\rho}\omega'_j.$$

For instance, the MC element associated to the identity is $\sum a_i\otimes\alpha_i+\sum b_i\otimes\beta_i$.

Example 12.24. Given a connected simplicial set X, consider the mapping spaces

$$\operatorname{Map}(X, S^n_{\mathbb{Q}}), \quad \text{for} \quad n \geq 1.$$

If A is a cdga model for X, then a Lie model of $\operatorname{Map}(X, S^n_{\mathbb{Q}})$ is given by the cdgl

$$(A \otimes (\mathbb{L}(y), 0), \quad \text{with} \quad |y| = n-1.$$

Whenever n is odd, $[y,y]=0$ and we immediately observe that an MC element of this cdgl is an element $a \otimes y$ of degree -1 such that $da = 0$. Moreover, two such elements $a \otimes y$ and $b \otimes y$ are gauge equivalent if and only if $[a] = [b]$. That is,

$$\widetilde{\operatorname{MC}}(A \otimes \mathbb{L}(y)) \cong H^n(A) \otimes \mathbb{Q}\, y.$$

Moreover, all perturbed differentials coincide with the original differential in $A \otimes \mathbb{L}(y)$. Hence, all the components have the same homotopy type [108, Theorem 2.1].

12.4.2 Lie models of pointed mapping spaces

We now describe how to obtain Lie models of pointed mapping spaces. Let X and Y be pointed simplicial sets and denote by

$$\operatorname{Map}^*(X, Y)$$

the sub-simplicial set of $\operatorname{Map}(X, Y)$ of pointed maps. Notice that $\operatorname{Map}^*(X, Y)$ is the homotopy fiber of the evaluation map at the base point of X,

$$\operatorname{Map}^*(X, Y) \longrightarrow \operatorname{Map}(X, Y) \xrightarrow{\text{ev}} Y.$$

Then, we have:

Proposition 12.25. *Let X be a connected simplicial set with cdga model A and let Y be a connected simplicial set with cdgl model L. Then, there is a commutative diagram of the form*

$$\begin{array}{ccccc}
\operatorname{Map}^*(X, \mathbb{Q}_\infty Y) & \longrightarrow & \operatorname{Map}(X, \mathbb{Q}_\infty Y) & \xrightarrow{\text{ev}} & \mathbb{Q}_\infty Y \\
\simeq \uparrow & & \simeq \uparrow & & \simeq \uparrow \\
\langle A^+ \widehat{\otimes} L \rangle & \longrightarrow & \langle A \widehat{\otimes} L \rangle & \longrightarrow & \langle L \rangle
\end{array}$$

where the vertical maps are homotopy equivalences.

Proof. The existence of a homotopy commutative square on the right is clear by all the material presented in this section. As ev is a fibration, we can then make it strictly commutative.

On the other hand, by Corollary 8.4, the bottom horizontal maps form a fibration sequence since it is the realization of the cdgl fibration sequence,

$$0 \longrightarrow A^+\widehat{\otimes}L \longrightarrow A\widehat{\otimes}L \longrightarrow L \longrightarrow 0.$$

Hence, the left vertical arrow exists and it is necessarily a homotopy equivalence. $\square$

As in the preceding section, any pointed map $f\colon X \to \mathbb{Q}_\infty Y$ corresponds in this case to an element $z \in \mathrm{MC}(A^+\widehat{\otimes}L)$. Then, the analogue of Corollary 12.20 reads:

Corollary 12.26. *There is a homotopy equivalence of simplicial sets,*

$$\langle (A^+\widehat{\otimes}L)^z\rangle \simeq \mathrm{Map}^*_f(X, \mathbb{Q}_\infty Y).$$

In particular,

$$\pi_n \,\mathrm{Map}^*_f(X, \mathbb{Q}_\infty Y) \cong H_{n-1}(A^+\widehat{\otimes}L, d_z). \qquad \square$$

12.4.3 Lie models of free loop spaces

The space of free loops on a given simplicial set is a particularly rich object which provides many interesting geometrical invariants of that simplicial set. Here, we describe in detail a Lie model of the free loops on the realization of any connected cdgl L.

By Theorem 12.18, $\langle L\rangle^{S^1} = \mathrm{Map}(S^1, \langle L\rangle)$ is homotopy equivalent to the realization of the cdgl

$$(\wedge x \otimes L, d)$$

with $|x| = -1$, $d(x\otimes a) = -x \otimes da$, and

$$[1\otimes a + x\otimes b, 1\otimes c + x\otimes e] = 1\otimes [a,c] + x\otimes [b,c] + (-1)^{|x||a|} x\otimes [a,e]\,.$$

As L is connected, one trivially obtains

$$\mathrm{MC}(\wedge x\otimes L, d) = \{x\otimes a,\ a\in L_0\}.$$

On the other hand, $(\wedge x\otimes L)_0 = \{1\otimes b_0 + x\otimes b_1,\ b_0\in L_0,\ b_1\in L_1\}$. Then, a direct computation shows that the gauge action is given by:

Lemma 12.27. $(1\otimes b_0 + x\otimes b_1)\,\mathcal{G}\,(x\otimes a) = x\otimes \big(e^{\mathrm{ad}_{b_0}}(a) + \frac{e^{\mathrm{ad}_{b_0}}-1}{\mathrm{ad}_{b_0}}(db_1)\big).$ $\square$

From this, we can explicitly determine the path components of $\langle L\rangle^{S^1}$.

Proposition 12.28. *There is a bijection:*

$$\mathrm{cl}\big(\pi_1\langle L\rangle\big) \cong \pi_0\langle L\rangle^{S^1}.$$

Here, $\mathrm{cl}(\pi_1\langle L\rangle)$ denotes the set of conjugacy classes of the group $\pi_1\langle L\rangle$.

Proof. As $\widetilde{\mathrm{MC}}(\wedge x\otimes L,d)=\pi_0\langle L\rangle^{S^1}$ and $\pi_1\langle L\rangle\cong H_0(L)$, the statement amounts to finding a bijective map

$$\phi\colon \mathrm{cl}\big(H_0(L)\big)\xrightarrow{\cong}\widetilde{\mathrm{MC}}(\wedge x\otimes L,d).$$

We set

$$\phi(\alpha * H_0(L)*\alpha^{-1})=x\otimes a,\quad \text{for}\quad \alpha*H_0(L)*\alpha^{-1}\in \mathrm{cl}\big(H_0(L)\big)\quad\text{and}\quad [a]=\alpha.$$

First, we see that this map is well defined: let $\alpha,\gamma\in H_0(L)$ be in the same conjugacy class. Write $[a]=\alpha$, $[c]=\gamma$ and $\gamma=\alpha*\beta*\alpha^{-1}$ with $\beta=[b_0]$. This translates to

$$a*b_0*a^{-1}=c-db',\quad\text{with}\quad b'\in L_1.$$

Define,

$$b_1=\frac{\mathrm{ad}_{b_0}}{e^{\mathrm{ad}_{b_0}}-1}(b').$$

Then, by Proposition 4.13 and taking into account that b_0 is a cycle, we see that

$$\begin{aligned}(1\otimes b_0+x\otimes b_1)\,\mathcal{G}\,(x\otimes a)&=x\otimes\Big(e^{\mathrm{ad}_{b_0}}(a)+\frac{e^{\mathrm{ad}_{b_0}}-1}{\mathrm{ad}_{b_0}}(db_1)\Big)\\&=x\otimes\left(a*b_0*a^{-1}+d\Big(\frac{e^{\mathrm{ad}_{b_0}}-1}{\mathrm{ad}_{b_0}}(b_1)\Big)\right)\\&=x\otimes(a*b_0*a^{-1}+db')=x\otimes c.\end{aligned}$$

That is, $x\otimes a$ and $x\otimes c$ are gauge related and thus ϕ is well defined.

A similar argument shows that ϕ is also injective, while the surjectivity is obvious. □

Example 12.29. Consider the minimal Lie model of a wedge of circles $S^1\vee \overset{n}{\cdots}\vee S^1$ which, by Lemma 12.1 is,

$$L=(\widehat{\mathbb{L}}(x_1,\dots,x_n),0),\quad\text{where}\quad |x_i|=0\quad\text{for all}\quad i=1,\dots,n.$$

By Theorem 12.18,

$$\langle L\rangle\cong\mathbb{Q}_\infty(S^1\vee\overset{n}{\cdots}\vee S^1),$$

and by Proposition 12.28 the path components of $\langle L\rangle^{S^1}$ correspond to the conjugacy classes of L with respect to the BCH product.

Let $\langle L\rangle^{S^1}_\omega$ be the component of $\langle L\rangle^{S^1}$ associated to the MC element $\omega\in\mathrm{MC}(\wedge x\otimes L,0)=\wedge^+x\otimes L$.

When $\omega=0$, $\langle L\rangle^{S^1}_\omega$ is the component of homotopically trivial maps, and

$$\pi_1\langle L\rangle^{S^1}_0=H_0(\wedge x\otimes L,0)\cong L.$$

On the other hand, when $\omega \neq 0$, write $\omega = x \otimes a$ and observe that, for any $c \in L$,

$$d_\omega(1 \otimes c) = x \otimes [a, c] \quad \text{and} \quad d_w(x \otimes c) = 0.$$

In particular, $d_\omega(1 \otimes c) = 0$ if and only if a is a multiple of c. Therefore,

$$\pi_1 \langle L \rangle_\omega^{S^1} \cong H_0(\wedge x \otimes L, d_\omega) \cong \mathbb{Q} \cdot (1 \otimes a).$$

12.4.4 Simplicial enrichment of cdgl and cdga

Here we notice that, as in the cdga setting, the category **cdgl** is simplicially enriched. With that in mind, we show that rationally, a mapping space of simplicial sets is homotopy equivalent to the simplicially enriched morphisms between cdgl models of the given simplicial sets.

Recall that a category is *simplicially enriched* if the set of morphisms between any two given objects is a simplicial set.

Definition 12.30. Given cdgl's L and L', define the simplicial set

$$\mathrm{Hom}_{\mathbf{cdgl}^{\Delta}}(L, L') = \mathrm{Hom}_{\mathbf{cdgl}}\big(L, A_{\mathrm{PL}}(\underline{\Delta}^\bullet)\widehat{\otimes} L'\big).$$

Similarly, given cdga's A and A', define the simplicial set

$$\mathrm{Hom}_{\mathbf{cdga}^{\Delta}}(A, A') = \mathrm{Hom}_{\mathbf{cdga}}(A, A_{\mathrm{PL}}\big(\underline{\Delta}^\bullet) \otimes A'\big).$$

We denote by $\mathbf{cdgl}^{\boldsymbol{\Delta}}$ and $\mathbf{cdga}^{\boldsymbol{\Delta}}$ the corresponding enriched categories.

The category **sset** of simplicial sets is obviously simplicially enriched by the usual simplicial mapping

$$\mathrm{Map}_\bullet(X, Y) = \mathrm{Hom}_{\mathbf{sset}}(X \times \underline{\Delta}^\bullet, Y).$$

We first observe that the classical Sullivan adjunction of Theorem 1.2 extends to the enriched categories. In other words:

Proposition 12.31. *Let X be a simplicial set and let A be a cdga. Then, there is an equivalence of simplicial sets*

$$\mathrm{Map}(X, \langle A \rangle^{\mathrm{S}}) \cong \mathrm{Hom}_{\mathbf{cdga}^{\Delta}}\big(A, A_{\mathrm{PL}}(X)\big).$$

Proof. This result follows from the naturality of the "unenriched" adjunction:

$$\begin{aligned}
\mathrm{Map}(X, \langle A \rangle^{\mathrm{S}}) &= \mathrm{Hom}_{\mathbf{sset}}(X \times \underline{\Delta}^n, \langle A \rangle^{\mathrm{S}}) \cong \mathrm{Hom}_{\mathbf{cdga}}\big(A, A_{\mathrm{PL}}(X \times \underline{\Delta}^n)\big) \\
&= \mathrm{Hom}_{\mathbf{cdga}}\big(A, A_{\mathrm{PL}}(X) \otimes A_{\mathrm{PL}}(\underline{\Delta}^n)\big) \\
&= \mathrm{Hom}_{\mathbf{cdga}^{\Delta}}\big(A, A_{\mathrm{PL}}(X)\big).
\end{aligned}$$

□

An important consequence, that the reader can also find in [16], is the following:

Theorem 12.32. *Let X and Y be connected simplicial sets of finite type with respective Sullivan minimal models $(\wedge V_X, d)$ and $(\wedge V_Y, d)$. Then, there is an equivalence of simplicial sets*

$$\mathrm{Map}(X, \mathbb{Q}_\infty Y) \cong \mathrm{Hom}_{\mathbf{cdga}^\Delta}\big((\wedge V_Y, d), (\wedge V_X, d)\big).$$

Proof. This is deduced from Proposition 12.31 and the following sequence of equivalences:

$$\begin{aligned}\mathrm{Map}(X, \mathbb{Q}_\infty Y) &= \mathrm{Hom}_{\mathbf{sset}^\Delta}(X, \mathbb{Q}_\infty Y) \simeq \mathrm{Hom}_{\mathbf{sset}^\Delta}(X, \langle \wedge V_Y, d\rangle^{\mathrm{S}})\\ &\cong \mathrm{Hom}_{\mathbf{cdga}^\Delta}\big((\wedge V_Y, d), A_{\mathrm{PL}}(X)\big)\\ &\simeq \mathrm{Hom}_{\mathbf{cdga}^\Delta}\big((\wedge V_Y, d), (\wedge V_X, d)\big). \qquad \square\end{aligned}$$

We now translate all of the above to the Lie setting. We begin by extending the adjunction given by the global model and realization functors to the homotopy enriched categories:

Theorem 12.33. *Let X be a connected simplicial set and let L be a connected cdgl for which $L/[L,L]$ has finite type. Then, there is a natural homotopy equivalence of simplicial sets*

$$\mathrm{Map}(X, \langle L\rangle) \simeq \mathrm{Hom}_{\mathbf{cdgl}^\Delta}(\mathfrak{L}_X, L).$$

Proof. By Theorems 7.4, 11.13 and 12.17, we have the following sequence of homotopy equivalences for each $n \geq 0$:

$$\begin{aligned}\mathrm{Hom}_{\mathbf{cdgl}^\Delta}(\mathfrak{L}_X, L)_n &= \mathrm{Hom}_{\mathbf{cdgl}}\big(\mathfrak{L}_X, A_{\mathrm{PL}}(\underline{\Delta}^n)\widehat{\otimes} L\big) \cong \mathrm{Hom}_{\mathbf{sset}}\big(X, \langle A_{\mathrm{PL}}(\underline{\Delta}^n)\widehat{\otimes} L\rangle\big)\\ &\simeq \mathrm{Hom}_{\mathbf{sset}}(X, \mathrm{MC}_\bullet(A_{PL}(\underline{\Delta}^n)\widehat{\otimes} L))\\ &\simeq \mathrm{Hom}_{\mathbf{sset}}(X, \mathrm{Map}(\underline{\Delta}^n, \mathrm{MC}_\bullet(L))\\ &= \mathrm{Map}_0(X, \mathrm{Map}(\underline{\Delta}^n, \langle L\rangle)) = \mathrm{Map}_0(X \times \underline{\Delta}^n, \langle L\rangle)\\ &\cong \mathrm{Hom}_{\mathbf{sset}}(X \times \underline{\Delta}^n, \mathrm{MC}_\bullet(L)) \cong \mathrm{Map}_n(X, \mathrm{MC}_\bullet(L))\\ &= \mathrm{Map}_n(X, \langle L\rangle).\end{aligned}$$

All the homotopy equivalences of this sequence are compatible with the cosimplicial structure of $\underline{\Delta}^\bullet$ and thus, with the simplicial structure on $A_{\mathrm{PL}}(\underline{\Delta}^\bullet)$. Therefore, it induces a homotopy equivalence of simplicial sets. □

Finally, we prove the following analogue of Theorem 12.32:

Theorem 12.34. *Let X and Y be connected simplicial sets of finite type with minimal Lie models L and L'. Then, there is a homotopy equivalence of simplicial sets*

$$\mathrm{Map}(X, \mathbb{Q}_\infty Y) \simeq \mathrm{Hom}_{\mathbf{cdgl}^\Delta}(L, L').$$

Proof. Indeed, using Theorems 12.17 and 12.33, we have

$$\operatorname{Map}(X, \mathbb{Q}_\infty Y) = \operatorname{Map}(X, \langle L' \rangle) \simeq \operatorname{Hom}_{\mathbf{cdgl}^\Delta}(\mathfrak{L}_X, L').$$

Since L' is connected, given any 0 simplex a of X,

$$\operatorname{Hom}_{\mathbf{cdgl}^\Delta}(\mathfrak{L}_X, L') \cong \operatorname{Hom}_{\mathbf{cdgl}^\Delta}(\mathfrak{L}_X/(a), L').$$

Finally, the quasi-isomorphism $L \xrightarrow{\simeq} \mathfrak{L}_x/(a)$ induces a homotopy equivalence of simplicial sets,

$$\operatorname{Hom}_{\mathbf{cdgl}^\Delta}(\mathfrak{L}_X/(a), L') \xrightarrow{\simeq} \operatorname{Hom}_{\mathbf{cdgl}^\Delta}(L, L'). \qquad \square$$

12.4.5 Complexes of derivations and homotopy groups of mapping spaces

Corollary 12.26 offers a way to compute the homotopy groups of a given component of a pointed mapping space. Here, we exhibit these groups as the homology of certain complexes of derivations.

Let $f\colon X \to \langle L \rangle$ be a pointed simplicial map from a connected simplicial set X to the realization of a connected cdgl L. By adjunction, f corresponds to a cdgl morphism $\ell\colon \mathfrak{L}_x \to L$, which, since L is connected, it restricts to a morphism $\ell\colon \mathfrak{L}_X^a \to L$.

Theorem 12.35. *Let X be a nilpotent simplicial set of finite type. For any $n \geq 1$, there is an isomorphism,*

$$\pi_n \operatorname{Map}_f^*(X, \langle L \rangle) \cong H_n \operatorname{Der}^\ell(\mathfrak{L}_X^a, L).$$

Recall that $\operatorname{Map}_f^*(X, \langle L \rangle)$ denotes the simplicial set of pointed maps from X to $\langle L \rangle$ that are homotopic to f.

On the other hand, $\operatorname{Der}^\ell(\mathfrak{L}_X^a, L)$ is the chain complex of ℓ-derivations, which we now carefully analyze.

Let $f\colon L \to L'$ be a cdgl morphism. A linear map $\theta\colon L \to L'$ of degree n is an f-derivation if for each $a, b \in L$, we have

$$\theta[a, b] = [\theta(a), f(b)] + (-1)^{n|a|}[f(a), \theta(b)].$$

Denote by $(\operatorname{Der}_*^f, [d, -])$ the chain complex in which Der_n^f is the vector space of the f-derivations of degree n and the differential is given as usual by

$$D = [d, -]\colon \operatorname{Der}_n^f \longrightarrow \operatorname{Der}_{n-1}^f, \quad \text{where} \quad [d, \theta] = d \circ \theta - (-1)^{|\theta|}\theta \circ d.$$

If $g\colon L' \to L''$ is another cdgl morphism, composition with g induces a morphism of chain complexes

$$g_*\colon \operatorname{Der}_*^f \longrightarrow \operatorname{Der}_*^{gf}, \qquad g_*(\theta) = g \circ \theta.$$

Lemma 12.36. *Let* $f\colon (\widehat{\mathbb{L}}(V),d)\to L'$ *be a cdgl morphism in which with* $V=V_{\geq r}$ *for some* $r\in\mathbb{Z}$*, and* $dV_n\subset\widehat{\mathbb{L}}(V_{<n})$*. If* $g\colon L'\xrightarrow{\simeq} L''$ *is a quasi-isomorphism then,* $g_*\colon \mathrm{Der}^f_*\xrightarrow{\simeq}\mathrm{Der}^{gf}_*$ *is also a quasi-isomorphism.*

Proof. First, let θ be a cycle in Der^{gf}_*. We construct by induction on n an f-derivation $\psi\colon \widehat{\mathbb{L}}(V_{\leq n})\to L'$ and a gf-derivation $\varphi\colon \widehat{\mathbb{L}}(V_{\leq n})\to L''$ such that $D\psi=0$ and $g_*(\psi)-\theta=D\varphi$. Suppose this is done for some integer n and let $\{v_i\}$ be a basis for V_{n+1}. Then ψdv_i is a cycle and

$$g\psi dv_i = d(\varphi dv_i+(-1)^{|\theta|}\theta v_i).$$

Therefore, there exist elements $a_i\in L'$ and $b_i\in L''$ with

$$\psi dv_i = da_i \qquad\text{and}\qquad ga_i=\varphi dv_i+(-1)^{|\theta|}\theta v_i+db_i.$$

We then define

$$\psi(v_i)=(-1)^{|\theta|}a_i \qquad\text{and}\qquad \varphi(v_i)=(-1)^{|\theta|}b_i.$$

Now, let θ be a cycle in Der^f_* such that $g\circ\theta=D\psi$. We construct by induction on the degree an f-derivation φ and a gf-derivation ρ such that $\theta=D\varphi$ and $g\varphi-\psi=D\rho$. Suppose this is done on $V_{\leq n}$ and let again $\{v_i\}$ be a basis of V_{n+1}. Then $\omega_i=\theta(v_i)+(-1)^{|\varphi|}\varphi dv_i$ is a cycle and, by the induction hypothesis,

$$g\omega_i = d\left(\psi v_i+(-1)^{|\varphi|}\rho dv_i\right).$$

Thus there exist elements $\beta_i\in L'$ and $\gamma_i\in L''$ such that $\omega_i=d\beta_i$ and

$$g\beta_i-(\psi v_i+(-1)^{|\varphi|}\rho dv_i)=d\gamma_i.$$

We define

$$\varphi(v_i)=\beta_i \qquad\text{and}\qquad \rho(v_i)=\gamma_i.$$

This completes the induction process. □

Lemma 12.37. *Let* $L=(\widehat{\mathbb{L}}(V),d)$ *be a free cdgl with* $V=V_{\geq r}$ *for some* $r\in\mathbb{Z}$*, and* $d(V_n)\subset\widehat{\mathbb{L}}(V_{<n})$*. If* $f,g\colon L\to L'$ *are homotopic, then* $H_*(\mathrm{Der}^f_*)\cong H_*(\mathrm{Der}^g_*)$*.*

Proof. Let

$$\Psi\colon L\longrightarrow L'^I$$

be a homotopy between f and g (see Definition 8.18). That is, $f=\varepsilon_0\circ h$ and $g=\varepsilon_1\circ h$. Since ε_0 and ε_1 are quasi-isomorphisms, by Lemma 12.36 we have the following quasi-isomorphisms

$$\mathrm{Der}^{\varepsilon_1\Psi}_*\xleftarrow{\simeq}\mathrm{Der}^{\Psi}_*\xrightarrow{\simeq}\mathrm{Der}^{\varepsilon_0\Psi}_*,$$

which imply the assertion. □

Lemma 12.38. *Let*

$$L \xrightarrow{i} L \,\widehat{\amalg}\, \widehat{\mathbb{L}}(R \oplus dR) \xrightarrow{f} L'$$

be a sequence of cdgl morphism where i is the canonical injection. Then, precomposition with i induces a quasi-isomorphism

$$i^*\colon \operatorname{Der}_*^f \xrightarrow{\simeq} \operatorname{Der}_*^{fi}.$$

Proof. Suppose that θ is a cycle in $\operatorname{Der}_*^{fi}$. By setting

$$\theta'(R) = \theta'(dR) = 0 \quad \text{and} \quad \theta'(x) = \theta(x) \quad \text{for} \quad x \in L,$$

we define a cycle in Der_*^f satisfying $i^*(\theta') = \theta$. Therefore $H(i^*)$ is surjective.

Suppose now that θ is a cycle in Der_*^f and that $i^*(\theta) = D\psi$. We define then the f-derivation φ by

$$\varphi(x) = \psi(x) \quad \text{for} \quad x \in L, \quad \varphi(r) = 0 \quad \text{and} \quad \varphi(dr) = (-1)^{|\theta|}\theta(r).$$

Then, it is clear that $\theta = D\varphi$. □

Now, let $f\colon L \to L'$ be a morphism of cdgl's and consider in $\mathscr{C}(L)$ the word length filtration:

$$\mathscr{C}(L) = \bigoplus_{q \geq 0} \mathscr{C}_q(L), \quad \text{with} \quad \mathscr{C}_q(L) = \wedge^q(sL).$$

As in Proposition 11.5, we associate to f a Maurer–Cartan element

$$\omega \in \operatorname{Hom}(\mathscr{C}_{\geq 1}(L), L')$$

defined by

$$\omega(sx) = -f(x), \quad \text{and} \quad \omega(\mathscr{C}_{\geq 2}(L)) = 0.$$

Then, we define a linear map of degree +1,

$$\Phi\colon \operatorname{Der}^f \longrightarrow \operatorname{Hom}(\mathscr{C}_{\geq 1}(L), L'),$$

$$\Phi(g)(sx) = -(-1)^{|g|} g(x), \qquad \Phi(g)(\mathscr{C}_{\geq 2}(L)) = 0.$$

Lemma 12.39. *With the above definitions and notation,*

$$\Phi\colon (\operatorname{Der}^f, D) \longrightarrow (\operatorname{Hom}(\mathscr{C}_{\geq 1}(L), L'), D_\omega)$$

is a morphism of complexes. Moreover, if $L = (\widehat{\mathbb{L}}(V), d)$ is a free cdgl, then Φ is a quasi-isomorphism.

Proof. For the first assertion we have only to verify that $D_\omega\Phi = -\Phi d$. On sL, this is immediate:

$$\begin{aligned}\big(D_\omega\Phi(g)\big)(sx) &= d\big(\Phi(g)(sx)\big) - (-1)^{|\Phi(g)|}\Phi(g)(dsx)\\ &= -(-1)^{|g|}dgx + gdx = -\big(\Phi(dg)\big)(sx).\end{aligned}$$

On elements in $\mathscr{C}_{\geq 3}(L)$, this is also clear, since both sides of the equation are 0. Now, taking into account that g is an f-derivation,

$$\begin{aligned}&\big(D_\omega\Phi(g)\big)(sx\wedge sy)\\ &\quad= d\big(\Phi(g)(sx\wedge sy)\big) - (-1)^{|\Phi(g)|}\Phi(g)\big(d(sx\wedge sy)\big) + [\omega,\Phi(g)](sx\wedge sy)\\ &\quad= -(-1)^{|\Phi(g)|}\Phi(g)\big((-1)^{|sx|}s[x,y]\big) + [\omega,\Phi(g)](sx\wedge sy)\\ &\quad= (-1)^{|x|}g[x,y] + (-1)^{|\Phi(g)||sx|}[\omega(sx),\Phi(g)(sy)]\\ &\qquad - (-1)^{|\Phi(g)|+|sx|}[\Phi(g)(sx),\omega(sy)] = 0.\end{aligned}$$

Now assume that $L = (\widehat{\mathbb{L}}(V), d)$ is a free cdgl and prove that Φ is a quasi-isomorphism. Write

$$L' = \varprojlim_n L'/M^n,$$

and introduce decreasing filtrations

$$\{F^n\}_{n\geq 1} \quad\text{and}\quad \{G^n\}_{n\geq 1}$$

on Der^f and on $\mathrm{Hom}(\mathscr{C}_{\geq 1}(L), L')$ by

$$F^n = \{g\in \mathrm{Der}^f \mid g(L)\subset M^n\},$$

$$G^n = \{h\in \mathrm{Hom}(\mathscr{C}_{\geq 1}(L), L') \mid h(\mathscr{C}(L))\subset M^n\}.$$

Clearly, the morphism Φ preserves the filtrations. On the other hand,

$$\mathrm{Der}^f = \varprojlim_n \mathrm{Der}^f/F^n \quad\text{and}\quad \mathrm{Hom}(\mathscr{C}_{\geq 1}(L), L') = \varprojlim_n \mathrm{Hom}(\mathscr{C}_{\geq 1}(L), L')/G^n.$$

We will prove that the quotient maps $\overline{\Phi}_n\colon F^n/F^{n+1}\to G^n/G^{n+1}$ are quasi-isomorphisms. By Lemma 3.14, this will finish the proof.

We are therefore reduced to the case where L' is abelian, and we have a short exact sequence of complexes

$$0 \longrightarrow \mathrm{Der}^f \longrightarrow \mathrm{Hom}(\mathscr{C}_{\geq 1}(L), L') \longrightarrow E \longrightarrow 0,$$

where E is the subset of $\mathrm{Hom}(\mathscr{C}_{\geq 1}(L), L')$ consisting of the maps h satisfying $h(sx) = 0$. The space Der^f is identified here with the space of maps $sV\to L'$ that are 0 on $s\mathbb{L}^{\geq 2}(V)$, and E can be identified with the space of maps $\mathscr{C}_{\geq 2}(L)\oplus$

$s\mathbb{L}^{\geq 2}(V) \to L'$. Since L' is abelian, the perturbed differential D_ω is equal to the original differential. Therefore, if $h \in E$,

$$D_\omega h = \overline{D}h + D'$$

with

$$\overline{D}h = d \circ h - (-1)^{|h|} h \circ d$$

and

$$D'(h) \in \mathrm{Der}^f, \quad D'(h)(x) = -(-1)^{|h|} hd(sx).$$

Recall from Lemma 2.2(ii) that $\mathscr{C}_{\geq 2}(\mathbb{L}(V)) \oplus s\mathbb{L}^{\geq 2}(V)$ is a contractible complex. It follows that $H(E, \overline{D}) = 0$ and so $\overline{\Phi}_n$ is a quasi-isomorphism for any n. $\square$

Proof of Theorem 12.35. By Proposition 12.25, there is a homotopy equivalence of simplicial sets,

$$\mathrm{Map}^*(X, \langle L \rangle) \simeq \langle \wedge^+ V \widehat{\otimes} L \rangle,$$

where $(\wedge V, d)$ denotes the Sullivan minimal model of X. Since this simplicial set is nilpotent and of finite type, we may apply Theorem 10.2 to conclude that its Neisendorfer model $\mathscr{L}(\wedge V, d)^\#$ is quasi-isomorphic to $\mathfrak{L}^a_X$. Hence, see Proposition 2.3, the following cdgc's are all quasi-isomorphic,

$$(\wedge V, d)^\# \simeq \mathscr{C}\mathscr{L}(\wedge V, d)^\# \simeq \mathscr{C}(\mathfrak{L}^a_X).$$

In particular, there is a homotopy equivalence,

$$\langle \wedge^+ V \widehat{\otimes} L \rangle \cong \langle \mathrm{Hom}\big((\wedge^+ V)^\#, L\big) \rangle \simeq \langle \mathrm{Hom}(\mathscr{C}_{\geq 1}(\mathfrak{L}^a_X), L) \rangle.$$

Then, denoting by ω the MC element corresponding to ℓ, we have a homotopy equivalence

$$\mathrm{Map}^*_f(X, \langle L \rangle) \simeq \langle \mathrm{Hom}(\mathscr{C}_{\geq 1}(\mathfrak{L}^a_X), L) \rangle_\omega$$

and now the result follows from Proposition 12.39 and Theorem 7.18. $\square$

12.5 Homotopy invariants of the realization functor

Up to this point we have given examples of modeling various geometrical constructions. Here, we reverse direction and characterize some homotopy invariants in the realization $\langle L \rangle$ of a given cdgl. Our aim is not to cover an exhaustive list. Rather, we just describe the most classical invariants, highlighting the methods used to help the reader developing the right intuition to go further in this approach when needed.

12.5.1 Action of $\pi_1\langle L\rangle$ on $\pi_*\langle L\rangle$

Given a connected cdgl L, we describe the action of $\pi_1\langle L\rangle$ on $\pi_*\langle L\rangle$.

Definition 12.40. Let L be a complete graded Lie algebra. Then, the sub-Lie algebra L_0 with the group structure given by the BCH product acts on L_n by

$$x \bullet v = e^{\mathrm{ad}_x}(v)\,,$$

where $x \in L_0$ and $v \in L_n$.

This is a group action since, by Corollary 4.12,

$$(x * y) \bullet v = e^{\mathrm{ad}_{x*y}}(v) = e^{\mathrm{ad}_x} e^{\mathrm{ad}_y}(v) = x \bullet (y \bullet v).$$

Note that

$$x \bullet v = v + \frac{e^{\mathrm{ad}_x} - 1}{\mathrm{ad}_x}[x, v].$$

Therefore, as the operator $\frac{e^{\mathrm{ad}_x}-1}{\mathrm{ad}_x}$ is invertible, $x \bullet v = v$ if and only if $[x, v] = 0$.

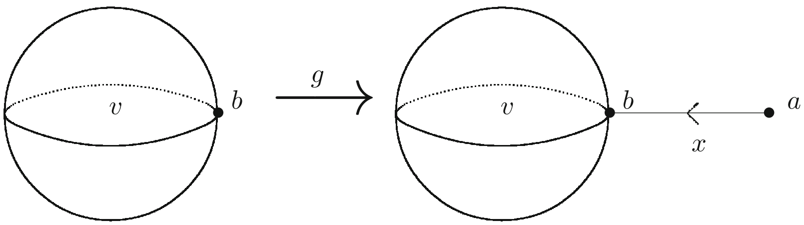
Observe also that whenever L is a cdgl, $\bullet$ also induces an action, which is denoted in the same way,

$$\bullet\colon H_0(L) \times H_n(L) \longrightarrow H_n(L), \qquad n \geq 1.$$

On the topological side, consider the wedge $S^n \vee [a, b]$ of the n-dimensional sphere with an interval. Choose b as the base point and let

$$f\colon S^n \vee [a, b] \to S^n$$

be obtained by contracting $[a, b]$ to b. There is clearly a continuous map $g\colon S^n \to S^n \vee [a, b]$ satisfying $g(b) = a$ and such that $f \circ g$ is homotopic to the identity.



Now, a homotopy class α of a given curve $c\colon [a, b] \to X$ in a connected topological space X induces a natural isomorphism $\pi_n\big(X, c(b)\big) \xrightarrow{\cong} \pi_n\big(X, c(a)\big)$ which assigns, to each $\beta \in \pi_n\big(X, c(b)\big)$ represented by $h\colon S^n \to X$, the class $\alpha \bullet \beta \in \pi_n\big(X, c(a)\big)$ represented by $(h \vee c) \circ g$.

In particular, if c is a loop having $x_0 \in X$ as endpoints, this construction gives the classical action $\bullet$ of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$.

Theorem 12.41. *Let L be a connected cdgl and let $n \geq 1$. Then, the following diagram commutes:*

$$\begin{array}{ccc} \pi_1\langle L\rangle \times \pi_n\langle L\rangle & \xrightarrow{\ \bullet\ } & \pi_n\langle L\rangle \\ {\scriptstyle \rho_1\times\rho_n}\downarrow\cong & & \cong\downarrow{\scriptstyle \rho_n} \\ H_0(L)\times H_{n-1}(L) & \xrightarrow{\ \bullet\ } & H_{n-1}(L) \end{array}$$

where ρ_1 and ρ_n are the isomorphisms of Theorem 7.18*.*

Proof. Recall that S^n can be thought of as the simplicial set with only two non-degenerate simplices v and b in dimension 0 and n, respectively. Therefore, by Proposition 7.8,

$$\mathfrak{L}_{S^n} = (\widehat{\mathbb{L}}(v,b), d),$$

where b is a Maurer–Cartan element, $|v| = n-1$ and $dv = -[b,v]$. On the other hand, since $\mathfrak{L}$ preserves colimits (see Proposition 7.5),

$$\mathfrak{L}_{S^n\vee[a,b]} = (\widehat{\mathbb{L}}(a,b,x,v), d),$$

where $(\widehat{\mathbb{L}}(a,b,x), d)$ is an LS interval. Moreover, as f is a retraction of the inclusion $S^n \hookrightarrow S^n \vee [a,b]$,

$$\mathfrak{L}_f\colon \mathfrak{L}_{S^n\vee[a,b]} \longrightarrow \mathfrak{L}_{S^n}$$

is necessarily of the form

$$\mathfrak{L}_f(v) = v, \quad \mathfrak{L}_f(b) = b, \quad \mathfrak{L}_f(a) = b \quad \text{and} \quad \mathfrak{L}_f(x) = 0.$$

On the other hand observe that, for the perturbed models,

$$H_q(\mathfrak{L}_{S^n}, d_b) = \begin{cases} 0, & \text{if } q \neq n-1, \\ \mathbb{Q}\cdot[v], & \text{if } q = n-1, \end{cases}$$

while, by Proposition 4.24,

$$H_q(\mathfrak{L}_{S^n\vee[a,b]}, d_a) = \begin{cases} 0, & \text{if } q \neq n-1, \\ \mathbb{Q}\cdot[e^{\mathrm{ad}_x}(v)], & \text{if } q = n-1. \end{cases}$$

Hence, the morphism

$$\mathfrak{L}_g\colon (\mathfrak{L}_{S^n}, d_b) \longrightarrow (\mathfrak{L}_{S^n\vee[a,b]}, d_a)$$

is given by

$$\mathfrak{L}_g(b) = a \quad \text{and} \quad \mathfrak{L}_g(v) = \lambda e^{\mathrm{ad}_x}(v) + d\Phi, \quad \text{with} \quad \lambda \in \mathbb{Q}.$$

However, as $\mathfrak{L}$ preserves homotopy (see (ii) of Corollary 8.2), we have $\mathfrak{L}_f \circ \mathfrak{L}_g \sim \mathrm{id}$ and therefore, it induces the identity in homology. In particular,

$$H(\mathfrak{L}_f) \circ H(\mathfrak{L}_g)[v] = \lambda[v]$$

and thus $\lambda = 1$. Hence, up to homotopy, we may write,

$$\mathfrak{L}_g \colon \mathfrak{L}_{S^n} \longrightarrow \mathfrak{L}_{S^n \vee [a,b]}, \qquad \mathfrak{L}_g(b) = a, \qquad \mathfrak{L}_g(v) = e^{\mathrm{ad}_x}(v).$$

This guarantees the commutativity of the square,

$$\begin{array}{ccc}
[\mathfrak{L}_{S^n \vee [a,b]}, L] & \xrightarrow{\mathfrak{L}_{g*}} & [\mathfrak{L}_{S^n}, L] \\
\cong \big\downarrow & & \big\downarrow \cong \\
H_0(L) \times H_{n-1}(L) & \xrightarrow{\bullet} & H_{n-1}(L)
\end{array}$$

where the vertical bijections send the homotopy classes of the maps $\alpha \colon \mathfrak{L}_{S^n \vee [a,b]} \to L$ and $\beta \colon \mathfrak{L}_n \to L$ to $\big(\alpha(x), \alpha(v)\big)$ and $\beta(v)$ respectively.

Insert this square as the bottom one in the following diagram, where the middle square is commutative by Corollary 8.2(iv), and the upper one trivially commutes:

$$\begin{array}{ccc}
\pi_1\langle L\rangle \times \pi_n\langle L\rangle & \xrightarrow{\bullet} & \pi_n\langle L\rangle \\
= \big\downarrow & & \big\downarrow = \\
[S^n \vee [a,b], \langle L\rangle] & \xrightarrow{g_*} & [S^n, \langle L\rangle] \\
\cong \big\downarrow & & \big\downarrow \cong \\
[\mathfrak{L}_{S^n \vee [a,b]}, L] & \xrightarrow{\mathfrak{L}_{g*}} & [\mathfrak{L}_{S^n}, L] \\
\cong \big\downarrow & & \big\downarrow \cong \\
H_0(L) \times H_{n-1}(L), & \xrightarrow{\bullet} & H_{n-1}(L).
\end{array}$$

To finish, simply check that the composition of the left (respectively right) vertical arrows is the morphism $\rho_1 \times \rho_n$ (respectively ρ_n). □

12.5.2 The rational homotopy Lie algebra of $\langle L\rangle$

In this section we show that the natural Lie algebra structure on the homology $H(L)$ of a connected cdgl coincides with the Lie algebra on $\pi_{\geq 1}\Omega\langle L\rangle$.

Recall that given a connected space X with base point a, the Whitehead product is a natural bilinear transformation

$$\pi_n(X) \times \pi_m(X) \longrightarrow \pi_{n+m-1}(X), \quad \text{for} \quad n, m \geq 2.$$

This gives a structure of graded Lie algebra on $\pi_{\geq 1}(\Omega X)\otimes\mathbb{Q}$. In [115] it is proved that, when X is simply connected, there is an isomorphism of Lie algebras,

$$\pi_*(\Omega X)\otimes\mathbb{Q}\cong H\big(\lambda(X)\big),$$

where $\lambda(X)$ denotes, as usual, the Quillen functor (see Sect. 1.2.2). We now prove:

Theorem 12.42. *Let L be a connected cdgl. Then, there is an isomorphism of graded Lie algebras*

$$H_{\geq 1}(L)\cong \pi_{\geq 1}\Omega\langle L\rangle.$$

Proof. Equivalently, we will show that, for any $n, m \geq 2$, there is a commutative diagram

$$\begin{array}{ccc}
\pi_n\langle L\rangle\times\pi_m\langle L\rangle & \xrightarrow{[\,,\,]} & \pi_{n+m-1}\langle L\rangle\\
\cong\downarrow & & \downarrow\cong\\
H_{n-1}(L)\times H_{m-1}(L) & \xrightarrow{[\,,\,]} & H_{n+m-2}(L)
\end{array}$$

where the upper bracket denotes the Whitehead product.

Let $\alpha\in\pi_n\langle L\rangle$ and $\beta\in\pi_m\langle L\rangle$ be represented by maps $f\colon S^n\to\langle L\rangle$ and $g\colon S^m\to\langle L\rangle$, which define $f\vee g\colon S^n\vee S^m\to\langle L\rangle$. The universal Whitehead bracket,

$$h=[\mathrm{id}_{S^n},\mathrm{id}_{S^m}]\colon S^{n+m-1}\longrightarrow S^n\vee S^m,$$

defines, by naturality, the Whitehead bracket $[\alpha,\beta]$ as the class of the composition

$$(f\vee g)\circ h\colon S^{n+m-1}\longrightarrow\langle L\rangle.$$

By adjunction, $f\vee g$ and h correspond to maps

$$(\mathbb{L}(a,b),0)\longrightarrow L \qquad\text{and}\qquad \varphi\colon(\mathbb{L}(u),0)\longrightarrow(\mathbb{L}(a,b),0),$$

where $|a|=n-1$, $|b|=m-1$ and $|u|=n+m-2$.

Since S^n and S^m are simply connected, we deduce from [115, Theorem I] that $\varphi(u)=[a,b]$. We then have the following commutative diagram where all vertical maps are bijections:

$$\begin{array}{ccc}
\pi_n\langle L\rangle\times\pi_m\langle L\rangle & \xrightarrow{[\,,\,]} & \pi_{n+m-1}\langle L\rangle\\
\cong\uparrow & & \uparrow\cong\\
[S^n\vee S^m,\langle L\rangle] & \xrightarrow{h_*} & [S^{n+m-1},\langle L\rangle]\\
\cong\uparrow & & \uparrow\cong\\
[\mathbb{L}(a,b),L] & \xrightarrow{\varphi_*} & [\mathbb{L}(u),L]\\
\cong\downarrow & & \downarrow\cong\\
H_{n-1}(L)\times H_{m-1}(L) & \xrightarrow{[\,,\,]} & H_{n+m-2}(L)
\end{array}$$

□

12.5.3 Postnikov decomposition of $\langle L\rangle$

Another direct application of this homotopy theoretical setting is the description of the Postnikov decomposition of the realization $\langle L\rangle$ of a given connected cdgl. For such a cdgl and for any $n \geq 0$, consider the quotient cdgl

$$L_{[n]} = L/(L_{>n} \oplus Z_n),$$

where $Z_n \subset L_n$ denotes the subspace of cycles. Note that $L_{[0]} = 0$ and for each $n \geq 1$ there is a commutative triangle of surjections,

$$\begin{array}{ccc} & & L[n] \\ & \nearrow & \\ L & & \big\downarrow \\ & \searrow & \\ & & L[n-1]. \end{array}$$

Proposition 12.43. *The induced sequence*

$$\begin{array}{ccccccccccc} & & & & \langle L\rangle & & & & & & \\ & & & \swarrow & \downarrow & & \searrow & & & & \\ \dots & \longrightarrow & \langle L_{[n]}\rangle & \longrightarrow & \langle L_{[n-1]}\rangle & \longrightarrow & \dots & \longrightarrow & L_{[1]} & \longrightarrow & * \end{array}$$

is the Postnikov decomposition of $\langle L\rangle$

Proof. As $\langle\,\cdot\,\rangle$ preserves fibrations all the maps in this diagram are Kan fibrations. Moreover, in view of the short exact sequence,

$$0 \longrightarrow L_{>n} \oplus Z_n \longrightarrow L \longrightarrow L_{[n]} \longrightarrow 0, \tag{12.14}$$

we deduce that

$$H_p(L_{[n]} = \begin{cases} H_p(L), & \text{if } p \leq n-1, \\ 0, & \text{if } p > n-1. \end{cases}$$

That is,

$$\pi_q\langle L_{[n]}\rangle = \begin{cases} \pi_q\langle L\rangle, & \text{if } q \leq n, \\ 0, & \text{if } q > n. \end{cases}$$

Finally, since $\langle\,\cdot\,\rangle$ preserves projective limits,

$$\langle L\rangle = \varprojlim_n \langle L_{[n]}\rangle.$$

These properties characterize the above sequence as the Postnikov decomposition of $\langle L\rangle$. □

In particular, see for instance Corollary 8.4, as the realization of (12.14),

$$\langle L_{>n} \oplus Z_n \rangle \to \langle L \rangle \to \langle L_{[n]} \rangle,$$

is a fibration sequence, we deduce:

Corollary 12.44. *For any $n \geq 1$, $\langle L_{>n} \oplus Z_n \rangle$ is the nth connected cover of $\langle L \rangle$.* □

Bibliographical notes

The first algebraic model of the rational homotopy type of mapping spaces was constructed by Haefliger in [71]. Later, Bousfield, Peterson and Smith gave in [14] a more natural presentation in the spirit of the Lannes functor, see also the work of Brown and Szczarba in [17]. Rational models of mapping spaces involving Lie algebras have been described later by Scheerer and Tanré [121, Proposition 6.5], by Buijs, Félix and Murillo [31, 20, 21], and by Lupton-Smith [95].

It is convenient to remark that Theorem 12.17 was proved by Berglund for an arbitrary connected cdgl in [8, Theorem 6.6]. However, to keep the text self-contained we have considered a slightly restricted version with a shorter proof. Hence, the first equivalence of Theorem 12.18, and all its consequences, hold for any connected simplicial set Y not necessarily of finite type. Whenever Y is nilpotent, this result was proved in [8, Theorem 1.4]. A presentation of the same statement in a language that does not use MC elements can be found in [20, Theorem 10].

Now suppose that Y is a nilpotent space with finite Betti numbers, X is a finite connected CW-complex, and $f\colon X \to Y$ is a continuous map. We denote by $(\wedge V, d)$ a minimal Sullivan model of Y, by (A, d) a cdga quasi-isomorphic to the Sullivan minimal model of X, and by $g\colon (\wedge V, d) \to (A, d)$ a Sullivan representative for f. The map g makes (A, d) a $(\wedge V, d)$-module. The Harrison(–André–Quillen) complex of $(\wedge V, d)$ with coefficients in A is defined as

$$C^n_{\mathrm{Harr}}(\wedge V, A) \subset \mathrm{Hom}\big((\wedge V)^{\otimes n}, A\big),$$

where $h \in C^*_{\mathrm{Harr}}(\wedge V, A)$ if h vanishes on all shuffle products. The differential is the usual Hochschild differential. Then, see [1, 10, 92], there is an isomorphism of complexes

$$C^*_{\mathrm{Harr}}(\wedge V, A) \cong (\mathscr{L}(\wedge V)^{\#} \widehat{\otimes} A, d_\omega),$$

where ω is an MC element that corresponds to f. Since, by Theorem 10.1,

$$\mathscr{E}(\wedge V, d)^{\#} \simeq \mathfrak{L}_Y/(a),$$

we deduce that there are isomorphisms

$$H^*_{\mathrm{Harr}}(\wedge V, A) \cong H_{*-1}(\mathfrak{L}_X/a)\widehat{\otimes} A) \cong \pi_* \mathrm{Map}(X, Y_{\mathbb{Q}}).$$

Notation Index

General notation

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Categories

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