Operads and monoidal categories

1. Operads' favorite vehicles
2. What is algebraic topology (again)?
3. Categories of quadratic data
4. Topological operads from graphs
5. Towards GT...

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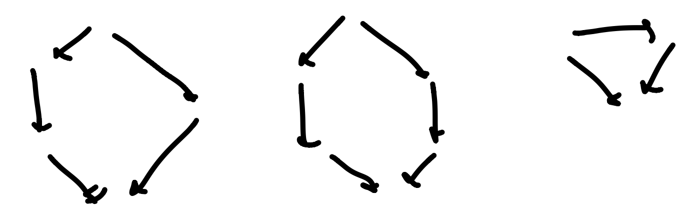
$\boxed{1}$ **Def**: $(\mathcal{C}, \otimes, I)$ braided mon. cat

- $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
- nat. isos. $\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$

  $\lambda_A : I \otimes A \to A \quad \rho_A : A \otimes I \to A$

  $\beta_{A,B} : A \otimes B \to B \otimes A$

verifying



symmetric if $\beta_{A,B} \circ \beta_{B,A} = id \quad \forall A, B \in Ob(\mathcal{C})$

THM [Maclane]: All diagrams commute.

Ex: $(\mathrm{Top}_*, \times)$ $(\mathrm{dgVect}, \otimes)$ $(\mathrm{Lie\text{-}alg}, \oplus)$

symmetry

$x \otimes y \mapsto (-1)^{|x||y|} y \otimes x$

unit $= 0$

direct sum

$\mathfrak{g} \oplus \mathfrak{h}$

$[x+y, x'+y'] := [x,x'] + [y,y']$

Prop: We can speak of operads! $\{P(n)\}_{n \in \mathbb{N}}$

Def: A functor is sym. monoidal if it preserves units and commutes with the $\alpha, \beta, \lambda, \rho$'s.

$$\boxed{\begin{array}{ll} \text{THM: } * \text{ Any cov. sym. mon. functor} & \text{operads} \to \text{operads} \\ & \text{Coop.} \to \text{coop.} \\ * \quad \text{contr.} & \text{op} \to \text{coop.} \\ & \text{coop} \to \text{op.} \end{array}}$$

Merkulov.

4.2.3. Exercise. Let $\mathcal{O}$ be an operad in a symmetric monoidal category C and $F : \mathsf{C} \to \mathsf{D}$ a symmetric monoidal functor to some other category. Show that the data

(i) the S-module structure, $F\mathcal{O} : \mathsf{S} \to \mathsf{D}$, given by the composition $\mathsf{S} \xrightarrow{\mathcal{O}} \mathsf{C} \xrightarrow{F} \mathsf{D}$, and
(ii) the operadic "insertions",

$$\bar{\circ}_i^{I,J} : F\mathcal{O}(I) \otimes_{\mathsf{C}} F\mathcal{O}(J) \longrightarrow F\mathcal{O}(I \setminus \{i\} \sqcup J),$$

given by the compositions

$$F\mathcal{O}(I) \otimes_{\mathsf{C}} F\mathcal{O}(J) \xrightarrow{\phi_{\mathcal{O}(I),\mathcal{O}(J)}} F(\mathcal{O}(I) \otimes_{\mathsf{C}} \mathcal{O}(J)) \xrightarrow{F(\circ_i^{I,J})} F(\mathcal{O}(I \setminus \{i\} \sqcup J)) = F\mathcal{O}(I \setminus \{i\} \sqcup J),$$

give us an operad $F\mathcal{O}$ in the symmetric monoidal category D. This fact is of an extreme importance in applications — starting with a "geometric" operad in the category, say, of topological spaces, and applying the chain or homology functor one arrives to an operad in the category of vector spaces. This particular property of operads is another manifestation of the *amazing unity of mathematics.*

$\boxed{2}$ Q: What is alg. top.?

$$\begin{tikzcd} & X \arrow[dl] \arrow[d] \arrow[dr] & \\ \pi_\bullet(X) & H_\bullet(X) & H^\bullet(X) \end{tikzcd}$$

lax (not symmetric but still preserves operads)

$\pi_\bullet(X)$	$H_\bullet(X)$	$H^\bullet(X)$
Lie-alg (Whitehead)	Com-coalg (Δ)	Com-alg ($\cup$)

R: The study of monoidal functors!

$\Longrightarrow$ they transport top. operads ...

$\underline{Q}$: How do we go from Lie-alg to Com-alg?

Koszul duality

$$\begin{tikzcd}
\mathrm{Lie} \arrow[r, "i"] \arrow[d] \arrow[rr, bend left, "!"] & \mathrm{Com}^{\vee} \arrow[r, "*"] \arrow[d] & \mathrm{Com} \arrow[d] \\
\text{Lie-alg} & \text{Com-coalg.} & \text{Com-alg.}
\end{tikzcd}$$

$\underline{Q}$: What about $\underline{\text{operads}}$ in the cat. of algebras?

$$\begin{tikzcd}
(QD^-, \otimes) \arrow[r, "i"] \arrow[d, "L"] \arrow[rr, bend left, "!"] & (QD^+, \underline{\otimes}) \arrow[r, "*"] \arrow[d, "SC"] & (QD^+, \vee) \arrow[d] \\
(\text{Lie-alg}, \otimes) & (\text{Com-coalg}, \otimes) & (\text{Com-alg}, \otimes)
\end{tikzcd}$$

Upshot: top. operad $\rightsquigarrow$ quadratic data $\rightsquigarrow$ recover π_*, H_*, H^*

(gentle top. space, work over $\mathbb{Q}$, finitude hypothesis, formality)

$\boxed{3}$ Let $V \in$ gVect.

$$V^{\otimes 2} \cong V^{\odot 2} \oplus V^{\wedge 2}$$

$\langle x \otimes y + y \otimes x \rangle \qquad \langle x \otimes y - y \otimes x \rangle$

$$\mathbb{k}[S_2] \cong \underset{\text{tr.}}{\mathbb{k}} \oplus \underset{\text{sgn.}}{\mathbb{k}}$$

<u>Def</u>: Categories of quadratic data

objects	QD	QD^+	QD^-
(V,R)	$R \subset V^{\otimes 2}$	$R \subset V^{\odot 2}$	$R \subset V^{\wedge 2}$
		sym.	skew sym.

<u>Morph</u>: $f : (V,R) \to (W,S)$ s.t. $f^{\otimes 2}(R) \subset S$

<u>Def</u>: Functors between them.

i) Realizations functors

$T(V)/(R) \qquad S(V)/(R) \qquad Lie(V)/(R) \qquad T^c(V,R) \qquad S^c(V,R)$

$$\begin{tikzcd} C \arrow[r] & T^c(V) \arrow[r] \arrow[l, phantom] & T^c(V)/(R) \end{tikzcd}$$

$$\begin{tikzcd} C \arrow[rr, bend left, "0"] \arrow[r] & T^c(V) \arrow[r] & T^c(V)/(R) \end{tikzcd}$$

$\exists !$ $\Downarrow$ // 0

$T^c(V,R)$

ii) Liftings

$\mathcal{U}$: Lie-alg $\longrightarrow$ Ass-alg $\quad \rightsquigarrow \quad \Lambda : QD^- \longrightarrow QD$, $(V,R) \longmapsto (V, \Lambda(R))$

i : Com-alg $\longrightarrow$ Ass-alg $\quad \rightsquigarrow \quad S : QD^+ \longrightarrow QD$, $(V,R) \longmapsto (V, \Sigma(R) \oplus V^{\otimes 2})$

iii) Duality functors

$\Lambda(x \otimes y) = \Lambda x \otimes \Lambda y \quad |s| = 1$

suspension

$i(V,R) := (sV, s^2R) \qquad *(V,R) := (V^*, R^\perp)$

Def: Monoidal structures $\quad [V,W]_\pm := \langle v \otimes w \pm w \otimes v \mid {}^{v \in V}_{w \in W} \rangle$

$(V,R)\ ,\ (W,S) \rightarrow (V \oplus W, ?)$

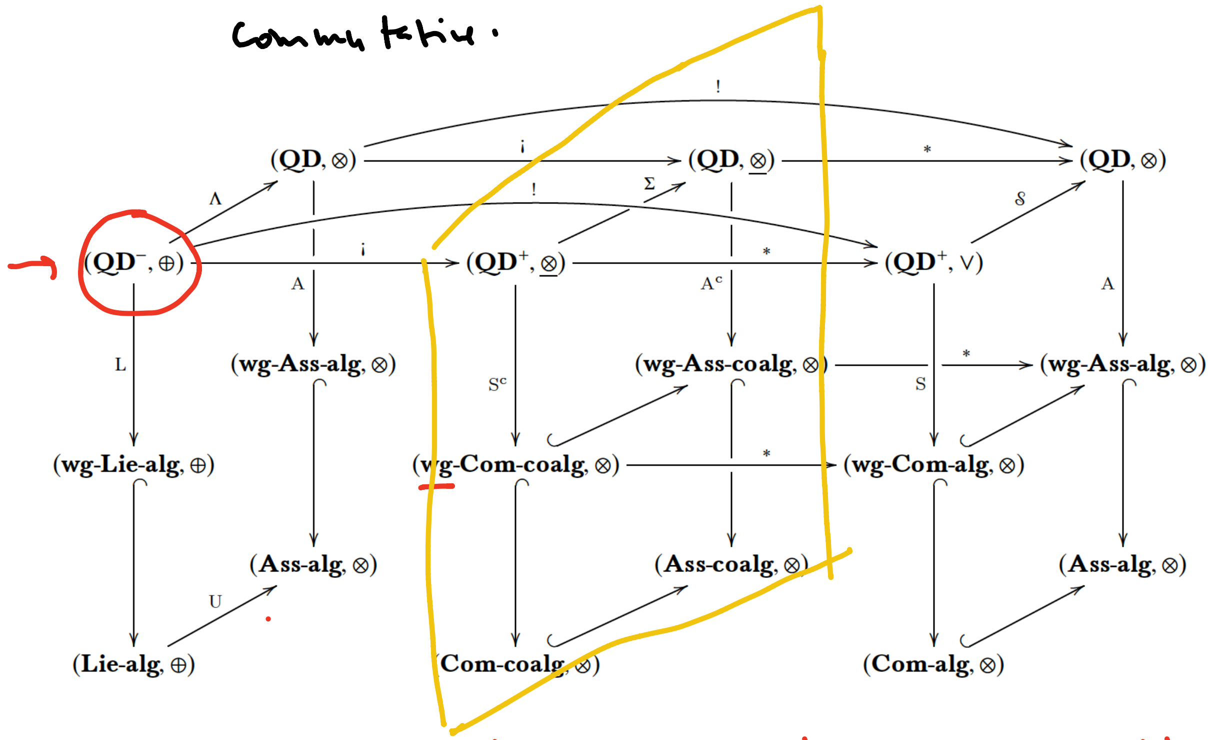
QD	QD+	QD-
$\otimes$ $R \oplus [V,W]_- \oplus S$	$\vee$ $R \oplus S$	$\oplus$ $\underbrace{R \oplus [V,W]_- \oplus S}$
$\underline{\otimes}$ $R \oplus [V,W]_+ \oplus S$	$\underline{\otimes}$ $R \oplus [V,W]_+ \oplus S$	

how to choose it s.t.

$L : (QD^-, \oplus) \longrightarrow (Lie, \oplus)$

be monoidal?

THM: The following diagram is made of sym. mon. categories, sym. mon. functors, and it is commutative.



$\Rightarrow$ 1 operad $\Rightarrow$ 11 (co)operads (!) $\quad A = \bigoplus_{n \geq 0} A^{(n)}$

$\boxed{4}$ We work over $\mathbb{Q}$. Let $\mathcal{O}$ be a operad in $(\mathrm{Top}_*, \times)$

CGH $\quad$ kelly

1) $H_\bullet(\mathcal{O})$ $\underbrace{\text{operad in } (\text{Com-coalg}, \otimes)}_{\text{cocom. Hopf operad}}$

2) $H^\bullet(\mathcal{O})$ $\underbrace{\text{cooperad in } (\text{Com-alg}, \otimes)}_{\text{com. Hopf operad}}$

3| What about $\pi_1(\Theta)$?

→ $*^n \circ_i *^m = *^{n+m-1}$

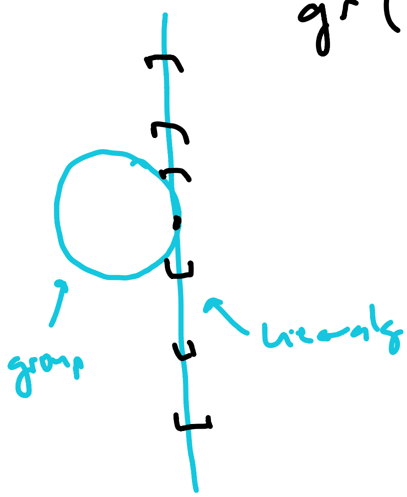
→ Magnus constr.

LCS

$T_1 G = G$

$T_{k+1} G = [T_k G, G]$

$$gr(G) := \bigoplus_{k \geq 1} T_k G \,/\, T_{k+1} G$$



Lie-alg over $\mathbb{Z}$!

$[\ ,\ ]$ induced by $xyx^{-1}y^{-1}$.

LEMMA: $(Top_*, \times) \xrightarrow{\pi_1} (Gr, \times) \xrightarrow{gr} (\text{Lie-alg}, \oplus)$

monoidal functors.

$\Longrightarrow$ $gr(\pi_1(\Theta))$ operad in Lie-alg $\mathbb{Z}$.

Q: Can we recover these 3 structures via QD?

restriction of

$\cup \ : \ \underline{H^1(\Theta(n))^{\otimes 2}} \subset H^1(\Theta(n))^{\otimes 2} \longrightarrow H^2(\Theta(n))$

cup product

$$\begin{tikzcd}
QD^- \arrow[r, "i"] & QD^+ \arrow[r, "*"] & QD^+ \\
(s^{-1}H_1(\theta), s^{-2}\,\mathrm{im}\,\Delta) \arrow[d, "L"] & (H_1(\theta), \mathrm{im}\,\Delta) \arrow[d, "S^c"] & (H^1(\theta), \ker \cup) \arrow[d, "S"] \\
\mathfrak{g}_\theta := L(\text{---}) \arrow[d, phantom, "\cong" rotate=90] & S^c(\text{---}) \arrow[d, phantom, "\cong" rotate=90] & S(\text{---}) \arrow[d, phantom, "\cong" rotate=90] \\
gr(\pi_1(\theta)) \otimes \mathbb{Q} & H_\bullet(\theta) & H^\bullet(\theta)
\end{tikzcd}$$

holonomy Lie-alg

THM: If, $\forall n$, $H^\bullet(\theta(n))$ admits f.g. homogeneous quadratic pres., with generators in $H^1(\theta(n))$

Then, $S^c(H_1(\theta), \mathrm{im}\,\Delta) \cong H_\bullet(\theta)$ and $H^\bullet(\theta) \cong S(-)$

THM [Sullivan]: Let X pointed, path. conn., 1-finite top. space. When X is $\mathbb{Q}$-formal,

$$\mathfrak{g}_X \cong gr(\pi_1(X)) \otimes \mathbb{Q}$$

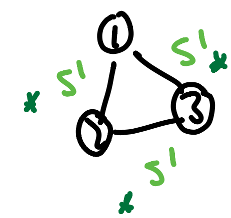
Operadic version?

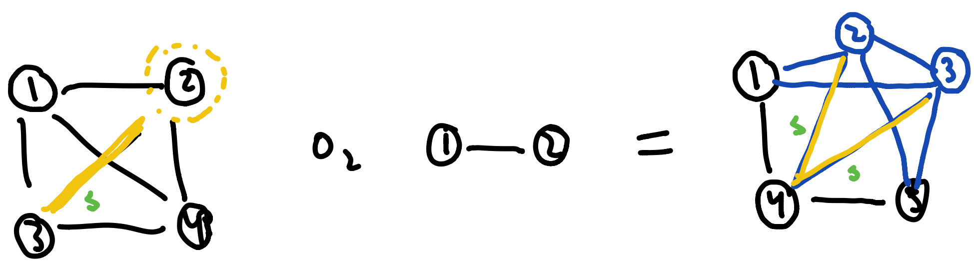
Ex 1 $\quad \mathcal{O} = \mathrm{Gra}_{S^1} \quad$ in $(\mathrm{Top}_*, \times)$

$n \leq 1 \qquad \mathrm{Gra}_{S^1}(n) = \{*\}$

$n \geq 2 \qquad \mathrm{Gra}_{S^1}(n) = (S^1)^{\binom{n}{2}}$

Γ_n = complete graph with n vertices





Rem: $S^1 \sim \mathbb{C} \setminus \{0\}$

Prop. $H_\bullet^{sing}(\mathrm{Gra}_{S^1}, \mathbb{Z}) \xrightarrow{\sim} C_\bullet^{sing}(\mathrm{Gra}_{S^1}, \mathbb{Z})$

proof: use the cross product [Dotsenko–Shadrin–Vallette] or Hodge formality [Cirici–Horel]

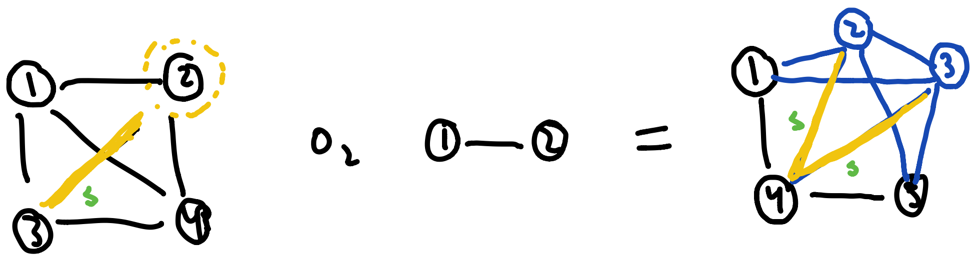
Def: Berger-Kontsevich-Willwacher quadratic data

$$\mathrm{BKW}(n) := \big(t^n_{ij} \,,\; t^n_{ij} \wedge t^n_{kl} \big) \in \mathrm{QD}^-$$

edges in Γ_n $\qquad$ all relations

Def: $\circ_k : BKW(n) \otimes BKW(m) \longrightarrow BKW(n+m-1)$

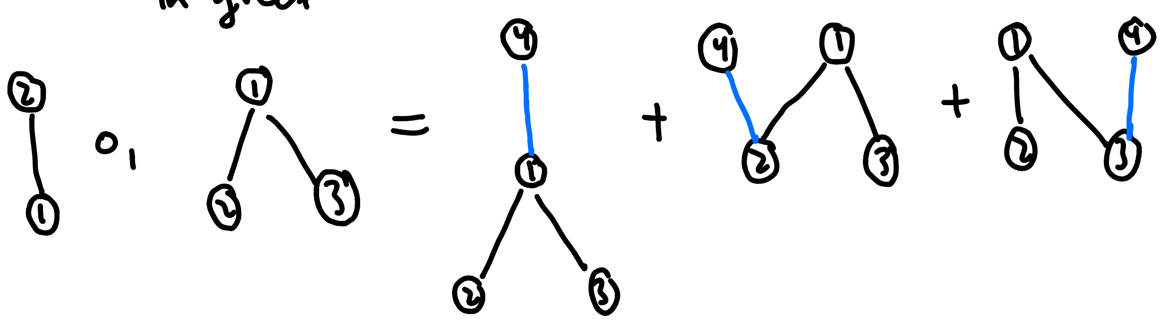
$$
\begin{aligned}
t^n_{ij} &\mapsto \begin{cases}
t^{n+m-1}_{i+m-1\, j+m-1} & \text{for } k < i, j\,, \\
t^{n+m-1}_{i\, j+m-1} + t^{n+m-1}_{i+1\, j+m-1} + \cdots + t^{n+m-1}_{i+m-2\, j+m-1} + t^{n+m-1}_{i+m-1\, j+m-1} & \text{for } k = i\,, \\
t^{n+m-1}_{i\, j+m-1} & \text{for } i < k < j\,, \\
t^{n+m-1}_{i\, j} + t^{n+m-1}_{i\, j+1} + \cdots + t^{n+m-1}_{i\, j+m-2} + t^{n+m-1}_{i\, j+m-1} & \text{for } k = j\,, \\
t^{n+m-1}_{ij} & \text{for } i, j < k\,,
\end{cases} \\
t^m_{ij} &\mapsto \quad t^{n+m-1}_{i+k-1\, j+k-1}\,.
\end{aligned}
$$



Prop. BKW is an operad in $(QD^-, \otimes)$

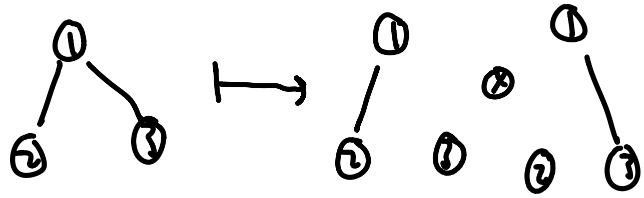
$\Rightarrow$ || (co)operads

Def [KW] Graph operad [polyvector fields] in gVect $\quad$ Gra(n) = graphs with n vertices



$\mathrm{Gra}(n)$ is a Com-coalg $\qquad \gamma \xrightarrow{\Delta} \sum \gamma' \otimes \gamma''$

" distributing the edges "



Gra is cocom. Hopf operad.

So, by the previous results,

$$\begin{tikzcd}
\mathrm{BKW} \arrow[r] \arrow[d, "L"] & \mathrm{BKW}^{i} \arrow[r] \arrow[d, "sc"] & \mathrm{BKW}^{!} \arrow[d, "s^{c}"] \\
gr(\pi_1(\mathrm{Gra}_{g,1})) \otimes \mathbb{Q} & H_\bullet(\mathrm{Gra}_{g,1}) & H^\bullet(\mathrm{Gra}_{g,\bullet})
\end{tikzcd}$$

since $\mathrm{Gra}_{g,1}$ is formal

$\cong$ THM Gra

Ex 2

Def: Drinfeld-Kohno quadratic data

$$DK(n) := \left(t^n_{ij},\ t^n_{ij} \wedge t^n_{kl} \text{ and } t^n_{ij} \wedge (t^n_{ik} + t^n_{jk}) \right) \in QD^-$$

Same ok as before

Prop: DK forms an operad in $(QD^-, \oplus)$

$\Rightarrow$ || (co)operads

THM: DK is the smallest suboperad of BKW.

Operads in $(QD^-, \oplus)$

* generators t_{ij}
* composition ok

BKW all relations

. . .

DK min relations

Consider a pointed model K_2 [Kontsevich '99] of the little discs D_2

Prop. K_2 is formal (as is D_2!)

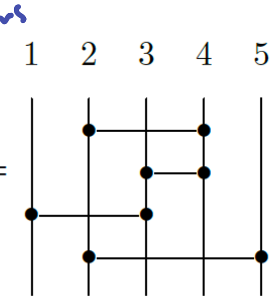
Since
$$\begin{tikzcd} K_2(n) \arrow[r, phantom, "\sim"] & D_2(n) \arrow[r, phantom, "\sim"] & \mathrm{Conf}_n(\mathbb{C}) \arrow[d, "\pi_1"] \\ & & PB_n \end{tikzcd}$$

We have the diagram

$$\begin{tikzcd} DK \arrow[r] \arrow[d, "L"] & DK^{i} \arrow[r] \arrow[d, "s^c"] & DK^{!} \arrow[d, "s"] \\ \mathrm{gr}(PB_n)\otimes\mathbb{Q} \arrow[d, "\mathcal{U}"] & H_\bullet(D_2) \arrow[d, phantom, "\cong" rotate=90] & H^\bullet(D_2) \\ CD & \mathrm{Gerst} & \end{tikzcd}$$

Lie-alg of inf. braids

op. in ass. alg. of "chord diagrams"

$t^5_{25}t^5_{13}t^5_{34}t^5_{24} =$  .

Rem: $\mathcal{D}_2$ is not well-pointed...

5.5. An operad of parenthesized braids. If we were able to make the operad of little disks $\mathcal{D}$ into an operad in the category of *based* topological spaces, then, by applying the monoidal functor (10), we would have obtained an operad $\pi_1(\mathcal{D})$ in the category of groups. However this is impossible as there is *no* $\mathbb{S}_n$ invariant configuration of little disks in $\mathcal{D}(n)$ for every $n \geq 2$.

— Merkulov

We need the fundamental groupoid functor!

$\boxed{5}$ Synthèse

PROFINITE	PRO-UNIPOTENT
$\widehat{GT}$	$GT := \mathrm{Aut}(\widehat{PaB})$ ← operad in groupoids compl. of $\pi_1(\mathcal{D}_2)$
$\uparrow$	$\downarrow \cong$ Drinfeld associator
$Gal(\bar{\mathbb{Q}}/\mathbb{Q})$	$GRT := \mathrm{Aut}(\widehat{PaCD})$ ← chord diagrams
	$\mid$

Willwacher

grt Lie-alg

THM: $\mathfrak{grt} = \mathrm{Def}(\mathrm{Gert} \hookrightarrow \mathrm{Gra})$

$$\begin{array}{c} \mathrm{Def}(\mathrm{Gert} \hookrightarrow \mathrm{Gra}) \\ \| \\ H_0(DK) \hookrightarrow H_0(BKW) \end{array}$$

To be continued ...

$$Q_2 \overset{?}{\hookrightarrow} \mathrm{Gra}_{S^1}$$