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Homotopy of Operads and Grothendieck–Teichmüller Groups

Part 1: The Algebraic Theory and its Topological Background

Benoit Fresse

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Un souffle ouvre des brèches opéradiques dans les cloisons, – brouille le pivotement des toits rongés, – disperse les limites des foyers, – éclipse les croisées.

Arthur Rimbaud
Nocturne vulgaire. Les Illuminations (1875)

Contents

This work is divided in two volumes, referred to as "Part 1: The Algebraic Theory and its Topological Background" and "Part 2: The Applications of (Rational) Homotopy Theory Methods", as indicated on the cover of the volumes. But our narrative is actually organized into three parts, numbered I-II-III, and three appendices, numbered A-B-C, which define the internal divisions of this book. This first volume comprises Part I, "From Operads to Grothendieck–Teichmüller Groups"; Appendix A, "Trees and the Construction of Free Operads"; and Appendix B, "The Cotriple Resolution of Operads". The second volume comprises Part II, "Homotopy Theory and its Applications to Operads"; Part III, "The Computation of Homotopy Automorphism Spaces of Operads"; and Appendix C, "Cofree Cooperads and the Bar Duality of Operads". Each volume includes a preface, a notation glossary, a bibliography, and a subject index.

References to chapters, sections, paragraphs, and statements of the book are given by §$x.y.z$ when these cross references are done within a part (I, II, and III) and by §$P.x.y.z$ where $P = \mathrm{I}, \mathrm{II}, \mathrm{III}$ otherwise. The cross references to the sections, paragraphs, and statements of the appendices are given by §$P.x.y$ throughout the book, where §$P = $ §A, §B, §C. The preliminary part of the first volume of this book also includes a Foundations and Conventions section, whose paragraphs, numbered §§0.1-0.16, give a summary of the main conventions used in this work.

Preliminaries

Preface

The first purpose of this work is to give an overall reference, starting from scratch, on applications of methods of algebraic topology to the study of operads in topological spaces. Most definitions, notably fundamental concepts of the theory of operads and of homotopy theory, are reviewed in this book in order to make our account accessible to graduate students and to researchers coming from the various fields of mathematics related to our subject. Then our ultimate objective is to give a homotopical interpretation of a deep relationship between operads and Grothendieck–Teichmüller groups. This connection, which has emerged from research on the deformation quantization process in mathematical physics, gives a new approach to understanding internal symmetries of structures that occur in various constructions of algebra and topology.

We review the definition of an operad at the beginning of this monograph. Simply recall for the moment that an operad is a structure, formed by collections of abstract operations, which is used to define a category of algebras. In our study, we mainly consider the example of E_n-operads, $n = 1, 2, \dots, \infty$, which are used to model a hierarchy of homotopy commutative structures, from fully homotopy associative but not commutative ($n = 1$), up to fully homotopy associative and commutative ($n = \infty$). Let us mention that the notion of an E_1-operad is synonymous to that of an A_∞-operad, which is used in the literature when one deals only with purely homotopy associative structures.

The notion of an E_n-operad formally refers to a class of operads rather than to a singled-out object. This class consists, in the initial definition, of topological operads which are homotopically equivalent to a reference model, the Boardman–Vogt operad of little n-discs D_n. The operad of little n-cubes, which is a simple variant of the little n-discs operad, is also used in the literature to provide an equivalent definition of the class of E_n-operads. We provide a detailed account of the definition of these notions in this book. Nevertheless, as we soon explain, our ultimate purpose is not to study E_n-operads themselves, but homotopy automorphism groups attached to these structures.

Before explaining this goal, we survey some motivating applications of E_n-operads which are not our main subject matter (we only give short introductions to these topics) but illustrate our approach of the subject.

The operads of little n-discs D_n were initially introduced to collect operations acting on iterated loop spaces. The first main application of these operads, which has motivated their definition, is the Boardman–Vogt and May recognition theorems of iterated loop spaces: any space Y equipped with an action of the operad D_n has the homotopy type of an n-fold loop space $\Omega^n X$ up to group-completion (see [27, 28] and [140]). Recall that the set of connected components of an n-fold loop space $\Omega^n X$ is identified with the nth homotopy group $\pi_n(X)$ of the space X. (Recall also that this group is abelian as soon as $n > 1$.) The action of D_n on $\Omega^n X$ includes a product operation $\mu : \Omega^n X \times \Omega^n X \to \Omega^n X$ which, at the level of connected components, gives the composition operation of the group $\pi_n(X)$ for any $n > 0$. The operad D_n carries the homotopies that make this product associative (and commutative for $n > 1$) and includes further operations, representing homotopy constraints, which we need to form a faithful picture of the structure of the n-fold loop space $\Omega^n X$.

This outline gives the initial topological interpretation of E_n-operads. But this topological picture has also served as a guiding idea for a study of E_n-operads in other domains. Indeed, new applications of E_n-operads, which have initiated a complete renewal of the subject, have been discovered in the fields of algebra and mathematical physics, mostly after the proof of the Deligne conjecture asserting that the Hochschild cochain complex $C^*(A, A)$ of an associative algebra A inherits an action of an E_2-operad. In this context, we deal with a chain version of the previously considered topological little 2-discs operad D_2.

The cohomology of the Hochschild cochain complex $C^*(A, A)$ is identified in degree 0 with the center $Z(A)$ of the associative algebra A. In a sense, the Hochschild cochain complex represents a derived version of this ordinary center $Z(A)$. From this point of view, the construction of an E_2-structure on $C^*(A, A)$ determines, as in the study of iterated loop spaces, the level of homotopical commutativity of the derived center which lies beyond the apparent commutativity of the ordinary center. The first proofs of the Deligne conjecture have been given by Kontsevich-Soibelman [109] and McClure-Smith [142]. The interpretation in terms of derived centers has been emphasized by Kontsevich [108] in order to formulate a natural extension of the conjecture for algebras over E_n-operads, where we now consider any $n \geq 1$ (we also refer to John Francis's work [64] for a solution of this problem in the framework of ∞-category theory).

The verification of the Deligne conjecture has yielded a second generation of proofs, promoted by Tamarkin [173] and Kontsevich [108], of the Kontsevich formality theorem on Hochschild cochains. Recall that this result implies the existence of deformation quantizations of arbitrary Poisson manifolds (we also refer to [38, 148] for higher dimensional generalizations of the deformation quantization problem involving the categories of algebras associated to E_n-operads for all $n \geq 1$). The new approaches of the Kontsevich formality theorem rely on the application of Drinfeld's associators to transport the E_2-structure yielded by the Deligne conjecture on the Hochschild cochain complex to the cohomology. In the final outcome, one obtains that each associator gives rise to a deformation quantization functor. This result has hinted the existence of a deep connection between the deformation quantization problem and the program, initiated in Grothendieck's famous "esquisse" [83], which aims to understand Galois groups through geometric actions on curves. The Grothendieck–Teichmüller groups are devices, introduced in this program, which encode the information that can be captured through the actions considered by Grothendieck. The correspondence between associators and deformation quantizations imply that a rational pro-unipotent version of the Grothendieck–Teichmüller group $GT(\mathbb{Q})$ acts on the moduli space of deformation quantizations. The initial motivation of our work was the desire to understand this connection from a homotopical viewpoint, in terms of homotopical structures associated to E_2-operads. The homotopy automorphisms of operads come into play at this point.

Recently, it has also been discovered that mapping spaces of E_n-operads can be used to compute the homotopy of the spaces of compactly supported embeddings of Euclidean spaces modulo immersions $\overline{\mathtt{Emb}}_c(\mathbb{R}^m, \mathbb{R}^n)$ (see notably the works of Dev Sinha [162], Lambrechts-Turchin-Volić [112], Arone-Turchin [9], Dwyer-Hess [59], and Boavida-Weiss [29] for various forms of this statement). We give more details on these developments in the concluding chapter of this book. In a related field, it has been observed that algebras over E_n-operads can be used to define multiplicative analogues of the classical singular homology theory for manifolds (different but equivalent constructions of such multiplicative homology theories are the "topological chiral homology", studied by Jacob Lurie in [127, 128], and the "factorization homology", studied by John Francis in [64] and by Costello-Gwilliam in [50]). These new developments give further motivations for the study of mapping spaces and homotopy automorphism spaces of E_n-operads.

Recall again that an operad is a structure which governs a category of algebras. The homotopy automorphisms of an operad P are transformations, defined at the operad level, which encode natural homotopy equivalences on the category of algebras associated to P. In this interpretation, the group of homotopy automorphism classes of an E_2-operad, which we actually aim to determine, represents the internal symmetries of the first level of homotopy commutative structures which E_2-operads serve to encode. To obtain our result, we mainly work in the setting of rational homotopy theory and we consider a rational version of the notion of an E_2-operad in topological spaces. We precisely establish that the group of rational homotopy automorphism classes of E_2-operads is isomorphic to the pro-unipotent Grothendieck–Teichmüller group $GT(\mathbb{Q})$. This result is new and represents the main outcome of our work.

In the conclusion of this monograph, we will give an overview of a sequel of this research [67], where the author, Victor Turchin, and Thomas Willwacher tackle the computation of the homotopy of the spaces of rational homotopy automorphisms of E_n-operads in dimension $n \geq 2$ (see § III.6). The main outcome of this study is that these spaces of rational homotopy automorphisms of general E_n-operads can be described in terms of the homology of a complex of graphs which, according to an earlier work of Willwacher [184], reduces to the Grothendieck–Teichmüller group in the case $n = 2$. (We have a similar description of the mapping spaces of E_n-operads which occur in the study of embedding spaces.)

To reach all of these results, we have to set up a new rational homotopy theory for topological operads beforehand and to give a sense to the rationalization of operads in topological spaces. We actually define an analogue of the Sullivan model of the rational homotopy of spaces for operads. We then consider cooperads, the structures which are dual to operads in the categorical sense. We precisely show that the rational homotopy of an operad in topological spaces is determined by an associated cooperad in commutative dg-algebras (a Hopf dg-cooperad). We have a small model of the operad of little 2-discs which is given by the Chevalley–Eilenberg cochain complex of the Drinfeld–Kohno Lie algebras (the Lie algebras of infinitesimal braids). We use this model in our proof that the group of rational homotopy automorphism classes of E_2-operads reduces to the pro-unipotent Grothendieck–Teichmüller group. In the course of our study, we also define a cosimplicial analogue of the Sullivan model of operads. This cosimplicial model remains well defined in the positive characteristic setting and gives, in this context, a model for the homotopy of the completion of topological operads at a prime.

The other main topics considered in our study include the application of homotopy spectral sequences and of Koszul duality techniques for the computation of mapping spaces attached to operads. We aim to give a detailed and comprehensive introduction to the applications of these methods for the study of operads from the point of view of homotopy theory. Besides, we thoroughly review the applications of Hopf algebras to the Malcev completion (the rationalization) of general groups. For the applications to operads, we actually consider an extension of the classical Malcev completion of groups to groupoids. Indeed, we will explain that the pro-unipotent Grothendieck–Teichmüller group can be defined as the group of automorphisms of the Malcev completion of a certain operad in groupoids which governs operations acting on braided monoidal categories. We use this observation and classical constructions of homotopy theory to define our correspondence between the Grothendieck–Teichmüller group and the space of homotopy automorphisms of E_2-operads. The previously mentioned homotopy spectral sequence techniques are used to check that this correspondence induces a bijection in homotopy. This first volume of this monograph is mainly devoted to the fundamental and algebraic aspects of our subject, from the definition of the notion of an operad to the definition of the Grothendieck–Teichmüller group. The applications of homotopy theory to operads and the proof of our isomorphism statement between the Grothendieck–Teichmüller group and the group of homotopy automorphisms classes of E_2-operads are explained in the second volume.

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Mathematical Objectives

The ultimate goal of this work, as we explain in the general introduction, is to prove that the Grothendieck–Teichmüller group represents, in the rational setting at least, the group of homotopy automorphism classes of E_2-operads. This objective can be taken as a motivation to read this book or as a guiding example of an application of our methods.

The definition of an operad is recalled with full details in the first part of this volume. In this introductory section, we only aim to give an idea of our main results. Let us simply recall that an operad P basically consists of a collection $\mathsf{P}(r)$, $r \in \mathbb{N}$, where each object $\mathsf{P}(r)$ parameterizes operations with r inputs $p = p(x_1, \dots, x_r)$, together with a multiplicative structure which models the composition of such operations. We can define operads in any category equipped with a symmetric monoidal structure $\mathcal{M}$. We then assume $\mathsf{P}(r) \in \mathcal{M}$, and we use the tensor product operation, given with this category $\mathcal{M}$, to define the composition structure attached to our operad. The operads in a base symmetric monoidal category form a category, which we denote by $\mathcal{M}\,\mathcal{O}p$, or more simply, by $\mathcal{O}p = \mathcal{M}\,\mathcal{O}p$, when this ambient category $\mathcal{M}$ is fixed by the context. An operad morphism $f : \mathsf{P} \to \mathsf{Q}$ naturally consists of a collection of morphisms in the base category $f : \mathsf{P}(r) \to \mathsf{Q}(r)$, $r \in \mathbb{N}$, which preserve the composition structure of our operads.

For technical reasons, we have to consider operads P_+ equipped with a distinguished element $* \in \mathsf{P}_+(0)$ (whenever the notion of an element makes sense), which represents an operation with zero input (a unitary operation in our terminology). In the context of sets, we moreover assume that $\mathsf{P}_+(0)$ is a one-point set reduced to this element. In the module context, we assume that $\mathsf{P}_+(0)$ is a one dimensional module over the ground ring. In a general setting, we assume that $\mathsf{P}_+(0)$ is the unit object given with the symmetric monoidal structure of our base category. We then say that P_+ forms a unitary operad. We use the notation $\mathcal{O}p_*$ to refer to the category of unitary operads. The subscript $*$ indicates the fixed arity zero component which we assign to the objects of this category of operads. We usually consider together both a non-unitary operad P, which has no term in arity 0, and an associated unitary operad P_+, where the arity zero term, spanned by the distinguished operation $* \in \mathsf{P}_+(0)$, is added. We therefore follow the convention to use a subscript $+$, marking the addition of this term, for the notation of the unitary operad P_+. We also say that the operad P_+ arises as a *unitary extension* of the non-unitary operad P. We often perform constructions on the non-unitary operad P first, and on the unitary operad P_+ afterwards, by assuming that the additional distinguished element (or unit term) of P_+ is preserved by the operations involved in our construction.

In topology, an E_2-operad is usually defined as an operad in the category of spaces which is equivalent to Boardman–Vogt's operad of little 2-discs D_2 in the homotopy category of operads. The spaces $D_2(r)$ underlying this operad have a trivial homotopy in dimension $* \neq 1$, and for $* = 1$, we have $\pi_1 D_2(r) = P_r$, where P_r denotes the pure braid group on r strands. Thus, the space $D_2(r)$ is an Eilenberg-MacLane space $K(P_r, 1)$ associated to the pure braid group P_r. For our purpose, we consider a rationalization of the little 2-discs operad $\widehat{D_2}$, for which we have $\pi_1 \widehat{D_2}(r) = \hat{P}_r$, where $\hat{P}_r$ denotes the Malcev completion of the group P_r. We give a general definition of the rationalization for operads in topological spaces in the second volume of this book. In the case of the little 2-discs operad D_2, we have a simple model of the rationalization $\widehat{D_2}$ which relies on the Eilenberg-MacLane space interpretation of the little 2-discs spaces. We give a brief outline of this approach soon.

Homotopy automorphisms can be defined in the general setting of model categories which provides a suitable axiomatic framework for the application of homotopy theory concepts to operads. In order to introduce our subject, we first explain a basic interpretation of the general definition of a homotopy automorphism in the context of topological operads.

We have a natural notion of homotopy $\sim$ for morphisms of operads in topological spaces. To a topological operad Q, we associate the collection of path spaces $Q^{\Delta^1}(r) = \mathtt{Map}_{\mathcal{T}op}([0,1], Q(r))$, which inherits an operad structure from Q and defines a path-object associated to Q in the category of topological operads. We explicitly define a homotopy between operad morphisms $f, g : P \to Q$ as an operad morphism $h : P \to Q^{\Delta^1}$ which satisfies $d_0 h = f$, $d_1 h = g$, where $d_0, d_1 : Q^{\Delta^1} \to Q$ are the natural structure morphisms (evaluation at the origin and at the end point) associated with our path-object Q^{Δ^1}. This homotopy h is intuitively equivalent to a continuous family of operad morphisms $h_t : P \to Q$ going from $h_0 = f$ to $h_1 = g$.

In a first approximation, we take the sets of homotopy classes of operad morphisms as the morphism sets of the homotopy category $\mathtt{Ho}(\mathcal{T}op\,\mathcal{O}p)$ which we associate to the category of topological operads $\mathcal{T}op\,\mathcal{O}p$. In principle, we have to deal with a suitable notion of cofibrant object in the category of operads and to replace any operad by a cofibrant resolution in order to use this definition of morphism set. But we will explain this issue later on. We focus on the basic definition of the morphism sets of the homotopy category for the moment.

The groups of homotopy automorphism classes, which we aim to determine, are the groups of automorphisms of the homotopy category $\mathtt{Ho}(\mathcal{T}op\,\mathcal{O}p)$. The automorphism group $\mathtt{Aut}_{\mathtt{Ho}(\mathcal{T}op\,\mathcal{O}p)}(P)$ associated to a given operad $P \in \mathcal{T}op\,\mathcal{O}p$ accordingly consists of homotopy classes of morphisms $f : P \to P$, which have a homotopy inverse $g : P \to P$ such that $fg \sim id$ and $gf \sim id$. We consider the operadic homotopy relation at each step of this definition.

Note that a topological operad P gives rise to an operad object in the homotopy category of topological spaces $\mathtt{Ho}(\mathcal{T}op)$, and we could also study the automorphism group $\mathtt{Aut}_{\mathtt{Ho}(\mathcal{T}op)\,\mathcal{O}p}(P)$ formed in this naive category of homotopy operads $\mathtt{Ho}(\mathcal{T}op)\,\mathcal{O}p$. But these automorphism groups differ from our groups of homotopy automorphisms and do not give the appropriate structure for the homotopy version of usual constructions of group theory (like homotopy fixed points). Indeed, an automorphism of the operad P in the homotopy category of spaces $\mathtt{Ho}(\mathcal{T}op)$ is just a collection of homotopy classes of maps $f \in [P(r), P(r)]$, invertible in the homotopy category of spaces, and which preserve the operadic structures up to homotopy, unlike our homotopy automorphisms that preserve operadic structures strictly. Moreover, actual operad morphisms $f, g : \mathsf{P} \to \mathsf{Q}$ define the same morphism of operads in the homotopy category of spaces $\mathtt{Ho}(\mathcal{T}op)$ as soon as we have a homotopy between the individual maps $f : \mathsf{P}(r) \to \mathsf{P}(r)$ and $g : \mathsf{P}(r) \to \mathsf{P}(r)$, for all $r \in \mathbb{N}$ (regardless of operad structures). Thus, operad morphisms which are homotopic in the strong operadic sense determine the same morphism of operads in the homotopy category of spaces $\mathtt{Ho}(\mathcal{T}op)$, but the converse implication does not hold. By associating the collection of homotopy classes of maps $f : \mathsf{P}(r) \to \mathsf{P}(r)$ to a homotopy automorphism $f \in \mathtt{Aut}_{\mathtt{Ho}(\mathcal{T}op\,\mathcal{O}p)}(\mathsf{P})$, we obtain a mapping $\mathtt{Aut}_{\mathtt{Ho}(\mathcal{T}op\,\mathcal{O}p)}(\mathsf{P}) \to \mathtt{Aut}_{\mathtt{Ho}(\mathcal{T}op)\,\mathcal{O}p}(\mathsf{P})$, from the group of homotopy classes of homotopy automorphisms towards the group of automorphisms of the operad in the homotopy category of spaces, but this mapping is neither an injection nor a surjection in general.

To apply methods of algebraic topology, we associate to any operad P a whole simplicial set of homotopy automorphisms $\mathtt{Aut}^h_{\mathcal{T}op\,\mathcal{O}p}(\mathsf{P})$ rather than a single group of homotopy automorphism classes. To be more precise, this group $\mathtt{Aut}_{\mathtt{Ho}(\mathcal{T}op\,\mathcal{O}p)}(\mathsf{P})$, which we primarily aim to determine, is identified with the set of connected components of our homotopy automorphism space $\pi_0(\mathtt{Aut}^h_{\mathcal{T}op\,\mathcal{O}p}(\mathsf{P}))$. In the second volume of this work, we explain the definition of these homotopy automorphism spaces in the general context of simplicial model categories. For the moment, we simply give a short outline of the definition in the context of topological operads.

First, we extend the definition of our path object and we consider, for each $n \in \mathbb{N}$, an operad P^{Δ^n} which is defined by the collection of function spaces $\mathsf{P}^{\Delta^n}(r) = \mathtt{Map}_{\mathcal{T}op}(\Delta^n, \mathsf{P}(r))$ on the n-simplex Δ^n. This operad sequence P^{Δ^n} inherits a simplicial structure from the topological simplices Δ^n. In particular, since we obviously have $\mathsf{P} = \mathsf{P}^{\Delta^0}$, we have a morphism $v^* : \mathsf{P}^{\Delta^n} \to \mathsf{P}$ associated to each vertex v of the n-simplex Δ^n. The simplicial set $\mathtt{Aut}^h_{\mathcal{T}op\,\mathcal{O}p}(\mathsf{P})$ precisely consists, in dimension $n \in \mathbb{N}$, of the morphisms of topological operads $f : \mathsf{P} \to \mathsf{P}^{\Delta^n}$ such that the composite $v^* f$ defines a homotopy equivalence of the operad $\mathsf{P} = \mathsf{P}^{\Delta^0}$, for each vertex $v \in \Delta^n$. From this definition, we immediately see that the 0-simplices of the simplicial set $\mathtt{Aut}^h_{\mathcal{T}op\,\mathcal{O}p}(\mathsf{P})$ are the homotopy equivalences of the operad P, the 1-simplices are the operadic homotopies $h : \mathsf{P} \to \mathsf{P}^{\Delta^1}$ between homotopy equivalences, and therefore, we have a formal identity $\mathtt{Aut}_{\mathtt{Ho}(\mathcal{T}op\,\mathcal{O}p)}(\mathsf{P}) = \pi_0\,\mathtt{Aut}^h_{\mathcal{T}op\,\mathcal{O}p}(\mathsf{P})$ between our group of homotopy automorphism classes $\mathtt{Aut}_{\mathtt{Ho}(\mathcal{T}op\,\mathcal{O}p)}(\mathsf{P})$ and the set of connected components of $\mathtt{Aut}^h_{\mathcal{T}op\,\mathcal{O}p}(\mathsf{P})$.

In what follows, we often consider simplicial sets as combinatorial models of topological spaces. In this situation, we adopt a common usage of homotopy theory to use the name ‘space’ for a simplicial set. Therefore we refer to the simplicial set $\mathtt{Aut}^h_{\mathcal{T}op\,\mathcal{O}p}(\mathsf{P})$ as the homotopy automorphism space associated to the operad P.

Besides homotopy equivalences, we consider a class of morphisms, called weak-equivalences, which are included in the definition of a model structure on the category of operads. We adopt the standard notation of the theory of model categories $\xrightarrow{\sim}$ to refer to this class of distinguished morphisms. The notion of a model category also includes the definition of a class of cofibrant objects, generalizing the cell complexes of topology, and which are well suited for the homotopy constructions we aim to address.

To be more specific, recall that a map of topological spaces $f : X \to Y$ is a weak-equivalence when this map induces a bijection on connected components $f_* : \pi_0(X) \xrightarrow{\simeq} \pi_0(Y)$ and an isomorphism on homotopy groups $f_* : \pi_*(X, x) \xrightarrow{\simeq} \pi_*(Y, f(x))$, for all $* > 0$ and for any choice of base point $x \in X$. We define a weak-equivalence of operads as an operad morphism $f : \mathsf{P} \to \mathsf{Q}$ of which underlying maps $f : \mathsf{P}(r) \to \mathsf{Q}(r)$ are weak-equivalences of topological spaces. In the context of topological spaces, a classical result asserts that any weak-equivalence between cell complexes is homotopically invertible as a map of topological spaces. In the context of topological operads, we similarly obtain that any weak-equivalence between cofibrant operads $f : \mathsf{P} \xrightarrow{\sim} \mathsf{Q}$ is homotopically invertible as an operad morphism: we have an operad morphism $g : \mathsf{Q} \to \mathsf{P}$ in the converse direction as our weak-equivalence $f : \mathsf{P} \xrightarrow{\sim} \mathsf{Q}$ such that $fg \sim id$ and $gf \sim id$, where we now consider the operadic homotopy relation (as in the definition of a homotopy automorphism for operads).

The proof of the model category axioms for operads includes the construction of a cofibrant resolution functor, which assigns a cofibrant operad R equipped with a weak-equivalence $\mathsf{R} \xrightarrow{\sim} \mathsf{P}$ to any given operad P. The definition of the homotopy category of topological operads in terms of homotopy class sets of morphisms is actually the right one when we replace each operad P by such a cofibrant model $\mathsf{R} \xrightarrow{\sim} \mathsf{P}$. In particular, when we form the group of homotopy automorphism classes of an operad $\mathtt{Aut}_{\mathtt{Ho}(\mathcal{T}op\,\mathcal{O}p)}(\mathsf{P})$, we have to assume that P is cofibrant as an operad, otherwise we tacitely assume that we apply our construction to a cofibrant resolution of P. The general theory of model categories ensures that the obtained group $\mathtt{Aut}_{\mathtt{Ho}(\mathcal{T}op\,\mathcal{O}p)}(\mathsf{P})$ does not depend, up to isomorphism, on the choice of this cofibrant resolution. We have similar results and we apply similar conventions for the homotopy automorphism spaces $\mathtt{Aut}^h_{\mathcal{T}op\,\mathcal{O}p}(\mathsf{P})$. To be precise, in the general context of the theory of model categories, we have a notion of fibrant object, which is dual to the notion of a cofibrant object, and we actually have to consider objects which are both cofibrant and fibrant when we use the above definition of the group of homotopy automorphism classes of an object. We have a similar observation for the definition of homotopy automorphism spaces. But we can neglect this issue for the moment, because all objects of our model category of topological operads are fibrant.

We go back to the case of the little 2-discs operad. We aim to determine the homotopy groups of the homotopy automorphism space $\mathtt{Aut}^h_{\mathcal{T}op\,\mathcal{O}p}(\mathsf{D}_{2+}^{\wedge})$ associated to the rationalization of D_2 and in the unitary operad context, which we mark by the addition of the subscript $+$ in our notation. Recall that the connected components of this space $\mathtt{Aut}^h_{\mathcal{T}op\,\mathcal{O}p}(\mathsf{D}_{2+}^{\wedge})$ correspond to homotopy classes of operad homotopy equivalences $f : \hat{\mathsf{R}}_{2+} \xrightarrow{\sim} \hat{\mathsf{R}}_{2+}$, where $\hat{\mathsf{R}}_2$ denotes a cofibrant model of the rationalized little 2-discs operad $\mathsf{D}_2^{\wedge}$. Our result reads:

Theorem A. *The homotopy automorphism space of the rationalization of the little* 2*-discs operad* $\mathsf{D}_{2+}^{\wedge}$ *satisfies*

$$\pi_* \, \mathtt{Aut}^h_{\mathcal{T}op\,\mathcal{O}p}(\mathsf{D}_{2+}^{\wedge}) = \begin{cases} GT(\mathbb{Q}), & \textit{for } * = 0, \\ \mathbb{Q}, & \textit{for } * = 1, \\ 0, & \textit{otherwise}, \end{cases}$$

where $GT(\mathbb{Q})$ denotes the rational pro-unipotent version of the Grothendieck–Teichmüller group, such as defined by V. Drinfeld in [57].

The identity established in this theorem is a new result. The ultimate goal of this work precisely consists in proving this statement. In fact, we will more precisely prove that the homotopy automorphism space $\mathtt{Aut}^h_{\mathcal{T}op\,\mathcal{O}p}(D_{2+}^{\wedge})$ is weakly-equivalent (as a monoid) to a semi-direct product $GT(\mathbb{Q}) \ltimes SO(2)^{\wedge}$, where we regard the Grothendieck–Teichmüller group $GT(\mathbb{Q})$ as a discrete group in topological spaces and $SO(2)^{\wedge}$ denotes a rationalization of the space of rotations of the plane $SO(2)$ which naturally acts on the little 2-discs spaces by operad automorphisms. We have $\pi_1 SO(2) = \mathbb{Z} \Rightarrow \pi_1 SO(2)^{\wedge} = \mathbb{Q}$ and the module $\mathbb{Q}$ in the above theorem comes from this factor $SO(2)^{\wedge}$ of our semi-direct product decomposition of the homotopy automorphism space $\mathtt{Aut}^h_{\mathcal{T}op\,\mathcal{O}p}(D_{2+}^{\wedge})$.

We mentioned, at the beginning of this survey, that the operad of little 2-discs D_2 consists of Eilenberg-MacLane spaces $K(P_r, 1)$, where P_r denotes the pure braid group on r strands, and the associated rationalized operad $D_2^{\wedge}$ consists of Eilenberg-MacLane spaces $K(\hat{P}_r, 1)$, where we now consider the Malcev completion of the group P_r. We have standard models of the Eilenberg-MacLane spaces $K(P_r, 1)$ which are given by the classifying spaces of the groups P_r. But these spaces do not form an operad. Nevertheless, we can adapt this classifying space approach to give a simple model of E_2-operad. Instead of the pure braid group P_r, we consider the classifying space of a groupoid of parenthesized braids $PaB(r)$. The morphisms of this groupoid are braids on r strands indexed by elements of the set $\{1, \dots, r\}$. The parenthesization refers to an extra structure, added to the contact points of the braids, which define the object sets of our groupoid. Unlike the pure braid groups P_r, the collection of groupoids $PaB(r)$ forms an operad in the category of groupoids, and the associated collection of classifying spaces $\mathtt{B}(PaB)(r) = \mathtt{B}(PaB(r))$ forms an operad in topological spaces. We check, by relying on an argument of Zbigniew Fiedorowicz, that this operad $\mathtt{B}(PaB)$ is weakly-equivalent to the operad of little 2-discs D_2 and, hence, forms an E_2-operad.

We can perform the Malcev completion of the groupoids $PaB(r)$ underlying the parenthesized braid operad PaB. We get an operad in groupoids $PaB^{\wedge}$. We will see that the collection of classifying spaces $\mathtt{B}(PaB^{\wedge})(r) = \mathtt{B}(PaB(r)^{\wedge})$ associated to this operad $PaB^{\wedge}$ forms an operad in topological spaces which defines a model for the rationalization of the operad of little 2-discs $D_2^{\wedge}$.

The Grothendieck–Teichmüller group $GT(\mathbb{Q})$ can actually be identified with a group of automorphisms associated to the (unitary) operad in groupoids $PaB_+^{\wedge}$ and an automorphism of topological operads $\mathtt{B}(\phi) : \mathtt{B}(PaB_+^{\wedge}) \xrightarrow{\sim} \mathtt{B}(PaB_+^{\wedge})$ can be associated to any element of this group $\phi \in GT(\mathbb{Q})$ by functoriality of the classifying space construction. We can lift any such automorphism to any chosen cofibrant model $R_{2+}^{\wedge}$ of the rationalization of the operad of little 2-discs $D_{2+}^{\wedge}$. We accordingly associate a well-defined rational homotopy automorphism of the E_2-operad $R_{2+}^{\wedge}$ to our element of the Grothendieck–Teichmüller group $\phi \in GT(\mathbb{Q})$. Our main theorem asserts that, when we work in the rational setting, this construction gives exactly all homotopy automorphism classes of E_2-operads.

We consider a pro-unipotent version of the Grothendieck–Teichmüller group in our theorem. We mostly study this group for the applications in the rational homotopy theory of operads, but we also have a pro-finite version of the

Grothendieck–Teichmüller group which is better suited for the purposes of the original Grothendieck's program in Galois theory. In fact, a pro-finite analogue of our result, which relates this pro-finite Grothendieck–Teichmüller group to a pro-finite completion of E_2-operads, has been obtained by Horel [90] during the writing of this monograph. We will give more explanations on this statement and on other generalizations of the result of Theorem A in the concluding chapter of the second volume of this monograph (§ III.6).

The (rational) homology of the little 2-discs spaces $\mathtt{H}_*(D_2(r)) = \mathtt{H}_*(D_2(r), \mathbb{Q})$, $r \in \mathbb{N}$, forms an operad in graded modules $\mathtt{H}_*(D_2)$. We will see that this homology operad $\mathtt{H}_*(D_2)$ is identified with an operad, defined in terms of generating operations and relations, and which we can associate to the category of Gerstenhaber algebras (a graded version of the notion of a Poisson algebra). We therefore use the notation Gerst_2 for this operad such that $\mathsf{Gerst}_2 = \mathtt{H}_*(D_2)$. In what follows, we consider the (rational) cohomology of the little 2-discs spaces $\mathtt{H}^*(D_2(r)) = \mathtt{H}^*(D_2(r), \mathbb{Q})$ rather than the homology $\mathtt{H}_*(D_2(r))$. We use that the cohomology of a space inherits a unitary commutative algebra structure and that the collection of the cohomology algebras $\mathtt{H}^*(D_2(r))$ associated to the little 2-discs spaces $D_2(r)$ forms a cooperad in the category of unitary commutative algebras in graded modules, where the name cooperad obviously refers to the structure dual to an operad in the categorical sense (we go back to this concept later on). We also use the phrase '(graded) Hopf cooperad' to refer to this particular case of the structure defined by a cooperad in the category of unitary commutative algebras (in graded modules).

For our purpose, we actually need a counterpart, in the category of graded Hopf cooperads, of the category of unitary operads. We have to adapt the definition of a cooperad in this case, because the consideration of an arity zero term in the context of a cooperad creates convergence difficulties in the definition of cofree objects. We work out this problem by integrating this part of the composition structure of our cooperads into a diagram structure. We will be more precise later on. For the moment, we just use the notation $gr\,\mathcal{H}opf\,\mathcal{O}p_*^c$, with our distinguishing $*$ mark, for this category of (graded) Hopf cooperads which we associate to unitary operads. We will adopt another notation as soon as we will be able to make the definition of this category more precise.

We have a general approach to compute the homotopy of mapping spaces in the category of operads $\mathtt{Map}_{\mathcal{T}op\,\mathcal{O}p}(P, Q)$. In short, the idea is to determine the homotopy of mapping spaces on free simplicial resolutions of our objects P, Q in the category of operads in topological spaces $\mathcal{T}op\,\mathcal{O}p$. We then get a spectral sequence whose second page reduces to the cohomology of a deformation complex associated to the cohomology cooperads $\mathtt{H}^*(P), \mathtt{H}^*(Q) \in gr\,\mathcal{H}opf\,\mathcal{O}p_*^c$. The definition of this deformation complex involves both the commutative algebra structure and the cooperad structure of these graded Hopf cooperads $\mathtt{H}^*(P), \mathtt{H}^*(Q) \in gr\,\mathcal{H}opf\,\mathcal{O}p_*^c$. To be explicit, at the deformation complex level, we deal with a free resolution of the Hopf cooperad $\mathtt{H}^*(Q)$ in the commutative algebra direction and with a cofree resolution of the object $\mathtt{H}^*(P)$ in the cooperad direction. Then our deformation complex precisely consists of modules of biderivations associated to these resolutions in the category of graded Hopf cooperads.

In fact, we only use this general approach in a follow-up (see [68]) as we can use a simplification of the free commutative algebra resolution when we establish our result about the homotopy automorphism space of the operad of little 2-discs.

To be explicit, instead of a resolution in the category of commutative algebras, we use a counterpart, for operads, of the classical Postnikov decomposition of spaces. To ease the definition of this Postnikov decomposition for the rationalization of the operad of little 2-discs $\mathsf{D}_2^{\widehat{}}$, we actually consider a classifying space $\mathtt{B}(\mathsf{CD}^{\widehat{}})$ on an operad of chord diagrams $\mathsf{CD}^{\widehat{}}$ which is equivalent to the (Malcev completion of the) operad of parenthesized braids $\mathsf{PaB}^{\widehat{}}$.

To perform our computation, we moreover decompose the general homotopy spectral sequences of operadic mapping spaces in two intermediate spectral sequences, where we deal with the resolution in the operad direction in a first step and with the obstruction problems associated to the Postnikov decomposition of our target object in the second step. We will see that these spectral sequences vanish in degree $* > 1$, reduce to the module of rank one $\mathbb{Q}$ in degree $* = 1$, and reduce to a graded Lie algebra $\mathfrak{grt}$ associated to the Grothendieck–Teichmüller group $GT(\mathbb{Q})$ in degree $* = 0$. We use this correspondence to check that all classes of degree $* = 0$ in the second page of our spectral sequence are hit by homotopy automorphisms of our operad which come from the Grothendieck–Teichmüller group. We conclude from this result that our mapping from the Grothendieck–Teichmüller group $GT(\mathbb{Q})$ to the space of homotopy automorphisms of the operad $\mathsf{D}_{2+}^{\widehat{}}$ induces a bijection in homotopy.

We tackle this verification in the third part of this work, in the second volume of this monograph. We review Drinfeld's definition of the pro-unipotent Grothendieck–Teichmüller group first. We explain that the pro-unipotent Grothendieck–Teichmüller group can be defined as a group of automorphisms associated to the Malcev completion of the parenthesized braid operad $\mathsf{PaB}^{\widehat{}}$. We then develop a new rational homotopy theory of operads before tackling the computation of the homotopy of the homotopy automorphism space of rational E_2-operads.

For this purpose, we notably define an analogue of the Sullivan model for the rational homotopy of operads in topological spaces. Briefly recall that the classical Sullivan model of a simplicial set X is defined by a commutative cochain differential graded algebra $\Omega^*(X)$ (a cochain commutative dg-algebra for short), which consists of piecewise linear differential forms on X. We consider cooperads in commutative cochain dg-algebras to define the Sullivan model of operads in topological spaces. We use the notation $dg^*\,\mathcal{H}opf\,\mathcal{O}p_*^c$ for this category of cooperads and, for short, we also call 'Hopf cochain dg-cooperads' the objects of this category. We also adopt the notation $dg^*\,\mathcal{C}om_+$ for the category of unitary commutative cochain dg-algebras in what follows. We already mentioned that a cooperad is a structure dual to an operad in the categorical sense. Briefly say for the moment that a cooperad C in a category $\mathcal{M}$ essentially consists of a collection of objects $\mathsf{C}(r) \in \mathcal{M}$ together with a comultiplicative structure of a form opposite to the composition operations of an operad. We simply take $\mathcal{M} = dg^*\,\mathcal{C}om_+$ in this general definition when we form our category of Hopf cochain dg-cooperads. Recall that we temporarily use the subscript mark $*$ in our notation of the category of Hopf cochain dg-cooperads $dg^*\,\mathcal{H}opf\,\mathcal{O}p_*^c$ in order to indicate that we actually consider a counterpart, in the cooperad context, of our category of unitary operads.

The Sullivan dg-algebra $\Omega^*(X)$ does not preserve operad structures, but we will explain in the second volume of this monograph that we can define an operadic enhancement of the Sullivan functor in order to assign a well-defined Hopf cochain dg-cooperad $\Omega_\sharp^*(\mathsf{P})$ to any operad in simplicial sets P. We will prove that the commutative cochain dg-algebras $\Omega^*_\sharp(P)(r)$ which define the components of this Hopf cochain dg-cooperad $\Omega^*_\sharp(P)$ are weakly-equivalent (quasi-isomorphic) to the Sullivan dg-algebras $\Omega^*(P(r))$ associated to the spaces $P(r)$ which underlie our operad P. We use this observation to check that this Hopf dg-cooperad $\Omega^*_\sharp(P)$ determines the operad in simplicial sets P up to rational equivalence.

We will also see that our functor $\Omega^*_\sharp : P \to \Omega^*(P)$, from the category of operads in simplicial sets to the category of Hopf cochain dg-cooperads, admits a left adjoint $\mathtt{G}_\bullet(-) : K \to \mathtt{G}_\bullet(K)$, which assigns an operad in simplicial sets $\mathtt{G}_\bullet(K)$ to any Hopf cochain dg-cooperad K. We consider the image of the Hopf cochain dg-cooperad $K = \Omega^*_\sharp(P)$ associated to an operad in simplicial sets P under a left derived functor of this left adjoint $\mathtt{L}\,\mathtt{G}_\bullet : K \to \mathtt{L}\,\mathtt{G}_\bullet(K)$. We will prove that this operad in simplicial sets $P^{\wedge} = \mathtt{L}\,\mathtt{G}(\Omega^*_\sharp(P))$ forms, under mild finiteness assumptions, a suitable model for the rationalization of the operad P in the sense that the components of this operad $P^{\wedge}(r)$ are equivalent, in the homotopy category of spaces, to the Sullivan rationalization $X^{\wedge} = P(r)^{\wedge}$ of the simplicial sets $X = P(r)$.

The Sullivan dg-algebra functor (and our operadic enhancement of this functor similarly) is defined on the category of simplicial sets $s\mathcal{S}et$. But we can use the classical singular complex functor $\mathtt{Sing}_\bullet(-)$, from topological spaces to simplicial sets, and the geometric realization functor which goes the other way round, in order to prolong our constructions on operads in simplicial sets to operads in topological spaces.

The category $dg^*\,\mathcal{H}opf\,\mathcal{O}p^c_*$ inherits a model structure, like the category of topological operads, and we can therefore apply the general theory of model categories to associate homotopy automorphism groups $\mathtt{Aut}_{\mathtt{Ho}(dg^*\,\mathcal{H}opf\,\mathcal{O}p^c_*)}(A)$, as well as homotopy automorphisms spaces $\mathtt{Aut}^h_{dg^*\,\mathcal{H}opf\,\mathcal{O}p^c_*}(A)$, to any object of the category of Hopf cochain dg-cooperads $A \in dg^*\,\mathcal{H}opf\,\mathcal{O}p^c_*$. In the case of topological operads, we already mentioned that homotopy automorphisms spaces are well defined for cofibrant objects only. In the case of Hopf cochain dg-cooperads, we have to perform both cofibrant and fibrant resolutions before applying the homotopy automorphism construction.

The results obtained in our study of the rational homotopy of operads imply that the group of homotopy automorphisms attached to the model $\Omega^*_\sharp(P)$ of an operad in spaces P is isomorphic to the group of homotopy automorphisms attached to the rational completion of this operad $P^{\wedge}$. Theorem A is therefore equivalent to the following statement:

THEOREM B. *Let E_2 be a (cofibrant) model of E_2-operad in the category of topological space. Let $K_2 = \Omega^*_\sharp(E_2)$ be the Hopf cochain dg-cooperad associated to this operad E_2. The homotopy automorphism space of this object in the category of Hopf cochain dg-cooperads $\mathtt{Aut}^h_{dg^*\,\mathcal{H}opf\,\mathcal{O}p^c_*}(K_2)$ has trivial homotopy groups*

$$\pi_*(\mathtt{Aut}^h_{dg^*\,\mathcal{H}opf\,\mathcal{O}p^c_*}(K_2)) = 0$$

in dimension $* > 1$, the $\mathbb{Q}$-module of rank one as homotopy group

$$\pi_1(\mathtt{Aut}^h_{dg^*\,\mathcal{H}opf\,\mathcal{O}p^c_*}(K_2)) = \mathbb{Q}$$

in dimension $* = 1$, and we have an isomorphism of groups

$$GT(\mathbb{Q})^{op} \xrightarrow{\simeq} \pi_0(\mathtt{Aut}^h_{dg^*\,\mathcal{H}opf\,\mathcal{O}p^c_*}(K_2))$$

in dimension $* = 0$.

The assertions of this theorem have been foreseen by M. Kontsevich in [108]. First results in the direction of Theorem B also occur in articles of D. Tamarkin [174] and T. Willwacher [184]. But these authors deal with operads within the category of differential graded modules, after forgetting about commutative algebra structures, and their results actually give a stable version (in the sense of homotopy theory) of our statements. The definition of a setting, where we can combine a model for operadic structures and a commutative algebra model for the topology underlying our objects, is a new contribution of this monograph. The proof of Theorem B in this context is also a new outcome of our work, like the result of Theorem A.

Recall that E_2-operads only give the second layer of a full sequence of homotopy structures, ranging from E_1, fully homotopy associative but non-commutative, up to E_∞, fully homotopy associative and commutative. In a follow-up [67], the author, Victor Turchin and Thomas Willwacher give a computation, in terms of graph complexes, of the homotopy of the homotopy automorphism spaces of E_n-operads for $2 < n < \infty$. We give an overview of these results in the concluding chapter of the second volume of this monograph. Let us mention that the group of homotopy automorphism classes of E_1-operads can easily be determined, but the result is trivial in this case. The group of homotopy automorphisms of an E_∞-operad is trivial too.

The proof of our result requires the elaboration of new theories, like the definition of a model for the rational homotopy of topological operads, and the first aim of this monograph is to work out such general problems. The purpose of this book is also to give a comprehensive introduction to our subject, heading to our main theorems as straight as possible and with minimal background, for mathematicians coming from other domains and for graduate students.

We heavily use the formalism of Quillen's model categories [152] which we apply to operads in order to form our model for the rational homotopy of topological operads. We rely on the modern reference books by Hirschhorn [89] and by Hovey [91] for the subject of Quillen's model categories. We also refer to the book [61] for a comprehensive introduction to the rational homotopy theory and to Bousfield-Gugenheim's memoir [36] for an interpretation of the Sullivan model in the formalism of model categories. We also refer to Sullivan's article [170] for the applications of rational homotopy theory to the study of homotopy automorphisms of spaces. We review these subjects thoroughly before tackling our own constructions.

We first explain the connections between little 2-discs operads and braided structures, as well as the definition of the Grothendieck–Teichmüller group in terms of automorphisms of operads in groupoids. We give a comprehensive account of these topics in the first volume of this monograph, after an introduction to the general theory of operads. We notably give an operadic formulation of the classical coherence theorems of monoidal categories, of braided monoidal categories, and of symmetric monoidal categories. We will explain that the previously considered operad of parenthesized braids PaB, which we define by using the fundamental groupoid of the little 2-discs operad, actually governs the structure of a braided monoidal category.

We also review the applications of the theory of Hopf algebras to the Malcev completion of groups with the aim of extending this completion process to groupoids and to operads. We focus on the study of a pro-unipotent Grothendieck–Teichmüller group in this work and we actually rely on the operadic Malcev completion construction in our definition of the Grothendieck–Teichmüller group. In passing, we will explain an operadic interpretation of the notion of a Drinfeld associator which was used by Tamarkin in order to prove the rational formality of E_2-operads.

We address the definition of our model for the rational homotopy of operads in the second volume of this monograph, after a survey of the general theory of model categories and of the rational homotopy theory of spaces. We mentioned in the introduction of this work that the Chevalley–Eilenberg cochain complex of the Drinfeld–Kohno Lie algebras (the Lie algebras of infinitesimal braids) can be used to define a small model of a rational E_2-operad. We also explain this construction in the second volume of this monograph. We review the already alluded proof of the rational formality of E_2-operads in the second volume too. We actually explain the definition of small models of E_n-operads for all $n \geq 2$ by using a graded version of the Drinfeld–Kohno Lie algebras. We tackle the computation of the homotopy automorphism space of rational E_2-operads afterwards, in the concluding part of the second volume.

Foundations and Conventions

The reader is assumed to be familiar with the language of category theory and to have basic knowledge about fundamental concepts like adjoint and representable functors, colimits and limits, categorical duality, which we freely use throughout this work. The reader is also assumed to be aware on the applications of colimits and limits in basic examples of categories (including sets, topological spaces, and modules). Nonetheless, we will review some specialized topics, like reflexive coequalizers and filtered colimits, which are considered in applications of category theory to operads.

We use single script letters (like $\mathcal{C}$, $\mathcal{M}$, ...) as general notation for abstract categories. We use script expressions (like $\mathcal{M}od$, $\mathcal{A}s$, $\mathcal{O}p$, ...) for particular instances of categories which we consider in this work (like modules, associative algebras, operads, ...). We are going to explain that the formal definition of many algebraic structures remains the same in any instance of base category $\mathcal{M}$ and essentially depends on a *symmetric monoidal structure* given with this category $\mathcal{M}$. We usually assume that the category $\mathcal{M}$, to which we assign the role of a base category, is equipped with enriched hom-bifunctors $\mathtt{Hom}_{\mathcal{M}}(-,-)$. We give a more detailed reminder on this notion in §§0.12-0.13.

In practice, we take our base category $\mathcal{M}$ among the category of sets $\mathcal{S}et$, the category of simplicial sets $s\mathcal{S}et$, the category of topological spaces $\mathcal{T}op$, a category of $\Bbbk$-modules $\mathcal{M}od$ (where $\Bbbk$ refers to a fixed ground ring), or among a variant of these categories. To be precise, besides plain $\Bbbk$-modules, we have to consider categories formed by differential graded modules $dg\,\mathcal{M}od$ (we usually speak about *dg-modules* for short), graded modules $gr\,\mathcal{M}od$, simplicial modules $s\,\mathcal{M}od$, and cosimplicial modules $c\,\mathcal{M}od$. The first purpose of this preliminary chapter is to quickly recall the definition of these categories (at least, in order to fix our conventions). By the way, we also recall the definition of the category of simplicial sets $s\mathcal{S}et$, which we use along with the familiar category of topological spaces $\mathcal{T}op$.

To complete our account, we recall the general definition of a symmetric monoidal category and we explain some general constructions which we associate to this notion. But we postpone our reminder on the definition of the monoidal structure of the category of dg-modules, simplicial modules and cosimplicial modules until the moment where we tackle the applications of these base categories.

In the module context, we assume that a ground ring $\Bbbk$ is given and is fixed once and for all. In certain constructions, we have to assume that this ground ring $\Bbbk$ is a field of characteristic 0.

0.1. *Graded and differential graded modules.* The category of *differential graded modules* $dg\,\mathcal{M}od$ (*dg-modules* for short) consists of $\Bbbk$-modules equipped with a decomposition $C = \bigoplus_{n\in\mathbb{Z}} C_n$, which ranges over $\mathbb{Z}$, and with a morphism $\delta : C \to C$ (the *differential*) such that we have $\delta^2 = 0$ and $\delta(C_n) \subset C_{n-1}$, for all $n \in \mathbb{Z}$. We obviously define a morphism of dg-modules as a morphism of $\Bbbk$-modules $f : C \to D$ which intertwines differentials and which satisfies the relation $f(C_n) \subset D_n$, for all $n \in \mathbb{Z}$.

In textbooks of homological algebra (like [181]), authors mostly deal with an equivalent notion of *chain complex*, of which components are split off into a sequence of $\Bbbk$-modules C_n linked by the differentials $\delta : C_n \to C_{n-1}$ rather than being put together in a single object. The idea of a dg-module (used for instance in [129]) is more natural for our purpose and is also more widely used in homotopy theory. In what follows, we rather reserve the phrase 'chain complex' for certain specific constructions of dg-modules.

The category of *graded modules* $gr\,\mathcal{M}od$ consists of $\Bbbk$-modules equipped with a decomposition $C = \bigoplus_{n\in\mathbb{Z}} C_n$, which ranges over $\mathbb{Z}$, but where we have no differential. We obviously define a morphism of graded modules as a morphism of $\Bbbk$-modules $f : C \to D$ which satisfies the relation $f(C_n) \subset D_n$, for all $n \in \mathbb{Z}$.

We have an obvious functor $(-)_\flat : dg\,\mathcal{M}od \to gr\,\mathcal{M}od$ defined by retaining the underlying graded structure of a dg-module and by forgetting about the differential. We notably consider the underlying graded module of dg-modules, which this forgetful process formalizes, when we define the notion of a quasi-free object (in the category of commutative algebras, in the category of operads, ...). The other way round, we can embed the category of graded modules $gr\,\mathcal{M}od$ into the category of dg-modules $dg\,\mathcal{M}od$ by viewing a graded module as a dg-module equipped with a trivial differential $\delta = 0$. We use this identification at various places.

Recall that the homology of a dg-module C is defined by the quotient $\Bbbk$-module $\mathtt{H}_*(C) = \ker\delta/\operatorname{im}\delta$ which inherits a natural grading from C. The homology defines a functor $\mathtt{H}_*(-) : dg\,\mathcal{M}od \to gr\,\mathcal{M}od$. In most references of homological algebra, authors use the phrase '*quasi-isomorphism*' for the class of morphisms of dg-modules which induce an isomorphism in homology. In what follows, we rather use the name '*weak-equivalence*' which we borrow from the general formalism of model categories (see §II.1 for this notion).

We generally use the mark $\xrightarrow{\sim}$ to refer to the class of weak-equivalences in a model category (see §II.1) and we naturally use the same notation in the dg-module context. We mostly use the notions introduced in this paragraph in the second part of this book and we review these definitions with full details in §II.5.

0.2. *Degrees and signs of dg-algebra.* The component C_n of a dg-module (respectively, graded module) C defines the homogeneous component of degree n of C. To specify the degree of a homogeneous element $x \in C_n$, we use the expression $\deg(x) = n$. We adopt the standard convention of dg-algebra to associate a sign $(-1)^{\deg(x)\deg(y)}$ to each transposition of homogeneous elements (x, y). We do not specify such a sign in general and we simply use the notation ± to refer to it. We will see that the introduction of these signs is forced by the definition of the symmetry isomorphism of the tensor product of dg-modules (see §II.5.2).

We usually consider lower graded dg-modules, but we also have a standard notion of dg-module equipped with a decomposition in upper graded components $C = \bigoplus_{n\in\mathbb{Z}} C^n$ such that the differential satisfies $\delta(C^n) \subset C^{n+1}$. Certain constructions (like the duality of $\Bbbk$-modules and the conormalized complex of cosimplicial spaces) naturally produce upper graded dg-modules. In what follows, we apply the relation $C_{-n} = C^n$ to identify an upper graded with a lower graded dg-module. We also review these concepts in §II.5.

0.3. *Simplicial objects and cosimplicial objects in a category.* The *simplicial category* Δ, which models the structure of simplicial and cosimplicial objects in a category, is defined by the collection of finite ordinals $\underline{n} = \{0 < \cdots < n\}$ as objects together with the non-decreasing maps between finite ordinals $u : \{0 < \cdots < m\} \to \{0 < \cdots < n\}$ as morphisms. We define a *simplicial object* X in a category $\mathcal{C}$ as a contravariant functor $X : \Delta^{op} \to \mathcal{C}$ which assigns an object $X_n \in \mathcal{C}$ to each $n \in \mathbb{N}$ and a morphism $u^* : X_n \to X_m$ to each non-decreasing map $u : \{0 < \cdots < m\} \to \{0 < \cdots < n\}$. We similarly define a *cosimplicial object* in $\mathcal{C}$ as a covariant functor $X : \Delta \to \mathcal{C}$ which assigns an object $X^n \in \mathcal{C}$ to each $n \in \mathbb{N}$ and a morphism $u_* : X^m \to X^n$ to each non-decreasing map $u : \{0 < \cdots < m\} \to \{0 < \cdots < n\}$. Naturally, we define a morphism of simplicial objects $f : X \to Y$ (and a morphism of cosimplicial object similarly) as a sequence of morphisms $f : X_n \to Y_n$ in the ambient category $\mathcal{C}$ which intertwine the action of the simplicial operators u^* on our objects X and Y.

We use the notation $s\,\mathcal{C}$ for the category of simplicial objects in a given ambient category $\mathcal{C}$ and the notation $c\,\mathcal{C}$ for the category of cosimplicial objects. The category of *simplicial sets $s\mathcal{S}et$*, for instance, formally consists of the simplicial objects in the category of sets $X : \Delta^{op} \to \mathcal{S}et$.

The *simplices* Δ^n, $n \in \mathbb{N}$, are the fundamental examples of simplicial sets which are given by the representable functors $\mathtt{Mor}_{\Delta}(-, \underline{n}) : \Delta^{op} \to \mathcal{S}et$, where we use the notation $\mathtt{Mor}_{\Delta}(\underline{m}, \underline{n})$ to refer to the morphism sets of the simplicial category Δ. The collection of the n-simplices Δ^n, $n \in \mathbb{N}$, forms a cosimplicial object in the category of simplicial sets itself, with the covariant action of non-decreasing maps $u_* : \Delta^m \to \Delta^n$ defined by the composition on the target in these morphism sets $\Delta^n = \mathtt{Mor}_{\Delta}(-, \underline{n})$.

In the case of a simplicial set X, an element $\sigma \in X_n$ is called an *n-dimensional simplex* (or more simply an *n-simplex*) in X. The definition of the n-simplex Δ^n as a representable functor $\Delta^n = \mathtt{Mor}_{\Delta}(-, \underline{n})$ implies that we have the relation $\mathtt{Mor}_{s\mathcal{S}et}(\Delta^n, X) = X_n$, for any simplicial set $X \in s\mathcal{S}et$, where we use the notation $\mathtt{Mor}_{s\mathcal{S}et}(-,-)$ for the morphism set of the category $s\mathcal{S}et$. To make this correspondence explicit, we consider the n-simplex, denoted by $\iota_n \in (\Delta^n)_n$, which corresponds to the identity of the object $\underline{n}$ in the simplicial category Δ. The morphism $\sigma_* : \Delta^n \to X$, associated to any n-simplex $\sigma \in X_n$, is characterized by the relation $\sigma_*(\iota_n) = \sigma$.

The topological simplices $\mathbb{A}^n = \{(t_0, \dots, t_n) | 0 \leq t_i \leq 1,\, t_0 + \cdots + t_n = 1\}$ form another fundamental instance of a cosimplicial object which is defined in the category of topological spaces. The cosimplicial structure map $u_* : \mathbb{A}^m \to \mathbb{A}^n$ associated to a morphism of the simplicial category $u \in \mathtt{Mor}_{\Delta}(\underline{m}, \underline{n})$ sends any element $(s_0, \dots, s_m) \in \mathbb{A}^m$ to the point $(t_0, \dots, t_n) \in \mathbb{A}^n$ such that $t_i = \sum_{u(k)=i} s_k$.

We mainly use simplicial objects and cosimplicial objects in the second volume of this book and we go back to the definitions of this paragraph in §§II.1-3. We also study simplicial and cosimplicial modules (simplicial and cosimplicial objects in module categories) in depth in §II.5.

0.4. *Faces and degeneracies in a simplicial object.* The maps $d^i : \{0 < \cdots < n-1\} \to \{0 < \cdots < n\}$, $i = 0, \dots, n$, such that

$$(1) \qquad d^i(x) = \begin{cases} x, & \text{for } x < i, \\ x+1, & \text{for } x \geq i, \end{cases}$$

and the maps $s^j : \{0 < \cdots < n+1\} \to \{0 < \cdots < n\}$, $j = 0, \dots, n$, such that

$$s^j(x) = \begin{cases} x, & \text{for } x \leq j, \\ x-1, & \text{for } x > j, \end{cases} \tag{2}$$

generate the simplicial category in the sense that any non-decreasing map $u : \{0 < \cdots < m\} \to \{0 < \cdots < n\}$ can be written as a composite of maps of that form. Moreover, any relation between these generating morphisms can be deduced from the following generating relations:

$$\begin{aligned} & d^j d^i = d^i d^{j-1}, \text{ for } i < j, \\ & s^j d^i = \begin{cases} d^i s^{j-1}, & \text{for } i < j, \\ id, & \text{for } i = j, j+1, \\ d^{i-1} s^j, & \text{for } i > j+1, \end{cases} \\ & s^j s^i = s^i s^{j+1}, \text{ for } i \leq j. \end{aligned} \tag{3}$$

The structure of a cosimplicial object is, as a consequence, fully determined by a sequence of objects $X^n \in \mathcal{C}$ together with morphisms $d^i : X^{n-1} \to X^n$, $i = 0, \dots, n$, and $s^j : X^{n+1} \to X^n$, $j = 0, \dots, n$, for which these relations (3) hold. The morphisms $d^i : X^{n-1} \to X^n$, $i = 0, \dots, n$, which represent the image of the maps d^i under the functor defined by X, are the *coface operators* of the cosimplicial object X (we may also speak about the *cofaces* of X for short). The morphisms $s^j : X^{n+1} \to X^n$, $j = 0, \dots, n$, which represent the image of the maps s^j are the *codegeneracy operators* of X (or, more simply, the *codegeneracies* of X).

Dually, the structure of a simplicial object is fully determined by a sequence of objects $X_n \in \mathcal{C}$ together with morphisms $d_i : X_n \to X_{n-1}$, $i = 0, \dots, n$, and $s_j : X_n \to X_{n+1}$, $j = 0, \dots, n$, for which relations

$$\begin{aligned} & d_i d_j = d_{j-1} d_i, \text{ for } i < j, \\ & d_i s_j = \begin{cases} s_{j-1} d_i, & \text{for } i < j, \\ id, & \text{for } i = j, j+1, \\ s_j d_{i-1}, & \text{for } i > j+1, \end{cases} \\ & s_i s_j = s_{j+1} s_i, \text{ for } i \leq j, \end{aligned} \tag{4}$$

opposite to (3), hold. The morphisms $d_i : X_n \to X_{n-1}$, $i = 0, \dots, n$, which represent the image of the maps d^i under the contravariant functor defined by X, are the *face operators* of the simplicial object X, and the morphisms $s_j : X_n \to X_{n+1}$, $j = 0, \dots, n$, which represent the image of the maps s^j, are the *degeneracy operators* of X. We also recall the definition of these operators in the course of our study (in §§II.1-3).

0.5. *Geometric realization of simplicial sets and singular complex of topological spaces.* Recall that a topological space $|K|$, traditionally called the geometric realization of K, is naturally associated to each simplicial set $K \in s\mathcal{S}et$. This space is defined by the coend

$$|K| = \int^{\underline{n} \in \Delta} K_n \times \mathbb{\Delta}^n .$$

where each set K_n is viewed as a discrete space and we consider the topological n-simplices $\mathbb{\Delta}^n$ (of which definition is recalled in §0.3). The coend which we consider in this construction is equivalent to the quotient object of the coproduct $\coprod_n K_n \times \mathbb{A}^n = \coprod_n \{\coprod_{\sigma \in K_n} \{\sigma\} \times \mathbb{A}^n\}$ under the relations

$$(u^*(\sigma), (t_0, \dots, t_m)) \equiv (\sigma, u_*(t_0, \dots, t_m)),$$

for $u \in \mathtt{Mor}_\Delta(\underline{m}, \underline{n})$, $\sigma \in K_n$, and $(t_0, \dots, t_m) \in \mathbb{A}^m$. The definition of the map $u_* : \mathbb{A}^m \to \mathbb{A}^n$ associated to each $u \in \mathtt{Mor}_\Delta(\underline{m}, \underline{n})$ involves the cosimplicial structure of the topological n-simplices $\mathbb{A}^n$. One easily checks that the realization of the n-simplex $\Delta^n = \mathtt{Mor}_\Delta(-, \underline{n})$ is identified with the topological n-simplex $\mathbb{A}^n$.

In the converse direction, we can use the singular complex construction to associate a simplicial set $\mathtt{Sing}_\bullet(X)$ to any topological space X. This simplicial set $\mathtt{Sing}_\bullet(X)$ consists in dimension n of the set of continuous maps $\sigma : \mathbb{A}^n \to X$ going from the topological n-simplex $\mathbb{A}^n$ to X. The composition of simplices $\sigma : \mathbb{A}^n \to X$ with the cosimplicial operator $u_* : \mathbb{A}^m \to \mathbb{A}^n$ associated to any $u \in \mathtt{Mor}_\Delta(\underline{m}, \underline{n})$ yields a map $u^* : \mathtt{Sing}_n(X) \to \mathtt{Sing}_m(X)$ so that the collection of sets $\mathtt{Sing}_n(X) = \mathtt{Mor}_{\mathcal{T}op}(\mathbb{A}^n, X)$, $n \in \mathbb{N}$, inherits a natural simplicial structure.

The geometric realization obviously gives a functor $| - | : s\mathcal{S}et \to \mathcal{T}op$. The singular complex construction gives a functor in the converse direction $\mathtt{Sing}_\bullet : \mathcal{T}op \to s\mathcal{S}et$, which is actually a right adjoint of the geometric realization functor (see [79, §I.2]). We study generalizations of these constructions in §II.3.

0.6. *Simplicial modules, cosimplicial modules, and homology.* The category of *simplicial modules* $s\,\mathcal{M}od$ is the category of simplicial objects in the category of $\Bbbk$-modules $\mathcal{M}od$. Thus, a simplicial module K can be defined either as a contravariant functor from the simplicial category Δ to the category of $\Bbbk$-modules $\mathcal{M}od$, or, equivalently, as a collection of $\Bbbk$-modules K_n, $n \in \mathbb{N}$ equipped with face operators $d_i : K_n \to K_{n-1}$, $i = 0, \dots, n$, and degeneracy operators $s_j : K_n \to K_{n+1}$, $j = 0, \dots, n$, which satisfy the simplicial relations.

The category of *cosimplicial modules* $c\,\mathcal{M}od$ similarly consists of the cosimplicial objects in the category of $\Bbbk$-modules.

To any simplicial module K, we associate a dg-module $\mathtt{N}_*(K)$, called the *normalized complex* of K, and defined by the quotient $\mathtt{N}_n(K) = K_n / s_0 K_{n-1} + \dots + s_{n-1} K_{n-1}$ in degree n, together with the differential $\delta : \mathtt{N}_n(K) \to \mathtt{N}_{n-1}(K)$ such that $\delta = \sum_{i=0}^n (-1)^i d_i$. This normalized chain complex construction naturally gives a functor $\mathtt{N}_* : s\,\mathcal{M}od \to dg\,\mathcal{M}od$. The homology of a simplicial module K is defined as the homology of the associated normalized complex $\mathtt{N}_*(K)$. For simplicity, we use the same notation for the homology functor on simplicial modules and on dg-modules. Hence, we set $\mathtt{H}_*(K) = \mathtt{H}_*(\mathtt{N}_*(K))$, for any $K \in s\,\mathcal{M}od$. We study simplicial and cosimplicial modules in depth in §II.5 and we recall the definition of the normalized complex construction at this moment.

0.7. *Normalized complex and homology of simplicial sets.* We will consider the functor $\Bbbk[-] : s\mathcal{S}et \to s\,\mathcal{M}od$ which maps a simplicial set X to the simplicial module $\Bbbk[X]$ generated by the set X_n in dimension n, for any $n \in \mathbb{N}$, and which inherits an obvious simplicial structure. We also have a contravariant functor $\mathtt{A} : s\mathcal{S}et^{op} \to c\,\mathcal{M}od$ which maps a simplicial set X to the cosimplicial module $\mathtt{A}(X) = \Bbbk^X$, dual to $\Bbbk[X]$, and defined in dimension n by the $\Bbbk$-module of functions $u : X_n \to \Bbbk$ on the set $X_n \in \mathcal{S}et$.

We use the notation $\mathtt{N}_*(X)$ for the normalized complex of the simplicial $\Bbbk$-module $\Bbbk[X]$ associated to a simplicial set X. We retrieve the classical homology of simplicial sets by considering the homology of these simplicial modules. We also use the notation $\mathtt{H}_*(-)$ for the homology functor on simplicial sets. We accordingly have the formula $\mathtt{H}_*(X) = \mathtt{H}_*(\mathtt{N}_*(X))$, for any $X \in s\mathcal{S}et$.

The normalized complexes of the simplices Δ^n, $n \in \mathbb{N}$, naturally form a simplicial object in the category of dg-modules $\mathtt{N}_*(\Delta^\bullet)$. For a given simplicial module K, we have a coend formula

$$\mathtt{N}_*(K) = \int^{\underline{n}\in\Delta} K_n \otimes \mathtt{N}_*(\Delta^n),$$

and the normalized complex construction of §0.6 can be regarded as a dg-module version of the geometric realization of simplicial sets (we explain this idea in §5.0.11).

0.8. *Symmetric monoidal categories and the structure of base categories.* In the introduction of this chapter, we briefly mentioned that our base categories, let $\mathcal{M} = \mathcal{S}et, \mathcal{T}op, \mathcal{M}od, \dots$, are all instances of *symmetric monoidal categories.*

By definition, a symmetric monoidal category is a category $\mathcal{M}$ equipped with a tensor product $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ that satisfies natural unit, associativity and symmetry relations which we express as follows:

(a) We have a unit object $\mathbb{1} \in \mathcal{M}$ such that we have a natural isomorphism $A \otimes \mathbb{1} \simeq A \simeq \mathbb{1} \otimes A$, for $A \in \mathcal{M}$.

(b) We have a natural isomorphism $(A \otimes B) \otimes C \simeq A \otimes (B \otimes C)$, for every triple of objects $A, B, C \in \mathcal{M}$, which satisfies a pentagonal coherence relation (Mac Lane's pentagon relation) when we put together the associativity isomorphisms associated to a 4-fold tensor product, and additional triangular coherence relations with respect to the unit isomorphism when we assume that one of our objects is the unit object of our category (we refer to [130, §XI.1] for the expression of these constraints).

(c) We have a natural symmetry isomorphism $A \otimes B \simeq B \otimes A$, associated to every pair of objects $A, B \in \mathcal{M}$, which satisfies hexagonal coherence relations (Drinfeld's hexagon relation) when we apply the symmetry isomorphism to a 3-fold tensor product, and additional triangular coherence relations with respect to the unit isomorphism when we assume that one of our objects is the unit object (see again [130, §XI.1] for details).

In the case of $\mathbb{k}$-modules $\mathcal{M}od$, the monoidal structure is given by the usual tensor product of $\mathbb{k}$-modules, taken over the ground ring, together with the ground ring itself as unit object. The definition of the tensor product of dg-modules, simplicial modules, cosimplicial modules is reviewed later on, when we tackle applications of these base categories. In the category of sets $\mathcal{S}et$ (respectively, topological spaces $\mathcal{T}op$, simplicial sets $s\mathcal{S}et$), the tensor product is simply given by the cartesian product $\otimes = \times$ together with the one-point set $\mathbb{1} = pt$ as unit object. In what follows, we also use the general notation $*$ for the terminal object of a category, and we may write $pt = *$ when we want to stress that the one point-set actually represents the terminal object of the category of sets (respectively, topological spaces, simplicial sets).

The unit object and the isomorphisms that come with the unit, associativity and commutativity relations of a symmetric monoidal category are part of the structure. Therefore, these morphisms have, in principle, to be given with the definition. But, in our examples, we can assume that the unit and associativity relations are identities, and in general, we just make explicit the definition of the symmetry isomorphism $c = c(A, B) : A \otimes B \xrightarrow{\simeq} B \otimes A$.

We make explicit the coherence constraints for the unit, associativity, and symmetry isomorphisms of symmetric monoidal categories in §I.6 (we use a braided analogue of the structure of a symmetric monoidal category in our definition of the Grothendieck–Teichmüller group). We also review the definition of several notions of structure preserving functors between symmetric monoidal categories in §I.3.

0.9. *Tensor products and colimits.* In many constructions, we consider symmetric monoidal categories $\mathcal{M}$ equipped with colimits and limits and whose the tensor product *distributes over colimits* in the sense that:

(a) The canonical morphism $\operatorname{colim}_{\alpha\in\mathcal{I}}(A_\alpha \otimes B) \to (\operatorname{colim}_{\alpha\in\mathcal{I}} A_\alpha)\otimes B$ associated to a diagram $A_\alpha \in \mathcal{M}$, $\alpha \in \mathcal{I}$, is an isomorphism for all $B \in \mathcal{M}$, and similarly as regards the canonical morphism $\operatorname{colim}_{\beta\in\mathcal{J}}(A \otimes B_\beta) \to A \otimes (\operatorname{colim}_{\beta\in\mathcal{J}} B_\beta)$ associated to a diagram $B_\beta \in \mathcal{M}$, $\beta \in \mathcal{J}$, where we now fix the object $A \in \mathcal{M}$.

This requirement is fulfilled by all categories which we take as base symmetric monoidal categories $\mathcal{M} = \mathcal{S}et, \mathcal{T}op, \mathcal{M}od, \dots$ and is required for the application of categorical constructions to operads and to algebras over operads. On the other hand, we will also consider instances of symmetric monoidal categories which do not satisfy this distribution relation. One simple example is given by taking the direct sum $\oplus : \mathcal{M}od \times \mathcal{M}od \to \mathcal{M}od$ (instead of the ordinary tensor product) as the tensor product operation of a symmetric monoidal structure on the category of $\mathbb{k}$-modules. We use this additive monoidal structure when we study a counterpart of the Postnikov decomposition of spaces in the category of operads.

0.10. *Symmetric groups and tensor permutations.* We use the notation Σ_r for the group of permutations of $\{1,\dots,r\}$. Depending on the context, we regard a permutation $s \in \Sigma_r$ as a bijection $s : \{1,\dots,r\} \to \{1,\dots,r\}$ or as a sequence $s = (s(1),\dots,s(r))$ equivalent to an ordering of the set $\{1,\dots,r\}$. In any case, we will use the notation $id = id_r$ for the identity permutation in Σ_r. We omit the subscript r which indicates the cardinal of our permutation when we do not need to specify this information.

In a symmetric monoidal category equipped with a strictly associative tensor product, we can form r-fold tensor products $T = X_1 \otimes \cdots \otimes X_r$ without care and omit unnecessary bracketing. Then we also have a natural isomorphism

$$X_1 \otimes \cdots \otimes X_r \xrightarrow{s^*} X_{s(1)} \otimes \cdots \otimes X_{s(r)},$$

associated to each permutation $s \in \Sigma_r$, and such that the standard unit and associativity relations $id^* = id$ and $t^* s^* = (st)^*$ hold. To construct this action, we use the classical presentation of Σ_r with the transpositions $t_i = (i\ i+1)$ as generating elements and the identities

$$t_i^2 = id, \quad \text{for } i = 1,\dots,r-1, \tag{1}$$

$$t_i t_j = t_j t_i, \quad \text{for } i,j = 1,\dots,r-1, \text{ with } |i-j| \geq 2, \tag{2}$$

$$t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}, \quad \text{for } i = 1,\dots,r-2, \tag{3}$$

as generating relations. We assign the morphism

$$X_1 \otimes \cdots \otimes X_i \otimes X_{i+1} \otimes \cdots \otimes X_r \xrightarrow{\simeq} X_1 \otimes \cdots \otimes X_{i+1} \otimes X_i \otimes \cdots \otimes X_r,$$

induced by the symmetry isomorphism $c(X_i, X_{i+1}) : X_i \otimes X_{i+1} \xrightarrow{\simeq} X_{i+1} \otimes X_i$, to the transposition $t_i = (i\ i+1)$. The axioms of symmetric monoidal categories imply that these morphisms satisfy the relations (1-3) attached to the elementary transpositions in Σ_r. Hence, we can use the presentation of Σ_r to coherently extend the action of the transpositions $t_i \in \Sigma_r$ on tensor powers to the whole symmetric group.

0.11. *Tensor products over arbitrary finite sets.* In our constructions, we often deal with tensor products $\bigotimes_{i\in\underline{r}} X_i$ that range over an arbitrary set $\underline{r} = \{i_1, \dots, i_r\}$ (not necessarily equipped with a canonical ordering). In fact, we effectively realize such a tensor product $\bigotimes_{i\in\underline{r}} X_i$ as an ordered tensor product $X_{u(1)} \otimes \cdots \otimes X_{u(r)}$, which we associate to the choice of a bijection $u : \{1 < \cdots < r\} \xrightarrow{\simeq} \underline{r}$. The tensor products associated to different bijection choices $u, v : \{1 < \cdots < r\} \xrightarrow{\simeq} \underline{r}$ differ by a canonical isomorphism $s^* : X_{u(1)} \otimes \cdots \otimes X_{u(r)} \xrightarrow{\simeq} X_{v(1)} \otimes \cdots \otimes X_{v(r)}$ which we determine from the permutation $s \in \Sigma_r$ such that $v = us$ by using the just defined action of symmetric groups on tensors.

In principle, the tensor product $\bigotimes_{i_k\in\underline{r}} X_{i_k}$ is only defined up to these canonical isomorphisms. However, we can adapt the general Kan extension process to make this construction more rigid. Formally, we define the unordered tensor product as the colimit $\bigotimes_{i_k\in\underline{r}} X_{i_k} = \operatorname{colim}_{u:\{1<\cdots<r\}\xrightarrow{\simeq}\underline{r}} X_{u(1)} \otimes \cdots \otimes X_{u(r)}$ that ranges over the category formed by the bijections $u : \{1 < \cdots < r\} \xrightarrow{\simeq} \underline{r}$ as objects and the permutations $s \in \Sigma_r$ such that $v = us$ as morphisms. The colimit process automatically performs the identification of the tensors associated to different bijection choices.

This construction can be regarded as an instance of a Kan extension process which we will apply to structures, called symmetric sequences, underlying operads (see §I.2.5).

0.12. *Enriched category structure of base categories.* The morphism sets of a category $\mathcal{C}$ will always be denoted by $\mathtt{Mor}_{\mathcal{C}}(A, B)$. But many categories which we consider come equipped with a hom-bifunctor $\mathtt{Hom}_{\mathcal{C}}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{M}$ with values in one of our base symmetric monoidal categories $\mathcal{M} = \mathcal{S}et, \mathcal{T}op, \mathcal{M}od, \dots$, and which provides $\mathcal{C}$ with an *enriched category structure.*

The structure of an enriched category includes operations that extend the classical composition operations attached to the morphism sets of ordinary categories. In the usual setting, the units of the composition are given by identity morphisms $id_A \in \mathtt{Mor}_{\mathcal{C}}(A, A)$ associated to all objects $A \in \mathcal{C}$. In the case of an enriched category, the units of the composition are morphisms

$$id_A : \mathbb{1} \to \mathtt{Hom}_{\mathcal{C}}(A, A), \tag{1}$$

given for all objects $A \in \mathcal{C}$, and defined on the tensor unit of the base category $\mathbb{1}$. The composition products are morphisms

$$\circ : \mathtt{Hom}_{\mathcal{C}}(B, C) \otimes \mathtt{Hom}_{\mathcal{C}}(A, B) \to \mathtt{Hom}_{\mathcal{C}}(A, C), \tag{2}$$

given for all $A, B, C \in \mathcal{C}$, and where we consider the tensor product of hom-objects in the base category instead of the cartesian product of morphism sets. These composition products are assumed to satisfy obvious analogues, now expressed in terms of commutative diagrams, of the unit and associativity relations of the composition in ordinary categories. Each of our base categories $\mathcal{M} = \mathcal{S}et, \mathcal{T}op, \mathcal{M}od, \dots$ is enriched over itself. In the case of sets $\mathcal{S}et$, we trivially take $\mathtt{Hom}_{\mathcal{S}et}(-,-) = \mathtt{Mor}_{\mathcal{S}et}(-,-)$. In the case of topological spaces $\mathcal{T}op$, the hom-objects $\mathtt{Hom}_{\mathcal{T}op}(A, B)$ are given by the morphism sets $\mathtt{Mor}_{\mathcal{T}op}(A, B)$ equipped with the usual compact-open topology. In the case of modules $\mathcal{M}od$, the hom-objects $\mathtt{Hom}_{\mathcal{M}od}(A, B)$ are similarly given by the morphism sets of the category $\mathtt{Hom}_{\mathcal{M}od}(A, B) = \mathtt{Mor}_{\mathcal{M}od}(A, B)$, which come naturally equipped with a module structure (the usual one). In our remaining fundamental examples $\mathcal{M} = s\mathcal{S}et, dg\,\mathcal{M}od, \dots$, the hom-objects $\mathtt{Hom}_{\mathcal{M}}(A, B)$ consist of maps which are given by an extension of the definition of the morphisms of our category $\mathcal{M}$. (We give the explicit definition of these hom-objects later on, when we begin to use these categories.)

In all these examples, we actually take hom-objects which fit an adjunction relation with respect to the symmetric monoidal structure (authors say that our base categories are instances of *closed monoidal categories*). We review this connection in a next paragraph.

0.13. *The general notion of an enriched category, morphisms and homomorphisms.* In what follows, we actually use enriched categories both as a natural framework to perform constructions on objects and as examples of structured objects. The base categories $\mathcal{M} = \mathcal{S}et, \mathcal{M}od, s\mathcal{S}et, dg\,\mathcal{M}od, \dots$ correspond to the first usage of enriched category structures, while the Hopf categories, which we consider in our definition of the Malcev completion of groupoids (see §I.9.0), correspond to the second form of applications of enriched categories.

In the first case, an enriched category structure is often given as an extra structure associated with an ordinary category $\mathcal{C}$. Then we deal with both morphism sets $\mathtt{Mor}_{\mathcal{C}}(-,-)$ and with hom-objects $\mathtt{Hom}_{\mathcal{C}}(-,-)$ with values in a given symmetric monoidal category $\mathcal{M}$ (not necessarily a base category). We say that our category $\mathcal{C}$ is *enriched over* $\mathcal{M}$ when we need to specify this category where our hom-objects are defined. We assume that the hom-objects are equipped with unit and composition morphisms §0.12(1-2) formed within our symmetric monoidal category $\mathcal{M}$.

In this context, where enriched categories arise as extra-structures associated with an underlying ordinary category $\mathcal{C}$, we also naturally assume that the hom-objects form a bifunctor $\mathtt{Hom}_{\mathcal{C}}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{M}$ so that we have morphisms

$$(1) \qquad f_* : \mathtt{Hom}_{\mathcal{C}}(-, A) \to \mathtt{Hom}_{\mathcal{C}}(-, B) \quad \text{and} \quad f^* : \mathtt{Hom}_{\mathcal{C}}(B, -) \to \mathtt{Hom}_{\mathcal{C}}(A, -),$$

for every $f \in \mathtt{Mor}_{\mathcal{C}}(A, B)$. The unit morphisms and the composition operations §0.12(1-2) have to be invariant under these actions of morphisms on hom-objects.

In our basic examples, where hom-objects are made from point-sets, we can identify the actual morphisms of the category $f \in \mathtt{Mor}_{\mathcal{C}}(A, B)$ with particular elements of the hom-objects $\mathtt{Hom}_{\mathcal{C}}(A, B)$. The general elements $u \in \mathtt{Hom}_{\mathcal{C}}(A, B)$ are conversely identified with maps $u : A \to B$ which satisfy some mild requirements, and these hom-objects $\mathtt{Hom}_{\mathcal{C}}(A, B)$ are generally given by an extension of the morphism sets of our category $\mathtt{Mor}_{\mathcal{C}}(A, B)$. In this setting, we use the name '*homomorphism*' to refer to the general elements of the hom-objects $\mathtt{Hom}_{\mathcal{C}}(A, B)$ as opposed to the '*morphisms*', which refer to the elements of the morphism sets $\mathtt{Mor}_{\mathcal{C}}(A, B)$. We may however use the arrow notation $u : A \to B$ when we want to regard such a homomorphism $u \in \mathtt{Hom}_{\mathcal{C}}(A, B)$ as a map. In this case, the belonging category of the arrow u is specified by the context. The composition on hom-objects also usually extends the composition on morphisms, and the morphisms (1), which make the hom-objects into a bifunctor, are generally identified with the left (respectively, right) composition with the homomorphism which we associate to any morphism $f \in \mathtt{Mor}_{\mathcal{C}}(A, B)$.

In a general setting, we can define a correspondence between morphisms and homomorphisms by using a natural transformation

$$(2) \qquad \iota_{\sharp} : \mathbb{1}[\mathtt{Mor}_{\mathcal{C}}(A, B)] \to \mathtt{Hom}_{\mathcal{C}}(A, B),$$

where the expression $\mathbb{1}[\mathtt{Mor}_{\mathcal{C}}(A,B)]$ denotes the coproduct, ranging over the set of morphisms $f \in \mathtt{Mor}_{\mathcal{C}}(A,B)$, of copies of the unit object $\mathbb{1}$.

0.14. *Closed symmetric monoidal categories.* In the case of our base categories $\mathcal{M} = \mathcal{S}et, \mathcal{T}op, \mathcal{M}od, \dots$, we actually take hom-bifunctors that fit in an adjunction relation $\mathtt{Mor}_{\mathcal{M}}(A \otimes B, C) \simeq \mathtt{Mor}_{\mathcal{M}}(A, \mathtt{Hom}_{\mathcal{M}}(B,C))$ with respect to the symmetric monoidal structure of the category[1]. The bijection which gives this adjunction relation is also assumed to be natural in $A, B, C \in \mathcal{M}$.

We generally say that a symmetric monoidal category $\mathcal{M}$ is *closed* when the tensor product $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ has a right adjoint $\mathtt{Hom}_{\mathcal{M}}(-,-) : \mathcal{M}^{op} \times \mathcal{M} \to \mathcal{M}$ so that we have an isomorphism

$$\mathtt{Mor}_{\mathcal{M}}(A \otimes B, C) \simeq \mathtt{Mor}_{\mathcal{M}}(A, \mathtt{Hom}_{\mathcal{M}}(B,C)), \tag{1}$$

for any $A, B, C \in \mathcal{M}$. Note that the existence of this adjoint implies that our tensor product distributes over colimits as we require in §0.9. The other way round, the hom-object bifunctor $\mathtt{Hom}_{\mathcal{M}}(-,-)$, which we define by an adjunction relation of this form (1), automatically satisfies the same distribution relations with respect to colimits and limits as the general morphism set bifunctor of our base category. Namely, we have the identity $\mathtt{Hom}_{\mathcal{M}}(\operatorname{colim}_\alpha A_\alpha, B) = \lim_\alpha \mathtt{Hom}_{\mathcal{M}}(A_\alpha, B)$, when we take a colimit on the source of our hom-object $A = \operatorname{colim}_\alpha A_\alpha$, and the identity $\mathtt{Hom}_{\mathcal{M}}(A, \lim_\beta B_\beta) = \lim_\beta \mathtt{Hom}_{\mathcal{M}}(A, B_\beta)$, when we consider a limit on the target $B = \lim_\beta B_\beta$.

The hom-objects $\mathtt{Hom}_{\mathcal{M}}(A,B)$ defined by an internal hom-functor naturally inherit an evaluation morphism

$$\epsilon : \mathtt{Hom}_{\mathcal{M}}(A,B) \otimes A \to B \tag{2}$$

which represents the augmentation of the adjunction (1) and which generalizes the usual evaluation of maps in the category of sets. The unit of our adjunction is given by a morphism

$$\iota : A \to \mathtt{Hom}_{\mathcal{M}}(B, A \otimes B) \tag{3}$$

associated to each pair of objects in our category $A, B \in \mathcal{M}$.

The hom-objects of a closed symmetric monoidal category automatically inherit composition units $id_A : \mathbb{1} \to \mathtt{Hom}_{\mathcal{C}}(A,A)$, given by the right adjoint of the unit isomorphisms $\mathbb{1} \otimes A \xrightarrow{\simeq} A$ of the symmetric monoidal structure, as well as composition operations $\circ : \mathtt{Hom}_{\mathcal{C}}(B,C) \otimes \mathtt{Hom}_{\mathcal{C}}(A,B) \to \mathtt{Hom}_{\mathcal{C}}(A,C)$, given by the right adjoint of the composite evaluation morphisms $\mathtt{Hom}_{\mathcal{M}}(B,C) \otimes \mathtt{Hom}_{\mathcal{M}}(A,B) \otimes A \xrightarrow{id \otimes \epsilon} \mathtt{Hom}_{\mathcal{M}}(B,C) \otimes B \xrightarrow{\epsilon} C$. Thus, any closed symmetric monoidal category is automatically enriched in the sense of §0.12.

Besides, we have tensor product operations $\mathtt{Hom}_{\mathcal{M}}(A,B) \otimes \mathtt{Hom}_{\mathcal{M}}(C,D) \xrightarrow{\otimes} \mathtt{Hom}_{\mathcal{M}}(A \otimes C, B \otimes D)$, which are given by the right adjoint of the composites $\mathtt{Hom}_{\mathcal{M}}(A,B) \otimes \mathtt{Hom}_{\mathcal{M}}(C,D) \otimes A \otimes C \simeq \mathtt{Hom}_{\mathcal{M}}(A,B) \otimes A \otimes \mathtt{Hom}_{\mathcal{M}}(C,D) \otimes C \xrightarrow{\epsilon \otimes \epsilon} B \otimes D$ where we apply the symmetry operator of $\mathcal{M}$ and we form the tensor product of the evaluation morphisms associated to the hom-objects. This tensor product operation gives an extension, at the level of enriched hom-objects, of the tensor product of morphisms and satisfies the same unit, associativity, and symmetry relations.

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[1] In the case $\mathcal{M} = \mathcal{T}op$, we just need to restrict ourselves to a good category of topological spaces, such as the usual category of compactly generated spaces, in order to ensure that such an adjunction relation holds (see for instance [130, §VII.8] for an overview of this subject).

The existence of the hom-object bifunctor $\mathtt{Hom}_{\mathcal{M}}(-,-)$ is notably useful for the study of algebras over operads and we give a short account of these applications in the first chapter of this book §I.1. In this book, we mostly study operad themselves and we mainly deal with the usual internal hom-objects associated to our fundamental examples of base model categories. To be specific, we use simplicial hom-objects, which we deduce from the standard internal hom-objects of the category of simplicial sets, when we study mapping spaces of operads in the second part of this work and we use hom-objects of dg-modules as an auxiliary device to compute the homotopy of these mapping spaces on the category of operads.

0.15. *Concrete (symmetric monoidal) categories.* Recall that a category $\mathcal{C}$ is *concrete* when we have a faithful functor $U : \mathcal{C} \to \mathcal{S}et$ from this category $\mathcal{C}$ to the category of sets $\mathcal{S}et$. Most usual categories, which are defined in terms of point sets equipped with extra structures, are naturally equipped with the structure of a concrete category.

In what follows, we will say that a base symmetric monoidal category $\mathcal{M}$ is *concrete (as a symmetric monoidal category)* when such a faithful functor $U : \mathcal{M} \to \mathcal{S}et$ is given by the representable functor $U = \mathtt{Mor}_{\mathcal{M}}(\mathbb{1},-)$ associated to the unit object of our category $\mathbb{1} \in \mathcal{M}$. The category of sets $\mathcal{M} = \mathcal{S}et$, the category of topological spaces $\mathcal{M} = \mathcal{T}op$, the category of modules $\mathcal{M} = \mathcal{M}od$, ... are examples of concrete symmetric monoidal categories. In this situation, we regard the morphism set $U(X) = \mathtt{Mor}_{\mathcal{M}}(\mathbb{1}, X)$ as a set of points, which we faithfully associate to any object of our category $X \in \mathcal{C}$, and we have a natural pointwise tensor product operation $x \otimes y \in U(X \otimes Y)$, which we define by the obvious composition operation $\mathbb{1} \xrightarrow{\simeq} \mathbb{1} \otimes \mathbb{1} \xrightarrow{x \otimes y} X \otimes Y$, for any $x \in U(X)$ and $y \in U(Y)$, where we consider the unit isomorphism $\mathbb{1} \simeq \mathbb{1} \otimes \mathbb{1}$ of our category $\mathcal{M}$. We just have $x \otimes y = (x, y) \in X \times Y$ in the case $\mathcal{M} = \mathcal{S}et, \mathcal{T}op$, and we retrieve the standard notion of a tensor product of elements in the case $\mathcal{M} = \mathcal{M}od$.

We mainly use the concept of a concrete symmetric monoidal category informally, in order to give a sense to set-theoretic tensor products which we define to illustrate some constructions of the theory. Let us mention that we can still form such set-theoretic tensor products (with some restriction) in the category of dg-modules, in the category of graded modules, in the category of simplicial modules and in the category of cosimplicial modules though these categories do not form concrete symmetric monoidal categories in the sense of our definition.

0.16. *The notation of colimits, limits and universal objects.* We adopt the following conventions for the notation of colimits, limits, and universal objects in categories. We generally use the unbased set notation $\varnothing$ for the initial object of a base category, the notation $\amalg$ for coproducts, and the notation $*$ for the terminal object. In certain situations, we use the empty set notation and we write $A = \varnothing$ to assert that an object A is undefined.

We use additive category notation when we deal with additive structures, or when our base category consists of modules. We then write 0 for the initial object of the category (the zero object). We also use $\oplus$ as a generic notation for the coproduct in the additive case.

When we deal with a category of objects equipped with a multiplicative structure (algebras, operads, ...), we generally adopt the base set notation $\vee$ for the coproduct, but we do not have any general convention for the notation of the initial object in this setting. In fact, we usually keep the notation of a particular object of the base category which we use to effectively realize the initial object of our categories of structured objects. We can use a similar convention for coproducts when we can deduce the definition of this categorical operation from structure operations of our base category. For instance, we generally use the tensor product notation to refer to a coproduct of unitary commutative algebras, because we will observe in §I.3.0.3 that the coproduct is realized by the tensor product in this case.

Reading Guide and Overview of this Volume

This monograph comprises three main parts, referred to as Part I–III, which form a progression up to our ultimate mathematical goal. Part I, "From Operads to Grothendieck–Teichmüller Groups", is mainly devoted to the algebraic foundations of our subject. In Part II, "Homotopy Theory and its Applications to Operads", we develop our rational homotopy theory of operads after a comprehensive review of the applications of methods of homotopy theory. In Part III, "The Computation of Homotopy Automorphism Spaces of Operads", we work out our problem of giving a homotopy interpretation of the Grothendieck–Teichmüller group.

These parts are widely independent from each others. Each part of this book is also divided into subparts which, by themselves, form self-contained groupings of chapters, devoted to specific topics, and organized according to an internal progression of the level of the chapters each. There is a progression in the level of the parts of the book too, but the chapters are written so that a reader with a minimal background could tackle any of these subparts straight away in order to get a self-contained reference and an overview of the literature on each of the subjects addressed in this monograph.

This volume comprises the first named part of the book, "From Operads to Grothendieck-Teichmüller Groups", and two appendices, "Trees and the Construction of Free Operads" and "The Cotriple Resolution of Operads", where we revisit with full details the definition of operads in terms of composition operations shaped on trees and we explain the applications of trees to the definition of universal objects in the category of operads.

The following overview is not intended for a linear reading but should serve as a guide each time the reader tackles new parts of this volume.

Part I. From Operads to Grothendieck–Teichmüller Groups. The first part of this book includes: an introduction to the fundamental concepts of the theory of operads; a survey on the definition of the little discs operads and of E_n-operads together with a detailed study of the connections between the little 2-disc operad and braided category structures; an introduction to the theory of Hopf algebras together with a study of the applications of Hopf algebras to the Malcev completion of groups, groupoids and operads; and a detailed account of the definition of the Grothendieck-Teichmüller group from the viewpoint of the theory of algebraic operads.

Part I(a). The General Theory of Operads. We give a detailed survey of the general definitions of the theory of operads in this part. The first chapter §1 is introductory and does not contain any original idea. We mainly explain the relationship between operads and algebras. In the second chapter §2, we explain our working definition of the notion of an operad and we give a new approach to handle unitary operads (the operads equipped with a distinguished arity zero operation which can be used to model categories of algebras with unit). In the third chapter §3, we study the applications of general concepts of the theory of monoidal categories to operads.

Chapter 1. *The Basic Concepts of the Theory of Operads.* In this first chapter, we explain May's definition of the notion of an operad as an object which governs the structure defined by collections of operations $p = p(x_1, \dots, x_r)$, where $r \in \mathbb{N}$ (see §1.1). We examine the applications of usual categorical constructions to operads and we study the categories of algebras associated to operads afterwards (see §§1.2-1.3). We also recall the definition of particular instances of colimits (filtered colimits and reflexive coequalizers) which we heavily use in the theory of operads in an appendix section of this chapter (§1.4).

Chapter 2. *The Definition of Operadic Composition Structures Revisited.* The definition of an operad depends on composition schemes which we associate to operations $p = p(x_1, \dots, x_r)$. In May's definition, which we recall in §1, we consider composition products where we can plug operations $q_i = q_i(x_1, \dots, x_{n_i})$ in all inputs $i = 1, \dots, r$ of a given operation $p = p(x_1, \dots, x_r)$. This definition is perfectly suited for an introduction of the subject and for the study of algebras associated to operads. However, to work with operads themselves, we need to revisit the definition of our objects in order to get more insights into the structure of the composition products. We devote this second chapter to this subject. In a first step (§2.1), we check that the composition products of an operad, are, according to an observation of Martin Markl, fully determined by composition products on two factors (these operations are also called the partial composition products in the operad literature). In a second step (§§2.2-2.4), we explain a new approach to handle unitary operads. In short, we will see that the compositions with an extra operation of arity zero of a unitary operad can be encoded in a diagram structure associated to our object. We crucially use this observation in our study of the (rational) homotopy of operads in the second part of this book.

In general, we assume that an operad consists of a sequence of terms $\mathsf{P}(r)$, indexed by non-negative integers $r \in \mathbb{N}$, and whose elements intuitively represent operations with r inputs indexed by the ordinal $\underline{\mathsf{r}} = \{1 < \dots < r\}$. To complete the account of this chapter, we explain an extension of the definition of an operad where terms $\mathsf{P}(\underline{\mathsf{r}})$ indexed by arbitrary finite sets $\underline{\mathsf{r}} = \{i_1, \dots, i_r\}$ are considered (§2.5). In general, we can use bijections $\{1 < \dots < r\} \xrightarrow{\simeq} \{i_1, \dots, i_r\}$ to make the indexing by an arbitrary finite set $\{i_1, \dots, i_r\}$ equivalent to an indexing by an ordinal $\{1 < \dots < r\}$. Nevertheless, certain constructions on operads produce operations with no canonical input numbering and the extension of the input indexing to arbitrary finite sets becomes useful in this case. (The construction of the free operad in §A gives a motivating application of this concept.) These ideas go back to Joyal's theory of species [98] and were first applied to operads in Getzler–Jones's paper [77] and in Ginzburg–Kapranov's paper [78].

Chapter 3. *Symmetric Monoidal Categories and Operads.* The third chapter of this part is devoted to applications of the theory of symmetric monoidal categories to the study of operads. The ideas of this chapter are not original, apart when we tackle the applications of our constructions to the new model of unitary operads which we introduced in the previous chapter.

We devote a preliminary section of the chapter (§3.0) to a survey of the definition of unitary commutative algebras and of counitary cocommutative coalgebras in symmetric monoidal categories. We examine the definition of operads in general symmetric monoidal categories afterwards (§3.1). We notably study the image of operads under functors between symmetric monoidal categories. We also survey the definition of the category of Hopf operads, which we define as the category of operads in the symmetric monoidal category of counitary cocommutative coalgebras (§3.2). The Hopf cooperads, which we consider in the summary of our mathematical objectives, are the dual structures of these Hopf operads.

We have various notions of functors associated to symmetric monoidal categories. We devote an appendix section of this chapter to a survey of this subject (§3.3).

Part I(b). Braids and E_2-operads. The main purpose of this part is to recall the general definition of an E_n-operad (by using the model of the little n-discs operads) and to study the connections between E_n-operads, Artin's braid groups, and braided monoidal categories in the case $n = 2$.

Chapter 4. *The Little Discs Model of E_n-operads.* We recall the definition of the little n-discs operad and we make explicit the definition of an E_n-operad in the first section of this chapter (§4.1). We devote the next section of the chapter (§4.2) to a survey on the computation of the cohomology and of the homology of the little n-discs operad. We then give an overview of several variants of the little discs operads in an outlook section (§4.3). This chapter is mostly a survey of the literature and does not contain any original result.

We will see that the homology of an operad in topological spaces forms an operad in graded modules. In the second part of this book, we will also use that the cohomology of a topological operad with coefficients in a field inherits the dual structure of a cooperad in graded modules (when our objects satisfy mild finiteness assumptions). We also have a natural unitary commutative algebra structure on the cohomology of a space. We actually get that the cohomology of a topological operad with coefficients in a field forms a cooperad in unitary commutative algebras in graded modules (a graded Hopf cooperad) when we put this unitary commutative algebra structure and the cooperad structure together. We just make explicit the definition of the homology operad associated to the operad of little n-discs and the commutative algebra structure of the spaces of little n-discs in the second section of this chapter. (We will see that the homology of the operad of little n-discs is identified with an operad governing graded Poisson algebra structures.)

We use that the category of graded modules inherits a symmetric monoidal structure to give a sense to these notions of commutative algebras and of operads in graded modules. We devote an appendix section of this chapter to an account of our conventions on graded modules and to a survey of the definition of this symmetric monoidal structure on the category of graded modules (§4.4).

Chapter 5. *Braids and the Recognition of E_2-operads.* We tackle the study of the relationship between E_2-operads and braids in this chapter. We recall the definition of the Artin braid groups and some conventions on braids in a preliminary section of the chapter (§5.0).

We then give an account of Fiedorowicz's definition of models of E_2-operads from contractible operads endowed with an action of braid groups (§5.1). We use this approach to prove that the classifying spaces of a certain operad in groupoids, the colored braid operad, define an E_2-operad (§5.2). This operad of colored braids is closely related to the operad of parenthesized braids which we consider in the summary of our mathematical objectives. For the moment, simply say that the operad of colored braids is formed by groupoids whose morphism sets are identified with cosets of the pure braid groups inside the full Artin braid groups.

We prove in a second part of the chapter that the operad of colored braids is equivalent to the operad in groupoids formed defined by the fundamental groupoids of the underlying spaces of the little 2-discs operad (§5.3). We then use the adjunction between the fundamental groupoid and the classifying space construction in homotopy theory to give a second proof that the classifying spaces of the colored braid operad define an operad which is weakly-equivalent to the operad of little 2-discs, and hence, define a model of E_2-operad.

We can regard Fiedorowicz construction as a recognition criterion for the class of E_2-operads in topological spaces. We give an overview of more general recognition methods, which address the problem of giving an intrinsic definition of E_n-operads for all $n \geq 1$, in the concluding section of this chapter (§5.4).

Chapter 6. *The Magma and Parenthesized Braid Operads.* In the introductory chapter (§1), we recalled a general correspondence between operads and categories of algebras. In the case of an operad in the category of small categories (or groupoids), like the operad of colored braids considered in §5, the algebras are objects of the category of categories, and our operad therefore governs a class of monoidal structures which can be associated to a category. The operad of colored braids of §5 actually encodes the structure of a strict braided monoidal category, where we have a tensor product which is associative in the strict sense. The main purpose of this chapter is to explain this correspondence with full details and to give the definition of a variant of the colored braid operad, the operad of parenthesized braids, which we associate to braided monoidal categories with general associativity isomorphisms.

We first give a definition of an operad governing general monoidal categories (where the tensor product is just associative up to coherently defined isomorphisms) by elaborating on the classical Mac Lane Coherence Theorem of which we give an operadic interpretation (§6.1). We explain the definition of the operad of parenthesized braids afterwards (§6.2). To complete the account of this chapter, we also explain the definition of an operad of parenthesized symmetries which is an analogue for symmetric monoidal categories of the operad of parenthesized braids (§6.3).

Part I(c). Hopf Algebras and the Malcev Completion. In this part, we revisit the fundamental results of the theory of Hopf algebras and we study the applications of Hopf algebras to the definition of a rationalization process, the Malcev completion, which extends the classical rationalization of abelian groups to general (possibly non-abelian) groups. We then check that the Malcev completion process applies to groupoids and to operads in groupoids. We use the Malcev completion of the parenthesized braid operad in our definition of the Grothendieck–Teichmüller group (in the next part).

Chapter 7. *Hopf Algebras.* We review the foundations of the theory of Hopf algebras first and we devote this first chapter of the part to this subject. We explain the definition of a Hopf algebra in the general context of additive symmetric monoidal categories enriched in $\mathbb{Q}$-modules and we check that the main results of the theory remain valid in this framework. In what follows, we mainly apply our constructions to Hopf algebras in a category of modules over a characteristic zero field and to Hopf algebras in complete filtered modules, but our framework covers other examples of categories where Hopf algebras are usually defined in the literature (for instance, the categories of motives).

We explain the general definition of a Hopf algebra in the first section of the chapter (§7.1). We review the relationship between Hopf algebras and Lie algebras in the second section of the chapter (§7.2). We recall the definition of the enveloping algebra of a Lie algebra in the course of this study and we revisit the proof of the classical structure theorems of the theory of Hopf algebras, namely the Poincaré-Birkhoff-Witt Theorem and the Milnor-Moore Theorem. We devote the third section of chapter (§7.3) to a thorough study of the structure of Hopf algebras in complete filtered modules. We then use the phrase 'complete Hopf algebras' to refer to a subcategory of the category of Hopf algebras in complete filtered modules formed by objects which satisfy a natural connectedness condition. We notably consider this subcategory of complete Hopf algebras when we define our Malcev completion functor on groups.

Chapter 8. *The Malcev Completion for Groups.* We just examine the applications of Hopf algebras to the Malcev completion of groups in this chapter.

We first explain the definition of a general completion process on Hopf algebras. We apply this completion process to group algebras in order to get a completed group algebra functor from the category of groups to the category of complete Hopf algebras. We have a natural (complete) group-like element functor which goes the other way round, from complete Hopf algebras to groups. We precisely define the Malcev completion of a group as the group of complete group-like elements in the completed group algebra of our group. We devote the first section of the chapter to the definition of these functors (§8.1).

We also say that a group is Malcev complete when this group occurs as the image of a complete Hopf algebra under the group-like element functor. We study the category of Malcev complete groups and the properties of the Malcev completion process in the second and third sections of the chapter (§§8.2-8.3). We will notably explain that the elements of a Malcev complete group can be represented as the exponential of elements of a complete Lie algebra associated to our group. We use this correspondence to check that, in a Malcev complete group, we can define power operations g^a with exponents in an arbitrary field of coefficients $a \in \Bbbk$.

We devote the rest of the chapter to the study of the Malcev completion of free groups and of semi-direct products (§§8.4-8.5). In the course of this study, we also recall the definition of a counterpart, for Hopf algebras and for complete Hopf algebras, of the classical semi-direct product of groups.

Chapter 9. *The Malcev Completion for Groupoids and Operads.* The (complete) Hopf algebras of §§7-8 can be identified with group objects in the category of (complete) counitary cocommutative algebras. In this chapter, we introduce a generalization of this notion, which we call (complete) Hopf groupoids (see §§9.0-9.1), in order to extend the Malcev completion process of the previous chapter from groups to groupoids (§9.1). Then we check that this Malcev completion functor on groupoids preserves symmetric monoidal category structures, and as a consequence, gives rise to a Malcev completion functor on the category of operads in groupoids (§9.2). By the way, we explain the definition of an operadic version of the classical notion of a local coefficient system. We naturally get such structures, which we call 'local coefficient system operads', when we study the tower decomposition of the Malcev completion of operads in groupoids.

We devote an appendix section of the chapter (§8.5) to the study of the existence of group-like elements in the complete counitary cocommutative algebras underlying a complete Hopf groupoid. We use the results of this appendix to formulate connectedness hypothesis which we naturally need for our study of complete Hopf groupoids.

Most of the statements explained in this chapter are new, though the completion of Hopf groupoids was already considered for the definition of motivic fundamental groupoids by Deligne and Deligne–Goncharov (see [52, 54]), and this chapter could also serve as a basic reference for the algebraic background of this subject.

Part I(d). The Operadic Definition of the Grothendieck–Teichmüller Group. The main goal of this part is to explain the definition of the pro-unipotent Grothendieck-Teichmüller group as a group of automorphisms associated to a Malcev completion of the parenthesized braid operad. We devote the first chapter of the part to a preliminary study of this Malcev complete operad of parenthesized braids. By the way, we explain the definition of a related operad, the operad of chord diagrams, and we give an operadic interpretation of the notion of a Drinfeld associator, which we use as equivalences of operads in Malcev complete groupoids between the Malcev completion of the parenthesized braid operad and the chord diagram operad. We also explain the definition of a graded version of the Grothendieck–Teichmüller group as a group of automorphisms associated to a parenthesized version of the chord diagram operad. We tackle the definition of the pro-unipotent Grothendieck-Teichmüller group itself in the second chapter of the part.

The Grothendieck–Teichmüller group has a pro-finite version too, which we do not really use in this work, but which is the version to be considered for the applications to the Grothendieck proposal, where the goal is to study of the absolute Galois group through geometric actions on curves. We just give an overview of this program in the concluding chapter of this part.

Most of the ideas used in this part are known to experts, but only partial references on the operadic interpretation of the pro-unipotent Grothendieck–Teichmüller group and on the Drinfeld associators were available in the literature so far.

Chapter 10. *The Malcev Completion of the Braid Operads and Drinfeld's Associators.* The operad of parenthesized braids is an operad in groupoids whose morphism sets consist (like the morphism sets of the colored braid operad of §5) of cosets of the pure braid groups inside the full Artin braid groups. We therefore study the Malcev completion of the pure braid groups in the first section of this chapter (§10.0) before studying the Malcev completion of the parenthesized braid operad (and of the colored braid operad). We notably recall the definition of Lie algebra counterparts of the braid groups, which we call the Drinfeld–Kohno Lie algebras and which we denote by $\mathfrak{p}(r)$ (some authors call these Lie algebras the 'Lie algebras of infinitesimal braids'). We will see that, by results of Drinfeld and Kohno, the Malcev completion $\hat{P}_r$ of the pure braid group on r-strands P_r is isomorphic to the group of exponential elements which we associate to a completion $\hat{\mathfrak{p}}(r)$ of the Drinfeld–Kohno Lie algebra $\mathfrak{p}(r)$.

We study the Malcev completion of the operad of parenthesized braids (and of the colored braid operad) afterwards (in §10.1). We then explain the definition of the chord diagram operad and we revisit the definition of a Drinfeld associator (§10.2). In short, we will see that the Drinfeld–Kohno Lie algebras form an operad and the chord diagram operad is an operad in Malcev complete groups which we associate to this operad in Lie algebras.

We devote the rest of the chapter to the definition of a graded version of the Grothendieck–Teichmüller group (§10.3) and to the applications of this object for the study of natural tower decompositions of the set of Drinfeld's associators (§10.4). In the course of this study, we explain the definition of the graded Grothendieck–Teichmüller Lie algebra $\mathfrak{grt}$ which we associate to this graded Grothendieck–Teichmüller group $GRT(\Bbbk)$. We use this analysis of the tower decomposition of the set of Drinfeld's associators when we examine the outcome of the homotopy spectral sequence associated to the space of homotopy automorphisms of E_2-operads in the third part of this book.

Chapter 11. *The Grothendieck–Teichmüller Group.* We explain the operadic definition of the pro-unipotent Grothendieck–Teichmüller group in the first section of this chapter (§11.1). We precisely check that this group $GT(\Bbbk)$, such as defined by Drinfeld, can be identified with a group of operad automorphisms associated to the Malcev completion of the operad of parenthesized braids. We study a natural action of the group $GT(\Bbbk)$ on the set of Drinfeld's associators afterwards (in §11.2). We use this action to check, after Drinfeld, that the pro-unipotent Grothendieck–Teichmüller group $GT(\Bbbk)$ is isomorphic to the graded Grothendieck–Teichmüller group studied in the previous chapter $GRT(\Bbbk)$.

Then we explain the definition of a tower decomposition of the Grothendieck–Teichmüller group $GT(\Bbbk) = \lim_m GT_{\langle m\rangle}(\Bbbk)$ (§11.3). This tower decomposition is associated to a filtration of the group $GT(\Bbbk)$ by normal subgroups $\mathtt{F}_m\, GT(\Bbbk)$. We explicitly have $GT_{\langle m\rangle}(\Bbbk) = GT(\Bbbk)/\mathtt{F}_m\, GT(\Bbbk)$ for all $m \geq 0$. We actually have a pro-unipotent structure on the first layer of this filtration $GT^1(\Bbbk) = \mathtt{F}_1\, GT(\Bbbk)$ (not on the whole group $GT(\Bbbk)$), which we can also identify with the kernel of a natural character map $\lambda : GT(\Bbbk) \to \Bbbk^\times$ on the group $GT(\Bbbk)$. To complete the account of this chapter, we check that the subquotients $\mathtt{E}^0_m\, GT(\Bbbk) = \mathtt{F}_m\, GT(\Bbbk)/\mathtt{F}_{m+1}(\Bbbk)$, $m \geq 1$, form a weight graded Lie algebra which is isomorphic to the graded Grothendieck–Teichmüller Lie algebra of the previous chapter $\mathfrak{grt}$. (We address this subject in §11.4.)

Chapter 12. *A Glimpse at the Grothendieck Program.* This chapter serves as a conclusion for this volume. We provide a brief introduction to the Grothendieck program in Galois theory and we give an overview of the literature about the connections between Grothendieck–Teichmüller groups, motivic Galois groups, and multizetas.

Appendix A. Trees and the Construction of Free Operads. In this appendix, we explain the applications of trees to the definition of universal constructions in the category of operads. We make our definition of a tree precise in a preliminary section (§A.1). We explain the definition of general treewise composition operations associated to operads afterwards (in §A.2). We then explain the applications of trees to the definition of free objects in the category of operads (§§A.3-A.4) and to the definition of coproducts with free objects (§A.5).

Most of the ideas which we use in this appendix are not original, as the applications of trees for the study of operads go back to Stasheff's work on the recognition of loop spaces [167] and to Boardman–Vogt's work on homotopy invariant structures [28]. The definition of free operads in terms of trees, in particular, is due to Ginzburg–Kapranov [78]. We just give a new definition of reduced free objects in the context of unitary operads.

Appendix B. The Cotriple Resolution of Operads. In this appendix, we explain the definition of a simplicial resolution functor on the category of operads, the cotriple resolution, which we use in our study of the homotopy of operads in the next parts of this book. We restrict our analysis to the case of connected operads for simplicity. In short, we prove that the cotriple resolution of an operad has an explicit description which we obtain by inserting extra structures, modelled by chains of tree morphisms, in our previous construction of free operads. We explain the definition of our notion of a tree morphism in the first section of this appendix (§B.0) and we tackle the applications to the cotriple construction afterwards (§B.1).

The free operad functor inherits a natural composition operation which makes this object a monad on the category of symmetric sequences (underlying the category of operads). We recall the definition of the concept of a monad in §B.2, and we prove that the structure of an operad can also be defined in terms of an action of this free operad monad on a symmetric sequence. In the language of category theory, this result asserts that the category of operads is monadic.

Most of the results explained in this appendix are known to experts (like the constructions of the previous appendix). We still just give a new definition of a reduced version of the cotriple resolution for unitary operads.

Part I

From Operads
to
Grothendieck–Teichmüller Groups

Part I(a)

The General Theory of Operads

CHAPTER 1

The Basic Concepts of the Theory of Operads

The main purpose of this chapter is to explain the definition of an operad. We make this definition explicit in the first section of the chapter (§1.1). We also explain the definition of an algebra over an operad and we give some basic examples in sets to illustrate this definition. We examine the application of standard constructions of category theory (like free objects, colimits, limits) to operads and to algebras over operads in the second and third sections of the chapter (§§1.2-1.3). We check in passing that the usual categories of algebras (associative algebras, commutative algebras, Lie algebras) are identified with categories of algebras associated to operads. We also devote an appendix section (§1.4) to a short survey of the definition of particular colimits (reflexive coequalizers and filtered colimits) which we use in our applications of category theory constructions to operads.

The basic definition of an operad, given in the next section, makes sense in the general setting of a symmetric monoidal category $\mathcal{M}$, where we only assume the existence of a tensor product $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ that satisfies the unit, associativity and symmetry axioms of §0.8. Nonetheless, we need the additional requirement that the tensor product distributes over colimits (see §0.9) when we tackle the definition of free objects in the category of operads. This distribution relation with respect to colimits is also needed to perform categorical constructions in the category of operads and in categories of algebras over operads. The existence of an internal-hom bifunctor $\mathtt{Hom}_{\mathcal{M}}(-,-) : \mathcal{M}^{op} \times \mathcal{M} \to \mathcal{M}$ which provides the base category $\mathcal{M}$ with a closed symmetric monoidal category structure (see §§0.8-0.14) is also useful for the study of algebras over operads. To simplify our account, we assume for the moment that we deal with a base category $\mathcal{M}$ which fulfills all these properties, and we take our examples of base categories among the category of sets $\mathcal{M} = \mathcal{S}et$, of simplicial sets $\mathcal{M} = s\mathcal{S}et$, of topological spaces $\mathcal{M} = \mathcal{T}op$, of modules over the ground ring $\mathcal{M} = \mathcal{M}od$, or among a variant of these categories. We just make a few remarks about minor issues which occur when the tensor product does not distribute over colimits. We revisit the setting of our definitions with more care in the next chapters.

Recall that we generically use the unbased set notation $\varnothing$ for the initial object of our base category, the notation $*$ for the terminal object, and the notation $\amalg$ for coproducts (see §0.16). We just pass to additive category notation when we deal with additive structures or when the base category consists of modules. We then write 0 for the zero object and $\oplus$ for the coproduct.

In §§1.2-1.3, we explain that the category of operads and the categories of algebras associated to an operad have all limits and colimits. The limits of operads are created in the underlying base category in general. This is also the case of some particular colimits, like filtered colimits and reflexive coequalizers, but coproducts in the category of operads do not reduce to coproducts in the base category and we have similar results for algebras over operads (see §§1.2-1.3). Therefore, we keep the notation of the base category for limits in the category of operads and for limits in categories of algebras over operads, but we will adopt another style of notation (the base set notation $\vee$) for coproducts.

The definition of an operad which we recall in this chapter is borrowed from May's monograph [140]. Besides this reference, we should cite Boardman–Vogt's work [28] for another approach of the notion of an operad, and Ginzburg–Kapranov's article [78], from which we borrow the definition of free operads and the definition of operads by generators and relations. Reference books on operads, emphasizing various aspects of the theory, include [66] about modules and algebra categories associated to operads, [117] about operads and higher categories, [126] which focuses on algebraic operads and the Koszul duality theory, and [138] which provides an overall introduction to operads and to the Koszul duality of operads. We also refer to the textbook [186] for a basic introduction to the objects of the theory of operads. Most definitions and statements of this introductory chapter are covered by these reference, and we do not make any claim of originality at this stage of our work.

1.1. The notion of an operad and of an algebra over an operad

The purpose of this section is, as we just explained, to make explicit the definition of an operad and of an algebra over an operad. We have several approaches available. In this introductory chapter, we mostly deal with May's definitions [140], which has the advantage of giving a direct and simple interpretation of operadic structures in terms of operations acting on algebras. In the next chapter (§2), we give a reduced definition of the structure of an operad. We then rely on an interpretation, in terms of trees, of the composition of operations in an operad. We give a first informal introduction to the applications of trees in the definition of operads in this section.

Intuitively, an operad P consists of a collection of objects $\mathsf{P}(r)$ which collect operations $p = p(x_1, \dots, x_r)$. The notion of an operad is formally defined as a structure formed by such a collection of objects $\mathsf{P}(r)$ together with composition products that model the composition of operations. From this viewpoint, an operad can be regarded as a particular instance of an analyzer, a notion introduced by Lazard in [114] in order to generalize the power series operations used in the theory of formal Lie groups. In what follows, we generally assume that the terms of our operads $\mathsf{P}(r)$ are indexed by non-negative integers $r \in \mathbb{N}$, and we follow this convention all through this section. Nevertheless, for some constructions on operads, we deal with an extension of the definition where terms $\mathsf{P}(\underline{\mathsf{r}})$ associated to all finite sets $\underline{\mathsf{r}} = \{i_1, \dots, i_r\}$ are allowed. This convention enables us to model operations $p = p(x_{i_1}, \dots, x_{i_r})$ with variables $(x_{i_1}, \dots, x_{i_r})$ indexed by any such finite set $\underline{\mathsf{r}} = \{i_1, \dots, i_r\}$ and not only by the standard ordered sets $\underline{\mathsf{r}} = \{1 < \dots < r\}$ which we consider in the basic definition of the notion of an operad. We explain this extension of the notion of an operad in §2.5.

In the literature, the number of variables r in an operation $p = p(x_1, \dots, x_r)$ (not necessarily related to an operad) is sometimes referred to as the *arity* of p. In the operadic context, we use the term of arity to refer to the number r that indexes the terms $\mathsf{P}(r)$ of an operad P and, more generally, for any structure shaped on an $\mathbb{N}$-indexed collection of objects which we relate to an operad. In the setting where

$$\begin{tikzcd}
P(r)\otimes P(n_1)\otimes\cdots\otimes P(n_r) \arrow[r, "id\otimes(t_1\otimes\cdots\otimes t_r)"] \arrow[d, "\mu"'] & P(r)\otimes P(n_1)\otimes\cdots\otimes P(n_r) \arrow[d, "\mu"] \\
P(n_1+\cdots+n_r) \arrow[r, "t_1\oplus\cdots\oplus t_r"'] & P(n_1+\cdots+n_r)
\end{tikzcd}$$

$$\begin{tikzcd}
P(r)\otimes P(n_1)\otimes\cdots\otimes P(n_r) \arrow[rr, "s\otimes(id\otimes\cdots\otimes id)"] \arrow[d, "id\otimes s^*"', "\simeq"] & & P(r)\otimes P(n_1)\otimes\cdots\otimes P(n_r) \arrow[d, "\mu"] \\
P(r)\otimes P(n_{s(1)})\otimes\cdots\otimes P(n_{s(r)}) \arrow[dr, "\mu"'] & & P(n_1+\cdots+n_r) \\
 & P(n_{s(1)}+\cdots+n_{s(r)}) \arrow[ur, "s_*(n_1,\dots,n_r)"'] &
\end{tikzcd}$$

FIGURE 1.1. The equivariance axioms of operads, which are required to hold for all arities $r \geq 0$, $n_1,\dots,n_r \geq 0$, and for all permutations $s \in \Sigma_r$ and $t_1 \in \Sigma_{n_1},\dots,t_r \in \Sigma_{n_r}$.

the terms of an operad $P(\underline{r})$ are indexed by all finite sets $\underline{r} = \{i_1,\dots,i_r\}$, we use the term of arity to refer to the cardinal r of these indexing sets $\underline{r} = \{i_1,\dots,i_r\}$ (either regarded as a non-negative integer or as a class of finite sets in bijection to each other).

The explicit definition of an operad, beyond the intuitive approach, is quite intricate. In fact, we implicitly rely on a primitive operad structure on permutations when we formulate this definition. In the logical order, we should explicitly define the operations underlying the composition structure of the permutation operad first and we should introduce the general definition of an operad afterwards. But we will proceed differently in order to bring out the ideas underlying the definition. In a first stage, we only define the shape of the structure of an operad. This incomplete account is enough to fully explain the intuitive interpretation of the operad formalism, which we do next. Then we give the missing part of our definition, which is the definition of the primitive operad structure on permutation groups.

1.1.1. *The notion of an operad.* Formally, an *operad* in a base category $\mathcal{M}$ consists of a sequence of objects $P(r) \in \mathcal{M}$, $r \in \mathbb{N}$, where $P(r)$ is equipped with an action of the symmetric group on r letters Σ_r, together with:

(1) a unit morphism $\eta : \mathbb{1} \to P(1)$,
(2) and composition products

$$\mu : P(r) \otimes P(n_1) \otimes \cdots \otimes P(n_r) \to P(n_1 + \cdots + n_r),$$

defined for any $r \geq 0$, for all $n_1,\dots,n_r \geq 0$,

and such that natural equivariance, unit and associativity relations, expressed by the commutativity of the diagrams of Figure 1.1, 1.2, and 1.3, hold. The permutations $t_1 \oplus \cdots \oplus t_r$ and $s_*(n_1,\dots,n_r)$ which occur in the equivariance relations of Figure 1.1, will be explicitly defined in §1.1.7.

$$\begin{tikzcd}
\mathbb{1}\otimes P(n) \arrow[r, "\eta\otimes id"] \arrow[dr, "\simeq"'] & P(1)\otimes P(n) \arrow[d, "\mu"] \\
& P(n)
\end{tikzcd}
\qquad
\begin{tikzcd}
P(r)\otimes\mathbb{1}^{\otimes r} \arrow[r, "id\otimes\eta^{\otimes r}"] \arrow[dr, "\simeq"'] & P(r)\otimes P(1)^{\otimes r} \arrow[d, "\mu"] \\
& P(r)
\end{tikzcd}$$

FIGURE 1.2. The unit axioms of operads, which are required to hold for all $r \geq 0$ and for all $n \geq 0$.

$$\begin{tikzcd}
\begin{array}{rl} (P(r)\otimes P(s_1)\otimes\cdots\otimes P(s_r)) & \otimes\ (P(n_1^1)\otimes\cdots\otimes P(n_1^{s_1})) \\ & \otimes\quad \cdots \\ & \otimes\ (P(n_r^1)\otimes\cdots\otimes P(n_r^{s_r})) \end{array} \arrow[rr, "\simeq"] \arrow[d, "\mu\otimes id\otimes\cdots\otimes id"'] & & \begin{array}{rl} P(r) & \otimes\ (P(s_1)\otimes P(n_1^1)\otimes\cdots\otimes P(n_1^{s_1})) \\ & \otimes\quad \cdots \\ & \otimes\ (P(s_r)\otimes P(n_r^1)\otimes\cdots\otimes P(n_r^{s_r})) \end{array} \arrow[d, "id\otimes\mu\otimes\cdots\otimes\mu"] \\
\begin{array}{rl} P(s_1+\cdots+s_r) & \otimes\ P(n_1^1)\otimes\cdots\otimes P(n_1^{s_1}) \\ & \otimes\quad \cdots \\ & \otimes\ P(n_r^1)\otimes\cdots\otimes P(n_r^{s_r}) \end{array} \arrow[dr, "\mu"'] & & \begin{array}{rl} P(r) & \otimes\ P(n_1^1+\cdots+n_1^{s_1}) \\ & \otimes\quad \cdots \\ & \otimes\ P(n_r^1+\cdots+n_r^{s_r}) \end{array} \arrow[dl, "\mu"] \\
& P(n_1^1+\cdots+n_1^{s_1}+\cdots+n_r^1+\cdots+n_r^{s_r}) &
\end{tikzcd}$$

FIGURE 1.3. The associativity axiom of operads, which is required to hold for all arities $r \geq 0$, $s_1, \dots, s_r \geq 0$, and for $n_i^j \geq 0$.

In principle, we assume that the symmetric groups acts on the left on the components of an operad and we formulate our equivariance axioms accordingly. This convention is used by most authors.

The morphism η in the above definition is referred to as the *unit morphism of the operad* and the morphisms μ as the *composition products*. In what follows, we also use the phrase '*full composition product*' for these morphisms μ, because we use the phrase '*composition product*' as a generic name for any class of composition operations which we associate to operads. In general, we specify an operad by the notation of the underlying collection P and we use the letters η and μ as generic notation for the corresponding unit and product morphisms. We simply add a subscript $\eta = \eta_P$ (respectively, $\mu = \mu_P$) in order to specify the operad to which this unit (respectively, product) morphism is attached whenever this precision is necessary.

1.1.2. *The category of operads.* We obviously define a morphism of operads $\phi : P \to Q$ as a sequence of morphisms in the base category $\phi : P(r) \to Q(r)$, $r \in \mathbb{N}$, which commute with the action of symmetric groups and preserve the unit and the composition structure of our objects. When we work within a fixed base category $\mathcal{M}$, we use the notation $\mathcal{O}p$ to refer to the category formed by operads in $\mathcal{M}$ and this natural class of morphisms. If we need to specify the base category in which our operads are defined, then we simply add this category as a prefix to our notation:

$$\mathcal{O}p = \mathcal{M}\,\mathcal{O}p\,.$$

To give an example, we use the notation $\mathcal{O}p = \mathcal{T}op\,\mathcal{O}p$ to refer to the category of operads in topological spaces (for short, we also speak about *topological operads*). We similarly write $\mathcal{O}p = s\mathcal{S}et\,\mathcal{O}p$ for the category of operads in simplicial sets. We also speak about *simplicial operads* in this case. We can actually identify an operad in simplicial sets with a simplicial object in the category of operads in sets (see II.§8.0). We can therefore use the name 'simplicial operad' without any confusion in order to refer to the objects of this category $s\mathcal{S}et\,\mathcal{O}p$.

The underlying collection of an operad defines a diagram over the category Σ formed by the coproduct of the symmetric groups Σ_r, $r \in \mathbb{N}$, in the category of categories (after identifying these groups with categories with a single object). In §2.2, we introduce a variant of the category of operads, where the action of the symmetric groups is replaced by another internal structure, encoded by an action of a certain category Λ such that $\Sigma \subset \Lambda$. In order to form the notation of any variant of the category of operads, our convention is to add the notation of the indexing category as a prefix to the expression $\mathcal{O}p$. We therefore use the notation $\Lambda\,\mathcal{O}p$ for this category of Λ-operads (see §2.2). We may similarly write $\Sigma\,\mathcal{O}p$ for the category of plain operads, such as defined in §1.1.1. We actually take $\mathcal{O}p$ as a short notation for this category $\mathcal{O}p = \Sigma\,\mathcal{O}p$.

The category of *non-symmetric operads*, which is also considered in the literature, is another variant of the category of operads defined by forgetting about the symmetric structure (and the equivariance axioms) which we consider in our definition. The phrase '*symmetric operads*', referring to the action of the symmetric groups, is used by some authors for the notion defined in §1.1.1. We do not use non-symmetric operads in this book. The category of symmetric operads is our category of plain operads. We therefore drop the adjective symmetric and we simply say 'operad' to refer to the category of symmetric operads, in the same way as we adopt the short notation $\mathcal{O}p = \Sigma\,\mathcal{O}p$ for the category of symmetric operads.

In what follows, we also consider variants of the category of operads where we forget about the term of arity zero. We use the name '*non-unitary operad*' to refer to this category of operads. When the tensor product distributes over colimits, we can identify a non-unitary operad with an operad whose term of arity zero is the initial object of the base category and we can therefore regard the category of non-unitary operads as a subcategory of the category of all operads $\mathcal{O}p$. We heavily use the concept of a non-unitary operad and we will devote subsequent paragraphs to a more thorough study of this notion.

1.1.3. *Miscellaneous remarks on the definition of an operad.* In the case $r = 0$, the composition product of §1.1.1(2) involves an empty set of factors $\mathsf{P}(n_i)$. This composition morphism therefore reduces to an endomorphism of the object $\mathsf{P}(0)$ for $r = 0$. The (right) unit axiom of Figure 1.2 actually forces this endomorphism to be the identity of $\mathsf{P}(0)$. Thus, the consideration of a composition product for $r = 0$ in §1.1.1 does not add anything to the structure of an operad. Nevertheless, the formulation of the associativity axiom in full generality in Figure 1.3 requires to integrate this degenerate case in our definition.

The commutative diagrams of Figure 1.1, which express the equivariance relations of the composition products of operads, can also be gathered in a single equivalent commutative diagram, displayed in Figure 1.4. The permutation $s(t_1, \dots, t_r)$, which occurs in this diagram, is given by the composite $s(t_1, \dots, t_r) = (t_1 \oplus \cdots \oplus t_r) \cdot s_*(n_1, \dots, n_r)$ of the permutations $t_1 \oplus \cdots \oplus t_r$ and $s_*(n_1, \dots, n_r)$

$$\begin{tikzcd}
P(r)\otimes P(n_1)\otimes\cdots\otimes P(n_r) \arrow[rr, "s\otimes(t_1\otimes\cdots\otimes t_r)"] \arrow[d, "id\otimes s^*"', "\simeq"] & & P(r)\otimes P(n_1)\otimes\cdots\otimes P(n_r) \arrow[d, "\mu"] \\
P(r)\otimes P(n_{s(1)})\otimes\cdots\otimes P(n_{s(r)}) \arrow[dr, "\mu"'] & & P(n_1+\cdots+n_r) \\
 & P(n_{s(1)}+\cdots+n_{s(r)}) \arrow[ur, "s(t_1,\dots,t_r)"'] &
\end{tikzcd}$$

FIGURE 1.4. The equivariance axioms of operads, put in a single diagram, where $s(t_1,\dots,t_r)\in\Sigma_{n_1+\cdots+n_r}$ is actually an operadic composite of the permutations $s\in\Sigma_r$ and $t_1\in\Sigma_{n_1},\dots,t_r\in\Sigma_{n_r}$.

which occur in our initial equivariance axioms. We will see that this composite permutation $s(t_1,\dots,t_r)$ is identified with the outcome of an operadic composition product on permutations (see Proposition 1.1.9).

Intuitively, the object $P(r)$ in the definition of an operad collects abstract operations $p=p(x_1,\dots,x_r)$ of a given arity $r\in\mathbb{N}$ (as we explain in the introduction of this section). The composition morphisms of §1.1.1(2) model composition operations which we naturally associate to operations of this form and the definition of the permutations $t_1\oplus\cdots\oplus t_r$ and $s_*(n_1,\dots,n_r)$ in our equivariance axioms reflects this interpretation of the composition operations of an operad. Therefore, we give detailed explanations on this interpretation of the definition of an operad first, and we explicitly define the permutations $t_1\oplus\cdots\oplus t_r$ and $s_*(n_1,\dots,n_r)$ occurring in our equivariance axioms afterwards.

1.1.4. *The interpretation of an operad structure.* In the case of a concrete symmetric monoidal category, we can use the notation $p(q_1,\dots,q_r)\in P(n_1,\dots,n_r)$ for the image of a tensor $p\otimes(q_1\otimes\cdots\otimes q_r)\in P(r)\otimes P(n_1)\otimes\cdots\otimes P(n_r)$ under the composition product (2) in our definition of the structure of an operad §1.1.1. The unit morphism of §1.1.1(1) is also equivalent to the definition of a distinguished element $1\in P(1)$ (the *unit of the operad*). In many constructions, we consider partial composition operations $\circ_i : P(m)\otimes P(n)\to P(m+n-1)$ which are determined from the (full) composition products by the formula $p\circ_i q = p(1,\dots,1,q,1,\dots,1)$ where we plug the operation $q\in P(n)$ in the ith input of $p\in P(m)$ and we assign operad units $1\in P(1)$ at the other inputs to complete the definition.

In the intuitive interpretation of elements $p\in P(r)$ in terms of abstract operations $p=p(x_1,\dots,x_r)$, the action of a permutation $s\in\Sigma_r$ on the component of an operad $P(r)$ models a permutation of inputs

$$sp = p(x_{s(1)},\dots,x_{s(r)}),$$

and the (full) composition products of an operad model the definition of composite operations of the form

$$\begin{aligned}
p(q_1,\dots,q_r) = p(&q_1(x_{k_1+1},\dots,x_{k_1+n_1}),\\
&q_2(x_{k_2+1},\dots,x_{k_2+n_2}),\\
&\qquad\vdots\\
&q_r(x_{k_r+1},\dots,x_{k_r+n_r})),
\end{aligned}$$

where we set $k_i = n_1 + \cdots + n_{i-1}$ (with $k_1 = 0$ by convention). Thus, in the expression of the composite $p(q_1, \ldots, q_r)$, we split the variables into groupings which we attach to each operation q_i, $i = 1, \ldots, r$. The operadic unit similarly represents an identity operation (of one variable) $1 = id(x_1)$ and a partial composite $p \circ_i q = p(1, \ldots, 1, q, 1, \ldots, 1)$ can be identified with a composite operation of the form

$$p \circ_i q = p(x_1, \ldots, x_{i-1}, q(x_i, \ldots, x_{i+n-1}), x_{i+n}, \ldots, x_{m+n-1}).$$

In this pointwise formalism, the unit axioms read $1(p) = p$, $p(1, \ldots, 1) = p$, and the associativity axiom reads

$$p(q_1, \ldots, q_r)(\theta_1^1, \ldots, \theta_1^{s_1}, \ldots, \theta_r^1, \ldots, \theta_r^{s_r}) = p(q_1(\theta_1^1, \ldots, \theta_1^{s_1}), \ldots, q_r(\theta_r^1, \ldots, \theta_r^{s_r})),$$

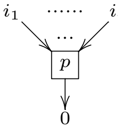
where we assume $p \in \mathsf{P}(r)$, $q_1 \in \mathsf{P}(s_1), \ldots, q_r \in \mathsf{P}(s_r)$ and $\theta_i^j \in \mathsf{P}(n_i^j)$. The equivariance axioms come from the identities

$$\begin{aligned}
&p(t_1 q_1, \ldots, t_r q_r) \\
&\qquad = p(q_1(x_{k_1+t_1(1)}, \ldots, x_{k_1+t_1(n_1)}), \ldots, q_r(x_{k_r+t_r(1)}, \ldots, x_{k_r+t_r(n_r)})), \\
&(sp)(q_1, \ldots, q_r) \\
&\qquad = p(q_{s(1)}(x_{k_{s(1)}+1}, \ldots, x_{k_{s(1)}+n_{s(1)}}), \ldots, q_{s(r)}(x_{k_{s(r)}+1}, \ldots, x_{k_{s(r)}+n_{s(r)}})).
\end{aligned}$$

The permutations $t_1 \oplus \cdots \oplus t_r$ and $s_*(n_1, \ldots, n_r)$ (which we formally define in §1.1.7) correspond to the input permutations that occur in these formulas.

Note that the full composition product of an operad is determined by the partial composition products. Indeed, the unit and associativity axioms imply that the composition product satisfies $p(q_1, \ldots, q_r) = (\cdots(p \circ_{k_1+1} q_1) \circ_{k_2+1} \cdots) \circ_{k_r+1} q_r$, for any $p \in \mathsf{P}(r)$ and for all $q_1 \in \mathsf{P}(n_1), \ldots, q_r \in \mathsf{P}(n_r)$, where we set $k_i = n_1 + \cdots + n_{i-1}$ for $i = 1, \ldots, r$. This observation is fully developed in §2.1, where we give another definition, in terms of the partial composition products, of the composition structure of an operad.

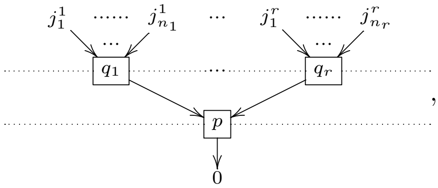
1.1.5. *The graphical representation of operad elements.* To get a better intuition of the definition of an operad, we also use a box picture



,

where $p \in \mathsf{P}(r)$ is any operation of the form collected by our operad P. The ingoing edges of the box materialize the inputs of such an operation and the outgoing edge is used to symbolize the output.

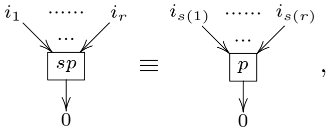
The composition products of an operad correspond to composition schemes of the following form:



where we plug the outputs of the upper level operations $q_1 \in \mathsf{P}(n_1), \ldots, q_r \in \mathsf{P}(n_r)$ in the inputs of the lower level operation $p \in \mathsf{P}(r)$ to obtain a composite operation $p(q_1, \ldots, q_r) \in \mathsf{P}(n_1 + \cdots + n_r)$ with as much inputs as the upper level operations together (and one final output). In the sequel, we use the above picture to represent the tensor $p \otimes (q_1 \otimes \cdots \otimes q_r) \in P(r) \otimes P(n_1) \otimes \cdots \otimes P(n_r)$ which we form to carry out our composition operation instead of the outcome of this process.

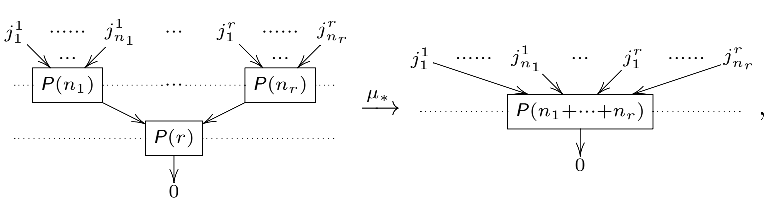
Recall that the elements of an operad $p \in P(r)$ represent operations $p = p(x_1, \dots, x_r)$ whose inputs are indexed by the elements of the set $\underline{r} = \{1 < \cdots < r\}$ (at least, in the setting of §1.1.1). In this context, we can assume that the ingoing edges of the box which represents our operation $p \in P(r)$ in the above picture are arranged in the plane according to the natural ordering of this indexing set $\underline{r} = \{1 < \cdots < r\}$. But we can also use the extra information specified by the indices of these ingoing edges in our picture to represent a permutation of inputs in our operation. For this purpose, we take the convention that these edges materialize a bijection, not necessarily the identity one, between an indexing set and the input set of our operation. In the picture of composite operations for instance, we associate the indices $j^i_k = n_1 + \cdots + n_{i-1} + k$, $k = 1, \dots, n_i$, to the ingoing edges of the boxes q_i, $i = 1, \dots, r$.

To identify equivalent indexing, we simply assume that we have the relation



when we apply a permutation $s \in \Sigma_r$ to the inner operation $p \in P(r)$. This formalism is explained with full details in §2.5 in the context where we consider operad components associated to all finite sets.

1.1.6. *The graphical representation of an operad structure.* The representation of the previous paragraphs can be applied to the abstract collection of objects which underlies an operad and to the tensor products of objects which we use in our definition of the composition structure of an operad. In this setting, the composition products of an operad can be depicted as morphisms



where the treewise arrangement materializes the tensor product which we consider in the definition of §1.1.1.

The composition schemes which occur in the unit and associativity relations of operads are represented in Figure 1.5-1.6. In these pictures, we identify the application of operadic units and operadic composition products with internal operations on some factors of our treewise tensor product. In general, we use the notation η_* and μ_*, which symbolizes the performance of internal operations on treewise tensors, for these mappings. The factors to which we apply the operation can in principle be determined from the internal structure of the trees which occur in the representation of our mapping.

1.1.7. *Fundamental operations on permutations.* We now define the permutations $t_1 \oplus \cdots \oplus t_r$ and $s_*(n_1, \ldots, n_r)$ which occur in the equivariance relations of Figure 1.1. We use the notation $k_i = n_1 + \cdots + n_{i-1}$ introduced in the previous paragraphs. To make our definition more explicit, we use that a permutation of $(1, \ldots, r)$ is equivalent to an ordered sequence $w = (w(1), \ldots, w(r))$ in which each value $k = 1, \ldots, r$ occurs once and only once. In some cases, we can also use the standard table representation:

$$w = \begin{pmatrix} 1 & \cdots & r \\ w(1) & \cdots & w(r) \end{pmatrix}.$$

The *direct sum* of permutations $t_1 \in \Sigma_{n_1}, \ldots, t_r \in \Sigma_{n_r}$ is the permutation of $\{1, \ldots, n_1 + \cdots + n_r\}$ given by the action of t_i on the interval $\{k_i + 1, \ldots, k_i + n_i\} \subset \{1, \ldots, n_1 + \cdots + n_r\}$ through the canonical bijection of ordered sets $\{1 < \cdots < n_i\} \xrightarrow{\simeq} \{k_i + 1 < \cdots < k_i + n_i\}$. This permutation is represented by the sequence $t_1 \oplus \cdots \oplus t_r = (k_1 + t_1(1), \ldots, k_1 + t_1(n_1), \ldots, k_r + t_r(1), \ldots, k_r + t_r(n_r))$ formed by the concatenation of the sequences $t_i = (t_i(1), \ldots, t_i(n_i))$ associated to the permutations t_i, $i = 1, \ldots, r$, together with the index shifts k_i. For instance, in the case of a pair of permutations $s \in \Sigma_m$ and $t \in \Sigma_n$, we obtain:

$$s \oplus t = \begin{pmatrix} 1 & \cdots & m & m+1 & \cdots & m+n \\ s(1) & \cdots & s(m) & m+t(1) & \cdots & m+t(n) \end{pmatrix}.$$

The *block permutation* $s_*(n_1, \ldots, n_r)$ associated to a permutation $s \in \Sigma_r$, where $n_1, \ldots, n_r \geq 0$ is any collection of natural numbers, is given by the permutation, under s, of the intervals $\underline{\mathsf{n}}_i = (k_i + 1, k_i + 2, \ldots, k_i + n_i)$ in the ambient set $\{1, \ldots, n_1 + \cdots + n_r\}$. In the sequence representation, the block permutation $s_*(n_1, \ldots, n_r)$ is defined by the sequence $s_*(n_1, \ldots, n_r) = (\underline{\mathsf{n}}_{s(1)}, \ldots, \underline{\mathsf{n}}_{s(r)})$ formed by the concatenation of the blocks $\underline{\mathsf{n}}_i$ which are ordered according to the order of the permutation $s = (s(1), \ldots, s(r))$. For instance, the block permutation $t_*(m, n)$ associated to a transposition $t = (1\ 2) \in \Sigma_2$ has the form:

$$\underbrace{(m+1, \ldots, m+n, 1, \ldots, m)}_{t_*(m,n)} = \begin{pmatrix} 1 & \cdots & n & n+1 & \cdots & n+m \\ m+1 & \cdots & m+n & 1 & \cdots & m \end{pmatrix}.$$

The following proposition follows from easy verifications:

Proposition 1.1.8. *Let $n_1, \ldots, n_r \geq 0$. In the symmetric group $\Sigma_{n_1 + \cdots + n_r}$, we have the relation*

$$(1) \qquad (s_1 \oplus \cdots \oplus s_r) \cdot (t_1 \oplus \cdots \oplus t_r) = (s_1 t_1) \oplus \cdots \oplus (s_r t_r)$$

for all r-tuples of permutations $(s_1, \ldots, s_r), (t_1, \ldots, t_r) \in \Sigma_{n_1} \times \cdots \times \Sigma_{n_r}$, the relation

$$(2) \qquad s_*(n_1, \ldots, n_r) \cdot t_*(n_{s(1)}, \ldots, n_{s(r)}) = (st)_*(n_1, \ldots, n_r),$$

for every $s, t \in \Sigma_r$, and the relation

$$(3) \qquad (t_1 \oplus \cdots \oplus t_r) \cdot s_*(n_1, \ldots, n_r) = s_*(n_1, \ldots, n_r) \cdot (t_{s(1)} \oplus \cdots \oplus t_{s(r)})$$

for every $s \in \Sigma_r$ and for all $(t_1, \ldots, t_r) \in \Sigma_{n_1} \times \cdots \times \Sigma_{n_r}$. □

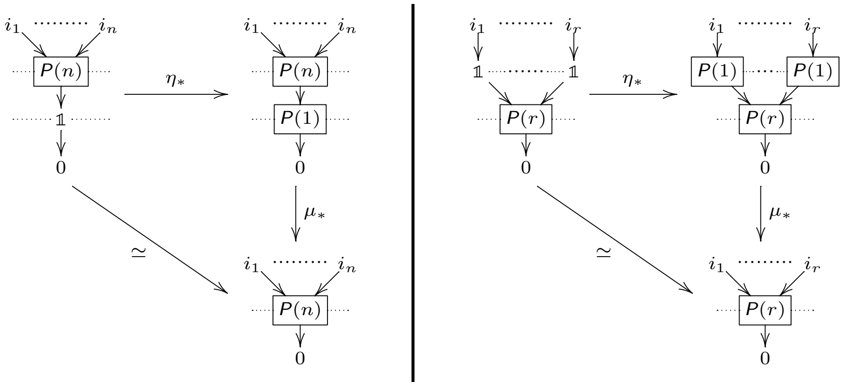


FIGURE 1.5. The treewise representation of the unit relations of operads.

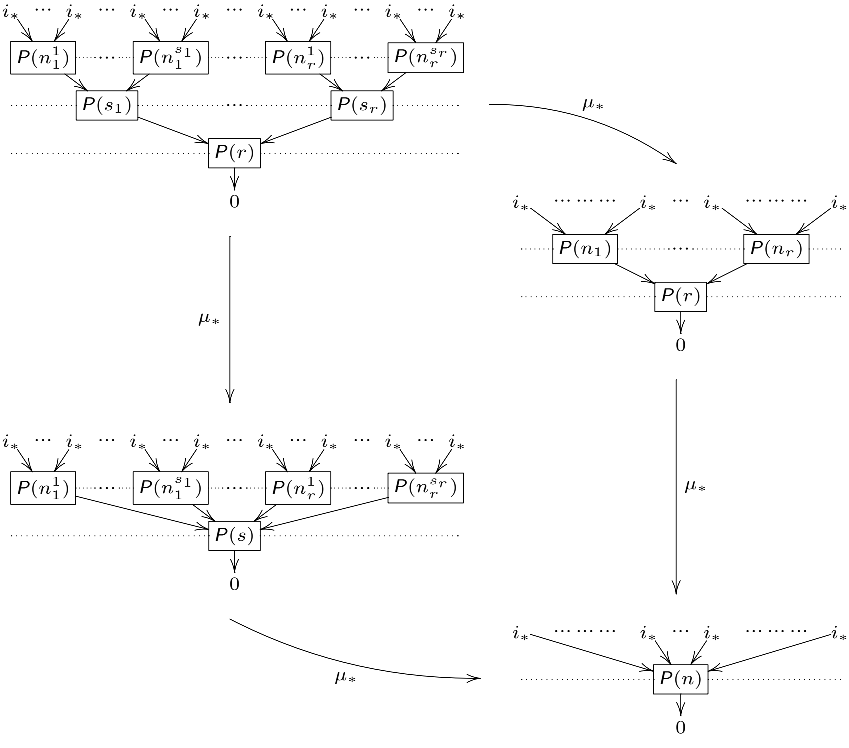


FIGURE 1.6. The treewise representation of the associativity relations of operads, where we set $n_i = n_i^1 + \cdots + n_i^{s_i}$ for $i = 1, \ldots, r$, and $s = s_1 + \cdots + s_r$, $n = n_1 + \cdots + n_r$ to shorten notation.

Then we obtain:

PROPOSITION 1.1.9. *The collection of symmetric groups Σ_n, $n \in \mathbb{N}$, forms an operad in sets such that:*

(0) *the action of the symmetric group on each Σ_n is given by left translations;*
(1) *the identity permutation on one element $id_1 \in \Sigma_1$ defines the operadic unit;*
(2) *and the composition product $\mu : \Sigma_r \times (\Sigma_{n_1} \times \cdots \times \Sigma_{n_r}) \to \Sigma_{n_1+\cdots+n_r}$ maps a collection $s \in \Sigma_r$, $(t_1, \dots, t_r) \in \Sigma_{n_1} \times \cdots \times \Sigma_{n_r}$ to the product permutation $s(t_1, \dots, t_r) = (t_1 \oplus \cdots \oplus t_r) \cdot s_*(n_1, \dots, n_r)$.*

PROOF. Easy verification from the relations of Proposition 1.1.8. ☐

This proposition explains our remark that the operations $t_1 \oplus \cdots \oplus t_r$ and $s_*(n_1, \dots, n_r)$, which occur in the general definition of an operad, come themselves from a primitive operad structure on the collection of symmetric groups. The definition of the composite $s(t_1, \dots, t_r)$ in Proposition 1.1.9 is forced by the equivariance axioms of operads and the requirement $id_r(id_{n_1}, \dots, id_{n_r}) = id_{n_1+\cdots+n_r}$, where we use the notation id_n for the identity permutation of the set $\{1, \dots, n\}$ (see §0.10). In this sense, the result of Proposition 1.1.9 expresses the internal coherence of the definition of an operad.

To give another (more) simple example, we can readily see that:

PROPOSITION 1.1.10. *The collection of one-point sets $pt(r) = pt$ forms an operad in sets. The action of the symmetric groups is trivial in each arity, and we take identities of one-point sets to define the composition unit and the composition products of the operad.* ☐

In what follows, we use the notation Π for the operad of Proposition 1.1.9, which we also call the *permutation operad*, and the notation Γ for the operad of Proposition 1.1.10, which we also call *one-point set operad*. To be more precise, when we use this notation Π, we actually refer to a version of the permutation operad where we forget about the term of arity zero of our object. We therefore assume $\Pi(r) = \Sigma_r$ for $r > 0$ and $\Pi(0) = \varnothing$. We use the notation Π_+, with the extra subscript mark $+$, when we keep this term $\Pi_+(0) = \Sigma_0 = pt$ in our object. We adopt similar conventions in the case of the one-point set operad Γ.

We soon explain that the permutation operad governs associative monoid structures while the one-point set operad governs commutative monoid structures. We will moreover see that both the permutation operad and the one-point set operad admit generalizations in the context of symmetric monoidal categories, as operads governing the category of associative algebras and the category of commutative algebras respectively (which we just call monoids in the context of sets). We use the name '*associative operad*' and the name '*commutative operad*' in order to refer to these generalizations of the permutation operad and of the one-point set operad. We also adopt the notation As and Com for these operads. We only use the notation Π and Γ in the context of the category of sets $\mathcal{M} = \mathcal{S}et$, in which we therefore have identities $\mathsf{As} = \Pi$ and $\mathsf{Com} = \Gamma$.

We explain the definition of a universal operad $\mathtt{End}_A$, which we associate to any object A of our base category $\mathcal{M}$, before explaining the definition of the category of algebras associated to an operad.

1.1.11. *Endomorphism operads.* This operad $\mathtt{End}_A$ which we associate to any object of the base category $A \in \mathcal{M}$ is called the *endomorphism operad* of A.

The definition of the endomorphism operad involves an internal hom-bifunctor $\mathtt{Hom}_{\mathcal{M}}(-,-) : \mathcal{M}^{op} \times \mathcal{M} \to \mathcal{M}$ and we therefore assume that our base category is closed (see §0.14) when we use this notion. We set $\mathtt{Hom}(-,-) = \mathtt{Hom}_{\mathcal{M}}(-,-)$ for short in what follows.

The endomorphism operad of $A \in \mathcal{M}$ is defined by the collection of hom-objects

$$\mathtt{End}_A(r) = \mathtt{Hom}(A^{\otimes r}, A),$$

where we form the tensor powers of A in the base category $\mathcal{M}$. We take the action of the symmetric groups Σ_r on the tensor powers $A^{\otimes r}$ to provide this collection of hom-objects $\mathtt{End}_A(r) = \mathtt{Hom}(A^{\otimes r}, A)$ with a symmetric structure. The composite morphisms

$$\begin{aligned} &\mathtt{Hom}(A^{\otimes r}, A) \otimes \Bigl(\bigotimes_{i=1}^{r} \mathtt{Hom}(A^{\otimes n_i}, A)\Bigr) \otimes A^{\otimes n} \\ &\qquad \xrightarrow{\simeq} \mathtt{Hom}(A^{\otimes r}, A) \otimes \Bigl(\bigotimes_{i=1}^{r} \mathtt{Hom}(A^{\otimes n_i}, A) \otimes A^{\otimes n_i}\Bigr) \\ &\qquad \xrightarrow{\epsilon} \mathtt{Hom}(A^{\otimes r}, A) \otimes A^{\otimes r} \xrightarrow{\epsilon} A, \end{aligned}$$

where we consider the natural evaluation morphisms attached to our hom-objects $\epsilon : \mathtt{Hom}(K, A) \otimes K \to A$, give operadic composition operations

$$\mu : \mathtt{Hom}(A^{\otimes r}, A) \otimes \Bigl(\bigotimes_{i=1}^{r} \mathtt{Hom}(A^{\otimes n_i}, A)\Bigr) \to \mathtt{Hom}(A^{\otimes n}, A)$$

by adjunction, for all $r \geq 0$, $n_1, \dots, n_r \geq 0$, and where we set $n = n_1 + \dots + n_r$. We use these operations to define the composition structure of the endomorphism operad. The symmetric monoidal unit $\mathbb{1} \otimes A \simeq A$ also gives a morphism

$$\eta : \mathbb{1} \to \mathtt{Hom}(A, A)$$

by adjunction and we use this morphism to define the unit of our operad.

The reader can easily check that these structure morphisms satisfy the axioms of §1.1.1 and hence do provide the object $\mathtt{End}_A$ with the structure of an operad.

1.1.12. *Endomorphism operads in basic ambient categories.* In the basic example of sets $\mathcal{M} = \mathcal{S}et$, the endomorphism operad of an object $X \in \mathcal{S}et$ consists of the mapping sets $\mathtt{End}_X(r) = \{f : X^{\times r} \to X\}$, for $r \in \mathbb{N}$. We then have the following pointwise formula for the action of permutations on our operad:

$$s f(x_1, \dots, x_n) = f(x_{s(1)}, \dots, x_{s(n)}),$$

where the variables x_k now refer to actual elements of X, and we have the formula

$$\begin{aligned} f(g_1, \dots, g_r)(x_1, \dots, x_{n_1 + \dots + n_r}) = f(&g_1(x_{k_1+1}, \dots, x_{k_1+n_1}), \\ &g_2(x_{k_2+1}, \dots, x_{k_2+n_2}), \\ &\vdots \\ &g_r(x_{k_r+1}, \dots, x_{k_r+n_r})), \end{aligned}$$

where we still set $k_i = n_1 + \dots + n_{i-1}$, for the composition products. The operadic unit is identified with the identity map $id : X \to X$.

In the context of topological spaces $\mathcal{M} = \mathcal{T}op$, we have the same explicit definition of the endomorphism operad $\mathtt{End}_X$ since the mapping sets $\mathtt{End}_X(r) = \{f :$

$X^{\times r} \to X\}$ are equipped with a topology which identify them with the internal hom-objects of the category of spaces $\mathcal{T}op$ (see §0.12 and §0.14).

In the context of a category of modules $\mathcal{M} = \mathcal{M}od$, the terms of the endomorphism operad $\mathtt{End}_K$, where $K \in \mathcal{M}od$, consist of morphisms $f : K^{\otimes r} \to K$ by construction of hom-objects in this category $\mathcal{M}od$ (see also §0.12). We may identify such morphisms $f : K^{\otimes r} \to K$ with r-linear maps $f : (x_1, \dots, x_r) \mapsto f(x_1, \dots, x_r)$. The action of permutations on such maps, as well as the operadic composition products, are given by the same pointwise formulas as in the context of sets.

1.1.13. *The notion of an algebra over an operad.* An *algebra over an operad* P (a P-algebra for short) is an object of the base category $A \in \mathcal{M}$ together with morphisms

$$(*) \qquad \lambda : \mathsf{P}(r) \otimes A^{\otimes r} \to A,$$

given for all $r \geq 0$, and such that equivariance, associativity and unit relations, expressed by the commutativity of the diagrams of Figure 1.7-1.9, hold. In applications of this definition, we usually say that the morphisms $(*)$ define the *action of the operad* P on the object $A \in \mathcal{M}$. We also say that these morphisms $(*)$ are the *evaluation morphisms* attached to the P-algebra A when we consider an object A equipped with a fixed P-action. In general, we refer to a P-algebra by the expression of the underlying object A and we use the letter λ as a generic notation for the morphisms $(*)$ which define the action of the operad on A. As in the operad case (see §1.1.1), we simply add the notation of the algebra as a subscript to this notation $\lambda = \lambda_A$ when we need to specify it.

The P-algebras form a category with, as morphisms, the morphisms of the base category $f : A \to B$ which preserve the P-actions on A and B. In what follows, we usually convert the notation of the operad P into calligraphic letters $\mathcal{P}$ in order to get the notation of the category of algebras associated to P. If necessary, then we write $\mathcal{P} = \mathcal{M}\,\mathcal{P}$ to specify the base category $\mathcal{M}$.

1.1.14. *The interpretation of the structure of an algebra over an operad in the context of a concrete category.* In the context of a concrete symmetric monoidal category, we can also use the pointwise notation $p(a_1, \dots, a_r) \in A$ for the image of a tensor $p \otimes (a_1 \otimes \dots \otimes a_r) \in \mathsf{P}(r) \otimes A^{\otimes r}$ under the evaluation morphism §1.1.13$(*)$. In the interpretation of operads given in §1.1.4, this evaluation morphism §1.1.13$(*)$ is equivalent to the evaluation of abstract operations $p = p(x_1, \dots, x_r)$ on actual elements $a_1, \dots, a_r \in A$.

The unit axiom is equivalent to the pointwise formula $1(a) = a$ for $a \in A$. The associativity axiom reads

$$p(q_1, \dots, q_r)(a_1^1, \dots, a_1^{n_1}, \dots, a_r^1, \dots, a_r^{n_r}) = p(q_1(a_1^1, \dots, a_1^{n_1}), \dots, q_r(a_r^1, \dots, a_r^{n_r}))$$

for all $p \in \mathsf{P}(r)$, $q_1 \in \mathsf{P}(n_1), \dots, q_r \in \mathsf{P}(n_r)$ and where $a_i^j \in A$. The equivariance axiom reads

$$sp(a_1, \dots, a_r) = p(a_{s(1)}, \dots, a_{s(r)}),$$

for all $p \in \mathsf{P}(r)$ and $a_1, \dots, a_r \in A$.

The graphical representation of §1.1.5 can also be applied to depict the action of operads on algebras. In short, we mark the ingoing edges of our boxes with algebra elements $a_1, \dots, a_r \in A$ (which we take as inputs for the operation represented by the box) and we mark the outgoing edge with the result of the operation

$$\begin{tikzcd}
P(r)\otimes A^{\otimes r} \arrow[r, "s\otimes id"] \arrow[d, "id\otimes s^*"', "\simeq"] & P(r)\otimes A^{\otimes r} \arrow[d, "\lambda"] \\
P(r)\otimes A^{\otimes r} \arrow[r, "\lambda"'] & A
\end{tikzcd}$$

Figure 1.7. The equivariance axiom for algebras over an operad, which is required to hold for every arity $r \geq 0$, and for all permutations $s \in \Sigma_r$.

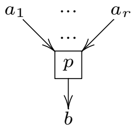
$$\begin{tikzcd}
\mathbb{1}\otimes A \arrow[r, "\eta\otimes id"] \arrow[dr, "\simeq"'] & P(1)\otimes A \arrow[d, "\lambda"] \\
 & A
\end{tikzcd}$$

Figure 1.8. The unit axiom for algebras over an operad.

$$\begin{tikzcd}
(P(r)\otimes P(n_1)\otimes\cdots\otimes P(n_r))\otimes A^{\otimes n} \arrow[rr, "\simeq"] \arrow[d, "\mu\otimes id^{\otimes n}"'] & & P(r)\otimes(P(n_1)\otimes A^{\otimes n_1})\otimes\cdots\otimes(P(n_r)\otimes A^{\otimes n_r}) \arrow[d, "id\otimes\lambda\otimes\cdots\otimes\lambda"] \\
P(n_1+\cdots+n_r)\otimes A^{\otimes n} \arrow[dr, "\lambda"'] & & P(r)\otimes A^{\otimes r} \arrow[dl, "\lambda"] \\
 & A &
\end{tikzcd}$$

Figure 1.9. The associativity axiom for algebras over an operad, which is required to hold for all $r \geq 0$, $n_1, \dots, n_r \geq 0$, and where we set $n = n_1 + \cdots + n_r$.

$b = p(a_1, \dots, a_r)$. Thus, we get the following picture:



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In the context of a closed monoidal category, the morphisms §1.1.13(∗), which define the action of an operad P on an object $A \in \mathcal{M}$, are equivalent to morphisms

$$\phi : P(r) \to \mathtt{Hom}_{\mathcal{M}}(A^{\otimes r}, A)$$

by adjunction, for all $r \geq 0$. The equivariance, unit and associativity axioms of operad actions in §1.1.13 are equivalent to the observation that these morphisms define an operad morphism from P towards the endomorphism operad associated to A. Hence, we have the following statement:

Proposition 1.1.15. *Providing an object $A \in \mathcal{M}$ with the structure of an algebra over an operad P amounts to giving an operad morphism $\phi : P \to \mathtt{End}_A$, where we consider the endomorphism operad of our object $\mathtt{End}_A$.* ☐

The evaluation morphisms $\epsilon : \mathtt{Hom}_{\mathcal{M}}(A^{\otimes r}, A) \otimes A^{\otimes r} \to A$ of the hom-objects $\mathtt{Hom}_{\mathcal{M}}(A^{\otimes r}, A)$ actually give an action of the endomorphism operad $\mathtt{End}_A$ on A. In the equivalence of Proposition 1.1.15, this action corresponds to the identity morphism $id : \mathtt{End}_A \to \mathtt{End}_A$. The assertion of the proposition can therefore be interpreted as the claim that the endomorphism operad $\mathtt{End}_A$ represents the universal operad which acts on A in the category $\mathcal{M}$.

In the context of a concrete symmetric monoidal category, the morphism $\phi : \mathsf{P} \to \mathtt{End}_A$ associates a homomorphism $p : A^{\otimes r} \to A$ to any operation $p \in \mathsf{P}(r)$. In the formalism of §1.1.14, we are simply considering the map $p : (a_1, \dots, a_r) \mapsto p(a_1, \dots, a_r)$ associated to a fixed element $p \in \mathsf{P}(r)$. The mapping ϕ is usually omitted in the notation of that map since the expression $p : A^{\otimes r} \to A$ indicates that we consider a map associated to $p \in \mathsf{P}(r)$ and not the abstract operation itself $p = p(x_1, \dots, x_r)$.

1.1.16. *Examples of operads associated to basic algebraic structures in sets.* We will prove that many usual algebraic structures, including associative algebras and commutative algebras, are governed by operads. The associative operad, the operad which we associate to the category of associative algebras, will be denoted by As. The commutative operad, the operad which we associate to the category of (associative and) commutative algebras, will be denoted by Com. In the context of sets, we also have identities $\Pi = \mathsf{As}$ and $\Gamma = \mathsf{Com}$, where we consider the permutation operad Π of Proposition 1.1.9 and the one-point set operad Γ of Proposition 1.1.10. (We give a first proof of these relations in Proposition 1.1.17-1.1.18.)

In general, we do not assume that an algebra is equipped with a unit (unless we explicitly assert the contrary) and we accordingly use the notation As (respectively Com) for the version of the associative (respectively, commutative) operad which governs the category of associative (respectively, commutative) algebras without unit. To refer to the operads governing algebras with unit, we add a subscript $+$ to the notation and we say that we deal with a *unitary* version of the operad. The connection between the operads governing the unitary and the non-unitary version of a structure is outlined in §§1.1.19-1.1.20, as a preparation for a more detailed study, which we address in §2.2. Simply mention for the moment that the operads As and As_+ agree in arity $r > 0$, but differ in arity $r = 0$, where we have $\mathsf{As}(0) = \varnothing$ (the initial object of the base category), in the non-unitary case, whereas we take $\mathsf{As}_+(0) = \mathbb{1}$ (the tensor unit) in the unitary case. We have the same identities in the case of the commutative operad.

We give a conceptual definition by generators and relations of the associative operad and of the commutative operad in the next section (see §1.2.6, §1.2.8). We can also give a direct construction of these operads, which makes sense in any symmetric monoidal category, by generalizing the definition of the permutation operad Π of Proposition 1.1.9 and the definition of the one-point set operad Γ of Proposition 1.1.10. We just explain this correspondence between these operads and the category of associative (respectively, commutative) algebras in the context of sets in order to complete the account of this section. We explain the definition of the operad associated to the category associative (respectively, commutative) algebras in a category of modules in the next section.

Recall that the (unitary version of the) permutation operad Π is defined by the symmetric group $\Pi(r) = \Sigma_r$ in each arity $r > 0$, while the one-point set operad Γ satisfies $\Gamma(r) = pt$ for all $r > 0$. We just take the extra arity zero term $\Pi_+(0) = \Gamma_+(0) = pt$ when we consider the unitary version of these operads Π_+ and Γ_+. In the context of sets, we speak about '*associative (respectively, commutative)* *monoids*' rather than about 'associative (respectively, commutative) algebras'. We have the following statements:

PROPOSITION 1.1.17. *The category of associative monoids with unit is isomorphic to the category of algebras over the permutation operad $\mathsf{\Pi}_+$, given in Proposition* 1.1.9*, and which is defined by the collection of symmetric groups $\mathsf{\Pi}_+(r) = \Sigma_r$ for $r \in \mathbb{N}$. The category of associative monoids without unit is isomorphic to the category of algebras over the operad $\mathsf{\Pi}$, which we form by removing the term of arity zero $\mathsf{\Pi}_+(0) = pt$ from this operad $\mathsf{\Pi}_+$.*

By removing the term of arity 0, we mean, again, that we consider the suboperad of the permutation operad such that $\mathsf{\Pi}(0) = \varnothing$ and $\mathsf{\Pi}(r) = \Sigma_r$ for $r > 0$. Thus, this proposition explains the difference, announced in §1.1.16, between the unitary case, where we consider an operad such that $\mathsf{As}_+(0) = \mathsf{\Pi}_+(0) = pt$ (in the context of sets), and the non-unitary case, where we take $\mathsf{As}(0) = \varnothing$.

PROOF. Let A be an associative monoid with unit. To a permutation $w \in \Sigma_r$, we can associate the operation $w : A^{\times r} \to A$ such that $w(a_1, \dots, a_r) = a_{w(1)} \cdot \ldots \cdot a_{w(r)}$. In plain terms, this operation is formed by the r-fold product of the sequence of elements $a_{w(1)}, \dots, a_{w(r)}$ in the monoid A. In the case $r = 0$, we use the unit morphism $\eta : pt \to A$ (equivalent to an empty product) to define the operation assigned to the degenerate permutation $id_0 \in \Sigma_0$. The verification of the axioms of §1.1.13 is the matter of an easy understanding exercise.

This process gives a functor from the category of associative monoids with unit to the category of algebras over the permutation operad.

In the converse direction, when A is an algebra over the permutation operad, we consider the unit operation $\eta : pt \to A$ associated to the degenerate permutation $id_0 \in \Sigma_0$ and the binary operation $\mu : A \times A \to A$ associated to the identity permutation $id_2 \in \Sigma_2$ in arity $r = 2$. The identity permutation in arity one $1 = id_1 \in \Sigma_1$ defines the unit of the permutation operad and, as such, is supposed to act as the identity operation on A. The unit operation $\eta : pt \to A$ is naturally equivalent to an element $e \in A$ which represents the image of the point pt under η. The identities $id_2(id_0, id_1) = id_1 = id_2(id_1, id_0)$ and $id_2(id_2, id_1) = id_3 = id_2(id_1, id_2)$ in the permutation operad are respectively equivalent to the unit relation $\mu(e, a) = a = \mu(a, e)$ and to the associativity relation $\mu(\mu(a_1, a_2), a_3) = \mu(a_1, \mu(a_2, a_3))$ in A. Hence, we have a monoid with unit naturally associated to each algebra over the permutation operad. This correspondence obviously gives a functor which is strictly inverse to the previously defined functor from the category associative monoids with unit to the category of algebras over the permutation operad. This assertion finishes the proof of the first assertion of the proposition.

The second assertion follows from the same verification (we simply forget about the degenerate permutation id_0 which corresponds to the unit operation $\eta : pt \to A$ in our arguments). ☐

PROPOSITION 1.1.18. *The category of commutative monoids with unit is isomorphic to the category of algebras over the one-point set operad $\mathsf{\Gamma}_+$, given in Proposition* 1.1.10*, and which is defined by the collection of one-point sets $\mathsf{\Pi}_+(r) = pt$ for $r \in \mathbb{N}$. The category of commutative monoids without unit is isomorphic to the category of algebras over the operad $\mathsf{\Gamma}$, which we form by removing the term of arity zero $\mathsf{\Gamma}_+(0) = pt$ from this operad $\mathsf{\Gamma}_+$.*

By removing the term of arity 0, we mean, again, that we consider the suboperad of the permutation operad such that $\Gamma(0) = \varnothing$ and $\Gamma(r) = pt$ for $r > 0$. Thus, we retrieve the same difference as in Proposition 1.1.17 between the unitary case, where we consider an operad such that $Com_{+}(0) = \Gamma_{+}(0) = pt$, and the non-unitary case, where we assume $Com(0) = \varnothing$.

Proof. The arguments are the same as in the case of algebras over the permutation operad (Proposition 1.1.17). The only difference is the following: the identity $(1\ 2)\, pt = pt$ in the one-point set operad implies, according to the equivariance axiom of operad actions (diagram of Figure 1.7), that the element $pt \in pt(2)$ represents a symmetric operation $\mu : A \times A \to A$, for any algebra over the one-point set operad. This explains that the structures associated to the one-point set operad are commutative. □

1.1.19. *Unitary and non-unitary operads.* In general, we say that an operad P_{+} is *unitary* when we have $P_{+}(0) = \mathbb{1}$, the unit object of the ambient symmetric monoidal category. Thus, we just have $P_{+}(0) = pt$ when we work in the category of sets $\mathcal{M} = \mathcal{S}et$. In the context of a concrete symmetric monoidal category, we use the notation $* \in P_{+}(0)$ for the distinguished arity-zero element of our operad which we associate to this unit object $P_{+}(0) = \mathbb{1}$. We also use this mark $*$ in a more general context to specify the factors $P_{+}(0) = \mathbb{1}$ occurring in abstract composition operations (see §2.2.9). We say that an operad P is *non-unitary* (as opposed to unitary) when we have $P(0) = \varnothing$, where $\varnothing$ represents the initial object of our base category. To be precise, when we use this definition, we assume that the base category has an initial object such that $X \otimes \varnothing = \varnothing \otimes X = \varnothing$. (These identities are particular cases of the distributivity relation of the tensor product with respect to colimits.) In the case where this distribution relation does not hold, we just define the notion of a non-unitary operad by forgetting about the terms of arity zero in the definition of §1.1.1. We then use the expression $P(0) = \varnothing$ to assert that our operad is not defined in arity 0.

The operad of unitary associative monoids As_{+} and the operad of unitary commutative monoids Com_{+}, which we define by using the identities $As_{+} = \Pi_{+}$ and $Com_{+} = \Gamma_{+}$ in the context of sets, are fundamental examples of unitary operads (in the category of sets). The operads $As = \Pi$ and $Com = \Gamma$, which we formed by removing the terms of arity zero from these unitary operads As_{+} and Com_{+}, are instances of non-unitary operads.

The terminology '*unitary operad*' refers to the observation that the evaluation morphism of a P_{+}-algebra gives a morphism $\lambda : P_{+}(0) \to A$, when we consider the term of arity zero of our operad. If we assume $P_{+}(0) = pt$ (in the point set context), then this morphism is equivalent to the definition of a distinguished element in A, which in usual examples (like associative or commutative monoids) represents a unit of the structure. Because of this interpretation, we also use the phrase '*unitary operation*' to refer to the elements of the term of arity zero of an operad. The non-unitary operads are operads which have no unitary operation.

In principle, our operads are supposed to be unital in the sense that they are equipped with a unit morphism $\eta : \mathbb{1} \to P(1)$ (which corresponds to a unit element $1 \in P(1)$ in the case of a concrete base symmetric monoidal category). This general assumption has not to be confused with the defining condition $P_{+}(0) = \mathbb{1}$ of a unitary operad P_{+}. The class of non-unitary operads, similarly, has not to be confused with the complement of the class of unitary operads. The unitary operations $p \in P_+(0)$ have not to be confused with the unit element $1 \in P_+(1)$ too.

In general, we reserve the adjective 'unital' to refer to the unit of the composition structure of an operad (as do many authors). To avoid confusion, we adopt the word 'unitary' to refer to the unit operations which we attach to an algebra and for the related structures which we get in our operads. Let us mention, however, that there is no fixed convention in the literature on this subject. (In particular, the expression '*unital operad*' is used in [140] for what we call a unitary operad.)

1.1.20. *Unitary extensions of operads.* We consider the category formed by the unitary operads as objects and the operad morphisms $\phi : P \to Q$ which are the identity of the unit object $\mathbb{1}$ in arity zero as morphisms. We adopt the convention to mark the consideration of fixed terms in operad categories by adding subscripts to our notation. We therefore use the notation $\mathcal{O}p_*$ for the category of unitary operads. The subscript $*$ in this expression obviously refers to our notation convention for the arity zero term of unitary operads in concrete categories.

We also adopt the notation $\mathcal{O}p_\varnothing$ for the category formed by the non-unitary operads of §1.1.19. We then use the lower script $\varnothing$ with two meanings. In general, we just define this category of operads $\mathcal{O}p_\varnothing$ by forgetting about arity-zero terms in the definition of an operad and in the definition of morphisms. We then write $P(0) = \varnothing$ to assert that our objects $P \in \mathcal{O}p_\varnothing$ are not defined in arity 0. If the tensor product of the base category distributes over colimits (as we assume all through this chapter), then we identify the category of non-unitary operads with the subcategory formed by the operads P such that $P(0) = \varnothing$, where we now use the notation $\varnothing$ to refer to the initial object of the base category $\mathcal{M}$. Note that a morphism $\phi : P(0) \to Q(0)$ is automatically the identity when we have $P(0) = Q(0) = \varnothing$.

We say that a non-unitary operad P admits a *unitary extension* when we have a unitary operad P_+ which agrees with P in arity $r > 0$ and of which composition operations extend the composition operations of P. If the tensor product of the base category distributes over colimits, then this condition implies that the canonical embedding $i_+ : P \to P_+$ defines a morphism in the category of operads. We may write $Q = P_+$ to assert that a given operad Q forms a unitary extension of another given (non-unitary) operad P.

We examine the definition of unitary extensions of operads more thoroughly in §2.2, after a comprehensive review of the definition of operadic composition structures which we carry out in §2.1. We immediately see that the underlying collection of a unitary extension P_+ is determined from the associated non-unitary operad P by the addition of the unit term $P_+(0) = \mathbb{1}$ in arity 0. In §2.2, we will more precisely explain that the composition structure of a unitary operad P_+ can be determined from the internal composition products of the associated non-unitary operad P and from extra operations which reflect composition products with the additional term of arity zero in the unitary extension but which we can still form inside the non-unitary part of our operad.

1.1.21. *Connected operads.* In subsequent constructions, we have to consider non-unitary operads P which satisfy the relation $P(1) = \mathbb{1}$ in addition to $P(0) = \varnothing$. We say that the operad P is *connected* (or that P forms a connected operad) when these conditions are satisfied. We adopt the notation $\mathcal{O}p_{\varnothing 1}$ (with the subscripts indicating the first terms of our operads) for the category of connected operads, which we can regard as a full subcategory of the category of non-unitary operads $\mathcal{O}p_{\varnothing}$ (observe that the preservation of operadic unit implies that any morphism of connected operads is the identity of the unit object in arity one).

We similarly consider operads P_{+} which are unitary in the sense of §1.1.19 and satisfy $P_{+}(1) = \mathbb{1}$. We then say that the operad P_{+} is *connected as a unitary operad* (or that P_{+} forms a *connected unitary operad*). We use the notation $\mathcal{O}p_{*1} \subset \mathcal{O}p_{*}$ for the full subcategory of the category of unitary operads generated by the connected unitary operads. We mostly deal with non-unitary operads in what follows. Therefore we generally use the short name '*connected operad*' for the category of connected non-unitary operads, while we keep the full name '*connected unitary operads*' when we deal with unitary operads. To give basic examples, we immediately see that the non-unitary associative operad As and the non-unitary commutative operad Com are instances of connected operads, whereas the unitary version of these operads As_{+} and Com_{+} are connected as unitary operads.

We go back to the definition of connected operads at the end of the next section.

1.2. Categorical constructions for operads

In this section, we explain the definition of free objects in the category of operads and the definition of operads by generators and relations. We also examine the application of usual categorical constructions, like colimits and limits, to operads.

For these purposes, we naturally have to consider the structure, underlying an operad, which is formed by a sequence $M = \{M(r), r \in \mathbb{N}\}$ whose terms $M(r)$ are objects of the base category equipped with an action of the symmetric groups Σ_r. We call this structure a *symmetric sequence*. We also adopt the notation $\mathcal{S}eq$ to refer to the category formed by these objects and where the morphisms $f : M \to N$ obviously consist of sequences of morphisms in the base category $f : M(r) \to N(r)$, $r \in \mathbb{N}$, which commute with the action of symmetric groups.

We adopt the notation Σ (with no decoration) to refer the category which has the standard finite ordered sets $\underline{\mathsf{n}} = \{1 < \dots < n\}$ as objects and whose morphism sets are defined by $\mathtt{Mor}_{\Sigma}(\underline{\mathsf{n}}, \underline{\mathsf{n}}) = \Sigma_n$ for $n \in \mathbb{N}$ and $\mathtt{Mor}_{\Sigma}(\underline{\mathsf{m}}, \underline{\mathsf{n}}) = \varnothing$ when $m \neq n$. The category of symmetric sequences is identified with the category of diagrams associated to this small category Σ. We may also use the phrase 'Σ-*sequence*' (rather than the name 'symmetric sequence' in plain words) to refer to the objects of this category. In this terminology, the word '*sequence*' just refers to the sequence of the finite ordered sets $\underline{\mathsf{n}} = \{1 < \dots < n\}$, $n \in \mathbb{N}$, which shapes the underlying collection of our objects. We adopt similar conventions for variants of the category of symmetric sequences which we introduce later on.

We have an obvious forgetful functor $\omega : \mathcal{O}p \to \mathcal{S}eq$ from the category of operads $\mathcal{O}p$ to the category of symmetric sequences $\mathcal{S}eq$. We have the following theorem:

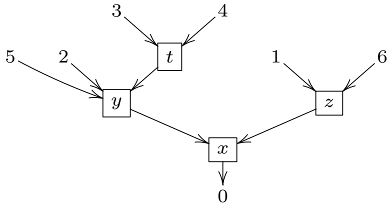
Theorem 1.2.1. *The forgetful functor $\omega : \mathcal{O}p \to \mathcal{S}eq$, from the category of operads to the category of symmetric sequences, has a left adjoint $\mathbb{O} : \mathcal{S}eq \to \mathcal{O}p$, which maps any symmetric sequence $M \in \mathcal{S}eq$ to an associated free object in the category of operads $\mathbb{O}(M) \in \mathcal{O}p$.*

Explanations. We explain the relationship between the definition of this statement and the usual definition of free objects (in terms of universal properties in categories) in the next proposition. We only explain the main ideas of our construction for the moment. We give more details on the proof of this theorem in §A.3. Intuitively, the free operad is the structure formed by all formal operadic composites of generating elements $\xi \in \mathsf{M}(n)$ with no relation between them apart from the universal relations which can be deduced from the axioms of operads.

To make the definition of the free operad explicit, we need a definition of the composition structure of an operad in terms of the partial composition operations of §1.1.4. In the construction of §A.3, we actually deal with a treewise representation of operadic composites which reflects the relations between these partial composition operations. The idea is that the level structures, which we need in the treewise representation of the full composition products of operads, can be forgotten when we depict elements of the free operad, because the unit and associativity relations of Figure 1.5-1.6 imply that the multi-fold composition products associated to different choices of level structure determine the same composite element in any operad.

To give a simple example, the multiple partial composite $p = (1\ 5)\cdot(((x \circ_1 y) \circ_4 z) \circ_3 t)$ such that $x \in \mathsf{M}(2)$, $y \in \mathsf{M}(3)$, $z \in \mathsf{M}(2)$, $t \in \mathsf{M}(2)$, and where we also consider an action of the transposition $(1\ 5) \in \Sigma_6$, defines an element of the free operad $\Theta(\mathsf{M})$ which we represent by the following picture:

(∗)

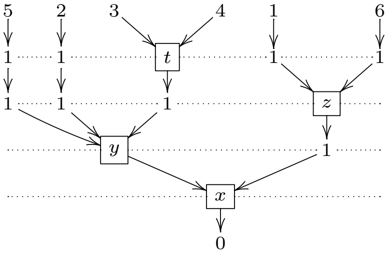


.

This treewise picture elaborates on a representation of partial composites which we introduce in §2.1.

For the moment, we can justify this picture by considering the formula $p = (1\ 5)\cdot x(y,1)(1,1,1,z)(1,1,t,1,1)$ which arises from the definition of partial composition operations in §1.1.4. The element p can also be determined by a 3-fold composition product $p = (1\ 5)\cdot x(y,z)(1,1,t,1,1)$, or equivalently, by $p = (1\ 5)\cdot x(y,1)(1,1,t,z)$. Each formula actually arises from the choice of a particular level structure on our tree representation. For instance, we get the following picture for our first decomposition formula:

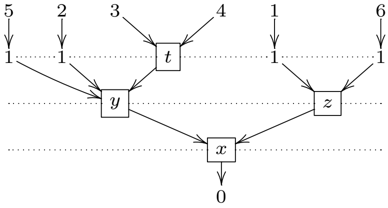
$$p = (1\ 5)\cdot x(y,1)(1,1,1,z)(1,1,t,1,1) \quad =$$



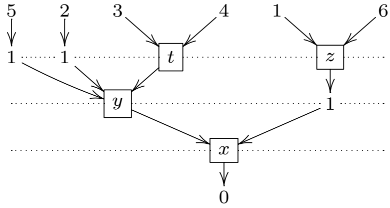
,

but we may also write:

$$p = (1\ 5) \cdot x(y,z)(1,1,t,1,1) \qquad =$$



$$= (1\ 5) \cdot x(y,1)(1,1,t,z) \qquad =$$



$$= \dots$$

by using our other expressions of this element p. (The factors 1 represent operadic units in all cases.) The identity between these representations follows (non trivially) from repeated applications of the unit and associativity relations of Figure 1.5-1.6.

In §A, we explicitly define the component of arity r of the free operad $\Theta(M)$ as a colimit

$$\Theta(M)(r) = \operatorname*{colim}_{\underline{T} \in \mathcal{T}ree(r)} M(\underline{T}) \tag{1}$$

which ranges over a category of trees with r ingoing edges $\mathcal{T}ree(r)$ and where $M(\underline{T})$ denotes a tensor product, along the vertex set of a tree $\underline{T}$, of components of the symmetric sequence M. We use these treewise tensor products $M(\underline{T})$ to formalize the general composition schemes that occur in an operad, as in our previous example $(*)$.

By definition of an adjunction, the free operad is characterized by the existence of a functorial bijection

$$\mathtt{Mor}_{\mathcal{O}p}(\Theta(M), P) \simeq \mathtt{Mor}_{\mathcal{S}eq}(M, P), \tag{2}$$

which holds for any pair (M, P) such that $M \in \mathcal{S}eq$ and $P \in \mathcal{O}p$. Together with this adjunction relation, we have:

- a morphism of symmetric sequences $\iota : M \to \Theta(M)$, the unit of the adjunction, naturally associated to any $M \in \mathcal{S}eq$, which corresponds to the identity of the free operad $id : \Theta(M) \to \Theta(M)$ under our bijection (2);
- an operad morphism $\lambda : \Theta(P) \to P$, called the adjunction augmentation, naturally associated to any operad $P \in \mathcal{O}p$, and which, under our bijection (2), corresponds to the identity of the operad P, viewed as an object of the category of symmetric sequences.

In the above expansion (1), the adjunction augmentation $\lambda : \Theta(P) \to P$ is given termwise by a morphism $\lambda_{\underline{T}} : P(\underline{T}) \to P(r)$ defined on each treewise tensor product $P(\underline{T})$, $\underline{T} \in \mathcal{T}ree(r)$, and for any $r \in \mathbb{N}$. We refer to these morphisms as the *treewise composition operations* (or as the *treewise composition products*) associated to our operad P. Intuitively, the morphism $\lambda_{\underline{T}} : P(\underline{T}) \to P(r)$ maps the formal operadic composites which span the treewise tensor product $P(\underline{T})$ to their evaluation in P. □

In §A.3, we give a direct construction of the free operad $\Theta(M)$ and of the morphism $\iota : M \to \Theta(M)$. We prove that the obtained object fulfills the required adjunction relation of free operads in a second step. We define our adjunction correspondence (2) by associating the composite morphism $\phi\iota \in \mathtt{Mor}_{\mathcal{S}eq}(M, P)$ to any operad morphism $\phi \in \mathtt{Mor}_{\mathcal{O}p}(\Theta(M), P)$ and the proof that this correspondence defines a bijection reduces to the following statement:

Proposition 1.2.2. *Any morphism of symmetric sequences $f : M \to P$, where P is an operad, admits a unique factorization*

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "\iota"'] & & P \\ & \Theta(M) \arrow[ur, dotted, "\exists!\phi_f"'] & \end{tikzcd}$$

such that ϕ_f is an operad morphism.

This proposition, proved in §A.3, expresses the adjunction relation of Theorem 1.2.1 in terms of an equivalent universal property which is usually given as the definition of a free object in the literature (we refer to [130, §IV.1] for the relationship between adjunctions and universals).

1.2.3. *The unit operad.* The purpose of the next paragraphs is to examine the definition of colimits and limits in the context of operads.

To start with, we consider the symmetric sequence such that

$$I(r) = \begin{cases} \mathbb{1}, & \text{if } r = 1, \\ \varnothing, & \text{otherwise,} \end{cases}$$

and whose terms reduce to a unit object $\mathbb{1}$ in arity $r = 1$. This symmetric sequence inherits an obvious operad structure, where the unit morphism $\eta : \mathbb{1} \to I(1)$ is the identity morphism of the unit object $\mathbb{1}$ and the composition products are forced by the unit axiom of Figure 1.2.

For any operad P, we have one and only one operad morphism from I to P, which is simply given by the unit morphism of our operad in arity one $I(1) = \mathbb{1} \xrightarrow{\eta} P(1)$. (The definition of this morphism is forced by the preservation of operad units.) Thus, the object I, which we call the *unit operad* in what follows, defines the initial object of the category of operads $\mathcal{O}p$. In general, we use the same letter η as in our notation of the unit morphism of an operad $\eta : \mathbb{1} \to P(1)$ for this initial morphism $\eta : I \to P$.

The category of operads has a terminal object too, which is identified with the constant operad such that $*(r) = *$ for any arity $r \in \mathbb{N}$, where $*$ denotes the terminal object of our base category $\mathcal{M}$.

The category of symmetric sequences, like any category of diagrams, has colimits and limits of any kind, which are formed termwise in the base category. In the context of operads, we have the following general proposition:

Proposition 1.2.4.

(a) *The forgetful functor from operads to symmetric sequences creates all small limits, the filtered colimits, and the coequalizers which are reflexive in the category of symmetric sequences.*

(b) *The category of operads admits coproducts too and, as a consequence, all small colimits, though the forgetful functor from operads to symmetric sequences does not preserve colimits in general.*

We refer to the appendix section §1.4 for a reminder on filtered colimits and reflexive coequalizers.

Proof. Let $\{P_\alpha, \alpha \in \mathcal{I}\}$ be any diagram in the category of operads. The collection

$$(\lim_{\alpha\in\mathcal{I}} P_\alpha)(r) = \lim_{\alpha\in\mathcal{I}} P_\alpha(r),$$

defined by the limit of the diagrams $\{P_\alpha(r), \alpha \in \mathcal{I}\}$ in the base category $\mathcal{M}$, for each arity $r \in \mathbb{N}$, inherits a natural operadic composition product

$$(\lim_{\alpha\in\mathcal{I}} P_\alpha(r)) \otimes (\lim_{\alpha\in\mathcal{I}} P_\alpha(n_1)) \otimes \dots \otimes (\lim_{\alpha\in\mathcal{I}} P_\alpha(n_r)) \to \lim_{\alpha\in\mathcal{I}} P_\alpha(n_1 + \dots + n_r),$$

for any $r \geq 0$ and $n_1, \dots, n_r \geq 0$, which is given by the composite of the morphism

$$\lim_{\alpha\in\mathcal{I}}(P_\alpha(r) \otimes P_\alpha(n_1) \otimes \dots \otimes P_\alpha(n_r)) \to \lim_{\alpha\in\mathcal{I}} P_\alpha(n_1 + \dots + n_r),$$

induced by the composition products of the operads P_α, with the natural morphism

$$\begin{aligned}(\lim_{\alpha\in\mathcal{I}} P_\alpha(r)) \otimes (\lim_{\alpha\in\mathcal{I}} P_\alpha(n_1)) \otimes \dots \otimes (\lim_{\alpha\in\mathcal{I}} P_\alpha(n_r))& \\ \to \lim_{\alpha\in\mathcal{I}}(P_\alpha(r) \otimes P_\alpha(n_1) \otimes \dots \otimes P_\alpha(n_r))&,\end{aligned}$$

which we deduce from the universal property of limits. The unit morphisms of our operads $\eta : \mathbb{1} \to P_\alpha(1)$ induce a unit morphism on the limit too, and we readily deduce from the uniqueness condition in the universal property of limits that these structure morphisms on $\lim_{\alpha\in\mathcal{I}} P_\alpha$ fulfill the axioms of operads. We also easily check that this operad, which we form by the aritywise limit of our objects P_α, represents the limit of the diagram $\{P_\alpha, \alpha \in \mathcal{I}\}$ in the category of operads. The requirement that the morphisms $P \to P_\alpha$ in the definition of the object $P = \lim_{\alpha\in\mathcal{I}} P_\alpha$ preserve operad structures clearly forces this definition of the structure of our operad P. In this sense, we obtain that the forgetful functor $\omega : \mathcal{O}p \to \mathcal{S}eq$ creates our limit in the category of operads (see [130] for the definition of this concept of creation).

In the case of a colimit, we can not use the above construction to provide the aritywise colimit of our diagram of operads

$$(\operatorname*{colim}_{\alpha\in\mathcal{I}} P_\alpha)(r) = \operatorname*{colim}_{\alpha\in\mathcal{I}} P_\alpha(r)$$

with an operadic composition structure in general, because the universal morphisms which we may deduce from the definition of a colimit go in the wrong direction:

$$\begin{aligned}(\operatorname*{colim}_{\alpha\in\mathcal{I}} P_\alpha(r)) \otimes (\operatorname*{colim}_{\alpha\in\mathcal{I}} P_\alpha(n_1)) \otimes \dots \otimes (\operatorname*{colim}_{\alpha\in\mathcal{I}} P_\alpha(n_r))& \\ \xleftarrow{(*)} \operatorname*{colim}_{\alpha\in\mathcal{I}}(P_\alpha(r) \otimes P_\alpha(n_1) \otimes \dots \otimes P_\alpha(n_r))&.\end{aligned}$$

Nevertheless, the results of Proposition 1.4.2 and Proposition 1.4.4 in the appendix section §1.4 imply that this morphism is an isomorphism when the diagrams $\{P_\alpha(n), \alpha \in \mathcal{I}\}$ are shaped on a filtered category or when we consider the coequalizer of a pair of morphisms which is reflexive in the base category. Hence, in these situations, we can form natural composition products

$$(\operatorname*{colim}_{\alpha\in\mathcal{I}} P_\alpha(r)) \otimes (\operatorname*{colim}_{\alpha\in\mathcal{I}} P_\alpha(n_1)) \otimes \dots \otimes (\operatorname*{colim}_{\alpha\in\mathcal{I}} P_\alpha(n_r)) \to \operatorname*{colim}_{\alpha\in\mathcal{I}} P_\alpha(n_1 + \dots + n_r)$$

by taking the composite of the morphisms

$$\operatorname*{colim}_{\alpha\in\mathcal{J}}(P_\alpha(r)\otimes P_\alpha(n_1)\otimes\cdots\otimes P_\alpha(n_r))\to\operatorname*{colim}_{\alpha\in\mathcal{J}}P_\alpha(n_1+\cdots+n_r)$$

induced by the composition products of the operads P_α with the inverse of the universal morphism (*). In this situation, we can also take the composite of the unit morphism $\eta:\mathbb{1}\to P_\alpha(1)$ of any object P_α in our diagram $\{P_\alpha,\alpha\in\mathcal{J}\}$ with the canonical morphism $P_\alpha(1)\to\operatorname{colim}_{\alpha\in\mathcal{J}}P_\alpha(1)$ corresponding to this object in our colimit in order to get a unit morphism with values in the object $\operatorname{colim}_{\alpha\in\mathcal{J}}P_\alpha(1)$. Let us simply observe that the unit morphism which we deduce from this construction does not depend on the choice of the term P_α when our colimit is shaped on a filtered category or when our colimit is the coequalizer of a parallel pair of operad morphisms. We easily check, again, that these structure morphisms on $\operatorname{colim}_{\alpha\in\mathcal{J}}P_\alpha$ fulfill the axioms of operads. We also readily check that this operad, which we form by the aritywise colimit of our objects P_α, represents the colimit of the diagram $\{P_\alpha,\alpha\in\mathcal{J}\}$ in the category of operads.

To realize a coproduct of a collection of operads P_α, $\alpha\in\mathcal{J}$, we form a reflexive coequalizer of the form

$$\begin{tikzcd}\Theta(\coprod_{\alpha\in\mathcal{J}}\Theta(P_\alpha)) \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & \Theta(\coprod_{\alpha\in\mathcal{J}}P_\alpha) \arrow[l, bend right, "s_0"'] \arrow[r, dotted] & Q\end{tikzcd},$$

where the morphisms (d_0,d_1) are determined on each generating summand $\Theta(P_\alpha)$ of the free operad $Q_1=\Theta(\coprod_{\alpha\in\mathcal{J}}\Theta(P_\alpha))$ by:

- the morphism $\Theta(\iota_\alpha):\Theta(P_\alpha)\to\Theta(\coprod_{\alpha\in\mathcal{J}}P_\alpha)$ induced by the canonical embedding $\iota_\alpha:P_\alpha\to\coprod_{\alpha\in\mathcal{J}}P_\alpha$ as regards d_0;
- the composite of the adjunction augmentation $\lambda:\Theta(P_\alpha)\to P_\alpha$ with the canonical embedding $\iota_\alpha:P_\alpha\to\coprod_{\alpha\in\mathcal{J}}P_\alpha$ and the adjunction unit of the free operad $\iota:\coprod_{\alpha\in\mathcal{J}}P_\alpha\to\Theta(\coprod_{\alpha\in\mathcal{J}}P_\alpha)$ as regards d_1.

The reflection s_0 is given by the adjunction unit of the free operad $\iota:P_\alpha\to\Theta(P_\alpha)$ on each generating summand of $Q_0=\Theta(\coprod_{\alpha\in\mathcal{J}}P_\alpha)$. By the result established in the first part of the proposition, the existence of this reflection s_0 implies that we can form the coequalizer $\operatorname{coeq}(d_0,d_1)$ in the category of operads by taking the aritywise coequalizer of this parallel pair of morphisms in the base category.

By the universal property of sums and free operads, any morphism $\phi_f:Q_0\to R$ with values in an operad R is fully determined by a collection of symmetric sequence morphisms $f_\alpha:P_\alpha\to R$. Moreover, we have $\phi_f d_0=\phi_f d_1$ if and only if the diagram

$$\begin{tikzcd}\Theta(P_\alpha) \arrow[r, "\lambda"] \arrow[dr, "\phi_{f_\alpha}"'] & P_\alpha \arrow[d, "f_\alpha"] \\ & R\end{tikzcd}$$

commutes for every α, where we consider the operad morphism ϕ_{f_α} associated to f_α. This condition is equivalent to the requirement that f_α preserves operadic composites and operadic units, because λ is given by the evaluation of the formal composition products of the free operad in P_α and maps the unit of the free operad to the unit of the operad P_α. Hence we have $\phi_f d_0=\phi_f d_1$ if and only if each $f_\alpha:P_\alpha\to R$ is an operad morphism, and this result implies that giving an operad morphism $\bar{\phi}_f:\operatorname{coeq}(d_0,d_1)\to R$ amounts to giving operad morphisms $f_\alpha:P_\alpha\to R$, for all $\alpha \in \mathcal{I}$. We conclude that our coequalizer $\mathsf{Q} = \operatorname{coeq}(d_0, d_1)$ represents the coproduct of the objects P_α, which therefore exists in the category of operads.

The last assertion of the proposition is an application of the result of Proposition 1.4.5, in the appendix section 1.4. □

1.2.5. *Operads defined by generators and relations.* The existence of free objects and coequalizers enables us to define operads by generators and relations. To start with, we explain the definition of such constructions in the case where the base category is the category of sets $\mathcal{M} = \mathcal{S}et$.

We start with a symmetric sequence $\mathsf{M} \in \mathcal{S}eq$, whose elements $\xi \in \mathsf{M}(r)$ represent generating operations, and with a collection of pairs $(w_0^\alpha, w_1^\alpha) \in \Theta(\mathsf{M})(n_\alpha)^{\times 2}$, $\alpha \in \mathcal{I}$, which we use to define generating relations $w_0^\alpha \equiv w_1^\alpha$ within the free operad $\Theta(\mathsf{M})$.

We set $\Xi(n) = \{e_\alpha, \alpha \in \mathcal{I} \,|\, n_\alpha = n\}$, where each e_α denotes an abstract generating element associated to the indexing variable $\alpha \in \mathcal{I}$. Let G be any group. In what follows, we use the notation $K[G]$, for the free G-object associated to any object K in a base category $\mathcal{C}$. We generally have $K[G] = \coprod_{g \in G} K$, where we take a coproduct of copies of the object K indexed by the elements of our group G. We just have $K[G] = G \times K$ when we work in the category of sets $\mathcal{C} = \mathcal{S}et$. We form the free Σ_n-set $\mathsf{R}(n) = \Sigma_n \times \Xi(n)$, for each $n \in \mathbb{N}$. The collection of these free Σ_n-sets defines a symmetric sequence R such that $\mathsf{R}(n) = \Sigma_n \times \Xi(n)$, for each $n \in \mathbb{N}$. We consider the symmetric sequence morphisms $\rho_0, \rho_1 : \mathsf{R} \rightrightarrows \Theta(\mathsf{M})$ such that $\rho_0(e_\alpha) = w_0^\alpha$ and $\rho_1(e_\alpha) = w_1^\alpha$ respectively. We form the morphisms of symmetric sequences $\delta_0, \delta_1 : \mathsf{M} \amalg \mathsf{R} \to \Theta(\mathsf{M})$ induced by $\rho_0, \rho_1 : \mathsf{R} \rightrightarrows \Theta(\mathsf{M})$ on R and by the universal morphism $\iota : \mathsf{M} \to \Theta(\mathsf{M})$ on M. We consider the morphisms of free operads $d_0, d_1 : \Theta(\mathsf{M} \amalg \mathsf{R}) \rightrightarrows \Theta(\mathsf{M})$ induced by these morphisms δ_0 and δ_1. We have an operad morphism in the converse direction $s_0 : \Theta(\mathsf{M}) \to \Theta(\mathsf{M} \amalg \mathsf{R})$, yielded by the composite $\mathsf{M} \hookrightarrow \mathsf{M} \amalg \mathsf{R} \xrightarrow{\iota} \Theta(\mathsf{M} \amalg \mathsf{R})$, and such that $d_0 s_0 = d_1 s_0 = id$. The coequalizer of this reflexive pair $\mathsf{P} = \operatorname{coeq}(\Theta(\mathsf{M} \amalg \mathsf{R}) \rightrightarrows \Theta(\mathsf{M}))$, created in the category of sets, defines the operad

$$\mathsf{P} = \Theta(\mathsf{M} : w_0^\alpha \equiv w_1^\alpha, \alpha \in \mathcal{I})$$

which we associate to the generating symmetric sequence M and to the generating relations $w_0^\alpha \equiv w_1^\alpha$, $\alpha \in \mathcal{I}$.

Intuitively, we perform the reflexive coequalizer $\operatorname{coeq}(\Theta(\mathsf{M} \amalg \mathsf{R}) \overset{\curvearrowleft}{\rightrightarrows} \Theta(\mathsf{M}))$ in the underlying category of sets by identifying any formal composite which involves a subfactor of the form w_0^α in the operad $\Theta(\mathsf{M})$ with the formal composite which we obtain by performing the substitution $w_0^\alpha \mapsto w_1^\alpha$, and conversely when we have a formal composite with a subfactor of the form w_1^α.

For an operad morphism $\phi_f : \Theta(\mathsf{M}) \to \mathsf{Q}$, we have:

$$\begin{aligned} & \phi_f d_0 = \phi_f d_1 \\ \Leftrightarrow \quad & \phi_f \rho_0 = \phi_f \rho_1 \\ \Leftrightarrow \quad & \phi_f(w_0^\alpha) = \phi_f(w_1^\alpha), \ \forall \alpha \in \mathcal{I}\,. \end{aligned}$$

Hence, defining a morphism $\bar{\phi}_f : \mathsf{P} \to \mathsf{Q}$ on the operad $\mathsf{P} = \Theta(\mathsf{M} : w_0^\alpha \equiv w_1^\alpha, \alpha \in \mathcal{I})$ amounts to giving a morphism of symmetric collections $f : \mathsf{M} \to \mathsf{Q}$ such that the extension of this morphism to the free operad $\phi_f : \Theta(\mathsf{M}) \to \mathsf{Q}$ maps the relations $w_0^\alpha \equiv w_1^\alpha$, $\alpha \in \mathcal{I}$, to actual identities in the target operad Q.

1.2.6. *Basic examples of operads in sets.* The most classical examples of operads can actually be defined by a presentation by generators and relations. To give first examples of application of this process in the context of sets, we make explicit a definition by generators and relations of the operad governing associative algebras (the associative operad As), and we give a similar description of the operad governing commutative algebras (the commutative operad Com). We check in the next proposition that this approach gives the same result as the direct construction of these operads of the introductory section §1.1. We focus on the definition of non-unitary operads for the moment. We will devote a subsequent chapter §2 to the definition of unitary operads (as we explain in §1.2.8).

To give a more intuitive interpretation of our construction, we define the generating symmetric sequence of our operads M by giving operations $p = p(x_1, \dots, x_n)$ which generate the terms of this sequence $\mathsf{M}(n)$ as Σ_n-sets. We use explicit variables to specify the arity of generating operations, unless this information has already been specified by the context. We also use variable permutations to denote operations which correspond to each other under the action of permutations in $\mathsf{M}(n)$, but this indication may not be sufficient to fully determine the symmetric structure of our object, which we therefore specify apart in our definition.

We define the associative operad in sets by the presentation:

$$\mathsf{As} = \Theta(\mu(x_1, x_2), \mu(x_2, x_1) : \mu(\mu(x_1, x_2), x_3) \equiv \mu(x_1, \mu(x_2, x_3))),$$

where we take a generating operation of arity two $\mu = \mu(x_1, x_2)$, on which the group Σ_2 operates freely, together with a single generating relation, given by the identity of composite operations $\mu(\mu, 1) \equiv \mu(1, \mu)$ which expresses the associativity of our generating operation. We define the commutative operad in sets by a similar presentation:

$$\mathsf{Com} = \Theta(\mu(x_1, x_2) : \mu(\mu(x_1, x_2), x_3) \equiv \mu(x_1, \mu(x_2, x_3))),$$

where we now consider a generating operation of arity two $\mu = \mu(x_1, x_2)$ equipped with a trivial (identical) action of the group Σ_2 together with the associativity relation $\mu(\mu, 1) \equiv \mu(1, \mu)$ as generating relation again.

In what follows, we may also use classical algebraic notation $\mu(x_1, x_2) = x_1 x_2$ for the generating operation of the associative (respectively, commutative) operad. We obviously retrieve the standard expression of the associativity relation $(x_1 x_2) x_3 \equiv x_1(x_2 x_3)$ from the operadic identity $\mu(\mu, 1) \equiv \mu(1, \mu)$ when we adopt these notation conventions. We notably use the algebraic formalism when we define a basis of the components of the associative (respectively, commutative) operad in §1.2.11.

In §1.1, we used the permutation operad (respectively, the one-point set operad) to give a direct construction of an operad governing associative (respectively, commutative) algebras. The next proposition establishes the identity between this construction and the definition by generators and relations of the previous paragraph:

PROPOSITION 1.2.7.
(a) *The associative operad* As*, such as defined in* §1.2.6*, satisfies*

$$\mathsf{As}(r) = \begin{cases} \varnothing, & \text{if } r = 0, \\ \Sigma_r, & \text{otherwise}, \end{cases}$$

and is isomorphic to the (non-unitary version of the) permutation operad of Proposition 1.1.9*.*

(b) *The commutative operad* Com*, such as defined in* §1.2.6*, satisfies*

$$\mathsf{Com}(r) = \begin{cases} \varnothing, & \text{if } r = 0, \\ pt, & \text{otherwise}, \end{cases}$$

and is isomorphic to the (non-unitary version of the) one-point set operad of Proposition 1.1.10*.*

In §1.2.11, we will explain that the permutations $w \in \Sigma_r$ in the result of this proposition actually correspond to monomials $p(x_1, \dots, x_r) = x_{w(1)} \cdot \dots \cdot x_{w(r)}$ when we use standard algebraic notation to represent the elements of the associative operad As (as we briefly explain §1.2.6). Recall that we also use the notation $\mathsf{\Pi}$ for the (non-unitary version of the) permutation operad of Proposition 1.1.9 and the notation $\mathsf{\Gamma}$ for the (non-unitary version of the) one-point set operad of Proposition 1.1.10. Thus, this proposition asserts that, in the context of sets, we have an identity $\mathsf{As} = \mathsf{\Pi}$ (respectively, $\mathsf{Com} = \mathsf{\Gamma}$), where we use the notation As (respectively, Com) for the operad defined by the presentation of §1.2.6.

PROOF. We focus on the example of the associative operad (a) as the case of the commutative operad (b) follows from similar arguments. We still use the notation $\mathsf{\Pi}$ for the permutation operad all through this proof. Recall that we consider the non-unitary version of this operad with the term in arity 0 withdrawn when we use this notation (as specified in the proposition).

To start with, we observe (as in the proof of Proposition 1.1.17) that the permutation $\mu = id \in \Sigma_2$ satisfies the generating relations of the associative operad As in the permutation operad. Hence, we have a well-defined operad morphism $\phi : \mathsf{As} \to \mathsf{\Pi}$ which maps the generating operation of As to this permutation. To prove that this morphism is an isomorphism, we form a morphism in the converse direction by assigning the composite operation $\psi(w) = w \cdot \mu(\cdots(\mu(\mu, 1), 1), \dots, 1)$ to any $w \in \Sigma_r$. We immediately see that we have the identity $\phi\psi = id$ and we easily check that we have the relation $id \equiv \psi\phi$ in the operad As. The conclusion follows. ☐

1.2.8. *The presentation of unitary operads.* To define unitary versions of the commutative operad and of the associative operad, we may simply add a generating operation e in arity 0 and relations of the form $\mu(e, 1) = 1 = \mu(1, e)$, which express the identities of neutral elements, to our presentations. Thus, we may set

$$\mathsf{As}_+ = \Theta(e, \mu(x_1, x_2), \mu(x_2, x_1) : \mu(\mu, 1) \equiv \mu(1, \mu), \mu(e, 1) \equiv 1 \equiv \mu(1, e)),$$
$$\mathsf{Com}_+ = \Theta(e, \mu(x_1, x_2) : \mu(\mu, 1) \equiv \mu(1, \mu), \mu(e, 1) \equiv 1 \equiv \mu(1, e)),$$

to define these operads. The result of Proposition 1.2.7 also extends to the unitary version of our operads, so that we have $\mathsf{As}_+(r) = \Sigma_r$ (respectively, $\mathsf{Com}_+(r) = pt$), for all r (including $r = 0$).

Note however that the unitary operation e is special (at least in our examples). Indeed, in the outcome of the presentation process, the terms of arity $r > 0$ of the operad As_+ (respectively, Com_+) agree with the terms of the non-unitary operad As (respectively, Com) which we form by dropping the unitary operation e from our presentation. In fact, we do not use the general approach of operads defined by generators and relations for unitary operations. We put these operations apart in our constructions. We therefore do not consider the case of unitary operads for the moment. (We put off this study until §2.)

1.2.9. *Operad ideals and presentations of operads in module categories.* The construction of §1.2.5 has an analogue in the context of modules over a ground ring. We simply have to replace the set $\Xi(n) = \{e_\alpha, \alpha \in \mathcal{I} \,|\, n_\alpha = n\}$ by the associated free module $K(n) = \Bbbk[e_\alpha, \alpha \in \mathcal{I} \,|\, n_\alpha = n]$, where $\Bbbk$ is our ground ring, and we also define the free G-object associated to a $\Bbbk$-module K, where G is any group, by the direct sum operation $K[G] = \bigoplus_{g \in G} K$.

The purpose of this paragraph is to explain that, in the setting of a category of modules, we can use an operadic version of the notion of an ideal in order to give another construction of operads by generators and relations. In the next part of the book, we apply an extension of this construction in the context of graded modules. For the moment, we focus on the case of plain modules.

In brief, an ideal of an operad in $\Bbbk$-modules P is a collection of submodules $S(n) \subset P(n)$, which are preserved by the action of the symmetric groups on the components of our operad $P(n)$, and where we have $p(q_1, \dots, q_r) \in S(n_1 + \dots + n_r)$, for any composite of elements $p \in P(r)$, $q_1 \in P(n_1), \dots, q_r \in P(n_r)$, as soon as we have $p \in S(r)$ or $q_i \in S(n_i)$ for some $i \in \{1 < \dots < r\}$. We immediately see that the collection $P / S(n) = P(n)/S(n)$, which we obtain by taking the quotient of an operad P by an ideal S, inherits an operad structure.

To a collection of elements $z^\alpha \in P(n_\alpha)$, $\alpha \in \mathcal{I}$, in an operad P, we associate the symmetric sequence $\langle z^\alpha, \alpha \in \mathcal{I} \rangle \subset P$ linearly spanned by the composites of the form $p(1, \dots, z^\alpha(q_1, \dots, q_{n_\alpha}), \dots, 1)$, where the factors p and $q_1, \dots, q_{n_\alpha}$ run over the whole operad P. We easily check, by using the axioms of operads, that this symmetric sequence $S = \langle z^\alpha, \alpha \in \mathcal{I} \rangle$ forms an ideal in P and is actually the smallest ideal which contains the elements z^α, $\alpha \in \mathcal{I}$. We also easily check that an operad morphism $\phi : P \to Q$ factors through the quotient $P / \langle z^\alpha, \alpha \in \mathcal{I} \rangle$ if and only if we have the relation $\phi(z^\alpha) = 0$ in Q, for all $\alpha \in \mathcal{I}$. Hence, in the case $P = \mathbb{O}(M)$, an operad morphism $\bar{\phi}_f : \mathbb{O}(M)/\langle z^\alpha, \alpha \in \mathcal{I} \rangle \to Q$ is uniquely determined by a morphism of symmetric collections $f : M \to Q$ whose extension to the free operad $\phi_f : \mathbb{O}(M) \to Q$ cancels the generating elements of the ideal z^α, $\alpha \in \mathcal{I}$. From this observation, we conclude that, in the module context, we can define operads by generators and relations as quotients

$$\mathbb{O}(M : w_0^\alpha = w_1^\alpha, \alpha \in \mathcal{I}) = \mathbb{O}(M)/\langle w_0^\alpha - w_1^\alpha, \alpha \in \mathcal{I} \rangle,$$

where we replace our relations by differences before forming our ideal $S = \langle w_0^\alpha - w_1^\alpha, \alpha \in \mathcal{I} \rangle$.

1.2.10. *Basic examples of operads in module categories.* We can adapt the construction of §1.2.6 to define the module version of the associative (respectively, commutative) operad. We simply replace the generating sets of §1.2.6 by associated free modules (as we explained §1.2.9). We explicitly have:

$$As = \mathbb{O}(\Bbbk\, \mu(x_1, x_2) \oplus \Bbbk\, \mu(x_2, x_1) : \mu(\mu(x_1, x_2), x_3) \equiv \mu(x_1, \mu(x_2, x_3))),$$
$$Com = \mathbb{O}(\Bbbk\, \mu(x_1, x_2) : \mu(\mu(x_1, x_2), x_3) \equiv \mu(x_1, \mu(x_2, x_3))),$$

where we take the same conventions regarding the notation of generating operations as in the context of sets. In what follows, we also use the classical algebraic notation $\mu(x_1, x_2) = x_1 x_2$ for the generating operation of these operads $P = As, Com$.

Formally, the generating symmetric sequence of the associative operad is defined by $M_{As}(2) = \Bbbk[\mu(x_1, x_2), \mu(x_2, x_1)] = \Bbbk[\Sigma_2]$ and $M_{As}(n) = 0$ for $n \neq 2$. The generating symmetric sequence of the commutative operad is defined by $M_{\mathit{Com}}(2) = \Bbbk[\mu(x_1, x_2)] = \Bbbk$ and $M_{\mathit{Com}}(n) = 0$ for $n \neq 2$. By the observations of §1.2.9, we can identify the associative operad and the commutative operad with quotients $\mathit{As} = \Theta(M_{\mathit{As}})/\langle\mu(\mu, 1) - \mu(1, \mu)\rangle$ and $\mathit{Com} = \Theta(M_{\mathit{Com}})/\langle\mu(\mu, 1) - \mu(1, \mu)\rangle$, where we consider the ideal generated by the difference $z(x_1, x_2, x_3) = \mu(\mu(x_1, x_2), x_3) - \mu(x_1, \mu(x_2, x_3))$ to implement the associativity relation.

The next classical example of operad which we consider in this chapter is the Lie operad, which we define by the presentation

$$\mathit{Lie} = \Theta(\Bbbk\,\lambda(x_1, x_2) : \lambda(\lambda(x_1, x_2), x_3) + \lambda(\lambda(x_2, x_3), x_1) + \lambda(\lambda(x_3, x_1), x_2) \equiv 0),$$

where we have a single generating operation of arity two $\lambda = \lambda(x_1, x_2)$ equipped with the action of the symmetric group such that $(1\ 2)\lambda = -\lambda$. The generating symmetric sequence of the Lie operad is accordingly defined by $M_{\mathit{Lie}}(2) = \Bbbk[\lambda(x_1, x_2)] = \Bbbk^{\pm}$, where $\pm$ refers to a twist of the action of permutations by the signature, and we take $M_{\mathit{Lie}}(n) = 0$ for $n \neq 2$. We can also realize this operad as a quotient of the free operad $\Theta(M_{\mathit{Lie}})$ by the ideal generated by the element $z(x_1, x_2, x_3) = \lambda(\lambda(x_1, x_2), x_3) + \lambda(\lambda(x_2, x_3), x_1) + \lambda(\lambda(x_3, x_1), x_2)$. The relation $z(x_1, x_2, x_3) \equiv 0$ corresponds to the classical Jacobi identity of Lie algebras. In what follows, we also use the classical Lie bracket notation $\lambda(x_1, x_2) = [x_1, x_2]$ for our generating operation of the Lie operad $\lambda = \lambda(x_1, x_2) \in \mathit{Lie}(2)$.

In Proposition 1.2.7, we established that the non-unitary associative (respectively, commutative) operad in sets is identified with the non-unitary version of the permutation (respectively, one-point set) operad. In a subsequent chapter (§3.1), we will explain that the free $\Bbbk$-module functor $\Bbbk[-] : \mathcal{S}et \to \mathcal{M}od$ induces a functor on operads and we can use this process to associate an operad in $\Bbbk$-modules to any operad in sets. We can easily adapt the arguments of Proposition 1.2.7 to identify the operad As (respectively, Com) given by our presentation by generators and relations in the category of $\Bbbk$-modules with the image of the non-unitary permutation (respectively, one-point set) operad under this functor $\Bbbk[-]$. Hence, we have the identity:

$$\mathit{As}(r) = \begin{cases} 0, & \text{if } r = 0, \\ \Bbbk[\Sigma_r], & \text{otherwise,} \end{cases}$$

for the associative operad in the category of $\Bbbk$-modules, and the identity:

$$\mathit{Com}(r) = \begin{cases} 0, & \text{if } r = 0, \\ \Bbbk, & \text{otherwise,} \end{cases}$$

for the commutative operad in the category of $\Bbbk$-modules.

1.2.11. *The underlying symmetric sequence of classical operads.* In this paragraph, we revisit the correspondence between operad elements and abstract operations in the module context, and for the associative, commutative and Lie operads $P = \mathit{As}, \mathit{Com}, \mathit{Lie}$. To simplify, we still focus on the case of the non-unitary version of the associative and commutative operads $P = \mathit{As}, \mathit{Com}$. We have $P(0) = 0$ for all these operads $P = \mathit{As}, \mathit{Com}, \mathit{Lie}$ and we therefore focus on the terms of arity $r > 0$ of our objects in what follows. This vanishing relation $P(0) = 0$ actually follows from the restriction of the generating operations to terms of arity $r > 0$ in our presentation of these operads.

In the case of the associative operad As, an element $p(x_1, \dots, x_r) \in \mathit{As}(r)$ is obtained by a multiple composition of an associative product $\mu(x_1, x_2) = x_1 x_2$

together with an appropriate shift of the variable indices which ensures that each variable x_i occurs once and only once in the expression of this operation $p = p(x_1, \dots, x_r)$. Hence, the term of arity r of the associative operad $\mathsf{As}(r)$ is identified with the module spanned by the monomials $p(x_1, \dots, x_r)$ on r non-commutative variables $(x_1, \dots, x_r)$ which have degree one with respect to each variable. In standard algebraic notation, such a monomial is written $p(x_1, \dots, x_r) = x_{i_1} \cdot \dots \cdot x_{i_r}$, and the degree requirement is equivalent to the assumption that the sequence $(i_1, \dots, i_r)$ forms a permutation of $(1, \dots, r)$. Hence, we have the identity:

$$(1) \qquad \mathsf{As}(r) = \bigoplus_{s \in \Sigma_r} \Bbbk\, x_{s(1)} \cdot \dots \cdot x_{s(r)} = \Bbbk[\Sigma_r], \quad \text{for all } r > 0,$$

and we retrieve the observation that $\mathsf{As}(r)$ is the regular representation of the symmetric group Σ_r from this expression.

The term of arity r of the commutative operad $\mathsf{Com}(r)$ is similarly identified with the module spanned by the operations $p = p(x_1, \dots, x_r)$ which form a monomial on r commutative variables $(x_1, \dots, x_r)$ and which have degree one with respect to each of these variables $(x_1, \dots, x_r)$. In standard algebraic notation, such a monomial is written $p(x_1, \dots, x_r) = x_1 \cdot \dots \cdot x_r$. Hence, we have the identity:

$$(2) \qquad \mathsf{Com}(r) = \Bbbk\, x_1 \cdot \dots \cdot x_r = \Bbbk, \quad \text{for all } r > 0,$$

from which we retrieve the identity between $\mathsf{Com}(r)$ and the free $\Bbbk$-module of rank one equipped with a trivial action of the symmetric group Σ_r.

In the case of the Lie operad Lie, we consider the module spanned by all Lie monomials $p(x_1, \dots, x_r)$ which have degree one with respect to each variable x_i. The determination of the module structure of $\mathsf{Lie}(r)$ is more intricate than in the case of the commutative and associative operads. Nevertheless, one can prove (see [155, §5.6.2] for instance) that $\mathsf{Lie}(r)$ has a basis of the form:

$$(3) \qquad \mathsf{Lie}(r) = \bigoplus_{\substack{s \in \Sigma_r \\ s(1)=1}} \Bbbk[\cdots[[x_{s(1)}, x_{s(2)}], x_{s(3)}], \dots, x_{s(r)}], \quad \text{for all } r > 0,$$

where we use the Lie bracket notation $\lambda(x_1, x_2) = [x_1, x_2]$ for the generating operation of our operad $\lambda = \lambda(x_1, x_2) \in \mathsf{Lie}(2)$. This object $\mathsf{Lie}(r)$ therefore forms a free module of rank $(r-1)!$. In the case $\mathbb{Q}[e^{2i\pi/r}] \subset \Bbbk$, we also have an identity between $\mathsf{Lie}(r)$ and the representation $\mathsf{Lie}(r) = \operatorname{Ind}_{C_r}^{\Sigma_r} \chi$, where C_r denotes the cyclic group generated by the r-cycle $(1\ 2\ \cdots\ r) \in \Sigma_r$ and χ denotes the one-dimensional representation of C_r associated to the character $\chi(1\ 2\ \cdots\ r) = e^{2i\pi/r}$ (see [155, §8.2] for a general reference on this subject).

1.2.12. *The example of the Poisson operad.* To complete our examples, we examine the definition of the Poisson operad. (We will see that a graded version of this operad occurs as the homology of E_n-operads.) We define this operad by the presentation

$$\begin{aligned} \mathsf{Pois} = \ \Theta(\Bbbk\, \mu(x_1, x_2) \oplus \Bbbk\, \lambda(x_1, x_2) :& \\ & \mu(\mu(x_1, x_2), x_3) \equiv \mu(x_1, \mu(x_2, x_3)), \\ & \lambda(\lambda(x_1, x_2), x_3) + \lambda(\lambda(x_2, x_3), x_1) + \lambda(\lambda(x_3, x_1), x_2) \equiv 0, \\ & \lambda(\mu(x_1, x_2), x_3) \equiv \mu(\lambda(x_1, x_3), x_2) + \mu(x_1, \lambda(x_2, x_3))), \end{aligned}$$

where $\mu = \mu(x_1, x_2)$ is a symmetric generating operation, fixed by the action of the transposition $(1\ 2)\mu = \mu$, and $\lambda = \lambda(x_1, x_2)$ is an antisymmetric generating operation, which the action of the transposition carries to its opposite $(1\ 2)\lambda = -\lambda$. From this construction, we see that the Poisson operad is a combination of the commutative operad $\mathsf{Com} = \mathbb{O}(\Bbbk\,\mu(x_1,x_2) : \mu(\mu(x_1,x_2),x_3) \equiv \mu(x_1,\mu(x_2,x_3)))$ and of the Lie operad $\mathsf{Lie} = \mathbb{O}(\Bbbk\,\lambda(x_1,x_2) : \lambda(\lambda(x_1,x_2),x_3) + \lambda(\lambda(x_2,x_3),x_1) + \lambda(\lambda(x_3,x_1),x_2) \equiv 0)$, together with an additional distribution relation

$$\lambda(\mu(x_1,x_2),x_3) \equiv \mu(\lambda(x_1,x_3),x_2) + \mu(x_1,\lambda(x_2,x_3)),$$

called the Poisson relation, which mixes the operations of both operads. In what follows, we may still use the standard product notation $\mu(x_1,x_2) = x_1x_2$ and the Lie bracket notation $\lambda(x_1,x_2) = [x_1,x_2]$ for the image of the generating operations of the commutative operad and of the Lie operad in Pois.

The commutative operad (respectively, the Lie operad) can be identified with the suboperad of the Poisson operad generated by the element $\mu \in \mathsf{Pois}(2)$ (respectively, $\lambda \in \mathsf{Pois}(2)$). The Poisson relation implies that each composite of a Lie operation with commutative operations (in that order) can be rewritten as a composite of a commutative operation with Lie operations. One can prove by elaborating on this remark that $\mathsf{Pois}(r)$ is identified with the $\Bbbk$-module spanned by formal products

$$p(x_1,\dots,x_r) = p_1(x_{11},\dots,x_{1r_1})\cdot\ldots\cdot p_m(x_{m1},\dots,x_{mr_m}),$$

whose factors $p_i(x_{i1},\dots,x_{ir_i})$ run over Lie monomials on r_i variables x_{ik}, $k \in \{1,\dots,r_i\}$, such that each variable x_{ik} occurs once and only once in this Lie monomial $p_i(x_{i1},\dots,x_{ir_i})$ and the sets $\{x_{i1},\dots,x_{ir_i}\}$, $i = 1,\dots,r$, represent the components of a partition of the total set of variables $\{x_1,\dots,x_r\}$ of our operation $p = p(x_1,\dots,x_r)$.

1.2.13. *The case of non-unitary operads and of connected operads.* We consider operads in a general base symmetric monoidal category again. We just assume that the tensor product of this base category distributes over colimits (as usual all through this chapter) so that the constructions of this section makes sense. Recall that a non-unitary operad is an operad P in our base category $\mathcal{M}$ such that $P(0) = \varnothing$. Recall also that we use the notation $\mathcal{O}p_{\varnothing}$ for the category of non-unitary operads (see §1.1.20).

We can easily adapt the definition of free operads in the context of non-unitary operads. We then consider the category $\mathcal{S}eq_{>0}$ whose objects are the symmetric sequences M such that $M(0) = \varnothing$, where we use the same conventions as in the context of operads (namely, we use this expression $M(0) = \varnothing$ to assert either that our symmetric sequence is not defined in arity zero, or that the component of arity zero of our symmetric sequence is the initial object of the base category $\varnothing \in \mathcal{M}$). We call *non-unitary symmetric sequences* the objects of this category $\mathcal{S}eq_{>0}$. We have an obvious forgetful functor $\omega : \mathcal{O}p_{\varnothing} \to \mathcal{S}eq_{>0}$ from the category of non-unitary operads $\mathcal{O}p_{\varnothing}$ to this category of non-unitary symmetric sequences $\mathcal{S}eq_{>0}$.

To be more precise, we used in §1.1.20 that we can identify the category of non-unitary operads $\mathcal{O}p_{\varnothing}$ with the full subcategory of the category of operads formed by the objects $P \in \mathcal{O}p$ whose component of arity zero is defined by the initial object of our base category $P(0) = \varnothing$ (as soon as we assume that the tensor product of our base category distributes over colimits). We can similarly identify our category of non-unitary symmetric sequences $\mathcal{S}eq_{>0}$ with the full subcategory of the category of symmetric sequences formed by the objects $M \in \mathcal{S}eq$ whose component of arity zero is defined by the initial object of our base category $M(0) = \varnothing$ (in the same context as in the case of operads). Our forgetful functor $\omega : \mathcal{O}p_{\varnothing} \to \mathcal{S}eq_{>0}$ defines an obvious restriction of the forgetful functor $\omega : \mathcal{O}p \to \mathcal{S}eq$ on the category of all operads $\mathcal{O}p$ which we consider in Theorem 1.2.1. We can check (by using the explicit construction of §A) that the free operad $\mathbb{F}(M) \in \mathcal{O}p$ associated to a non-unitary symmetric sequence $M \in \mathcal{S}eq_{>0}$ is non-unitary $\mathbb{F}(M)(0) = \varnothing$. The free operad functor of Theorem 1.2.1 therefore induces a functor $\mathbb{F} : \mathcal{S}eq_{>0} \to \mathcal{O}p_{\varnothing}$ from the category of non-unitary symmetric sequences $\mathcal{S}eq_{>0}$ to the category of non-unitary operads $\mathcal{O}p_{\varnothing}$ and this functor fits in an obvious restriction of the adjunction relation of Theorem 1.2.1.

Recall that a non-unitary operad P is connected if we have the relation $P(1) = \mathbb{1}$ in addition to $P(0) = \varnothing$. If the base category is pointed, in the sense that initial and terminal objects coincide in $\mathcal{M}$, then any connected operad P inherits a natural augmentation $\epsilon : P \to I$, which is given by the identity morphism of the unit object $\mathbb{1}$ in arity one and by the terminal morphism on the object $P(r)$ otherwise. This augmentation obviously defines a morphism in the category of operads, and accordingly, the unit operad I forms a terminal object in the category of connected operads in addition to define an initial object. This observation implies that the category of connected operads is pointed (unlike the whole category of operads) whenever the base category is so. We need to modify our definitions in order to give a sense to the notion of a free object in this context. We proceed as follows.

First, to a connected operad P, we associate the symmetric sequence $\bar{P}$ such that

$$\bar{P}(r) = \begin{cases} \varnothing, & \text{if } r = 0, 1, \\ P(r), & \text{otherwise.} \end{cases}$$

We call this symmetric sequence the *augmentation ideal* of P, because we can identify this object with the kernel of the augmentation morphism $\epsilon : P \to I$ when the base category is pointed. (Nonetheless, we may consider the symmetric sequence $\bar{P}$ outside the pointed category context, where the definition of the augmentation morphism $\epsilon : P \to I$ does not make sense.)

Recall that we denote the category of connected operads by $\mathcal{O}p_{\varnothing 1}$ and that we can also identify this category with a full subcategory of the category of non-unitary operads $\mathcal{O}p_{\varnothing}$. For our purpose, we also consider the category $\mathcal{S}eq_{>1}$ formed by the symmetric sequences such that $M(0) = M(1) = \varnothing$. We call *connected symmetric sequences* the objects of this category $\mathcal{S}eq_{>1}$. The mapping $\bar{\omega} : P \mapsto \bar{P}$ gives a functor, denoted by $\bar{\omega} : \mathcal{O}p_{\varnothing 1} \to \mathcal{S}eq_{>1}$, from the category of connected operads $\mathcal{O}p_{\varnothing 1}$ towards the category of connected symmetric sequences $\mathcal{S}eq_{>1}$. Now, we have the following statement:

Theorem 1.2.14. *The free operad $\mathbb{F}(M)$ associated to a connected symmetric sequence $M \in \mathcal{S}eq_{>1}$ forms a connected (non-unitary) operad and the map $\mathbb{F} : M \mapsto \mathbb{F}(M)$ defines a left adjoint of the functor $\bar{\omega} : \mathcal{O}p_{\varnothing 1} \to \mathcal{S}eq_{>1}$ which maps any connected operads $P \in \mathcal{O}p_{\varnothing 1}$ to its augmentation ideal $\bar{P} \in \mathcal{S}eq_{>1}$.*

This theorem, which is essentially a follow-up of Theorem 1.2.1, is formally established in §A.4. In our account of the construction of free operads, we briefly explain that the components of the free operad $\mathbb{F}(M)$ are given by colimits over tree categories. In the case of a connected sequence $M \in \mathcal{S}eq_{>1}$, our expansion has a reduced expression of the form:

$$\Theta(M)(r) = \operatorname*{colim}_{\underline{\mathrm{T}} \in \widetilde{\mathcal{T}ree}(r)} M(\underline{\mathrm{T}}),$$

for each $r \in \mathbb{N}$, where we now consider a category $\widetilde{\mathcal{T}ree}(r)$ formed by trees $\underline{\mathrm{T}} \in \widetilde{\mathcal{T}ree}(r)$ of which all vertices have at least two ingoing edges (see §A.4). We refer to this subcategory of trees $\widetilde{\mathcal{T}ree}(r)$ as the category of *reduced trees* with r ingoing edges (for short, we also speak about *reduced r-trees*).

In general, an operad defined by a presentation by generators and relations $P = \Theta(M : w_0^\alpha = w_1^\alpha, \alpha \in \mathcal{I})$ is connected (in our sense) if and only if the generating sequence of this operad M vanishes in arity $r = 0, 1$ (and hence, is connected as a symmetric sequence). From this observation, we retrieve that the (non-unitary) associative operad As is connected, like the (non-unitary) commutative operad Com, and the Lie operad Lie.

1.2.15. *The adjunctions between connected operads, non-unitary operads, and the category of all operads.* Recall that the category of connected operads $\mathcal{O}p_{\varnothing 1}$, which we characterize by $P(0) = \varnothing$ and $P(1) = \mathbb{1}$, forms a full subcategory of the category of non-unitary operads $\mathcal{O}p_{\varnothing}$.

We easily see that the category embedding $\iota : \mathcal{O}p_{\varnothing 1} \hookrightarrow \mathcal{O}p_{\varnothing}$ has a right adjoint $\tau : \mathcal{O}p_{\varnothing} \to \mathcal{O}p_{\varnothing 1}$ which maps any non-unitary operad $P \in \mathcal{O}p_{\varnothing}$ to a connected operad τP such that $\tau P(0) = \varnothing$, $\tau P(1) = \mathbb{1}$, and $\tau P(r) = P(r)$ for $r > 1$. We just use the unit morphism $\eta : \mathbb{1} \to P(1)$ to restrict the composition products of the operad P to this object τP when we deal with composition products which involve the term of arity one $\tau P(1) = \mathbb{1}$. We accordingly get that this collection τP forms a suboperad of P and the proof that this object τP fits our adjunction relation between operads and connected operads is immediate.

Recall that the category of non-unitary operads $\mathcal{O}p_{\varnothing}$ is identified with a full subcategory of general operads $\mathcal{O}p$ when the tensor product of the base category distributes over colimits. In this situation, we also get that the category embedding $\iota : \mathcal{O}p_{\varnothing} \to \mathcal{O}p$ has a right adjoint which is given by an obvious truncation functor on the category of operads $\mathcal{O}p$. We use the same notation as in the connected case τ for this truncation functor from the category of general operads $\mathcal{O}p$ to the category of non-unitary operads $\mathcal{O}p_{\varnothing}$.

From our construction of colimits of operads, we can readily check that:

Proposition 1.2.16. *The category embeddings $\mathcal{O}p_{\varnothing 1} \hookrightarrow \mathcal{O}p_{\varnothing} \hookrightarrow \mathcal{O}p$ create colimits.* □

1.3. Categorical constructions for algebras over operads

In the previous section, we focused on the application of categorical constructions to operads. We now study the applications of such constructions to the categories of algebras associated to operads. We first explain that the construction of operads by generators and relations reflects the usual definition of algebras in terms of generating operations $\xi : A^{\otimes r} \to A$ and relations.

We also give a version of the categorical constructions of §1.2 (free objects, colimits and limits) for the categories of algebras associated to operads. We are precisely going to observe (following [140]) that the categories of algebras associated to operads are identified with categories of algebras equipped which free objects of a particular form. By the way, we check that any morphism of operads determine adjoint extension and restriction functors between the categories of algebras associated to our operads. We will see that classical functors which connect the categories of associative, commutative, and Lie algebras, are identified with functors of this form.

1.3.1. *Basic examples of categories of algebras associated to operads.* Recall (see Proposition 1.1.15) that defining an action of an operad P on an object $A \in \mathcal{M}$ amounts to giving an operad morphism $\phi : P \to \mathtt{End}_A$, where $\mathtt{End}_A$ denotes the endomorphism operad of A. In the case of an operad defined by a presentation by generators and relations

$$P = \Theta(M : w_0^\alpha = w_1^\alpha, \alpha \in \mathcal{I}),$$

we deduce, from the observations of §1.2.5, that such a morphism $\phi_f : P \to \mathtt{End}_A$ is uniquely determined by a morphism of symmetric sequences $f : M \to \mathtt{End}_A$, which maps the abstract generating operations $\xi \in M(r)$ to actual maps $\xi : A^{\otimes r} \to A$ such that the identities $w_0^\alpha \equiv w_1^\alpha$ hold in $\mathtt{End}_A$.

For our basic examples of (non-unitary) operads in the category of $\mathbb{k}$-modules $P = \mathit{Com}, \mathit{As}, \mathit{Lie}$, we obtain the following statements:

(a) An algebra over the commutative operad Com is a module A equipped with a product $\mu : A \otimes A \to A$, which satisfies the symmetry relation

$$\mu(a_1, a_2) = \mu(a_2, a_1),$$

for all $a_1, a_2 \in A$, and the associativity relation

$$\mu(\mu(a_1, a_2), a_3) = \mu(a_1, \mu(a_2, a_3)),$$

for all $a_1, a_2, a_3 \in A$.

(b) An algebra over the associative operad As is a module A equipped with a product $\mu : A \otimes A \to A$ which satisfies the associativity relation

$$\mu(\mu(a_1, a_2), a_3) = \mu(a_1, \mu(a_2, a_3)),$$

for all $a_1, a_2, a_3 \in A$ (but no symmetry requirement).

(c) An algebra over the Lie operad Lie is a module $\mathfrak{g}$ equipped with an operation $\lambda : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ which satisfies the antisymmetry relation

$$\lambda(x_1, x_2) = -\lambda(x_2, x_1),$$

for all $x_1, x_2 \in \mathfrak{g}$, and the Jacobi identity

$$\lambda(\lambda(x_1, x_2), x_3) + \lambda(\lambda(x_2, x_3), x_1) + \lambda(\lambda(x_3, x_1), x_2) = 0,$$

for all $x_1, x_2, x_3 \in \mathfrak{g}$.

We can give a similar description of the category of algebras associated to the Poisson operad (see §1.2.12).

In characteristic 2, we do not necessarily assume that the generating operation of a Lie-algebra $\mathfrak{g}$ satisfies the relation $\lambda(x, x) = 0$. To associate a category of algebras which satisfy this condition to the Lie operad Lie, we have to modify the definition of an algebra over an operad (see [65, §§1.2.12-1.2.16]).

The result of Proposition 1.1.17-1.1.18 (in the non-unitary context) is equivalent to the combination of the above observations (a-b) with the result of Proposition 1.2.7.

1.3.2. *The category of algebras associated to an operad and free algebras.* Recall that the algebras associated to a given operad P form a category $\mathcal{P}$ with, as morphisms, the morphisms of the base category $f : A \to B$ which preserve the P-actions on A and B. We have an obvious forgetful functor $\omega : \mathcal{P} \to \mathcal{M}$ from the category of P-algebras $\mathcal{P}$ towards the base category $\mathcal{M}$.

We can form a functor in the converse direction by considering a generalized symmetric tensor algebra

$$\mathbb{S}(\mathsf{P}, X) = \coprod_{n=0}^{\infty} (\mathsf{P}(n) \otimes X^{\otimes n})_{\Sigma_n},$$

for any object $X \in \mathcal{M}$, where the notation $(-)_{\Sigma_n}$ refers to the performance of a coinvariant construction in order to identify the natural action of the symmetric group Σ_n on the tensor power $X^{\otimes n}$ with the internal Σ_n-structure of the object $\mathsf{P}(n)$. In the case of a concrete symmetric monoidal category, we can define the object $(\mathsf{P}(n) \otimes X^{\otimes n})_{\Sigma_n}$ as the quotient of the tensor product $\mathsf{P}(n) \otimes X^{\otimes n}$ under the relations such that:

$$p \otimes (x_{s(1)} \otimes \cdots \otimes x_{s(n)}) \equiv s \cdot p \otimes (x_1 \otimes \cdots \otimes x_n),$$

for $p \in \mathsf{P}(n)$, $x_1 \otimes \cdots \otimes x_n \in X^{\otimes n}$, and where we assume that s runs over the symmetric group Σ_n.

We have natural evaluation morphisms

$$\lambda : \mathsf{P}(r) \otimes \mathbb{S}(\mathsf{P}, X)^{\otimes r} \to \mathbb{S}(\mathsf{P}, X)$$

which we define termwise by the morphisms

$$\begin{aligned} \mathsf{P}(r) \otimes (\mathsf{P}(n_1) \otimes X^{\otimes n_1})_{\Sigma_{n_1}} \otimes \cdots \otimes (\mathsf{P}(n_r) \otimes X^{\otimes n_r})_{\Sigma_{n_r}} & \\ \to (\mathsf{P}(n_1 + \cdots + n_r) \otimes X^{\otimes n_1 + \cdots + n_r})_{\Sigma_{n_1 + \cdots + n_r}} & \end{aligned}$$

induced by the composition products of our operad P. The axioms of operads imply that these morphisms satisfy the equivariance, unit and associativity relations of §1.1.13. We therefore obtain that the object $\mathbb{S}(\mathsf{P}, X) \in \mathcal{M}$ forms a P-algebra, naturally associated to any object $X \in \mathcal{M}$, so that the mapping $\mathbb{S}(\mathsf{P}) : X \mapsto \mathbb{S}(\mathsf{P}, X)$ defines a functor $\mathbb{S}(\mathsf{P}) : \mathcal{M} \to \mathcal{P}$.

The evaluation morphisms of a P-algebra A induce morphisms $\lambda : (\mathsf{P}(n) \otimes A^{\otimes n})_{\Sigma_n} \to A$ by equivariance, for all $n \geq 0$. We can put these morphisms together to get a single natural morphism $\lambda : \mathbb{S}(\mathsf{P}, A) \to A$ defined on the object $\mathbb{S}(\mathsf{P}, A) = \coprod_{n=0}^{\infty} (\mathsf{P}(n) \otimes A^{\otimes n})_{\Sigma_n}$. From the associativity axiom of operad actions, we easily check that this morphism $\lambda : \mathbb{S}(\mathsf{P}, A) \to A$ preserves the P-algebra structures attached to our objects, and hence, defines a morphism in the category of P-algebras. In the converse direction, for any object $X \in \mathcal{M}$, we have a natural morphism $\iota : X \to \mathbb{S}(\mathsf{P}, X)$ which is given by the composite

$$X \xrightarrow{\simeq} \mathbb{1} \otimes X \xrightarrow{\eta \otimes X} \mathsf{P}(1) \otimes X = (\mathsf{P}(1) \otimes X)_{\Sigma_1} \hookrightarrow \coprod_{n=0}^{\infty} (\mathsf{P}(n) \otimes X^{\otimes n})_{\Sigma_n},$$

where η refers to the unit morphism of the operad P.

We have the following statement:

Proposition 1.3.3. *The functor $\mathbb{S}(\mathsf{P}) : \mathcal{M} \to \mathcal{P}$ is left adjoint to the forgetful functor $\omega : \mathcal{P} \to \mathcal{M}$. The morphism $\iota : X \to \mathbb{S}(\mathsf{P}, X)$ (respectively, $\lambda : \mathbb{S}(\mathsf{P}, A) \to A$) defines the unit (respectively, the augmentation) of this adjunction relation.*

Explanations and proof. We aim to prove that we have a natural bijection:

$$\mathtt{Mor}_{\mathcal{P}}(\mathbb{S}(\mathsf{P}, X), A) \simeq \mathtt{Mor}_{\mathcal{M}}(X, A),$$

for any $X \in \mathcal{M}$ and for any $A \in \mathcal{P}$. In one direction, to a morphism of P-algebras $\phi : \mathbb{S}(\mathsf{P}, X) \to A$ we associate the morphism $f = \phi\iota$ in the base category. In the other direction, to a morphism in the base category $f : X \to A$ we associate the morphism $\phi_f = \lambda \cdot \mathbb{S}(\mathsf{P}, f)$ in the category of P-algebras. The adjunction augmentation itself $\lambda : \mathbb{S}(\mathsf{P}, A) \to A$ is the morphism of P-algebras ϕ_{id} associated to the identity of A, regarded as an object of the base category $\mathcal{M}$.

By a general result of category theory (see [130, §IV.1]), we just have to check that the composites

$$A \xrightarrow{\iota} \mathbb{S}(\mathsf{P}, A) \xrightarrow{\lambda} A \quad \text{and} \quad \mathbb{S}(\mathsf{P}, X) \xrightarrow{\mathbb{S}(\mathsf{P},\iota)} \mathbb{S}(\mathsf{P}, \mathbb{S}(\mathsf{P}, X)) \xrightarrow{\lambda} \mathbb{S}(\mathsf{P}, X)$$

are both identity morphisms to conclude that our mappings $\phi \mapsto \phi\iota$ and $f \mapsto \phi_f$ are converse to each other, and hence do give an adjunction bijection. This result follows from the unit axiom of operad actions as regards the first of our composites and from the unit axiom of operads as regards the second one. □

The result of Proposition 1.3.3 has, like Theorem 1.2.1, an equivalent formulation in terms of universal properties. From this point of view, the functor $\mathbb{S}(\mathsf{P}) : \mathcal{M} \to \mathcal{P}$ represents the mapping which associates a free object in the category of P-algebras to any object X of the base category $\mathcal{M}$. We make the universal property of this free object explicit in the next proposition:

Proposition 1.3.4. *Any morphism in the base category $f : X \to A$, where A is a P-algebra, admits a unique factorization*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\iota"'] & & A \\ & \mathbb{S}(\mathsf{P}, X) \arrow[ur, dotted, "\exists! \phi_f"'] & \end{tikzcd}$$

such that ϕ_f is a morphism of P-algebras.

Explanations. The morphism ϕ_f such that $\phi_f\iota = f$ represents the morphism of P-algebras ϕ_f which we associate to any given morphism of the base category $f : X \to A$ in the correspondence of Proposition 1.3.3. Thus, the uniqueness of our factorization is equivalent to the claim that the correspondence of Proposition 1.3.3 defines a bijection (as required to define an adjunction relation). □

1.3.5. *Basic examples of free algebras.* The aritywise expression of our fundamental examples of operads $\mathsf{P} = \mathsf{Com}, \mathsf{As}, \mathsf{Lie}$ can be retrieved from the classical expansions of free objects in the categories of algebras associated to these operads:

(a) The free commutative algebra (without unit) is identified with (the augmentation ideal of) the symmetric algebra

$$\overline{\mathbb{S}}(K) = \bigoplus_{n=1}^{\infty} (K^{\otimes n})_{\Sigma_n}$$

together with the commutative product yielded by the process of tensor concatenation. From this statement, we retrieve the identity $\mathsf{Com}(n) = \mathbb{k}$ of §1.2.11(2).

(b) The free associative algebra (without unit) is identified with (the augmentation ideal of) the tensor algebra

$$\overline{\mathbb{T}}(K) = \bigoplus_{n=1}^{\infty} K^{\otimes n}$$

together with an associative product defined by the concatenation of tensors. We can also retrieve the identity $\mathsf{As}(n) = \Bbbk[\Sigma_n]$ of §1.2.11(1) from this statement since we have $K^{\otimes n} = (\Bbbk[\Sigma_n] \otimes K^{\otimes n})_{\Sigma_n}$ for any $n \in \mathbb{N}$.

(c) The structure of free Lie algebras is more intricate. Nevertheless, in characteristic 0, we can apply the Milnor-Moore theorem to identify the free Lie algebra $\mathbb{L}(K)$ with the primitive part $\mathit{Prim}\,\mathbb{T}(K)$ of the (unitary version of the) tensor algebra $\mathbb{T}(K)$, which we equip with the shuffle coproduct of tensors (we recall the definition of these notions of the theory of Hopf algebras with full details in §7.2 and we explain this relation $\mathbb{L}(K) = \mathbb{P}\,\mathbb{T}(K)$ in Proposition 7.2.6 and in Proposition 7.2.14). Moreover, we have versions of the Milnor-Moore theorem which enable us to deduce an expansion of the form

$$\mathbb{L}(K) = \bigoplus_{n=1}^{\infty} (\mathsf{Lie}(n) \otimes K^{\otimes n})_{\Sigma_n}$$

from this relation $\mathbb{L}(K) = \mathbb{P}\,\mathbb{T}(K)$.

We keep focusing on non-unitary algebras in this paragraph, but the identifications of (a-b) obviously extend to the unitary setting.

The result of Proposition 1.2.4 (about the definition of limits and colimits in the category of operads) has the following analogue for the category of algebras associated to an operad:

Proposition 1.3.6. *Let P be any operad.*

(a) *The forgetful functor from P-algebras to the base category creates all small limits, the filtered colimits, and the coequalizers which are reflexive in the base category.*

(b) *The category of P-algebras admits coproducts too and, as a consequence, all kinds of small colimits, though the forgetful functor from P-algebras to the base category does not preserve colimits in general.*

Recall that we devote an appendix section §1.4 to a reminder on filtered colimits and reflexive coequalizers.

Proof. We deduce this result from the same argument lines as in the proof of Proposition 1.2.4 (see also [66, §3.3] or [156, Proposition 2.3.5] for a detailed proof of this proposition). ☐

1.3.7. *Restriction functors.* If an operad morphism $\phi : \mathsf{P} \to \mathsf{Q}$ is given, then we can define comparison functors that connect the category of P-algebras $\mathcal{P}$ and the category of Q-algebras $\mathcal{Q}$. First, we immediately get that any Q-algebra B inherits a natural P-algebra structure since the operad P acts on B through the morphism $\phi : \mathsf{P} \to \mathsf{Q}$. Thus we have a natural functor $\phi^* : \mathcal{Q} \to \mathcal{P}$, from the category of Q-algebras $\mathcal{Q}$ to the category of P-algebras $\mathcal{P}$. We call this functor ϕ^* the *restriction functor* associated to our operad morphism ϕ. The existence of reflexive coequalizers can be used to define a functor in the converse direction:

Proposition 1.3.8. *The restriction functor* $\phi^* : \mathcal{Q} \to \mathcal{P}$*, associated to any operad morphism* $\phi : \mathsf{P} \to \mathsf{Q}$*, has a left adjoint* $\phi_! : \mathcal{P} \to \mathcal{Q}$*. We call this functor* $\phi_!$ *the* extension functor *associated to* $\phi : \mathsf{P} \to \mathsf{Q}$*.*

Proof. Let $A \in \mathcal{P}$. Let $\phi_! A$ be the Q-algebra defined by the reflexive coequalizer such that

$$\begin{tikzcd} \mathbb{S}(\mathsf{Q}, \mathbb{S}(\mathsf{P}, A)) \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & \mathbb{S}(\mathsf{Q}, A) \arrow[l, bend right, "s_0"'] \arrow[r, dotted] & \phi_! A \end{tikzcd} ,$$

where:

- the morphism d_0 is the morphism of free Q-algebras induced by the adjunction augmentation $\lambda : \mathbb{S}(\mathsf{P}, A) \to A$ associated to the P-algebra A;
- the morphism d_1 is induced by the mapping $\mathbb{S}(\phi, A) : \mathbb{S}(\mathsf{P}, A) \to \mathbb{S}(\mathsf{Q}, A)$ which we define by using the functoriality of the generalized symmetric algebra construction with respect to the coefficients;
- and the reflection s_0 is the morphism of free Q-algebras induced by the universal morphism $\iota : A \to \mathbb{S}(\mathsf{P}, A)$ of the free P-algebra $\mathbb{S}(\mathsf{P}, A)$.

We easily check, by using the universal property of free Q-algebras, that giving a morphism of Q-algebras $g : \phi_! A \to B$ amounts to giving a morphism in the base category $f : A \to B$ which preserves the action of the operad Q on our objects. Therefore the mapping $\phi_! : A \mapsto \phi_! A$ defines a left adjoint of the restriction functor $\phi^* : B \mapsto \phi^* B$. ☐

1.3.9. *Basic examples of extension and restriction functors.* The commutative, associative and Lie operads are connected by morphisms

$$\mathsf{Lie} \xrightarrow{\iota} \mathsf{As} \xrightarrow{\alpha} \mathsf{Com}$$

which we determine on the generating operations of our operads $\lambda \in \mathsf{Lie}(2)$, $\mu \in \mathsf{As}(2)$ and $\mu \in \mathsf{Com}(2)$ by the formulas $\iota(\lambda) = \mu - (1\ 2)\mu$ and $\alpha(\mu) = \mu$.

The restriction functor $\alpha^* : \mathcal{C}om \to \mathcal{A}s$ is identified with the obvious embedding of the category of commutative algebras into the category of associative algebras. The restriction functor $\iota^* : \mathcal{A}s \to \mathcal{L}ie$ is identified with the standard functor which maps an associative algebra A to the Lie algebra $\iota^* A = A_-$ with the same underlying module as A and the commutator $\lambda(a_1, a_2) = \mu(a_1, a_2) - \mu(a_2, a_1) = a_1 a_2 - a_2 a_1$ as Lie bracket $[a_1, a_2] = \lambda(a_1, a_2)$.

The extension functor $\alpha_! : \mathcal{A}s \to \mathcal{C}om$, defined as the left adjoint of $\alpha^* : \mathcal{C}om \to \mathcal{A}s$, can be identified with the functor which maps an associative algebra A to the quotient $A/\langle [A, A] \rangle$, where $\langle [A, A] \rangle$ denotes the two-sided ideal of A generated by the commutators $[a_1, a_2] = \lambda(a_1, a_2) = \mu(a_1, a_2) - \mu(a_2, a_1)$, for $a_1, a_2 \in A$. The extension functor $\iota_! : \mathcal{L}ie \to \mathcal{A}s$, which we define as the left adjoint of the restriction functor $\iota^* : \mathcal{A}s \to \mathcal{L}ie$, can be identified with the functor which maps a Lie algebra $\mathfrak{g}$ to the augmentation ideal $\bar{\mathbb{U}}(\mathfrak{g})$ of the enveloping algebra $\mathbb{U}(\mathfrak{g})$ associated to $\mathfrak{g}$. Briefly recall that the enveloping algebra $\mathbb{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is the quotient of the (unitary) tensor algebra $\mathbb{T}(\mathfrak{g})$ associated to $\mathfrak{g}$ by the two-sided ideal generated by the differences $a_1 a_2 - a_2 a_1 - [a_1, a_2]$, $a_1, a_2 \in \mathfrak{g}$, where $\mu(a_1, a_2) = a_1 a_2$ refers to the product of $\mathbb{T}(\mathfrak{g})$ and $\lambda(a_1, a_2) = [a_1, a_2]$ refers to the internal Lie bracket of our Lie algebra $\mathfrak{g}$ (we review this definition with full details in §7.2.9). We easily check that these functors $\alpha_! : A \mapsto A/\langle [A, A] \rangle$ and $\iota_! : \mathfrak{g} \mapsto \bar{\mathbb{U}}(\mathfrak{g})$ satisfy the adjunction relation of extension functors and hence are isomorphic to the operadic extension functors of Proposition 1.3.8 which we associate to our morphisms $\alpha : \mathsf{As} \to \mathsf{Com}$ and $\iota : \mathsf{Lie} \to \mathsf{A}$.

1.3.10. *Algebras over connected operads.* The structure of an algebra over the unit operad I (see §1.2.3) reduces to an identity operation. Hence, the category of I-algebras is simply nothing but the base category $\mathcal{M}$. In the context of a pointed category (see §1.2.13), the existence of an augmentation $\epsilon : \mathsf{P} \to \mathsf{I}$, when P is a connected operad, implies that any object $X \in \mathcal{M}$ inherits a P-algebra structure, which is simply given by a trivial action in arity $r > 1$. In the context of $\Bbbk$-modules, the application of this construction to the classical examples of operads $\mathsf{P} = \mathsf{Com}, \mathsf{As}, \mathsf{Lie}$ identifies a module with an abelian commutative algebra (respectively, an abelian associative algebra, an abelian Lie algebra) on which the structure product (respectively, Lie bracket) is identically zero.

The extension functor $\epsilon_! : \mathsf{P} \to \mathcal{M}$ associated to an augmentation $\epsilon : \mathsf{P} \to \mathsf{I}$ is identified with an indecomposable functor which, in the module context, just kills the image of the operations $b = p(a_1, \dots, a_r)$ such that $r > 1$ in our P-algebra A. In the case $\mathsf{P} = \mathsf{As}$ (and in the case $\mathsf{P} = \mathsf{Com}$ similarly), this indecomposable functor $\epsilon_! : \mathcal{As} \to \mathcal{M}$ can be defined by the standard construction $\epsilon_! A = A/A^2$ where A^2 refers to the submodule of A spanned by the products $\mu(a_1, a_2) = a_1 a_2 \in A$, for $a_1, a_2 \in A$. In the case $\mathsf{P} = \mathsf{Lie}$, we obtain $\epsilon_!\, \mathfrak{g} = \mathfrak{g} \,/\, \Gamma_2(\mathfrak{g})$, where $\Gamma_2(\mathfrak{g})$ refers to the submodule of $\mathfrak{g}$ spanned by the Lie brackets $\lambda(a_1, a_2) = [a_1, a_2] \in \mathfrak{g}$, for $a_1, a_2 \in \mathfrak{g}$.

1.3.11. *Further remarks: operads and monads.* The use of the functor $\mathbb{S}(\mathsf{P})$ in operad theory goes back to [140], where it is observed that (a pointed space variant of) this functor $\mathbb{S}(\mathsf{P})$ defines a monad on the base category. The category of P-algebras is defined in terms of this monad $\mathbb{S}(\mathsf{P})$ in [140]. This definition is formally equivalent to the definition of §1.1.13 where we just consider (in the point of view of [140]) an expansion of the action of the monad $\mathbb{S}(\mathsf{P})$ on A. In the approach of [140], the result of Proposition 1.3.3 is a consequence of a general statement about algebras over monads (see [130, §VI.2]).

In the point of view of [140], the operads are exactly the symmetric sequences P such that $\mathbb{S}(\mathsf{P})$ inherits a monad structure. In fact, the definition of $\mathbb{S}(\mathsf{M}) : X \mapsto \mathbb{S}(\mathsf{M}, X)$ as a functor from the base category to itself makes sense for any symmetric collection M and not only for operads. The category of symmetric sequence comes also equipped with structures, like a composition product, that reflect pointwise operations on functors (see [66] for an overall reference on this subject). These observations are the source of abstract categorical definitions for the notion of an operad. These definitions are not used in this book, but we can give a sketch of the ideas.

The category of functors $F : \mathcal{M} \to \mathcal{M}$ is equipped with a natural monoidal structure, defined by the pointwise composition operation, and monads can be defined abstractly as monoid objects in that category. In parallel, we can interpret the definition of the composition structure of an operad in §1.1.1 as the definition of an abstract monoid structure in the category of symmetric sequences with respect to the composition operation reflecting the composition structure of functors. In that respect, the correspondence between operads and monads follows from the relationship between the composition of symmetric sequences and the composition of functors (we refer to [164] for the introduction of this idea, to the book [66] for a general study, based on this definition, of the category of operads and of the categories of algebras associated operads).

This definition of operads in terms of the correspondence with generalized symmetric algebra functors supposes that the tensor product of the base category distributes over colimits. But we consider categories for which this colimit requirement is not valid soon, and therefore we can not rely on this approach in what follows.

1.4. Appendix: Filtered colimits and reflexive coequalizers

The existence of colimits in the category of operads (and in categories of algebras over an operad) relies on the existence of particular colimits (filtered colimits and reflexive coequalizers), which we create in the base category. The purpose of this appendix section is to recall the definition of these fundamental colimits in a general context. For applications to operads, we also study the image of filtered colimits and reflexive coequalizers under a multifunctor $T : \mathcal{C}^{\times r} \to \mathcal{C}$, where $\mathcal{C}$ is any category, with the example of the r-fold tensor products $T(X_1, \dots, X_r) = X_1 \otimes \dots \otimes X_r$ in mind.

1.4.1. *Filtered colimits.* Recall (see [130, §IX.1]) that a small category $\mathcal{I}$ is *filtered* when:

(a) For any pair of objects $\alpha, \beta \in \mathcal{I}$, we have morphisms

$$\begin{tikzcd}
\alpha \arrow[dr, dotted, "u"] & \\
& \gamma \\
\beta \arrow[ur, dotted, "v"'] &
\end{tikzcd}$$

which meet at the same target object γ in $\mathcal{I}$.

(b) For any pair of parallel morphisms $u, v : \alpha \rightrightarrows \beta$, we have a coequalizing morphism

$$\begin{tikzcd}
\alpha \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & \beta \arrow[r, dotted, "t"] & \gamma
\end{tikzcd}$$

such that $tu = tv$ in $\mathcal{I}$.

We say that a colimit $\operatorname{colim}_{\alpha \in \mathcal{I}} X_\alpha$ is *filtered* when the indexing category $\mathcal{I}$ of our diagram X_α is filtered.

We have the following observation:

Proposition 1.4.2. *Suppose that the multifunctor $T : \mathcal{C}^{\times r} \to \mathcal{C}$ preserves filtered colimits on each input in the sense that the natural morphism*

$$\operatorname*{colim}_{\alpha \in \mathcal{I}} T(X^1, \dots, X^k_\alpha, \dots, X^r) \to T(X^1, \dots, \operatorname*{colim}_{\alpha \in \mathcal{I}} X^k_\alpha, \dots, X^r)$$

is an isomorphism for any diagram $\{X^k_\alpha, \alpha \in \mathcal{I}\}$ over a filtered category $\mathcal{I}$ and for all $X^i \in \mathcal{C}$, $i = 1, \dots, \widehat{k}, \dots, r$. Then the functor $T : \mathcal{C}^{\times r} \to \mathcal{C}$ preserves filtered colimits on the product category $\mathcal{C}^{\times r}$ in the sense that the natural morphism

$$\operatorname*{colim}_{\alpha \in \mathcal{I}} T(X^1_\alpha, \dots, X^r_\alpha) \to T(\operatorname*{colim}_{\alpha \in \mathcal{I}} X^1_\alpha, \dots, \operatorname*{colim}_{\alpha \in \mathcal{I}} X^r_\alpha)$$

is an isomorphism for any collection of diagrams $\{X^i_\alpha, \alpha \in \mathcal{I}\}$, $i = 1, \dots, r$, over the same given filtered category $\mathcal{I}$.

Proof. This proposition mainly follows from the observation that the diagonal functor $\Delta : \mathcal{I} \to \mathcal{I}^{\times r}$ is final when $\mathcal{I}$ is filtered and that a pullback along this functor does not change the value of a colimit (see [130, §IX.3] for the definition of the notion of a final functor). We accordingly have $\operatorname{colim}_{\alpha \in \mathcal{I}} T(X^1_\alpha, \dots, X^r_\alpha) \simeq \operatorname{colim}_{(\alpha_1, \dots, \alpha_r) \in \mathcal{I}^{\times r}} T(X^1_{\alpha_1}, \dots, X^r_{\alpha_r})$ and our claim follows from the assumption that our multifunctor preserves filtered colimits on each input and from the usual Fubini decomposition for colimits over a cartesian product of categories. We refer to [66, Proposition 1.2.2] or [156, Lemma 2.3.2] for further details on this proposition. □

1.4.3. *Reflexive coequalizers.* Recall that a coequalizer is the colimit of a diagram formed by a parallel pair of morphisms $d_0, d_1 : X_1 \rightrightarrows X_0$. We use the notation $\mathrm{coeq}(d_0, d_1 : X_1 \rightrightarrows X_0)$ for this kind of colimit.

In many examples, a parallel pair of morphisms is given together with an extra morphism $s_0 : X_0 \to X_1$ such that $d_0 s_0 = id = d_1 s_0$. In this situation, we say that the object $C = \mathrm{coeq}(d_0, d_1 : X_1 \rightrightarrows X_0)$ forms a *reflexive coequalizer* of the pair $d_0, d_1 : X_1 \rightrightarrows X_0$. We refer to the morphism $s_0 : X_0 \to X_1$ as the reflection of the parallel pair $d_0, d_1 : X_1 \rightrightarrows X_0$. We also write

$$C = \mathrm{coeq}(X_1 \overset{\curvearrowleft}{\rightrightarrows} X_0)$$

in order to stress the existence of this reflection $s_0 : X_0 \to X_1$ when we form our coequalizer. Note that the addition of a reflection $s_0 : X_0 \to X_1$ to the diagram of a parallel pair $X_1 \rightrightarrows X_0$ does not change the result of colimits. We explicitly have $\mathrm{colim}(X_1 \overset{\curvearrowleft}{\rightrightarrows} X_0) = \mathrm{coeq}(d_0, d_1 : X_1 \rightrightarrows X_0)$ for any reflection choice.

The significance of reflexive coequalizers lies in the following stability assertion:

PROPOSITION 1.4.4. *Suppose that the multifunctor $T : \mathcal{C}^{\times r} \to \mathcal{C}$ preserves reflexive coequalizers on each input in the sense that the natural morphism*

$$\begin{aligned}\mathrm{coeq}(T(X^1, \dots, X^k_1, \dots, X^r) \overset{\curvearrowleft}{\rightrightarrows} T(X^1, \dots, X^k_0, \dots, X^r))\\ \to T(X^1, \dots, \mathrm{coeq}(X^k_1 \overset{\curvearrowleft}{\rightrightarrows} X^k_0), \dots, X^r)\end{aligned}$$

is an isomorphism for any reflexive diagram $\{X^k_1 \overset{\curvearrowleft}{\rightrightarrows} X^k_0\}$ and for all $X^i \in \mathcal{C}$, $i = 1, \dots, \widehat{k}, \dots, r$. Then the functor $T : \mathcal{C}^{\times r} \to \mathcal{C}$ preserves reflexive coequalizers on the product category $\mathcal{C}^{\times r}$ in the sense that the natural morphism

$$\begin{aligned}\mathrm{coeq}(T(X^1_1, \dots, X^r_1) \overset{\curvearrowleft}{\rightrightarrows} T(X^1_0, \dots, X^r_0))\\ \to T(\mathrm{coeq}(X^1_1 \overset{\curvearrowleft}{\rightrightarrows} X^1_0), \dots, \mathrm{coeq}(X^r_1 \overset{\curvearrowleft}{\rightrightarrows} X^r_0))\end{aligned}$$

is an isomorphism for any collection of reflexive diagrams $\{X^i_1 \overset{\curvearrowleft}{\rightrightarrows} X^i_0\}$, $i = 1, \dots, r$, in the base category $\mathcal{C}$.

PROOF. Exercise or see [66, Proposition 1.2.1] or [156, Lemma 2.3.2]. In fact, we may establish this proposition (see §1.4.6) by using the same argument line as in the proof of Proposition 1.4.2 after observing that reflexive coequalizers are examples of colimits shaped on a category $\mathcal{I}$ such that the diagonal functor $\Delta : \mathcal{I} \to \mathcal{I}^{\times r}$ is final (we go back to this observation in §1.4.6). □

The fundamental role of reflexive coequalizers is also asserted by the following proposition:

PROPOSITION 1.4.5. *If coproducts and reflexive coequalizers exist in a category $\mathcal{C}$, then so does any kind of small colimit in $\mathcal{C}$.*

PROOF. Exercise (see also [30, §2] and [31, §4.3]). □

This proposition is applied in §§1.2-1.3 in order to prove the existence of colimits (of any shape) in the category of operads and in the categories of algebras associated to an operad.

1.4.6. *Remark.* Filtered colimits and reflexive coequalizers are both instances of *sifted colimits* (see [2]), a class of colimits which was studied by P. Gabriel and F. Ulmer in [71] (without being named) and by C. Lair independently in [111] (who used the adjective "tamisante"). In short, we say that a small category $\mathcal{I}$ is *sifted* when:

(a) For any pair of objects $\alpha, \beta \in \mathcal{I}$, we have morphisms

$$\begin{tikzcd}
\alpha \arrow[dr, dotted, "u"] & \\
& \gamma \\
\beta \arrow[ur, dotted, "v"'] &
\end{tikzcd}$$

which meet at the same target object γ in $\mathcal{I}$ (as in the definition of a filtered category).

(b) Every pair of zigzags as in (a) can be connected by a chain of zigzags of the same shape, so that we have a commutative diagram

$$\begin{tikzcd}
& & \alpha \arrow[dll] \arrow[drr] \arrow[dl, dotted] \arrow[d, dotted] & & \\
\gamma \arrow[r, dotted] & \cdot & \cdot \arrow[l, dotted] & \cdots \arrow[l, dotted] \arrow[r, dotted] & \gamma' \\
& & \beta \arrow[ull] \arrow[urr] \arrow[ul, dotted] \arrow[u, dotted] & &
\end{tikzcd}$$

in $\mathcal{I}$.

The *sifted colimits* are the colimits of diagrams shaped on a sifted category.

The above conditions are equivalent to the requirement that the diagonal functor $\Delta : \mathcal{I} \to \mathcal{I} \times \mathcal{I}$ is final in the sense of [130, §IX.3]. The statements of Proposition 1.4.2-1.4.4 actually extend to all kind of colimits shaped on a sifted category.

CHAPTER 2

The Definition of Operadic Composition Structures Revisited

In the introductory chapter §1, we gave a first definition of the notion of an operad, which we used to explain the relationship between operads and algebras. In this second chapter, we go deeper into the study of the internal structures of operads themselves.

The first outcome of this second examination, which we explain in the first section of the chapter (§2.1), is a new definition, in terms of partial composition operations, of the composition structure of an operad. The equivalence between May's definition [140], considered in §1, and this definition in terms of partial composition operations is due to Martin Markl [135, 136] and is also used in the work of Ginzburg–Kapranov on the Koszul duality of operads [78]. In what follows, we mostly use the definition of operads in terms of partial composition operations, which is more appropriate as soon as we study operads themselves, while the definition given in the previous chapter is generally better suited for the study of algebras over operads. To be specific, the partial composition operations have the important feature to satisfy homogeneous (quadratic) relations, unlike the full composition products considered in the definition of §1. The existence of this homogeneous structure is the crux of the Koszul reduction process of §III.3. Let us mention that examples of partial composition products were considered before the development of the theory of operads in the work of Murray Gerstenhaber on the Hochschild cochain complex (see [74]).

In a second part of this chapter (§§2.2-2.4), we examine the definition of operads such that $P_+(0) = \mathbb{1}$, where $\mathbb{1}$ is the tensor unit of the base category $\mathcal{M}$. In §1.2.8, we mentioned that such operads, which we call unitary operads, can be produced by the addition of the unit object $\mathbb{1}$ to the arity 0 component of a non-unitary operad P. To determine the composition structure of a unitary operad P_+ from the underlying non-unitary operad P, we have to assume that P is equipped with extra operations which reflect composition products with the additional unitary term $P_+(0) = \mathbb{1}$. In §2.2, we give a conceptual interpretation, in terms of an extension of the underlying symmetric structure of an operad, of these unitary composition operations. In §2.3, we use the result of this analysis to give a reduced version of the categorical constructions of §1.2 in the context of unitary operads. In §2.4, we study the structure of connected unitary operads and the applications of categorical constructions in the context of connected unitary operads.

The first instances of operads considered in May's monograph [140] are unitary (*unital* in the terminology of that reference) and May actually uses a unitary variant of free algebras over operads for the study of iterated loop spaces. We give a short survey of this subject in §2.2.

We have considered, so far, that the components of an operad $\mathsf{P}(r)$ are indexed by non-negative integers $r \in \mathbb{N}$. Recall that an element $p \in \mathsf{P}(r)$ intuitively represents an operation on r variables $p = p(x_1, \dots, x_r)$ indexed by the elements of the finite ordered set $\underline{\mathrm{r}} = \{1 < \dots < r\}$. But in the construction of free operads, we may take advantage of considering operad components $\mathsf{P}(\underline{\mathrm{r}})$ associated to all finite sets $\underline{\mathrm{r}} = \{i_1, \dots, i_r\}$ so that we can deal with operations $p = p(x_{i_1}, \dots, x_{i_r})$ whose variables are indexed by any such collection of indices $i_k = i_1, \dots, i_r$. We explain this extension of the definition of an operad in the concluding section of the chapter §2.5.

We assume all through this chapter that we work within a base symmetric monoidal category $\mathcal{M}$. When we examine the application of categorical constructions to unitary operads in §§2.3-2.4, we assume that the tensor product of this category distributes over colimits (see §0.9), but we do not use more than the general axioms of symmetric monoidal categories until this moment, and our statements are valid in this setting.

2.1. The definition of operads from partial composition operations

Recall that the partial composition products of an operad are defined by the formulas $p \circ_k q = p(1, \dots, 1, q, 1, \dots, 1)$, where $q \in \mathsf{P}(n)$ is plugged in the kth input of the operation $p \in \mathsf{P}(m)$ and we take operad units $1 \in \mathsf{P}(1)$ otherwise. In §1.1.4, we observed that the unit and associativity axioms of operads imply that the full composition products of §1.1.1 satisfy $p(q_1, \dots, q_r) = (\cdots (p \circ_{k_1+1} q_1) \circ_{k_2+1} \cdots) \circ_{k_r+1} q_r$, for any $p \in \mathsf{P}(r)$ and all $q_1 \in \mathsf{P}(n_1), \dots, q_r \in \mathsf{P}(n_r)$, where we set $k_i = n_1 + \dots + n_{i-1}$ for $i = 1, \dots, r$. This result still holds in a general categorical framework as we can obviously replace our pointwise relations by equivalent identities of morphisms. In any case, we obtain that the composition products of an operad can be fully determined by giving the partial composition products $\circ_k : \mathsf{P}(m) \otimes \mathsf{P}(n) \to \mathsf{P}(m + n - 1)$, where $k = 1, \dots, m$. The purpose of this section is to specify relations on partial composition operations which are equivalent to the equivariance, unit, and associativity axioms of §1.1.1. The first outcome of this study, as we announced in the introduction of this chapter, is a new representation of the structure of an operad which will serve as working definition in our subsequent constructions.

To start with, we give the formal definition, in categorical terms, of the partial composition operations.

2.1.1. *The partial composition products associated to an operad.* Let P be an operad (in the sense of the basic definition of §1.1.1). The partial composition products associated to P

$$\circ_k : \mathsf{P}(m) \otimes \mathsf{P}(n) \to \mathsf{P}(m + n - 1)$$

are formally defined as composites

$$\begin{aligned} \mathsf{P}(m) \otimes \mathsf{P}(n) \xleftarrow[(1)]{\simeq} {} & \mathsf{P}(m) \otimes \mathbb{1} \otimes \dots \otimes \mathsf{P}(n) \otimes \dots \otimes \mathbb{1} \\ \xrightarrow[(2)]{} {} & \mathsf{P}(m) \otimes \mathsf{P}(1) \otimes \dots \otimes \mathsf{P}(n) \otimes \dots \otimes \mathsf{P}(1) \\ \xrightarrow[(3)]{\mu} {} & \mathsf{P}(m + n - 1), \end{aligned}$$

where we consider a tensor product of operad units $\eta : \mathbb{1} \to \mathsf{P}(1)$, putting the factor $\mathsf{P}(n)$ at the kth position of the tensor grouping $\mathsf{P}(1) \otimes \dots \otimes \mathsf{P}(n) \otimes \dots \otimes \mathsf{P}(1)$, followed by the appropriate component of the full composition product of P. The range of definition of the full composition products in §1.1.1 implies that a partial composition operation of this form can be associated to any pair $m, n \in \mathbb{N}$ and for each composition index $k \in \{1 < \cdots < m\}$. (But, since the choice of composition index is empty for $m = 0$, we can assume $m > 0$ when we apply partial composition operations.) In the context of a concrete symmetric monoidal category, we just retrieve the formula

$$p \circ_k q = p(1, \dots, q, \dots, 1)$$

recalled in the introduction of this section.

From this definition, we can already readily deduce the equivariance relations of the partial composition products:

Proposition 2.1.2. *The equivariance axiom of operads, expressed by the commutativity of the diagram of Figure* 1.4*, implies that the partial composition operations of an operad* P *make the following diagrams commute*

$$\begin{tikzcd} \mathsf{P}(m) \otimes \mathsf{P}(n) \arrow[r, "\circ_k"] \arrow[d, "s\otimes t"'] & \mathsf{P}(m+n-1) \arrow[d, "s\circ_{s(k)} t"] , \\ \mathsf{P}(m) \otimes \mathsf{P}(n) \arrow[r, "\circ_{s(k)}"] & \mathsf{P}(m+n-1) \end{tikzcd}$$

for all $s \in \Sigma_m$*,* $t \in \Sigma_n$*, and where* $s \circ_{s(k)} t \in \Sigma_{m+n-1}$ *refers to the partial composite, within the permutation operad, of the permutations* $s \in \Sigma_m$*,* $t \in \Sigma_n$*.*

Proof. The equivariance relation of this proposition immediately follows from the commutativity of the diagram of Figure 1.4 (which is one form of the equivariance axiom of operads), where we take $r = m$ and $n_1 = \cdots = n_{s(k)-1} = 1$, $n_{s(k)} = n$, $n_{s(k)+1} = \cdots = n_r = 1$.

Simply observe that the permutation $s(id, \dots, t, \dots, id)$, occurring in this application of the axiom, with t plugged in the $s(k)$th composition position of s, defines the partial composite $s \circ_{s(k)} t$ of the permutations $s \in \Sigma_m$, $t \in \Sigma_n$. ☐

Before going further, we review the definition of the operadic composition of permutations. We aim to give an explicit definition of the permutation $s \circ_{s(k)} t \in \Sigma_{m+n-1}$ which occurs in the above proposition. We only use formal properties of the partial composition of permutations in what follows. We do not really need this explicit description therefore, but this inspection will enable us to illustrate the definition of the partial composition operations of operads.

2.1.3. *Partial composites of permutations.* The composite permutation $s \circ_{s(k)} t \in \Sigma_{m+n-1}$ which occurs in the previous proposition can be determined from the construction of the composition structure on permutations in §§1.1.7-1.1.9. Indeed, we have by definition:

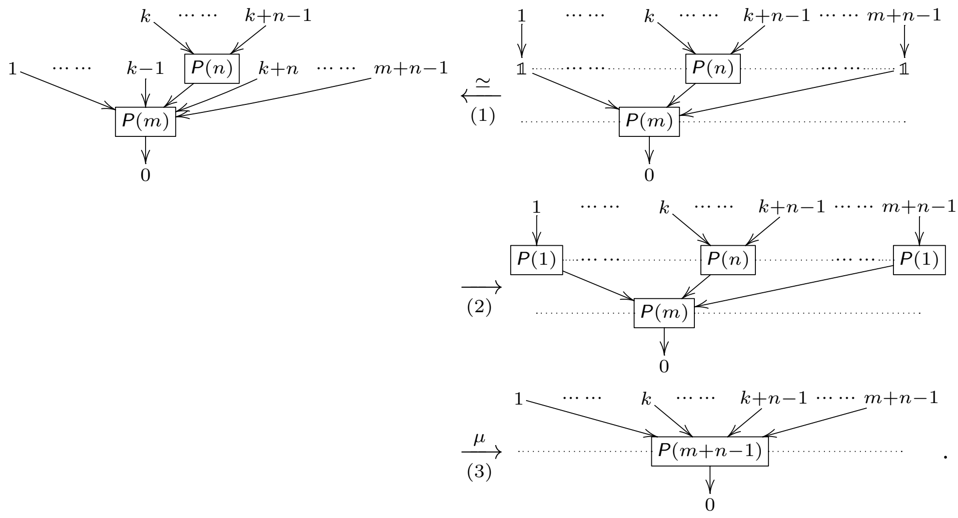
$$s \circ_{s(k)} t = s(id_1, \dots, t, \dots, id_1) = (id_1 \oplus \cdots \oplus t \oplus \cdots \oplus id_1) \cdot s(1, \dots, n, \dots, 1).$$

In the sequence representation of permutations (see §1.1.7), we readily see that the sequence associated to the composite $s \circ_{s(k)} t$ is defined by substituting the sequence associated to the permutation t to the occurrence of the composition index $s(k)$ in the sequence representing s. We also perform the standard value shift which reflects the interpretation of partial composites in terms of a composition of operations (see §1.1.4). To be explicit, if we set $s = (s(1), \dots, s(m))$ and $t = (t(1), \dots, t(n))$, then the result of this substitution process reads:

$$s \circ_{s(k)} t = (s(1)', \dots, s(k-1)', t(1)', \dots, t(n)', s(k+1)', \dots, s(m)'),$$

where we have $s(i)' = s(i)$ when $s(i) < s(k)$, we have $s(i)' = s(i)+n-1$ when $s(i) > s(k)$, and we set $t(j)' = t(j)+n-1$ in all cases. For instance, for the permutations $s = (1,3,5,4,2) \in \Sigma_5$ and $t = (3,1,2) \in \Sigma_3$, we obtain $s \circ_4 t = (1,3,7,6,4,5,2)$.

2.1.4. *The graphical representation of partial composition products.* In the picture of §1.1.6, the definition of the partial composition operations from the full composition products reads:

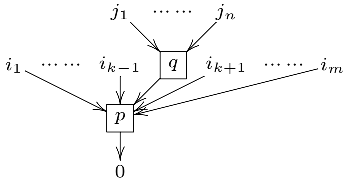


Recall (see §1.1.6) that an arrangement of operad components (or elements) on a tree represents a tensor product. The removal of the unit factors 𝟙 in the isomorphism (1) corresponds to the application of the unit isomorphisms in the formal definition of §2.1.1. This withdrawal operations gives the composition pattern, shaped on a tree with two vertices, which we depict in our figure. The morphisms (2-3) correspond to the unit insertion operation and to the full composition operation which we consider in the definition of §2.1.1.

2.1.5. *The partial composition scheme, input indexing and equivariance.* In the case of a concrete symmetric monoidal category, we can use the picture of §2.1.4 to represent the partial composition of operad elements $p \in \mathsf{P}(m)$ and $q \in \mathsf{P}(n)$. In the two-vertex tree which defines the source of our composition operation, we replace the objects $\mathsf{P}(m)$ and $\mathsf{P}(n)$ by these elements $p \in \mathsf{P}(m)$ and $q \in \mathsf{P}(n)$, and this picture gives the composition pattern which we associate to our operation $\circ_k : p \otimes q \mapsto p \circ_k q$.

In the picture of §2.1.4, we use the natural indexing of the inputs of the composite operation $p \circ_k q \in \mathsf{P}(m+n-1)$ to index the inputs of this composition pattern. In §1.1.5, we mention that we can perform re-indexing operations in order to materialize the action of permutations on operadic composites. In the context of the partial composition operation, this extension of our treewise representation makes us deal with composition patterns of the following general form:

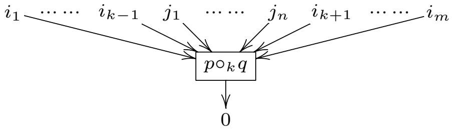
(1)



.

The partial composition operation $\circ_k : p \otimes q \mapsto p \circ_k q$ carries this treewise tensor to the element:

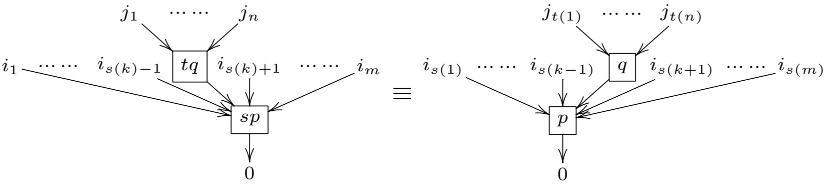
(2)



.

In §1.1.5, we also introduced relations to identify the action of permutations on operations with an input re-indexing of tree edges. In the case of the two-vertex tree (1), we consider relations of the following form:

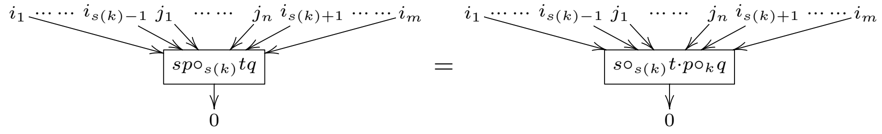
(3)



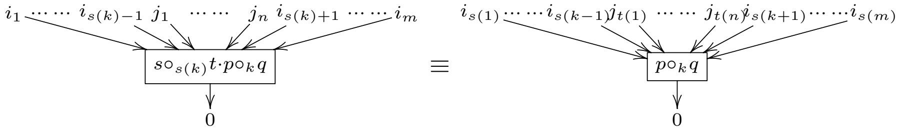
.

Recall that the ingoing edges of a box labeled by an operation $p \in P(r)$ are in bijection with the inputs of this operation. Moreover, as long as we assume that the inputs of such operations are indexed by the elements of the standard ordered sets $\underline{r} = \{1 < \cdots < r\}$, we adopt the convention that this bijection is materialized by the ordering of the edges in the plane. Hence, the application of identification rules in the above picture moves the outgoing edge of $q \in P(n)$ from the $s(k)$th position in the set of ingoing edges of the element $sp \in P(m)$ to the kth position in the set of in the ingoing edges of the element $p \in P(m)$.

The equivariance relation of Proposition 2.1.2 implies the coherence of our mapping (1)$\mapsto$(2) with respect to these identifications (3). Indeed, this equivariance relation reads



and, for a one-vertex treewise tensor, we have the relation



.

Hence, the map (1)$\mapsto$(2) equalizes both sides of our relation (3) as requested.

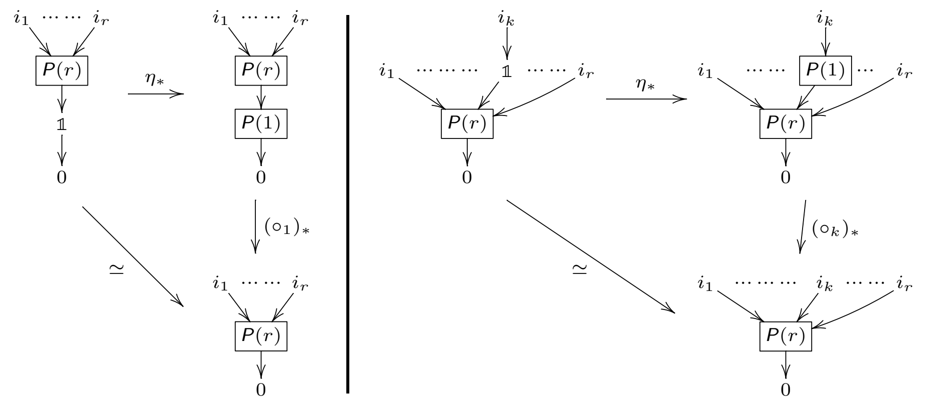
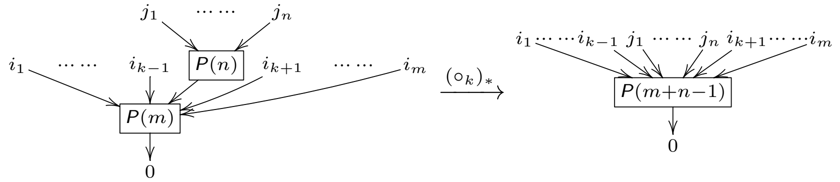


FIGURE 2.1. The unit relations of partial composition products, which hold for all $r \in \mathbb{N}$ and $k = 1, \dots, r$.

The other way round, as soon as we use the tree picture of the morphisms (1)$\mapsto$(2), we implicitly assume that these morphisms carry the relations (3) to identities, and this requirement trivially implies that our morphisms fulfill the equivariance relation of partial composition products in Proposition 2.1.2. Thus, whenever we use the treewise picture of the partial composition operations, we implicitly assume that our composition operations satisfy these equivariance relations.

2.1.6. *The graphical representation of the partial composition products of operads.* In general, we use the picture



to represent the partial composition products of an operad (as soon as we can assume that the equivariance relations of Proposition 2.1.2 are satisfied). In this representation, we identify the application of the partial composition product $\circ_k$ with the performance of an internal operation in our treewise tensor product (as in §1.1.6) and we use the notation $(\circ_k)_*$ to refer to this morphism. This internal operation has not to be confused with an external partial composition product, which we define in §A.2.7 and for which we use the notation $\circ_{i_k}$ (with the labeling of an input of our tree as composition index i_k).

In §A, we elaborate on this picture of the partial composition operations in order to give an explicit description of the free operad. We roughly deal with composition schemes, modeled on trees with an arbitrary number of vertices, which represent multiple applications of partial composition products. We have already given an example of this representation in our introduction of free operad structures in §1.2. In Figure 2.2-2.3, we give fundamental examples of such (multi-fold) composition schemes which give the shape of the associativity relations satisfied by the partial composition operations. In Figure 2.1, we also give a representation of the

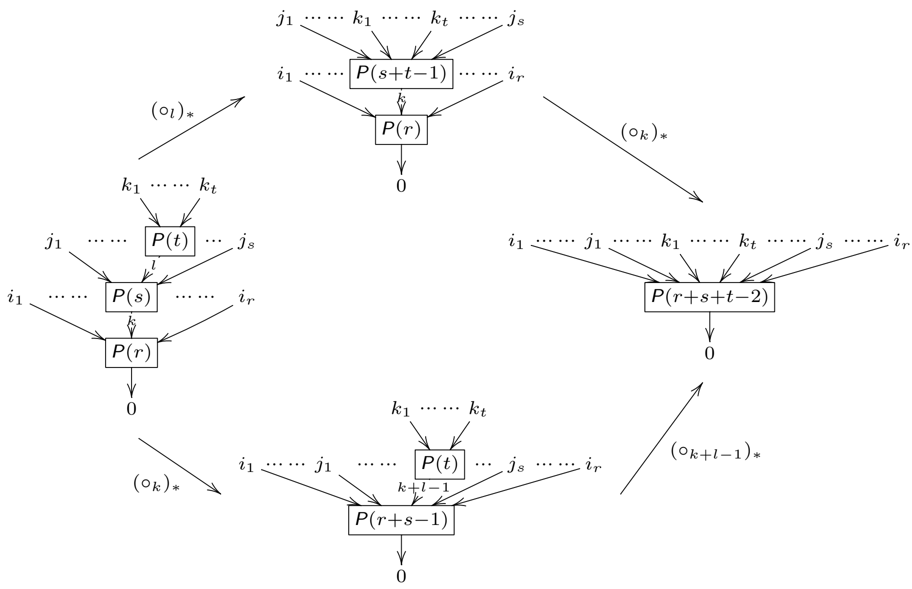


FIGURE 2.2. The associativity relation of partial composition products for a sequential arrangement of factors, where we assume $r, s, t \in \mathbb{N}$, and $k \in \{1 < \cdots < r\}$, $l \in \{1 < \cdots < s\}$.

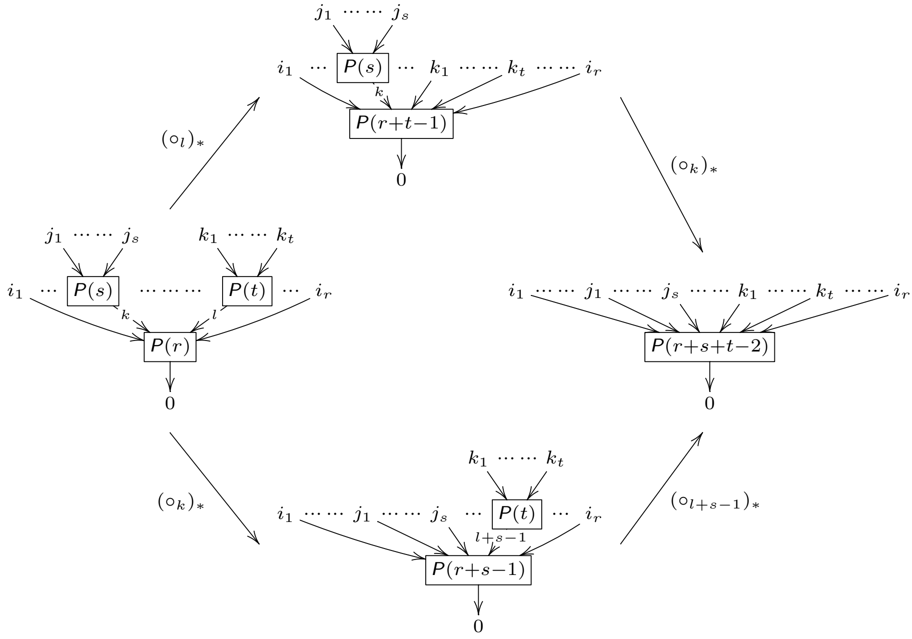


FIGURE 2.3. The associativity relation of partial composition products for a ramified arrangement of factors, where we assume $r, s, t \in \mathbb{N}$, and $\{k < l\} \subset \{1 < \cdots < r\}$.

unit relations which the partial composition operations satisfy. The verification of these unit and associativity relations from our initial definition of the composition structure of an operad in §1.1.1 is the goal of the next proposition:

PROPOSITION 2.1.7. *The partial composition operations*

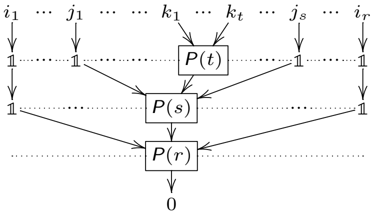
$$\circ_k : \mathsf{P}(m)\otimes\mathsf{P}(n)\to\mathsf{P}(m+n-1),\qquad k=1,\dots,m,$$

defined from the full composition products of an operad in §2.1.1, *fulfill the unit relations expressed by the commutativity of the diagrams of Figure* 2.1 *and the associativity relations expressed by the commutativity of the diagrams of Figure* 2.2-2.3.

PROOF. To establish this proposition, we use the treewise interpretation of the full composition products of operads and the corresponding representation of the unit and associativity axioms of operads in Figure 1.5-1.6. The unit relations of the proposition are immediate consequences of the unit axiom of full composition products, as expressed by the commutative diagrams of Figure 1.5. In one relation, we deal with a partial composite on an arity 1 component. But in this degenerate case, the partial composite is formally the same as a full composition product. In the other unit relation, we readily get that the partial composition operation $\circ_k$ with an operadic unit reduces to the composite morphism $\mathsf{P}(r)\otimes\mathbb{1}^{\otimes r}\to\mathsf{P}(r)\otimes\mathsf{P}(1)^{\otimes r}\xrightarrow{\mu}\mathsf{P}(r)$, where μ denotes the full composition product of our operad.

The first associativity relation of partial composition products, expressed in Figure 2.2, is also immediate from the associativity axiom of the full composition products. Indeed, we simply have to apply the diagram of Figure 1.6 to a configuration of the form

(1)

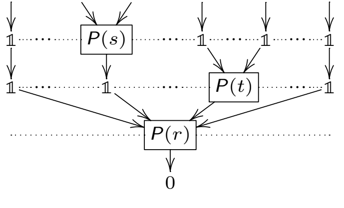


which, under the construction of partial composites in §§2.1.1-2.1.5, corresponds to the composition of partial composition operations represented in Figure 2.2.

In this process (and in the next constructions as well), the unit factors $\mathbb{1}$ correspond to the (delayed) application of unit morphisms $\eta:\mathbb{1}\to\mathsf{P}(1)$. The unit axiom of Figure 1.5 implies that the evaluation of an operadic composite on a grouping of such unit factors is equal to the insertion of a unit morphism $\eta:\mathbb{1}\to\mathsf{P}(1)$ at the place resulting from the composition operation. In our picture, we just keep unit factors at the positions associated to such groupings of operadic units.

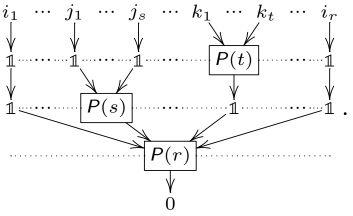
To check the second associativity relation, we examine the associativity diagram of Figure 1.6 for configurations of the form:

(2)



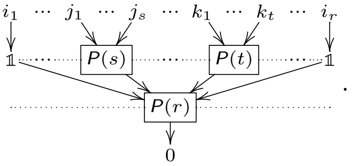
and:

(3)



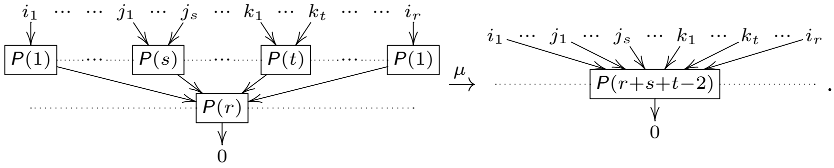
The definition of partial composites implies that the composites of partial composition operations represented in Figure 2.3 are identified with the composite composition products of (2-3) when the composition of the lower rows is performed first. On the other hand, if we perform the composition of the upper rows in (2-3), then we obtain in both cases a configuration of the form

(4)



These composition operations reduce to the application of unit relations and define isomorphisms (2) $\xrightarrow{\simeq}$ (4) $\xleftarrow{\simeq}$ (3). From this identification, we deduce, by applying the associativity axiom of operads, that the composites of the partial composition operations of Figure 2.3 are both equal to a three-fold composition operation of the form

(4) →



This identification finishes the proof of the proposition. ☐

2.1.8. *The pointwise formulas for the equivariance, unit, and associativity relations of partial composition products.* In general, we use the graphical picture to express the relations satisfied by the partial composition products of an operad. But we can easily give a representation of our relations in terms of formulas on elements when our base category forms a concrete symmetric monoidal category. The equivariance relation of partial composition products, stated in Proposition 2.1.2, is equivalent to the identity $sp \circ_{s(k)} tq = s \circ_{s(k)} t \cdot p \circ_k q$, for $p \in \mathsf{P}(m)$, $q \in \mathsf{P}(n)$, $s \in \Sigma_m$, $t \in \Sigma_n$, and $k = 1, \dots, m$. The unit relations, given by the diagrams of Figure 2.1, are equivalent to the formulas

$$1 \circ_1 p = p \quad \text{and} \quad p \circ_k 1 = p,$$

for all $p \in \mathsf{P}(r)$, and $k = 1, \dots, r$. The associativity relation of Figure 2.2 reads:

$$(a \circ_k b) \circ_{k+l-1} c = a \circ_k (b \circ_l c),$$

for $a \in \mathsf{P}(r)$, $b \in \mathsf{P}(s)$, $c \in \mathsf{P}(t)$, and $k \in \{1 < \cdots < r\}$, $l \in \{1 < \cdots < s\}$, while the associativity relation of Figure 2.3 reads:

$$(a \circ_k b) \circ_{s+l-1} c = (a \circ_l c) \circ_k b,$$

for $a \in P(r)$, $b \in P(s)$, $c \in P(t)$, and $\{k < l\} \subset \{1 < \dots < r\}$.

2.1.9. *The definition of operads in terms of partial composition operations.* The result of Proposition 2.1.7 gives natural axioms for the definition of operads in terms of partial composition products.

To be explicit, we temporarily call *operad shaped on partial composition schemes* the structure defined by a sequence of objects $P(n) \in \mathcal{M}$, $n \in \mathbb{N}$, where each $P(n)$ is equipped with an action of the symmetric group Σ_n (as in the definition of §1.1.1), together with:

(1) a unit morphism $\eta : \mathbb{1} \to P(1)$,
(2) and partial composition products $\circ_k : P(m) \otimes P(n) \to P(m+n-1)$, defined for any $m, n \in \mathbb{N}$, for each $k \in \{1 < \dots < m\}$, and which satisfy the equivariance relations of Proposition 2.1.2, as well as the unit relations of Figure 2.1 and the associativity relations of Figure 2.2–2.3.

Recall that we assume the equivariance relation to give a sense to the treewise representation which we use in our figures.

The definition of a connected operad in §1.1.21 has an obvious analogue for operads shaped on partial composition schemes. In this case, we forget about arity zero components in our definition and we set $P(1) = \mathbb{1}$. We then see that the partial composition operations (2) such that $m = 1$ or $n = 1$ are determined by the unit axioms of Figure 2.1. Hence, the composition structure of a connected operad shaped on partial composition schemes can be fully determined by partial composition products (2) such that $m, n > 1$.

The operads shaped on partial composition schemes form a category with, as morphisms, the morphisms of symmetric sequences $\phi : P \to Q$ which preserve operadic units and the internal partial composition operations of our operads. The result of Proposition 2.1.7 implies that we have an obvious functor from the standard category of operads towards the category of operads shaped on partial composition schemes. Our claim is that:

Theorem 2.1.10. *The correspondence of* §2.1.1*, between the partial composition operations and the standard full composition products of operads, defines an isomorphism of categories between the standard category of operads, such as defined in* §1.1.1*, and the category of operads shaped on partial composition schemes, such as defined in* §2.1.9. □

This result follows from coherence statements of §A, where we explain a general definition of treewise composition operations which include the full composition products of §1.1.1 and the partial composition products considered in this section as particular examples.

2.1.11. *The commutative operad.* To complete the survey of this section, we revisit the definition of the commutative operad, which is one of the main examples of operad introduced in the previous chapter (in the context of sets and modules). We just check that the direct definition of this operad works in the setting of a general symmetric monoidal category $\mathcal{M}$. We still consider both a non-unitary version and a unitary version of the commutative operad, which we respectively denote by Com and Com_+ as usual.

In the context of a general symmetric monoidal category $\mathcal{M}$, we can define the commutative operad Com_+ by $\mathsf{Com}_+(r) := \mathbb{1}$ for any arity $r \in \mathbb{N}$, where we consider the unit object $\mathbb{1}$ of our category $\mathcal{M}$. We provide each object $\mathsf{Com}_+(r)$ with a trivial (in the sense of identical) action of the symmetric group, for any

$r \in \mathbb{N}$. We take the identity morphism of the unit object to define the operadic unit $\eta : \mathbb{1} \to \mathsf{Com}_+(1)$, and we take the unit isomorphisms of our symmetric monoidal structure $\mathbb{1} \otimes \mathbb{1} \simeq \mathbb{1}$ to define the partial composition products of this operad $\circ_k : \mathsf{Com}_+(m) \otimes \mathsf{Com}_+(n) \to \mathsf{Com}_+(m+n-1)$, for any $m, n \in \mathbb{N}$ and for each $k = 1, \dots, m$. We easily check that these composition operations fulfill our equivariance, unit and associativity relations.

We just drop the term of arity zero $\mathsf{Com}_+(0) = \mathbb{1}$ when we deal with the non-unitary version of the commutative operad Com. We may also set $\mathsf{Com}(0) = \varnothing$, where $\varnothing$ represents the initial object of our base category (see §1.1.19).

In the next chapter, we will observe that the unitary commutative operad Com_+ is associated to a general notion of unitary commutative algebra, which can be defined in any symmetric monoidal category, and we have a similar statement for the non-unitary version of this operad Com. But our main motivation for the study of the commutative operad (for the moment at least) comes from the observation that the operad Com_+ represents the terminal object of the category of unitary operads $\mathcal{O}p_*$. We check this assertion in the next section.

2.2. The definition of unitary operads

Recall that we use the name '*unitary operad*' to refer to operads P_+ such that $\mathsf{P}_+(0) = \mathbb{1}$, whereas we say that an operad is *non-unitary* when this operad is void in arity zero. In the case where the tensor product of the base category distributes over colimits (see §0.9), we can identify non-unitary operads with operads P such that $\mathsf{P}(0) = \varnothing$, where $\varnothing$ denotes the initial object of the base category (as we did in the case of the commutative operad $\mathsf{P} = \mathsf{Com}$ in the previous paragraph). In general, we just forget about arity zero terms to define the structure of a non-unitary operad. In this case, we also write $\mathsf{P}(0) = \varnothing$ to assert that our operad is not defined in arity zero (see §1.1.19).

The operad of unitary associative monoids As_+, defined in §1 (in the set context and in the module context), and the operad of unitary commutative monoids Com_+, studied in the previous paragraph §2.1.11, are our basic examples of unitary operads. The operads As and Com, which we obtain by forgetting about the terms of arity zero of these unitary operads, form basic instances of non-unitary operads.

Recall that we use the notation $\mathcal{O}p_*$ for the category which has the unitary operads P_+ as objects and the operad morphisms $\phi : \mathsf{P}_+ \to \mathsf{Q}_+$ which are the identity of the unit object $\mathbb{1}$ in arity zero as morphisms. We use the notation $\mathcal{O}p_\varnothing$ for the category of non-unitary operads.

In §1.1.20, we call '*unitary extension*' a unitary operad P_+ which is obtained by the addition of a unit term $\mathsf{P}_+(0) = \mathbb{1}$ to a given non-unitary operad P. The main purpose of this section is to check that the composition structure of a unitary extension P_+ is determined by the internal structure of the underlying non-unitary operad P and extra operators which reflect the composition operations with the unit term $\mathsf{P}_+(0) = \mathbb{1}$ in this unitary operad P_+.

In applications, we generally use the partial composition products of the previous section to determine the composition structure of non-unitary operads. But, we put the composition products with the arity zero term $\mathsf{P}_+(0) = \mathbb{1}$ apart when we deal with unitary operads. In short, our idea is to regard these special instances of composition products as part of an internal diagram structure which we associate to our object. We will see that fixing $\mathsf{P}_+(0) = \mathbb{1}$ transforms these composition products into additive operations, while the partial composition products are quadratic, and this observation motivates the use of a different approach to handle the composition products with the arity zero term of unitary operads.

We go back to the analysis of the previous section in order to carry out our program. In a first step, we study the composition products which we form by taking the arity zero term $\mathsf{P}_+(0) = \mathbb{1}$ as composition factors in a unitary operad P_+.

2.2.1. *The restriction operators associated to a unitary operad structure.* We assume that P_+ is a unitary operad, so that $\mathsf{P}_+(0) = \mathbb{1}$, and we use the notation P to refer to the non-unitary operad which agrees with P_+ in arity $r > 0$.

We consider composition operations of the form

$$\begin{aligned} \mathsf{P}_+(n) &\xleftarrow[(1)]{\simeq} \mathsf{P}_+(n) \otimes \mathsf{P}_+(0) \otimes \cdots \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \cdots \otimes \mathsf{P}_+(0) \\ &\xrightarrow[(2)]{\eta_*} \mathsf{P}_+(n) \otimes \mathsf{P}_+(0) \otimes \cdots \otimes \mathsf{P}_+(1) \otimes \cdots \otimes \mathsf{P}_+(1) \otimes \cdots \otimes \mathsf{P}_+(0) \\ &\xrightarrow[(3)]{\mu} \mathsf{P}_+(m), \end{aligned}$$

where η_* is given by the application of operadic units $\eta : \mathbb{1} \to \mathsf{P}_+(1)$ at places specified by an increasing sequence $1 \leq k_1 < \cdots < k_m \leq n$, and μ denotes the full composition product of the unitary operad P_+ that corresponds to this composition scheme.

We can associate such an increasing sequence $1 \leq k_1 < \cdots < k_m \leq n$ to any increasing map $u : \{1 < \cdots < m\} \to \{1 < \cdots < n\}$ by using the relation $u(i) = k_i$, for $i = 1, \dots, m$. We accordingly have a morphism $u^* : \mathsf{P}_+(n) \to \mathsf{P}_+(m)$, associated any such increasing map $u : \{1 < \cdots < m\} \to \{1 < \cdots < n\}$, which we determine by the above composite (1-3). We call this morphism the *restriction operator* associated to the map $u : \{1 < \cdots < m\} \to \{1 < \cdots < n\}$. In the case $m, n > 0$, we identify this restriction operator with an internal operation of the non-unitary operad P underlying P_+:

$$u^* : \mathsf{P}(n) \to \mathsf{P}(m)$$

In the case $m = 0$, $n > 0$, our construction returns augmentation morphisms with values in the unit object of the base category:

$$\epsilon : \mathsf{P}(n) \to \mathbb{1}\,.$$

We also write $o^* = \epsilon$ for this augmentation, which represents the restriction operator associated to the initial map $o : \underline{0} \to \underline{n}$.

In the context of a concrete category, we can define these restriction maps $u^* : \mathsf{P}(n) \to \mathsf{P}(m)$ by the formula $u^*(p) = p(*, \dots, *, 1, *, \dots, *, \dots, *, 1, *, \dots, *)$, for any $p \in \mathsf{P}(n)$, where we use the notation $*$ to refer to the distinguished element of the operad P_+ in arity 0. The augmentation is similarly defined by $\epsilon(p) = p(*, \dots, *)$.

In the case where we take a single unitary factor $\mathsf{P}_+(0) = \mathbb{1}$ in our composition operation (1-3), we retrieve the definition of a partial composition operation $\mathsf{P}(n) \xleftarrow{\simeq} \mathsf{P}_+(n) \otimes \mathsf{P}_+(0) \xrightarrow{\circ_k} \mathsf{P}_+(n-1)$. The restriction operator $\partial_k : \mathsf{P}(n) \to \mathsf{P}(n-1)$ which corresponds to this composite is associated to the increasing map

$$\partial^k : \{1 < \cdots < n-1\} \to \{1 < \cdots < n\}$$

such that:

$$\partial^k(x) = \begin{cases} x, & \text{for } x = 1, \dots, k-1, \\ x+1, & \text{for } x = k, \dots, n-1, \end{cases}$$

for any $k = 1, \dots, n$. In the context of a concrete category, we can define this restriction operator by the formula $\partial_k(p) = p \circ_k *$.

We immediately see that all restriction operators associated to a unitary operad structure occur as composites of these particular restriction operators $\partial_k : \mathsf{P}(n) \to \mathsf{P}(n-1)$, for $k = 1, \dots, n$, as we observed in §2.1 that the full composition products of an operad are composites of partial restriction operators. This assertion can also be deduced from the observation that all increasing maps $u : \{1 < \dots < m\} \to \{1 < \dots < n\}$ are composites of maps of the form ∂^k as soon as we establish that the action of restriction operators satisfies a natural associativity relation (see Lemma 2.2.4).

2.2.2. *The category of finite ordinals and injections.* We aim to establish that the restriction operators $u^* : \mathsf{P}(n) \to \mathsf{P}(m)$ defined in the previous paragraph can be embodied in an extension of the internal symmetric structure of operads. For this purpose, we consider the category $\Lambda_{>0}$ which has the finite ordered sets $\underline{\mathtt{n}} = \{1 < \dots < n\}$, where $n > 0$, as objects, and all injective maps $f : \{1 < \dots < m\} \to \{1 < \dots < n\}$ (not necessarily monotonous) as morphisms. This category contains a distinguished subcategory $\Lambda^+_{>0} \subset \Lambda_{>0}$ with the same objects as $\Lambda_{>0}$, but of which morphisms reduce to the increasing maps of §2.2.1.

The lower script > 0 in the above notation refers to the restriction to ordered sets $\underline{\mathtt{n}} = \{1 < \dots < n\}$ such that $n > 0$ in the set of objects of the categories $\Lambda^+_{>0} \subset \Lambda_{>0}$. We actually use the notation Λ (with no subscript) for the variant of the category $\Lambda_{>0}$ whose object set includes a zero object $\underline{\mathtt{0}}$ which corresponds to the empty set. We refer to this category Λ as the category of *(finite) ordinals and injections*, since any object of this category $\underline{\mathtt{n}} = \{1 < \dots < n\}$ is equivalent to an ordinal. We also use the notation Λ^+ for the category which has the same objects as the category Λ but where we only take the increasing injections as morphisms. We trivially have $\Lambda^+_{>0} = \Lambda_{>0} \cap \Lambda^+$.

We do not really deal with diagrams over the whole category Λ in this book, but we generally use the notation Λ (with no extra decoration) as a qualifier for objects of which structure includes an action of morphisms $f \in \mathtt{Mor}_{\Lambda}(\underline{\mathtt{m}}, \underline{\mathtt{n}})$. In particular, we use the phrase '*non-unitary* Λ*-sequence*' to refer to the category of contravariant diagrams over the category $\Lambda_{>0}$. In §2.4, we similarly coin the phrase '*connected* Λ*-sequence*' for the category of contravariant diagrams over the full subcategory $\Lambda_{>1}$ of the category Λ generated by the ordinals $\underline{\mathtt{n}} = \{1 < \dots < n\}$ such that $n > 1$.

We also use the notation of the whole category Λ (rather than the notation of a specific full subcategory) in the expression of morphism sets. We adopt similar conventions for the sets of increasing maps on any subcategory of the category $\Lambda^+ \subset \Lambda$.

2.2.3. *The decomposition of morphisms in the category of finite ordinals and injections.* The symmetric group Σ_n is identified with the group of automorphisms of the object $\underline{\mathtt{n}}$ in the category $\Lambda_{>0}$. We readily see that any morphism $f \in \mathtt{Mor}_{\Lambda}(\underline{\mathtt{m}}, \underline{\mathtt{n}})$ has a unique decomposition $f = \rho\sigma$, such that $\rho \in \mathtt{Mor}_{\Lambda^+}(\underline{\mathtt{m}}, \underline{\mathtt{n}})$ and $\sigma \in \Sigma_m$. The map which occurs in this decomposition ρ is characterized by the relation $\{f(1),\dots,f(m)\} = \{\rho(1) < \dots < \rho(m)\}$ and the permutation $\sigma = (\sigma(1),\dots,\sigma(m))$ is characterized by the equation $\rho(\sigma(i)) = f(i)$, for any $i \in \underline{\mathtt{m}}$.

In the particular case of a composite $f = su$, where $u \in \mathtt{Mor}_{\Lambda^+}(\underline{\mathtt{m}},\underline{\mathtt{n}})$, and $s \in \Sigma_n$, the existence of our decomposition is equivalent to a commutation formula $su = \rho\sigma$, where $\rho \in \mathtt{Mor}_{\Lambda^+}(\underline{\mathtt{m}},\underline{\mathtt{n}})$ and $\sigma \in \Sigma_m$ is a permutation associated to $s \in \Sigma_n$. This permutation $\sigma \in \Sigma_m$ is actually identified with the image of s under the application of the restriction operator ρ^* on the permutation operad $\Pi(n) = \Sigma_n$. We make the restriction operators of the permutation operad explicit in §§2.2.7-2.2.8. We will give a proof of this identity $\sigma = \rho^*(s)$ at this moment.

In §1.2, we define the permutation category Σ as the category formed by the finite ordered sets $\underline{\mathtt{n}} = \{1 < \dots < n\}$ as objects, where $n \in \mathbb{N}$, together with the morphisms sets such that $\mathtt{Mor}_{\Sigma}(\underline{\mathtt{n}},\underline{\mathtt{n}}) = \Sigma_n$ and $\mathtt{Mor}_{\Sigma}(\underline{\mathtt{m}},\underline{\mathtt{n}}) = \varnothing$ for $m \neq n$. In parallel to our category $\Lambda_{>0}$, we consider the full subcategory of the permutation category $\Sigma_{>0} \subset \Sigma$ generated by the ordered sets $\underline{\mathtt{n}} = \{1 < \dots < n\}$ such that $n > 0$. This category $\Sigma_{>0}$ is identified with the isomorphism subcategory of the category $\Lambda_{>0}$. We write $\Lambda_{>0} = \Lambda^+_{>0}\Sigma_{>0}$ to express the decomposition $f = us$ of the morphisms in the category $\Lambda_{>0}$. The following lemma provides a first motivation for the introduction of the category $\Lambda_{>0}$:

Lemma 2.2.4. *Let P_+ be any unitary operad with P as underlying non-unitary operad.*

(a) *The restriction operators $\mathsf{P}(t) \xrightarrow{v^*} \mathsf{P}(s) \xrightarrow{u^*} \mathsf{P}(r)$ associated to any sequence of increasing maps $\{1 < \dots < r\} \xrightarrow{u} \{1 < \dots < s\} \xrightarrow{v} \{1 < \dots < t\}$ such that $r,s,t > 0$ satisfy the relation $u^*v^* = (vu)^*$.*

(b) *The restriction operators $\mathsf{P}(n) \xrightarrow{u^*} \mathsf{P}(m)$ associated to increasing maps $\{1 < \dots < m\} \xrightarrow{u} \{1 < \dots < n\}$ also satisfy equivariance relations, expressed by the commutativity of the diagrams*

$$\begin{tikzcd}
\mathsf{P}(n) \arrow[r, "s"] \arrow[d, "u^*"'] & \mathsf{P}(n) \arrow[d, "\rho^*"] \\
\mathsf{P}(m) \arrow[r, "\sigma"] & \mathsf{P}(m)
\end{tikzcd} ,$$

for all $s \in \Sigma_n$, where ρ denotes the increasing map and $\sigma \in \Sigma_m$ denotes the permutation such that we have the relation $su = \rho\sigma$ in $\mathtt{Mor}_{\Lambda}(\underline{\mathtt{m}},\underline{\mathtt{n}})$.

Proof. The first assertion follows from an application the associativity relation of Figure 1.3 to the unitary operad P_+. In the proof of this assertion, we also use the unit axiom of Figure 1.2.

To be explicit, we consider a composition pattern formed by three rows of operad components, with the single factor $\mathsf{P}_+(t)$ on the lower row, and a t-fold (respectively, s-fold) tensor product of the form $\mathsf{P}_+(0)\otimes\dots\otimes\mathbb{1}\otimes\dots\otimes\mathbb{1}\otimes\dots\otimes\mathsf{P}_+(0)$ on the second (respectively, third) row. The unit factors $\mathbb{1}$ are set at positions $v(1) < \dots < v(s)$ (respectively, $u(1) < \dots < u(r)$) on the second (respectively, third row). We take the image of these factors under the unit morphism $\eta : \mathbb{1} \to \mathsf{P}_+(1)$ before performing composition operations. We readily see that, for such a composition scheme, the commutativity of the diagram of Figure 1.3 gives the identity between the morphisms u^*v^* and $(vu)^*$ considered in our proposition.

The morphism u^*v^* occurs when we perform the operadic composite of the factors of the first row with the factors of the second row first and the operadic composite of the outcome of this composition operation with the factors of the third row afterwards. The other way round, when we perform the operadic composites of the second and third rows at first, we get operadic composites of operadic units with arity zero factors of our operad. More explicitly, we consider composite morphisms of the form $\mathbb{1}\otimes P_{+}(0)\xrightarrow{\eta\otimes id}P_{+}(1)\otimes P_{+}(0)\xrightarrow{\mu}P_{+}(0)$, where we still use the notation $\eta:\mathbb{1}\to P_{+}(1)$ for the unit of our unitary operad. We use the unit axiom of Figure 1.2 to identify these composites with the canonical isomorphism $\mathbb{1}\otimes P_{+}(0)\xrightarrow{\simeq}P_{+}(0)$. We readily check that we get the composition scheme of the restriction operator $(vu)^{*}$ after the performance of this first composition operation. The conclusion of assertion (a) follows.

The second assertion of the proposition is a consequence of the second equivariance axiom of Figure 1.1, where we take $n_k=1$ if $k\in\{s(u(1)),\dots,s(u(m))\}$ and $n_k=0$ otherwise. The permutation s^{*} moves the factors $P_{+}(1)$ in the tensor product $P_{+}(0)\otimes\cdots\otimes P_{+}(1)\otimes\cdots\otimes P_{+}(1)\otimes\cdots\otimes P_{+}(0)$ to the positions $1\leq u(1)<\cdots<u(m)\leq n$. Hence, the composite $\mu\cdot(id\otimes s^{*})$ occurring in our application of the equivariance axiom gives the restriction operator associated to u. On the other hand, the composition product $P_{+}(n)\otimes P_{+}(0)\otimes\cdots\otimes P_{+}(1)\otimes\cdots\otimes P_{+}(1)\otimes\cdots\otimes P_{+}(0)\xrightarrow{\mu}P_{+}(m)$ with the factors $P_{+}(1)$ at the initial positions $k\in\{s(u(1)),\dots,s(u(m))\}$ of our tensor product gives the restriction operator associated to the increasing map ρ such that $\{\rho(1)<\cdots<\rho(m)\}=\{s(u(1)),\dots,s(u(m))\}$. From the constructions of decompositions in §2.2.3, we immediately deduce that this map ρ is identified with the increasing map ρ such that $su\in\rho\Sigma_m$. Thus, the equivariance relation gives a commutative diagram of the form considered in our statement, but where σ denotes the block permutation $\sigma=s_{*}(0,\dots,0,1,0,\dots,0,\dots,0,1,0,\dots,0)$ associated to our lengths $n_k\in\{0,1\}$.

The definition of §1.1.7 implies that this block permutation is represented by the sequence $(s(u(1))',\dots,s(u(m))')$ which we obtain by withdrawing the values $k\notin\{s(u(1)),\dots,s(u(m))\}$ from $(s(1),\dots,s(n))$ and where we perform an appropriate index shift, marked by the symbol $'$, in order to retrieve a permutation of $(1,\dots,m)$. To be precise, we just jump over the values $k\notin\{s(u(1)),\dots,s(u(m))\}$ in order to carry out this re-indexing process. Performing this operation obviously amounts to applying an inverse of our increasing map ρ to the sequence $s(u(1)),\dots,s(u(m))$. We consequently have $(\rho(s(u(1))'),\dots,\rho(s(u(m))'))=(s(u(1)),\dots,s(u(m)))$ and this observation immediately implies that our block permutation is identified with the permutation σ which occurs in the decomposition $su=\rho\sigma$ of the map $f=su$. □

The following proposition is a consequence of Lemma 2.2.4:

Proposition 2.2.5. *Let P be a non-unitary operad. We assume that P is associated to a unitary operad P_{+}. Then the underlying symmetric sequence of this non-unitary operad P inherits a $\Lambda_{>0}^{op}$-diagram structure. The restriction operators of* §2.2.1 *give the action of the subcategory $\Lambda_{>0}^{+}\subset\Lambda_{>0}$ on our object P, while the action of the isomorphism subcategory $\Sigma_{>0}\subset\Lambda_{>0}$ such that $\Lambda_{>0}=\Lambda_{>0}^{+}\Sigma_{>0}$ is yielded by the natural symmetric structure of our operad.*

In §2.2.2, we coin the phrase '*non-unitary Λ-sequence*' for the $\Lambda_{>0}^{op}$-diagram structures which we consider in this proposition. In what follows, we use this terminology (rather than the name '$\Lambda_{>0}^{op}$-diagram') in order to stress the parallelism between this category of diagrams and the category of symmetric sequences. The idea is that the structure of a (non-unitary) Λ-sequence occurs as an enrichment of the structure of a (non-unitary) symmetric sequence.

Explanations. To obtain the contravariant action $s^* : \mathsf{P}(n) \to \mathsf{P}(n)$ of a permutation $s \in \Sigma_n$, regarded as a morphism of the category $\Lambda_{>0}$, we actually consider the action of the permutation $s^{-1} : \mathsf{P}(n) \to \mathsf{P}(n)$, inverse to s, in the natural symmetric structure of the operad. The inversion operation enables us to retrieve a contravariant action, as required, from the natural left action of the symmetric group Σ_n on $\mathsf{P}(n)$.

In general, the morphism $f^* : \mathsf{P}(n) \to \mathsf{P}(m)$ associated to a map $f \in \mathtt{Mor}_\Lambda(\underline{\mathsf{m}}, \underline{\mathsf{n}})$ such that $f = us$, where $u \in \mathtt{Mor}_{\Lambda^+}(\underline{\mathsf{m}}, \underline{\mathsf{n}})$ and $s \in \Sigma_m$, is explicitly defined by the composite

$$\mathsf{P}(n) \xrightarrow{u^*} \mathsf{P}(m) \xrightarrow{s^{-1}} \mathsf{P}(m),$$

where we take the restriction operator associated to u, followed by the action of the inverse of the permutation s on P.

We obviously have $id^* = id$ for the action of identity morphisms and the associativity relation of the action $f^* g^* = (gf)^*$ for general morphisms of the category $\Lambda_{>0}$ is an immediate consequence of the results of Lemma 2.2.4. ☐

To complete our analysis of the structure on the underlying sequence of unitary operads, we give a categorical interpretation of the augmentations of §2.2.1:

Proposition 2.2.6. *The augmentations* $\epsilon : \mathsf{P}(n) \to \mathbb{1}$, $n > 0$, *which we deduce from the structure of a unitary operad* P_+, *define a morphism of* $\Lambda^{op}_{>0}$*-diagrams* $\epsilon : \mathsf{P} \to \mathsf{Cst}$ *from the non-unitary operad* P *towards the constant diagram such that* $\mathsf{Cst}(n) = \mathbb{1}$, *for all* $n > 0$.

Proof. We easily check, by using the same arguments as in the proof of the functoriality relation $u^* v^* = (vu)^*$ in Lemma 2.2.4, that the augmentations $\epsilon : \mathsf{P}(n) \to \mathbb{1}$, $n > 0$, make commute the diagrams

$$\begin{tikzcd}
\mathsf{P}(n) \arrow[rr, "u^*"] \arrow[dr, "\epsilon"'] & & \mathsf{P}(m) \arrow[dl, "\epsilon"] \\
& \mathbb{1} &
\end{tikzcd},$$

where we consider the restriction operator associated to any map $u \in \mathtt{Mor}_{\Lambda^+}(\underline{\mathsf{m}}, \underline{\mathsf{n}})$. We similarly see, by using the equivariance axiom of §1.1.1, that the augmentation $\epsilon : \mathsf{P}(n) \to \mathbb{1}$ carries the action of a permutation $s \in \Sigma_n$ on $\mathsf{P}(n)$ to the identity of the operad term $\mathsf{P}_+(0) = \mathbb{1}$. This verification completes the proof of our proposition. ☐

The constant diagram Cst actually represents the underlying non-unitary $\Lambda^{op}_{>0}$-sequence of the commutative operad Com (we examine this connection more thoroughly in §2.2.19). The augmentation morphism $\epsilon : \mathsf{P} \to \mathsf{Cst}$ can therefore be identified with a morphism towards this object Com. We use the notation of the commutative operad Com, rather than the notation of the constant object Cst, in our subsequent applications of the result of Proposition 2.2.6.

In what follows, we generally call '*augmented non-unitary* Λ*-sequence*' the structure formed by a $\Lambda^{op}_{>0}$-diagrams equipped with an augmentation over the object $\mathsf{Cst} = \mathsf{Com}$. We may just forget about the augmentation when $\mathsf{Com}(r) = \mathsf{Cst}(r) = \mathbb{1}$

represents the terminal object of our base category (for instance, when we work in the category of sets $\mathcal{M} = \mathcal{S}et$), so that every non-unitary Λ-sequence is tautologically equipped with an augmentation over this diagram Com.

2.2.7. *The example of the permutation operad.* We can easily make explicit the restriction operators of the permutation operad. We just go back to the definition of the operadic composition of permutations in §§1.1.7-1.1.9.

Let $s \in \Sigma_n$. Let $u : \{1 < \cdots < m\} \to \{1 < \cdots < n\}$ be any increasing map. In the permutation operad, we have $* = id_0 \in \Sigma_0$, $1 = id_1 \in \Sigma_1$, and the composite

$$u^*(s) = s(*, \dots, *, 1, *, \dots, *, 1, *, \dots, *)$$

is given by the block permutation $s_*(0, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0)$, where we take blocks of length 1 at the positions specified by the values of our injection $1 \leq u(1) < \cdots < u(m) \leq n$ and we fill the remaining positions with blocks of length 0. This block permutation can be identified with the permutation which we obtain by withdrawing the terms $s(k) \notin \{u(1) < \cdots < u(m)\}$ in the sequence representation of the permutation $s = (s(1), \dots, s(n))$, and where we also perform an index shift to retrieve a permutation of $(1, \dots, m)$ instead of $(u(1), \dots, u(m))$. This index shift operation, which carries the set $\{u(1) < \cdots < u(m)\}$ to $\{1 < \cdots < m\}$, can formally be identified with the application of the mapping $u^{-1} : \{u(1) < \cdots < u(m)\} \xrightarrow{\simeq} \{1 < \cdots < m\}$ converse to our increasing map.

For instance, in the case of the map $u : \{1, 2, 3\} \to \{1, 2, 3, 4, 5\}$ such that

$$u(1) = 1, \quad u(2) = 4, \quad u(3) = 5,$$

and of the permutation $s = (3, 1, 5, 2, 4)$, we perform the withdrawal operation $(3, 1, 5, 2, 4) \mapsto (1, 5, 4)$, followed by the normalization operation $(1, 5, 4) \mapsto (1, 3, 2)$, to get:

$$u^*(3, 1, 5, 2, 4) = (1, 3, 2).$$

2.2.8. *Remark: decomposition of injective maps and the image of permutations under restriction operators.* In §2.2.3, we mention that the permutation $\sigma \in \Sigma_m$ which fits in the decomposition $su = \rho\sigma$ of the composite map $f = su : \{1, \dots, m\} \to \{1, \dots, n\}$ is identified with the image of the permutation $s \in \Sigma_n$ under the restriction operator $\rho^* : \Sigma_n \to \Sigma_m$ associated to the increasing map ρ on the permutation operad.

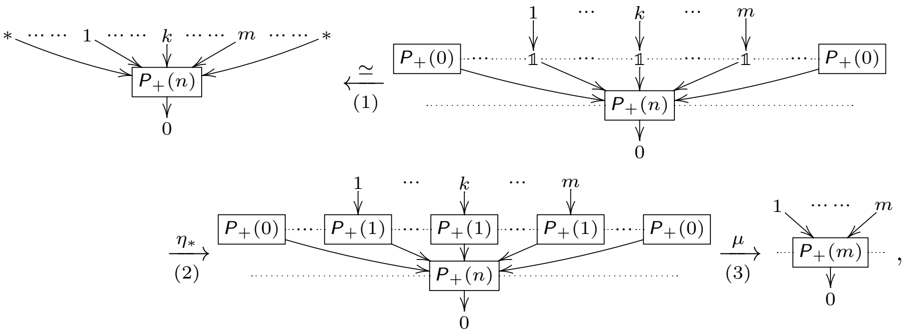
This identity $\sigma = \rho^*(s)$ actually follows from an application the equivariance relation of Lemma 2.2.4 to the identity permutation $id_n \in \Sigma_n$ in the permutation operad $\mathsf{\Pi}$. Indeed, in this case, the equivariance relation reads $\sigma \cdot u^*(id_n) = \rho^*(s)$, and this relation immediately gives our identity $\sigma = \rho^*(s)$ since we trivially have $u^*(id_n) = id_m$ for the identity permutation $id_n \in \Sigma_n$.

By applying a similar argument to the inverse permutation s^{-1}, we also obtain that the permutation $u^*(s) \in \Sigma_m$ is determined by the equation

$$s^{-1} \cdot u = \rho \cdot u^*(s)^{-1}$$

in the mapping set $\mathtt{Mor}_{\Lambda}(\underline{\mathtt{m}}, \underline{\mathtt{n}})$, where we consider the decomposition $f = \rho\sigma$ of the map $f = s^{-1}u$ as an increasing injection $\rho \in \mathtt{Mor}_{\Lambda^+}(\underline{\mathtt{m}}, \underline{\mathtt{n}})$ followed by a permutation $\sigma \in \Sigma_m$.

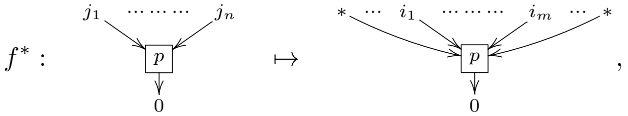
2.2.9. *The graphical definition of restriction operators.* We generally use the symbol $*$ to mark the positions of unitary factors $P_+(0) = \mathbb{1}$ in the picture of a restriction operator. When we use this convention, the definition of the restriction operators from the full composition products of a unitary operad in §2.2.1 reads:



where (2) is the morphism η_*, considered in §2.2.1, which is given by the insertion of operadic units $\eta : \mathbb{1} \to P_+(1)$ on the inputs $k = 1, \dots, m$ of our composition scheme, and (3) is the full composition product of the unitary operad P_+.

2.2.10. *The graphical representation of restriction operators.* We elaborate on our representation of the action of permutations in §1.1.5 to get a graphical interpretation of our restriction operations on treewise tensors.

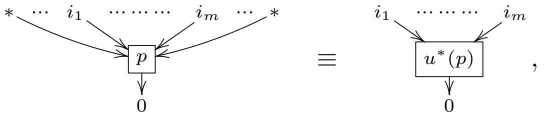
In a first step, we again use a re-indexing operation to depict the external action of any map $f : \{1 < \dots < m\} \to \{1 < \dots < n\}$ of the category Λ on a treewise tensor of the form considered in §1.1. In the context of a concrete base category, we explicit write:



for any $p \in P(n)$. To form this picture, we replace the input indices satisfying $j \in \{f(1), \dots, f(m)\}$ by their pre-image under the map $f : \{1 < \dots < m\} \to \{1 < \dots < n\}$ in the representation of the operation $p \in P(n)$, and we replace the remaining input indices $j \notin \{f(1), \dots, f(m)\}$ by the symbol $*$.

Recall that the object on the source of our map in the above picture also represents the image of the element $p \in P(n)$ under the action of the permutation $\tau \in \Sigma_n$ such that $\tau(k) = j_k$, for $k = 1, \dots, n$, on the operad P. We now consider a decomposition of the form $\tau^{-1} f = u\sigma^{-1}$ in the category Λ, where u is an increasing map and $\sigma \in \Sigma_m$. We readily see that the target of our map, which gives the result of our re-indexing operation, represents the composition pattern associated to the restriction operator $u^* : P(n) \to P(m)$, where we also apply the permutation σ to re-index the inputs associated to the element $u^*(p) \in P(m)$ in the outcome of this restriction operator. Indeed, we clearly have $\sigma(k) = i_k$, for all $k = 1, \dots, m$, when we take the orientation of our figure to enumerate these indices $(i_1, \dots, i_m)$ and the mapping $u : \{1 < \dots < m\} \to \{1 < \dots < n\}$ simply determines the positions of these inputs within the inputs of the element $p \in P(n)$.

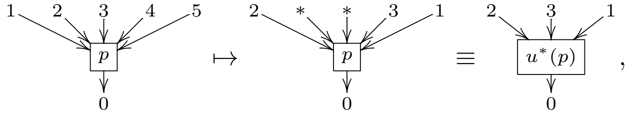
In a second step, we can use the restriction operator $u^* : P(n) \to P(m)$, associated to any increasing map $u : \{1 < \dots < m\} \to \{1 < \dots < n\}$, and which we regard as an internal operation of the operad P, in order to define a reduced form of the treewise tensors returned by our re-indexing process. We explicitly set:



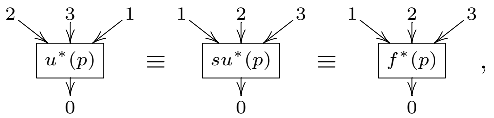
where we consider the increasing map $u : \{1 < \cdots < m\} \to \{1 < \cdots < n\}$ which fits in the decomposition $\tau^{-1} f = u\sigma^{-1}$ of our morphism f in the category Λ. The set $\{u(1) < \cdots < u(m)\} \subset \{1 < \cdots < n\}$ which determines this map simply corresponds to the position of the inputs labeled by an index $i_k \in \{1 < \cdots < m\}$, in our picture.

In §1.1.5, we also introduced an equivariance relation on treewise tensors in order to identify an input re-indexing with the internal action of a permutation on the operad. We can elaborate on this representation in order to extend the above reduction operations to restriction operators $f^* : \mathsf{P}(n) \to \mathsf{P}(m)$ associated to any map $f : \{1 < \cdots < m\} \to \{1 < \cdots < n\}$ (possibly not monotonous) in the category $\Lambda_{>0}$. In our picture, we move the index i_k attached to an input $k = 1, \dots, m$ to the edge at position $f(k)$ on the source treewise tensor product, and we mark the remaining edges with the symbol of the unitary composite $*$, as in the increasing map case. We readily see that the permutation of the index positions $(i_1, \dots, i_m)$ involved in this process corresponds to the action of the permutation σ^{-1} which occurs in the decomposition $f = u\sigma^{-1}$ of our mapping. We use the equivariance relation of Lemma 2.2.4(b) to establish the coherence of this extended reduction process with respect to the composition of restriction operators and the action of permutations.

To give a simple example, for the map $f : \underline{3} \to \underline{5}$ such that $f(1) = 5$, $f(2) = 1$, $f(3) = 4$, the image of an element $p \in \mathsf{P}(5)$ under the restriction operator $f^* : \mathsf{P}(5) \to \mathsf{P}(3)$ is given by the following picture:

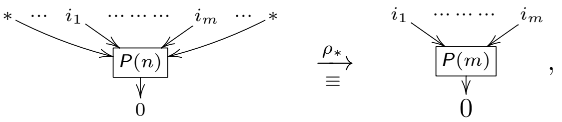


where u denotes the increasing map such that $u(1) = 1$, $u(2) = 4$, $u(3) = 5$. We also have:



where s denotes the permutation such that $s(1) = 2$, $s(2) = 3$, $s(3) = 1$, and for which also we get $f = us^{-1} \Rightarrow su^* = f^*$.

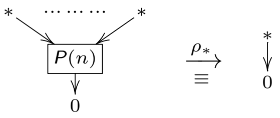
In what follows, we use an extension of this graphical representation to a general categorical setting where we deal with abstract morphisms not necessarily defined by maps on concrete tensors. We then adopt the generic notation ρ_* for the restriction operators



which we consider in our reduction process.

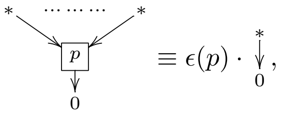
The treewise restriction operators can also be applied to treewise tensor products shaped on trees with several vertices. This natural extension of our picture is used in Proposition 2.2.16, when we formulate an associativity relation between the restriction operators and the partial composition products of operads.

2.2.11. *The graphical representation of augmentations.* We adapt the conventions of the previous paragraph to represent the augmentations $\epsilon : \mathsf{P}(n) \to \mathbb{1}$ which we deduce from the structure of a unitary operad P_+. We use the augmentation morphism to extend the reduction procedure of the previous paragraph when all inputs are marked by the unitary symbol $*$. We get the following picture



for this reduction process. The degenerate tree $* \to 0$ which occurs in this picture formally represents a copy of the unit object $\mathbb{1}$.

If we work within a category of modules, then we have:



for any operad element $p \in \mathsf{P}(n)$, where we now regard the degenerate tree $* \to 0$ as a canonical generator for the free $\mathbb{k}$-module of rank 1, and we identify the image of the element $p \in \mathsf{P}(n)$ under the augmentation $\epsilon : p \mapsto \epsilon(p)$ with a multiplicative scalar $\epsilon(p) \in \mathbb{k}$.

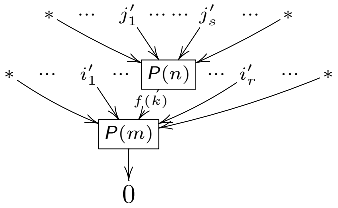
In what follows, we also deal with an obvious extension of this construction for treewise tensors shaped on trees with several vertices.

In §2.2.1, we focus on composition products of a unitary operad P_+ which only involve the arity zero term $\mathsf{P}_+(0) = \mathbb{1}$ and operadic unit as composition factors. But we can still consider partial composition products $\circ_k : \mathsf{P}_+(m) \otimes \mathsf{P}_+(n) \to \mathsf{P}_+(m+n-1)$ defined by the composition scheme of §2.1.4. In the cases $m, n > 0$, which exclude the composites with the unitary factor $\mathsf{P}_+(0) = \mathbb{1}$, these composition operations are identified with internal composition operations of the non-unitary operad P underlying P_+ and they satisfy the equivariance, unit, and associativity relations of §2.1 within this non-unitary operad.

To complete our results, we make explicit associativity relations which combine our restriction operators and the partial composition products of §2.1. We regard these mixed associativity relations as part of an equivariance property of the partial composition products with respect to the action of category $\Lambda_{>0}$ on the underlying collection and to the action of the augmentations. We actually establish an equivariance relation of the same shape as the relation of Proposition 2.1.2, but we now deal with an action of general injective maps instead of permutations. In a preliminary step, we extend the definition of the partial composition product of permutations, involved in this equivariance relation, to injective maps.

2.2.12. *Partial composition products for ordinal injections.* Let $f : \{1 < \dots < r\} \to \{1 < \dots < m\}$ and $g : \{1 < \dots < s\} \to \{1 < \dots < n\}$ be any pair of injective maps. Let $k \in \{1 < \dots < r\}$. Our purpose is to define a map $f \circ_{f(k)} g : \{1 < \dots < r+s-1\} \to \{1 < \dots < m+n-1\}$ that reflects the input indexing and the distribution of unitary symbols $*$ in the treewise tensor representation of a partial composition operation:

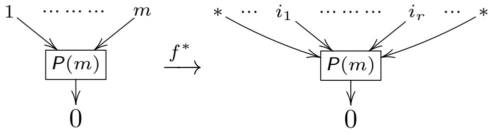
(1)



,

where we perform the re-indexing process determined by the map f on the inputs of the lower factor:

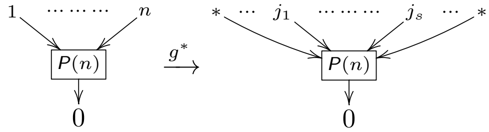
(2)



,

and the re-indexing process determined by the map g on the inputs of the upper factor:

(3)



.

To simplify our layout, we assume that we start with treewise tensors equipped with a canonical input indexing, as indicated in the above pictures (2-3). In this situation, the value $f(k) \in \{1 < \cdots < m\}$ of our mapping $f : \{1 < \cdots < r\} \to \{1 < \cdots < m\}$ marks the position of the ingoing edge labeled by the index $k \in \{1 < \cdots < r\}$ in the outcome of our re-indexing process (2) and similarly in (3).

The indices $(i_1, \dots, i_r)$ in (2) form a permutation of $(1, \dots, r)$ and the indices $(j_1, \dots, j_s)$ in (3) form a permutation of $(1, \dots, s)$. When we form the composite of our re-indexed operations to get the picture of Equation (1), we perform our standard index shift in order to obtain a composite operation with inputs labeled by the index set $\{1, \dots, r + s - 1\}$. In our figure, we use the notation i'_* (respectively, j'_*) to denote the image of the indices i_* (respectively, j_*) under this canonical re-indexing operation.

Thus, our goal is essentially to determine the position of the edge marked by each value $x \in \{1, \dots, r + s - 1\}$ in the outcome of this re-indexing process. This position gives the value of our composite map $f \circ_{f(k)} g$ on $x \in \{1, \dots, r + s - 1\}$. The determination of this position follows from a straightforward inspection of our figure. We obtain that the map $f \circ_{f(k)} g$ can be determined by the following procedure. We first set:

$$f(x)' = \begin{cases} f(x), & \text{if } f(x) < f(k), \\ f(x) + n - 1, & \text{if } f(x) > f(k), \end{cases}$$

for $x = 1, \dots, k - 1, k + 1, \dots, r$, and:

$$g(y)' = g(y) + f(k) - 1,$$

for $y = 1, \dots, s$. This index shift is equivalent to the shuffle $\{1 < \dots < \widehat{f(k)} < \dots < m\} \amalg \{1 < \dots < n\} \xrightarrow{\simeq} \{1 < \dots < m+n-1\}$ which reflects the planar ordering of the ingoing edges of the lower and upper vertices in our first picture (1). Then we define our map $f \circ_{f(k)} g : \{1 < \dots < r+s-1\} \to \{1 < \dots < m+n-1\}$ by the formula:

$$(f \circ_{f(k)} g)(l) = \begin{cases} f(l)', & \text{for } l = 1, \dots, k-1, \\ g(l-k+1)', & \text{for } l = k, \dots, k+s-1, \\ f(l-s+1)', & \text{for } l = k+s, \dots, r+s-1. \end{cases}$$

Equivalently, we can determine our map $f \circ_{f(k)} g$ by replacing the occurrence of the term $f(k)$ in the sequence $(f(1), \dots, f(r))$ by the sequence $(g(1), \dots, g(s))$ and by performing the above shift operations $f(x) \mapsto f(x)'$ and $g(y) \mapsto g(y)'$ in the outcome of this substitution process.

2.2.13. *Remarks: Partial composition of increasing injective maps and associativity relations.* In the case of increasing maps $u \in \mathtt{Mor}_{\Lambda^+}(\underline{\mathtt{r}}, \underline{\mathtt{m}})$ and $u \in \mathtt{Mor}_{\Lambda^+}(\underline{\mathtt{s}}, \underline{\mathtt{n}})$, which are determined by sequences of values of the form:

$$1 \leq u(1) < \dots < u(r) \leq m \quad \text{and} \quad 1 \leq v(1) < \dots < v(s) \leq n,$$

we immediately see, from our definition, that the map $u \circ_{u(k)} v : \{1 < \dots < r+s-1\} \to \{1 < \dots < m+n-1\}$ is the increasing map represented by the sequence:

$$\begin{aligned} 1 \leq u(1) < \dots < u(k-1) < v(1) + u(k) - 1 < \dots < v(s) + u(k) - 1 \\ < u(k+1) + n - 1 < \dots < u(r) + n - 1 \leq m + n - 1. \end{aligned}$$

If we assume that u and v represent the ordered sequence of the remaining input positions after our re-indexing operation §2.2.12(2-3), then we readily see that this sequence, associated to $u \circ_{u(k)} v$, corresponds to the input positions of our composition scheme §2.2.12(1).

Let $f \in \mathtt{Map}_{\Lambda}(\underline{\mathtt{r}}, \underline{\mathtt{m}})$ be a map such that $f = u\sigma$, where we assume $u \in \mathtt{Map}_{\Lambda^+}(\underline{\mathtt{k}}, \underline{\mathtt{m}})$ and $\sigma \in \Sigma_r$. Let $g \in \mathtt{Map}_{\Lambda}(\underline{\mathtt{s}}, \underline{\mathtt{n}})$ be a map such that $g = v\tau$, for some $v \in \mathtt{Map}_{\Lambda^+}(\underline{\mathtt{l}}, \underline{\mathtt{n}})$ and some $\tau \in \Sigma_s$. We easily check that we have the associativity relation $f \circ_{f(k)} g = (u \circ_{u\sigma(k)} v) \cdot (\sigma \circ_{\sigma(k)} \tau)$, where we consider the partial composition $\sigma \circ_{\sigma(k)} \tau$ of the permutations $\sigma \in \Sigma_r$ and $\tau \in \Sigma_s$, such as defined in §2.1.3. We can use this identity to establish the correspondence of §2.2.12 in two steps: we check the case of increasing maps first and we use the coherence of our definition with respect to the action of permutations in order to establish our general formula afterwards.

2.2.14. *Remarks: Partial composition with the empty map.* The definition of the partial composition operation $f \circ_{g(k)} g$ in §2.2.12 remains valid when g is an empty map $g = o : \underline{\mathtt{0}} \to \underline{\mathtt{n}}$. We still assume that $f : \underline{\mathtt{r}} \to \underline{\mathtt{m}}$ is any injective map and we fix $k \in \underline{\mathtt{r}}$. In this setting, our composition operation returns a map $f \circ_{f(k)} o : \{1 < \dots < r-1\} \to \{1 < \dots < m+n-1\}$ such that:

$$(f \circ_{f(k)} o)(l) = \begin{cases} f(l)', & \text{for } l = 1, \dots, k-1, \\ f(l+1)', & \text{for } l = k, \dots, r-1, \end{cases}$$

where we still perform the following index shift on the values of our function $f : \underline{\mathsf{r}} \to \underline{\mathsf{m}}$ at the points $x = 1, \dots, \widehat{k}, \dots, r$:

$$f(x)' = \begin{cases} f(x), & \text{when } f(x) < f(k), \\ f(x) + n - 1, & \text{when } f(x) > f(k). \end{cases}$$

In the case of an increasing map $u : \{1 < \cdots < r\} \to \{1 < \cdots < m\}$, represented by a sequence of the form $1 \leq u(1) < \cdots < u(r) \leq r$ this construction returns the increasing map $u \circ_{u(k)} o : \{1 < \cdots < r - 1\} \to \{1 < \cdots < m + n - 1\}$ represented by the sequence:

$$1 \leq u(1) < \cdots < u(k-1) < u(k+1) + n - 1 < \cdots < u(r) + n - 1 \leq m + n - 1$$

which we obtain by removing the value $u(k)$ and by shifting the terms $u(l) > u(k)$ by $n - 1$.

The following observation is used in §II.8.4 and in §II.11.2 in our construction of a model for the rational homotopy of unitary operads:

Proposition 2.2.15. *We fix a pair of finite ordered sets* $\underline{\mathsf{m}} = \{1 < \cdots < m\}$, $\underline{\mathsf{n}} = \{1 < \cdots < n\}$, *and a composition index* $i \in \{1 < \cdots < m\}$. *Each injective map* $h \in \mathtt{Mor}_{\Lambda}(\underline{\mathsf{t}}, \underline{\mathsf{m} + \mathsf{n} - 1})$ *admits a decomposition* $h = f \circ_{f(k)} g$, *for uniquely determined maps* $f \in \mathtt{Mor}_{\Lambda}(\underline{\mathsf{r}}, \underline{\mathsf{m}})$ *and* $g \in \mathtt{Mor}_{\Lambda}(\underline{\mathsf{s}}, \underline{\mathsf{n}})$, *uniquely determined ordered sets* $\underline{\mathsf{r}} = \{1 < \cdots < r\}$ *and* $\underline{\mathsf{s}} = \{1 < \cdots < s\}$ *(possibly,* $s = 0$*), and a composition index* $k \in \{1 < \cdots < r\}$ *such that* $f(k) = i$.

Proof. Exercise. ☐

We have the following statement:

Proposition 2.2.16. *Let* P *be the underlying non-unitary operad of a unitary operad* P_+.

(a) *We have a commutative diagram:*

$$\begin{tikzcd} \mathsf{P}(m) \otimes \mathsf{P}(n) \arrow[r, "\circ_k"] \arrow[d, "\epsilon\otimes\epsilon"'] & \mathsf{P}(m+n-1) \arrow[d, "\epsilon"] \\ \mathbb{1} \otimes \mathbb{1} \arrow[r, "\simeq"] & \mathbb{1} \end{tikzcd} ,$$

for any $m, n > 0$, $k = 1, \dots, m$, *where we consider the partial composition products* $\circ_k : \mathsf{P}(m) \otimes \mathsf{P}(n) \to \mathsf{P}(m + n - 1)$ *of the non-unitary operad* P *and the augmentation morphisms* $\epsilon : \mathsf{P}(r) \to \mathbb{1}$, $r > 0$, *which we determine from the composition operations of our unitary operad* P_+.

(b) *We also have a commutative diagram of the form:*

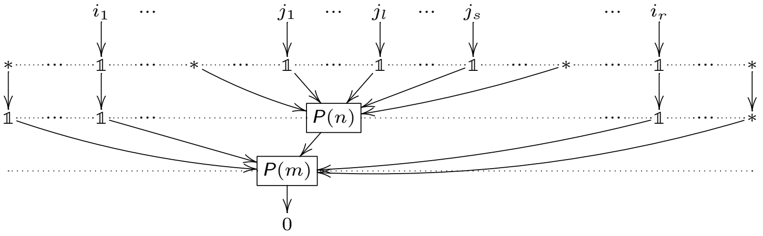
$$\begin{tikzcd} \mathsf{P}(m) \otimes \mathsf{P}(n) \arrow[r, "\circ_{f(k)}"] \arrow[d, "f^*\otimes g^*"'] & \mathsf{P}(m+n-1) \arrow[d, "(f\circ_{f(k)} g)^*"] \\ \mathsf{P}(r) \otimes \mathsf{P}(s) \arrow[r, "\circ_k"] & \mathsf{P}(r+s-1) \end{tikzcd} ,$$

for any $m, n > 0$, $k = 1, \dots, m$, *where we consider the restriction operators associated to any pair of maps* $f \in \mathtt{Mor}_{\Lambda}(\underline{\mathsf{r}}, \underline{\mathsf{m}})$, $g \in \mathtt{Mor}_{\Lambda}(\underline{\mathsf{s}}, \underline{\mathsf{n}})$ *on our operad* P, *and we* *have a commutative diagram of the form:*

$$\begin{tikzcd}
\mathsf{P}(m)\otimes\mathsf{P}(n) \arrow[rr, "\circ_{f(k)}"] \arrow[d, "f^*\otimes\epsilon"'] & & \mathsf{P}(m+n-1) \arrow[d, "(f\circ_{f(k)}o)^*"] & , \\
\mathsf{P}(r)\otimes\mathbb{1} \arrow[r, "\simeq"] & \mathsf{P}(r) \arrow[r, "\partial_k"] & \mathsf{P}(r-1) &
\end{tikzcd}$$

for any $m,n>0$, $k=1,\dots,m$, where we consider an empty map $o\in\mathtt{Mor}_{\Lambda}(\underline{0},\underline{\mathtt{n}})$, we still assume $f\in\mathtt{Mor}_{\Lambda}(\underline{\mathtt{r}},\underline{\mathtt{m}})$, $k\in\{1<\dots<r\}$, and ∂_k denotes the restriction operator associated to the increasing map $\partial^k:\{1<\dots<r-1\}\to\{1<\dots<r\}$ such that $\partial^k(x)=x$ for $x=1,\dots,k-1$ and $\partial^k(x)=x+1$ for $x=k,\dots,r-1$ (see §2.2.1*).*

PROOF. This proposition follows from a simple variation of the argument line of Proposition 2.1.7, where we establish the associativity relations of the partial composition products. To get our result, we just replace the composition schemes considered in the proof of this proposition by composition schemes of the form:



with unitary factors $*$ on the vertices of the upper level. The claim of assertion (a) corresponds to the case where all these vertices are marked by the symbol $*$. The proof of our statements reduces to straightforward verifications. Note that we can restrict ourselves to the case of increasing maps. The general result then follows from the associativity of the partial composition operation with respect to the composition of maps in the category Λ and from the result of Proposition 2.1.2, where we check the equivariance of the partial composition products of an operad with respect to the action of permutations. ☐

2.2.17. *The definition of unitary operads as extensions of non-unitary operads.* In §2.1, we established that the composition structure of an operad is determined by giving the partial composition products of §2.1.1. In the unitary context, we obtain from the results obtained in this section that the composition structure of a unitary extension P_+ of a non-unitary operad P can be determined by giving:

(1) the internal partial composition products of the non-unitary operad P, which, in our unitary extension, correspond to the partial composition products
$$\underbrace{\mathsf{P}_+(m)\otimes\mathsf{P}_+(n)}_{=\mathsf{P}(m)\otimes\mathsf{P}(n)}\xrightarrow{\circ_k}\underbrace{\mathsf{P}_+(m+n-1)}_{=\mathsf{P}(m+n-1)}$$
such that $m,n>0$,

(2) the restriction operators $u^*:\mathsf{P}(n)\to\mathsf{P}(m)$, for $u\in\mathtt{Mor}_{\Lambda^+}(\underline{\mathtt{m}},\underline{\mathtt{n}})$ and $m,n>0$, which are equivalent to composition operations of the form
$$\underbrace{\mathsf{P}_+(n)\otimes\mathsf{P}_+(0)\otimes\cdots\otimes\mathbb{1}\otimes\cdots\otimes\mathbb{1}\otimes\cdots\otimes\mathsf{P}_+(0)}_{=\mathsf{P}(n)}\xrightarrow{\mu\eta_*}\underbrace{\mathsf{P}_+(m)}_{=\mathsf{P}(m)},$$

with operadic units and unitary terms as composition factors,

(3) and the augmentations $\epsilon : P(n) \to \mathbb{1}$, $n > 0$, which yield composition products

$$\underbrace{P_+(n) \otimes P_+(0) \otimes \cdots \otimes P_+(0)}_{=P(n)} \xrightarrow{\mu} \underbrace{P_+(0)}_{=\mathbb{1}}$$

with the unitary term $P_+(0) = \mathbb{1}$ as target.

These structure morphisms also satisfy the following assertions:

(a) The partial composition products (1) fulfill the equivariance, unit, and associativity axioms of §2.1.9 within the non-unitary operad P.

(b) The restriction operators (2) and the augmentation (3) also fulfill equivariance and internal associativity relations, formulated in Lemma 2.2.4, so that (2-3) actually provide the underlying sequence of the non-unitary operad P with the structure of an augmented Λ-diagram (in the terms of §2.2.2).

(c) The restriction operators (2) and the augmentation (3) fulfill associativity relations, which we express by the commutativity of the diagrams of Proposition 2.2.16, with respect to the partial composition operations (1).

(d) The component of arity one of the augmentation (3) defines a retraction of the operadic unit $\eta : \mathbb{1} \to P(1)$.

We call *augmented non-unitary* Λ*-operad* the general structure defined by a non-unitary operad P equipped with restriction operators (2) and augmentations (3) that satisfy the above requirements. We also adopt the notation $\Lambda\,\mathcal{O}p_{\varnothing}\,/\,\mathit{Com}$ for this category of augmented non-unitary Λ-operads, where we obviously take the morphisms of non-unitary operads that preserve the restriction operators and the augmentation of our objects as morphisms.

We already briefly mentioned that the augmentations $\epsilon : P(n) \to \mathbb{1}$ which we associate to our unitary operad structure in (3) actually define a morphism with values in the underlying $\Lambda^{op}_{>0}$-diagram of the commutative operad Com_+. We soon check that these morphisms define a morphism of augmented non-unitary Λ-operads from P to the augmented non-unitary Λ-operad Com underlying Com_+, so that the (non-unitary) commutative operad Com actually represents the terminal object of our category of augmented non-unitary Λ-operads. This observation motivates the notation $\Lambda\,\mathcal{O}p_{\varnothing}\,/\,\mathit{Com}$ which we give to this category of operads.

The notion of an augmented Λ-operad has a natural extension to operads with no fixed arity zero component. But in what follows we only consider the non-unitary version of this notion, such as defined in this paragraph. For this reason, we often omit to specify that we restrict ourselves to non-unitary operads and we just use the name 'augmented Λ-operad' to refer to objects of the category of augmented non-unitary Λ-operads.

In certain cases (when the unit object of the category is the terminal object), an operad comes automatically equipped with augmentation morphisms (3), which are canonically determined by the structure of the ambient category. In this context, we use the abridged notation $\Lambda\,\mathcal{O}p_{\varnothing} = \Lambda\,\mathcal{O}p_{\varnothing}\,/\,\mathit{Com}$ for the category of augmented non-unitary Λ-operads. Furthermore, we just use the name 'Λ-operad', where we omit to specify the existence of the augmentation, for the objects of this category $P \in \Lambda\,\mathcal{O}p_{\varnothing}$.

We can summarize the previous results of this section and the correspondence of §2.2.17 as the definition of a functor $\tau : \mathcal{O}p_* \to \Lambda\,\mathcal{O}p_\varnothing\,/\,\mathsf{Com}$ from the category of unitary operads $\mathcal{O}p_*$ towards the category of augmented non-unitary Λ-operad $\Lambda\,\mathcal{O}p_\varnothing\,/\,\mathsf{Com}$. We check that:

Theorem 2.2.18. *The correspondence of* §2.2.17 *gives an isomorphism of categories between the category of unitary operads* $\mathcal{O}p_*$ *and the category of augmented non-unitary* Λ*-operads* $\Lambda\,\mathcal{O}p_\varnothing\,/\,\mathsf{Com}$.

Proof. In what follows, we use general restriction operators (and the correspondence of this theorem) to handle the composition operations with the unitary term $\mathsf{P}_+(0) = \mathbb{1}$ of unitary operads P_+, while we consider partial composition operations when we deal with composition structures attached to the non-unitary operad P underlying our object P_+. Nevertheless, we can apply the result of Theorem 2.1.10, where we explain the definition of arbitrary operads in terms of partial composition operations, to unitary operads P_+, and we temporarily focus on the restriction operators corresponding to the partial composition operations with the unitary term $\mathsf{P}_+(0) = \mathbb{1}$ in the proof of this theorem.

Thus, we consider the particular restriction operators $\partial_k : \mathsf{P}(n) \to \mathsf{P}(n-1)$, defined in §2.2.1, which correspond to the partial composition operations with a unitary factor $\circ_k : \mathsf{P}_+(n) \otimes \mathsf{P}_+(0) \to \mathsf{P}_+(n-1)$, for any $n > 1$. The terminal augmentation $\epsilon : \mathsf{P}(1) \to \mathbb{1}$, on the component of arity one of our operad $\mathsf{P}(1)$, is also clearly identified with the unique partial composition product $\circ_1 : \mathsf{P}_+(1) \otimes \mathsf{P}_+(0) \to \mathsf{P}_+(0)$, which has the unitary component as target $\mathsf{P}_+(0) = \mathbb{1}$. The structure of an augmented non-unitary Λ-operad P accordingly includes all partial composition products which we have to fix in order to determine this unitary operad P_+. The equivariance, unit and associativity requirements of the restriction operators in the definition of an augmented non-unitary Λ-operad also cover all equivariance, unit and associativity relations which these partial composition products with a unitary factor $\circ_k : \mathsf{P}_+(n) \otimes \mathsf{P}_+(0) \to \mathsf{P}_+(n-1)$, $k = 1, \dots, n$, have to satisfy within the unitary operad P_+. We therefore obtain that the mapping $\tau : \mathsf{P}_+ \mapsto \mathsf{P}$ defines an isomorphism of categories $\tau : \mathcal{O}p_* \to \Lambda\,\mathcal{O}p_\varnothing\,/\,\mathsf{Com}$ as claimed in our theorem. ☐

The unitary commutative operad Com_+, such as defined in §2.1.11, provides a natural example of unitary operad. We check that this operad Com_+ represents the terminal object of the category of unitary operads $\mathcal{O}p_*$ (as we already mentioned in this section) and, equivalently, that the non-unitary commutative operad Com underlying Com_+ represents the terminal object of the category of augmented non-unitary Λ-operads $\Lambda\,\mathcal{O}p_\varnothing\,/\,\mathsf{Com}$.

2.2.19. *The unitary structure of the commutative operad.* Recall that the operad Com_+ is defined by the constant symmetric sequence $\mathsf{Com}_+(n) = \mathbb{1}$, that the action of the symmetric group Σ_n on $\mathsf{Com}_+(n)$ is trivial, and that the composition products $\circ_k : \mathsf{Com}_+(m) \otimes \mathsf{Com}_+(n) \to \mathsf{Com}_+(m+n-1)$ are given by the canonical unit isomorphisms $\mathbb{1} \otimes \mathbb{1} \simeq \mathbb{1}$ in our base symmetric monoidal category $\mathcal{M}$. The composition products $\mu : \mathsf{Com}_+(r) \otimes \mathsf{Com}_+(n_1) \otimes \dots \otimes \mathsf{Com}_+(n_r) \to \mathsf{Com}_+(n_1 + \dots + n_r)$, which we associate to this composition structure, are obviously given by the canonical unit isomorphisms $\mathbb{1} \otimes \mathbb{1} \otimes \dots \otimes \mathbb{1} \simeq \mathbb{1}$ too, and we easily get from this definition that the restriction operators of the non-unitary operad Com underlying Com_+ are given by identity morphisms of the unit object. Hence, in the result of Proposition 2.2.5, we obtain that Com inherits a constant $\Lambda^{op}_{>0}$-diagram structure (in arity $n > 0$). The augmentations $\epsilon : \mathsf{Com}(n) \to \mathbb{1}$ associated to this operad are given by the identity of the unit object too.

In Proposition 2.2.6, we prove that the $\Lambda_{>0}^{op}$-diagram formed by the non-unitary operad P underlying a unitary operad P_+ is canonically augmented over this constant diagram. In fact, at the operad level, we have the following result:

Proposition 2.2.20. *The augmentations $\epsilon : \mathsf{P}_+(n) \to \mathbb{1}$, which we define by considering the components $\mathsf{P}_+(n) = \mathsf{P}_+(n) \otimes \mathsf{P}_+(0)^{\otimes n} \xrightarrow{\mu} \mathsf{P}_+(0)$ of the composition products of a unitary operad P_+ (see* §2.2.1*), form a morphism of unitary operads $\epsilon : \mathsf{P}_+ \to \mathsf{Com}_+$.*

Proof. The equivariance, unit, and associativity axioms of operads, as formulated in the definition of §1.1.1, imply that the augmentations $\epsilon : \mathsf{P}_+(n) \to \mathbb{1}$ carry all structure morphisms associated to the unitary operad P_+ to the identity of the unit object $\mathbb{1}$. We immediately conclude that these augmentations define a morphism towards the commutative operad Com_+ as asserted. ☐

From this proposition, we readily conclude that:

Proposition 2.2.21. *The unitary commutative operad Com_+, such as defined in* §3.1*, represents the terminal object of the category of unitary operads.* ☐

In the non-unitary setting, we obtain the following counterpart of these statements:

Proposition 2.2.22. *The augmentation morphisms $\epsilon : \mathsf{P}(n) \to \mathbb{1}$ attached to an augmented non-unitary Λ-operad $\mathsf{P} \in \Lambda\,\mathcal{O}p_{\varnothing} \,/\, \mathsf{Com}$ define a morphism of augmented non-unitary Λ-operads $\epsilon : \mathsf{P} \to \mathsf{Com}$, where we assume that the commutative operad Com is equipped with the constant $\Lambda_{>0}^{op}$-diagram structure of Proposition* 2.2.6 *and with the identity of the unit object $\mathbb{1}$ as augmentation morphisms $\epsilon : \mathsf{Com}(n) \to \mathbb{1}$.*

The commutative operad Com accordingly represents the terminal object of the category of augmented non-unitary Λ-operads $\Lambda\,\mathcal{O}p_{\varnothing} \,/\, \mathsf{Com}$. ☐

Recall that the structure morphisms which we associate with the commutative operad Com in this proposition correspond to the composition structure attached to the unitary version of the commutative operad Com_+. The claim of this proposition is therefore equivalent, in view of the result of Theorem 2.2.18, to the assertions of Proposition 2.2.20-2.2.21.

2.2.23. *Free algebras over unitary operads.* We assume throughout this paragraph that we work in the category of topological spaces. The operads consider in May's monograph [140] are actually unitary operads (called unital operads in that reference).

We already observed that giving the arity 0 operation $\lambda : \mathsf{P}_+(0) \to A$, for an algebra A over a unitary operad P_+, amounts to fixing a unit element in A. In the topological setting, we can identify this unit element with a base point associated to our space A and we therefore naturally deal with the category of pointed spaces when we consider unitary operads P_+ in topological spaces. Then we can use the action of the category of finite ordinals and injections to give a reduced version of the free P_+-algebra functor. We proceed as follows. In this paragraph, we consider the whole category of finite ordinals and injections Λ rather than the subcategory $\Lambda_{>0}$. The (contravariant) action of the truncated category $\Lambda_{>0}$ on the non-unitary

P, which we have considered all through this section, extends to a (contravariant) action of the category Λ on the unitary operad P_+ associated to P.

The basic observation is that the cartesian powers $X^{\times n}$, $n \in \mathbb{N}$, of a pointed space X inherits a (covariant) Λ-diagram structure: to any injective map $f : \{1 < \cdots < m\} \to \{1 < \cdots < n\}$, which defines a morphism in the category Λ, we associate the mapping $f_* : X^{\times m} \to X^{\times n}$ which maps any m-tuple $(x_1, \dots, x_m) \in X^{\times m}$ to the n-tuple $(y_1, \dots, y_n) \in X^{\times n}$ such that:

$$y_j = \begin{cases} x_i, & \text{if } j = f(i), \text{ for some } i, \\ *, & \text{otherwise.} \end{cases}$$

We then form the coend

$$\mathbb{S}_*(\mathsf{P}_+, X) = \int^{\underline{\mathrm{n}} \in \Lambda} \mathsf{P}_+(n) \times X^{\times n}$$

to define the (reduced free) P_+-algebra associated to our based space X. Intuitively, performing this coend amounts to implementing identities

$$p f_*(x_1, \dots, x_m) = f^* p(x_1, \dots, x_m),$$

in the free algebra structures of §1.3. One can readily check that this reduced free algebra functor $\mathbb{S}_*(\mathsf{P}_+) : X \mapsto \mathbb{S}_*(\mathsf{P}_+, X)$ gives a left adjoint of the reduced forgetful functor $\omega : \mathcal{P}_+ \to \mathcal{T}op_*$, where we retain the base point, defined by the map $\lambda : \mathsf{P}_+(0) \to A$, from the structure of our P_+-algebras $A \in \mathcal{P}_+$.

May's approximation theorem [140] deals with the reduced free algebras associated to the little n-cubes operads C_{n+} (see §4.1). May's result precisely asserts that, when X is a connected space (and more generally, when X is group-like), the free algebra $\mathbb{S}_*(\mathsf{C}_{n+}, X)$ is weakly-equivalent to the iterated loop spaces $\Omega^n \Sigma^n X$, where Ω^n refers to the n-fold loop space functor on pointed spaces, and Σ^n refers to the n-fold suspension. In this construction, which gives the starting point of the iterated loop space theory of [140], the little n-cubes operad C_{n+} can be replaced by any Λ-object weakly-equivalent to C_{n+} and which satisfies some mild cofibration conditions with respect to the action of symmetric groups.

The most classical example of such an equivalence, occurring in the case $n = 1$, is given by the associative (permutation) operad As_+. In this case, the reduced free associative monoid, which our construction represents, can be identified with the construction $J(X)$ introduced by James in [96]. May's approximation theorem actually occurred as a generalization of a result established by James for the combinatorial construction $J(X)$.

The abstract structure defined by restriction operators is notably used to give a model of generalized James-Hopf's maps in [44] and in Berger's recognition criterion of E_n-operads (the operads which are weakly-equivalent to the little n-cubes) in [23, 24]. We go back to the definition of the little cubes operads and to the subject of iterated loop space theory in §4. We give further references on these topics in this subsequent chapter §4.

2.3. Categorical constructions for unitary operads

We now assume that the tensor product of the base category distributes over colimits, as we require in §0.9, and we revisit the definition of the categorical constructions of §1.2 in the context of unitary operads. The idea is to use the category isomorphism of Theorem 2.2.18 and to formulate our constructions in terms of augmented non-unitary Λ-operads rather than in terms of unitary operads.

Our first purpose is to explain the definition of a reduced version of free unitary operads. Let P_+ be any unitary operad. By Proposition 2.2.5 and Proposition 2.2.6, the collection $\mathsf{P}(n)$, $n > 0$, where we forget about the arity zero term $\mathsf{P}_+(0) = \mathbb{1}$ of this unitary operad P_+, inherits the structure of a $\Lambda_{>0}^{op}$-diagram and is also equipped with an augmentation over the constant $\Lambda_{>0}^{op}$-diagram Cst, which represents the $\Lambda_{>0}^{op}$-diagram associated to the underlying non-unitary operad Com of the unitary commutative operad Com_+. Recall that we rather use the phrase 'augmented non-unitary Λ-sequence' to refer to this structure (see §2.2.2). Thus, the mapping $\omega_+ : \mathsf{P}_+ \mapsto \mathsf{P}$ actually gives a functor $\omega_+ : \mathcal{O}p_* \to \Lambda\,\mathcal{S}eq_{>0}\,/\,\mathsf{Com}$ from the category of unitary operads $\mathcal{O}p_*$ to the category of augmented non-unitary Λ-sequences, which we denote by $\Lambda\,\mathcal{S}eq_{>0}\,/\,\mathsf{Com}$. This functor reduces to the obvious forgetful functor $\omega : \Lambda\,\mathcal{O}p_{\varnothing}\,/\,\mathsf{Com} \to \Lambda\,\mathcal{S}eq_{>0}\,/\,\mathsf{Com}$ when we pass to the category of augmented non-unitary Λ-operads $\mathsf{P} \in \Lambda\,\mathcal{O}p_{\varnothing}\,/\,\mathsf{Com}$ equivalent to the category of unitary operads $\mathsf{P}_+ \in \mathcal{O}p_*$.

We elaborate on the free operad construction of the introductory chapter §1 to define a left adjoint of this forgetful functor $\mathbb{\Theta} : \Lambda\,\mathcal{S}eq_{>0}\,/\,\mathsf{Com} \to \Lambda\,\mathcal{O}p_{\varnothing}\,/\,\mathsf{Com}$. We then use the isomorphism of Theorem 2.2.18 to go back to the category of unitary operads $\mathcal{O}p_*$ and to define the reduced unitary operad $\mathbb{\Theta}(\mathsf{M})_+$ which we associate to any object $\mathsf{M} \in \Lambda\,\mathcal{S}eq_{>0}\,/\,\mathsf{Com}$. In comparison with the construction of §1.2, we already fix a subpart of the composition structure of the free unitary operad $\mathbb{\Theta}(\mathsf{M})_+$ in the definition of the augmented non-unitary Λ-sequence M. This free unitary operad $\mathbb{\Theta}(\mathsf{M})_+$ is therefore smaller than the standard free operad of §1.2.

The forgetful functor from the category of augmented non-unitary Λ-operads to the category of augmented non-unitary Λ-sequences fits in a diagram

$$\begin{tikzcd}
\Lambda\,\mathcal{S}eq_{>0}\,/\,\mathsf{Com} \arrow[d] & \Lambda\,\mathcal{O}p_{\varnothing}\,/\,\mathsf{Com} \arrow[l, "\omega"'] \arrow[d] \\
\mathcal{S}eq_{>0} \arrow[r, dotted, shift left, "\mathbb{\Theta}"] & \mathcal{O}p_{\varnothing} \arrow[l, shift left, "\omega"]
\end{tikzcd},$$

where the vertical arrows are the obvious forgetful functors from the category of augmented non-unitary Λ-operads (respectively, Λ-sequences) to the category of ordinary non-unitary operads (respectively, symmetric sequences) and we consider the standard forgetful functor from the category of ordinary non-unitary operads to the category of non-unitary symmetric sequences on the bottom row. We actually prove that our free object functor $\mathbb{\Theta} : \Lambda\,\mathcal{S}eq_{>0}\,/\,\mathsf{Com} \to \Lambda\,\mathcal{O}p_{\varnothing}\,/\,\mathsf{Com}$ defines a lifting of the free object functor of §1.2 on the bottom row of this diagram.

To be more precise, let M be any object in the category of augmented non-unitary Λ-sequences $\Lambda\,\mathcal{S}eq_{>0}\,/\,\mathsf{Com}$. Let $\mathbb{\Theta}(\mathsf{M})$ be the free operad associated to the symmetric sequence underlying our object M, where we forget about the restriction operators and the augmentation morphisms. We have the following proposition:

PROPOSITION 2.3.1.

(a) *The free operad* $\mathbb{\Theta}(\mathsf{M})$ *inherits the additional structure of an augmented* Λ-*operad which is also uniquely determined by requiring that the canonical embedding* $\iota : \mathsf{M} \to \mathbb{\Theta}(\mathsf{M})$ *defines a morphism of augmented non-unitary* Λ-*sequences.*

(b) *Let* $f : \mathsf{M} \to \mathsf{P}$ *be a morphism of augmented non-unitary* Λ*-sequences with values in an augmented non-unitary* Λ*-operad* P*. The operad morphism* $\phi_f : \mathbb{\Theta}(\mathsf{M}) \to \mathsf{P}$ *associated to* f *in the adjunction relation of Theorem* 1.2.1 *and Proposition* 1.2.2 *preserves the additional augmented* Λ*-operad structure attached to our objects and hence, defines a morphism in the category of augmented non-unitary* Λ*-operads.*

Explanations. We refer to the appendix part (Proposition A.3.12) for a detailed proof of this statement. We just outline the main ideas of our construction for the moment.

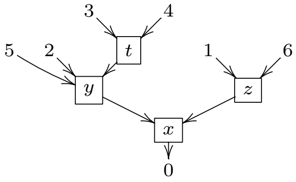
The requirement that the canonical embedding $\iota : \mathsf{M} \to \mathbb{\Theta}(\mathsf{M})$ defines a morphism of augmented non-unitary Λ-sequences is equivalent to the following assumptions:

- The restriction operators $u^* : \mathbb{\Theta}(\mathsf{M})(n) \to \mathbb{\Theta}(\mathsf{M})(m)$, for $u \in \mathtt{Mor}_\Lambda(\underline{\mathtt{m}}, \underline{\mathtt{n}})$ and $m, n > 0$, are determined on $\mathsf{M}(n) \subset \mathbb{\Theta}(\mathsf{M})(n)$ by the internal restriction operators $u^* : \mathsf{M}(n) \to \mathsf{M}(m)$ attached to our object M.
- The augmentations $\epsilon : \mathbb{\Theta}(\mathsf{M})(r) \to \mathbb{1}$, for $r > 0$, are similarly determined on $\mathsf{M}(r) \subset \mathbb{\Theta}(\mathsf{M})(r)$ by the augmentation morphisms $\epsilon : \mathsf{M}(r) \to \mathbb{1}$ given with M.

In §1.2, we explained that the free operad $\mathbb{\Theta}(\mathsf{M})$ intuitively consists of formal composites of elements of the generating symmetric sequence M (when we work in a concrete base symmetric monoidal category). Recall that we represent such formal composites by tensors arranged on trees. The idea is to use the associativity relations of Proposition 2.2.16 in order to extend the restriction operators and the augmentations of our generating symmetric sequence to these formal operadic composites.

To illustrate our constructions, we consider the same formal composite $p = (1\ 5) \cdot (((x \circ_1 y) \circ_4 z) \circ_3 t)$ as in our definition of the free operad in §1.2. This element p is represented by the following treewise tensor:

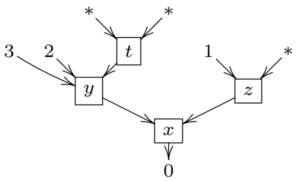
$p =$



(see §1.2). We assume that we work within a category of modules, so that the application of the augmentation $\epsilon : \mathsf{M}(n) \to \Bbbk$ to an element $\xi \in \mathsf{M}(n)$ returns a multiplicative scalar $\epsilon(\xi) \in \Bbbk$.

Let $u : \underline{3} \to \underline{6}$ be the map such that $u(1) = 1$, $u(2) = 2$, $u(3) = 5$. We use the conventions of §2.2.10 in order to represent the application of the restriction operator u^* on our operation p. We replace the indices attached to the inputs of our element in the above picture by their corresponding counterimage $u^{-1}(k)$ when $k \in \{u(1), u(2), u(3)\}$ and by the mark $*$ otherwise:

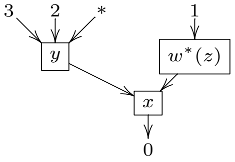
$u^*(p) =$



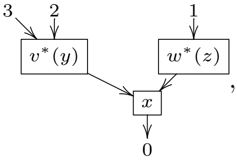
.

We then get the following reductions:

$u^*(p) \equiv \epsilon(t) \cdot$



$\equiv \epsilon(t) \cdot$



,

where $v^*(y)$ denotes the image of the element $y \in M(3)$ under the restriction operator associated to the increasing map $v : \underline{2} \to \underline{3}$ such that $v(1) = 1$, $v(2) = 2$, and $w^*(z)$ denotes the image of the element $z \in M(2)$ under the restriction operator associated to the increasing map $w : \underline{1} \to \underline{2}$ such that $w(1) = 1$.

We can also give an algebraic formulation, in terms of partial composition products, of this restriction process. We then consider a decomposition of our map $u : \underline{3} \to \underline{6}$ of the same shape $u = ((a \circ_1 b) \circ_4 c) \circ_3 d$ as our formal composite $p = (1\ 5) \cdot (((x \circ_1 y) \circ_4 z) \circ_3 t)$ in the free operad. We can easily determine this decomposition from the picture of our restriction operator in the treewise representation of the element $u^*(p)$. We actually have $a = id = id_2$ (the identity map of the ordered set $\underline{2} = \{1 < 2\}$) and $b = id = id_3$ (the identity map of the ordered set $\underline{3} = \{1 < 2 < 3\}$). We have $c = w : \underline{1} \to \underline{2}$, where w is the map such that $w(1) = 1$. We also get the empty map $o : \underline{0} \to \underline{2}$ for the remaining factor d of our decomposition $u = ((a \circ_1 b) \circ_4 c) \circ_3 d$. We have $u^* \cdot (1\ 5) = (1\ 3) \cdot u^*$, and in this formalism, we deduce from the equivariance relations of Proposition 2.2.16 that we have the formulas:

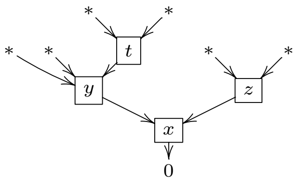
$$\begin{aligned} u^*(p) &= \epsilon(t) \cdot (1\ 3) \cdot \partial_3((x \circ_1 y) \circ_4 w^*(z)) \\ &= \epsilon(t) \cdot (1\ 3) \cdot ((x \circ_1 v^*(y)) \circ_3 w^*(z)), \end{aligned}$$

where we again consider the map $v : \underline{2} \to \underline{3}$ such that $v(1) = 1$, $v(2) = 2$. We exactly retrieve the composite represented in our treewise picture of the restriction operator $u^* : p \mapsto u^*(p)$.

In §1.2, we briefly explain that the treewise composites which span the free operad $\Theta(M)$ are formally defined as elements of treewise tensor products $M(\underline{T})$, where $\underline{T}$ runs over a category of trees $\mathcal{T}ree(r)$, so that the components of the free operad have an expansion of the form $\Theta(M)(r) = \operatorname{colim}_{\underline{T} \in \mathcal{T}ree(r)} M(\underline{T})$, for all $r \in \mathbb{N}$. The restriction operators $u^* : \Theta(M)(n) \to \Theta(M)(m)$ are defined termwise by morphisms of the form $u^* : M(\underline{T}) \to M(u^* \underline{T})$, for $\underline{T} \in \mathcal{T}ree(n)$, where $u^* \underline{T}$ determines the composition schemes associated to the image of our formal composites under the restriction operator $u^* : p \mapsto u^*(p)$. In §A.1.11, we will explain that these maps $u^* : \underline{T} \mapsto u^* \underline{T}$ also represent the restriction operators of a Λ-operad structure on the categories of trees $\mathcal{T}ree(r)$, $r \in \mathbb{N}$.

The augmentation morphisms $\epsilon : \Theta(M)(n) \to Com(n)$ can either be obtained by the same explicit construction, where we just assign a unitary symbol to all inputs of our treewise tensor products, or can be determined as the components of the unique operad morphism $\epsilon : \Theta(M) \to Com$ that extends the augmentation of M. In fact, to determine the image of a treewise tensor under the augmentation morphism of the free operad, we just have to apply the augmentation of our Λ-sequence $\epsilon : M(r) \to \mathbb{1}$ to each factor of our tensor product. This operation lands in the tensor product of the unit object $\mathbb{1}$ over our tree. We just use the canonical isomorphism $\mathbb{1} \otimes \cdots \otimes \mathbb{1} \xrightarrow{\simeq} \mathbb{1}$ in order to get a result in the unit object $\mathbb{1}$. In our example (where we also assume $\mathbb{1} = \mathbb{k}$), we get:

$\epsilon(p) =$



$\equiv \epsilon(x) \cdot \epsilon(y) \cdot \epsilon(z) \cdot \epsilon(t).$

We immediately obtain that the morphism $\phi_f : \Theta(\mathsf{M}) \to \mathsf{P}$ which extends our morphism of augmented non-unitary Λ-sequences $f : \mathsf{M} \to \mathsf{P}$ in assertion (b) preserves restriction operators and augmentations, because we use universal relations, valid in any given unitary extension by Proposition 2.2.16, to define our structures on the free operad $\Theta(\mathsf{M})$. We deduce the conclusion of assertion (b) from this observation. ☐

Then we check that:

Theorem 2.3.2.
(a) *The construction of Proposition* 2.3.1 *gives a functor*

$$\Theta(-) : \Lambda\,\mathcal{S}eq_{>0}\,/\,\mathsf{Com} \to \Lambda\,\mathcal{O}p_{\varnothing}\,/\,\mathsf{Com},$$

which defines a left adjoint of the obvious forgetful functor $\omega : \Lambda\,\mathcal{O}p_{\varnothing}\,/\,\mathsf{Com} \to \Lambda\,\mathcal{S}eq_{>0}\,/\,\mathsf{Com}$ *from the category of augmented non-unitary* Λ*-operads* $\Lambda\,\mathcal{O}p_{\varnothing}\,/\,\mathsf{Com}$ *towards the category of augmented non-unitary* Λ*-sequences* $\Lambda\,\mathcal{S}eq_{>0}\,/\,\mathsf{Com}$*, and which maps any augmented non-unitary* Λ*-sequence* $\mathsf{M} \in \Lambda\,\mathcal{S}eq_{>0}\,/\,\mathsf{Com}$ *to an associated free object in the category of augmented non-unitary* Λ*-operads* $\Theta(\mathsf{M}) \in \Lambda\,\mathcal{O}p_{\varnothing}\,/\,\mathsf{Com}$*.*

(b) *If we take the unitary extension* $\Theta(\mathsf{M})_+$ *of this free augmented connected* Λ*-operad* $\Theta(\mathsf{M})$ *which we associate to any augmented non-unitary* Λ*-sequence* $\mathsf{M} \in \Lambda\,\mathcal{S}eq_{>0}\,/\,\mathsf{Com}$ *in assertion (*a*), then we get a functor*

$$\Theta(-)_+ : \Lambda\,\mathcal{S}eq_{>0}\,/\,\mathsf{Com} \to \mathcal{O}p_*$$

which is left adjoint to our extended forgetful functor $\omega_+ : \mathcal{O}p_* \to \Lambda\,\mathcal{S}eq_{>0}\,/\,\mathsf{Com}$ *from the category of unitary operads* $\mathcal{O}p_*$ *towards the category of augmented non-unitary* Λ*-sequences* $\Lambda\,\mathcal{S}eq_{>0}\,/\,\mathsf{Com}$*.*

Explanations. We naturally gain the result of this theorem in the category of augmented non-unitary Λ-operads and assertion (b) is an immediate corollary of the result of assertion (a). We elaborate on the definition of the adjunction relation for the ordinary free operad functor in §1.2 and on the claims of Proposition 2.3.1 to get our statement.

To be explicit, recall that the canonical embedding of the free operad $\iota : \mathsf{M} \to \Theta(\mathsf{M})$ represents the unit morphism of our adjunction in Theorem 1.2.1, while the augmentation of this adjunction $\lambda : \Theta(\mathsf{P}) \to \mathsf{P}$ is identified with the morphism of the free operad $\lambda = \phi_{id}$ associated to the identity morphism of the object P, for each operad P. The assertions of Proposition 2.3.1 imply that the morphism $\iota : \mathsf{M} \to \Theta(\mathsf{M})$ lies in the category of augmented non-unitary Λ-sequences when M belongs to this category and $\lambda = \phi_{id}$ similarly defines a morphism of augmented Λ-operads when we assume that P is so. The relations of adjunction units and of adjunction augmentations remain obviously valid for this restriction of our morphisms to augmented non-unitary Λ-sequences and augmented non-unitary Λ-operads, which therefore define the unit and the augmentation of an adjunction relation for our upgraded free operad functor with values in the category of augmented non-unitary Λ-operads.

The morphism of augmented Λ-operads $\phi_f : \mathbb{\Theta}(M) \to P$ yielded by the result of Proposition 2.3.1(b) also represents image of the morphism of augmented non-unitary Λ-sequences $f : M \to P$ under the correspondence of our adjunction relation since we have $\phi_f \iota = f$ by definition of this morphism in the category of ordinary operads (see Proposition 1.2.2). □

The unit operad I admits an obvious unitary extension I_+ and we immediately see that this operad I_+ defines the initial object of the category of unitary operads. The restriction operators $u^* : I(n) \to I(m)$, which we determine from this unitary operad structure, reduce to the identity of the initial object when $m, n > 1$, to the initial morphism of the unit object $\varnothing \to \mathbb{1}$ when $m > 1$, $n = 1$, and to the identity of the unit object $\mathbb{1} \to \mathbb{1}$ when $m = n = 1$. We have a similar trivial representation of the augmentations $\epsilon : I(n) \to \mathbb{1}$ attached to the unit operad. We may also directly check that the unit operad I, equipped with this extra structure, defines the initial object of the category of augmented non-unitary Λ-operad.

Then we can establish the following result, which parallels the statement of Proposition 1.2.4 about the construction of limits and colimits in the category of ordinary operads:

PROPOSITION 2.3.3.
(a) *The forgetful functor* $\omega : \Lambda\,\mathcal{O}p_{\varnothing} / \mathit{Com} \to \Lambda\,\mathcal{S}eq_{>0} / \mathit{Com}$ *creates all small limits, the filtered colimits, and the coequalizers which are reflexive in the category of non-unitary* Λ*-sequences.*

(b) *The category of augmented non-unitary* Λ*-operads also admits coproducts and, as a byproduct, all small colimits.*

PROOF. Exercise: check that the arguments of §1.2.4 extend to the setting of augmented non-unitary Λ-operads. (We go back to the definition of coproducts in the proof of the next statement.) □

To complete the statement of this Proposition 2.3.3, we may note that limits in the category of augmented non-unitary Λ-sequences are created aritywise as limits of objects of the ambient category equipped with an augmentation over the unit object $\mathit{Com}(r) = \mathbb{1}$. We have a similar statement regarding colimits (we can forget about augmentations in this case). We elaborate on these observations to compare limits and colimits in the category of augmented non-unitary Λ-operads and in the category of ordinary non-unitary operads. We get the following result:

PROPOSITION 2.3.4. *The functor* $\tau : \Lambda\,\mathcal{O}p_{\varnothing} / \mathit{Com} \to \mathcal{O}p_{\varnothing} / \mathit{Com}$*, which forgets about the restriction operators in the structure of an augmented* Λ*-operad, preserves colimits and limits.*

We can also consider the prolongment of the forgetful functor of this proposition to the category of plain non-unitary operad $\mathcal{O}p_{\varnothing}$, where we also forget about the augmentation morphisms attached to our objects. This functor actually creates the colimits in the category of augmented non-unitary Λ-operads, but we do not use this more precise result.

PROOF. The case of limits follows from the observation that the forgetful functor $\omega : \Lambda\,\mathcal{S}eq_{>0}\,/\,\mathsf{Com} \to \mathcal{S}eq_{>0}\,/\,\mathsf{Com}$ creates limits (because limits in diagram categories are created termwise) and from the results of Proposition 1.2.4 and Proposition 2.3.3, where we check that limits of operads are created aritywise in the base category (see also Proposition 1.2.16).

We can check that the forgetful functor $\tau : \Lambda\,\mathcal{O}p_{\varnothing}\,/\,\mathsf{Com} \to \mathcal{O}p_{\varnothing}$ creates the coequalizers which are reflexive in the base category by the same argument. We examine the construction of coproducts with more details in order to check that this forgetful functor preserves coproducts as well and this verification will be enough to conclude that our functor preserves all colimits.

In Proposition 1.2.4, we establish that a coproduct of operads $Q = \bigvee_{\alpha} P_{\alpha}$ is given by a reflexive coequalizer of free operads

$$\begin{tikzcd}
\Theta(\coprod_{\alpha\in\mathcal{J}} \Theta(P_{\alpha})) \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & \Theta(\coprod_{\alpha\in\mathcal{J}} P_{\alpha}) \arrow[l, bend right=40, "s_0"'] \arrow[r, dotted] & Q
\end{tikzcd}$$

If we assume that each P_{α} is equipped with an augmented Λ-operad structure, then the free operad $\Theta(\coprod_{\alpha\in\mathcal{J}} P_{\alpha})$ inherits an augmented Λ-operad structure, and we have the same result for the free operad $\Theta(\coprod_{\alpha\in\mathcal{J}} \Theta(P_{\alpha}))$. The structure morphisms of our coequalizer belong to the category of augmented non-unitary Λ-operads too. From this result, we deduce that our coequalizer Q admits an augmented Λ-operad structure (since we observed that reflexive coequalizers are created termwise in all operadic categories) and this augmented non-unitary Λ-operad also represents the coproduct of the objects P_{α} in the category of augmented non-unitary Λ-operads. The conclusion follows. ☐

We can use Proposition 2.3.3 and the isomorphism of Theorem 2.2.18 to get the existence of limits and colimits in the category unitary operads $\mathcal{O}p_{*}$.

We can also rephrase the result of Proposition 2.3.4 in terms of the truncation functor $\tau : P_{+} \mapsto P$ from unitary operads to non-unitary operads. To be precise, recall that any unitary operad P_{+} comes equipped with an augmentation over the commutative operad Com_{+} (see Proposition 2.2.20) and the augmentation morphism of the augmented non-unitary Λ-operad equivalent to P_{+} reflects this structure attached to our unitary operad. For the study of limits, we have to deal with the functor $\tau : \mathcal{O}p_{*} \to \mathcal{O}p_{\varnothing}\,/\,\mathsf{Com}$ which retains this augmentation $\epsilon : P \to \mathsf{Com}$ from the structure of the unitary operad P_{+}.

We get the following statement:

PROPOSITION 2.3.5. *The obvious truncation functor* $\tau_{+} : \mathcal{O}p_{*} \to \mathcal{O}p_{\varnothing}$ *from the category of unitary operads to the category of non-unitary operads* $\mathcal{O}p_{\varnothing}$ *preserves colimits and the extension of this forgetful functor* $\tau_{+} : \mathcal{O}p_{*} \to \mathcal{O}p_{\varnothing}\,/\,\mathsf{Com}$, *where we keep the natural augmentation of unitary operads, preserves limits.* ☐

2.4. The definition of connected unitary operads

In the previous sections, we explained the definition of general unitary operads, but in applications we often deal with operads which satisfy an extra connectedness condition. To be more precise, recall that a non-unitary operad P is called connected when this operad satisfies $P(1) = \mathbb{1}$ in addition to the condition $P(0) = \varnothing$ of the definition of the category of non-unitary operads. In the unitary setting, we say that P is connected when have $\mathsf{P}(0) = \mathsf{P}(1) = \mathbb{1}$ (see §§1.1.19-1.1.20). The connected unitary operads P_+ are obviously identified with the unitary extensions of the non-unitary operads P which are connected as non-unitary operads. Hence, the category isomorphism of Theorem 2.2.18, between unitary operads and augmented non-unitary Λ-operads, has an obvious restriction to connected operads.

Recall that a symmetric sequence M is called connected when we have $\mathsf{M}(0) = \mathsf{M}(1) = \varnothing$ and we write $\mathcal{S}eq_{>1}$ for the category of symmetric sequences which satisfy this condition. The category of connected (non-unitary) operads is denoted by $\mathcal{O}p_{\varnothing 1}$ and the category of connected unitary operads by $\mathcal{O}p_{*1}$. We also write $\Lambda\,\mathcal{O}p_{\varnothing 1} / \mathsf{Com}$ for the category of augmented connected Λ-operads, which consists of the augmented non-unitary Λ-operads which are connected as non-unitary operads. If we restrict ourselves to connected operads, then the result of Theorem 2.2.18 precisely implies that we have an isomorphism

$$\tau_+ : \mathcal{O}p_{*1} \xrightarrow{\simeq} \Lambda\,\mathcal{O}p_{\varnothing 1} / \mathsf{Com}$$

between this subcategory of the category of augmented non-unitary Λ-operads $\Lambda\,\mathcal{O}p_{\varnothing 1} / \mathsf{Com} \subset \Lambda\,\mathcal{O}p_{\varnothing} / \mathsf{Com}$ and the category of connected unitary operads $\mathcal{O}p_{*1}$.

We aim to adapt the construction of §2.3 in order to get a reduced definition of free objects in the context connected unitary operads. We again define a free object functor with values in the category of augmented connected Λ-operads first. We use the isomorphism of Theorem 2.2.18 afterwards in order to get unitary operads from this construction. We still assume that the tensor product of our base category distributes over colimits since we need this property in our definition of free objects.

In Theorem 1.2.14, we observed that the standard free operad functor $\mathbb{O}(-)$ gives rise to a functor from the category of connected symmetric sequences to the category of connected (non-unitary) operads. In this theorem, we also proved that this free connected operad functor is left adjoint to the augmentation ideal functor $\bar{\omega} : \mathsf{P} \mapsto \bar{\mathsf{P}}$ which maps a connected operad P to the symmetric sequence such that $\bar{\mathsf{P}}(n) = \varnothing$, if $n = 0, 1$, and $\bar{\mathsf{P}}(n) = \mathsf{P}(n)$, otherwise. The definition of free augmented non-unitary Λ-operads in §2.3, and our subsequent definition of free unitary operads, is not well suited for this connected version of the adjunction relation, because when we form the augmentation ideal of an operad $\bar{\mathsf{P}}$, we have to discard the restriction operators $u^* : \mathsf{P}(n) \to \mathsf{P}(m)$ which have the arity 1 component of the operad as target. Hence, we introduce a new subcategory $\Lambda_{>1}$ of the category $\Lambda_{>0}$ where the object $\underline{1} = \{1\}$, which corresponds to this arity $r = 1$, is removed.

2.4.1. *The connected version of the category of finite ordinals and injections.* We basically consider the full subcategory of the category of ordinals and injections generated by the ordered sets $\underline{n} = \{1 < \cdots < n\}$ such that $n > 1$. We use the notation $\Lambda_{>1}$ for this truncated category, and as in §2.2.2, we adopt the notation $\Lambda^+_{>1}$, with a + superscript, to refer to the subcategory of $\Lambda_{>1}$ which has the same objects, but whose morphisms consist of the increasing injections only.

We also use the notation $\Sigma_{>1}$ for the isomorphism subcategory of $\Lambda_{>1}$. We still have $\mathtt{Mor}_{\Sigma}(\underline{n}, \underline{n}) = \Sigma_n$ and $\mathtt{Mor}_{\Sigma}(\underline{m}, \underline{n}) = \varnothing$ when $m \neq n$. We have obvious identities $\Lambda^+_{>1} = \Lambda^+ \cap \Lambda_{>1}$ and $\Sigma_{>1} = \Sigma \cap \Lambda_{>1}$, as well as a decomposition $\Lambda_{>1} = \Lambda^+_{>1}\Sigma_{>1}$ as in the case of the category $\Lambda_{>0}$. Recall that we use the notation of the whole category Λ (respectively, Λ^+, Σ) rather than the notation of a specific subcategory in the expression of morphism sets.

We call *connected* Λ*-sequence* the structure formed by a connected symmetric sequence M equipped with restriction operators $u^* : M(n) \to M(m)$, associated to all maps $u \in \mathtt{Mor}_{\Lambda^+}(\underline{\mathrm{m}}, \underline{\mathrm{n}})$, where $m, n > 1$, and which fulfill the associativity and equivariance relations of Proposition 2.2.4. We immediately see that giving this structure amounts to providing the sequence M with a contravariant action of the category $\Lambda_{>1}$ such that the restriction operators $u^* : M(n) \to M(m)$ give the action of the subcategory $\Lambda^+_{>1}$ on M and the internal symmetric structure of the collection $M(n)$ provides the action of the isomorphism subcategory $\Sigma_{>1} \subset \Lambda_{>1}$ (see Proposition 2.2.5). We use the notation $\Lambda\,\mathcal{S}eq_{>1}$ (and the same conventions as in §2.2.2) to denote this category of diagrams. We moreover adopt the notation $\Lambda\,\mathcal{S}eq_{>1}/\overline{Com}$ to refer to the category formed by the connected Λ-sequences $M \in \Lambda\,\mathcal{S}eq_{>1}$ which are equipped with an augmentation over the augmentation ideal of the commutative operad $\overline{Com}$. We can also identify this object $\overline{Com}$ with the constant $\Lambda^{op}_{>1}$-diagram such that $\overline{Com}(r) = \mathbb{1}$, for all $r > 1$.

2.4.2. *The augmentation ideal of connected* Λ*-operads.* Let P an augmented non-unitary Λ-operad which satisfies our connectedness condition $P(1) = \mathbb{1}$. The action of the category $\Lambda_{>0}$ on P obviously restricts to an action of the truncated category $\Lambda_{>1}$ on the augmentation ideal $\bar{P}$. We moreover have an augmentation morphism $\epsilon : \bar{P} \to \overline{Com}$, with the augmentation ideal of the commutative operad as codomain, which is given by the obvious restriction of the augmentation morphism $\epsilon : P \to Com$ attached to our operad P.

The mapping $\bar{\omega} : P \mapsto \bar{P}$ therefore gives a functor

$$\bar{\omega} : \Lambda\,\mathcal{O}p_{\varnothing 1}/\,Com \to \Lambda\,\mathcal{S}eq_{>1}/\overline{Com}$$

from the category of augmented connected Λ-operads $\Lambda\,\mathcal{O}p_{\varnothing 1}/\,Com$ towards the category of augmented connected Λ-sequences $\Lambda\,\mathcal{S}eq_{>1}/\overline{Com}$.

In §§1.2.13-1.2.14, we prove that the restriction of the standard free operad functor $\Theta : \mathcal{S}eq \to \mathcal{O}p$ to the category of connected symmetric sequences $\mathcal{S}eq_{>1}$ defines a left adjoint for the augmentation ideal functor $\bar{\omega} : P \mapsto \bar{P}$ on the category of ordinary connected operads $\mathcal{O}p_{\varnothing 1}$. We now check that this adjunction relation lifts to augmented connected Λ-operads.

Let M be any object in the category of augmented connected Λ-sequences $\Lambda\,\mathcal{S}eq_{>1}/\overline{Com}$. Let $\Theta(M)$ be the free operad associated to the symmetric sequence underlying M, where we still forget about the restriction and the augmentation morphisms attached to our object. Recall again that the assumption $M(0) = M(1) = \varnothing$ implies that this free operad $\Theta(M)$ is connected. In particular, we have $\Theta(M)(1) = \mathbb{1}$, with a unit morphism $\eta : \mathbb{1} \to \Theta(M)(1)$ given by the identity morphism of the unit object $\mathbb{1} \in \mathcal{M}$. Recall also that we write $\bar{\Theta}(M)$ for the augmentation ideal of this free connected operad $\Theta(M)$ and that the unit of our adjunction $\Theta : \mathcal{S}eq_{>1} \rightleftarrows \mathcal{O}p_{\varnothing 1} : \bar{\omega}$ in §§1.2.13-1.2.14 is the morphism $\iota : M \to \bar{\Theta}(M)$ defined by the obvious restriction, to the components of arity $r > 1$, of the canonical embedding $\iota : M \to \Theta(M)$ of the object M in the free operad $\Theta(M)$.

We get the following proposition:

Proposition 2.4.3.

(a) *The free connected operad* $\Theta(M)$ *inherits the additional structure of an augmented* Λ*-operad which is also uniquely determined by requiring that the canonical morphism* $\iota : M \to \bar{\Theta}(M)$ *defines a morphism of augmented connected* Λ*-sequences.*

(b) *Let* $f : \mathsf{M} \to \bar{\mathsf{P}}$ *be a morphism of augmented connected* Λ*-sequences with values in the augmentation ideal of an augmented connected* Λ*-operad* P*. The operad morphism* $\phi_f : \Theta(\mathsf{M}) \to \mathsf{P}$ *associated to* f *in the adjunction relation of Theorem* 1.2.1 *and Proposition* 1.2.2 *preserves the additional augmented* Λ*-operad structures of our objects and hence, defines a morphism in the category of augmented connected* Λ*-operads.*

Explanations. We again give short explanations for the proof of this statement and we refer to the appendix part (Proposition A.4.7) for details. We first adapt the construction of Proposition 2.3.1 in order to get an augmented Λ-operad structure on the free operad $\Theta(\mathsf{M})$. In this previous statement, we assumed that the restriction (respectively, augmentation) morphisms of the free operad are determined on $\mathsf{M} \subset \Theta(\mathsf{M})$ by the restriction (respectively, augmentation) morphisms attached to our collection M. In the present case, we use the following requirements, which are equivalent to the assumption that the morphism $\iota : \mathsf{M} \to \bar{\Theta}(\mathsf{M})$ defines a morphism of augmented connected Λ-sequences:

- The restriction operator $u^* : \Theta(\mathsf{M})(n) \to \Theta(\mathsf{M})(m)$, associated to any injective map $u : \{1 < \cdots < n\} \to \{1 < \cdots < m\}$, for $n, m > 0$, is determined on $\mathsf{M}(n) \subset \Theta(M)(n)$ by:
  - the augmentation
$$\mathsf{M}(n) \xrightarrow{\epsilon} \mathbb{1} = \Theta(\mathsf{M})(1)$$
    when $n > m = 1$;
  - the internal restriction operator attached to our object
$$\mathsf{M}(n) \xrightarrow{u^*} \mathsf{M}(m) \subset \Theta(\mathsf{M})(m)$$
    when $n \geq m \geq 2$.
- The augmentation morphism $\epsilon : \Theta(\mathsf{M})(n) \to \mathbb{1}$ is determined on $\mathsf{M}(n) \subset \Theta(M)(n)$ by the natural augmentation morphism
$$\epsilon : \mathsf{M}(n) \to \mathbb{1}$$
  given with our object M.

In each condition, we implicitly assume $n > 1$ when we consider the restriction of our structure morphisms on the free operad to the subobject $\mathsf{M}(n)$.

The definition of restriction operators requires some explanations. In the case $n, m > 1$, we retrieve the same requirements as in the construction of Proposition 2.3.1. In the case $m = 1$, the construction of this previous statement does not make sense, but the unit relation of §2.2.17 and the invariance of the augmentation with respect to restriction operators imply that we have a commutative diagram

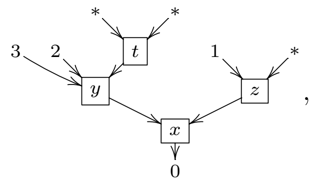
$$\begin{tikzcd}
\mathsf{M}(n) \arrow[r, "\iota"] \arrow[drr, bend right, "\epsilon"'] & \Theta(\mathsf{M})(n) \arrow[d, dotted, "u^*"] \arrow[dr, "\epsilon"] & \\
 & \Theta(\mathsf{M})(1) \arrow[r, "\epsilon", "="'] & \mathbb{1}
\end{tikzcd}$$

which forces the definition of our restriction operator $u^* : \Theta(\mathsf{M})(n) \to \Theta(\mathsf{M})(1)$ on the subobject $\mathsf{M}(n) \subset \Theta(\mathsf{M})(n)$.

We use the associativity relations of Proposition 2.2.16 (as in the proof of Proposition 2.3.1) to extend these restriction and augmentation morphisms to the treewise tensor products which span the free operad. The single difference occurs when the application of our restriction operator to treewise tensor products makes appear a factor of arity 1. In this case, our construction produces a unit factor $\mathbb{1}$, returned by the application of the augmentation $\epsilon : \mathsf{M}(r) \to \mathbb{1}$, which we reduce further to get the result of our operation.

We go back to the example given in the verification of Proposition 2.3.1 (and to the module setting) in order to illustrate this process. We then have the following picture:

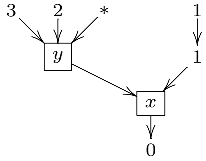
$$u^*(p) =$$



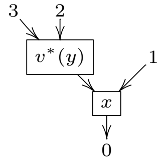
,

from which we obtain

$$u^*(p) \equiv \epsilon(t) \cdot \epsilon(z) \cdot$$



$$\equiv \epsilon(t) \cdot \epsilon(z) \cdot$$



,

where $v^*(y)$ denotes the image of the element $y \in \mathsf{M}(3)$ under the restriction operator associated to the increasing map $v : \underline{2} \to \underline{3}$ such that $v(1) = 1$, $v(2) = 2$ (as in the verification of Proposition 2.3.1). In this construction, we apply the augmentation $\epsilon : \mathsf{M}(2) \to \mathbb{1}$ to both $z \in \mathsf{M}(2)$ and $t \in \mathsf{M}(2)$, but we perform this operation in order to retrieve a multiple of the unitary factor $*$ in the case of $t(*,*) = \epsilon(t)*$ and a multiple of the operadic unit 1 in the case of $z(1,*) = \epsilon(z)1$. For this factor $z \in \mathsf{M}(2)$, we equivalently use the identity $w^*(z) = \epsilon(z) \cdot 1$ in the free operad $\mathbb{O}(\mathsf{M})$, where we consider the restriction operator $w^* : z \mapsto w^*(z)$ associated to the map $w : \underline{1} \to \underline{2}$ such that $w(1) = 1$ (compare with Proposition 2.3.1).

We can also identify the treewise tensor product in our first picture of the restriction operator $u^*(p)$ with a representation of the composite

$$u^*(p) = \epsilon(t) \cdot (1\ 3) \cdot ((x \circ_1 v^*(y)) \circ_3 w^*(z)),$$

already considered in the proof of Proposition 2.3.1, and where we perform the additional reduction

$$\begin{aligned} u^*(p) &= \epsilon(t) \cdot \epsilon(z) \cdot (1\ 3) \cdot ((x \circ_1 v^*(y)) \circ_3 1) \\ &= \epsilon(t) \cdot \epsilon(z) \cdot (1\ 3) \cdot ((x \circ_1 v^*(y)) \end{aligned}$$

by using the identity $w^*(z) = \epsilon(z) \cdot 1$ in the connected free operad $\mathbb{O}(\mathsf{M})$.

The definition of the augmentation morphism $\epsilon : \mathbb{O}(\mathsf{M}) \to \mathsf{Com}$ is the same as in the setting of general augmented Λ-operads of Proposition 2.3.1.

In §2.3, we briefly explain that the restriction operators of the free operad $\mathbb{O}(\mathsf{M})$ associated to a (possibly non-connected) Λ-sequence M are defined by termwise operations $u^* : \mathsf{M}(\underline{\mathrm{T}}) \to \mathsf{M}(u^* \underline{\mathrm{T}})$ on the treewise tensor products $\mathsf{M}(\underline{\mathrm{T}})$ which span our object. In the connected setting, we have a similar identity, but we now consider a restriction operator on reduced r-trees $u^* : \underline{\mathrm{T}} \mapsto u^* \underline{\mathrm{T}}$ modeling the composition schemes which arise from our reduced construction of restriction operators on connected free operads. In §A.4, we will explain that these maps $u^* : \underline{\mathrm{T}} \mapsto u^* \underline{\mathrm{T}}$ also represent the restriction operators of a connected unitary operad structure on our categories of reduced trees.

The remaining assertions of our lemma follow from the same arguments as in Proposition 2.3.1. □

Then we have the following statement:

THEOREM 2.4.4.
(a) *The construction of Proposition* 2.4.3 *gives a functor*

$$\mathbb{O}(-) : \Lambda\,\mathcal{S}eq_{>1}\,/\overline{\mathsf{Com}} \to \Lambda\,\mathcal{O}p_{\varnothing 1}\,/\,\mathsf{Com},$$

which defines a left adjoint of the upgraded augmentation ideal functor $\bar{\omega} : \mathsf{P} \mapsto \bar{\mathsf{P}}$ *from the category of augmented connected* Λ*-operads* $\Lambda\,\mathcal{O}p_{\varnothing 1}\,/\,\mathsf{Com}$ *towards the category of augmented connected* Λ*-sequences* $\Lambda\,\mathcal{S}eq_{>1}\,/\overline{\mathsf{Com}}$ *(see* §2.4.2*), and which maps any augmented connected* Λ*-sequence* $\mathsf{M} \in \Lambda\,\mathcal{S}eq_{>1}\,/\overline{\mathsf{Com}}$ *to an associated free object in the category of augmented connected* Λ*-operads* $\mathbb{O}(\mathsf{M}) \in \Lambda\,\mathcal{O}p_{\varnothing 1}\,/\,\mathsf{Com}$.

(b) *If we take the unitary extension* $\mathbb{O}(\mathsf{M})_+$ *of this free augmented connected* Λ*-operad* $\mathbb{O}(\mathsf{M})$ *which we associate to any augmented connected* Λ*-sequence* $\mathsf{M} \in \Lambda\,\mathcal{S}eq_{>1}\,/\overline{\mathsf{Com}}$ *in assertion (*a*), then we get a functor*

$$\mathbb{O}(-)_+ : \Lambda\,\mathcal{S}eq_{>0}\,/\,\mathsf{Com} \to \mathcal{O}p_*$$

which is left adjoint to the prolongment of the augmentation ideal functor $\bar{\omega} : \mathsf{P} \mapsto \bar{\mathsf{P}}$ *to the category of connected unitary operads* $\mathcal{O}p_{*1}$.

EXPLANATIONS. We use the same arguments as in the proof of Theorem 2.3.2 to deduce this statement from the results of Proposition 2.4.3. We still obtain that the canonical morphism $\iota : \mathsf{M} \to \bar{\mathbb{O}}(\mathsf{M})$, which we associate to the free operad on a connected symmetric sequence in §§1.2.13-1.2.14, determines the unit of our adjunction between augmented connected Λ-sequences and augmented connected Λ-operads.

We also get that the augmentation morphism of our adjunction $\lambda : \mathbb{O}(\bar{\mathsf{P}}) \to \mathsf{P}$ is yielded by the augmentation morphism of the adjunction of §§1.2.13-1.2.14 between connected symmetric sequences and ordinary connected operads. This adjunction is identified with the morphism $\lambda = \phi_\iota$ on the free operad $\mathbb{O}(\bar{\mathsf{P}})$ which we associate to the obvious embedding $\iota : \bar{\mathsf{P}} \to \mathsf{P}$. When we work with unitary operads, we just take the unitary extension of this morphism $\lambda : \mathbb{O}(\bar{\mathsf{P}})_+ \to \mathsf{P}_+$ to get the augmentation of our adjunction in assertion (b). □

We observed in §1.2.15 that the category embedding $\iota : \mathcal{O}p_{\varnothing 1} \hookrightarrow \mathcal{O}p_{\varnothing}$ has an obvious right adjoint $\tau : \mathcal{O}p_{\varnothing} \to \mathcal{O}p_{\varnothing 1}$, which maps a non-unitary operad $\mathsf{P} \in \mathcal{O}p_{\varnothing}$ to the connected operad such that $\tau\,\mathsf{P}(0) = \varnothing$, $\tau\,\mathsf{P}(1) = \mathbb{1}$, and $\tau\,\mathsf{P}(n) = \mathsf{P}(n)$ for $n > 1$. We have a version of this truncation functor construction in the setting of augmented non-unitary Λ-operads, and as a byproduct, we have an analogue of the adjunction of §1.2.15 in the setting of unitary operads:

PROPOSITION 2.4.5.
(a) *The obvious category embedding* $\iota : \Lambda\,\mathcal{O}p_{\varnothing 1}\,/\,\mathsf{Com} \hookrightarrow \Lambda\,\mathcal{O}p_{\varnothing}\,/\,\mathsf{Com}$ *has a right adjoint* $\tau : \Lambda\,\mathcal{O}p_{\varnothing}\,/\,\mathsf{Com} \to \Lambda\,\mathcal{O}p_{\varnothing 1}\,/\,\mathsf{Com}$.

(b) *If we take the unitary extension* $\tau(P_+) = \tau(P)_+$ *of the augmented non-unitary* Λ*-operad associated to any* $P \in \Lambda\,\mathcal{O}p_{\varnothing}\,/\,\mathit{Com}$ *in (*a*), then we get a functor* $\tau : \mathcal{O}p_* \to \mathcal{O}p_{*1}$ *on the category of unitary operads* $P_+ \in \mathcal{O}p_*$*, and this functor defines a right adjoint of the category embedding* $\iota : \mathcal{O}p_{*1} \hookrightarrow \mathcal{O}p_*$ *which naturally occurs in the setting of unitary operads.*

Proof. Let $P \in \Lambda\,\mathcal{O}p_{\varnothing}\,/\,\mathit{Com}$. We define the components of the operad $\tau\,P$ by the formula:

$$\tau\,P(r) = \lim_{u:\underline{\mathsf{1}}\to\underline{\mathsf{n}}} \mathbb{1}\times_{P(1)} P(r), \tag{1}$$

for any fixed arity $r > 0$, where the limit ranges over the category of maps $u \in \mathtt{Mor}_{\Lambda}(\underline{\mathsf{1}},\underline{\mathsf{n}})$ as objects together with the factorizations $v = su$ such that $s \in \Sigma_r$ as morphisms, and we consider the pullback of the restriction operators $u^* : P(r) \to P(1)$ along the unit morphism of our operad in the base category:

$$\begin{tikzcd}
\mathbb{1}\times_{P(1)} P(r) \arrow[r, dotted] \arrow[d, dotted] & P(r) \arrow[d, "u^*"] \\
\mathbb{1} \arrow[r, "\eta"] & P(1)
\end{tikzcd} . \tag{2}$$

We use that the permutation $s \in \Sigma_r$ in a factorization $v = su$ induces a morphism $\mathbb{1}\times_{P(1)} s : \mathbb{1}\times_{P(1)} P(r) \to \mathbb{1}\times_{P(1)} P(r)$ between the base change associated to the restriction operators $u^*, v^* : P(r) \to P(1)$ when we form our limit. These objects $\tau\,P(r)$ are endowed with a canonical augmentation $\epsilon : \tau\,P(r) \to \mathbb{1}$ by construction, and fit in commutative diagrams

$$\begin{tikzcd}
\tau\,P(r) \arrow[r, dotted] \arrow[d, dotted, "\epsilon"'] & P(r) \arrow[d, "u^*"] \\
\mathbb{1} \arrow[r, "\eta"] & P(1)
\end{tikzcd} , \tag{3}$$

for all $u \in \mathtt{Mor}_{\Lambda}(\underline{\mathsf{1}},\underline{\mathsf{n}})$. If we take the category of sets as base category, then we can identify this object $\tau\,P(r)$ with the subset of operations $p \in P(r)$ such that $u^*(p) = 1$, for all restriction operators $u^* : P(r) \to P(1)$, where $1 \in P(1)$ denotes the unit of our operad.

Each map $f \in \mathtt{Mor}_{\Lambda}(\underline{\mathsf{m}},\underline{\mathsf{n}})$ induces a restriction operator $f^* : \tau\,P(n) \to \tau\,P(m)$ which we determine on our limit termwise by using the commutative diagrams

$$\begin{tikzcd}
\tau\,P(n) \arrow[r, dotted] \arrow[dr, dotted] & P(n) \arrow[rr, "f^*"] \arrow[dr, dotted, "(fu)^*"'] & & P(m) \arrow[dl, "u^*"] \\
& \mathbb{1} \arrow[r, dotted] & P(1) &
\end{tikzcd} , \tag{4}$$

for all $u \in \mathtt{Mor}_{\Lambda}(\underline{\mathsf{1}},\underline{\mathsf{m}})$. The augmentations $\epsilon : \tau\,P(r) \to \mathbb{1}$ which we associate to our objects in (3) clearly satisfy the invariance relation $\epsilon f^* = \epsilon$ with respect to the action of our restriction operators $f^* : \tau\,P(n) \to \tau\,P(m)$. The collection $\tau\,P = \{\tau\,P(r), r > 0\}$ accordingly inherits the structure of an augmented Λ-sequence. Moreover, we clearly have $\tau\,P(1) = \mathbb{1}$ since our limit reduces to the single term $\mathbb{1}\times_{P(1)} P(1) = \mathbb{1}$ in this case.

We also use a termwise construction to define the products $\circ_i : \tau\, P(m) \otimes \tau\, P(n) \to \tau\, P(m+n-1)$ of the operadic composition structure on $\tau\, P$. Let $u \in \mathtt{Mor}_{\Lambda}(\underline{1}, \underline{\mathsf{m}+\mathsf{n}-1})$. We use that we have $u = f \circ_{f(1)} g$, for some maps $f \in \mathtt{Mor}_{\Lambda}(\underline{1}, \underline{\mathsf{m}})$ and $g \in \mathtt{Mor}_{\Lambda}(\underline{1}, \underline{\mathsf{n}})$, which we can explicitly determine by $f(1) = i$ and $g(1) = u(1) - i + 1$. We simply form the diagram

$$\begin{tikzcd}
\tau\, P(m) \otimes \tau\, P(n) \arrow[r, dotted] \arrow[d, dotted] & P(m) \otimes P(n) \arrow[r, "\circ_i"] \arrow[d, dotted, "f^* \otimes g^*"] & P(m+n-1) \arrow[d, "u^*"] \, , \\
\mathbb{1} \otimes \mathbb{1} \arrow[r, dotted] & P(1) \otimes P(1) \arrow[r, dotted, "\circ_1"] & P(1)
\end{tikzcd} \tag{5}$$

to define the value of our composition operation on the term associated to the map $u \in \mathtt{Mor}_{\Lambda}(\underline{1}, \underline{\mathsf{m}+\mathsf{n}-1})$ in (1). We easily check that these operations fulfill the equivariance, unit and associativity relations of augmented Λ-operads. Hence, our construction does give a functor $\tau : \Lambda\, \mathcal{O}p_{\varnothing} / \mathit{Com} \to \Lambda\, \mathcal{O}p_{\varnothing 1} / \mathit{Com}$ from the category of augmented non-unitary Λ-operads $\Lambda\, \mathcal{O}p_{\varnothing} / \mathit{Com}$ to the category of augmented connected operads $\Lambda\, \mathcal{O}p_{\varnothing 1} / \mathit{Com}$. We easily check that this functor is left adjoint to the category embedding $\iota : \Lambda\, \mathcal{O}p_{\varnothing 1} / \mathit{Com} \to \Lambda\, \mathcal{O}p_{\varnothing} / \mathit{Com}$ too. The extension of this adjunction relation to categories of unitary operads follows from an immediate application of the isomorphism result of Theorem 2.2.18. ☐

We can use the result of this proposition to extend the adjunction relation of the free augmented connected Λ-operad of Theorem 2.4.4 to morphisms $\phi_f : \Theta(M) \to P$ with values in any augmented non-unitary Λ-operad $P \in \Lambda\, \mathcal{O}p_{\varnothing} / \mathit{Com}$ (possibly not connected) and we have an obvious counterpart of this result in the context of unitary operads. We can also apply this observation to the construction of morphisms on connected unitary operads defined by a presentation by generators and relations. We tackle this subject soon (see §2.4.8).

We easily check that the construction of colimits in the category of all operads (see Proposition 1.2.4) and in the category of augmented non-unitary Λ-operads (see Proposition 2.3.3) works in the category of augmented connected Λ-operads as well. The result of Proposition 2.4.5 implies that the category embedding $\iota : \Lambda\, \mathcal{O}p_{\varnothing 1} / \mathit{Com} \to \Lambda\, \mathcal{O}p_{\varnothing} / \mathit{Com}$ preserves colimits.

We also have limits in the category of augmented connected Λ-operads which are created aritywise in the base category as in the case of general augmented non-unitary Λ-operads (see Proposition 2.3.3). We immediately deduce from this construction that the functor $\iota : \Lambda\, \mathcal{O}p_{\varnothing 1} / \mathit{Com} \to \Lambda\, \mathcal{O}p_{\varnothing} / \mathit{Com}$ preserves limits besides colimits. We have similar statements for the category embedding $\iota : \mathcal{O}p_{\varnothing 1} / \mathit{Com} \to \mathcal{O}p_{\varnothing} / \mathit{Com}$ (compare with Proposition 1.2.16). The obvious forgetful functor $\omega : \Lambda\, \mathcal{O}p_{\varnothing} / \mathit{Com} \to \mathcal{O}p_{\varnothing} / \mathit{Com}$ preserves limits and colimits too by Proposition 2.3.5.

We summarize our statements concerning colimits in the next proposition. We should note that in the colimit case, we can forget about augmentations in categories of ordinary operads, because this structure is not essential for the definition of our objects and does not affect the result of our constructions.

Proposition 2.4.6. *The functors of the diagram*

$$\begin{tikzcd}
\Lambda\,\mathcal{O}p_{\varnothing 1}\,/\,\mathsf{Com} \arrow[r, "\iota"] \arrow[d, "\tau"] & \Lambda\,\mathcal{O}p_{\varnothing}\,/\,\mathsf{Com} \arrow[d, "\tau"] \\
\mathcal{O}p_{\varnothing 1} \arrow[r, "\iota"] & \mathcal{O}p_{\varnothing}
\end{tikzcd}$$

preserve colimits.

We may actually see that all functors considered in this diagram create colimits, but we do not use this more precise statement.

Proof. We are left to check that the forgetful functor on the left-hand side $\tau : \Lambda\,\mathcal{O}p_{\varnothing 1}\,/\,\mathsf{Com} \to \mathcal{O}p_{\varnothing 1}\,/\,\mathsf{Com}$ preserves colimits, but this result is an immediate consequence of the other cases, of the commutativity of our diagram, and of the observation that the category embedding $\iota : \mathcal{O}p_{\varnothing 1}\,/\,\mathsf{Com} \hookrightarrow \mathcal{O}p_{\varnothing}\,/\,\mathsf{Com}$ actually creates colimits. ☐

The result of Proposition 2.4.6 has the following unitary counterpart:

Proposition 2.4.7. *The functors of the diagram*

$$\begin{tikzcd}
\mathcal{O}p_{*1} \arrow[r, "\iota"] \arrow[d, "\tau_+"] & \mathcal{O}p_{*} \arrow[d, "\tau_+"] \\
\mathcal{O}p_{\varnothing 1} \arrow[r, "\iota"] & \mathcal{O}p_{\varnothing}
\end{tikzcd}$$

preserve colimits. ☐

We can also formulate a similar result for limits. We just have to retain the augmentation over the commutative operad in this case in order to get the correct limit construction in the category of non-unitary operads. We more explicitly get that the extended truncation functor $\tau_+ : \mathcal{O}p_{*1} \to \mathcal{O}p_{\varnothing 1}\,/\,\mathsf{Com}$ preserves limits.

We now focus on the case where we take a category of modules as base category $\mathcal{M} = \mathcal{M}od$. We explain in §1.2.9 that, in this setting, operads can be defined by generators and relations as quotients $P = \mathbb{O}(M)/\langle z^{\alpha}, \alpha \in \mathcal{I}\rangle$, where we consider an ideal $\langle z^{\alpha}, \alpha \in \mathcal{I}\rangle$ in a free operad $\mathbb{O}(M)$. We elaborate on this construction and we use our definition of free objects in the category of augmented connected Λ-operads given in Theorem 2.4.4 in order to define a reduced construction of operads by generators and relations in the context of unitary operads. We explain our approach in the next paragraph.

2.4.8. *The definition of unitary operads by generators and relations.* We first assume that M is an augmented connected Λ-sequence (in $\mathbb{k}$-modules). We apply the construction of Proposition 2.4.3 to provide the free operad associated to M with the structure of an augmented Λ-operad.

Let $S = \langle z^{\alpha}, \alpha \in \mathcal{I}\rangle$ be an ideal generated by a collection of elements $z^{\alpha} \in S(n_{\alpha})$ in this free operad $\mathbb{O}(M)$. We assume that we have the vanishing relation

$$\epsilon(z^{\alpha}) = 0 \tag{1}$$

when we apply the augmentation $\epsilon : \mathbb{O}(M)(n_{\alpha}) \to \mathbb{k}$ to any such element $z^{\alpha} \in S(n_{\alpha})$ in our ideal S. We also assume that we have the relation

$$u^{*}(z^{\alpha}) \equiv 0 \mod S(m), \tag{2}$$

for all restriction operators $u^* : \Theta(\mathsf{M})(n) \to \Theta(\mathsf{M})(m)$, where $n = n_\alpha$. We easily check that the symmetric sequence $\mathsf{S} = \langle z^\alpha, \alpha \in \mathcal{J}\rangle$ forms an operadic ideal in the unitary extension of the free operad $\Theta(\mathsf{M})$ as soon as these conditions are satisfied (use the correspondence of §2.2.17 between augmentations, restriction operators and composition operations). Therefore, we can form a quotient of the free unitary operad $\Theta(\mathsf{M})_+$ by our ideal $\mathsf{S} = \langle z^\alpha, \alpha \in \mathcal{J}\rangle$ to get an object $\Theta(\mathsf{M})_+/\langle z^\alpha, \alpha \in \mathcal{J}\rangle$ in the category of unitary operads. We immediately check that this operad defines a unitary extension of the basic quotient operad $\Theta(\mathsf{M})/\langle z^\alpha, \alpha \in \mathcal{J}\rangle$ which we considered in the study of §1.2.9.

The morphisms of unitary operads $\bar{\phi}_{f+} : \Theta(\mathsf{M})_+/\langle z^\alpha, \alpha \in \mathcal{J}\rangle \to \mathsf{Q}_+$ are clearly in bijection with the morphisms of augmented non-unitary Λ-operads $\phi_f : \Theta(\mathsf{M}) \to \mathsf{Q}$ such that $\phi_f(z^\alpha) = 0$ for each generating element of the ideal $z^\alpha \in \mathsf{S}(n_\alpha)$.

In applications, we can reduce the verification of our vanishing condition (2) to the restriction operators $\partial_k : \Theta(\mathsf{M})(n) \to \Theta(\mathsf{M})(n-1)$, $k = 1, \dots, n$, associated to the partial composition products $\partial_k(p) = p \circ_k *$ since we observed in §2.2.1 that all restriction operators on a unitary operad occur as composites of these particular restriction operators.

2.4.9. *Examples of unitary operads constructed by generators and relations.* We explain the application of the construction of §2.4.8 to the basic examples of unitary operads considered in §1.2.10, namely the associative operad As_+ and the commutative operad Com_+. We also address the definition of a unitary version of the Poisson operad Pois_+. We consider the case of the associative operad first.

Recall that we have $\mathsf{As} = \Theta(\Bbbk\,\mu(x_1,x_2) \oplus \Bbbk\,\mu(x_1,x_2))/\langle \mu(\mu,1) - \mu(1,\mu)\rangle$, for a generating symmetric sequence such that $\mathsf{M}_{\mathsf{As}}(2) = \Bbbk\,\mu(x_1,x_2) \oplus \Bbbk\,\mu(x_1,x_2)$ and $\mathsf{M}_{\mathsf{As}}(r) = 0$ for $r \neq 2$. Since M_{As} vanishes in arity $r > 2$, we only have to specify an augmentation $\epsilon : \mathsf{M}_{\mathsf{As}}(2) \to \Bbbk$ in order to provide this symmetric sequence with the structure of an augmented Λ-sequence. We take $\epsilon(\mu) = 1$ to reflect the idempotence relations $\mu(e,e) = e$ for the unit element of an associative algebra. By applying the associativity of restriction operators with respect to operadic composition structures, we obtain $\partial_1(\mu(\mu,1) - \mu(1,\mu)) = \mu(\mu(*,1),1) - \mu(1(*),\mu) = \mu - 1(\mu) = 0$ and similarly $\partial_2(\mu(\mu,1) - \mu(1,\mu)) = \partial_3(\mu(\mu,1) - \mu(1,\mu)) = 0$. Hence, the assumptions of §2.4.8 are fulfilled, so that the operad As inherits restriction operators, and as a consequence, admits a unitary extension As_+ such that $\mathsf{As}_+(0) = \Bbbk$ and $\mathsf{As}_+(r) = \mathsf{As}(r) = \Bbbk[\Sigma_r]$ for $r > 0$. This operad As_+ is actually identified with the image of the permutation operad under the functor $\Bbbk[-] : \mathcal{S}et\,\mathcal{O}p \to \mathcal{M}od\,\mathcal{O}p$ (see §3.1.2).

The case of the commutative operad is similar. We take the same expression as in the associative case for the value of the augmentation $\epsilon : \mathsf{M}_{\mathsf{Com}}(2) \to \Bbbk$ on the generating operation $\mu \in \mathsf{M}_{\mathsf{Com}}(2)$. We see that the assumptions of §2.4.8 are also fulfilled for the commutative operad, which therefore admits a unitary extension Com_+ such that $\mathsf{Com}_+(0) = \Bbbk$ and $\mathsf{Com}_+(r) = \mathsf{Com}(r) = \Bbbk$ for $r > 0$. This operad Com_+ is actually identified with the image of the one-point set operad under our functor $\Bbbk[-] : \mathcal{S}et\,\mathcal{O}p \to \mathcal{M}od\,\mathcal{O}p$ (see the concluding discussion of §3.1).

The unitary extension process can also be applied to the Poisson operad Pois. Recall that this operad has a generating symmetric sequence such that $\mathsf{M}_{\mathsf{Pois}}(2) = \Bbbk\,\mu(x_1,x_2) \oplus \Bbbk\,\lambda(x_1,x_2)$, where $\mu = \mu(x_1,x_2)$ represents a (symmetric) commutative product and $\lambda = \lambda(x_1,x_2)$ represents an (anti-symmetric) Lie bracket. We take $\epsilon(\mu) = 1$ (as usual) and $\epsilon(\lambda) = 0$. We check again that the generating relations of the Poisson operad (see §1.2.12) are canceled by the restriction operators so that the operad *Pois* admits a unitary extension Pois_+ such that $\mathsf{Pois}_+(0) = \Bbbk$ and $\mathsf{Pois}_+(r) = \mathsf{Pois}(r)$ for $r > 0$. In fact, the definition $\epsilon(\lambda) = 0$ for the Lie bracket operation $\lambda \in \mathsf{Pois}(2)$ is forced by the cancelation condition for the Jacobi relation, or by the invariance of augmentation under the action of the transposition $(1\ 2) \in \Sigma_2$ on this operation $\lambda \in \mathsf{Pois}(2)$.

Note that we can also define a unitary extension for the Lie operad *Lie*. We then necessarily set $\epsilon(\lambda) = 0$ as explained in the case of the Poisson operad, so that we have $\lambda(*, x_1) = \lambda(x_1, *) = 0$ in the unitary operad Lie_+. For an algebra $\mathfrak{g}$ over this operad Lie_+, we simply get that the unitary operation $*$ determines a central element e which satisfies the relation $\lambda(e, -) = 0$ in our Lie algebra $\mathfrak{g}$.

2.5. The definition of operads shaped on finite sets

In the definition of §1.1, and in the definition of §2.1 similarly, we assumed that the terms of an operad $P(r)$ are indexed by non-negative integers $r \in \mathbb{N}$. Intuitively, we assume that the elements of an operad (in the context of a concrete category) represent operations whose inputs are indexed by elements of the standard finite ordered sets $\underline{\mathsf{r}} = \{1 < \cdots < r\}$. In the graphical representation of §1.1.5 and §2.1.5, this input ordering is used to determine the planar arrangement of the ingoing edges of a box associated to an operation. In §1.1.6 (and in §2.1.5 similarly), we observed that the operadic composition operations are left invariant when we perform a change of planar arrangement in our representation. This observation motivates us to give a new definition of the notion of an operad which reflects this invariance property of the operadic composition operations. For this aim, we use that any symmetric sequence, underlying the structure of an operad, extends to a functor on the category which has the finite sets as objects and the bijections of finite sets as morphisms.

This extension of the definition of an operad intuitively amounts to the introduction of operations $p = p(x_{i_1}, \dots, x_{i_r})$ with variables indexed by an arbitrary set $\underline{\mathsf{r}} = \{i_1, \dots, i_r\}$ rather than by the standard ordered set $\underline{\mathsf{r}} = \{1 < \cdots < r\}$ (see the introduction of the chapter). In this setting, we consider partial composites of the form $p \circ_{i_k} q = p(x_{i_1}, \dots, q(x_{j_1}, \dots, x_{j_s}), \dots, x_{i_r})$, for any composition index $i_k \in \{i_1, \dots, i_r\}$, and for operations $p = p(x_{i_1}, \dots, x_{i_r})$, $q = q(x_{j_1}, \dots, x_{j_s})$. The main purpose of this section is to reformulate the definition of an operad in terms of these numbering-free extensions of the partial composition products and to make explicit the expression of the equivariance, unit and associativity axioms of operads when we work in this setting. By the way, we will observe that the graphical representation of §§2.1.5-2.1.6 gives the picture of the partial composition of operations with inputs indexed by finite sets (we simply forget about the planar embedding of our figures).

To begin with, we explain the equivalence between the category of symmetric sequences and the category of functors on the category of finite sets.

2.5.1. *Symmetric collections.* We use the notation $\mathcal{B}ij$ for the category which has the finite sets as objects and the bijections of finite sets as morphisms. We adopt the convention to denote a finite set, regarded as an object of $\mathcal{B}ij$, by an underlined sans serif letter $\underline{\mathsf{r}}$. We use the italic letter $r = \mathrm{card}(\underline{\mathsf{r}})$ corresponding to the notation of our set $\underline{\mathsf{r}}$ to refer to the cardinal of this set. We can regard this cardinal either as a non-negative integer, or as an isomorphism class of objects in the category of finite sets.

We call *symmetric collection* a functor $\mathsf{M} : \mathcal{B}ij \to \mathcal{M}$ which maps any finite set $\underline{\mathrm{r}} \in \mathcal{B}ij$ to an object of the base category $\mathsf{M}(\underline{\mathrm{r}}) \in \mathcal{M}$ and any bijection $u : \underline{\mathrm{r}} \xrightarrow{\simeq} \underline{\mathrm{s}}$ to a morphism $u_* : \mathsf{M}(\underline{\mathrm{r}}) \to \mathsf{M}(\underline{\mathrm{s}})$.

We use the notation $\mathcal{C}oll$ for the category of symmetric collections, where a morphism of symmetric collections $f : \mathsf{M} \to \mathsf{N}$ obviously consists of a collection of morphisms in the base category $f : \mathsf{M}(\underline{\mathrm{r}}) \to \mathsf{N}(\underline{\mathrm{r}})$ which preserve the action of bijections on our objects.

The category of finite sets $\mathcal{B}ij$ has a small skeleton with the standard ordered sets $\underline{\mathrm{r}} = \{1 < \dots < r\}$, $r \in \mathbb{N}$, as objects and the permutations $w \in \Sigma_r$, viewed as bijections $w : \{1 < \dots < r\} \xrightarrow{\simeq} \{1 < \dots < r\}$, as morphisms. The following proposition is a consequence of this fact:

PROPOSITION 2.5.2. *The category of symmetric collections $\mathcal{C}oll$ is equivalent to the category of symmetric sequences $\mathcal{S}eq$ considered in* §1.2.

CONSTRUCTION AND PROOF. In one direction, to a symmetric collection $\mathsf{M} \in \mathcal{C}oll$, we associate the symmetric sequence such that $\mathsf{M}(r) = \mathsf{M}(\{1, \dots, r\})$. (We just identify a permutation $w \in \Sigma_r$ with a bijection $w : \{1, \dots, r\} \xrightarrow{\simeq} \{1, \dots, r\}$ in order to provide this object $\mathsf{M}(r)$ with an action of the symmetric group Σ_r, for each $r \in \mathbb{N}$.)

If the base category has small colimits, then we can use a general Kan extension process to obtain a functor in the converse direction, from symmetric sequences to symmetric collections, and to get a left adjoint of the above functor $i^* : \mathcal{C}oll \to \mathcal{S}eq$ therefore. Let $\mathsf{M} = \{\mathsf{M}(r), r \in \mathbb{N}\}$ be any given symmetric sequence. We use the relative tensor product notation

$$\mathsf{M}(\underline{\mathrm{r}}) = \mathtt{Mor}_{\mathcal{B}ij}(\{1, \dots, r\}, \underline{\mathrm{r}}) \otimes_{\Sigma_r} \mathsf{M}(r), \quad \text{where } r = \mathrm{card}(\underline{\mathrm{r}}),$$

to symbolize this Kan extension process.

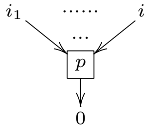
If we work in a category of modules $\mathcal{M} = \mathcal{M}od$, then we can define this relative tensor product as the module spanned by formal tensors $u \otimes \xi$, where $u \in \mathtt{Mor}_{\mathcal{B}ij}(\{1, \dots, r\}, \underline{\mathrm{r}})$ and $\xi \in \mathsf{M}(r)$, modulo the relations $us \otimes \xi \equiv u \otimes s\xi$ which identify the action of permutations $s \in \Sigma_r$ by right translation on bijections $u \in \mathtt{Mor}_{\mathcal{B}ij}(\{1, \dots, r\}, \underline{\mathrm{r}})$ with the internal Σ_r-structure of the object $\mathsf{M}(r)$. The verification that this mapping gives an inverse equivalence of the canonical functor $\mathcal{C}oll \to \mathcal{S}eq$ is straightforward. In a general context, we can replace the set of tensors $u \otimes \xi$ by a coproduct of copies of the object $\mathsf{M}(r)$ over the set $\mathtt{Mor}_{\mathcal{B}ij}(\{1, \dots, r\}, \underline{\mathrm{r}})$, and we perform an appropriate coend construction to implement the identities $us \otimes \xi \equiv u \otimes s\xi$ in our object. Recall, by the way, that we generally use the notation $S \otimes K$, where S is a set and K is any object in a category $\mathcal{C}$, to refer to a coproduct of copies of the object K over the set S.

In fact, the category equivalence of this proposition still holds when the base category is not equipped with colimits. To avoid the colimit construction, we just pick a bijection $u_{\underline{\mathrm{r}}} : \{1 < \dots < r\} \to \underline{\mathrm{r}}$ for each finite set $\underline{\mathrm{r}}$ of cardinal $r = \mathrm{card}(\underline{\mathrm{r}})$. We then set $\mathsf{M}(\underline{\mathrm{r}}) = \mathsf{M}(r)$, and we define the morphism $f_* : \mathsf{M}(\underline{\mathrm{r}}) \to \mathsf{M}(\underline{\mathrm{s}})$ associated to any $f \in \mathtt{Mor}_{\mathcal{B}ij}(\underline{\mathrm{r}}, \underline{\mathrm{s}})$ by the action of the composite bijection $u_{\underline{\mathrm{s}}}^{-1} \cdot f \cdot u_{\underline{\mathrm{r}}} : \{1 < \dots < r\} \to \{1 < \dots < r\}$, which defines a permutation of $\{1 < \dots < r\}$, on the object $\mathsf{M}(r)$. ☐

In the case of the permutation groups $\mathit{\Pi}(r) = \Sigma_r$, which represent the underlying symmetric sequence of the permutation operad (§§1.1.7-1.1.9), we have an identity $\mathit{\Pi}(\underline{\mathsf{r}}) = \mathtt{Mor}_{\mathcal{B}ij}(\{1,\dots,r\},\underline{\mathsf{r}})$.

In what follows, we often use that the bijections $u \in \mathtt{Mor}_{\mathcal{B}ij}(\{1,\dots,r\},\underline{\mathsf{r}})$ which occur in the construction of Proposition 2.5.2, are equivalent to orderings $i_1 < \dots < i_r$ of the set $\underline{\mathsf{r}}$. The kth term of such an ordering i_k gives the value of the corresponding bijection $u(k) = i_k$ on the kth element of the set $\{1 < \dots < r\}$. Thus, in the case of the permutation operad $\mathit{\Pi}$, we can identify the object $\mathit{\Pi}(\underline{\mathsf{r}})$ with the set of orderings of the set $\underline{\mathsf{r}}$.

2.5.3. *Back to the graphical representation of symmetric sequences.* The construction of the collection associated to a symmetric sequence in Proposition 2.5.2 can be materialized by using our graphical representation of operads in §1.1.5. Indeed, in the picture



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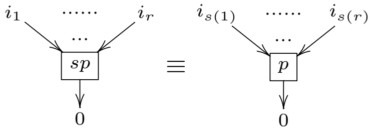
where p is any operad element, we can obviously assume that $(i_1,\dots,i_r)$ are the elements of an arbitrary finite set, and not necessarily a permutation of $(1,\dots,r)$. The relation



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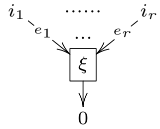
used to identify equivalent elements in the construction of §1.1.5, corresponds to the quotient process involved in the tensor product $\mathtt{Mor}_{\mathcal{B}ij}(\{1,\dots,r\},\underline{\mathsf{r}}) \otimes_{\Sigma_r} \mathsf{P}(r)$ (see the proof of Proposition 2.5.2). We only applied this formalism to operads in §1.1.5 but this interpretation of our construction obviously works for arbitrary symmetric sequences (and not only for the underlying symmetric sequence of an operad).

2.5.4. *The graphical representation of symmetric collections.* The box representation, recalled in the previous paragraph, has a natural extension in the context of symmetric collections. In the picture of §1.1.5, we use the planar arrangement of the ingoing edges of the box to materialize the bijection between the global inputs $\{i_1,\dots,i_r\}$ and the inputs of operad elements. To be more precise, we can use the ordering, determined by the orientation of the ambient plane of our figure, to get a canonical bijection between the set of ingoing edges $\{e_1,\dots,e_r\}$ and the finite ordered set $\underline{\mathsf{r}} = \{1 < \dots < r\}$ whose terms correspond to the inputs of our operation $p \in \mathsf{P}(r)$. In the setting of symmetric collections, we just forget about the planar embedding of our figure. We consider abstract trees, and we assume that ingoing edges form an abstract set $\{e_1,\dots,e_r\}$ (not necessarily equipped with a linear ordering).

In the context of a concrete base category, we represent any element of our collection $\xi \in \mathsf{M}(\underline{\mathsf{r}})$ by a box labeled by ξ together with one outgoing edge e_0, whose target is usually marked by the symbol 0, and a set of ingoing edges $\{e_{i_1},\dots,e_{i_r}\}$, whose source are usually labeled by the elements of the indexing set $\underline{\mathsf{r}} = \{i_1,\dots,i_r\}$, as in the following picture:



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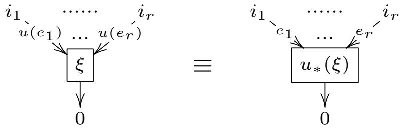
The edge set $\underline{e} = \{e_1, \dots, e_r\}$ may be distinguished from the external indexing set $\underline{r} = \{i_1, \dots, i_r\}$ and we assume that ξ belongs to $M(\underline{e})$. The edge indexing is equivalent to a bijection between $\underline{r} = \{i_1, \dots, i_r\}$ and the edge set $\underline{e} = \{e_1, \dots, e_r\}$. Thus, when we form our representation, we formally consider pairs (s, ξ), where $s : \underline{e} \xrightarrow{\simeq} \underline{r}$ and $\xi \in M(\underline{e})$. The isomorphism $s_* : M(\underline{e}) \xrightarrow{\simeq} M(\underline{r})$ induced by the bijection can be used to associate an element of $M(\underline{r})$ to $\xi \in M(\underline{e})$. To make this correspondence faithful, we simply set $(su, \xi) \equiv (s, u_*(\xi))$ whenever we apply a bijection $u : \underline{e} \xrightarrow{\simeq} \underline{f}$ to change the edge set. Graphically, this identity $(su, \xi) \equiv (s, u_*(\xi))$ reads:



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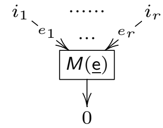
The natural action of bijections $v : \underline{r} \xrightarrow{\simeq} \underline{s}$ corresponds, in the graphical representation, to the obvious reindexing operation on the input labels of ingoing edges.

The information of the input set $\underline{e}$ is obviously redundant with respect to the information fixed by the indexing set $\underline{r}$. Performing our identification process amounts to reducing the unnecessary pieces of information. The consideration of redundant information in our picture is motivated by certain constructions where we are naturally lead to delay the reduction process (when we represent composite operations for instance).

We can also apply our representation to objects and not only to elements. To be explicit, we use the picture



to represent a copy of the object $M(\underline{e})$ which is naturally identified with $M(\underline{r})$ when we apply the isomorphism $s_* : M(\underline{e}) \xrightarrow{\simeq} M(\underline{r})$ induced by the bijection $s : \underline{e} \xrightarrow{\simeq} \underline{r}$ such that $s(e_k) = i_k$, for $k = 1, \dots, r$, where we still assume $\underline{r} = \{i_1, \dots, i_r\}$ and $\underline{e} = \{e_1, \dots, e_r\}$.

2.5.5. *The operadic composition of finite sets.* To an operad P, we now associate a collection $P(\underline{r})$ indexed by the finite sets $\underline{r}$. The unit morphism of the operad P is obviously equivalent to a morphism $\eta : \mathbb{1} \to P(\underline{1})$ with values in the component of this collection associated to any one-point set $\underline{1} = \{1\}$. This collection also inherits composition operations which extend the partial composition of our operads. We need to introduce composition operations on finite sets in order to define the scheme of our composition operations.

Let $\underline{r} = \{i_1, \dots, i_r\}$. Let $\underline{s} = \{j_1, \dots, j_s\}$. For any $i_k \in \underline{r}$, we set

$$\underline{r} \circ_{i_k} \underline{s} = \{i_1, \dots, \widehat{i_k}, \dots, i_r\} \amalg \{j_1, \dots, j_s\},$$

where we use the notation $\widehat{i_k}$ to mark the removal of the element i_k from $\underline{\mathsf{r}}$. To bijections $u : \underline{\mathsf{r}} \xrightarrow{\simeq} \underline{\mathsf{m}}$, $v : \underline{\mathsf{s}} \xrightarrow{\simeq} \underline{\mathsf{n}}$, and to any composition index $i_k \in \underline{\mathsf{r}}$, we can associate a bijection, denoted by $u \circ_{u(i_k)} v : \underline{\mathsf{r}} \circ_{i_k} \underline{\mathsf{s}} \to \underline{\mathsf{m}} \circ_{u(i_k)} \underline{\mathsf{n}}$, which is given by the map u on the set $\{i_1, \dots, \widehat{i_k}, \dots, i_r\}$ and by the map v on the set $\{j_1, \dots, j_s\}$.

We readily check that the partial composition of finite sets fulfills analogues of the unit and associativity relations of §2.1. To be explicit, we have the following assertions:

(a) For any finite set $\underline{\mathsf{s}}$, we have an identity $\underline{\mathsf{1}} \circ_1 \underline{\mathsf{s}} = \underline{\mathsf{s}}$. For a finite set $\underline{\mathsf{r}}$ equipped with a distinguished element $i_k \in \underline{\mathsf{r}}$, the set $\underline{\mathsf{r}} \circ_{i_k} \underline{\mathsf{1}}$ is not equal to $\underline{\mathsf{r}}$ in the strict sense, but we have an obvious bijection $\underline{\mathsf{r}} \circ_{i_k} \underline{\mathsf{1}} \simeq \underline{\mathsf{r}}$ naturally associated to the pair $(\underline{\mathsf{r}}, i_k)$.

(b) For a triple $(\underline{\mathsf{r}}, \underline{\mathsf{s}}, \underline{\mathsf{t}})$, we have associativity identities

$$(\underline{\mathsf{r}} \circ_{i_k} \underline{\mathsf{s}}) \circ_{j_l} \underline{\mathsf{t}} = \underline{\mathsf{r}} \circ_{i_k} (\underline{\mathsf{s}} \circ_{j_l} \underline{\mathsf{t}}),$$
$$(\underline{\mathsf{r}} \circ_{i_k} \underline{\mathsf{s}}) \circ_{i_l} \underline{\mathsf{t}} = (\underline{\mathsf{r}} \circ_{i_l} \underline{\mathsf{t}}) \circ_{i_k} \underline{\mathsf{s}},$$

where we assume $i_k \in \underline{\mathsf{r}}$ and $j_l \in \underline{\mathsf{s}}$ in the first case, while we take $\{i_k \neq i_l\} \subset \underline{\mathsf{r}}$ in the second case.

(c) The bijections of (a) are coherent with respect to the associativity relations of (b) in the sense that all diagrams which we may form by combining a unit bijection $\underline{\mathsf{r}} \circ_{i_k} \underline{\mathsf{1}} \simeq \underline{\mathsf{r}}$ with an associativity identity (in which we take a unit set $\underline{\mathsf{1}}$ for one of the objects $\underline{\mathsf{r}}$, $\underline{\mathsf{s}}$, or $\underline{\mathsf{t}}$) commute.

This is enough to formalize the definition of partial composition operations shaped on the composition of finite sets:

Proposition 2.5.6. *The definition of morphisms*

$$P(m) \otimes P(n) \xrightarrow{\circ_k} P(m+n-1) \tag{1}$$

for all $m, n \in \mathbb{N}$, $k = 1, \dots, m$, *and such that the equivariance relation of Proposition* 2.1.2 *holds, is equivalent to the definition of morphisms*

$$P(\underline{\mathsf{m}}) \otimes P(\underline{\mathsf{n}}) \xrightarrow{\circ_{i_k}} P(\underline{\mathsf{m}} \circ_{i_k} \underline{\mathsf{n}}), \tag{2}$$

for all finite sets $\underline{\mathsf{m}}$, $\underline{\mathsf{n}}$, *and for each* $i_k \in \underline{\mathsf{m}}$, *such that the diagram*

$$\begin{tikzcd} P(\underline{\mathsf{r}}) \otimes P(\underline{\mathsf{s}}) \arrow[r, "\circ_{i_k}"] \arrow[d, "u_* \otimes v_*"'] & P(\underline{\mathsf{r}} \circ_{i_k} \underline{\mathsf{s}}) \arrow[d, "(u \circ_{u(i_k)} v)_*"] \\ P(\underline{\mathsf{m}}) \otimes P(\underline{\mathsf{n}}) \arrow[r, "\circ_{u(i_k)}"] & P(\underline{\mathsf{m}} \circ_{u(i_k)} \underline{\mathsf{n}}) \end{tikzcd}$$

commutes, for any pair of bijections $u : \underline{\mathsf{r}} \xrightarrow{\simeq} \underline{\mathsf{m}}$, $v : \underline{\mathsf{s}} \xrightarrow{\simeq} \underline{\mathsf{n}}$.

The main purpose of this proposition is to make explicit the relationship between the plain partial composition operations (1) and the extended ones (2).

Proof. For standard ordered sets $\underline{\mathsf{m}} = \{1 < \dots < m\}$ and $\underline{\mathsf{n}} = \{1 < \dots < n\}$, we consider the bijection $\{1 < \dots < m\} \circ_{i_k} \{1 < \dots < n\} = \{1 < \dots < \widehat{i_k} < \dots, m\} \amalg \{1 < \dots < n\} \xrightarrow{\simeq} \{1 < \dots < m+n-1\}$ which maps the interval $\{1 < \dots < i_k - 1\} \subset \{1 < \dots < \widehat{i_k} < \dots, m\}$ to the same interval $\{1 < \dots < i_k - 1\}$ in $\{1 < \dots < m+n-1\}$, the summand $\{1 < \dots < n\}$ to $\{i_k < \dots < i_k + n - 1\}$ and the remaining elements $\{i_k + 1 < \dots < m\}$ of the summand $\{1 < \dots < \widehat{i_k} <$

$\dots < m\}$ to $\{i_k + n < \dots < m + n - 1\}$. The desired correspondence between our partial composition operations is deduced from the commutativity of the following diagram:

$$\begin{tikzcd}
\mathsf{P}(m) \otimes \mathsf{P}(n) \arrow[rr, dotted, "\circ_{i_k}"] \arrow[dd, "="'] & & \mathsf{P}(m+n-1) \arrow[d, "="] \\
& & \mathsf{P}(\{1,\dots,m+n-1\}) \\
\mathsf{P}(\{1,\dots,m\}) \otimes \mathsf{P}(\{1,\dots,n\}) \arrow[rr, "\circ_{i_k}"] & & \mathsf{P}(\{1,\dots,m\} \circ_{i_k} \{1,\dots,n\}) \arrow[u, "\simeq"']
\end{tikzcd} .$$

This diagram enables us to define the collection of partial composition operations (1) from partial composition operations of the form (2) by identification. In the other direction, given finite sets $\underline{\mathsf{m}}$ and $\underline{\mathsf{n}}$, we can pick bijections $\{1 < \dots < m\} \xrightarrow{\simeq} \underline{\mathsf{m}}$, $\{1 < \dots < n\} \xrightarrow{\simeq} \underline{\mathsf{n}}$, and use the equivariance diagram of the proposition to retrieve the partial composite (2) associated to the sets $\underline{\mathsf{m}}$ and $\underline{\mathsf{n}}$ from a partial composite of the form occurring in our diagram:

$$\mathsf{P}(\{1 < \dots < m\}) \otimes \mathsf{P}(\{1 < \dots < n\}) \xrightarrow{\circ_{i_k}} \mathsf{P}(\{1 < \dots < m\} \circ_{i_k} \{1 < \dots < n\}).$$

This process makes the correspondence between partial composition operations of the form (1) and (2) fully explicit. The equivalence between the equivariance relations for (1) and (2) follows from straightforward verifications. □

2.5.7. *The example of the permutation operad.* In the case of the permutation operad $\mathit{\Pi}(r) = \Sigma_r$, the elements of $\mathit{\Pi}(\underline{\mathsf{r}})$ are identified with orderings $u = \{i_1 < \dots < i_r\}$ of the unordered set $\underline{\mathsf{r}}$ (see our explanations after Proposition 2.5.2). We can use this representation to give a simple definition of the partial composition operations associated to this operad. We just describe the final result of this composition process again and we leave the verification of our claim as an exercise to the readers.

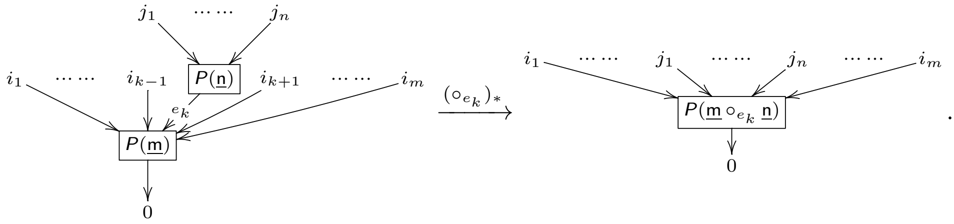
In short, the sequence corresponding to the composite $u \circ_{i_k} v$ can be obtained by replacing the occurrence of the composition index i_k in the sequence representing u by the sequence representing v. For elements $u = \{i_1 < \dots < i_m\} \in \mathit{\Pi}(\underline{\mathsf{m}})$ and $v = \{j_1 < \dots < j_n\} \in \mathit{\Pi}(\underline{\mathsf{n}})$, we explicitly obtain a result of the form

$$u \circ_{i_k} v = \{i_1 < \dots < i_{k-1} < j_1 < \dots < j_n < i_{k+1} < \dots < i_m\}.$$

In comparison with the process of §2.1.3, we simply have to forget the value shifts, which actually correspond to the bijection considered in the proof of Proposition 2.5.6.

2.5.8. *Operads with terms indexed by finite sets.* We can readily adapt the representation of §2.1.6 to get the picture of partial composition products in the context of operads with terms indexed by finite sets:

(∗)



On the source of this mapping, we consider indexing sets $\underline{m}$ and $\underline{n}$ that represent the sets of ingoing edges attached to the boxes of our treewise structure. The labeling of the tree inputs is equivalent to bijections $\{i_1,\dots,i_{k-1},i_{k+1},\dots,i_m\}\xrightarrow{\simeq}\underline{m}\setminus\{e_k\}$ and $\{j_1,\dots,j_l\}\xrightarrow{\simeq}\underline{n}$.

We readily see that these partial composites satisfy an obvious generalizations of the unit and associativity relations of §2.1. In the expression of the unit relation in the diagram of Figure 2.1, we consider composition operations of the form $\circ_{i_k}: P(\underline{r})\otimes P(\underline{1})\to P(\underline{r}\circ_{i_k}\underline{1})$, together with the composition operation $\circ_1: P(\underline{1})\otimes P(\underline{r})\to P(\underline{1}\circ_1\underline{r})$, and we use the bijection $\underline{r}\circ_{i_k}\underline{1}\simeq\underline{r}$ and the identity $\underline{1}\circ_1\underline{r}=\underline{r}$ to retrieve the object $P(\underline{r})$ in our diagram. In the diagrams of Figure 2.2-2.3, which express the associativity of our composition operations, we similarly have to replace the ordinal numbers $r+s-1$, $s+t-1$, $r+t-1$ and $r+s+t-2$ by appropriate composites of finite sets and we use the associativity of the composition of finite sets.

We consider the general structure formed by a symmetric collection $P=\{P(\underline{r}),r>0\}$ equipped with treewise composition operations of the form (∗) such that these extensions of the unit and associativity relations of operads hold. We deduce from Proposition 2.5.2 and Proposition 2.5.6 that this category of operads, shaped on the structure of a symmetric collection, is equivalent to the category of plain operads, where we index the terms of our objects by the non-negative integers. We use the definition of operads with terms indexed by finite sets as working definition of the structure of an operad in the appendix chapters §§A-B.

2.5.9. *Unitary operads with terms indexed by finite sets.* The constructions of §2.2, about the definition of unitary operads, can be extended to operads with terms indexed by finite sets. We then consider the category $\mathcal{I}nj_{>0}$ formed by the non-empty finite sets as objects and the injective maps as morphisms. We also consider the complete variant of this category $\mathcal{I}nj$. We moreover consider the subcategory $\mathcal{I}nj_{>1}\subset\mathcal{I}nj$ generated by the finite sets $\underline{r}$ of cardinal $r>1$. We now have an identity between the category of bijection $\mathcal{B}ij$ and the isomorphism subcategory of $\mathcal{I}nj$. Note however that we can hardly give a sense to the decomposition $\Lambda=\Lambda^+\Sigma$ of §2.2.3 in this category $\mathcal{I}nj$.

We use the name '*non-unitary* Λ-*collection*' (which parallels the phrase 'non-unitary Λ-sequence') for the category of contravariant functors from $\mathcal{I}nj_{>0}$ to any base category $\mathcal{M}$. We similarly use the name '*connected* Λ-*collection*' for the category of contravariant functors from $\mathcal{I}nj_{>1}$ to $\mathcal{M}$. We can easily extend the construction of Proposition 2.5.2 to define an equivalence of categories between the category of non-unitary (respectively, connected) Λ-collections and the category of non-unitary (respectively, connected) Λ-sequences. We have a similar result when we deal with objects equipped with an augmentation over the constant diagram underlying the commutative operad Com. We can also define an analogue of the notion of an augmented non-unitary Λ-operad in the context of operads with terms indexed by finite sets and we can extend the correspondence of the previous paragraphs to this category of augmented non-unitary Λ-operads.

We can also use this category of augmented non-unitary Λ-operads with terms indexed by finite sets in order to model the structure of unitary operads (as we do in §§2.2.5-2.2.18). We can easily make our correspondence between the structure of an augmented non-unitary Λ-operad and the structure of a unitary operad entirely explicit in the context of operads with terms indexed by finite sets too. Indeed, we readily see that the non-unitary operad P underlying a unitary operad P_+ inherits restriction operators $f^* : \mathsf{P}(\underline{\mathsf{s}}) \to \mathsf{P}(\underline{\mathsf{r}})$ which we associate to the injections $f : \underline{\mathsf{r}} \to \underline{\mathsf{s}}$ and which extend the restriction operators of §2.2.1. Each component of our operad $\mathsf{P}(\underline{\mathsf{r}})$ inherits an augmentation $\epsilon : \mathsf{P}(\underline{\mathsf{r}}) \to \mathbb{1}$ too. The collection P therefore forms a contravariant diagram over the category $\mathcal{I}nj_{>0}$ and comes equipped with an augmentation with values in the underlying collection of the commutative operad Com.

We extend the partial composition operations of §2.5.7 to injections in order to write down an analogue of the equivariance relations of Proposition 2.2.16. We proceed as follows. Let $\underline{\mathsf{r}} = \{i_1, \dots, i_r\}$ and $\underline{\mathsf{s}} = \{j_1, \dots, j_s\}$ be finite sets. Let $i_k \in \underline{\mathsf{r}}$. To any pair of injective maps $f : \underline{\mathsf{r}} \to \underline{\mathsf{m}}$ and $g : \underline{\mathsf{s}} \to \underline{\mathsf{n}}$, we associate the map $f \circ_{f(i_k)} g : \underline{\mathsf{r}} \circ_{i_k} \underline{\mathsf{s}} \to \underline{\mathsf{m}} \circ_{f(i_k)} \underline{\mathsf{n}}$ such that:

$$(f \circ_{f(i_k)} g)(x) = \begin{cases} f(x), & \text{when } x \in \{i_1, \dots, \widehat{i_k}, \dots, i_r\}, \\ g(x), & \text{when } x \in \{j_1, \dots, j_s\}. \end{cases}$$

In the case where $s = 0$ and g is an empty map $o : \underline{\mathsf{0}} \to \underline{\mathsf{n}}$, we get the following expression for our partial composition operations:

$$(f \circ_{f(i_k)} o)(x) = f(x), \quad \text{for } x \in \{i_1, \dots, \widehat{i_k}, \dots, i_r\}.$$

We readily see that we retrieve the partial composition operations of injective maps of §§2.2.12-2.2.13 when we apply the correspondence of Proposition 2.5.6 to these partial composition operations. We just replace the partial composites of ordinal injections in the expression of the equivariance relations of Proposition 2.2.16 by these extended composition operations in order to get the expression of the equivariance of the partial composition products with respect to the injective maps between finite sets.

We use these relations as axioms (together with the unit and associativity relations of the previous paragraph §2.5.8) in order to extend the definition of our notion of an augmented non-unitary Λ-operad to the context of operads with terms indexed by finite sets. The equivalence between this category of augmented non-unitary Λ-operads and the category of unitary operads is immediate from our definition.

CHAPTER 3

Symmetric Monoidal Categories and Operads

In the introductory chapter §1, we have worked in the setting of a base category $\mathcal{M}$ equipped with a tensor product $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ which distributes over colimits. This assumption is required for the application of categorical constructions (like colimits, free objects) to operads (§§1.2-1.3) and is also implicitly used as soon as we deal with endomorphisms operads (see §1.1). Nonetheless, we also observed that the definition of an operad in §1.1.1 makes sense in any symmetric monoidal category without assuming that the tensor product satisfies any other requirement than the fundamental unit, associativity and symmetry axioms of §0.8(a-c). In §2, we observed that the definition of operads in terms of partial composition operations makes sense in this general setting too. The isomorphism between the category of (connected) augmented non-unitary Λ-operads and the category of (connected) unitary operads (see Theorem 2.2.18) is defined in any symmetric monoidal category as well.

In this third chapter, we study the application of general symmetric monoidal category constructions to operads (regardless of any colimit requirement). In §3.1, we study the image of operads under functors between symmetric monoidal categories. In §3.2, we study the category of operads in counitary cocommutative coalgebras (called Hopf operads in what follows) as an application of the definition of the notion of an operad in a general symmetric monoidal category. In an appendix section §3.3, we review the definition of various notions of structure preserving functors associated to symmetric monoidal categories.

Throughout this chapter, we deal with a generalization of the notion of a commutative algebra and of the notion of a cocommutative coalgebra which we formalize by using the notion of a symmetric monoidal category. We devote a preliminary section to a survey of this subject. We tackle our main topics afterwards.

Most definitions of this chapter are not original. Our first purpose is to give a comprehensive and detailed survey of concepts and constructions scattered over the literature. In particular, the definition of the notion of an operad in the axiomatic setting of symmetric monoidal categories was apparently first considered in a report of G. Kelly, now published in [103] (in the case where the tensor product distributes over colimits). The notion of a Hopf operad was introduced by E. Getzler and J. Jones in [77]. These authors notably observed that the homology of an operad in topological spaces inherits a Hopf operad structure (we go back to this statement in the next chapter). By the way, we also check in this chapter that the classical constructions on Hopf operads extend to Λ-operads, the notion introduced in the previous chapter for the study of unitary operads.

3.0. Commutative algebras and cocommutative coalgebras in symmetric monoidal categories

The purpose of this preliminary section is to make explicit the definition of the notion of a unitary commutative algebra and of the dual notion of a counitary cocommutative coalgebra in the context of symmetric monoidal categories. We address the case of commutative algebras first.

3.0.1. *The category of unitary commutative algebras in a symmetric monoidal category.* Let $\mathcal{M}$ be any symmetric monoidal category. We define a *unitary commutative algebra in* $\mathcal{M}$ as a structure formed by an object $A \in \mathcal{M}$ together with morphisms $\eta : \mathbb{1} \to A$ and $\mu : A \otimes A \to A$ which make the following diagrams commute:

$$\begin{tikzcd}
A \otimes \mathbb{1} \arrow[r, "id \otimes \eta"] \arrow[dr, "\simeq"'] & A \otimes A \arrow[d, "\mu"] & \mathbb{1} \otimes A \arrow[l, "\eta \otimes id"'] \arrow[dl, "\simeq"] \\
 & A &
\end{tikzcd} , \quad
\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "id \otimes \mu"] \arrow[d, "\mu \otimes id"'] & A \otimes A \arrow[d, "\mu"] \\
A \otimes A \arrow[r, "\mu"] & A
\end{tikzcd} ,$$

and
$$\begin{tikzcd}
A \otimes A \arrow[rr, "(1\ 2)^*"] \arrow[dr, "\mu"'] & & A \otimes A \arrow[dl, "\mu"] \\
 & A &
\end{tikzcd} .$$

The morphism η (respectively, μ) represents the unit (respectively, the product) which we associate to our object A. The above diagrams express the unit, associativity and commutativity relations that govern the structure of a unitary commutative algebra.

In the basic case where $\mathcal{M}$ is the category of sets $\mathcal{M} = \mathcal{S}et$ (respectively, the category of modules $\mathcal{M} = \mathcal{M}od$ over a ground ring $\Bbbk$), we obviously retrieve the classical notion of a commutative monoid with unit (respectively, of a commutative $\Bbbk$-algebra with unit).

In general, we refer to a unitary commutative algebra by the notation of the underlying object of the base category $A \in \mathcal{M}$, and we abusively assume that the unit morphism η and the product μ are part of the internal structure attached to this object A. We adopt the letter η (respectively, μ) as a generic notation for all unit (respectively, product) morphisms associated to a unitary commutative algebra. If necessary, then we just use a subscript $\eta = \eta_A$ (respectively, $\mu = \mu_A$) in order to specify the algebra $A \in \mathcal{M}$ associated to this unit (respectively, product) morphism.

The unitary commutative algebras in $\mathcal{M}$ form a category, which we denote by $\mathcal{M}\,\mathcal{C}om_+$, or just by $\mathcal{C}om_+ = \mathcal{M}\,\mathcal{C}om_+$ when the monoidal category $\mathcal{M}$ is fixed by the context. We obviously define a morphism of unitary commutative algebras as a morphism of the base category $f : A \to B$ which makes the following diagrams commute:

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "="] \arrow[d, "\eta_A"'] & \mathbb{1} \arrow[d, "\eta_B"] \\
A \arrow[r, "f"] & B
\end{tikzcd} , \quad
\begin{tikzcd}
A \otimes A \arrow[r, "f \otimes f"] \arrow[d, "\mu_A"'] & B \otimes B \arrow[d, "\mu_B"] \\
A \arrow[r, "f"] & B
\end{tikzcd} .$$

Recall that we use the lower script + to mark the consideration of unitary structures (as in §1.1.16). The category of non-unitary commutative algebras, which we denote by $\mathcal{M}\,\mathcal{C}om$ (or just by $\mathcal{C}om = \mathcal{M}\,\mathcal{C}om$), is obviously defined by forgetting about the unit morphisms in our definitions.

Note that the unit object of the underlying category $\mathbb{1}$ inherits a natural commutative algebra structure, and represents the initial object of the category of unitary commutative algebras $\mathcal{M}\,\mathcal{C}om_{+}$. One can prove that the obvious forgetful functor $\omega : \mathcal{M}\,\mathcal{C}om_{+} \to \mathcal{M}$ creates limits in unitary commutative algebras (whenever limits exist in $\mathcal{M}$). But the forgetful functor $\omega : \mathcal{M}\,\mathcal{C}om_{+} \to \mathcal{M}$ does not preserve colimits in general. (To give a simple example, we have already observed that the unit object $\mathbb{1}$, which generally differs from the initial object of $\mathcal{M}$, is the initial object of $\mathcal{M}\,\mathcal{C}om_{+}$.)

In the case where the tensor product of $\mathcal{M}$ distributes over colimits (see §0.9), one can prove that colimits of any shape exist in the category of unitary commutative algebras. This statement is a particular case of the general result of Proposition 1.3.6, where we prove the existence of colimits in any category of algebras over an operad (we check in the next paragraphs that unitary commutative algebras are equivalent to algebras over the commutative operad).

This general construction implies that the filtered colimits of the category unitary commutative algebras are created in the base category (when the tensor product of $\mathcal{M}$ distributes over colimits), and we have the same result for the coequalizers of parallel pairs of unitary commutative algebra morphisms which are reflexive in the base category. But we can simplify the general construction of Proposition 1.3.6 when we need to define coproducts in the category of unitary commutative algebras. Indeed, we will see that a tensor product of unitary commutative algebras inherits a natural unitary commutative algebra structure and represents the coproduct of our objects in the category of unitary commutative algebras (see §3.0.3).

3.0.2. *The equivalence with the category of algebras over the commutative operad.* In the introductory chapter §1.1, we generally assume that the tensor product of our base category $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ distributes over colimits. Nevertheless, we already observed that the definition of an operad in §1.1.1 makes sense as soon as the unit, associativity and symmetry axioms of symmetric monoidal categories are satisfied. This is also the case of the definition of an algebra over an operad in §1.1.13 though the statement of Proposition 1.1.15, which gives an interpretation of operad actions in terms of endomorphism operads, does not make sense when the tensor product is not compatible with colimits (since endomorphism operads are not defined in this case).

In §2.1.11, we check that the definition of the commutative operad extends to arbitrary symmetric monoidal categories. We then set $\mathsf{Com}_{+}(r) = \mathbb{1}$ for any $r \in \mathbb{N}$, where we consider the unit object of our symmetric monoidal category $\mathbb{1} \in \mathcal{M}$ and we take a trivial action of the symmetric group in each arity. We define the operadic unit $\eta : \mathbb{1} \to \mathsf{Com}_{+}(1)$ by the identity morphism of the unit object $id : \mathbb{1} \xrightarrow{=} \mathbb{1}$ and the partial composition products $\circ_k : \mathsf{Com}_{+}(m) \otimes \mathsf{Com}_{+}(n) \to \mathsf{Com}_{+}(m+n-1)$ by the unit isomorphisms of our symmetric monoidal structure $\mathbb{1} \otimes \mathbb{1} \simeq \mathbb{1}$. We can also identify the full composition products $\mu : \mathsf{Com}_{+}(r) \otimes \mathsf{Com}_{+}(n_1) \otimes \cdots \otimes \mathsf{Com}_{+}(n_r) \to \mathsf{Com}_{+}(n_1 + \cdots + n_r)$ with the canonical isomorphisms $\mathbb{1} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \simeq \mathbb{1}$ which we deduce from the unit relations of the tensor product in our symmetric monoidal category. We just forget about the arity zero term $\mathsf{Com}_+(0) = \mathbb{1}$ when we consider the non-unitary version of the commutative operad Com.

We easily see that the category of unitary commutative algebras $\mathcal{C}om_+ = \mathcal{M}\,\mathcal{C}om_+$, such as defined in the previous paragraph, is isomorphic to the category of algebras associated to (this generalization of) the commutative operad Com_+, and we have a similar statement in the case of the category of non-unitary commutative algebras $\mathcal{C}om = \mathcal{M}\,\mathcal{C}om$. The proof of this observation follows from a formal extension, in the context of a general symmetric monoidal category, of the arguments of Proposition 1.1.17-1.1.18.

3.0.3. *The symmetric monoidal structure of the category of unitary commutative algebras.* The category of unitary commutative algebras in a symmetric monoidal category $\mathcal{M}\,\mathcal{C}om_+$ actually inherits a symmetric monoidal structure from the base category $\mathcal{M}$.

First, we readily see that a tensor product of unitary commutative algebras $A \otimes B$ inherits a natural unitary commutative algebra structure, with the composite morphism

$$\mathbb{1} \xrightarrow{\simeq} \mathbb{1} \otimes \mathbb{1} \xrightarrow{\eta_A \otimes \eta_B} A \otimes B$$

as unit, and the morphism

$$A \otimes B \otimes A \otimes B \xrightarrow{(2\ 3)^*} A \otimes A \otimes B \otimes B \xrightarrow{\mu \otimes \mu} A \otimes B$$

as product.

For the unit object $\mathbb{1}$, which represents the initial object of the category of commutative algebras $\mathcal{M}\,\mathcal{C}om_+$, the isomorphisms $A \otimes \mathbb{1} \xrightarrow{\simeq} A \xleftarrow{\simeq} \mathbb{1} \otimes A$, formed in the underlying monoidal category $\mathcal{M}$, are isomorphisms of unitary commutative algebras. Hence, the unit relations of the tensor product hold within the category $\mathcal{M}\,\mathcal{C}om_+$. The associativity and symmetry relations of the tensor product remain valid in the category of unitary commutative algebras as well. Thus, we have a whole symmetric monoidal structure on $\mathcal{M}\,\mathcal{C}om_+$, as claimed at the beginning of this paragraph.

We can easily check that the tensor product $A \otimes B$ represents the coproduct of A and B in $\mathcal{C}om_+$ (and therefore coproducts exist in $\mathcal{C}om_+$ without any assumption on the tensor product of the base category). The universal morphisms $A \xrightarrow{i} A \otimes B \xleftarrow{j} B$ are given by the tensor products $i = id_A \otimes \eta_B$ and $j = \eta_A \otimes id_B$, where we consider the unit morphisms of our algebras $\eta_A : \mathbb{1} \to A$ and $\eta_B : \mathbb{1} \to B$.

3.0.4. *The category of counitary cocommutative coalgebras in a symmetric monoidal category.* The structure of a counitary cocommutative coalgebra in a symmetric monoidal category is defined by duality from the definition of a unitary commutative algebra.

In brief, a counitary cocommutative coalgebra in $\mathcal{M}$ consists of an object $C \in \mathcal{M}$ equipped with morphisms $\epsilon : C \to \mathbb{1}$ and $\Delta : C \to C \otimes C$ such that the following diagrams commute:

$$\begin{tikzcd} & C \arrow[dl, "\simeq"'] \arrow[d, "\Delta"] \arrow[dr, "\simeq"] & \\ C \otimes \mathbb{1} & C \otimes C \arrow[l, "id \otimes \epsilon"] \arrow[r, "\epsilon \otimes id"'] & \mathbb{1} \otimes C \end{tikzcd} , \qquad \begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\Delta \otimes id"] \\ C \otimes C \arrow[r, "id \otimes \Delta"'] & C \otimes C \otimes C \end{tikzcd} ,$$

and

$$\begin{tikzcd} & C \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\ C \otimes C \arrow[rr, "(1\ 2)^*"'] & & C \otimes C \end{tikzcd} .$$

The morphism ϵ (respectively, Δ) is called the counit or augmentation (respectively, the coproduct or diagonal) of the cocommutative algebra C. The above diagrams express the counit, coassociativity and cocommutativity relations that govern the structure of a counitary cocommutative coalgebra.

We refer to a counitary cocommutative coalgebra by the notation of its underlying object $C \in \mathcal{M}$ (as in the algebra case). We use the letter ϵ (respectively, Δ) as a generic notation for all counit (respectively, coproduct) morphisms attached to a counitary cocommutative coalgebra structure. If necessary, then we just use a subscript $\epsilon = \epsilon_C$ (respectively, $\Delta = \Delta_C$) in order to specify the coalgebra $C \in \mathcal{M}$ associated to this counit (respectively, coproduct) morphism.

The counitary cocommutative coalgebras in $\mathcal{M}$ form a category, which we denote by $\mathcal{M}\,\mathcal{C}om_+^c$, or just by $\mathcal{C}om_+^c = \mathcal{M}\,\mathcal{C}om_+^c$, where we use a superscript c to mark the consideration of coalgebra structures. We obviously define a morphism of counitary cocommutative coalgebras as a morphism of the base category $f : C \to D$ which makes the following diagrams commute:

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\epsilon_C"'] & D \arrow[d, "\epsilon_D"] \\ \mathbb{1} \arrow[r, "="] & \mathbb{1} \end{tikzcd} , \qquad \begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\Delta_C"'] & D \arrow[d, "\Delta_D"] \\ C \otimes C \arrow[r, "f\otimes f"] & D \otimes D \end{tikzcd} .$$

The usual notion of counitary cocommutative coalgebra corresponds to the case where $\mathcal{M} = \mathcal{M}od$ is a category of modules over a ground ring $\Bbbk$. In the case where $\mathcal{M}$ is the category of sets $\mathcal{M} = \mathcal{S}et$ (and more generally when the symmetric monoidal structure operation of our category is defined by the cartesian product), any object $X \in \mathcal{S}et$ inherits a counit $\epsilon : X \to *$, because the unit object is the final object of our category $*$ (the one-point set in the case $\mathcal{M} = \mathcal{S}et$), as well as a coproduct $\Delta : X \to X \times X$ (the diagonal) and this operation trivially fulfills our counit, coassociativity and cocommutativity relations. Hence, any set $X \in \mathcal{S}et$ inherits a tautological counitary cocommutative coalgebra structure in $\mathcal{S}et$. The definition of the coproduct on X is actually forced by the counit relation and we therefore have an identity of categories $\mathcal{S}et\,\mathcal{C}om_+^c = \mathcal{S}et$.

The tensor unit $\mathbb{1}$ of a symmetric monoidal category $\mathcal{M}$ generally inherits a coalgebra structure (invert the orientation of arrows in the definition of the algebra structure of $\mathbb{1}$ in §3.0.1) and represents the terminal object of the category of counitary cocommutative coalgebras. We can also dualize the definition of the tensor product of algebras in §3.0.3 to obtain that a tensor product of counitary cocommutative coalgebras $C \otimes D$ inherits a counitary cocommutative coalgebra structure, with the composite morphism

$$C \otimes D \xrightarrow{\epsilon_C \otimes \epsilon_D} \mathbb{1} \otimes \mathbb{1} \xrightarrow{\simeq} \mathbb{1}$$

as counit, and the morphism

$$C \otimes D \xrightarrow{\Delta_C \otimes \Delta_D} C \otimes C \otimes D \otimes D \xrightarrow{(2\ 3)^*} C \otimes D \otimes C \otimes D$$

as coproduct. We provide the category of counitary cocommutative coalgebras with the symmetric monoidal structure determined by this tensor product operation.

This tensor product $C \otimes D$ also represents the cartesian product of C and D in the category of counitary cocommutative coalgebras. The universal morphisms $C \xleftarrow{p} C \otimes D \xrightarrow{q} D$ are given by the tensor products $p = id \otimes \epsilon_D$ and $q = \epsilon_C \otimes id$, where we consider the counit morphisms of our coalgebras $\epsilon_C : C \to \mathbb{1}$ and $\epsilon_D : D \to \mathbb{1}$. (Thus, we can fully dualize the observations of §3.0.3 about the categorical interpretation of the tensor product of unitary commutative algebras.) We can also easily check that the forgetful functor $\omega : \mathcal{M}\,\mathcal{C}om^c_+ \to \mathcal{M}$ creates colimits whenever colimits exist in $\mathcal{M}$ (just like the dual forgetful functor on the category of commutative algebras creates limits).

3.0.5. *The image of algebras and coalgebras under functors between underlying symmetric monoidal categories.* To complete this account, we study the image of algebras and coalgebras under functors between symmetric monoidal categories.

First, we consider the case where we have a lax symmetric monoidal functor $F : \mathcal{M} \to \mathcal{N}$ between symmetric monoidal categories $\mathcal{M}$ and $\mathcal{N}$. In this situation, the object $F(A) \in \mathcal{N}$, where A is a unitary commutative algebra in $\mathcal{M}$, forms a unitary commutative algebra in $\mathcal{N}$.

Recall that a functor is *lax symmetric monoidal* when we have a unit morphism $\eta : \mathbb{1} \to F(\mathbb{1})$ and a natural transformation $\theta : F(X) \otimes F(Y) \to F(X \otimes Y)$ which satisfy natural coherence constraints with respect to the unit, associativity and symmetry isomorphisms of our symmetric monoidal categories (see §3.3.1). In what follows, we generally assume that we have the relation $F(\mathbb{1}) = \mathbb{1}$ in the category $\mathcal{N}$ and that our unit morphism $\eta : \mathbb{1} \to F(\mathbb{1})$ is given by the identity morphism of the unit object $\mathbb{1} \in \mathcal{N}$. We then say that F is a *unit-preserving functor* and that $\theta : F(X) \otimes F(Y) \to F(X \otimes Y)$ defines a symmetric monoidal transformation on F. We do not use this stronger notion for the moment.

If we assume that A is a unitary commutative algebra in $\mathcal{M}$, then we just form the composites

$$\mathbb{1} \to F(\mathbb{1}) \xrightarrow{F(\eta)} F(A) \quad \text{and} \quad F(A) \otimes F(A) \xrightarrow{\theta} F(A \otimes A) \xrightarrow{F(\mu)} F(A)$$

in order to define a unit morphism and a product on the object $F(A) \in \mathcal{N}$. We easily check that these operations satisfy the unit, associativity, and commutativity axioms of §3.0.1 as soon as the unit morphism $\eta : \mathbb{1} \to F(\mathbb{1})$ and the natural transformation $\theta : F(X) \otimes F(Y) \to F(X \otimes Y)$ fulfill the coherence constraints of §3.3.1. We therefore get that the object $F(A) \in \mathcal{N}$ forms a unitary commutative algebra in the category $\mathcal{N}$.

This construction is obviously functorial and the mapping $F : A \mapsto F(A)$ therefore induces a functor from the category of unitary commutative algebras in $\mathcal{M}$ towards the category of unitary commutative algebras in $\mathcal{N}$. Furthermore, we easily check that the symmetric monoidal transformation $\theta : F(A) \otimes F(B) \to F(A \otimes B)$, inherited from F, defines a morphism in the category of unitary commutative algebras when we assume $A, B \in \mathcal{M}\,\mathcal{C}om_+$ and we consider the unitary commutative algebra in $\mathcal{N}$ associated to the tensor product $A \otimes B \in \mathcal{M}$. The unit morphism $\eta : \mathbb{1} \to F(\mathbb{1})$ associated to our functor F tautologically defines a morphism of unitary commutative algebras in $\mathcal{N}$ as well. Thus, the functor $F : \mathcal{M}\,\mathcal{C}om_+ \to \mathcal{N}\,\mathcal{C}om_+$, induced by $F : \mathcal{M} \to \mathcal{N}$, is also lax symmetric monoidal with respect to the symmetric monoidal structures which our categories of unitary commutative algebras inherit from the base category (see §3.0.3). The functor $F : \mathcal{M}\,\mathcal{C}om_+ \to \mathcal{N}\,\mathcal{C}om_+$ is obviously unit-preserving too as soon as F is so.

These observations can be dualized in the context of coalgebras. We then assume that $F : \mathcal{M} \to \mathcal{N}$ is a lax symmetric comonoidal functor, with a coaugmentation $\epsilon : F(\mathbb{1}) \to \mathbb{1}$ and a natural transformation $\theta : F(X \otimes Y) \to F(X) \otimes F(Y)$ which again fulfill natural coherence constraints with respect to the unit, associativity and symmetry isomorphisms of our symmetric monoidal categories (see §3.3.1). In this situation, the object $F(C) \in \mathcal{N}$, where C is a counitary cocommutative coalgebra in $\mathcal{M}$, inherits the structure of a counitary cocommutative coalgebra in $\mathcal{N}$, and this mapping $F : C \mapsto F(C)$ gives a functor from the category of counitary cocommutative coalgebras in $\mathcal{M}$ towards the category of counitary cocommutative coalgebras in $\mathcal{N}$. This functor $F : \mathcal{M}\,\mathcal{C}om_+^c \to \mathcal{N}\,\mathcal{C}om_+^c$, induced by $F : \mathcal{M} \to \mathcal{N}$, is also lax symmetric comonoidal with respect to the symmetric monoidal structures which our categories of counitary cocommutative coalgebras inherit from the base category. The functor $F : \mathcal{M}\,\mathcal{C}om_+^c \to \mathcal{N}\,\mathcal{C}om_+^c$ is obviously unit-preserving too as soon as F is so.

If our functor $F : \mathcal{M} \to \mathcal{N}$ is strongly symmetric monoidal in the sense that the morphisms which we use to compare our symmetric structures are isomorphisms $\eta : \mathbb{1} \xrightarrow{\simeq} F(\mathbb{1})$ and $\theta : F(X) \otimes F(Y) \xrightarrow{\simeq} F(X \otimes Y)$ (see §3.3.1), then we have a functor induced by F both on the category of unitary commutative algebras $F : \mathcal{M}\,\mathcal{C}om_+ \to \mathcal{N}\,\mathcal{C}om_+$ and on the category of counitary cocommutative coalgebras $F : \mathcal{M}\,\mathcal{C}om_+^c \to \mathcal{N}\,\mathcal{C}om_+^c$. These functors are both symmetric monoidal (in the strong sense) too. If we have a pair of functors $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ that form a symmetric monoidal adjunction in the sense of §3.3.3, then we have an induced symmetric monoidal adjunction at the level of our categories of algebras $F : \mathcal{M}\,\mathcal{C}om_+ \rightleftarrows \mathcal{N}\,\mathcal{C}om_+ : G$ and at the level of categories of coalgebras $F : \mathcal{M}\,\mathcal{C}om_+^c \rightleftarrows \mathcal{N}\,\mathcal{C}om_+^c : G$ as well. Indeed, we readily see that the unit $\eta : X \to G(F(X))$ and the augmentation $\epsilon : G(F(A)) \to A$ of such an adjunction define morphisms of unitary commutative algebras (respectively, counitary cocommutative coalgebras) when X (respectively, A) is equipped with such a structure. Therefore, these morphisms define the unit and the augmentation morphism of an adjunction at the algebra (respectively, coalgebra) level.

3.0.6. *The basic example of the free module functor.* To give a simple example of symmetric monoidal functor construction, we consider the functor $\Bbbk[-] : \mathcal{S}et \to \mathcal{M}od$ which maps any object of the category of sets $X \in \mathcal{S}et$ to the associated free $\Bbbk$-module, which we denote by $\Bbbk[X]$, for any fixed ground ring $\Bbbk$. We generally write $[x]$ for the generating element of this $\Bbbk$-module $\Bbbk[X]$ associated to any $x \in X$.

This functor $\Bbbk[-] : \mathcal{S}et \to \mathcal{M}od$ is symmetric monoidal (see §3.3.2), and hence, induces a symmetric monoidal functor both from the category of unitary commutative monoids (the category of unitary commutative algebras in sets) to the category of unitary commutative algebras in $\Bbbk$-modules and from the category of counitary cocommutative coalgebras in sets (which reduces to the category of sets by an observation of §3.0.4) to the category of counitary cocommutative coalgebras.

The counit and coproduct which define the counitary cocommutative coalgebra structure of a free $\mathbb{k}$-module $\mathbb{k}[X]$ can be defined by the explicit formula

$$\epsilon[x] = 1 \quad \text{and} \quad \Delta[x] = [x] \otimes [x],$$

for each element $x \in X$. In what follows, we generally say that an element $c \in C$ in a counitary cocommutative coalgebra in $\mathbb{k}$-modules C is *group-like* when it satisfies the same relations $\epsilon(c) = 1$ and $\Delta(c) = c \otimes c$ with respect to the counit and the coproduct of our coalgebra C as such an element $c = [x] \in \mathbb{k}[X]$ in the counitary cocommutative coalgebra associated to a set $C = \mathbb{k}[X]$. We use the notation $\mathbb{G}(C)$ for the set formed by the group-like elements in any counitary cocommutative coalgebra C. We can easily check that the mapping $\mathbb{G} : C \mapsto \mathbb{G}(C)$ defines a right-adjoint of our functor $\mathbb{k}[-] : \mathcal{S}et \to \mathcal{C}om_+^c$ from sets to counitary cocommutative coalgebras $\mathcal{C}om_+^c = \mathcal{M}od\,\mathcal{C}om_+^c$. The unit of this adjunction is the obvious embedding $\iota : X \to \mathbb{k}[X]$ and the augmentation is identified with the obvious morphism of $\mathbb{k}$-modules $\rho : \mathbb{k}[\mathbb{G}(C)] \to C$ induced by the tautological set inclusion $\mathbb{G}(C) \subset C$ on the basis of $\mathbb{k}[\mathbb{G}(C)]$.

We deduce from the general observations of §3.0.5 that the functor $\mathbb{k}[-] : \mathcal{S}et \to \mathcal{C}om_+^c$ is (strongly) symmetric monoidal since our initial functor from sets to $\mathbb{k}$-modules $\mathbb{k}[-] : \mathcal{S}et \to \mathcal{M}od$ is so. We immediately see that the group-like element functor $\mathbb{G} : \mathcal{C}om_+^c \to \mathcal{S}et$ is symmetric monoidal too, because this functor, as a right-adjoint, preserves final objects and cartesian products which we respectively identify with the unit and the tensor product of our symmetric monoidal category of coalgebras (see §3.0.4). We easily check that the unit morphism and the augmentation morphism of the adjunction $\mathbb{k}[-] : \mathcal{S}et \rightleftarrows \mathcal{C}om_+^c : \mathbb{G}$ are also symmetric monoidal transformations, so that our adjoint functors define a symmetric monoidal adjunction in the sense of §3.3.3.

We also have $\mathbb{G}(C) = \mathtt{Mor}_{\mathcal{C}om_+^c}(\mathbb{k}, C)$, where we consider the natural counitary coalgebra structure associated to the unit object $\mathbb{1} = \mathbb{k}$ (see §3.0.4). We can use this identity $\mathbb{G}(C) = \mathtt{Mor}_{\mathcal{C}om_+^c}(\mathbb{k}, C)$ to retrieve the claim that the group-like element functor $\mathbb{G} : \mathcal{C}om_+^c \to \mathcal{S}et$ preserves final objects and cartesian products.

3.1. Operads in general symmetric monoidal categories

In this section, we study the dependence of the definition of an operad from the underlying symmetric monoidal category. We mainly prove that operads are preserved by (lax) symmetric monoidal functors and that any symmetric monoidal adjunction between symmetric monoidal categories gives rise to an adjunction at the level of operad categories. We also explain a construction of functors on operads when we have an adjunction relation where only the right adjoint functor is (lax) symmetric monoidal.

Recall that a functor $F : \mathcal{M} \to \mathcal{N}$ between symmetric monoidal categories $\mathcal{M}$ and $\mathcal{N}$ is lax symmetric monoidal when we have a unit morphism $\eta : \mathbb{1} \to F(\mathbb{1})$ and a natural transformation $\theta : F(X) \otimes F(Y) \to F(X \otimes Y)$ which are compatible with the unit, associativity and symmetry isomorphisms of our symmetric monoidal categories (see §3.3.1). Most examples of lax symmetric monoidal functors which we consider in this book satisfy $F(\mathbb{1}) = \mathbb{1}$ (we then say that F is *unit-preserving*) and our unit morphism $\eta : \mathbb{1} \to F(\mathbb{1})$ is given by the identity morphism of the unit object $\mathbb{1} \in \mathcal{N}$. In this situation, we also say that the natural transformation

$\theta : F(X) \otimes F(Y) \to F(X \otimes Y)$ which we associate to our lax monoidal functor defines a *symmetric monoidal transformation* on F. We have the following result:

Proposition 3.1.1.
(a) *If $F : \mathcal{M} \to \mathcal{N}$ is a lax symmetric monoidal functor, then the collection of objects $F(\mathsf{P}(r)) \in \mathcal{N}$, $r \in \mathbb{N}$, defined by applying F aritywise to the underlying collection of an operad P in $\mathcal{M}$, forms an operad $F(\mathsf{P})$ in $\mathcal{N}$, so that F induces a functor from the category of operads in $\mathcal{M}$ to the category of operads in $\mathcal{N}$:*

$$F : \mathcal{M}\,\mathcal{O}p \to \mathcal{N}\,\mathcal{O}p\,.$$

(b) *If we moreover assume that $F : \mathcal{M} \to \mathcal{N}$ is unit-preserving $F(\mathbb{1}) = \mathbb{1}$, then this functor on operads preserves unitary extensions in the sense that we have the identity $F(\mathsf{P}_+) = F(\mathsf{P})_+$, for any unitary operad $\mathsf{P}_+ \in \mathcal{M}\,\mathcal{O}p_*$ (see* §1.1.20*), so that the mapping $F : \mathsf{P}_+ \mapsto F(\mathsf{P}_+)$ defines a functor from the category of unitary operads in $\mathcal{M}$ to the category of unitary operads $\mathcal{N}$:*

$$F : \mathcal{M}\,\mathcal{O}p_* \to \mathcal{N}\,\mathcal{O}p_*\,.$$

Explanations. The definition of the operad structure on the collection of objects $F(\mathsf{P}(r)) \in \mathcal{N}$, $r \in \mathbb{N}$, is immediate:

- each object $F(\mathsf{P}(r)) \in \mathcal{N}$ trivially inherits an action of the symmetric group Σ_r by functoriality;
- the collection $F(\mathsf{P})(r) = F(\mathsf{P}(r))$ also inherits a unit morphism
$$\mathbb{1} \to F(\mathbb{1}) \xrightarrow{F(\eta)} F(\mathsf{P}(1))$$
as well as partial composition operations
$$F(\mathsf{P}(m)) \otimes F(\mathsf{P}(n)) \xrightarrow{\theta} F(\mathsf{P}(m) \otimes \mathsf{P}(n)) \xrightarrow{\circ_k} F(\mathsf{P}(m+n-1))$$
defined for all $m, n \in \mathbb{N}$, $k = 1, \dots, m$, and which clearly satisfy the equivariance, unit and associativity relations of operads.

This construction is obviously functorial in $\mathsf{P} \in \mathcal{M}\,\mathcal{O}p$.

For a unitary operad P_+ (in the sense of §1.1.19), we have $F(\mathsf{P}_+(0)) = F(\mathbb{1}) = \mathbb{1}$ as soon as the functor F is unit-preserving, and $F(\mathsf{P}_+)$ clearly forms a unitary operad therefore. This operad $F(\mathsf{P}_+)$ has $F(\mathsf{P})$ as underlying non-unitary operad and this verification proves the second assertion of the proposition. We may also check that our functor $F : \mathcal{M} \to \mathcal{N}$ induces a functor on the category of augmented non-unitary Λ-operads $F : \mathcal{M}\,\Lambda\,\mathcal{O}p_\varnothing \,/\, \mathsf{Com} \to \mathcal{N}\,\Lambda\,\mathcal{O}p_\varnothing \,/\, \mathsf{Com}$ by adapting our definition of the functor on the category of ordinary operads $F : \mathcal{M}\,\mathcal{O}p \to \mathcal{N}\,\mathcal{O}p$. We just need the identity $F(\mathbb{1}) = \mathbb{1}$ in this case in order to establish that each object $F(\mathsf{P}(r)) \in \mathcal{N}$, $r > 0$, inherits an augmentation $\epsilon : F(\mathsf{P}(r)) \to F(\mathbb{1}) = \mathbb{1}$ from the operad $\mathsf{P} \in \mathcal{M}\,\Lambda\,\mathcal{O}p_\varnothing \,/\, \mathsf{Com}$. We can then use the equivalence between augmented non-unitary Λ-operads and unitary operads (in Theorem 2.2.18) in order to retrieve the relation $F(\mathsf{P}_+) = F(\mathsf{P}_+)$ for any unitary operad P_+. □

In the context of this proposition, we may also observe that the image of any P-algebra under our functor $F : \mathcal{M} \to \mathcal{N}$ inherits an $F(\mathsf{P})$-algebra structure, so that the mapping $F : A \mapsto F(A)$ defines a functor from the category of algebras over $\mathsf{P} \in \mathcal{M}\,\mathcal{O}p$ to the category of algebras over the operad $F(\mathsf{P})$ associated to P in the category $\mathcal{N}$.

3.1.2. *Examples of functors between operads in symmetric monoidal categories.* The functors considered in §3.3.2 give examples of situations where we can use the result of Proposition 3.1.1:

(a) Let us begin with the simplest example, namely the functor $\Bbbk[-] : \mathcal{S}et \to \mathcal{M}od$ which maps a set $X \in \mathcal{S}et$ to the associated free $\Bbbk$-module $\Bbbk[X] \in \mathcal{M}od$. Proposition 3.1.1 implies that this functor induces a functor $\Bbbk[-] : \mathcal{S}et\,\mathcal{O}p \to \mathcal{M}od\,\mathcal{O}p$, from the category of operads in sets towards the category of operads in $\Bbbk$-modules, and we have a similar statement for the extension of this functor to simplicial sets $\Bbbk[-] : s\mathcal{S}et \to s\,\mathcal{M}od$.

If we apply this functor $\Bbbk[-] : \mathcal{S}et\,\mathcal{O}p \to \mathcal{M}od\,\mathcal{O}p$ to the permutation (respectively, one-point set) operad of §1.1, then we clearly get a model of the associative (respectively, commutative) operad in $\Bbbk$-modules. In the case of the permutation operad, we just get $\mathsf{As}_+(r) = \Bbbk[\Sigma_r]$ for $r \in \mathbb{N}$ (unitary case). In the case of the one-point set operad, we get $\mathsf{Com}_+(r) = \Bbbk[pt] = \Bbbk$ for $r \in \mathbb{N}$. In the non-unitary setting, we simply replace the arity 0 component of these operads by the null module. In each case, we exactly retrieve the expansion of §§1.2.10-1.2.11 for the operads defined by generators and relations in §1.2.10. This identification gives an analogue of the results of Proposition 1.2.7 in the context of $\Bbbk$-modules. Note that $\mathsf{Com}_+(r) = \Bbbk$ can also be identified with a particular instance of the commutative operad of §2.1.11 since $\Bbbk$ represents the unit object of the category of $\Bbbk$-modules.

(b) The geometric realization functor $|-| : s\mathcal{S}et \to \mathcal{T}op$ similarly induces a functor $|-| : s\mathcal{S}et\,\mathcal{O}p \to \mathcal{T}op\,\mathcal{O}p$ from the category of operads in simplicial sets $s\mathcal{S}et\,\mathcal{O}p$ towards the category of topological operads $\mathcal{T}op\,\mathcal{O}p$. In the converse direction, the singular complex functor $\mathtt{Sing}_{\bullet}(-) : \mathcal{T}op \to s\mathcal{S}et$ induces a functor $\mathtt{Sing}_{\bullet}(-) : \mathcal{T}op\,\mathcal{O}p \to s\mathcal{S}et\,\mathcal{O}p$ from the category of topological operads towards the category of operads in simplicial sets.

Recall that the geometric realization and singular complex functors define an instance of a symmetric monoidal adjunction (see §3.3.2). In such a situation, we have the following additional result:

Proposition 3.1.3. *The functors on operads $F : \mathcal{M}\,\mathcal{O}p \leftrightarrows \mathcal{N}\,\mathcal{O}p : G$ induced by the functors of a symmetric monoidal adjunction $F : \mathcal{M} \leftrightarrows \mathcal{N} : G$ are still adjoint to each other. The augmentation $\epsilon : F(G(\mathsf{Q})) \to \mathsf{Q}$ and the unit $\eta : \mathsf{P} \to G(F(\mathsf{P}))$ of this adjunction (at the operad level) are given by the aritywise application of the augmentation and of the unit morphism of the underlying adjunction between the categories $\mathcal{M}$ and $\mathcal{N}$.*

Proof. The augmentation $\epsilon : F(G(Y)) \to Y$ and the unit $\eta : X \to G(F(X))$ of the adjunction $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ are symmetric monoidal transformations by definition of the notion of a symmetric monoidal adjunction. This observation immediately implies that we can apply these morphisms to operads aritywise in order to get operad morphisms. The structure relations between the adjunction augmentation and the adjunction unit remain obviously valid for these induced operad morphisms, and therefore, we still have an adjunction relation at the level of operad categories, with the unit and augmentation morphisms specified in the proposition. □

Thus, in the particular case of the geometric realization and singular complex functors, we obtain the following proposition:

PROPOSITION 3.1.4. *The functors* $|-| : s\mathcal{S}et\,\mathcal{O}p \leftrightarrows \mathcal{T}op\,\mathcal{O}p : \mathtt{Sing}_{\bullet}(-)$ *induced by the realization of simplicial sets and by the singular complex functor on operads are adjoint to each other. The augmentation* $\epsilon : |\mathtt{Sing}_{\bullet}(Q)| \to Q$ *(respectively, the unit* $\eta : P \to \mathtt{Sing}_{\bullet}(|P|)$*) of this adjunction is given by the aritywise application of the augmentation (respectively, of the unit) of the underlying adjunction between simplicial sets and topological spaces.* □

The result of Proposition 3.1.3 also applies to the adjunction of §3.0.6 between sets and counitary cocommutative coalgebras $\Bbbk[-] : \mathcal{S}et \to \mathcal{C}om_{+}^{c} : \mathbb{G}$, where we consider the lifting of the free module functor $\Bbbk[-] : \mathcal{S}et \to \mathcal{M}od$ to the category of coalgebras $\mathcal{C}om_{+}^{c}$.

In the sequel, we often deal with adjunction relations $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ such that the right adjoint functor G is symmetric monoidal, but not the left adjoint F. In this situation, we still have a functor $G : \mathcal{N}\,\mathcal{O}p \to \mathcal{M}\,\mathcal{O}p$ given by the result of Proposition 3.1.1 but we can not apply the construction of this proposition to get a functor on operads from F. Nevertheless, in the case where we deal with symmetric monoidal categories equipped with colimits and limits, and if we moreover assume that the tensor product distributes over colimits (see §0.9), then we have the following result:

PROPOSITION 3.1.5.

(a) *Let* $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ *be a pair of adjoint functors between symmetric monoidal categories, such that* G *(but not necessarily* F*) is lax symmetric monoidal. If the category* $\mathcal{M}$ *is equipped with colimits and is equipped with a tensor product that distributes over colimits (so that we can define free operads and form colimits of operads in that category), then the functor on operads* $G : \mathcal{N}\,\mathcal{O}p \to \mathcal{M}\,\mathcal{O}p$ *which we obtain by the aritywise application of* $G : \mathcal{N} \to \mathcal{M}$ *(see Proposition* 3.1.3*) admits a left adjoint* $F_{\sharp} : \mathcal{M}\,\mathcal{O}p \to \mathcal{N}\,\mathcal{O}p$.

(b) *If* G *is unit-preserving, so that the functor* $G : \mathcal{N}\,\mathcal{O}p \to \mathcal{M}\,\mathcal{O}p$ *preserves unitary operad structures (see Proposition* 3.1.1*), then we also have a functor on unitary operads* $F_{\sharp} : \mathcal{M}\,\mathcal{O}p_{*} \to \mathcal{N}\,\mathcal{O}p_{*}$ *which is left adjoint to* $G : \mathcal{N}\,\mathcal{O}p_{*} \to \mathcal{M}\,\mathcal{O}p_{*}$. *This functor satisfies the relation* $F_{\sharp}(A_{+}) = F_{\sharp}(A)_{+}$, *for any unitary extension* $A_{+} \in \mathcal{M}\,\mathcal{O}p_{*}$ *of an operad* $A \in \mathcal{M}\,\mathcal{O}p_{\varnothing}$ *in the category* $\mathcal{M}$.

PROOF. We focus on the first assertion of this proposition for the moment. We adapt a general construction of adjoint functors, namely the adjoint lifting theorem (see [31, §4.5] and [97]), to get the functor $F_{\sharp} : \mathcal{M}\,\mathcal{O}p \to \mathcal{N}\,\mathcal{O}p$ adjoint to $G : \mathcal{N}\,\mathcal{O}p \to \mathcal{M}\,\mathcal{O}p$. We just note that this functor $G : \mathcal{N}\,\mathcal{O}p \to \mathcal{M}\,\mathcal{O}p$ preserves limits since limits of operads are created aritywise in the base category and our functor G preserves limits at this level by adjunction.

We consider the case of a free operad $P = \mathbb{O}(M)$ first. We then set:

$$F_{\sharp}(\mathbb{O}(M)) := \mathbb{O}(F(M)), \tag{1}$$

where $F(M)$ denotes the symmetric sequence in $\mathcal{N}$ formed by the image of the components of the collection $M(r) \in \mathcal{M}\,\mathcal{S}eq$, $r \in \mathbb{N}$, under the functor F on the base category $\mathcal{M}$. We also take the image of the morphisms $s_{*} : M(r) \to M(r)$ which define the action of permutations $s \in \Sigma_{r}$ on $M(r) \in \mathcal{M}$ under our functor F to determine the symmetric structure of this collection $F(M)(r) = F(M(r))$, $r \in \mathbb{N}$, in the category $\mathcal{N}$. We have a chain of adjunction relations

$$(2)\quad \mathtt{Mor}_{\mathcal{N}\,\mathcal{O}p}(\Theta(F(\mathsf{M})),\mathsf{Q}) = \mathtt{Mor}_{\mathcal{N}\,\mathcal{S}eq}(F(\mathsf{M}),\mathsf{Q}) \\ = \mathtt{Mor}_{\mathcal{M}\,\mathcal{S}eq}(\mathsf{M},G(\mathsf{Q})) = \mathtt{Mor}_{\mathcal{M}\,\mathcal{O}p}(\Theta(\mathsf{M}),G(\mathsf{Q})),$$

which we deduce from the adjunction relation of free operads and from the aritywise adjunction relation between our functors on base categories $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$.

Let $\phi : \Theta(\mathsf{M}) \to \Theta(\mathsf{N})$ be a morphism of free operads in the category $\mathcal{M}$. The adjunction relations in Equation (2) are functorial in $\mathsf{Q} \in \mathcal{N}\,\mathcal{O}p$. By the Yoneda Lemma, we have an operad morphism $\phi_\sharp : \Theta(F(\mathsf{M})) \to \Theta(F(\mathsf{N}))$ which we associate to the dotted natural transformation of morphism sets in the following diagram:

$$(3)\qquad \begin{tikzcd}
\mathtt{Mor}_{\mathcal{N}\,\mathcal{O}p}(\Theta(F(\mathsf{M})),\mathsf{Q}) \arrow[r, "\simeq"] & \mathtt{Mor}_{\mathcal{M}\,\mathcal{O}p}(\Theta(\mathsf{M}),G(\mathsf{Q})) \, . \\
\mathtt{Mor}_{\mathcal{N}\,\mathcal{O}p}(\Theta(F(\mathsf{N})),\mathsf{Q}) \arrow[r, "\simeq"] \arrow[u, dotted, "\phi_\sharp^*"'] & \mathtt{Mor}_{\mathcal{M}\,\mathcal{O}p}(\Theta(\mathsf{N}),G(\mathsf{Q})) \arrow[u, "\phi^*"']
\end{tikzcd}$$

If we assume that $\phi = \phi_f$ is associated to a morphism of symmetric sequences $f : \mathsf{M} \to \Theta(\mathsf{N})$, then we can determine $\phi_\sharp$ as the free operad morphism associated to the morphism of symmetric sequences defined by the composite:

$$F(\mathsf{M}(r)) \xrightarrow{F(f)} F(\Theta(\mathsf{N})(r)) \xrightarrow{F(\phi_{G(\iota)\eta})} FG(\Theta(F(\mathsf{N}))(r)) \xrightarrow{\epsilon} \Theta(F(\mathsf{N})(r)),$$

for each arity $r \in \mathbb{N}$, where we consider:

- the morphism $\epsilon : FG(\Theta(F(\mathsf{N}))(r)) \to \Theta(F(\mathsf{N}))(r)$ determined by the augmentation of the adjunction $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$,
- the operad morphism $\phi_{G(\iota)\eta} : \Theta(\mathsf{N}) \to G(\Theta(F(\mathsf{N})))$, determined on the symmetric sequence N by the morphisms
  $$\mathsf{N}(r) \xrightarrow{\eta} GF(\mathsf{N}(r)) \xrightarrow{G(\iota)} G(\Theta(F(\mathsf{N}))(r))$$
  which we obtain by composing the unit of our adjunction in the base category $\eta : \mathsf{N}(r) \to GF(\mathsf{N}(r))$ with the canonical embedding $\iota : F(\mathsf{N}(r)) \hookrightarrow \Theta(F(\mathsf{N}))(r)$ of the symmetric sequence $F(\mathsf{N})(r) = F(\mathsf{N}(r))$ in the free operad $\Theta(F(\mathsf{N}))$, for each $r \in \mathbb{N}$.

To extend our adjoint functor to the whole category of operads $\mathcal{M}\,\mathcal{O}p$, we use that any object $\mathsf{P} \in \mathcal{M}\,\mathcal{O}p$ fits in a reflexive coequalizer of the form:

$$(4)\qquad \begin{tikzcd}
\Theta(\Theta(\mathsf{P})) \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & \Theta(\mathsf{P}) \arrow[l, bend right=40, "s_0"'] \arrow[r, dotted, "\epsilon"] & \mathsf{P} \, .
\end{tikzcd}$$

Recall that we write $\lambda = \phi_{id}$ for the operad morphism $\lambda : \Theta(\mathsf{P}) \to \mathsf{P}$ induced by the identity of the operad P. This morphism represents the augmentation of the adjunction between the forgetful functor on the category of operads and the free operad functor $\Theta : \mathcal{S}eq \to \mathcal{O}p$. The embedding $\iota : \mathsf{M} \to \Theta(\mathsf{M})$, already considered in this proof, represents the unit of this adjunction relation. To define our reflexive coequalizer (4), we explicitly consider:

- the morphism of free operads $d_0 = \phi_{id} : \Theta(\Theta(\mathsf{P})) \to \Theta(\mathsf{P})$ associated to the identity of the object $\Theta(\mathsf{P})$;
- the morphism of free operads $d_1 = \Theta(\phi_{id}) : \Theta(\Theta(\mathsf{P})) \to \Theta(\mathsf{P})$ induced by the just considered morphism $\lambda = \phi_{id} : \Theta(\mathsf{P}) \to \mathsf{P}$;

- and the morphism of free operads $s_0 = \mathbb{\Theta}(\iota) : \mathbb{\Theta}(P) \to \mathbb{\Theta}(\mathbb{\Theta}(P))$ induced by the embedding $\iota : P \to \mathbb{\Theta}(P)$.

We also set $\epsilon = \phi_{id}$ to get our coequalizing morphism with values in the object P. We clearly have $\epsilon d_0 = \epsilon d_1$ and the identity between this morphism $\epsilon = \phi_{id}$ and the coequalizer of our diagram (4) is a general result on adjoint functors (see [130, §§VI.6-7]). We also refer to §B for further details on this statement.

To define the image of our operad P under the functor $F_\sharp$, we just set:

$$F_\sharp(P) := \mathrm{coeq}(\underbrace{\mathbb{\Theta}(F(\mathbb{\Theta}(P)))}_{=F_\sharp(\mathbb{\Theta}(\mathbb{\Theta}(P)))} \overset{\curvearrowleft}{\rightrightarrows} \underbrace{\mathbb{\Theta}(F(P))}_{=F_\sharp(\mathbb{\Theta}(P))}), \tag{5}$$

where we take the image of the coequalizer diagram (4) under our functor on free operads. We easily deduce from the case of free operads that this object $F_\sharp(P)$ fulfills the adjunction relation:

$$\mathtt{Mor}_{\mathcal{N}\,\mathcal{O}p}(F_\sharp(P), Q) = \mathtt{Mor}_{\mathcal{M}\,\mathcal{O}p}(P, G(Q)), \tag{6}$$

for all $Q \in \mathcal{N}\,\mathcal{O}p$. The definition of the morphism $F_\sharp(\phi) : F_\sharp(P) \to F_\sharp(B)$ associated to any morphism $\phi : P \to B$ in the category of operads in $\mathcal{M}$ can easily be deduced from the Yoneda Lemma. We can also observe that our coequalizer diagram (4) is functorial and we use the construction of our functor on free operads to get an explicit definition of this morphism $F_\sharp(\phi)$.

Let us observe that this functor on operads $F_\sharp : \mathcal{M}\,\mathcal{O}p \to \mathcal{N}\,\mathcal{O}p$ preserves the category of non-unitary operads, regarded as a full subcategory of the category of all operads, because the identity $F(\varnothing) = \varnothing$ (which follows from our adjunction relation) implies that the functor $F_\sharp$ carries the free non-unitary operad $\mathbb{\Theta}(M) \in \mathcal{M}\,\mathcal{O}p_\varnothing$, which we may associate to any non-unitary symmetric sequence $M \in \mathcal{M}\,\mathcal{S}eq_{>0}$, to a non-unitary operad in the category $\mathcal{N}$. The functor G, on the other hand, does not preserve connected operads in general, unless we assume $G(\varnothing) = \varnothing$. We can however form a functor on connected operads $G : \mathcal{N}\,\mathcal{O}p_\varnothing \to \mathcal{M}\,\mathcal{O}p_\varnothing$ from $G : \mathcal{N} \to \mathcal{M}$ by forgetting about the arity zero component of our objects in the construction of Proposition 3.1.1. We still get that the restriction of our functor $F_\sharp : \mathcal{M}\,\mathcal{O}p \to \mathcal{N}\,\mathcal{O}p$ to non-unitary operads forms a left adjoint of this functor on non-unitary operads $G : \mathcal{N}\,\mathcal{O}p_\varnothing \to \mathcal{M}\,\mathcal{O}p_\varnothing$.

In the proof Proposition 3.1.1, we still observe that $G : \mathcal{N} \to \mathcal{M}$ induces a functor on the category of augmented non-unitary Λ-operads $G : \mathcal{N}\,\Lambda\,\mathcal{O}p_\varnothing\,/\,\mathit{Com} \to \mathcal{M}\,\Lambda\,\mathcal{O}p_\varnothing\,/\,\mathit{Com}$ as soon as we assume that G is unit-preserving $G(\mathbb{1}) = \mathbb{1}$. Recall that the definition of this functor reflects the identity $G(Q_+) = G(Q)_+$ for the unitary extension Q_+ of an operad $Q \in \mathcal{N}\,\mathcal{O}p_\varnothing$. We just retrieve the previous functor on non-unitary operads $G : \mathcal{N}\,\mathcal{O}p_\varnothing \to \mathcal{M}\,\mathcal{O}p_\varnothing$ when we forget about the extra structures attached to the objects of the category of augmented non-unitary Λ-operads. We can easily adapt the previous construction to define a left adjoint $F_\sharp : \mathcal{M}\,\Lambda\,\mathcal{O}p_\varnothing\,/\,\mathit{Com} \to \mathcal{N}\,\Lambda\,\mathcal{O}p_\varnothing\,/\,\mathit{Com}$ of this functor on augmented non-unitary Λ-operads $G : \mathcal{N}\,\Lambda\,\mathcal{O}p_\varnothing\,/\,\mathit{Com} \to \mathcal{M}\,\Lambda\,\mathcal{O}p_\varnothing\,/\,\mathit{Com}$. We just consider the extension of the free operad functor to the category of augmented non-unitary Λ-operads in the first step of our process. We use the equivalence between augmented non-unitary Λ-operads and unitary operads in order to define a left adjoint of the functor $G : \mathcal{N}\,\mathcal{O}p_* \to \mathcal{M}\,\mathcal{O}p_*$ induced by $G : \mathcal{N} \to \mathcal{M}$ from this functor on augmented non-unitary Λ-operads $F_\sharp : \mathcal{M}\,\Lambda\,\mathcal{O}p_\varnothing\,/\,\mathit{Com} \to \mathcal{N}\,\Lambda\,\mathcal{O}p_\varnothing\,/\,\mathit{Com}$.

Let us observe that this functor defines a lifting of our previously defined functor on the category of ordinary (non-unitary) operads:

$$\begin{tikzcd} \mathcal{M}\,\Lambda\,\mathcal{O}p_{\varnothing}\,/\,\mathsf{Com} \arrow[r, dotted, "F_{\sharp}"] \arrow[d, "\omega"'] & \mathcal{N}\,\Lambda\,\mathcal{O}p_{\varnothing}\,/\,\mathsf{Com} \arrow[d, "\omega"] \\ \mathcal{M}\,\mathcal{O}p_{\varnothing} \arrow[r, "F_{\sharp}"] & \mathcal{M}\,\mathcal{O}p_{\varnothing} \end{tikzcd}$$

because we have a similar result for the free operad functor $\Theta : \Lambda\,\mathcal{S}eq_{>0}\,/\overline{\mathsf{Com}} \to \Lambda\,\mathcal{O}p_{\varnothing}\,/\,\mathsf{Com}$ which we use in the construction of this adjoint. We use this correspondence to gives a sense to the relation $F(P_+) = F(P)_+$ given in our theorem when $P_+ \in \mathcal{M}\,\mathcal{O}p_*$ is the unitary extension of an operad $P \in \mathcal{M}\,\mathcal{O}p_{\varnothing}$. □

In §II.10, we rely on the construction of this theorem in order to produce an operadic enhancement of the Sullivan cochain dg-algebra functor $\Omega^* : X \mapsto \Omega^*(X)$.

3.2. The notion of a Hopf operad

We devote this section to the study of operads in the symmetric monoidal category of counitary cocommutative coalgebras. One of our aims is to check that operads in counitary cocommutative coalgebras are equivalent to counitary cocommutative coalgebra objects in the category of operads. The existence of these multiple equivalent definitions motivates us to adopt specific conventions for these operads. To be explicit, we generally use the name '*Hopf operad*' (rather than the phrase 'operad in counitary cocommutative coalgebras') to refer to these objects, unless we want to emphasize a particular definition of our structure. We also use the notation $\mathcal{H}opf\,\mathcal{O}p$, rather than $\mathcal{C}om^c_+\,\mathcal{O}p$, to refer to the category of Hopf operads.

We actually adopt the general convention to use the name 'Hopf' as a prefix for any category of structured objects which we may form in a category of counitary cocommutative coalgebras (or in a category of unitary commutative algebras). We stress that the coalgebra (respectively, algebra) underlying a Hopf object is always supposed to be cocommutative (respectively, commutative) under our convention.

The constructions of the next paragraphs §§3.2.1-3.2.5 are valid in any ambient symmetric monoidal category $\mathcal{M}$ in which we define our category of counitary cocommutative coalgebras $\mathcal{C}om^c_+ = \mathcal{M}\,\mathcal{C}om^c_+$. We fix such a base category $\mathcal{M}$ all through this section.

3.2.1. *The definition of Hopf operads as operads in counitary cocommutative coalgebras.* The symmetric monoidal structure of the category of counitary cocommutative coalgebras $\mathcal{C}om^c_+ = \mathcal{M}\,\mathcal{C}om^c_+$ is defined in §3.0.4. Recall simply that the tensor product of coalgebras $A, B \in \mathcal{C}om^c_+$ is obtained by providing the tensor product of A and B in the underlying symmetric monoidal category with a natural coalgebra structure. The unit, associativity and symmetry isomorphisms of the tensor product of coalgebras are inherited from the ambient symmetric monoidal category and the forgetful functor $\omega : \mathcal{M}\,\mathcal{C}om^c_+ \to \mathcal{M}$ is, as a consequence, symmetric monoidal in the sense of §3.3.1.

To define operads in counitary cocommutative coalgebras, we simply apply the general definition of §1.1.1 to the symmetric monoidal category of coalgebras $\mathcal{C}om^c_+$. Under this approach, an operad in counitary cocommutative coalgebras (a Hopf operad in our terminology) consists of a collection of counitary cocommutative coalgebras $P(r)$ together with an action of the symmetric group Σ_r

on $P(r)$, for each $r \in \mathbb{N}$, a unit morphism $\eta : \mathbb{1} \to P(1)$, and composition operations $\circ_k : P(m) \otimes P(n) \to P(m+n-1)$, defined for all $m, n \in \mathbb{N}$, $k = 1, \ldots, m$. We assume that these structure morphisms are formed in the category of counitary cocommutative coalgebras and satisfy the equivariance, unit, and associativity relations of operads §2.1.9 in this category $\mathcal{C}om_+^c$. If we use the definition of §1.1 (rather than the definition of operads in terms or partial composition operations), then we equivalently assume that the total composition products of our operad $\mu : P(r) \otimes P(n_1) \otimes \cdots \otimes P(n_r) \to P(n_1 + \cdots + n_r)$ are morphisms of counitary cocommutative coalgebras.

3.2.2. *The internal structure of Hopf operads.* An operad in counitary cocommutative coalgebras forms an operad in the base category since, as we just observed, the forgetful functor $\omega : \mathcal{M}\,\mathcal{C}om_+^c \to \mathcal{M}$ is symmetric monoidal by construction. As such, an operad in counitary cocommutative coalgebras P can be identified with an operad in $\mathcal{M}$ such that the symmetric group Σ_r acts on $P(r)$ by morphisms of cocommutative coalgebras, for each $r \in \mathbb{N}$, and the unit morphism $\eta : \mathbb{1} \to P(1)$, as well as the composition operations $\circ_k : P(m) \otimes P(n) \to P(m+n-1)$ preserve coalgebra structures.

We go back to the definition of the coalgebra structure on the unit object $\mathbb{1}$ and on the tensor product $P(m) \otimes P(n)$ in order to make explicit the conditions which these coalgebra morphisms η and μ have to satisfy. We obtain that the preservation of coalgebra structures by the operadic unit $\eta : \mathbb{1} \to P(1)$ is equivalent to the commutativity of the diagrams

(1)
$$\begin{tikzcd}
\mathbb{1} \arrow[r, "\eta"] \arrow[dr, "="'] & P(1) \arrow[d, "\epsilon"] \\
& \mathbb{1}
\end{tikzcd}, \qquad
\begin{tikzcd}
\mathbb{1} \arrow[r, "\eta"] \arrow[d, "\simeq"'] & P(1) \arrow[d, "\Delta"] \\
\mathbb{1} \otimes \mathbb{1} \arrow[r, "\eta \otimes \eta"] & P(1) \otimes P(1)
\end{tikzcd},$$

where we use the notation ϵ (respectively, Δ) to refer to the counit (respectively, coproduct) associated to each coalgebra $P(r)$. We similarly get that the preservation of coalgebra structures by the composition products $\circ_k : P(m) \otimes P(n) \to P(m+n-1)$ is equivalent to the commutativity of the diagrams

(2)
$$\begin{tikzcd}
P(m) \otimes P(n) \arrow[r, "\circ_k"] \arrow[d, "\epsilon \otimes \epsilon"'] & P(m+n-1) \arrow[d, "\epsilon"] \\
\mathbb{1} \arrow[r, "="] & \mathbb{1}
\end{tikzcd},$$

$$\begin{tikzcd}
P(m) \otimes P(n) \arrow[rr, "\circ_k"] \arrow[d, "\Delta \otimes \Delta"'] & & P(m+n-1) \arrow[dd, "\Delta"] \\
(P(m) \otimes P(m)) \otimes (P(n) \otimes P(n)) \arrow[d, "\simeq"'] & & \\
(P(m) \otimes P(n) \otimes (P(m) \otimes P(n)) \arrow[rr, "\circ_k \otimes \circ_k"] & & P(m+n-1) \otimes P(m+n-1)
\end{tikzcd},$$

for all $m, n \in \mathbb{N}$, $k = 1, \ldots, m$.

In the case where $\mathcal{M}$ is the category of $\mathbb{k}$-modules so that we have $\mathbb{1} = \mathbb{k}$, the requirement that $\eta : \mathbb{1} \to P(1)$ is a morphism of coalgebras is equivalent to the assumption that the operadic unit element $1 \in \mathsf{P}(1)$ (determining η) is group-like, because the unit 1 is so in the ground ring $\Bbbk$ (regarded as a coalgebra). The relations which we express by the commutativity of the diagrams (2) are also equivalent to the equations:

$$\epsilon(p \circ_k q) = \epsilon(p) \cdot \epsilon(q) \quad \text{and} \quad \Delta(p \circ_k q) = \sum_{(p),(q)} (p_{(1)} \circ_k q_{(1)}) \otimes (p_{(2)} \circ_k q_{(2)}),$$

for all $p \in \mathsf{P}(m)$, $q \in \mathsf{P}(n)$, where we use the notation $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$ to represent the expansion of the coproduct of any element x in a coalgebra. We have an analogous pointwise expression of our relations in the context of the symmetric monoidal categories of graded modules, of differential graded modules, of simplicial modules and of cosimplicial modules, which we consider later on in this work.

The observations of this paragraph imply that we can define operads in counitary cocommutative coalgebras as operads in the base category P, where each $\mathsf{P}(r)$ is equipped with a counit $\epsilon : \mathsf{P}(r) \to \mathbb{1}$ and a coproduct $\Delta : \mathsf{P}(r) \to \mathsf{P}(r) \otimes \mathsf{P}(r)$, which define a counitary cocommutative coalgebra structure on $\mathsf{P}(r)$, such that the diagrams (1-2) commute, for all $m, n \in \mathbb{N}$, $k = 1, \dots, m$.

To give an abstract interpretation of the compatibility conditions expressed by these commutative diagrams, we will check that the category of operads inherits a tensor product from the base category $\boxtimes : \mathcal{O}p \times \mathcal{O}p \to \mathcal{O}p$ such that the doubled factors in the tensor products of (1-2) can be interpreted as the components of a tensor square $\mathsf{P}^{\boxtimes 2}$ in $\mathcal{O}p$. We devote the next paragraphs to this subject. This tensor product $\boxtimes : \mathcal{O}p \times \mathcal{O}p \to \mathcal{O}p$ will be called the *aritywise tensor product* of operads.

3.2.3. *The aritywise tensor product of operads.* Let $\mathsf{P}, \mathsf{Q} \in \mathcal{O}p$. The components of the operad $\mathsf{P} \boxtimes \mathsf{Q}$ are given by the obvious formula $(\mathsf{P} \boxtimes \mathsf{Q})(r) = \mathsf{P}(r) \otimes \mathsf{Q}(r)$ in each arity $r \in \mathbb{N}$, where we form the tensor product of the objects $\mathsf{P}(r)$ and $\mathsf{Q}(r)$ in the ground symmetric monoidal category $\mathcal{M}$. The diagonal action of permutations $w \in \Sigma_r$ on the tensor product $\mathsf{P}(r) \otimes \mathsf{Q}(r)$ provides the object $(\mathsf{P} \boxtimes \mathsf{Q})(r) = \mathsf{P}(r) \otimes \mathsf{Q}(r)$ with an action of the symmetric group Σ_r, for each $r \in \mathbb{N}$. The unit of the operad $\mathsf{P} \boxtimes \mathsf{Q}$ is given by the composite morphism

$$\mathbb{1} \xrightarrow{\simeq} \mathbb{1} \otimes \mathbb{1} \xrightarrow{\eta_{\mathsf{P}} \otimes \eta_{\mathsf{Q}}} \mathsf{P}(1) \otimes \mathsf{Q}(1)$$

where we consider the unit morphisms of the operads P and Q. The partial composition products of $\mathsf{P} \boxtimes \mathsf{Q}$ are defined by the composite morphisms

$$\begin{aligned}(\mathsf{P}(m) \otimes \mathsf{Q}(m)) \otimes (\mathsf{P}(n) \otimes \mathsf{Q}(n)) &\xrightarrow{\simeq} (\mathsf{P}(m) \otimes \mathsf{P}(n)) \otimes (\mathsf{Q}(m) \otimes \mathsf{Q}(n)) \\ &\xrightarrow{\circ_k \otimes \circ_k} \mathsf{P}(m+n-1) \otimes \mathsf{Q}(m+n-1),\end{aligned}$$

where we apply an appropriate tensor permutation to gather the factors attached to each operad P and Q and we apply the composition products of these operads. We immediately check that these structure morphisms satisfy the equivariance, unit and associativity axioms of operads. Thus, our construction, which is also obviously natural with respect to $\mathsf{P}, \mathsf{Q} \in \mathcal{O}p$, yields a bifunctor $\boxtimes : \mathcal{O}p \times \mathcal{O}p \to \mathcal{O}p$.

We readily see that the commutative operad Com_+, which consists of the unit object $\mathbb{1}$ in all arities $\mathsf{Com}_+(r) = \mathbb{1}$, forms a unit for the aritywise tensor product of operads. We also have a natural associativity (respectively, symmetry) isomorphism on $\boxtimes$ which is given by the aritywise application of the associativity (respectively, symmetry) isomorphism of the tensor product $\otimes$ in the ambient category $\mathcal{M}$. We simply have to check that these aritywise associativity (respectively, symmetry) isomorphisms preserve the internal structure of operads, but this assertion follows from formal verifications. We conclude that the bifunctor $\boxtimes : \mathcal{O}p \times \mathcal{O}p \to \mathcal{O}p$ is the tensor product of a symmetric monoidal structure on $\mathcal{O}p$.

A counitary cocommutative coalgebra in $\mathcal{O}p$ formally consists of an operad $\mathsf{P} \in \mathcal{O}p$ equipped with a counit (an augmentation) $\epsilon : \mathsf{P} \to \mathsf{Com}_+$ and a coproduct $\Delta : \mathsf{P} \to \mathsf{P} \boxtimes \mathsf{P}$, both formed in the category of operads, such that the counit, coassociativity and cocommutativity relations of §3.0.4 hold. We immediately see that giving these structure morphisms amounts to providing each object $\mathsf{P}(r)$ with a counitary cocommutative coalgebra structure which is preserved by the action of permutations on our object. Furthermore, for the morphisms $\epsilon : \mathsf{P} \to \mathsf{Com}_+$ and $\Delta : \mathsf{P} \to \mathsf{P} \boxtimes \mathsf{P}$, the preservation of operadic units and of composition products is equivalent to the commutativity of the diagrams (1-2) in §3.2.2. Hence, we have the following result:

PROPOSITION 3.2.4. *The Hopf operads, initially defined as operads in counitary cocommutative coalgebras in* §3.2.1*, can equivalently be defined as counitary cocommutative coalgebras in operads, where we take the aritywise tensor product of* §3.2.3 *to provide the category of operads with a symmetric monoidal structure.* □

We crucially need the equivalence of this proposition for the definition of Hopf operads by generators and relations (see Proposition 3.2.10).

In §3.0.4, we mention that the tensor unit $\mathbb{1}$ represents the terminal object of the category of counitary cocommutative coalgebras and the tensor product represents the cartesian product in that category. The same results hold in the operad context:

PROPOSITION 3.2.5.

(a) *The unitary commutative operad Com_+, which defines the unit of the aritywise tensor product of operads, inherits a natural Hopf operad structure and defines the terminal object of the category of Hopf operads.*

(b) *The aritywise tensor product of Hopf operads inherits a natural Hopf operad structure. The aritywise tensor product therefore induces a bifunctor $\boxtimes : \mathcal{H}opf\,\mathcal{O}p \times \mathcal{H}opf\,\mathcal{O}p \to \mathcal{H}opf\,\mathcal{O}p$ which provides the category of Hopf operads with a symmetric monoidal structure with the unitary commutative operad Com_+ as unit object.*

(c) *The tensor product of Hopf operads $\mathsf{P} \boxtimes \mathsf{Q} \in \mathcal{H}opf\,\mathcal{O}p$, considered in (*b*), actually represents the cartesian product of P and Q in $\mathcal{H}opf\,\mathcal{O}p$. The structure projections $\mathsf{P} \xleftarrow{p} \mathsf{P} \boxtimes \mathsf{Q} \xrightarrow{q} \mathsf{Q}$, which characterize this cartesian product, are identified with the tensor products $p = id \boxtimes \epsilon$ and $q = \epsilon \boxtimes id$, where we consider the counit morphisms $\epsilon : \mathsf{P} \to \mathsf{Com}_+$ (respectively, $\epsilon : \mathsf{Q} \to \mathsf{Com}_+$) of the Hopf operad structure on P (respectively, Q).*

PROOF. This result follows from the identity $\mathcal{H}opf\,\mathcal{O}p = \mathcal{O}p\,\mathcal{C}om_+^c$ established in Proposition 3.2.4 and from the observations of §3.0.4 concerning the categorical interpretation of the tensor product of coalgebras in a symmetric monoidal category which we apply to the category of operads $\mathcal{M} = \mathcal{O}p$. □

The assertions of this proposition can also be deduced from the result of Proposition 1.2.4 which asserts that limits of operads are created in the underlying category. We simply note that Proposition 1.2.4 holds as soon as limits exist in the base category and we use the observations of §3.0.4 to get the definition of terminal objects and cartesian products in categories of counitary cocommutative coalgebras.

We now examine the adjunction between symmetric sequences and operads in the context of Hopf operads. We assume for the construction of free operads that the base category $\mathcal{M}$ is equipped with colimits and with a tensor product which satisfies the distribution relation of §0.9(a) with respect to colimits.

In parallel to the name 'Hopf operad', we use the phrase '*Hopf symmetric sequence*' to refer to the category of symmetric sequences in counitary cocommutative coalgebras. We also use the notation $\mathcal{H}opf\,\mathcal{S}eq$, instead of $\mathcal{C}om_+^c\,\mathcal{S}eq$, to refer to the category of Hopf symmetric sequences. We revisit the definition of the structure of a Hopf symmetric sequence in the next paragraph (just as we did in the case of Hopf operads).

3.2.6. *Hopf symmetric sequences and the definition of free Hopf operads.* We can obviously extend the definition of the aritywise tensor product of operads to symmetric sequences. We then obtain a bifunctor $\boxtimes : \mathcal{S}eq \times \mathcal{S}eq \to \mathcal{S}eq$ which provides $\mathcal{S}eq$ with the structure of a symmetric monoidal category (we just keep the action of symmetric groups in the construction of §3.2.3 and we forget about the operadic unit and the composition operations). The tensor unit in the category $\mathcal{S}eq$ is still given by the unitary commutative operad Com_+, of which we forget the operadic composition structure.

We can readily identify a Hopf symmetric sequence with a symmetric sequence in the base category $M \in \mathcal{S}eq$ equipped with a counit morphism $\epsilon : M \to \mathsf{Com}_+$ and a coproduct $\Delta : M \to M \boxtimes M$ in the category of symmetric sequences such that the counit, coassociativity, and cocommutativity relations of §3.0.4 are satisfied in this symmetric monoidal category $\mathcal{S}eq$. We therefore have an identity between the category of Hopf symmetric sequences and the category of counitary cocommutative coalgebras in $\mathcal{S}eq$. In our notation, this identity reads $\mathcal{H}opf\,\mathcal{S}eq = \mathcal{C}om_+^c\,\mathcal{S}eq = \mathcal{S}eq\,\mathcal{C}om_+^c$.

We can apply the construction of the free operad to the symmetric monoidal category of counitary cocommutative coalgebras whenever the base category $\mathcal{M}$ is equipped with colimits and has a tensor product which distributes over colimits. (Recall that the category of counitary cocommutative coalgebras has colimits as well, which are created in the base category $\mathcal{M}$.) We then get a Hopf operad $\mathbb{O}(M)$, naturally associated to any Hopf symmetric sequence M, and which satisfies the universal property of Proposition 1.2.2 in the category of Hopf operads.

We have already observed that the forgetful functor $\omega : \mathcal{C}om_+^c \to \mathcal{M}$, from counitary cocommutative coalgebras to the base category, is symmetric monoidal by construction, and as a consequence, induces a functor $\omega : \mathcal{H}opf\,\mathcal{O}p \to \mathcal{O}p$ from Hopf operads to operads. According to the discussion of §§3.2.1-3.2.4, we can also identify this functor with a forgetful functor which retains the operad structure in Hopf operads and forgets about the coalgebra structure on each component of our object. We also have an obvious forgetful functor $\omega : \mathcal{H}opf\,\mathcal{S}eq \to \mathcal{S}eq$ from the category of Hopf symmetric sequences $\mathcal{H}opf\,\mathcal{S}eq$ to the category of plain symmetric sequences $\mathcal{S}eq$. We study the interplay between these forgetful functors and the free object functors which we attach to the category of Hopf operads and to the category of operads.

The explicit construction of the free operad $\mathbb{O}(M)$ in §A involves a combination of colimits and tensor products. We mentioned in §3.0.4 that the forgetful functor $\omega : \mathcal{C}om_+^c \to \mathcal{M}$ creates colimits (in addition to tensor products). From this observation, we immediately deduce that the forgetful functor $\omega : \mathcal{H}opf\,\mathcal{O}p \to \mathcal{O}p$ preserves free operads. But we are going to use another approach to prove this statement. Namely, we rely on our interpretation of Hopf operads as coalgebras in operads, which we use in the following observations:

LEMMA 3.2.7. *Let M be a Hopf symmetric sequence. Let $\mathbb{O}(M)$ be the free operad associated to M, and formed in the base category after forgetting the internal cocommutative coalgebra structure of this object M.*

(a) *The counit morphisms $\epsilon : M(r) \to \mathbb{1}$ and the coproduct operations $\Delta : M(r) \to M(r) \otimes M(r)$, which define the counitary cocommutative coalgebra structure of the objects $M(r)$, extend to operad morphisms $\epsilon : \mathbb{O}(M) \to \mathsf{Com}_+$ and $\Delta : \mathbb{O}(M) \to \mathbb{O}(M) \boxtimes \mathbb{O}(M)$ which provide the free operad $\mathbb{O}(M)$ with the structure of a Hopf operad.*

(b) *Let $f : M \to P$ be a morphism of Hopf symmetric sequences, where P is a Hopf operad. Let $\phi_f : \mathbb{O}(M) \to P$ be the unique morphism factorizing f in the category of operads. The free operad $\mathbb{O}(M)$ inherits a Hopf operad structure by assertion (*a*). The morphism ϕ_f automatically preserves this additional coalgebra structure which we attach to the object $\mathbb{O}(M)$ and as a consequence defines a factorization of our morphism f in the category of Hopf operads.*

(c) *In the construction of (*a*), the universal morphism of the free operad $\iota : M \to \mathbb{O}(M)$ defines a morphism of Hopf symmetric sequences. In the construction of (*b*), if we consider the morphism $\lambda : \mathbb{O}(P) \to P$, associated to the identity of P, and which defines the augmentation of the free operad adjunction, then we obtain a morphism of Hopf operads.*

PROOF. Recall that the counits $\epsilon : M(r) \to \mathbb{1}$, which we associate to each coalgebra $M(r)$, can be viewed as a morphism of symmetric sequences with values in the unitary commutative operad Com_+. The existence of the operad morphism extending these counit morphisms $\epsilon : \mathbb{O}(M) \to \mathsf{Com}_+$ immediately follows from the universal property of the free operad, such as stated in Proposition 1.2.2.

By composing the diagonals $\Delta : M(r) \to M(r) \otimes M(r)$ with a tensor product of the universal morphisms $\iota : M(r) \to \mathbb{O}(M)(r)$ in each arity $r \in \mathbb{N}$, we also obtain a morphism $\Delta : M \to \mathbb{O}(M) \boxtimes \mathbb{O}(M)$. By applying the universal property of the free operad, we obtain again an operad morphism $\Delta : \mathbb{O}(M) \to \mathbb{O}(M) \boxtimes \mathbb{O}(M)$ which extends this morphism of symmetric sequences.

The uniqueness requirement in the universal property of free operads (see Proposition 1.2.2 again) implies that the just defined morphisms fulfill the counit, coassociativity and cocommutativity relations of coalgebras on the free operad $\mathbb{O}(M)$.

The universal morphism $\iota : M \to \mathbb{O}(M)$ forms a morphism of Hopf symmetric sequences by construction of the coalgebra structure on $\mathbb{O}(M)$. Thus, the first assertion of (c) is immediate. The uniqueness requirement in the universal property of free operads also implies that the morphism $\phi_f : \mathbb{O}(M) \to P$ associated to a morphism of Hopf symmetric sequences in (b) intertwines coalgebra structures and hence, forms a morphism of Hopf operads. The second assertion of (c) about the adjunction augmentation $\lambda : \mathbb{O}(P) \to P$ is also immediate from this result. □

Then we obtain:

Proposition 3.2.8. *The free operad* $\mathbb{\Theta}(\mathsf{M})$*, together with the Hopf structure constructed in the previous lemma, forms the free object associated to* M *in the category of Hopf operads.*

Proof. This proposition is a formal consequence of the results of assertions (b-c) in Lemma 3.2.7. □

Lemma 3.2.7 also implies the following result on the free operad adjunction:

Proposition 3.2.9. *The free Hopf operad functor of Proposition* 3.2.8 *fits in a commutative diagram*

$$\begin{tikzcd}
\mathcal{H}opf\,\mathcal{S}eq \arrow[r, dotted, shift left, "\mathbb{\Theta}"] \arrow[d, "\omega"'] & \mathcal{H}opf\,\mathcal{O}p \arrow[l, shift left, "\omega"] \arrow[d, "\omega"] \\
\mathcal{S}eq \arrow[r, shift left, "\mathbb{\Theta}"] & \mathcal{O}p \arrow[l, shift left, "\omega"]
\end{tikzcd} ,$$

where we consider the obvious forgetful functor from the category of Hopf operads (respectively, symmetric sequences) to the category of plain operads (respectively, symmetric sequences). We also have a commutative diagram

$$\begin{tikzcd}
\mathtt{Mor}_{\mathcal{H}opf\,\mathcal{O}p}(\mathbb{\Theta}(\mathsf{M}), \mathsf{P}) \arrow[r, "\simeq"] \arrow[d, "\omega"'] & \mathtt{Mor}_{\mathcal{H}opf\,\mathcal{S}eq}(\mathsf{M}, \mathsf{P}) \arrow[d, "\omega"] \\
\mathtt{Mor}_{\mathcal{O}p}(\mathbb{\Theta}(\mathsf{M}), \mathsf{P}) \arrow[r, "\simeq"] & \mathtt{Mor}_{\mathcal{S}eq}(\mathsf{M}, \mathsf{P})
\end{tikzcd} ,$$

where the horizontal maps are defined by the adjunction relations of free operads, while the vertical maps are given by our forgetful functors from Hopf operads (respectively, symmetric sequences) to plain operads (respectively, symmetric sequences).

Proof. The assertion of Proposition 3.2.8 implies that the forgetting of coalgebra structures preserves free operads. In Lemma 3.2.7, assertion (c) similarly implies that the forgetting of coalgebra structures preserves the unit morphism and the augmentation morphism of the free operad adjunction. From this observation, we immediately conclude that the forgetting of coalgebra structures also intertwines the adjunction correspondence on morphisms. □

In §1.2, we briefly explain that the free operad $\mathbb{\Theta}(\mathsf{M})$ intuitively consists of formal operadic composites of elements $\xi \in \mathsf{M}(n)$ (when we work in a concrete base symmetric monoidal category). In this interpretation, the construction of Lemma 3.2.7 gives an extension of the counit (respectively, coproduct) of M to such composites by using the pointwise commutation relations of §3.2.2. We use this idea soon in order to determine the counit and the coproduct of composite elements in operads defined by generators and relations (see §3.2.11).

We now focus on the case where we take a category of modules as base category $\mathcal{M} = \mathcal{M}od$. We explain in §1.2.9 that operads in module categories can be defined by generators and relations as quotients $\mathsf{P} = \mathbb{\Theta}(\mathsf{M})/\langle z^{\alpha}, \alpha \in \mathcal{I}\rangle$, where we consider an ideal $\langle z^{\alpha}, \alpha \in \mathcal{I}\rangle$ in a free operad $\mathbb{\Theta}(\mathsf{M})$. In the context of Hopf operads, we have the following result:

PROPOSITION 3.2.10. *Let M be a Hopf symmetric sequence (in $\Bbbk$-modules). We apply the construction of Lemma 3.2.7 to obtain a Hopf structure on the free operad associated to M. Let $\mathsf{S} = \langle z^{\alpha}, \alpha \in \mathcal{I}\rangle$ be the ideal generated by a collection of elements $z^{\alpha} \in \mathsf{S}(n_{\alpha})$ in the free operad $\mathbb{\Theta}(\mathsf{M})$. If we have*

$$\epsilon(z^{\alpha}) = 0 \quad \textit{and} \quad \Delta(z^{\alpha}) \in \mathsf{S}(n_{\alpha}) \otimes \mathbb{\Theta}(\mathsf{M})(n_{\alpha}) + \mathbb{\Theta}(\mathsf{M})(n_{\alpha}) \otimes \mathsf{S}(n_{\alpha})$$

for each $z^{\alpha} \in \mathsf{S}(n_{\alpha})$, then:

(a) *The operad $\mathsf{P} = \mathbb{\Theta}(\mathsf{M})/\langle z^{\alpha}, \alpha \in \mathcal{I}\rangle$ inherits a quotient Hopf operad structure from the free operad $\mathbb{\Theta}(\mathsf{M})$.*

(b) *The morphisms of Hopf operads $\bar{\phi}_f : \mathbb{\Theta}(\mathsf{M})/\langle z^{\alpha}, \alpha \in \mathcal{I}\rangle \to \mathsf{Q}$ defined on this quotient are in obvious bijection with the morphisms of Hopf operads $\phi_f : \mathbb{\Theta}(\mathsf{M}) \to \mathsf{Q}$ such that $\phi_f(z^{\alpha}) = 0$ for each generating element of the ideal $z^{\alpha} \in \mathsf{S}(n_{\alpha})$.*

In the situation of this proposition, we also say that the ideal $\mathsf{S} = \langle z^{\alpha}, \alpha \in \mathcal{I}\rangle$ forms a *Hopf ideal* in the operad $\mathbb{\Theta}(\mathsf{M})$.

PROOF. The requirement $\epsilon(z^{\alpha}) = 0$ implies that ϵ induces a morphism on the quotient $\mathbb{\Theta}(\mathsf{M})/\mathsf{S}$, and hence provides this quotient operad with a counit

$$\epsilon : \mathbb{\Theta}(\mathsf{M})/\langle z^{\alpha}, \alpha \in \mathcal{I}\rangle \to \mathsf{Com}_{+}\,.$$

The requirement $\Delta(z^{\alpha}) \in \mathsf{S}(n_{\alpha}) \otimes \mathbb{\Theta}(\mathsf{M})(n_{\alpha}) + \mathbb{\Theta}(\mathsf{M})(n_{\alpha}) \otimes \mathsf{S}(n_{\alpha})$ is equivalent to the vanishing of $\Delta(z^{\alpha})$ in the module:

$$\begin{aligned}((\mathbb{\Theta}(\mathsf{M})/\mathsf{S}) \boxtimes (\mathbb{\Theta}(\mathsf{M})/\mathsf{S}))(n_{\alpha}) &= \big(\mathbb{\Theta}(\mathsf{M})(n_{\alpha})/\mathsf{S}(n_{\alpha})\big) \otimes \big(\mathbb{\Theta}(\mathsf{M})(n_{\alpha})/\mathsf{S}(n_{\alpha})\big)\\ &= \big(\mathbb{\Theta}(\mathsf{M})(n_{\alpha}) \otimes \mathbb{\Theta}(\mathsf{M})(n_{\alpha})\big)/\big(\mathsf{S}(n_{\alpha}) \otimes \mathbb{\Theta}(\mathsf{M})(n_{\alpha}) + \mathbb{\Theta}(\mathsf{M})(n_{\alpha}) \otimes \mathsf{S}(n_{\alpha})\big),\end{aligned}$$

and implies that $\Delta : \mathbb{\Theta}(\mathsf{M}) \to \mathbb{\Theta}(\mathsf{M}) \boxtimes \mathbb{\Theta}(\mathsf{M})$ induces a morphism

$$\Delta : \mathbb{\Theta}(\mathsf{M})/\mathsf{S} \to (\mathbb{\Theta}(\mathsf{M})/\mathsf{S}) \boxtimes (\mathbb{\Theta}(\mathsf{M})/\mathsf{S})$$

on the quotient operad $\mathbb{\Theta}(\mathsf{M})/\mathsf{S} = \mathbb{\Theta}(\mathsf{M})/\langle z^{\alpha}, \alpha \in \mathcal{I}\rangle$. These morphisms, obtained by a quotient process, naturally satisfy the counit, coassociativity, and cocommutativity relations of coalgebras and hence, provide the operad $\mathbb{\Theta}(\mathsf{M})/\mathsf{S}$ with a well-defined Hopf structure.

To check the second assertion of the proposition, we simply observe that the morphism $\bar{\phi}_f : \mathbb{\Theta}(\mathsf{M})/\langle z^{\alpha}, \alpha \in \mathcal{I}\rangle \to \mathsf{Q}$, induced by the morphism of Hopf operads $\phi_f : \mathbb{\Theta}(\mathsf{M}) \to \mathsf{Q}$, naturally preserves coalgebra structures as well, and hence, defines a morphism of Hopf operads. □

3.2.11. *First examples of Hopf operads defined by a presentation.* In Proposition 3.2.5, we check that the unitary commutative operad Com_{+} has a natural Hopf structure. The same result holds for the non-unitary version of the commutative operad Com. To illustrate our constructions, we check that this structure result can be retrieved from the statement of Proposition 3.2.10 and from the presentation commutative operad in §1.2.10. We then assume that the ground symmetric monoidal category is a category of modules over a ring.

Recall that the generating symmetric sequence of the commutative operad is defined by $\mathsf{M}_{\mathsf{Com}}(2) = \Bbbk[\mu(x_1, x_2)] = \Bbbk$, where $\mu = \mu(x_1, x_2)$ denotes an operation on which Σ_2 acts trivially, and we have $\mathsf{M}_{\mathsf{Com}}(r) = 0$ for $r \neq 2$. We provide the module $\mathsf{M}_{\mathsf{Com}}(2) = \Bbbk[\mu(x_1, x_2)]$ with the coalgebra structure such that $\epsilon(\mu) = 1$ and $\Delta(\mu) = \mu \otimes \mu$ for this generating operation. We use the preservation of operadic composition structures to determine the image of the generating relations of *Com* under the counit and the coproduct in the free operad:

$$\begin{aligned}\epsilon(\mu(\mu,1)-\mu(1,\mu)) &= 1-1=0,\\ \Delta(\mu(\mu,1)-\mu(1,\mu)) &= (\mu\otimes\mu)(\mu\otimes\mu,1\otimes 1)-(\mu\otimes\mu)(1\otimes 1,\mu\otimes\mu)\\ &= \mu(\mu,1)\otimes\mu(\mu,1)-\mu(1,\mu)\otimes\mu(1,\mu)\\ &= (\mu(\mu,1)-\mu(1,\mu))\otimes\mu(\mu,1)+\mu(1,\mu)\otimes(\mu(\mu,1)-\mu(1,\mu)).\end{aligned}$$

We see, from this computation, that the generating relations of the commutative operad generate a Hopf ideal. Hence, the assumptions of Proposition 3.2.10 are satisfied, and we retrieve that *Com* inherits a well-defined Hopf operad structure such that $\epsilon(\mu)=1$ and $\Delta(\mu)=\mu\otimes\mu$ for the generating operation $\mu=\mu(x_1,x_2)$.

The unitary and the non-unitary version of the associative operad also inherits a Hopf structure. Let us see how to retrieve this structure result from our presentation again. The generating symmetric sequence of the associative operad is given by $M_{As}(2)=\Bbbk[\mu(x_1,x_2),\mu(x_2,x_1)]=\Bbbk[\Sigma_2]$, where $\mu=\mu(x_1,x_2)$ denotes an operation on which Σ_2 acts regularly, and $M_{As}(r)=0$ for $r\neq 2$. We provide the module $M_{As}(2)$ with the coalgebra structure such that $\epsilon(\mu)=1$ and $\Delta(\mu)=\mu\otimes\mu$. The definition of the counit and of the coproduct of the transposed operation $(1\ 2)\cdot\mu=\mu(x_2,x_1)$ is then forced by the equivariance requirement. We check, as in the case of the commutative operad, that $\mu(\mu,1)-\mu(1,\mu)$ generates a Hopf ideal, from which we conclude again that the operad *As* inherits a well-defined Hopf structure.

In the case of the Lie operad, we have a generating symmetric sequence such that $M_{Lie}(2)=\Bbbk[\lambda(x_1,x_2)]=\Bbbk^{\pm}$ where $\Bbbk^{\pm}$ denotes the signature representation. We have in this case no possibility of fixing a counit $\epsilon(\lambda)\in\Bbbk$ and a coproduct $\Delta(\lambda)\in \mathit{Lie}(2)\otimes \mathit{Lie}(2)$ such that the counit relations hold, the equivariance requirements of operad morphisms are satisfied, and the Jacobi relation is canceled by the counit in $\Bbbk$ and by the coproduct in $\mathit{Lie}(3)\otimes\mathit{Lie}(3)$. Hence, we have no Hopf structure on the Lie operad.

3.2.12. *The example of the Poisson operad.* Though we have no Hopf structure on the Lie operad, we can define an appropriate counit and coproduct for the corresponding generating operation λ in the Poisson operad. Recall that the Poisson operad *Pois* is defined by a presentation of the form

$$\begin{aligned}\mathit{Pois} = \Theta(\ \Bbbk\,\mu(x_1,x_2)\oplus\Bbbk\,\lambda(x_1,x_2)\ :\ &\mu(\mu(x_1,x_2),x_3)\equiv\mu(x_1,\mu(x_2,x_3)),\\ &\lambda(\lambda(x_1,x_2),x_3)+\lambda(\lambda(x_2,x_3),x_1)+\lambda(\lambda(x_3,x_1),x_2)\equiv 0,\\ &\lambda(\mu(x_1,x_2),x_3)\equiv\mu(\lambda(x_1,x_3),x_2)+\mu(x_1,\lambda(x_2,x_3))\ ),\end{aligned}$$

where the action of the symmetric group in arity 2 is determined by $(1\ 2)\cdot\mu=\mu$ and $(1\ 2)\cdot\lambda=-\lambda$. We extend the formula of the commutative operad to define the counit and the coproduct of the product operation $\mu=\mu(x_1,x_2)$. We define the counit and the coproduct of the Lie bracket operation $\lambda=\mu(x_1,x_2)$ by $\epsilon(\lambda)=0$ and $\Delta(\lambda)=\lambda\otimes\mu+\mu\otimes\lambda$. We easily check again that the generating relations of the Poisson operad form a Hopf ideal (adapt the verifications performed in §3.2.11 for the commutative operad) and we have a well-defined Hopf structure on the Poisson operad therefore. We use a graded variant of this Hopf structure in our study of the homology of E_n-operads (see §4.2).

3.2.13. *Remark: Tensor product of algebras over Hopf operads.* The existence of a Hopf structure on an operad P implies that the associated category of algebras $\mathcal{P}$ inherits a symmetric monoidal structure from the underlying symmetric monoidal category $\mathcal{M}$. Indeed, the tensor product of P-algebras $A, B \in \mathcal{P}$ inherits an action of P, given by the composite morphisms

$$\begin{aligned} \mathsf{P}(r) \otimes (A \otimes B)^{\otimes r} &\xrightarrow{\Delta} (\mathsf{P}(r) \otimes \mathsf{P}(r)) \otimes (A \otimes B)^{\otimes r} \\ &\xrightarrow{\simeq} (\mathsf{P}(r) \otimes A^{\otimes r}) \otimes (\mathsf{P}(r) \otimes B^{\otimes r}) \xrightarrow{\lambda_A \otimes \lambda_B} A \otimes B, \end{aligned}$$

for any $r \in \mathbb{N}$, where we consider the coproduct of P followed by the obvious tensor permutation and the tensor product of the evaluation morphisms attached to the P-algebras. The tensor unit $\mathbb{1}$ also inherits an action of the operad P by restriction of the natural commutative algebra structure of this object $\mathbb{1}$ through the counit morphism $\epsilon : \mathsf{P} \to \mathsf{Com}_+$. The counit, coassociativity and cocommutativity relations at the level of the coalgebra structure of the Hopf operad P imply that the unit, associativity and symmetry isomorphisms of the base category define P-algebra morphisms when we deal with tensor products of P-algebras. Hence, we have a whole symmetric monoidal structure on the category of P-algebras.

In the case of algebras over the commutative operad, we retrieve the symmetric monoidal structure of §3.0.3. In the case of algebras over the associative operad, we retrieve the similarly defined symmetric monoidal structure of the category of unitary associative algebras (see the introduction of §3.0).

3.2.14. *The case of connected operads.* Recall that the category of connected operads $\mathcal{O}p_{\varnothing 1}$, such as defined in §1.1.21, consists of the operads P such that $\mathsf{P}(0) = \varnothing$ and $\mathsf{P}(1) = \mathbb{1}$.

The constructions of §§3.2.3-3.2.5 can readily be adapted in the context of connected operads. We actually have $(\mathsf{P} \boxtimes \mathsf{Q})(0) = \varnothing$ and $(\mathsf{P} \boxtimes \mathsf{Q})(1) = \mathbb{1}$ so that the category $\mathcal{O}p_{\varnothing 1}$ is equipped with a well-defined aritywise tensor product inherited from the category of operads. We accordingly have a symmetric monoidal structure on $\mathcal{O}p_{\varnothing 1}$. We just need to observe that the unit object of this category is the non-unitary version of the commutative operad Com (defined in §2.1.11).

The result of Proposition 3.2.4 remains valid for connected operads, and so does the result of Proposition 3.2.5 (provided that we replace the unitary version of the commutative operad Com_+ in this statement by its non-unitary counterpart Com).

3.2.15. *Unitary Hopf operads and non-unitary Hopf Λ-operads.* The description of unitary operads in terms of augmented (connected) non-unitary Λ-operad structures given in the previous chapter makes sense in any base category. Hence, we can apply these ideas without change within the category of counitary cocommutative coalgebras in order to give a description of the category of unitary Hopf operads in terms of augmented non-unitary Λ-operads in counitary cocommutative coalgebras. We use the phrase ‘ *non-unitary Hopf Λ-operad*’ to refer this category of augmented (connected) non-unitary Λ-operads.

We can also rely on the observations of §3.2 in order to identify unitary Hopf operads with counitary cocommutative coalgebras in the category of unitary operads, because the aritywise tensor products $(\mathsf{P} \boxtimes \mathsf{Q})(r) = \mathsf{P}(r) \otimes \mathsf{Q}(r)$, such as defined in §3.2.3, clearly preserves the category of unitary operads. We can equivalently check that the aritywise tensor product of operads lifts to a tensor product operation on the category of augmented non-unitary Λ-operads. We explicitly get that the action of the restriction operator u^* associated to a map $u \in \mathtt{Mor}_{\Lambda}(\underline{\mathtt{m}}, \underline{\mathtt{n}})$ on a tensor product of operads $\mathsf{P} \boxtimes \mathsf{Q}$ is given by the diagonal action of the restriction operators associated to our map u on P and Q. The augmentation morphisms $\epsilon : (\mathsf{P} \boxtimes \mathsf{Q})(r) \to \mathbb{1}$ are similarly defined by taking the tensor products of the augmentation morphisms of the operads P and Q. We moreover see that the preservation of counitary cocommutative coalgebra structures by the action of the restriction operators u^* on this tensor product $\mathsf{P} \boxtimes \mathsf{Q}$ is equivalent to the assumption that the augmentation $\epsilon : \mathsf{P} \to \mathsf{Com}$ and the diagonal $\Delta : \mathsf{P} \to \mathsf{P} \boxtimes \mathsf{P}$ of the Hopf operad underlying P preserve restriction operators. We have similar assertions for the augmentation morphisms of our augmented Λ-operad structure. These observations imply that we can identify the category of non-unitary Hopf Λ-operads, which we use to model unitary Hopf operads, with the category of counitary cocommutative coalgebras in the category of augmented Λ-operads equipped with the aritywise tensor structure. We have similar observations for the category of non-unitary Hopf Λ-sequences which we can equivalently define either as augmented non-unitary Λ-sequences in the category of counitary cocommutative coalgebras or as counitary cocommutative coalgebras in the category of augmented Λ-sequences (where we take the obvious extension of the aritywise tensor products of non-unitary Hopf Λ-operads).

Let us observe that the augmentation morphisms $\epsilon : \mathsf{P}(r) \to \mathbb{1}$ in the definition of an augmented non-unitary Λ-operad necessarily reduce to the counit morphism of the coalgebras $\mathsf{P}(r)$ which underlie our object P when we work in the category of counitary cocommutative coalgebras. We therefore omit to mention the augmentation when we deal with (connected) non-unitary Hopf Λ-operads. We similarly see that the commutative operad Com defines the terminal object of the category of (connected) non-unitary Hopf Λ-operads. We therefore adopt the following short notation for the category of non-unitary Hopf Λ-operads:

$$\mathcal{H}opf\,\Lambda\,\mathcal{O}p_{\varnothing} = \mathcal{H}opf\,\Lambda\,\mathcal{O}p_{\varnothing}\,/\,\mathsf{Com}\,.$$

We use an analogous abridged notation for the category of non-unitary (respectively, connected) Hopf Λ-operads and for the category of non-unitary (respectively, connected) Hopf Λ-sequences which underlies this category of Λ-operads.

We can combine the results of Proposition 2.3.1 (respectively, Proposition 2.4.3) and Proposition 3.2.7 to determine the free operad associated to a non-unitary Hopf Λ-sequence (respectively, to a connected Hopf Λ-sequence). We can also combine the observations of §2.4.8 and the results of Proposition 3.2.10 to get a definition of unitary Hopf operads by generators and relations.

The associative operad As_{+}, the commutative operad Com_{+}, and the Poisson operad Pois_{+}, give examples of connected unitary Hopf operads which we can define by a presentation by generators and relations of this form. In fact, we simply have to check that the augmentation morphisms defined in §2.4.9 preserve the coalgebra structure on the generating collection M_{P} of these operads $\mathsf{P} = \mathsf{As}, \mathsf{Com}, \mathsf{Pois}$ (see §§3.2.11-3.2.12) to conclude that $\mathsf{P} = \mathsf{As}, \mathsf{Com}, \mathsf{Pois}$ all have a unitary extension as a Hopf operad.

3.3. Appendix: Functors between symmetric monoidal categories

In various constructions of the previous sections, we have to transport structures (like commutative algebras) from one symmetric monoidal category $\mathcal{M}$ to another $\mathcal{N}$. For this aim, we deal with functors that preserve the internal structures of our symmetric monoidal categories in a strict or relaxed sense. The purpose of this appendix section is to review the definition of the extra structures, consisting of natural equivalences or natural transformations, which we use to govern the preservation of tensor products by such functors $F : \mathcal{M} \to \mathcal{N}$ between symmetric monoidal categories $\mathcal{M}$ and $\mathcal{N}$.

3.3.1. *(Lax) symmetric (co)monoidal functors.* Recall that a functor $F : \mathcal{M} \to \mathcal{N}$ between symmetric monoidal categories $\mathcal{M}$ and $\mathcal{N}$ is *lax symmetric monoidal* when we have a morphism $\eta : \mathbb{1} \to F(\mathbb{1})$ and a natural transformation $\theta : F(A) \otimes F(B) \to F(A \otimes B)$, such that natural unit, associativity and symmetry constraints, expressed by the commutativity of the following diagrams, hold:

$$\begin{tikzcd}
F(A) \otimes F(\mathbb{1}) \arrow[r, "\theta"] & F(A \otimes \mathbb{1}) \arrow[d, "\simeq"] \\
F(A) \otimes \mathbb{1} \arrow[u, "\eta"] \arrow[r, "\simeq"] & F(A)
\end{tikzcd} ,
\qquad
\begin{tikzcd}
F(\mathbb{1}) \otimes F(A) \arrow[r, "\theta"] & F(\mathbb{1} \otimes A) \arrow[d, "\simeq"] \\
\mathbb{1} \otimes F(A) \arrow[u, "\eta"] \arrow[r, "\simeq"] & F(A)
\end{tikzcd} ,$$

$$\begin{tikzcd}
(F(A) \otimes F(B)) \otimes F(C) \arrow[r, "\theta \otimes id"] \arrow[d, swap, "\simeq"] & F(A \otimes B) \otimes F(C) \arrow[d, "\theta"] \\
F(A) \otimes (F(B) \otimes F(C)) \arrow[d, swap, "id \otimes \theta"] & F((A \otimes B) \otimes C) \arrow[d, "\simeq"] \\
F(A) \otimes F(B \otimes C) \arrow[r, "\theta"] & F(A \otimes (B \otimes C))
\end{tikzcd}$$

and

$$\begin{tikzcd}
F(A) \otimes F(B) \arrow[r, "\theta"] \arrow[d, swap, "\simeq"] & F(A \otimes B) \arrow[d, "\simeq"] \\
F(B) \otimes F(A) \arrow[r, "\theta"] & F(B \otimes A)
\end{tikzcd} .$$

In applications to operads, we often have to assume that our functor satisfies $F(\mathbb{1}) = \mathbb{1}$ and that our natural morphism $\eta : \mathbb{1} \to F(\mathbb{1})$ reduces to the identity morphism of the unit object $\mathbb{1} \in \mathcal{N}$. In this situation, we say that the functor F is *unit-preserving* and that θ defines a *symmetric monoidal transformation* on F. Most lax symmetric monoidal functors which we consider in this book satisfies this extra condition $F(\mathbb{1}) = \mathbb{1}$. We therefore only consider this subclass of the class of lax symmetric monoidal functors in the sequel.

We have a dual situation where our functor F is equipped with a morphism $\epsilon : F(\mathbb{1}) \to \mathbb{1}$ and with a natural transformation $\theta : F(A \otimes B) \to F(A) \otimes F(B)$ which satisfy the dual of the above unit, associativity and symmetry constraints. We then say that F defines a *lax symmetric comonoidal functor*. If we have in addition $F(\mathbb{1}) = \mathbb{1}$, so that the augmentation morphism $\epsilon : F(\mathbb{1}) \to \mathbb{1}$ associated to F reduces to the identity morphism of the unit object $\mathbb{1} \in \mathcal{N}$, then we also say that F is unit-preserving and that θ defines a *symmetric comonoidal transformation* on F. We will still see that most lax symmetric comonoidal functors which we consider in this book are unit-preserving. We therefore only consider this subclass of the class of lax symmetric comonoidal functors in what follows (as in the case of lax symmetric monoidal functors).

We may also deal with a nicer situation, where we have both $F(\mathbb{1}) = \mathbb{1}$ and our symmetric monoidal transformation θ defines an isomorphism $\theta : F(A) \otimes F(B) \xrightarrow{\simeq} F(A \otimes B)$, for every pair of objects $A, B \in \mathcal{M}$ (or dually in the case of a symmetric comonoidal transformation). We say in this case that θ forms a *symmetric monoidal equivalence* and that $F : \mathcal{M} \to \mathcal{N}$ is a *symmetric monoidal functor* from $\mathcal{M}$ to $\mathcal{N}$ (some authors say that F is '*strongly symmetric monoidal*' in this context).

3.3.2. *Fundamental examples of symmetric monoidal functors.* The geometric realization functor

$$|-| : s\mathcal{S}et \to \mathcal{T}op$$

(see §0.5) is a fundamental example of a functor which is (strongly) symmetric monoidal. Recall that the tensor product operation on simplicial sets and topological spaces is defined by the cartesian product of these categories. In this context, the canonical projections $K \xleftarrow{p} K \times L \xrightarrow{q} L$ induce morphisms $|K| \xleftarrow{p} |K \times L| \xrightarrow{q} |L|$ which we can put together to define a natural transformation from $|K \times L|$ to $|K| \times |L|$ in the category of topological spaces. This natural transformation actually defines a homeomorphism

$$\theta : |K \times L| \xrightarrow{\simeq} |K| \times |L|,$$

for all $K, L \in s\mathcal{S}et$ (see for instance [141, §III]), This result follows from a topological interpretation, in terms of simplicial decompositions of prisms, of the classical Eilenberg-Zilber equivalence (we refer to *loc. cit.* for details). For a point, we obviously have $|\,pt\,| = pt$, and the definition of the natural transformation $\theta : |K \times L| \to |K| \times |L|$ from universal categorical constructions automatically ensures that the unit, associativity and symmetry constraints of §3.3.1 are fulfilled.

The singular complex functor

$$\mathtt{Sing}_\bullet : \mathcal{T}op \to s\mathcal{S}et,$$

which defines the right adjoint of the geometric realization functor $|-| : s\mathcal{S}et \to \mathcal{T}op$ (see §0.5), is also symmetric monoidal. In this case, the identity $\mathtt{Sing}_\bullet(pt) = pt$ and the existence of an isomorphism $\mathtt{Sing}_\bullet(K \times L) \xrightarrow{\simeq} \mathtt{Sing}_\bullet(K) \times \mathtt{Sing}_\bullet(L)$ immediately follows from the definition of $\mathtt{Sing}_\bullet : \mathcal{T}op \to s\mathcal{S}et$ as a right adjoint.

To give another simple example, the functor $\mathbb{k}[-] : \mathcal{S}et \to \mathcal{M}od$, which maps any set $X \in \mathcal{S}et$ to the associated free $\mathbb{k}$-module $\mathbb{k}[X]$, is symmetric monoidal because we have an obvious identity $\mathbb{k}[pt] = \mathbb{k}$ for the one-point set $pt \in \mathcal{S}et$, and a natural isomorphism $\mathbb{k}[X] \otimes \mathbb{k}[Y] \xleftarrow{\simeq} \mathbb{k}[X \times Y]$, for any cartesian product of sets $X, Y \in \mathcal{S}et$. (We easily check that this natural transformation fulfills our unit, associativity and symmetry constraints.) We go back to this example in §3.0.6.

The simplicial extension of the free $\mathbb{k}$-module functor $\mathbb{k}[-] : s\mathcal{S}et \to s\,\mathcal{M}od$ (considered in §0.3) is also symmetric monoidal (the symmetric monoidal structure of simplicial modules will be studied in §II.5.2).

The normalized chain complex functor $\mathtt{N}_* : s\mathcal{S}et \to dg\,\mathcal{M}od$, of which we recall the definition in §II.5.0.5, is an instance of functor which is lax but not strongly symmetric monoidal. In the case of this functor, we have a natural transformation $\theta : \mathtt{N}_*(X) \times \mathtt{N}_*(Y) \to \mathtt{N}_*(X \times Y)$, called the *Eilenberg-MacLane morphism*, which satisfies our unit, associativity and symmetry constraints, but this morphism is only a weak-equivalence and is not an isomorphism (see [129, §§VIII.6-8]). We give a detailed survey of this subject in §II.5.2.

3.3.3. *Symmetric monoidal adjunctions.* Suppose now we have a pair of adjoint functors $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ between symmetric monoidal categories $\mathcal{M}$ and $\mathcal{N}$ such that both F and G are symmetric monoidal. We then say that our adjunction is *symmetric monoidal* if the augmentation morphisms $\epsilon : F(G(X)) \to X$ and the unit $\eta : A \to G(F(A))$ of our adjunction are identity morphisms on unit objects and make commute the diagrams

$$\begin{tikzcd}
F(G(X)) \otimes F(G(Y)) \arrow[r, "\epsilon\otimes\epsilon"] \arrow[d, "\simeq"'] & X \otimes Y \\
F(G(X) \otimes G(Y)) \arrow[r, "\simeq"] & F(G(X \otimes Y)) \arrow[u, "\epsilon"']
\end{tikzcd},$$

and

$$\begin{tikzcd}
A \otimes B \arrow[r, "\eta"] \arrow[d, "\eta\otimes\eta"'] & G(F(A \otimes B)) \\
G(F(A)) \otimes G(F(B)) \arrow[r, "\simeq"] & G(F(A) \otimes F(B)) \arrow[u, "\simeq"']
\end{tikzcd},$$

where we consider the symmetric monoidal transformations associated to F and G.

One can check (exercise) that the augmentation $\epsilon : |\,\mathtt{Sing}_\bullet(X)| \to X$ and the unit $\eta : K \to \mathtt{Sing}_\bullet(|K|)$ of the adjunction between the geometric realization $|-| : s\mathcal{S}et \to \mathcal{T}op$ and the singular complex functor $\mathtt{Sing}_\bullet(-) : \mathcal{T}op \to s\mathcal{S}et$ satisfy these relations. Hence, this adjunction $|-| : s\mathcal{S}et \rightleftarrows \mathcal{T}op : \mathtt{Sing}_\bullet(-)$ is symmetric monoidal in the sense defined in this paragraph.

Part I(b)

Braids and E_2-operads

CHAPTER 4

The Little Discs Model of E_n-operads

We explain the definition of the operad of little n-discs D_n and the definition of the notion of an E_n-operad in the first section of this chapter (§4.1). We review classical results on the homology of the little disc operads in the second section (§4.2).

The homology functor goes from spaces to graded modules. In good cases, the homology of a space also inherits a coalgebra structure, dual to the standard commutative algebra structure of the cohomology, and the homology defines a symmetric monoidal functor from the category of spaces towards the category of counitary cocommutative coalgebras in graded modules. This observation implies that the homology of an operad forms an operad in the category of counitary cocommutative coalgebras in graded modules. Thus, we get that the homology of an operad forms a Hopf operad in graded modules in the terminology of §3.2. In this case, we also speak about *graded Hopf operads* for short. We make explicit the graded Hopf operad structure of the homology of the little n-discs operads in §4.2.

To complete our account, we provide an introduction to geometrical variants of the little discs operads: the operad of framed little discs, obtained by adding a rotation parameter in the definition of the ordinary little discs operad; and the Fulton–MacPherson operad, which is a model of E_n-operad obtained by a compactification of the configuration spaces of points in $\mathbb{R}^n$. We tackle these subjects in an outlook section (§4.3). We also briefly explain the relationship between the little 2-discs operad and an operad defined by another compactification of configuration spaces, the Deligne–Mumford–Knudsen compactification, whose terms represent the moduli spaces of stable marked curves of genus zero.

We devote an appendix section (§4.4) to a short account of our conventions on graded modules. We notably explain the definition of a symmetric monoidal structure on the category of graded modules. We work within this base category when we study the homology of the little n-discs operads.

In this monograph, we deal with non-unitary operad structures as soon as we perform in-depth constructions on operads, and for technical reasons, we systematically regard a unitary operad as a unitary extension of an underlying non-unitary operad. Therefore, in contrast with standard conventions, we assume that the little n-discs operad satisfies $\mathsf{D}_n(0) = \varnothing$ in the basic case. The unitary version of the operad of little n-discs, which is more usually considered in the literature, is denoted by D_{n+} and is obtained by adding an arity zero term $\mathsf{D}_{n+}(0) = pt$ to this non-unitary operad D_n.

The results and concepts surveyed in this chapter come from [27, 28, 140] as regards the definition of the little discs operads and the applications to iterated loop spaces, and from [8, 45, 46] as regards the homology of the little discs operads.

We deal with operads in topological spaces from now on. (Recall that we also use the phrase 'topological operad' for this category of operads.) We use basic concepts of homotopy theory in order to formulate some statements which we obtain for such operads. To be specific, recall that a map of topological spaces $f : X \to Y$ is a weak-equivalence if this map induces a bijection at the level of the sets of connected components $f_* : \pi_0 X \xrightarrow{\simeq} \pi_0 Y$ and an isomorphism of homotopy groups $f_* : \pi_n(X, x_0) \xrightarrow{\simeq} \pi_n(X, f(x_0))$, in every dimension $n \geq 1$ and for any choice of base point $x_0 \in X$. We say that a morphism of operads in topological spaces $\phi : \mathsf{P} \to \mathsf{Q}$ is a weak-equivalence if each component of this morphism $\phi : \mathsf{P}(r) \to \mathsf{Q}(r)$ defines a weak-equivalence of topological spaces.

We consider various categories equipped with such a class of weak-equivalences. We generally use the notation $\xrightarrow{\sim}$ to distinguish this class of morphisms in our category. Recall that a homotopy equivalence of topological spaces is automatically a weak-equivalence. The converse implication holds for cell complexes, but fails in general. In the operad case, we will consider homotopy equivalences in the operadic sense, which are invertible up to homotopy in the category of operads (as we briefly explained in the Mathematical Objectives chapter in the Preliminaries of this volume). Let us observe that a morphism of operads in topological spaces $\phi : \mathsf{P} \to \mathsf{Q}$ whose components $\phi : \mathsf{P}(r) \to \mathsf{Q}(r)$ are homotopy equivalences of spaces for all $r \in \mathbb{N}$ is a weak-equivalence of operads, but is not necessarily a homotopy equivalence of operads, because the homotopy inverses of the maps $\phi : \mathsf{P}(r) \to \mathsf{Q}(r)$ do not necessarily form an operad morphism.

We also consider the homotopy category of the category of operads in topological spaces $\mathtt{Ho}(\mathcal{T}op\,\mathcal{O}p)$ in what follows. We explain the definition of the notion of a homotopy category in detail in §II.1. We can simply assume that $\mathtt{Ho}(\mathcal{T}op\,\mathcal{O}p)$ is the category defined by formally inverting the weak-equivalences of the category of operads for the moment.

4.1. The definition of the little discs operads

The purpose of this section is to recall the definition of the little n-discs operad and of the derived notion of an E_n-operad, as we explained in the introduction of the chapter. To complete our account, we provide a short survey on the initial applications of the little discs operads to the study of iterated loop spaces.

To begin this account, we explain what the little discs are. We assume that n is a positive (finite) integer $n = 1, 2, \dots$ for the moment.

4.1.1. *The little discs.* Let $\mathbb{D}^n$ denote the standard unit n-disc, defined as the space $\mathbb{D}^n = \{(t_1, \dots, t_n) \in \mathbb{R}^n \,|t_1^2 + \dots + t_n^2 \leq 1\}$ inside the Euclidean space $\mathbb{R}^n$. The little n-discs, which give the name of the little n-discs operad, are affine embeddings $c : \mathbb{D}^n \to \mathbb{D}^n$ of the form

$$c(t_1, \dots, t_n) = (a_1, \dots, a_n) + R \cdot (t_1, \dots, t_n),$$

for some translation vector $(a_1, \dots, a_n) \in \mathbb{D}^n$ and a multiplicative scalar $R > 0$ such that $R^2 < 1 - (a_1^2 + \dots + a_n^2)$. In what follows, we use the graphical representation of the image of such maps $c(\mathbb{D}^n)$ to illustrate our constructions on little discs. By abuse of notation, we also set $c = c(\mathbb{D}^n)$ and we use the same letter c to denote both our mapping from $\mathbb{D}^n$ to $\mathbb{D}^n$ and the corresponding subspace in $\mathbb{D}^n$. Let us mention that this space $c = c(\mathbb{D}^n)$ forms an n-disc inside $\mathbb{D}^n$ with $(a_1, \dots, a_n) = c(0, \dots, 0) \in \mathbb{D}^n$ as center and $R > 0$ as radius. Thus, we can retrieve these parameters which

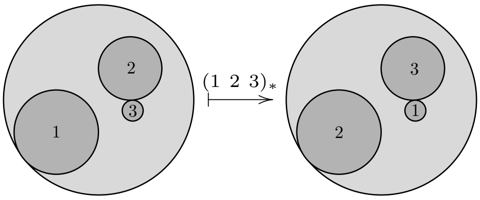


FIGURE 4.1. The representation of an element in the little 2-disc operad, and the action of the cyclic permutation (1 2 3) on this element.

determine our embedding $c : \mathbb{D}^n \to \mathbb{D}^n$ from our picture of this space $c = c(\mathbb{D}^n)$ inside $\mathbb{D}^n$.

The boundary of the unit n-disc $\mathbb{D}^n$, defined as the space of points $(t_1, \dots, t_n) \in \mathbb{D}^n$ such that $t_1^2 + \dots + t_n^2 = 1$, will be denoted by $\partial\,\mathbb{D}^n$. The interior of $\mathbb{D}^n$, defined as the complement of the subspace $\partial\,\mathbb{D}^n$ in $\mathbb{D}^n$ or equivalently as the space of points $(t_1, \dots, t_n) \in \mathbb{D}^n$ such that $t_1^2 + \dots + t_n^2 < 1$, will be denoted by $\mathring{\mathbb{D}}^n$. We define the boundary of a little n-disc c as the subspace $\partial c = c(\partial\,\mathbb{D}^n)$ of $c = c(\mathbb{D}^n)$, and the interior as $\mathring{c} = c(\mathring{\mathbb{D}}^n)$.

4.1.2. *The spaces of little discs.* The *space of little n-discs* $D_n(r)$ is the space of r-tuples $\underline{c} = (c_1, \dots, c_r)$, whose terms c_i are affine embeddings $c_i : \mathbb{D}^n \to \mathbb{D}^n$ of the form considered in §4.1.1 and such that $\mathring{c}_i \cap \mathring{c}_j = \varnothing$ for all pairs $i \neq j$.

The space $D_n(r)$ is equipped with the compact-open topology since the collection of affine maps $\underline{c} = (c_1, \dots, c_r)$ is naturally identified with an element of the mapping space $\mathtt{Map}_{\mathcal{T}op}(\coprod_{i=1}^r \mathbb{D}^n, \mathbb{D}^n)$. Equivalently, we can use some parameters that determine these maps, like the centers $(a_1, \dots, a_n) = c_i(0, \dots, 0) \in \mathbb{D}^n$ and the radius $R > 0$, to specify the topology of $D_n(r)$. The first definition of the topology of $D_n(r)$ is more convenient when we deal with applications of little discs to iterated loop spaces. The second definition is more convenient when we examine the connections of little discs with configuration spaces (see §4.2.1).

Figure 4.1 gives the representation of an element $\underline{c} \in D_n(3)$. In this picture, we use that the definition of $\underline{c}$ as an r-tuple $\underline{c} = (c_1, \dots, c_r)$ is equivalent to the assumption that the little n-discs $c_1, \dots, c_r \subset \mathbb{D}^n$ are indexed by a number $i = 1, \dots, r$. Thus, we regard an element of the space of little n-discs $\underline{c} \in D_n(r)$ as a configuration of r-little discs in $\mathbb{D}^n$, numbered by $i = 1, \dots, r$, and whose interiors do not overlap.

We have a natural map $s_* : D_n(r) \to D_n(r)$, associated to each permutation $s \in \Sigma_r$, such that $s_*(c_1, \dots, c_r) = (c_{s^{-1}(1)}, \dots, c_{s^{-1}(r)})$, for any $\underline{c} = (c_1, \dots, c_r) \in D_n(r)$. Pictorially, this map $s_* : D_n(r) \to D_n(r)$ is given by an obvious reindexing operation of the little discs of a configuration: we apply $s \in \Sigma_r$ to the index $i = 1, \dots, r$ associated to each little n-disc of the configuration $\underline{c} = (c_1, \dots, c_r) \in D_n(r)$ in order to get the picture of the permuted configuration of little n-discs $s_*(\underline{c}) = (c_{s^{-1}(1)}, \dots, c_{s^{-1}(r)}) \in D_n(r)$ (see Figure 4.1 for an example of application of this process).

The collection $D_n = \{D_n(r), r > 0\}$, where we equip each space $D_n(r)$ with this action of the symmetric group Σ_r, forms a symmetric sequence. In certain

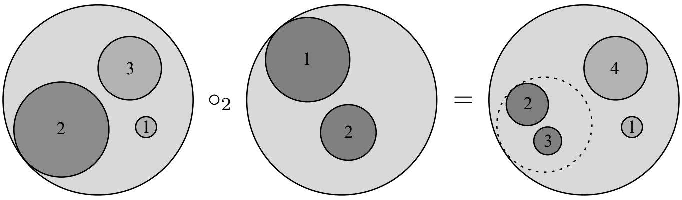


FIGURE 4.2. The composition of elements in the little 2-disc operad.

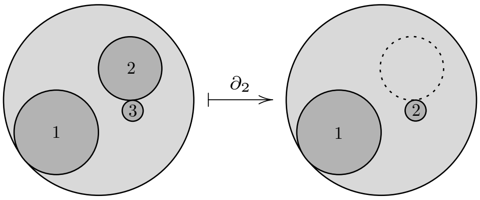


FIGURE 4.3. The representation of a restriction operator in the little 2-disc operad.

applications, we may prefer to consider the symmetric collection associated to D_n, of which terms $D_n(\underline{r})$ are indexed by arbitrary finite sets $\underline{r}$, rather than this symmetric sequence. The elements of a space $D_n(\underline{r})$ in this symmetric collection are identified with collections of little n-discs $\underline{c} = \{c_{i_1}, \dots, c_{i_r}\}$ indexed by the elements of our set $\underline{r} = \{i_1, \dots, i_r\}$ rather than by the numbers $i = 1, \dots, r$. The action of finite set bijections $u \in \mathtt{Mor}_{\mathcal{B}ij}(\underline{r}, \underline{s})$ on the symmetric collection $D_n(\underline{r})$ is the obvious extension of the reindexing process given by the action of permutations.

4.1.3. *The operad of little n-discs.* The component of arity one of the symmetric sequence of little n-discs is equipped with a natural unit element $1 \in D_n(1)$, given by the 1-tuple $1 = (id)$, where we consider the identity mapping $id : \mathbb{D}^n \to \mathbb{D}^n$. In our graphical description of the space of little n-discs, we can also associate this map to the subspace $\mathbb{D}^n = id(\mathbb{D}^n) \subset \mathbb{D}^n$ (the full unit n-disc). We also have natural composition operations $\circ_i : D_n(k) \times D_n(l) \to D_n(k+l-1)$, $i = 1, \dots, k$, which are defined by:

$$\underline{a} \circ_i \underline{b} = (a_1, \dots, a_{i-1}, a_i \circ b_1, \dots, a_i \circ b_l, a_{i+1}, \dots, a_k) \in D_n(k+l-1),$$

for any pair of elements of the little discs spaces $\underline{a} = (a_1, \dots, a_k) \in D_n(r)$ and $\underline{b} = (b_1, \dots, b_l) \in D_n(s)$, where the expression $a_i \circ b_j$ refers to the composite of the maps $a_i : \mathbb{D}^n \to \mathbb{D}^n$ and $b_j : \mathbb{D}^n \to \mathbb{D}^n$. Note that such a composite $a_i \circ b_j$ is still an embedding of the form specified in §4.1.1. Intuitively, the configuration of little n-discs $\underline{a} \circ_i \underline{b} \in D_n(k+l-1)$ is obtained by putting the configuration $\underline{b} = (b_1, \dots, b_l)$ in the ith little disc of the configuration $\underline{a} = (a_1, \dots, a_k)$, as in Figure 4.2. In this process, we apply the affine mapping $a_i : \mathbb{D}^n \to \mathbb{D}^n$, equivalent to the little n-disc $a_i = a_i(\mathbb{D}^n)$, in order to put the little n-disc configuration $\underline{b}$ at the appropriate position and scale.

The (non-unitary) operad of little n-discs D_n, for $n = 1, 2, \dots$, is the operad formed by the symmetric sequence of the little discs spaces $D_n = \{D_n(r), r > 0\}$, such as defined in §4.1.2, together with the unit element $1 = (id) \in D_n(1)$, and the just defined composition operations $\circ_i : D_n(k) \times D_n(l) \to D_n(k+l-1)$, which clearly satisfy the equivariance, unit and associativity axioms of operads, as we can see by a straightforward inspection of our definitions.

4.1.4. *The unitary version of the little n-disc operad.* We assume $D_n(0) = \varnothing$ by convention and we adopt the notation D_n for a non-unitary version of the little n-discs operad in this book (as we explained in the introduction of this chapter). We have, on the other hand, a unitary version of the little n-discs operad whose unique element of arity zero represents, by convention, an empty collection of little n-discs inside the unit n-discs. We use the notation D_{n+} for this operad such that $D_{n+}(0) = *$, where $*$ refers to both the one-point set and the element of this set (which represents the distinguished element of arity zero of our operad).

This *unitary operad of little n-discs* D_{n+} forms a unitary extension of the non-unitary little n-discs operad D_n (in the sense of §1.1.20). Recall that, in such a unitary extension, we associate the partial composites with the extra element of arity zero of our operad $* \in D_{n+}(0)$ to restriction operators $\partial_k : D_n(r) \to D_n(r-1)$ such that $\partial_k(\underline{c}) = \underline{c} \circ_k *$, for any $r > 1$ (see §2.2.1). The image of a configuration of little n-discs $\underline{c} = (c_1, \dots, c_r)$ under this restriction operator $\partial_k : D_{n+}(r) \to D_{n+}(r-1)$ can readily be identified with the $r-1$-tuple $\partial_k(\underline{c}) = (c_1, \dots, \widehat{c_k}, \dots, c_r)$, where the kth term of $\underline{c}$ has been removed (see Figure 4.3 for an example). Recall that the action of permutations and the restriction operators determine a contravariant action of the category of (non-empty finite) ordinal and injections $\Lambda_{>0}$ (see §2.2.2) on the non-unitary operad D_n underlying D_{n+} (see §2.2). The operation $u^* : D_n(l) \to D_n(k)$, which we associate to any injective map $u : \{1 < \dots < k\} \to \{1 < \dots < l\}$ in this category $\Lambda_{>0}$, is simply given by $u^*(\underline{c}) = (c_{u(1)}, \dots, c_{u(k)})$, for any $\underline{c} = (c_1, \dots, c_l) \in D_n(l)$, and the just considered basic restriction operator $\partial_k : D_n(r) \to D_n(r-1)$ corresponds to the increasing map $\partial^k : \{1 < \dots < r-1\} \to \{1 < \dots < r\}$ such that $\partial^k(x) = x$ for $x = 1, \dots, k-1$ and $\partial^k(x) = x+1$ for $x = k, \dots, r-1$ (see §2.2.1).

In the general study of unitary operads in §2.2, we also consider augmentation morphisms which reflect the operadic composites $\epsilon(p) = p(*, \dots, *)$ where we plug the unitary element $*$ in all inputs of our operation $p = p(x_1, \dots, x_r)$. In the case of operads in topological spaces such as the little n-discs operad $P = D_n$, these augmentations reduce to the obvious canonical maps $\epsilon : D_n(r) \to pt$ with values in the one-point set pt.

The unitary operad D_{n+} naturally occurs in applications to iterated loop spaces. We also use the restriction operators $u^* : D_n(s) \to D_n(r)$ in our description of the homology of the little discs operads in the next section.

4.1.5. *The operads of little discs as a nested sequence of operads.* The operads of little discs actually form a nested sequence of topological operads

$$D_1 \hookrightarrow D_2 \hookrightarrow \cdots \hookrightarrow D_n \hookrightarrow \cdots .$$

The embedding $\iota : D_{n-1} \hookrightarrow D_n$ is defined as follows. We first use the map $\iota : \mathbb{D}^{n-1} \to \mathbb{D}^n$ such that $\iota(t_1, \dots, t_{n-1}) = (t_1, \dots, t_{n-1}, 0)$ (the equatorial embedding of the $n-1$-disc $\mathbb{D}^{n-1}$ into the n-disc $\mathbb{D}^n$) to regard $\mathbb{D}^{n-1}$ as a subspace of $\mathbb{D}^n$. To a little $n-1$-disc $c : \mathbb{D}^{n-1} \to \mathbb{D}^{n-1}$, we then associate the little n-disc $\iota(c) : \mathbb{D}^n \to \mathbb{D}^n$ with the same center as c in the equatorial disc $\mathbb{D}^{n-1} \subset \mathbb{D}^n$ and the same radius. Thus, if we assume $c(t_1, \dots, t_{n-1}) = (a_1, \dots, a_{n-1}) + R \cdot (t_1, \dots, t_{n-1})$, then this little n-disc $\iota(c)$ is formally defined by $\iota(c)(t_1, \dots, t_{n-1}, t_n) = (a_1, \dots, a_{n-1}, 0) + R \cdot$

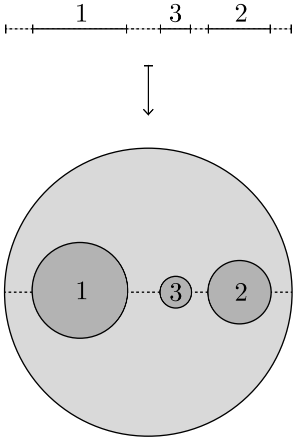


FIGURE 4.4. The image of a little 1-disc configuration in the little 2-disc operad.

$(t_1, \ldots, t_{n-1}, t_n)$. Finally, we set $\iota(\underline{c}) = (\iota(c_1), \ldots, \iota(c_r))$, for any collection of little n-discs $\underline{c} = (c_1, \ldots, c_r) \in \mathsf{D}_{n-1}(r)$ and for each $r > 0$, to define our operad map $\iota : \mathsf{D}_{n-1} \hookrightarrow \mathsf{D}_n$ (see Figure 4.4 for the graphical representation of this map). We readily see that these maps $\iota : \mathsf{D}_{n-1}(r) \hookrightarrow \mathsf{D}_n(r)$ preserve the internal structure of our operads and hence do define operad morphisms $\iota : \mathsf{D}_{n-1} \hookrightarrow \mathsf{D}_n$, for all $n > 1$. We see that these morphisms admit an obvious extension to the unitary version of the little discs operads too. We can check further that our mappings $\iota : \mathsf{D}_{n-1}(r) \hookrightarrow \mathsf{D}_n(r)$, are topological inclusions, for all $r > 0$, and hence, the little $n-1$-discs space $\mathsf{D}_{n-1}(r)$ can really be identified with a subspace of $\mathsf{D}_n(r)$.

We also set $\mathsf{D}_\infty = \operatorname{colim}_n \mathsf{D}_n$ to add a terminal term to our sequence of operads and to define an infinite dimensional version of the little disc operads. We have an obvious extension of this construction in the unitary setting.

To complete our definitions, we record the following result (already mentioned in the chapter introduction) about the operad of little 1-discs D_1 and this infinite dimensional little discs operad D_∞:

PROPOSITION 4.1.6.

(a) *We have $\pi_0\, \mathsf{D}_1(r) = \Sigma_r$, for $r > 0$, and the canonical maps $\mathsf{D}_1(r) \to \pi_0\, \mathsf{D}_1(r)$ define a weak-equivalence of topological operads $\mathsf{D}_1 \xrightarrow{\sim} \mathsf{As}$ between the little 1-disc operad D_1 and the associative operad in sets As (which we view as a discrete topological operad). In the unitary setting, we similarly have $\pi_0\, \mathsf{D}_{1+} \simeq \mathsf{As}_+$.*

(b) *We have $\pi_0\, \mathsf{D}_\infty(r) = *$, for $r > 0$, and the canonical maps $\mathsf{D}_\infty(r) \to \pi_0\, \mathsf{D}_\infty(r)$ define a weak-equivalence of topological operads $\mathsf{D}_\infty \xrightarrow{\sim} \mathsf{Com}$ between D_∞ and the commutative operad in sets Com (which we view as a discrete topological operad). In the unitary setting, we similarly have $\pi_0\, \mathsf{D}_{\infty+} \simeq \mathsf{Com}_+$.*

PROOFS AND EXPLANATIONS. Let P be any operad in spaces. We consider the sets of path-connected components $\pi_0\, \mathsf{P}(r)$ associated to the topological spaces $\mathsf{P}(r)$ underlying this operad P. We immediately see that the collection of these sets $\pi_0\, \mathsf{P}(r)$ inherits an operad structure from P. This assertion formally follows from the obvious observation that the mapping $\pi_0 : X \mapsto \pi_0 X$ defines a symmetric monoidal functor from topological spaces to sets (see §§3.1.1-3.1.4). If we assume that the spaces $\mathsf{P}(r)$ are locally path-connected and we regard the sets $\pi_0\,\mathsf{P}(r)$ as discrete topological spaces (as in the proposition), then we can moreover consider the collection of the canonical maps $\mathsf{P}(r) \to \pi_0\,\mathsf{P}(r)$ to define a morphism of topological operads between P and this operad in sets $\pi_0\,\mathsf{P}$. To establish our proposition, we determine the operad $\pi_0\,\mathsf{P}$ for $\mathsf{P} = \mathsf{D}_1, \mathsf{D}_\infty$ and we check that the morphism $\mathsf{P} \to \pi_0\,\mathsf{P}$ defines a weak-equivalence of topological operads in the case of these operads $\mathsf{P} = \mathsf{D}_1, \mathsf{D}_\infty$. This claim is equivalent to the assertion that the path-connected components of the spaces $\mathsf{P}(r) = \mathsf{D}_1(r), \mathsf{D}_\infty(r)$ are weakly-contractible.

In the case $\mathsf{P} = \mathsf{D}_1$, the embedding of a collection of little intervals (of little 1-discs) $\underline{c} = (c_1, \dots, c_r) \in \mathsf{D}_1(r)$ in the one dimensional space $\mathbb{D}^1$ determines an order relation between the intervals. To be explicit, we set $c_i < c_j$ when we have $c_i(0) < c_j(0)$ (equivalently, when we have $c_i(s) \leq c_j(t)$ for all $s, t \in \mathbb{D}^1$). The obtained ordering $c_{i_1} < \dots < c_{i_r}$ determines a permutation $(i_1, \dots, i_r)$ of the indices $(1, \dots, r)$ which we associate to the configuration of little 1-discs $(c_1, \dots, c_r)$. (To give an example, for the little configuration of Figure 4.4, we obtain the permutation $(1, 3, 2)$.)

This assignment gives a map $p : \mathsf{D}_1(r) \to \Sigma_r$. We can easily define a map in the converse direction $i : \Sigma_r \to \mathsf{D}(r)$ such that $pi = id$ and a homotopy $ip \sim id$ to establish that this map $p : \mathsf{D}_1(r) \to \Sigma_r$ is a homotopy equivalence of topological spaces, for each arity $r > 0$. From this verification, we conclude that we have an identity $\pi_0\,\mathsf{D}_1(r) = \Sigma_r$ and that the path-connected components of the space $\mathsf{D}_1(r)$ are contractible, as asserted. Recall that the permutation groups $\mathsf{\Pi}(r) = \Sigma_r$ define the underlying collection of the associative operad As when we work in the category of sets. By inspection of definitions, we also easily check that the relation $\pi_0\,\mathsf{D}_1 = \mathsf{As}$ holds as an identity of operads. We easily check that we can extend this relation to the unitary operad $\pi_0\,\mathsf{D}_{1+}$ when we add the one-point set $\mathsf{D}_{1+}(0) = *$ to the components of the little 1-discs operad D_{1+}.

We refer to [28, Lemma 2.50] for the detailed proof that the spaces $\mathsf{D}_\infty(r)$ are contractible. We then have $\pi_0\,\mathsf{D}_\infty(r) = *$, for each $r > 0$, where we use the notation $*$ for the one-point set (viewed as the terminal object of the category of sets). Recall that the commutative operad in sets Com is also given by $\mathsf{Com}(r) = pt = *$, for all $r > 0$. In the case of one-point sets, the existence of the relation $\pi_0\,\mathsf{D}_\infty(r) = *$ for each $r > 0$ automatically implies that the identity $\pi_0\,\mathsf{D}_\infty = \mathsf{Com}$ holds in the category of operads. We similarly get an identity of operads $\pi_0\,\mathsf{D}_{\infty+} = \mathsf{Com}_+$ when we consider the unitary extension of the operad D_∞. □

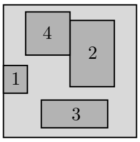
The operads D_n, where $1 < n < \infty$, are not weakly-equivalent to discrete operads (unlike D_1 and D_∞). This observation can be deduced from the homology computations of the next section. Nonetheless, we readily see that the spaces $\mathsf{D}_n(r)$ are path-connected for $n > 1$. The identity of the theorem $\pi_0\,\mathsf{D}_n = \mathsf{Com}$ in assertion (b) accordingly holds as soon as $n > 1$, and we similarly have the relation $\pi_0\,\mathsf{D}_{n+} = \mathsf{Com}_+$ when we consider the unitary extension of the little n-discs operad D_n.

4.1.7. *Relationship with the little n-cubes operad.* The little n-cubes operad, denoted by C_n, is a variant of the little n-discs operad D_n of which elements consist of configurations of cubes (rather than discs) inside a fixed unit cube. To be precise, we first define a little cube c as a map $c : [0,1]^n \hookrightarrow [0,1]^n$ of the form

$$c(t_1, \dots, t_n) = (a_1 + (b_1 - a_1)t_1, \dots, a_n + (b_n - a_n)t_n),$$

for each point $(t_1, \dots, t_n) \in [0,1]^n$, where $(a_1, \dots, a_n), (b_1, \dots, b_n) \in [0,1]^n$ are given parameters such that $0 \leq a_k < b_k \leq 1$, for each $k = 1, \dots, n$. The space $c = c([0,1]^n)$ accordingly defines an n-dimensional cube in $[0,1]^n$ with a non-empty interior $\mathring{c}$ and faces parallel to the faces of the ambient unit cube. The n-tuples $(a_1, \dots, a_n), (b_1, \dots, b_n) \in [0,1]^n$ represent the extremal vertices of this little cube.

The spaces $\mathcal{C}_n(r)$, which form the little n-cubes operad $\mathcal{C}_n$, consist of r-tuples of little n-cubes $\underline{c} = (c_1, \dots, c_r)$ with disjoint interiors. Thus, a typical element of the little n-cubes operad is represented by a picture of the following form:



.

The definition of the symmetric structure and of the composition operations of the operad of little n-cubes follows from an obvious variation of the definition of the symmetric structure and of the composition operations of the operad of little n-discs.

The operad of little n-discs is weakly-equivalent to the operad of little n-cubes as an operad in topological spaces (we refer to [23, 169] for proofs of this statement). The operads of little cubes can be used to define models of suspension maps in iterated loop space theory (see [140, Proposition 5.4]), while the operads of little discs can not. But the little discs operads make some of our constructions more natural and we therefore prefer to use this model.

4.1.8. *Iterated loop spaces.* The little n-discs are used to represent composition schemes of continuous maps $\alpha : \mathbb{D}^n \to X$ with values in a space X equipped with a fixed base point x_0 and such that $\alpha\,|_{\partial\,\mathbb{D}^n} = x_0$. The space formed by these maps

$$\Omega^n X = \{\alpha \in \mathtt{Map}_{\mathcal{T}op}(\mathbb{D}^n, X) \mid \alpha\,|_{\partial\,\mathbb{D}^n} = x_0\},$$

together with the topology inherited from $\mathtt{Map}_{\mathcal{T}op}(\mathbb{D}^n, X)$, is one of the possible equivalent definitions of the n-fold loop space associated to X. In the case $n = 1$, we retrieve with this construction the basic definition of the space of loops $\alpha : \mathbb{D}^1 \to X$ based at x_0. This 1-fold loop space is more usually denoted by ΩX (with the dimension exponent withdrawn from the notation).

The pairs (X, x_0), consisting of a topological space X together with a distinguished base point $x_0 \in X$, form the objects of the category of pointed spaces $\mathcal{T}op_*$. The morphisms of this category are the morphisms of topological spaces that preserve base points. By abuse of notation, we generally use the notation of the space X rather than the pair (X, x_0) to denote an object of this category $\mathcal{T}op_*$. In general, we also use the notation $*$ to refer to the base point which we associate to any such space $X \in \mathcal{T}op_*$ (we only make this point explicit when this precision is necessary). When we use this convention, we abusively assume that our space X, regarded as an object of the category of pointed spaces $\mathcal{T}op_*$, comes together with a base point which is part of its internal structure.

The loop space $\Omega^n X$ is equipped with a natural base point, which is defined by the constant map $\alpha \equiv x_0$ with values in the base point x_0 of the space X. The assignment $\Omega^n : X \mapsto \Omega^n X$ accordingly gives a functor $\Omega^n : \mathcal{T}op_* \to \mathcal{T}op_*$ with values in the category of pointed spaces $\mathcal{T}op_*$. The n-fold loop space functor $\Omega^n : \mathcal{T}op_* \to \mathcal{T}op_*$ can formally be identified with the n-fold composite of the basic single loop space functor $\Omega : \mathcal{T}op_* \to \mathcal{T}op_*$. This observation motivates the name 'iterated loop space' for the spaces of this form $Y = \Omega^n X$.

4.1.9. *Operations on iterated loop spaces associated to little discs.* Each element $\underline{c} \in \mathsf{D}_{n+}(r)$ in the unitary operad of little n-discs D_{n+} determines an r-fold operation $\underline{c} : \Omega^n X \times \cdots \times \Omega^n X \to \Omega^n X$. Let us recall this construction.

Let $\underline{c} = (c_1, \ldots, c_r) \in \mathsf{D}_{n+}(r)$. The assumption that each little disc c_i has a radius > 0 in the definition of the little n-discs operad implies that the map $c_i : \mathbb{D}^n \to \mathbb{D}^n$ induces an affine isomorphism between $\mathbb{D}^n$ and $c_i = c_i(\mathbb{D}^n)$. To a collection of n-fold loop space elements $\alpha_1, \ldots, \alpha_r \in \Omega^n X$, we then associate the map $\alpha : \mathbb{D}^n \to X$ such that

$$\alpha(t_1, \ldots, t_n) = \begin{cases} \alpha_i(c_i^{-1}(t_1, \ldots, t_n)), & \text{when } (t_1, \ldots, t_n) \text{ belongs to the image} \\ & \text{of a small disc } c_i = c_i(\mathbb{D}^n), \\ * \quad (\text{the base point of } X), & \text{otherwise.} \end{cases}$$

The assumption $\alpha_i \mid_{\partial \mathbb{D}^n} = *$ for the elements of $\Omega^n X$ ensures that this map is well-defined and continuous over $\mathbb{D}^n$. Moreover, we clearly have $\alpha \mid_{\partial \mathbb{D}^n} = *$. Thus, the map $\alpha : \mathbb{D}^n \to X$ defines an element of the n-fold loop space $\alpha = \underline{c}(\alpha_1, \ldots, \alpha_r) \in \Omega^n X$ naturally associated to $\alpha_1, \ldots, \alpha_r \in \Omega^n X$ and this mapping $\underline{c} : (\alpha_1, \ldots, \alpha_r) \mapsto \underline{c}(\alpha_1, \ldots, \alpha_r)$ gives the operation $\underline{c} : \Omega^n X \times \cdots \times \Omega^n X \to \Omega^n X$ associated to our operad element $\underline{c} \in \mathsf{D}_{n+}(r)$.

Intuitively, the composite $\alpha = \underline{c}(\alpha_1, \ldots, \alpha_r) : \mathbb{D}^n \to \Omega^n X$ is obtained by applying the maps α_i to the little n-discs of the configuration $\underline{c}$ by using the mappings $c_i^{-1} : c_i(\mathbb{D}^n) \xrightarrow{\simeq} \mathbb{D}^n$ associated to our little n-discs $c_i = c_i(\mathbb{D}^n)$. The complement of the little n-discs inside $\mathbb{D}^n$ is sent to the base point.

We easily see that the action of the symmetric groups and the composition products of the little n-discs operad reflect the action of the symmetric groups and the composition of such operations on n-fold loop spaces. We accordingly have the following statement:

PROPOSITION 4.1.10. *The construction of* §4.1.9 *provides any n-fold loop space $\Omega^n X$ with an action of the (unitary) little n-discs operad D_{n+} so that $\Omega^n X$ forms an algebra over this operad.* □

To summarize, this proposition gives the construction of an algebraic structure (an algebra over D_{n+}) from a topological object (an n-fold loop space). The following recognition theorem, which gave the first motivation for the introduction of operads in topology, proves that this algebraic structure provides a faithful picture of our object:

THEOREM 4.1.11 (J. Boardman, R. Vogt [27, 28], P. May [140]). *Let Y be a space equipped with an action of the (unitary) operad of little n-discs D_{n+}, so that this space Y forms a D_{n+}-algebra (in the category of topological spaces). We then have a pointed space $B_n Y$, naturally associated to Y, together with a chain of weak-equivalences of D_{n+}-algebras $\Omega^n B_n Y \xleftarrow{\sim} \cdot \xrightarrow{\sim} Y$, which connect Y to the n-fold loop space on $B_n Y$ when $\pi_0 Y$ forms a group.*

EXPLANATIONS AND REFERENCES. Recall that we have $\pi_0 \mathsf{D}_{1+} = \mathsf{As}_+$ and $\pi_0 \mathsf{D}_{n+} = \mathsf{Com}_+$ for $n > 0$. We deduce from this result that the set of connected components $\pi_0 Y$ of any space Y equipped with an action of the operad D_{n+} inherits a natural monoid structure. We just consider the operation $\mu = \underline{c} : \pi_0 Y \times \pi_0 Y \to$

$\pi_0 Y$ induced by the natural action $\underline{c} : Y \times Y \to Y$ of any element of arity two $\underline{c} \in D_{n+}(2)$ of our operad on our space Y to define the multiplication operation of this monoid structure on $\pi_0 Y$. (We go back to this construction in the next paragraph.) In our statement, we precisely assume that every element in this monoid is invertible, so that $\pi_0 Y$ forms a group. In the next paragraph, we will see that, in the case $Y = \Omega^n X$, we can identify the set $\pi_0 Y$ with the nth homotopy group of the space X. Thus, we get that $\pi_0 Y$ forms necessarily a group when $Y = \Omega^n X$, and this group condition is therefore necessary in our statement. We can actually associate a space $\Omega^n B_n Y$ to any space Y equipped with an action of the operad D_{n+}. We just get that this space forms a group completion of Y when $\pi_0 Y$ does not form a group (we refer to [3] for the definition of the notion of group completion in topology and for further references on this subject).

The references cited in our statement provide different approaches of this theorem. The arguments of [140] rely on an approximation theorem (see Theorem 2.7 in *loc. cit.*) which asserts that free algebras over D_{n+} are weakly-equivalent to iterated loop spaces of suspensions $\Omega^n \Sigma^n X$. The method of this reference [140] returns the n-fold delooping $B_n Y$ in one step whereas the arguments of [27, 28] rely on an inductive delooping process. □

To complete the account of this section, we explain that the action of the little n-discs operad on n-fold loop spaces represents a fine level of homotopical structures which underlies the classical definition of the homotopy groups of pointed spaces. We will not go much further into the applications of operads to iterated loop spaces. We refer to the literature, notably the already cited monographs [28, 140], for a thorough study of this subject.

4.1.12. *Basic motivations: the definition of homotopy groups.* The nth homotopy group $\pi_n(X, x_0)$ of a space X equipped with a base point $x_0 \in X$ can be defined as the set of homotopy classes of maps $u : \mathbb{D}^n \to X$ which are identical to the base point $x_0 \in X$ on $\partial \mathbb{D}^n \subset \mathbb{D}^n$. Simply recall that a homotopy between any such maps $u^0, u^1 : \mathbb{D}^n \to X$ consists of a map $h : [0,1] \times \mathbb{D}^n \to \mathbb{D}^n$ such that $h(0,\cdot) = u^0$, $h(1,\cdot) = u^1$ and $h(s,\cdot) \mid_{\partial \mathbb{D}^n} = x_0$, for all $s \in [0,1]$.

The group $\pi_1(X, x_0)$ is identified with the fundamental group of X because a based loop on the pointed space X is nothing but a map $\alpha : \mathbb{D}^1 \to X$ such that $\alpha \mid_{\partial \mathbb{D}^1} = x_0$ and we have a similar identification for homotopies. Recall that the fundamental group $\pi_1(X, x_0)$ is not abelian in general whereas the higher homotopy groups $\pi_n(X, x_0)$, $n \geq 2$, are. We aim to revisit the definition of the group structure on $\pi_n(X, x)$ from the operadic viewpoint.

We have a formal identity between the group $\pi_n(X, x_0)$ and the set of path-connected components of the n-fold loop space $\Omega^n X$. We can moreover identify the usual multiplication operation of this group $\pi_n(X, x_0)$ with an operation $\mu : \Omega^n X \times \Omega^n X \to \Omega^n X$, formed at the loop space level, which we deduce from the action of the little n-discs operad on the space $\Omega^n X$. We actually need to consider a model of the n-fold loop space $\Omega^n X$ as a set of maps on a cube $[0,1]^n$ instead of a disc $\mathbb{D}^n$, and to deal with an action of the operad of little n-cubes on $\Omega^n X$ instead of an action of the operad of little n-discs, in order to retrieve the exact picture of the mutiplication operation associated to this group $\pi_n(X, x_0)$ (we refer to [182, §IV] for this picture). We get an equivalent result, however, if we continue with the action of the little n-discs operad on our disc model of the n-fold loop space $\Omega^n X$. Then we precisely consider the operation $\mu = \underline{c} : \Omega^n X \times \Omega^n X \to \Omega^n X$, determined by the action of any element of arity 2 of the little n-discs operad $\underline{c} \in D_n(2)$ on our space $\Omega^n X$.

If we assume $n > 1$, then all operations $\underline{c} : \Omega^n X \times \Omega^n X \to \Omega^n X$ associated to such a little n-disc configuration $\underline{c} \in D_n(2)$ are the same up to homotopy. Indeed, since $D_n(2)$ is path-connected, any pair of configurations of little n-discs $\underline{c}^0, \underline{c}^1 \in D_n(2)$ are connected by a path $\underline{c}^s$, $s \in [0,1]$, in the space $D_n(2)$ and the collection of operations $\underline{c}^s : \Omega^n X \times \Omega^n X \to \Omega^n X$, $s \in [0,1]$, associated to this path determines a homotopy between the operations associated to $\underline{c}^0$ and $\underline{c}^1$ on $\Omega^n X$. This argument line also implies that the multiplication operation defined by an element $\underline{c} \in D_n(2)$ is homotopy equivalent to the multiplication operation determined by the image of this element under the action of the transposition on the space of little 2-discs $(1\ 2) \cdot \underline{c} \in D_n(2)$. Hence, we obtain that the multiplication of $\pi_n(X, x_0)$ is equal to the opposite operation, and as a consequence, that the group $\pi_n(X, x_0)$ is commutative.

In the case of $n = 1$, we have two choices of multiplications in homotopy, corresponding to the two path-connected components of the space $D_1(2)$, and these multiplications are transposed to each other. Thus we retrieve the non-commutativity of the fundamental group $\pi_1(X, x_0)$ from the identity $\pi_0 D_1(2) = As(2)$. The homotopy, which gives the associativity of the multiplication on homotopy groups, can also be defined by a one parameter family of triple operations $\mu_3^s(\cdot,\cdot,\cdot) : \Omega^n X \times \Omega^n X \times \Omega^n X \to \Omega^n X$, $s \in [0,1]$, associated to a path in the little n-discs space $D_n(3)$. The inversion operation is apart because the homotopies which give this operation are not included in the structure determined by the operad of little n-discs.

By pushing our operadic analysis further, we can regard the associativity (respectively, commutativity) of the group structure on $\pi_n(X, x_0)$ as a consequence of the operad identity $\pi_0 D_{1+} = As_+$ (respectively, $\pi_0 D_{n+} = Com_+$ for $n > 1$). We mention after Proposition 4.1.6 that the operads D_{n+} are not componentwise contractible for $1 < n < \infty$. We will check that the space $D_n(2)$ is homotopy equivalent to a sphere $\mathbb{S}^{n-1}$. We will also see that each space $D_n(r)$ has a non-trivial homology (we tackle this subject in the next section). Fine structures arising from the operad little n-discs operad can be revealed by studying homology groups $\mathrm{H}_*(\Omega^n X, \Bbbk)$ rather than restricting our consideration to the set of connected components $\pi_n(X, x_0) = \pi_0(\Omega^n X)$. The monograph [45] gives a complete description of these homological structures in the case where the coefficient ring of the homology is a field.

4.1.13. *The notion of an E_n-operad.* To conclude this section we just make explicit the formal definition of the notion of an E_n-operad. In brief, a non-unitary (respectively, unitary) E_n-operad in topological spaces is a topological operad $P \in \mathcal{T}op\,\mathcal{O}p$ which is isomorphic to the operad of little n-discs D_n (respectively, D_{n+}) in the homotopy category of topological operads $\mathrm{Ho}(\mathcal{T}op\,\mathcal{O}p)$. Equivalently, we assume that P is connected to D_n by a chain of morphisms of topological operads

$$P \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xrightarrow{\sim} D_n$$

which form weak-equivalences of spaces aritywise.

The existence of a model structure on $\mathcal{T}op\,\mathcal{O}p$ (see §II.1) implies that such a chain can be reduced to a zigzag of two weak-equivalences

$$P \xleftarrow{\sim} \cdot \xrightarrow{\sim} D_n\,.$$

The same observations hold in the unitary context.

In many applications, authors take the additional assumption that E_n-operads are cofibrant as symmetric sequences (we explain the definition of this concept in §II.1.4 and in §II.8.1) in order to ensure that the category of algebras associated with different models of E_n-operads are Quillen equivalent (see §II.1.4). The interesting reader can notice that all instances of E_n-operads considered in this work (including the reference model of little n-discs by the way) are cofibrant as symmetric sequences. But we will not pay attention to this technical point. Furthermore, as soon as we consider homotopy automorphism groups, we need to deal with cofibrant models of E_n-operads and this requirement is actually stronger than being cofibrant as a symmetric sequence (see for instance [25]).

In the case where P is cofibrant, the model category axioms imply that we can reduce our chain of weak-equivalences between P and D_n to a single morphism $\mathsf{P} \xrightarrow{\sim} \mathsf{D}_n$, but we usually do not need to make this weak-equivalence explicit too.

In the case $n = 1, \infty$, the result of Proposition 4.1.6 immediately implies:

Proposition 4.1.14.

(a) *A non-unitary operad P whose underlying spaces $\mathsf{P}(r)$ are locally path-connected is an E_1-operad if and only if we have $\pi_0 \mathsf{P}(r) = \Sigma_r$, for all $r > 0$, and the canonical maps $\mathsf{P}(r) \to \pi_0 \mathsf{P}(r)$ define a weak-equivalence of topological operads $\mathsf{P} \xrightarrow{\sim} \mathsf{As}$, where we regard the associative operad As, formed in the category of sets, as a discrete topological operad. A similar result holds in the unitary context, with the non-unitary associative operad As replaced by the unitary one As_+.*

(b) *A non-unitary operad P whose underlying spaces $\mathsf{P}(r)$ are locally path-connected is an E_∞-operad if and only if we have $\pi_0 \mathsf{P}(r) = *$, for all $r > 0$, and the canonical maps $\mathsf{P}(r) \to \pi_0 \mathsf{P}(r)$ define a weak-equivalence of topological operads $\mathsf{P} \xrightarrow{\sim} \mathsf{Com}$, where we regard the commutative operad Com, formed in the category of sets, as a discrete topological operad. A similar result holds in the unitary context, with the non-unitary commutative operad Com replaced by the unitary one Com_+.* □

Since the operads D_n are not equivalent to discrete operads for $1 < n < \infty$, we do not have such a simple characterization of E_n-operads in general. On the other hand, the existence of weak-equivalences $\mathsf{P} \xleftarrow{\sim} \cdot \xrightarrow{\sim} \mathsf{D}_n$ implies that E_n-operads have the same homology as the operad of little n-discs (and similarly in the unitary context). This observation gives a simple necessary condition which E_n-operads have to satisfy and we study the homology of E_n-operads in the next section.

4.2. The homology (and the cohomology) of the little discs operads

The goal of this section is to give a description of the homology of the little n-discs operad D_n, and as a byproduct of any E_n-operad.

We fix a ground ring $\Bbbk$ and we consider the homology $\mathtt{H}_*(X) = \mathtt{H}_*(X, \Bbbk)$ and the cohomology $\mathtt{H}^*(X) = \mathtt{H}^*(X, \Bbbk)$ with coefficients in this ring all through this section. We are going to use that the homology of a space inherits a coalgebra structure as soon as this homology $\mathtt{H}_*(X) = \mathtt{H}_*(X, \Bbbk)$ forms a free module over the ground ring. We can see that the homology of the space of little n-discs $\mathtt{H}_*(\mathsf{D}_n(r))$ does admit a free module structure, for any choice of ground ring $\Bbbk$, and hence, does inherit such a coalgebra structure, for any arity $r > 0$.

For simplicity, we make explicit some conditions on the ground ring which ensure that our general statements hold, but we do not insist on the optimal assumptions which we could use in the case of the little discs operads. In fact, we do not need more than the case of a field for the subsequent applications of the homology of operads which we consider in this book. In our study of the rational homotopy of operads, for instance, we only consider the case of the homology with coefficients in the field of rational numbers $\Bbbk = \mathbb{Q}$.

We naturally deal with objects defined in the category of graded modules, denoted by $gr\,\mathcal{M}od$. We have a symmetric monoidal structure on the category of graded modules (see the appendix section §4.4 for a reminder on the definition of this symmetric monoidal structure). The homology of a space forms a counitary cocommutative coalgebra in the symmetric monoidal category of graded modules (at least when we take a field as coefficient ring as we just explained). When we take the homology of a topological operad, we get an operad in the category of counitary cocommutative coalgebras in graded modules. We also speak about *graded Hopf operads* for short in this case. The main goal of this section is to determine the graded Hopf operad structure of the homology of the little discs operads.

We generally use the prefix graded to refer to any category of structured objects defined within the category of graded modules. For instance, we speak about *graded counitary cocommutative coalgebras*, *graded operads*, ... Recall that we also use the prefix 'Hopf' to refer to any category of structured objects defined in a category of counitary cocommutative coalgebras (see §3.2). We combine both conventions when we deal with operads in the category of counitary cocommutative coalgebras in graded modules.

In mathematical formulas, we similarly use the notation $gr\,\mathcal{C}om^c_+$ (rather than $gr\,\mathcal{M}od\,\mathcal{C}om^c_+$) for the category of graded counitary cocommutative coalgebras, we use the notation $gr\,\mathcal{O}p$ (rather than $gr\,\mathcal{M}od\,\mathcal{O}p$) for the category of graded operads, and we use the notation $gr\,\mathcal{H}opf\,\mathcal{O}p$ for the category of graded Hopf operads. In Proposition 3.2.4, we observed that Hopf operads can be identified with counitary cocommutative coalgebras in operads. In the graded context, this identity reads $gr\,\mathcal{H}opf\,\mathcal{O}p = gr\,\mathcal{C}om^c_+\,\mathcal{O}p = gr\,\mathcal{O}p\,\mathcal{C}om^c_+$.

The homology of the little n-discs operad is trivial in degree $* > 0$ when $n = 1, \infty$ since the topological spaces underlying these operads have contractible connected components (and similarly in the unitary context). Therefore we generally assume $1 < n < \infty$ in what follows.

In a first stage, we forget about operadic composition structures, and we give a description of the cohomology of each space $\mathsf{D}_n(r)$ as a graded unitary commutative algebra. We then work with the configuration spaces $\mathsf{F}(\mathring{\mathbb{D}}^n, r)$ which are homotopy equivalent to the spaces of little n-discs $\mathsf{D}_n(r)$ and which carry all the necessary information for this homology computation. We recall the definition of these spaces first.

4.2.1. *Configuration spaces.* The *space of configurations* of r points in a topological space $M \in \mathcal{T}op$ is defined by:

$$\mathsf{F}(M, r) = \{(a_1, \ldots, a_r) \in M^r | a_i \neq a_j \text{ for all pairs } i \neq j\},$$

for any $r > 0$. In what follows, we mostly consider the configuration space associated to the open n-discs $M = \mathring{\mathbb{D}}^n$. The configuration space associated to the Euclidean space $M = \mathbb{R}^n$ is more usually considered in the literature on the little discs operads, but the standard homeomorphism between the Euclidean space and the open n-disc induces a homeomorphism at the configuration space level. Therefore, we can deduce results involving one of these configuration spaces from results involving the other.

To an element of the little n-discs operad $\underline{c} \in D_n(r)$, we associate the configuration of points $(c_1(0), \dots, c_r(0)) \in F(\mathring{\mathbb{D}}^n, r)$ defined by the centers of the little n-discs c_i of our collection $\underline{c} = (c_1, \dots, c_r)$. We then get a map $\omega : D_n(r) \to F(\mathring{\mathbb{D}}^n, r)$, which we call the *disc center mapping*. We have the following result:

Proposition 4.2.2. *The disc center mapping defines a homotopy equivalence of topological spaces $\omega : D_n(r) \xrightarrow{\sim} F(\mathring{\mathbb{D}}^n, r)$, for each $r > 0$.*

Proof. Exercise or see [140, §4]. ☐

Though we can not define general operadic composition operations on configuration spaces, we do have restriction operators (and augmentation maps) which model the composition operations with an operation of arity zero in a unitary operad. In what follows, we use our notion of augmented non-unitary Λ-sequence to formalize this structure (see §2.2). Recall that the little n-discs operad D_n inherits its restriction operators (and augmentation maps) which correspond to the composition operations with the distinguished operation of arity zero in the unitary extension of this operad (see §4.1.4). We have the following statement:

Proposition 4.2.3. *The collection of configuration spaces $F(\mathring{\mathbb{D}}^n, r)$, $r > 0$, is equipped with the structure of an (augmented) non-unitary Λ-sequence and the collection of disc center mappings $\omega : D_n(r) \xrightarrow{\sim} F(\mathring{\mathbb{D}}^n, r)$, $r > 0$, define a morphism in the category of (augmented) non-unitary Λ-sequences.*

Explanations. The action of an injective map $u : \{1 < \dots < k\} \to \{1 < \dots < l\}$ on an element $\underline{a} = (a_1, \dots, a_l) \in F(\mathring{\mathbb{D}}^n, l)$ is defined by $u^*(\underline{a}) = (a_{u(1)}, \dots, a_{u(k)})$. This construction clearly gives an action of our category Λ on the collection of configuration spaces and we have an obvious canonical augmentation $\epsilon : F(\mathring{\mathbb{D}}^n, r) \to pt$, for each $r > 0$, so that the collection $F(\mathring{\mathbb{D}}^n, -) = \{F(\mathring{\mathbb{D}}^n, r), r > 0\}$ forms an augmented non-unitary Λ-sequence as stated in the proposition. We readily see that the disc center mappings $\omega : D_n(r) \xrightarrow{\sim} F(\mathring{\mathbb{D}}^n, r)$, $r > 0$, define a morphism of augmented non-unitary Λ-sequences, when we equip the collection $F(\mathring{\mathbb{D}}^n, -)$ with this Λ-sequence structure and we equip the collection of the little n-discs spaces $D_n(r)$, $r > 0$, with the Λ-sequence structure of §4.1.4, ☐

We now examine the topological structure of the configuration spaces $F(\mathring{\mathbb{D}}^n, r)$ with the aim of determining the cohomology of these spaces. We begin with the following simple observation:

Proposition 4.2.4. *We have a homotopy equivalence $F(\mathring{\mathbb{D}}^n, 2) \xrightarrow{\sim} \mathbb{S}^{n-1}$ between the configuration space of two points $F(\mathring{\mathbb{D}}^n, 2)$ and the $n-1$-sphere $\mathbb{S}^{n-1}$.*

Proof. We easily check that the mapping which associates the normalized vector $\overrightarrow{ab}/||\overrightarrow{ab}|| \in \mathbb{S}^{n-1}$ to each pair $(a, b) \in F(\mathring{\mathbb{D}}^n, 2)$ defines a homotopy equivalence. ☐

4.2.5. *The definition of fundamental classes.* For $n > 1$, the result of Proposition 4.2.4 implies that we have an identity:

$$\mathrm{H}_*(F(\mathring{\mathbb{D}}^n, 2)) = \mathrm{H}_*(\mathbb{S}^{n-1}) = \begin{cases} \Bbbk, & \text{for } * = 0, n-1, \\ 0, & \text{otherwise,} \end{cases}$$

and we have a similar result for the cohomology $\mathrm{H}^*(F(\mathring{\mathbb{D}}^n, 2))$. We use the notation $[\mathbb{S}^{n-1}]$ for the fundamental class of the sphere (equipped with a suitable orientation) which defines a generator of the module $\mathrm{H}_{n-1}(\mathbb{S}^{n-1})$ and which we transport to $\mathrm{H}_*(F(\mathring{\mathbb{D}}^n, 2))$ by using the homotopy equivalence of Proposition 4.2.4. We will also use the notation $[pt]$ for the canonical generator of the degree 0 component of the homology module $H_*(F(\mathring{\mathbb{D}}^n, 2))$. In the cohomological context, we consider the element $\omega \in \mathrm{H}^{n-1}(F(\mathring{\mathbb{D}}^n, 2))$, dual to $[\mathbb{S}^{n-1}]$, in order to obtain a canonical generator of $\mathrm{H}^{n-1}(F(\mathring{\mathbb{D}}^n, 2))$.

Let now $r \geq 2$. For each pair $1 \leq i < j \leq r$, we consider the map $\phi_{ij} : F(\mathring{\mathbb{D}}^n, r) \to F(\mathring{\mathbb{D}}^n, 2)$ such that $\phi_{ij}(a_1, \dots, a_r) = (a_i, a_j)$ and we set $\omega_{ij} = \phi_{ij}^*(\omega)$ for the image of $\omega \in \mathrm{H}^{n-1}(F(\mathring{\mathbb{D}}^n, 2))$ under the morphism $\phi_{ij}^* : \mathrm{H}^{n-1}(F(\mathring{\mathbb{D}}^n, 2)) \to \mathrm{H}^{n-1}(F(\mathring{\mathbb{D}}^n, r))$ induced by this map. Observe that ϕ_{ij} is the restriction operator associated to the injection $\rho_{ij} : \{1 < 2\} \to \{1 < \dots < r\}$ such that $\rho_{ij}(1) = i$ and $\rho_{ij}(2) = j$. We can use the same construction to associate a class ω_{ij} to any pair (i, j) (not necessarily well-ordered). We actually have $\omega_{ij} = (-1)^n \omega_{ji}$, for any pair $i \neq j$, since the change of the ordering (i, j) in the definition of our map ϕ_{ij} corresponds to the application of an antipode map on the sphere in the correspondence of Proposition 4.2.4.

Let $\mathbb{S}(\omega_{ij}, i < j)$ be the graded symmetric algebra generated by the classes ω_{ij} in degree $n-1$. We have the following result:

THEOREM 4.2.6 (See V. Arnold [8], F. Cohen [45]). *Let $n > 1$. Let $r \geq 2$.*

(a) *In $\mathrm{H}^*(F(\mathring{\mathbb{D}}^n, r))$, we have the relation $\omega_{ij}^2 = 0$, for each pair $i < j$, as well as the relation $\omega_{ij}\omega_{jk} - \omega_{ik}\omega_{jk} - \omega_{ij}\omega_{ik} = 0$, for each triple $i < j < k$.*

(b) *The morphism $\mathbb{S}(\omega_{ij}, i < j) \to \mathrm{H}^*(F(\mathring{\mathbb{D}}^n, r))$, which maps the generator ω_{ij} to the corresponding cohomology class in $\mathrm{H}^*(F(\mathring{\mathbb{D}}^n, r))$, induces an isomorphism*

$$\frac{\mathbb{S}(\omega_{ij}, i < j)}{(\omega_{ij}^2, \omega_{ij}\omega_{jk} - \omega_{ik}\omega_{jk} - \omega_{ij}\omega_{ik})} \xrightarrow{\simeq} \mathrm{H}^*(F(\mathring{\mathbb{D}}^n, r)),$$

*when we form the quotient of the symmetric algebra $\mathbb{S}(\omega_{ij}, i < j)$ by the ideal generated by the relations of (*a*).* □

This theorem is established in the cited references by using Euclidean spaces $\mathbb{R}^n$ instead of open discs $\mathring{\mathbb{D}}^n$. This choice does not change the result since the homeomorphism between the Euclidean n-space $\mathbb{R}^n$ and the open n-disc $\mathring{\mathbb{D}}^n$ induces a homeomorphism at the configuration space level.

In the reference [8], only the case $n = 2$ of the above theorem is treated. In this case, we can take the complex differential form $d\log(z_i - z_j)$ as a representative of the class ω_{ij} in the de Rham complex of the configuration space of the complex plane $F(\mathbb{C}, r) = \{(z_1, \dots, z_r) \in \mathbb{C}^r \,|z_i \neq z_j\}$ (we just consider the de Rham complex with coefficients in the field of complex numbers $\Bbbk = \mathbb{C}$). The general case of the theorem is addressed in the reference [45]. The computation involves the Leray-Serre spectral sequence associated to the map

$$f : F(\mathbb{R}^n, r) \to F(\mathbb{R}^n, r-1)$$

which forgets the last point of a configuration. We also refer to the article [163] for a comprehensive and accessible survey of the homological computations which give the result of Theorem 4.2.6.

For our applications, we need to determine the morphisms $\partial_k^* : \mathtt{H}^*(F(\mathring{\mathbb{D}}^n, r-1)) \to \mathtt{H}^*(F(\mathring{\mathbb{D}}^n, r))$ induced by the map $\partial_k : F(\mathring{\mathbb{D}}^n, r) \to F(\mathring{\mathbb{D}}^n, r-1)$ such that $\partial_k(a_1, \dots, a_r) = (a_1, \dots, \widehat{a_k}, \dots, a_r)$, for any $\underline{a} = (a_1, \dots, a_r) \in F(\mathring{\mathbb{D}}^n, r)$. Recall that these maps represent the restriction operators associated to morphisms $\partial^k : \{1 < \dots < r-1\} \to \{1 < \dots < r\}$ in the category Λ (see Proposition 4.2.3). We have the following straightforward result:

PROPOSITION 4.2.7. *Let $n > 1$ again. Let $r > 1$. Fix $k \in \{1 < \dots < r\}$. The morphism $\partial_k^* : \mathtt{H}^*(F(\mathring{\mathbb{D}}^n, r-1)) \to \mathtt{H}^*(F(\mathring{\mathbb{D}}^n, r))$ induced by the restriction operator $\partial_k : F(\mathring{\mathbb{D}}^n, r) \to F(\mathring{\mathbb{D}}^n, r-1)$ is determined by the formula:*

$$\partial_k^*(\omega_{ij}) = \begin{cases} \omega_{ij}, & \text{if } k \notin \{i < j\}, \\ 0, & \text{otherwise,} \end{cases}$$

for each generating cohomology class ω_{ij} of the cohomology algebra $\mathtt{H}^*(F(\mathring{\mathbb{D}}^n, r-1))$.

PROOF. Exercise. □

4.2.8. *Homology and monoidal structures.* We can use the homology isomorphism $\omega_* : \mathtt{H}_*(D_n(r)) \xrightarrow{\simeq} \mathtt{H}_*(F(\mathring{\mathbb{D}}^n, r))$ induced by the disc center mapping $\omega : D_n(r) \xrightarrow{\sim} F(\mathring{\mathbb{D}}^n, r)$ and the duality pairing

$$\mathtt{H}^*(F(\mathring{\mathbb{D}}^n, r)) \otimes \mathtt{H}_*(D_n(r)) \xrightarrow{\simeq} \mathtt{H}^*(F(\mathring{\mathbb{D}}^n, r)) \otimes \mathtt{H}_*(F(\mathring{\mathbb{D}}^n, r)) \xrightarrow{\langle -,- \rangle} \Bbbk$$

to determine the homology of each component of the little n-discs operad from our description of the cohomology of the configuration space $F(\mathring{\mathbb{D}}^n, r)$ in Theorem 4.2.6. We now aim to give a description of this collection of homology modules $\mathtt{H}_*(D_n) = \{\mathtt{H}_*(D_n(r)), r > 0\}$ as an operad.

We have already used that the cohomology defines a functor from spaces to commutative algebras. We examine the definition of a coalgebra structure on the homology of spaces first. We explain the general definition of operad structures on the homology of operads and we address the applications of this construction to the operad of little n-discs afterwards. We use the formalism of symmetric monoidal functors (see §3.3.1).

We obviously have $\mathtt{H}_*(pt) = \Bbbk$ by definition of ordinary homology so that the mapping $\mathtt{H}_* : X \mapsto \mathtt{H}_*(X)$ defines a unit-preserving functor from topological spaces to graded modules. We consider the Künneth morphism $\kappa : \mathtt{H}_*(X) \otimes \mathtt{H}_*(Y) \to \mathtt{H}_*(X \times Y)$. We have the following classical statement:

PROPOSITION 4.2.9 (See [129, §VIII] or [166, §5.3]).

(a) *The Künneth morphism defines a symmetric monoidal transformation on the homology functor $\mathtt{H}_* : X \mapsto \mathtt{H}_*(X)$ regarded as a functor from the symmetric monoidal category of spaces $\mathcal{T}op$ to the symmetric monoidal category of graded modules $gr\,\mathcal{M}od$.*

(b) *If the coefficient ring is a field, then the Künneth morphism is an isomorphism. Hence, the homology defines a (strongly) symmetric monoidal functor* $\mathtt{H}_* : \mathcal{T}op \to gr\,\mathcal{M}od$ *in this case.* ☐

We can therefore apply the general constructions of §3.0.5 to obtain:

PROPOSITION 4.2.10. *If the coefficient ring is a field, then the homology functor* $\mathtt{H}_* : \mathcal{T}op \to gr\,\mathcal{M}od$ *induces a functor from the category of topological spaces* $\mathcal{T}op$ *to the category of counitary cocommutative coalgebras in graded modules* $gr\,\mathcal{C}om^c_+$, *and this functor* $\mathtt{H}_* : \mathcal{T}op \to gr\,\mathcal{C}om^c_+$ *is also symmetric monoidal.*

EXPLANATIONS. In §3.0.5, we deal with the general case of a functor between symmetric monoidal categories. In the context of Proposition 4.2.10, we consider the homology functor $\mathtt{H}_* : \mathcal{T}op \to gr\,\mathcal{M}od$ between topological spaces and graded modules. The first result of this proposition, the existence of a counitary cocommutative coalgebra structure on the homology, follows from Proposition 4.2.9 and from the observation that any space X naturally forms a counitary cocommutative coalgebra in the category of spaces (with the constant map $\epsilon : X \to pt$ as counit and the diagonal map $\Delta : X \to X \times X$ as coproduct). The second result of the proposition, namely the definition of the symmetric monoidal functor $\mathtt{H}_* : \mathcal{T}op \to gr\,\mathcal{C}om^c_+$, arises from the observations of §3.0.5.

To prepare our subsequent study of the homology of little discs, we examine this applications of the general construction of §3.0.5 with more details. First, the graded counitary cocommutative coalgebra structure on the homology of a space $\mathtt{H}_*(X)$ is formed as follows:

- to define the counit of this coalgebra, we simply consider the morphism $\mathtt{H}_*(X) \to \mathtt{H}_*(pt) = \Bbbk$, associated to the constant map $X \to pt$;
- to define the coproduct, we form the composite

$$\mathtt{H}_*(X) \xrightarrow{\Delta_*} \mathtt{H}_*(X \times X) \xleftarrow{\simeq} \mathtt{H}_*(X) \otimes \mathtt{H}_*(X)$$

  where we consider the morphism induced by the diagonal of the space X followed by the Künneth isomorphism.

The unit, associativity and symmetry properties of the Künneth isomorphism ensure that the coproduct, which we obtain in this construction, satisfies the counit, coassociativity, and cocommutativity relations of graded counitary cocommutative coalgebras (see §3.0.5).

This coproduct on the homology of a space $\mathtt{H}_*(X)$ represents the dual morphism of the product $\mu : \mathtt{H}^*(X) \otimes \mathtt{H}^*(X) \to \mathtt{H}^*(X)$ that defines the commutative algebra structure of the cohomology $\mathtt{H}^*(X)$, because this product can also be defined as a composite

$$\mathtt{H}^*(X) \otimes \mathtt{H}^*(X) \xrightarrow{\kappa} \mathtt{H}^*(X \times X) \xrightarrow{\Delta^*} \mathtt{H}^*(X),$$

where we consider a cohomological version of the Künneth morphism, followed by the morphism induced by the diagonal of the space X. Note that the commutative algebra structure of the cohomology is still defined when the Künneth morphism is not an isomorphism (in contrast with the coalgebra structure of the homology). To give a more explicit expression of this duality between the product and the coproduct, we consider the natural pairing $\langle -, - \rangle : \mathtt{H}^*(X) \otimes \mathtt{H}_*(X) \to \Bbbk$ between the cohomology and the homology of X. If we set $\Delta(c) = \sum_i a_i \otimes b_i$ for the coproduct of an element c in $\mathtt{H}_*(X)$, then we have the adjunction relation

$$\langle\alpha\cdot\beta, c\rangle = \sum_i \pm\langle\alpha, a_i\rangle\cdot\langle\beta, b_i\rangle,$$

for every $\alpha,\beta\in\mathtt{H}^*(X)$, where the sign ± is produced by the commutation of the factors α and a_i in this expression.

Recall that we equip any category of counitary cocommutative coalgebras in a base category $\mathcal{M}\,\mathcal{C}om_+^c$ with the tensor product $\otimes : \mathcal{M}\,\mathcal{C}om_+^c\times\mathcal{M}\,\mathcal{C}om_+^c\to\mathcal{M}\,\mathcal{C}om_+^c$ inherited from the base category $\mathcal{M}$ (see §3.0.4). We just take $\mathcal{M} = gr\,\mathcal{M}od$ in the case of the category of graded counitary cocommutative coalgebras $gr\,\mathcal{C}om_+^c$. We readily check that the Künneth morphism $\mathtt{H}_*(X)\otimes\mathtt{H}_*(Y)\to\mathtt{H}_*(X\times Y)$ defines a morphism of graded counitary cocommutative coalgebras, and satisfies the unit, associativity and symmetry constraints of §3.3.1 in this category $gr\,\mathcal{C}om_+^c$ (see §3.0.5). We can therefore improve the assertion of Proposition 4.2.9. We explicitly obtain that the homology functor defines a symmetric monoidal functor $\mathtt{H}_* : \mathcal{T}op\to gr\,\mathcal{C}om_+^c$ from the category of spaces $\mathcal{T}op$ to the category of graded counitary cocommutative coalgebras $gr\,\mathcal{C}om_+^c$, as asserted in our statement. □

We then obtain:

Proposition 4.2.11. *Let P be any operad in topological spaces.*

(a) *The collection of graded modules $\mathtt{H}_*(\mathsf{P}) = \{\mathtt{H}_*(\mathsf{P}(r)), r\in\mathbb{N}\}$ defined by the homology of the spaces $\mathsf{P}(r)$ with coefficients in any ground ring $\Bbbk$ forms an operad in graded modules (a graded operad) naturally associated to P.*

(b) *If the ground ring is a field, then this operad $\mathtt{H}_*(\mathsf{P})$ is actually an operad in graded counitary cocommutative coalgebras (a graded Hopf operad), where we use the coalgebra structure of the homology modules $\mathtt{H}_*(\mathsf{P}(r))$ provided by the result of Proposition* 4.2.10.

Explanations. This proposition is a corollary of the results of §3.1, where we examine the image of operads under functors between symmetric monoidal categories. We consider the homology functor $\mathtt{H}_* : X\mapsto\mathtt{H}_*(X)$ from the category of spaces to the category of graded modules (respectively, to the category of graded counitary cocommutative coalgebras) and we use the result of Proposition 3.1.1 to get the definition of an operad structure on the homology $\mathtt{H}_*(\mathsf{P})$. We obtain the following result:

- the morphisms $w_* : \mathtt{H}_*(\mathsf{P}(r))\to\mathtt{H}_*(\mathsf{P}(r))$, induced by the action of permutations $w\in\Sigma_r$ at the topological level, give the action of permutations on the homology of the operad;
- the morphism $\Bbbk = \mathtt{H}_*(pt)\xrightarrow{\eta_*}\mathtt{H}_*(\mathsf{P}(1))$, induced by the operadic unit of the topological operad P, gives an operadic unit at the homology level;
- the partial composition products of the topological operad P induce morphisms

  $$\mathtt{H}_*(\mathsf{P}(m))\otimes\mathtt{H}_*(\mathsf{P}(n))\to\mathtt{H}_*(\mathsf{P}(m)\times\mathsf{P}(n))\xrightarrow{(\circ_i)_*}\mathtt{H}_*(\mathsf{P}(m+n-1))$$

  which give the partial composition products of the homology operad $\mathtt{H}_*(\mathsf{P})$;
- and the preservation of unit, associativity and symmetry isomorphisms by symmetric monoidal functors ensures that these structure morphisms fulfill the equivariance, unit and associativity axioms of operads (see §3.1).

Depending on the context (a-b). we can form these structure morphisms (and define the operad structure of the object $\mathtt{H}_*(\mathsf{P})$) in the category of graded modules or in the category of counitary cocommutative coalgebras.

To complete this analysis, recall that such a functor on operads $\mathtt{H}_* : \mathsf{P} \mapsto \mathtt{H}_*(\mathsf{P})$ preserves unitary extensions: we have the identity $\mathtt{H}_*(\mathsf{P}_+) = \mathtt{H}_*(\mathsf{P})_+$ for any unitary extension P_+ of a non-unitary operad P. □

Recall that we use the phrase '*graded Hopf operad*' to refer to an operad in graded counitary cocommutative coalgebras and we adopt the notation $gr\,\mathcal{H}opf\,\mathcal{O}p$ (instead of $gr\,\mathcal{C}om^c_+\,\mathcal{O}p$) for this category of operads. Thus, the result of Proposition 4.2.11(b) asserts that the homology functor $\mathtt{H}_* : \mathcal{T}op \to gr\,\mathcal{C}om^c_+$ induces a functor $\mathtt{H}_* : \mathcal{T}op\,\mathcal{O}p \to gr\,\mathcal{H}opf\,\mathcal{O}p$ from the category of topological operads $\mathcal{T}op\,\mathcal{O}p$ to the category of graded Hopf operads $gr\,\mathcal{H}opf\,\mathcal{O}p$.

For $\mathsf{P} = \mathsf{D}_1$ (respectively, $\mathsf{P} = \mathsf{D}_\infty$), the existence of a weak-equivalence between our operad and the discrete operad of associative (respectively, commutative) monoids implies:

Proposition 4.2.12.
(a) *We have an identity of graded Hopf operads* $\mathtt{H}_*(\mathsf{D}_1) = \mathsf{As}$, *where we consider the associative operads in* $\Bbbk$*-modules* As *(which we regard as a graded operad concentrated in degree* 0*) together with the coproduct inherited from the corresponding operad in sets (see the concluding paragraph of* §3.1*). In the unitary setting, we similarly have* $\mathtt{H}_*(\mathsf{D}_{1+}) = \mathsf{As}_+$.
(b) *We have an identity of graded Hopf operads* $\mathtt{H}_*(\mathsf{D}_\infty) = \mathsf{Com}$, *where we consider the commutative operads in* $\Bbbk$*-modules* Com *(which we regard as a graded operad concentrated in degree* 0*) together with the coproduct inherited from the corresponding operad in sets (see the concluding paragraph of* §3.1 *again). In the unitary setting, we similarly have* $\mathtt{H}_*(\mathsf{D}_{\infty+}) = \mathsf{Com}_+$. □

Recall that our main objective is to give the description of the graded Hopf operad $\mathtt{H}_*(\mathsf{D}_n)$ when $1 < n < \infty$. We define a graded Hopf operad by generators and relations in the next paragraph (the *n-Gerstenhaber operad* Gerst_n). We explain afterwards that this operad represents the homology of the operad of little n-discs.

4.2.13. *The Gerstenhaber operads.* The *n-Gerstenhaber operad* Gerst_n is actually a graded version of the Poisson operad of §1.2.12, and some authors use the phrase '*n-Poisson operad*', or the phrase '*Poisson operad of degree $n-1$*', rather than the name 'n-Gerstenhaber operad', to refer to this object. We actually define Gerst_n by the same presentation as the Poisson operad:

$$\begin{aligned}\mathsf{Gerst}_n = \ & \Theta\big(\Bbbk\,\mu(x_1,x_2) \oplus \Bbbk\,\lambda(x_1,x_2) : \\ & \qquad \mu(\mu(x_1,x_2),x_3) \equiv \mu(x_1,\mu(x_2,x_3)), \\ & \qquad \lambda(\lambda(x_1,x_2),x_3) + \lambda(\lambda(x_2,x_3),x_1) + \lambda(\lambda(x_3,x_1),x_2) \equiv 0, \\ & \qquad \lambda(\mu(x_1,x_2),x_3) \equiv \mu(\lambda(x_1,x_3),x_2) + \mu(x_1,\lambda(x_2,x_3))\big),\end{aligned}$$

where we still consider a generating operation $\mu = \mu(x_1,x_2)$ of degree 0 such that $(1\ 2)\cdot\mu = \mu$, but we now assume that $\lambda = \lambda(x_1,x_2)$ is an operation of degree $n-1$ such that $(1\ 2)\cdot\lambda = (-1)^n\lambda$, with a sign $(-1)^n$ that depends on the dimension n.

The operation μ defines an associative and commutative product which generates a suboperad isomorphic to the commutative operad Com inside the n-Gerstenhaber operad (see [77, 134]). The operation λ is a graded analogue of the Lie bracket operation of the Poisson operad. The suboperad of the n-Gerstenhaber operad generated by this operation λ is isomorphic to an operadic suspension of the Lie operad *Lie* (see [77]). (We only use operadic suspensions in §III.4 and we explain the definition of this concept in this subsequent chapter.)

The distribution relation $\lambda(\mu(x_1, x_2), x_3) = \mu(\lambda(x_1, x_3), x_2) + \mu(x_1, \lambda(x_2, x_3))$ implies, as in the Poisson case, that any composite of products and Lie brackets in the n-Gerstenhaber operad is equal to a product of Lie monomials. To be more precise, one can prove that each module $\mathit{Gerst}_n(r)$ is spanned by formal products

$$p(x_1, \ldots, x_r) = p_1(x_{11}, \ldots, x_{1r_1}) \cdot \ldots \cdot p_m(x_{m1}, \ldots, x_{mr_m}),$$

whose factors $p_i(x_{i1}, \ldots, x_{ir_i})$ run over Lie monomials on r_i variables x_{ik}, $k \in \{1, \ldots, r_i\}$, such that each variable x_{ik} occurs once and only once in this Lie monomial $p_i(x_{i1}, \ldots, x_{ir_i})$ and the sets $\{x_{i1}, \ldots, x_{ir_i}\}$, $i = 1, \ldots, r$, represent the components of a partition of the total set of variables $\{x_1, \ldots, x_r\}$ of our operation $p = p(x_1, \ldots, x_r)$. (We assume that the Lie bracket λ is homogeneous of degree $n-1$ when we form our Lie monomials.) The description of the Lie operad in §1.2.10, remains also valid in this context and any monomial $p_i = p_i(x_{i1}, \ldots, x_{ir_i})$ in the above expansion has a reduced form

$$p_i(x_{i1}, \ldots, x_{ir_i}) = \lambda(\cdots \lambda(\lambda(x_{i1}, x_{i2}), x_{i3} \cdots), x_{ir_i}),$$

such that we have the order relation $x_{i1} < x_{ik}$ for all $1 < k$ with respect to the natural ordering inherited from the full set of variables $\{x_1 < \cdots < x_r\}$ of our operation $p = p(x_1, \ldots, x_r)$.

We provide the operad Gerst_n with a Hopf structure such that $\epsilon(\mu) = 1$ and $\Delta(\mu) = \mu \otimes \mu$ for the commutative product operation $\mu \in \mathit{Gerst}_n(2)$, and $\epsilon(\lambda) = 1$ and $\Delta(\lambda) = \lambda \otimes \mu + \mu \otimes \lambda$ for the Lie bracket operation $\lambda \in \mathit{Gerst}_n(2)$. We can readily see, as in the Poisson case (see §3.2.12), that the ideal of generating relations forms a Hopf ideal, so that this Hopf structure is well-defined.

4.2.14. *The unitary Gerstenhaber operad.* We have considered a non-unitary version of the n-Gerstenhaber operad in the construction of the previous paragraph. We can also define a unitary n-Gerstenhaber operad by observing that the operad Gerst_n inherits restriction operators such that $\partial_1 \mu = \partial_2 \mu = 1$ and $\partial_1 \lambda = \partial_2 \lambda = 0$ (as in the Poisson case). We easily check that the application of these restriction operators cancels the generating relations of Gerst_n. We then use the process of §2.4.8 to obtain the definition of our unitary extension Gerst_{n+} of the operad Gerst_n.

The Hopf structure of the n-Gerstenhaber operad is clearly preserved by our restriction operators so that our construction yields a unitary extension of the n-Gerstenhaber operad in the category of Hopf operads.

In the computation of the homology of the operad of little discs, we use the Hopf structure of the Gerstenhaber operad and the restriction operators associated to this unitary extension of our object. The main result reads:

THEOREM 4.2.15 (F. Cohen [45]). *Let $n > 1$.*

(a) *The elements $\mu = [pt] \in \mathrm{H}_0(D_n(2))$ and $\lambda = [\mathbb{S}^{n-1}] \in \mathrm{H}_{n-1}(D_n(2))$ satisfy the graded symmetry relations $(1\ 2) \cdot \mu = \mu$ and $(1\ 2) \cdot \lambda = (-1)^n \lambda$ as well as the* *generating relations of the Poisson operad*

$$\begin{aligned} &\mu(\mu(x_1,x_2),x_3) = \mu(x_1,\mu(x_2,x_3)),\\ &\lambda(\lambda(x_1,x_2),x_3) + \lambda(\lambda(x_2,x_3),x_1) + \lambda(\lambda(x_3,x_1),x_2) = 0\\ \textit{and}\quad &\lambda(\mu(x_1,x_2),x_3) = \mu(\lambda(x_1,x_3),x_2) + \mu(x_1,\lambda(x_2,x_3)) \end{aligned}$$

in the homology of the little n-discs operad $\mathtt{H}_*(\mathsf{D}_n)$.

(b) *The counit and coproduct of these elements in the homology of the space $\mathsf{D}_n(2)$ are given by the formulas:*

$$\begin{aligned} &\epsilon[pt] = 1, \quad \Delta[pt] = [pt]\otimes[pt],\\ &\epsilon[\mathbb{S}^{n-1}] = 0, \quad \Delta[\mathbb{S}^{n-1}] = [\mathbb{S}^{n-1}]\otimes[pt] + [pt]\otimes[\mathbb{S}^{n-1}]. \end{aligned}$$

The restriction operators $\partial_k : \mathtt{H}_*(\mathsf{D}_n(2)) \to \mathtt{H}_*(\mathsf{D}_n(1))$, $k = 1,2$, induced by the restriction operators of the little n-discs operad in homology can be determined by:

$$\partial_1[pt] = \partial_2[pt] = 1, \quad \partial_1[\mathbb{S}^{n-1}] = \partial_2[\mathbb{S}^{n-1}] = 0,$$

where we use the obvious identity $\mathtt{H}_*(\mathsf{D}_n(1)) = \mathtt{H}_*(\mathsf{F}(\mathring{\mathbb{D}}^n,1)) = \Bbbk$.

(c) *The mapping $\mu \mapsto [pt] \in \mathtt{H}_0(\mathsf{D}_n(2))$ and $\lambda \mapsto [\mathbb{S}^{n-1}] \in \mathtt{H}_{n-1}(\mathsf{D}_n(2))$ induces an isomorphism of graded Hopf operads*

$$h : \mathsf{Gerst}_n \xrightarrow{\simeq} \mathtt{H}_*(\mathsf{D}_n),$$

which also admits a unitary extension $h : \mathsf{Gerst}_{n+} \xrightarrow{\simeq} \mathtt{H}_*(\mathsf{D}_{n+})$.

EXPLANATIONS AND REFERENCES. We refer to [45] for the proof of the identities of (a) in the homology of the little discs operad (see also [163] for another nice reference on this topic). The identities of (b) are obvious.

This preliminary verification is used to check that we have a well-defined morphism of graded operads $h : \mathsf{G}_n \to \mathtt{H}_*(\mathsf{D}_n)$ which maps the operation $\mu \in \mathsf{Gerst}_n(2)$ (respectively, $\lambda \in \mathsf{Gerst}_n(2)$) in the n-Gerstenhaber operad to the class $[pt] \in \mathtt{H}_0(\mathsf{D}_n(2))$ (respectively $[\mathbb{S}^{n-1}] \in \mathtt{H}_{n-1}(\mathsf{D}_n(2))$) in the homology of the little n-discs operad (as specified in the theorem). The coproduct of the homology classes $[pt]$ and $[\mathbb{S}^{n-1}]$ agrees with the coproduct of the corresponding generating operations in the n-Gerstenhaber operad. We deduce from this observation that our morphism preserves coproducts and forms a morphism of graded Hopf operads therefore.

We still have to check that this morphism is an isomorphism. We can deduce this claim from the computation of the cohomology of configuration spaces in Theorem 4.2.6 and from the duality formula of the next proposition. (We use the explicit construction of our comparison morphism in this proposition, but we do not use any further result on this morphism to check the given duality formula.) We refer to [163] for details on this proof of the isomorphism claim.

The result of this theorem also follows from the computation of [45] which gives the expression of the homology $\mathtt{H}_*(S_*(\mathsf{D}_{n+},X))$ as a functor in $\mathtt{H}_*(X)$, for any pointed space X, where we consider the unitary operad of little n-discs D_{n+} and $S_*(\mathsf{D}_{n+},X)$ refers to the free D_{n+}-algebra associated to X modulo the identification of the unit of this D_{n+}-algebra structure with the base of point of X (see *loc. cit.* for details). In the simplest case where we take the field of rational numbers $\Bbbk = \mathbb{Q}$ as ground ring and we consider the homology with rational coefficients $\mathtt{H}_*(-) = \mathtt{H}_*(-,\mathbb{Q})$, the result of [45] asserts that this functor is precisely given by the free Gerst_{n+}-algebra on the (reduced) homology $\tilde{\mathtt{H}}_*(X)$ of the space X. Thus, we have the functor identity $\mathtt{H}_*(S_*(\mathsf{D}_{n+}, X)) = S(\mathsf{Gerst}_{n+}, \tilde{\mathtt{H}}_*(X)))$. which implies the relation $\mathsf{Gerst}_n = \mathtt{H}_*(\mathsf{D}_n)$ at the operad level. The description of the homology $\mathtt{H}_*(S_*(\mathsf{D}_n, X))$ as a functor in $\tilde{\mathtt{H}}_*(X)$ is more complicated otherwise.

The preservation of restriction operators implies that our morphism h extends to a morphism of unitary operads h_+ which is obviously an isomorphism too since h is. □

Proposition 4.2.16. *Let $\omega_{ij} \in \mathtt{H}^*(\mathsf{F}(\mathring{\mathbb{D}}^n, r))$ be any of the generating elements of the cohomology algebra $\mathtt{H}^*(\mathsf{F}(\mathring{\mathbb{D}}^n, r))$ (as in* §4.2.5*). Let $p = p(x_1, \dots, x_r) \in \mathsf{Gerst}_n(r)$ be a product of Lie monomials which represents a basis element of the module $\mathsf{Gerst}_n(r)$ as we explain in* §4.2.13*. We apply the morphism of Theorem* 4.2.15 *to regard p as an element of $\mathtt{H}_*(\mathsf{D}_n(r))$. Then we have the duality relation*

$$\langle \omega_{ij}, p\rangle = \begin{cases} 1, & \textit{in the case } p = x_1 \cdot \ldots \cdot \lambda(x_i, x_j) \cdot \ldots \cdot \widehat{x_j} \cdot \ldots \cdot x_r, \\ 0, & \textit{otherwise,} \end{cases}$$

with respect the pairing $\langle -, - \rangle : \mathtt{H}_*(\mathsf{F}(\mathring{\mathbb{D}}^n, r)) \otimes \mathtt{H}_*(\mathsf{D}_n(r)) \to \Bbbk$ considered in §4.2.8.

Proof. We use that the disc center mapping $\omega : \mathsf{D}_n(r) \to \mathsf{F}(\mathring{\mathbb{D}}^n, r)$ defines a morphism of non-unitary Λ-sequences. We have by definition $\omega_{ij} = \phi_{ij}^*(\omega)$, where $\omega \in \mathtt{H}^{n-1}(\mathsf{F}(\mathring{\mathbb{D}}^n, 2))$ is the dual element of the class $[\mathbb{S}^{n-1}]$ which represents the Lie bracket operation $\lambda = \lambda(x_1, x_2)$ in $\mathtt{H}_*(\mathsf{D}_n)$. Recall that the map $\phi_{ij} : \mathsf{F}(\mathring{\mathbb{D}}^n, r) \to \mathsf{F}(\mathring{\mathbb{D}}^n, 2)$ considered in the definition of this element ω_{ij} is the restriction operator associated to the map $\rho_{ij} : \{1 < 2\} \to \{1 < \dots < r\}$ such that $\rho(1) = i$ and $\rho(2) = j$. By functoriality of the pairing between cohomology and homology, and by the preservation of restriction operators, we obtain the relation:

$$\langle \omega_{ij}, p\rangle = \langle (\phi_{ij})^*(\omega), p\rangle = \langle \omega, (\rho_{ij})^*(p)\rangle,$$

for any $p = p(x_1, \dots, x_r) \in \mathsf{Gerst}_n(r)$. The result of the proposition accordingly follows from the expression of restriction operators on products of Lie monomials in the Gerstenhaber operad §4.2.13 and from the duality formula $\langle \omega, \lambda\rangle = 1$. □

The expression of the pairing $\langle \pi, p\rangle$ associated to any monomial $\pi = \omega_{i_1 j_1} \cdot \ldots \cdot \omega_{i_r j_r}$ can be obtained from the result of this proposition and from the adjunction relation between the product of $\mathtt{H}^*(\mathsf{F}(\mathring{\mathbb{D}}^n, r))$ and the coproduct of $\mathtt{H}_*(\mathsf{D}_n(r))$ (see §4.2.8). The combinatorial formula that arises from this process is made explicit in [163] and implies that our construction yields a non-degenerate pairing in each arity $r > 0$ between the component of the n-Gerstenhaber operad $\mathsf{G}_n(r)$ and the cohomology algebra of Theorem 4.2.6. This argument provides a proof that the map of Theorem 4.2.15 defines an isomorphism between the n-Gerstenhaber operad and the homology of the little n-discs operad (as we mention in the proof of this statement).

4.3. Outlook: Variations on the little discs operads

The little n-discs operad of §4.1 is our reference model of E_n-operad, and we mostly deal with structures which we directly obtain from the consideration of this topological object. Nonetheless, this operad is not universal. We have other instances of E_n-operads and, depending on the considered application, one model of E_n-operad may be more appropriate than another. We may also consider additional

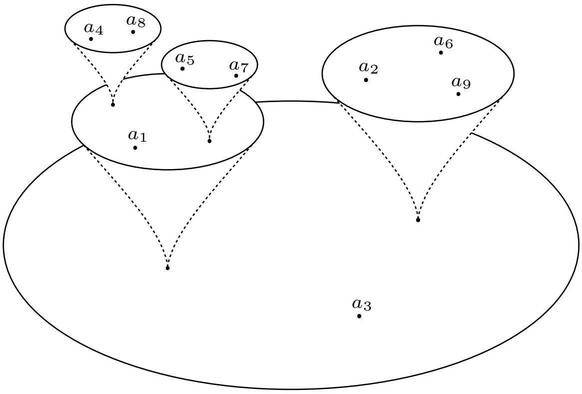


FIGURE 4.5. The picture of an element in the Fulton–MacPherson compactification of the configuration space.

structures in our definition in order to get variants of the notion of an E_n-operad. The purpose of this section is to give an overview of geometric constructions that yield such operads related to little discs.

In the first instance, we provide an outline of the definition of the Fulton–MacPherson operad FM_n, an instance of E_n-operad introduced by E. Getzler and J. Jones in [77] which arises from a compactification of the configuration spaces $\mathsf{F}(\mathbb{R}^n, r)$. Intuitively, we may regard a configuration of points as a configuration of discs equipped with a zero radius. The idea of the Fulton–MacPherson operad is to use the compactification process in order to extend the composition of little discs to this degenerate case. We then obtain a picture of the form of Figure 4.5, where we consider a scale of microscopic configurations organized along a tree which define (free) operadic composites in our spaces. We outline the definition of these topological spaces which form our operad first.

4.3.1. *The Fulton–MacPherson compactifications.* In the approach of [77], we first consider a compactified space $\overline{\mathsf{F}}(\mathbb{R}^n, r)$ defined by performing real blow-ups of the diagonal subspaces $x_{i_1} = x_{i_2} = \cdots = x_{i_k}$ in the product space $(\mathbb{R}^n)^r$ and by taking the closure of the configuration space $\mathsf{F}(\mathbb{R}^n, r)$ in the cartesian product of all these blow-ups, for each $r > 0$. This compactification process is actually a real analogue of the construction which Fulton–MacPherson introduced for the study of configuration spaces of points in complex varieties (see [69]). The real version of the compactification process which we use to define the Fulton–MacPherson operad was initially introduced by Axelrod-Singer, in [12], for the study of the perturbative expansion of Chern-Simons quantum field theories.

In summary, the real blow-up of the small diagonal $\Delta = \{x_1 = x_1 = \cdots = x_1\}$ in a product space $(\mathbb{R}^n)^k$ is a space $\mathtt{Bl}_{\Delta}(\mathbb{R}^n)^k \subset (\mathbb{R}^n)^k \times (\mathbb{R}^n)^k$ such that:

- the mapping
$$\pi : \mathtt{Bl}_{\Delta}(\mathbb{R}^n)^k \to (\mathbb{R}^n)^k$$
induced by the projection $\pi(x_1, \dots, x_k, v_1, \dots, v_k) = (x_1, \dots, x_k)$ is one-to-one over the complement of the diagonal Δ in $(\mathbb{R}^n)^k$,

- and we have

  $$\pi^{-1}(a,\dots,a) = \{(a,\dots,a)\} \times ((\Delta^{\perp} \setminus 0)/\mathbb{R}_{>0}),$$

  for the points of the diagonal $(a,\dots,a) \in \Delta$, where $(\Delta^{\perp} \setminus 0)/\mathbb{R}_{>0}$ is the space of open half lines $\mathbb{R}_{>0}\, v$ in the vector space $\Delta^{\perp} = \{x_1+x_2+\dots+x_k = 0\}$.

The spaces $\overline{\mathsf{F}}(\mathbb{R}^n, r)$ returned by the real Fulton–MacPherson compactification process are manifolds with corners, and the canonical embeddings $\mathsf{F}(\mathbb{R}^n, r) \hookrightarrow \overline{\mathsf{F}}(\mathbb{R}^n, r)$ define weak-equivalences of topological spaces. We refer to the cited articles [12, 69], or to [157], for further details on the construction of these spaces $\overline{\mathsf{F}}(\mathbb{R}^n, r)$.

The configuration space $\mathsf{F}(\mathbb{R}^n, r)$ inherits an action of the group $\mathbb{R}_{>0} \ltimes \mathbb{R}^n$ which we determine by the transformations of the Euclidean space $\mathbb{R}^n$ such that:

$$\phi : (x_1,\dots,x_n) \mapsto \lambda \cdot (x_1,\dots,x_n) + (a_1,\dots,a_n),$$

where $\lambda \in \mathbb{R}_{>0}$ and $(a_1,\dots,a_n) \in \mathbb{R}^n$. The Fulton–MacPherson compactification process can be performed equivariantly in order to get a compactification $\overline{\mathsf{C}}(\mathbb{R}^n, r)$ of the quotient space $\mathsf{C}(\mathbb{R}^n, r) = \mathsf{F}(\mathbb{R}^n, r)/\mathbb{R}_{>0} \ltimes \mathbb{R}^n$. The spaces $\overline{\mathsf{C}}(\mathbb{R}^n, r)$ have the structure of a manifold with corners as well and the composite map $\mathsf{F}(\mathbb{R}^n, r) \to \mathsf{C}(\mathbb{R}^n, r) \hookrightarrow \overline{\mathsf{C}}(\mathbb{R}^n, r)$ defines a weak-equivalence too (we refer to [157] for a detailed proof of this assertion).

4.3.2. *The Fulton–MacPherson operad.* The spaces $\mathsf{FM}_n(r) = \overline{\mathsf{C}}(\mathbb{R}^n, r)$ form the underlying collection of the Fulton–MacPherson operad FM_n. The structure of this operad is defined as follows. To start with, we immediately see that the symmetric group Σ_r acts on $\mathsf{FM}_n(r)$, for each r, so that our collection of spaces forms a symmetric sequence. We also have $\mathsf{FM}_n(1) = pt$, so that FM_n inherits a canonical operadic unit too.

Let $\mathring{\mathsf{FM}}_n(r) = \mathsf{C}(\mathbb{R}^n, r) = \mathsf{F}(\mathbb{R}^n, r)/\mathbb{R}_{>0} \ltimes \mathbb{R}^n$. For simplicity, we just explain the definition of the operadic composites of (equivariance classes of) points of the configuration spaces $\underline{a} = (a_1,\dots,a_k) \in \mathsf{F}(\mathbb{R}^n, k)$ and $\underline{b} = (b_1,\dots,b_l) \in \mathsf{F}(\mathbb{R}^n, l)$ which correspond to elements of the inner subspaces $\mathring{\mathsf{FM}}_n(-)$ of the Fulton–MacPherson operad $\mathsf{FM}_n(-) = \overline{\mathsf{C}}(\mathbb{R}^n, -)$. We can assume $a_1 + \dots + a_k = 0$ and $b_1 + \dots + b_l = 0$ by equivariance with respect the action of translations. We define the operadic composite $\underline{a} \circ_i \underline{b} \in \mathsf{FM}_n(k+l-1)$ as the point of the compactified space $\mathsf{FM}_n(k+l-1) = \overline{\mathsf{C}}(\mathbb{R}^n, k+l-1)$ represented by the collection

$$(a_1,\dots,a_{i-1},((a_i,\dots,a_i),(b_1,\dots,b_l)),a_{i+1},\dots,a_{k+l-1}) \\ \in \mathbb{R}^{i-1} \times \mathtt{Bl}_{\Delta}(\mathbb{R}^n)^l \times \mathbb{R}^{k-i},$$

where we consider the point

$$((a_i,\dots,a_i),(b_1,\dots,b_l)) \in \{(a_i,\dots,a_i)\} \times ((\Delta^{\perp} \setminus 0)/\mathbb{R}_{>0})$$

in the blow-up of the space $\{x_i = x_{i+1} = \dots = x_{i+l-1}\} \subset (\mathbb{R}^n)^{k+l-1}$.

This process has a natural extension to the whole spaces $\mathsf{FM}_n(-) = \overline{\mathsf{C}}(\mathbb{R}^n, -)$ and returns well-defined operadic composition operations $\circ_i : \mathsf{FM}_n(k) \times \mathsf{FM}_n(l) \to \mathsf{FM}_n(k+l-1)$, for all $k, l > 0$ and for each $i = 1,\dots,k$.

We already mentioned that the spaces $\mathsf{FM}_n(r)$ are weakly-equivalent to the configuration spaces $\mathsf{F}(\mathbb{R}^n, r)$ (see §4.3.1). We accordingly have a weak-equivalence $\mathsf{D}_n(r) \xrightarrow{\sim} \mathsf{F}(\mathbb{R}^n, r) \xrightarrow{\sim} \mathsf{FM}_n(r)$ between the spaces of little n-discs $\mathsf{D}_n(r)$ and the components of the Fulton–MacPherson operad $\mathsf{FM}_n(r)$. These maps do not form an operad morphism, but one can lift them to get a weak-equivalence of operads $\mathbb{W}(\mathsf{D}_n) \xrightarrow{\sim} \mathsf{FM}_n$, where $\mathbb{W}(\mathsf{D}_n)$ is the Boardman–Vogt construction of D_n (see [28]), an operad defined by formal composites of configurations of little n-discs arranged on trees equipped with a metric structure. This operad $\mathbb{W}(\mathsf{D}_n)$ is also equipped with a natural weak-equivalence $\mathbb{W}(\mathsf{D}_n) \xrightarrow{\sim} \mathsf{D}_n$, and hence, we have a chain of weak-equivalences of operads

$$\mathsf{D}_n \xleftarrow{\sim} \mathbb{W}(\mathsf{D}_n) \xrightarrow{\sim} \mathsf{FM}_n,$$

from which we conclude that the Fulton–MacPherson operad FM_n defines an E_n-operad. We refer to [157] for the explicit construction of the operad morphism $\mathbb{W}(\mathsf{D}_n) \xrightarrow{\sim} \mathsf{FM}_n$.

We have not been explicit about the terms of arity zero in the construction of the operad FM_n. We use the notation FM_n for a non-unitary version of the Fulton–MacPherson operad in general, so that we have $\mathsf{FM}_n(0) = \varnothing$ by convention. But we also have an obvious extension of the definition of these spaces $\mathsf{FM}_n(r)$ in arity zero, and we can use this observation to form a unitary version of the Fulton–MacPherson operad FM_{n+}.

The Boardman–Vogt construction is a general construction used to define cofibrant resolutions of operads (see §II.1.4 for the definition of this notion). In the case of the Fulton–MacPherson operad, we actually have an isomorphism of topological operads $\mathsf{FM}_n \simeq \mathbb{W}(\mathsf{FM}_n)$ and one can deduce from this relation that the Fulton–MacPherson operad forms a cofibrant model of E_n-operad (see [157]).

4.3.3. *Trees and the underlying structure of the Fulton–MacPherson operad.* The relation $\mathsf{FM}_n \simeq \mathbb{W}(\mathsf{FM}_n)$ mentioned in the previous paragraph implies that the operad FM_n is free as an operad in sets. If we forget about the topology, then we can actually identify the Fulton–MacPherson operad FM_n with the free operad generated by the symmetric sequence $\overset{\circ}{\mathsf{FM}}_n(r)$. This free operad structure reflects the geometry of the spaces $\mathsf{FM}_n(r)$ in the blow-up construction of §4.3.1.

To be more explicit, one can observe that each space $\mathsf{FM}_n(r)$ has a decomposition of the same shape as the components of the free operad

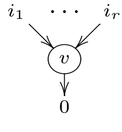
$$\mathsf{FM}_n(r) = \coprod_{\underline{\mathsf{T}} \in \mathcal{T}ree(r)} \overset{\circ}{\mathsf{FM}}_n(\underline{\mathsf{T}}),$$

where we use the formalism of the appendix chapter §A. Simply say for the moment that $\mathcal{T}ree(r)$ denotes the category of r-trees, and the space $\overset{\circ}{\mathsf{FM}}_n(\underline{\mathsf{T}})$, is formed by a cartesian product

$$\overset{\circ}{\mathsf{FM}}_n(\underline{\mathsf{T}}) = \prod_{v \in V(\underline{\mathsf{T}})} \overset{\circ}{\mathsf{FM}}_n(\underline{\mathsf{r}}_v),$$

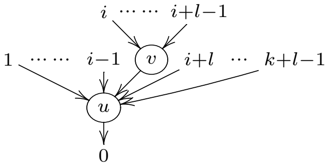
representing an arrangement of factors $\overset{\circ}{\mathsf{FM}}_n(\underline{\mathsf{r}}_v)$ on the vertices $v \in V(\underline{\mathsf{T}})$ of a tree $\underline{\mathsf{T}}$.

The open space $\overset{\circ}{\mathsf{FM}}_n(r) = \mathsf{C}(\mathbb{R}^n, r) = \mathsf{F}(\mathbb{R}^n, r) / \mathbb{R}_{>0} \ltimes \mathbb{R}^n$ inside the compactification $\mathsf{FM}_n(r) = \overline{\mathsf{C}}(\mathbb{R}^n, r)$ is identified with the space $\overset{\circ}{\mathsf{FM}}_n(\underline{\mathsf{Y}})$ associated to the corolla:

$\underline{\mathsf{Y}} =$ 

.

The operadic composite of configurations $\underline{a} = (a_1, \dots, a_k) \in \mathsf{F}(\mathbb{R}^n, k)$ and $\underline{b} = (b_1, \dots, b_l) \in \mathsf{F}(\mathbb{R}^n, l)$, of which we have made the definition explicit in §4.3.2, lies in the subspace $\mathring{\mathsf{FM}}_n(\underline{\mathsf{T}})$ associated to a tree with two vertices

$\underline{\mathsf{T}} =$  .

The spaces $\mathring{\mathsf{FM}}_n(\underline{\mathsf{T}})$, which we associate to r-trees $\underline{\mathsf{T}}$ such that $\operatorname{card}(V(\underline{\mathsf{T}})) \geq 2$, define the facets of the manifold with corners $\mathsf{FM}_n(r) = \overline{\mathsf{C}}(\mathbb{R}^n, r)$.

4.3.4. *Some variations on the Fulton–MacPherson compactification.* In [108], Kontsevich deals with a simpler definition of compactifications from the quotients $\mathsf{C}(\mathbb{R}^n, r) = \mathsf{F}(\mathbb{R}^n, r)/\, \mathbb{R}_{>0} \ltimes \mathbb{R}^n$ of the configuration spaces $\mathsf{F}(\mathbb{R}^n, r)$. For each pair $1 \leq i < j \leq r$, we consider the mapping $\theta_{ij} : \mathsf{C}(\mathbb{R}^n, r) \to \mathbb{S}^{n-1}$ which sends the equivariance class of a configuration $\underline{a} = (a_1, \dots, a_r)$ to the unit vector

$$\theta_{ij}(a_1, \dots, a_r) = \overrightarrow{a_i a_j}/||\overrightarrow{a_i a_j}||.$$

For each triple $1 \leq i < j < k \leq r$, we consider the mapping $\delta_{ijk} : \mathsf{C}(\mathbb{R}^n, r) \to [0, \infty]$ such that:

$$\delta_{ijk}(a_1, \dots, a_r) = ||\overrightarrow{a_i a_j}||/||\overrightarrow{a_i a_k}||.$$

We then form the map

$$\iota : \mathsf{C}(\mathbb{R}^n, r) \to (\mathbb{S}^{n-1})^{\binom{r}{2}} \times [0, \infty]^{\binom{r}{3}}$$

such that $\iota(a_1, \dots, a_r) = ((\theta_{ij}(a_1, \dots, a_r))_{ij}, (\delta_{ijk}(a_1, \dots, a_r))_{ijk})$. We readily see that this map is an embedding. We can actually identify the compactification $\overline{\mathsf{C}}(\mathbb{R}^n, r)$ of §4.3.1 with the closure of the image of the space $\mathsf{C}(\mathbb{R}^n, r)$ in the product space $(\mathbb{S}^{n-1})^{\binom{r}{2}} \times [0, \infty]^{\binom{r}{3}}$ (see [72] for a detailed proof of this claim). We refer to [72] for a detailed study of the relationship between Kontsevich's approach and the blow-up construction of §4.3.1.

We have a variant of this construction defined by considering the closure of the image of the space $\mathsf{C}(\mathbb{R}^n, r)$ under the map $\tilde{\iota} : \mathsf{C}(\mathbb{R}^n, r) \to (\mathbb{S}^{n-1})^{\binom{r}{2}}$ such that $\iota(a_1, \dots, a_r) = (\theta_{ij}(a_1, \dots, a_r))_{ij}$. Let $\tilde{\mathsf{C}}(\mathbb{R}^n, r)$ be the space obtained by this compactification construction. We still have an operad structure, which is studied in details in [162], on the collection of the spaces $\mathsf{FM}_n^{\sim}(r) = \tilde{\mathsf{C}}(\mathbb{R}^n, r)$. We see however that the map $\tilde{\iota}$ is not injective, and therefore, the space $\mathsf{FM}_n^{\sim}(r) = \tilde{\mathsf{C}}(\mathbb{R}^n, r)$ differs from the previously considered compactification $\mathsf{FM}_n(r) = \overline{\mathsf{C}}(\mathbb{R}^n, r)$.

Kontsevich is not precise about the operads used in his work. In [108], he calls $\mathsf{FM}_n^{\sim}$ the Fulton–MacPherson operad, though this operad differs from the standard Fulton–MacPherson operad FM_n.

This operad FM_n is better suited for Kontsevich's proof of the formality of the operad of little n-discs (we go back to this subject in §II.14 and in the concluding chapter of Part III), while the variant $\mathsf{FM}_n^{\sim}$ is used by Dev Sinha in [161, 162] in order to define a cosimplicial space model for knot spaces (we also go back to this subject in the concluding chapter of Part III).

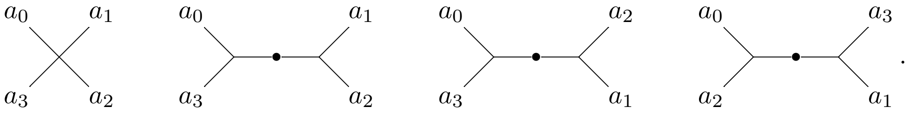
4.3.5. *The Deligne–Mumford–Knudsen compactification of the moduli spaces of curves.* We now consider the case $n = 2$ of the configuration spaces $\mathsf{F}(\mathbb{R}^n, r)$ and of the Fulton–MacPherson operad FM_n. In §4.3.1, we consider the action of the group of real similarities $\mathbb{R}_{>0} \ltimes \mathbb{R}^n$ on the configuration space $\mathsf{F}(\mathbb{R}^n, r)$, but when we deal with configuration of points in the plane $\mathbb{R}^2 = \mathbb{C}$, we can also consider an action of the group of complex similarities $\mathbb{C}^\times \ltimes \mathbb{C}$ (which consists of the transformations of the complex plane $\phi : z \mapsto az + b$ such that $a \in \mathbb{C}^\times$ and $b \in \mathbb{C}$).

The quotient space $\mathsf{F}(\mathbb{C}, r)/\, \mathbb{C}^\times \ltimes \mathbb{C}$ is identified with the quotient $\mathsf{F}(\mathbb{CP}^1, r + 1)/\, PGL(2, \mathbb{C})$ of the configuration space of $r + 1$ points in the projective line $\mathbb{CP}^1$ under the diagonal action of the group of homographies $PGL(2, \mathbb{C})$. This quotient $\mathsf{F}(\mathbb{CP}^1, r + 1)/\, PGL(2, \mathbb{C})$ also represents the moduli space $\mathcal{M}_{0r+1}$ which classifies the isomorphism classes of genus zero smooth curves $C = \mathbb{CP}^1$ with $r + 1$ marked points $a_0, \dots, a_r \in C$.

We have a compactification of this space, the Deligne–Mumford–Knudsen compactification $\overline{\mathcal{M}}_{0r+1}$, which we define by considering certain singular curves at the infinity of our moduli space $\mathcal{M}_{0r+1}$. To be explicit, we say that C is a *stable curve of genus zero with* $r + 1$ *marked points* when:

(1) this curve C admits at most a finite number of singularities, all of which consist of double points, and has $r+1$ marked points $a_0, \dots, a_r \in C$ which are distinct from the singular points;
(2) each irreducible component of C contains at least three special points (singularities and marked points);
(3) the dual graph of our curve C, which has the irreducible components as vertices and the local branches passing through special points as half edges (possibly glued on double points), is a tree.

In the case $r = 3$ for instance, we get the following shapes of dual graphs associated to our curves:



We precisely define the Deligne–Mumford–Knudsen compactification $\overline{\mathcal{M}}_{0r+1}$ of the space $\mathcal{M}_{0r+1}$ as the moduli space of the stable curves of genus zero with $r+1$ marked points. We refer to the work of Deligne and Mumford [53] and Knudsen [105] for a definition of these space $\overline{\mathcal{M}}_{0r+1}$ which relies on general ideas of stack theory. We refer to the work of Keel [102] for a combinatorial definition of these compactifications $\overline{\mathcal{M}}_{0r+1}$ which relies on iterated blow-up constructions (similar to the ideas used in the Fulton–MacPherson compactification process).

The collection such that

$$\overline{\mathcal{M}}(r) = \begin{cases} pt, & \text{for } r = 1, \\ \overline{\mathcal{M}}_{0r+1}, & \text{for } r > 1, \end{cases}$$

inherits an operad structure. The composition products of this operad $\circ_i : \overline{\mathcal{M}}_{0k+1} \times \overline{\mathcal{M}}_{0l+1} \to \overline{\mathcal{M}}_{0k+l}$, $i = 1, \dots, k$, are defined by gluing curves at marked points. Each space $\overline{\mathcal{M}}_{0r+1}$ is also equipped with a stratification. The strata are indexed by the dual graphs of curves and this stratum decomposition reflects the composition

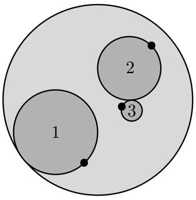


FIGURE 4.6. The representation of an element in the framed little 2-disc operad.

structure of the operad $\overline{\mathcal{M}}$. We refer to [76] for the detailed definition of this correspondence between the composition products of the operad $\overline{\mathcal{M}}$ and the stratification of the spaces $\overline{\mathcal{M}}_{0r+1}$.

The article [102] gives a description of the cohomology ring of the space $\overline{\mathcal{M}}_{0r+1}$. The Fulton–MacPherson compactification of the configuration space $\mathsf{F}(\mathbb{CP}^1, r+1)$ in [69] contains a divisor which is isomorphic to the cartesian product of the space $\overline{\mathcal{M}}_{0r+1}$ with the affine line $\mathbb{A}^1$. This divisor is used to get a representation of the classes (and of the product) of the cohomology ring $\mathtt{H}^*(\overline{\mathcal{M}}_{0r+1})$.

The homology of the spaces $\overline{\mathcal{M}}_{0r+1}$ also forms an operad in the category of graded modules $\mathtt{H}_*(\overline{\mathcal{M}})$ (like the homology of the little 2-discs spaces). The structure of this operad is determined in [76] in terms of a presentation by generators and relations. In short, the homology operad $\mathtt{H}_*(\overline{\mathcal{M}})$ is identified with an operad HyCom, called the hypercommutative operad in [76], which has a symmetric generating operation $\mu_r \in \mathsf{HyCom}(r)$ in each arity $r \geq 2$ and higher associativity relations as generating relations. We also refer to the book [133] for an account of this computation and for a study of a correspondence between the operations encoded by the operad HyCom and operations occurring in the quantum cohomology of projective algebraic varieties.

4.3.6. *The operad of framed little discs.* We can actually relate the Deligne–Mumford–Knudsen operad to a variant of the little 2-discs operad. We more precisely consider a framed version of the little 2-discs operad to express this relationship.

To be explicit, recall that we define a little n-disc as an affine embedding $c : \mathbb{D}^n \hookrightarrow \mathbb{D}^n$ of the form $c(t_1, \dots, t_n) = (a_1, \dots, a_n) + r \cdot (t_1, \dots, t_n)$, for a translation term $(a_1, \dots, a_n) \in \mathring{\mathbb{D}}$ and a scaling factor $r > 0$ (see §4.1.1). The framed little n-discs, which we consider to define the framed little n-discs operad fD_n, are embeddings $c : \mathbb{D}^n \to \mathbb{D}^n$ of the form $c(t_1, \dots, t_n) = (a_1, \dots, a_n) + r \cdot q(t_1, \dots, t_n)$, where in comparison with the definition of §4.1.1 we consider an additional rotation parameter $q \in SO(n)$. The space $\mathsf{fD}_n(r)$ precisely consists of collections of embeddings of this form $\underline{c} = (c_1, \dots, c_r)$ with the same non-intersection condition $i \neq j \Rightarrow \mathring{c}_i \cap \mathring{c}_j = \varnothing$ as in the definition of the ordinary little n-discs operad D_n. The symmetric structure, the unit, and the composition structure of this operad fD_n are defined by an obvious extension of the construction of §4.1.3. The framed little discs operad fD_n has also a natural unitary extension fD_{n+} such that $\mathsf{fD}_{n+}(0) = pt$.

In the 2-dimensional case, we add a mark to the picture of the little 2-discs in order to represent the angle of the rotation that occurs in the definition of the framed little discs (the horizontal axis defines the zero angle). Figure 4.6 for instance gives the picture of a configuration of framed little discs in the space $\mathsf{fD}_2(3)$.

We now focus on the case $n = 2$ of the construction. Let S be the operad such that $\mathsf{S}(1) = SO(2) = \mathbb{S}^1$ and $\mathsf{S}(r) = \varnothing$ for $r \neq 1$. We have an obvious operad morphism $\mathsf{S} \to \mathsf{fD}_2$. By a result of Drummond-Cole [58], the moduli space operad $\overline{\mathcal{M}}$ of §4.3.5 is identified with the result of a homotopy pushout in the category of topological operads:

$$\begin{tikzcd} \mathsf{S} \arrow[r] \arrow[d] & \mathsf{I} \arrow[d, dotted] \\ \mathsf{fD}_2 \arrow[r, dotted] & \overline{\mathcal{M}} \end{tikzcd} .$$

This operadic homotopy pushout is formally defined by replacing the morphism $\mathsf{S} \to \mathsf{fD}_2$ by a cofibration in the category of operads, and by taking the pushout of this cofibration along the morphism $\mathsf{S} \to \mathsf{I}$. We refer to §II.1 for the definition of this notion of a cofibration in the category of operads

The operad $\mathtt{H}_*(\mathsf{S})$ has a single generating operation Δ, given by the fundamental class of the sphere in arity one $\mathtt{H}_*(\mathsf{S}(1)) = \mathtt{H}_*(\mathbb{S}^1)$, and we have $\Delta \circ_1 \Delta = 0$. The operad $\mathtt{H}_*(\mathsf{fD}_2)$ is identified with an operadic semi-direct product of the Gerstenhaber operad Gerst_2 and of the operad $\mathtt{H}_*(\mathsf{S})$. We refer to [75] for a full description of this homology operad $\mathtt{H}_*(\mathsf{fD}_2)$, which is usually called the Batalin–Vilkovisky operad in the literature. We also refer to [158] for a description of the homology operads $\mathtt{H}_*(\mathsf{fD}_n)$ for all $n \geq 2$.

4.4. Appendix: The symmetric monoidal category of graded modules

Let $\Bbbk$ be any fixed a ground ring. In §0.1, we define the category of graded modules $gr\,\mathcal{M}od$ as the category formed by $\Bbbk$-modules K equipped with a splitting $K = \bigoplus_{n\in\mathbb{Z}} K_n$. A morphism of graded modules is a morphism of $\Bbbk$-modules $f : K \to L$ such that $f(K_n) \subset L_n$, for all $n \in \mathbb{Z}$. We say that an element $x \in K$ is *homogeneous of degree* $n \in \mathbb{Z}$ and we write $\deg(x) = n$ when we have $x \in K_n$.

The main purpose of this appendix section is to explain the definition of our symmetric monoidal structure on the category graded modules. By the way, we also check the existence of a hom-bifunctor $\mathtt{Hom}_{gr\,\mathcal{M}od}(-,-) : gr\,\mathcal{M}od^{op} \times gr\,\mathcal{M}od \to gr\,\mathcal{M}od$ which implies that the category $gr\,\mathcal{M}od$ is closed symmetric monoidal.

4.4.1. *The symmetric monoidal structure of the category of graded modules.* The tensor product of $K, L \in gr\,\mathcal{M}od$ in the category of graded modules is the tensor product of K and L as $\Bbbk$-modules, which we equip with the decomposition such that $(K \otimes L)_n = \bigoplus_{p+q=n} K_p \otimes L_q$. This construction obviously gives a bifunctor

$$\otimes : gr\,\mathcal{M}od \times gr\,\mathcal{M}od \to gr\,\mathcal{M}od$$

with the ground ring $\Bbbk$ regarded as a graded module concentrated in degree 0 as unit object. We also have an obvious associativity isomorphism $(K \otimes L) \otimes M \simeq K \otimes (L \otimes M)$ inherited from $\Bbbk$-modules.

We then provide the category of graded modules with a symmetry isomorphism which reflects the commutation rules of differential graded algebra. We precisely define the symmetry isomorphism of a tensor product of graded modules $c : K \otimes L \to L \otimes K$ by the formula $c(x \otimes y) = (-1)^{pq} y \otimes x$, for any pair of homogeneous elements $x \in K_p$ and $y \in L_q$, where we consider the sign $(-1)^{pq}$ determined by the rules of §0.2. We generally use the symbol ± to mark the occurrence of such a sign in our formula (see §0.2)

We immediately see that the tensor product of graded modules satisfies the distribution relation of §0.9 with respect to colimits. We mention in §0.14 that this extra condition is related to the existence of an internal hom in the category of graded modules. We make this internal hom explicit in the next paragraph.

4.4.2. *The internal hom of graded modules.* We basically define the internal hom of graded modules $L, M \in gr\,\mathcal{M}od$ as the graded module $\mathtt{Hom}_{gr\,\mathcal{M}od}(L, M)$ spanned in degree n by the morphisms of $\Bbbk$-modules $f : L \to M$ such that $f(L_p) \subset M_{p+n}$. Thus, we set $\mathtt{Hom}_{gr\,\mathcal{M}od}(L, M)_n = \prod_p \mathtt{Hom}_{gr\,\mathcal{M}od}(L_p, M_{p+n})$, for each $n \in \mathbb{Z}$. The adjunction relation $\mathtt{Mor}_{gr\,\mathcal{M}od}(K \otimes L, M) \simeq \mathtt{Mor}_{gr\,\mathcal{M}od}(K, \mathtt{Hom}_{gr\,\mathcal{M}od}(L, M))$ easily follows from the adjunction relation of $\Bbbk$-modules. Note that a morphism of graded modules is identified with a homomorphism of degree 0 where, according to the conventions of §0.13, we use the term “homomorphism” to refer to an element of the graded hom $\mathtt{Hom}_{gr\,\mathcal{M}od}(L, M)$.

In §0.14, we mention that, for general reasons, the internal hom-objects of a closed symmetric monoidal category inherit a composition product, an internal tensor product operation, and evaluation morphisms. In the context of graded modules, the evaluation morphism is identified with the morphism of graded modules $\epsilon : \mathtt{Hom}_{gr\,\mathcal{M}od}(L, M) \otimes L \to M$ which maps any tensor $f \otimes x \in \mathtt{Hom}_{gr\,\mathcal{M}od}(L, M) \otimes L$ to the element $f(x) \in M$ defined by applying the $\Bbbk$-module map $f : L \to M$ to $x \in L$. Note that $\mathtt{Hom}_{gr\,\mathcal{M}od}(L, M) \otimes L$ refers to the tensor product of graded modules in this construction. The composition product $\circ : \mathtt{Hom}_{gr\,\mathcal{M}od}(L, M) \otimes \mathtt{Hom}_{gr\,\mathcal{M}od}(K, L) \to \mathtt{Hom}_{gr\,\mathcal{M}od}(K, M)$ is induced by the obvious composition operation on $\Bbbk$-module morphisms. The tensor product operation $\otimes : \mathtt{Hom}_{gr\,\mathcal{M}od}(K, L) \otimes \mathtt{Hom}_{gr\,\mathcal{M}od}(M, N) \to \mathtt{Hom}_{gr\,\mathcal{M}od}(K \otimes M, L \otimes N)$ maps (homogeneous) homomorphisms $f : K \to L$ and $g : M \to N$ to the homomorphism $f \otimes g : K \otimes L \to M \otimes N$ such that $(f \otimes g)(x \otimes y) = \pm f(x) \otimes g(y)$, for any pair of (homogeneous) elements $x \in K$ and $y \in L$, where the sign ± is produced by the commutation of g and x.

CHAPTER 5

Braids and the Recognition of E_2-operads

Recall that P is an E_n-operad when we have weak-equivalences of topological operads $\mathsf{P} \xleftarrow{\sim} \cdot \xrightarrow{\sim} \mathsf{D}_n$ which connect P to the operad of little n-discs D_n. In this situation, we also say that P is weakly-equivalent to D_n. In many problems the issue is to prove that a given object P does form an E_n-operad. The usual method is to apply an appropriate recognition criterion that builds the required weak-equivalences from internal structures of P.

In the previous chapter, we observed that a topological operad P is an E_1-operad if and only if each space $\mathsf{P}(r)$ has contractible components which form an operad in sets $\pi_0\,\mathsf{P}$ isomorphic to the operad of associative monoids As (see Proposition 4.1.14). This criterion actually implies that P is weakly-equivalent to the operad in sets As, viewed as a discrete operad in topological spaces. The existence of the weak-equivalence with the little 1-discs operad follows in this context from the observation that the operad D_1 is itself weakly-equivalent to As. In Proposition 4.1.14, we also observed that a topological operad P forms an E_∞-operad if and only if each space $\mathsf{P}(r)$ is contractible. This criterion implies that P is weakly-equivalent to the discrete operad of commutative monoids Com. The weak-equivalence with D_∞ follows, again, from the observation that D_∞ consists of contractible spaces and is itself weakly-equivalent to Com.

The main objective of this chapter is to explain a similar characterization, due to Z. Fiedorowicz [62], of the class of E_2-operads.

We start with the observation that each space $\mathsf{D}_2(r)$ is an Eilenberg-MacLane space $K(P_r, 1)$, where $P_r = \pi_1\,\mathsf{D}_2(r)$ denotes the pure braid group on r strands. We then consider the universal covering $\check{\mathsf{D}}_2(r)$ of the little 2-discs space $\mathsf{D}_2(r)$, which is contractible and comes equipped with an action of the pure braid group such that $\check{\mathsf{D}}_2(r)/P_r = \mathsf{D}_2(r)$. This action of the group P_r on the covering space $\check{\mathsf{D}}_2(r)$ actually extends to an action of the entire braid group B_r which lifts the action of the symmetric group Σ_r on the little 2-discs space $\mathsf{D}_2(r)$. The crux of Fiedorowicz's idea relies on the observation that the collection of spaces $\check{\mathsf{D}}_2 = \{\check{\mathsf{D}}_2(r), r > 0\}$ inherits the same structure as an operad, except that we have to replace the symmetric group actions of the standard definition §1.1.1 by braid group actions. We also use the phrase '*braided operad*' for this variant of the notion of an operad. We regard the quotient construction $\check{\mathsf{D}}_2(r)/P_r = \mathsf{D}_2(r)$, which gives the connection between the little 2-discs space $\mathsf{D}_2(r)$ and the associated universal covering space $\check{\mathsf{D}}_2(r)$, as an instance of a general symmetrization process which enables us to retrieve a symmetric operad from any braided operad structure. The recognition theorem of Z. Fiedorowicz precisely asserts that any operad P obtained by symmetrization $\mathsf{P}(r) = \check{\mathsf{P}}(r)/P_r$ of a contractible braided operad $\check{\mathsf{P}}$ is E_2.

We use this recognition method to check that the classifying spaces of a certain operad in groupoids, the operad of colored braids, forms an instance of an E_2-operad.

In a preliminary section §5.0, we survey basic concepts of braid theory and we recall the definition of the braid groups B_r. In §5.1, we explain the definition of a braided operad and we state Fiedorowicz's recognition criterion. In §5.2, we give the definition of the operad of colored braids, and we explain our construction of a model of E_2-operad from the classifying spaces of this operad in groupoids. In §5.3, we explain that the operad of colored braids is also equivalent to an operad in groupoids, naturally associated to the little 2-discs operad, which is formed by the fundamental groupoids of the little 2-discs spaces. In a concluding section §5.4, we give a brief introduction to more general recognition theorems which aim to give a characterization of E_n-operads for any $n \geq 1$.

The ideas of §§5.1-5.2 are mostly borrowed from [62]. The preprint [188] provides a generalization of this approach for the recognition of operads built from Eilenberg-MacLane spaces. In §5.3, we outline another approach of Fiedorowicz's criterion, which relies on the adjunction between classifying spaces and fundamental groupoids.

5.0. Braid groups

In the previous chapter, we used the configuration spaces $\mathsf{F}(\mathring{\mathbb{D}}^n, r)$, $r > 0$, for our study of the homology of the little n-discs operad D_n. By the way, we observed that, in the case $n = 1$, the configuration spaces $\mathsf{F}(\mathring{\mathbb{D}}^1, r)$ have contractible connected components, indexed by the permutations of the sequence $(1, \dots, r)$ (like the little 1-discs spaces $\mathsf{D}_1(r)$). To begin this chapter, we record the following preliminary observation about the homotopy of the spaces $\mathsf{F}(\mathring{\mathbb{D}}^n, r)$ for $n \geq 2$:

Proposition 5.0.1. *The spaces $\mathsf{F}(\mathring{\mathbb{D}}^n, r)$ are connected for all $n \geq 2$. If $n > 2$, then we moreover have $\pi_1 \mathsf{F}(\mathring{\mathbb{D}}^n, r) = 0$. If $n = 2$, then we have in contrast $\pi_* \mathsf{F}(\mathring{\mathbb{D}}^2, r) = 0$, for all $* \neq 1$.*

Proof. In the previous chapter, we recalled that the map $f : \mathsf{F}(\mathring{\mathbb{D}}^n, r) \to \mathsf{F}(\mathring{\mathbb{D}}^n, r-1)$ which forgets about the last point of a configuration is a fibration. The idea is to prove this proposition by induction on r, by using the homotopy exact sequence associated to these fibrations:

$$\cdots \to \pi_*(f^{-1}(\underline{b}), \underline{a}) \to \pi_*(\mathsf{F}(\mathring{\mathbb{D}}^n, r), \underline{a}) \xrightarrow{f_*} \pi_*(\mathsf{F}(\mathring{\mathbb{D}}^n, r-1), \underline{b}) \to \cdots$$
$$\cdots \to \pi_1(f^{-1}(\underline{b}), \underline{a}) \to \pi_1(\mathsf{F}(\mathring{\mathbb{D}}^n, r), \underline{a}) \xrightarrow{f_*} \pi_1(\mathsf{F}(\mathring{\mathbb{D}}^n, r-1), \underline{b}) \to \underbrace{\pi_0(f^{-1}(\underline{b}), \underline{a})}_{=*},$$

where $\underline{a} = (a_1, \dots, a_r)$ is a fixed base point in the configuration space $\mathsf{F}(\mathring{\mathbb{D}}^n, r)$ and we set $\underline{b} = (a_1, \dots, a_{r-1}) = f(\underline{a})$. The fiber of the map f at this base point $\underline{b}$ is identified with the punctured space

$$f^{-1}(\underline{b}) = \{(a_1, \dots, a_{r-1}, b) \in \mathring{\mathbb{D}}^n | b \neq a_1, \dots, a_{r-1}\} = \mathring{\mathbb{D}}^n \setminus \{a_1, \dots, a_{r-1}\},$$

which is connected as soon as $n > 1$. Hence, we have the identity $\pi_0(f^{-1}(\underline{a}), a_r) = *$ as indicated in our formula.

The connectedness of this space $f^{-1}(\underline{b})$ implies, by induction on r, that the spaces $\mathsf{F}(\mathring{\mathbb{D}}^n, r)$ are connected as well, for all $n > 1$. In the case $n > 2$, we have besides $\pi_1(f^{-1}(\underline{b}), \underline{a}) = \pi_1(\mathring{\mathbb{D}}^n \backslash \{a_1, \dots, a_{r-1}\}, a_r) = *$, and by an immediate induction again, we deduce from the terms of degree one of the homotopy exact sequence that the spaces $\mathsf{F}(\mathring{\mathbb{D}}^n, r)$ are simply connected too. In the case $n = 2$, we have $\pi_*(f^{-1}(\underline{b}), \underline{a}) = \pi_*(\mathring{\mathbb{D}}^2 \setminus \{a_1, \dots, a_{r-1}\}, a_r) = *$ for all $* > 1$, and we use the higher terms of the homotopy exact sequence to conclude that $\pi_*(\mathsf{F}(\mathring{\mathbb{D}}^2, r), \underline{a})$ vanishes for all $* > 1$. ☐

We have the same assertions as in this proposition for the little disc spaces $\mathsf{D}_n(r)$ since we have a homotopy equivalence $\omega : \mathsf{D}_n(r) \xrightarrow{\sim} \mathsf{F}(\mathring{\mathbb{D}}^n, r)$ (see Proposition 4.2.2) which induces an isomorphism on homotopy groups. Briefly recall that this homotopy equivalence, which we call the disc center mapping, sends an r-tuple of little n-discs $\underline{c} = (c_1, \dots, c_r)$, which represents an element of the space $\mathsf{D}_n(r)$, to the configuration of points defined by the centers $c_i(0, \dots, 0) \in \mathring{\mathbb{D}}^n$ of the discs $c_i : \mathbb{D}^n \to \mathbb{D}^n$, $i = 1, \dots, r$.

The previous proposition implies that the configuration spaces $\mathsf{F}(\mathring{\mathbb{D}}^2, r)$, and hence the little 2-discs spaces $\mathsf{D}_2(r)$, are Eilenberg-MacLane spaces $K(P_r, 1)$, where we set $P_r = \pi_1(\mathsf{F}(\mathring{\mathbb{D}}^2, r), *)$. Recall that this group P_r is called the *pure braid group* on r strands. The Artin braid group B_r, which we also consider in our study of E_2-operads, sits in a short exact sequence $1 \to P_r \to B_r \to \Sigma_r \to 1$. The purpose of this preliminary section is to recall the classical interpretation of these groups, in terms of isotopy classes of braids on r strands, and the geometric representation which arises from this interpretation. The identity between the little 2-discs spaces $\mathsf{D}_2(r)$ and the Eilenberg-MacLane spaces $K(P_r, 1)$ is used in the next sections.

To begin with, we explain the definition of the Artin braid group B_r as the fundamental group of a space.

5.0.2. *Braid groups.* Recall that the space $\mathsf{F}(\mathring{\mathbb{D}}^2, r)$ is equipped with an action of the symmetric group Σ_r, which is given by the formula

$$w_*(a_1, \dots, a_r) = (a_{w^{-1}(1)}, \dots, a_{w^{-1}(r)}),$$

for each $(a_1, \dots, a_r) \in \mathsf{F}(\mathring{\mathbb{D}}^2, r)$, and for any permutation $w \in \Sigma_r$ (see Proposition 4.2.3). The braid group on r strands B_r is precisely defined as the fundamental group of the quotient of the configuration space $\mathsf{F}(\mathring{\mathbb{D}}^2, r)$ under this action:

$$B_r = \pi_1(\mathsf{F}(\mathring{\mathbb{D}}^2, r)/\Sigma_r, *).$$

The quotient map $q : \mathsf{F}(\mathring{\mathbb{D}}^2, r) \to \mathsf{F}(\mathring{\mathbb{D}}^2, r)/\Sigma_r$ induces a morphism $q_* : P_r \to B_r$. Now we easily check that:

LEMMA 5.0.3. *The symmetric group Σ_r acts freely and properly on $\mathsf{F}(\mathring{\mathbb{D}}^2, r)$ so that the quotient map $q : \mathsf{F}(\mathring{\mathbb{D}}^2, r) \to \mathsf{F}(\mathring{\mathbb{D}}^2, r)/\Sigma_r$ is a covering map.* ☐

Then we can apply standard results of covering theory to obtain:

PROPOSITION 5.0.4. *The morphism $q_* : P_r \to B_r$ fits in an exact sequence of groups $1 \to P_r \xrightarrow{q_*} B_r \xrightarrow{p_*} \Sigma_r \to 1$, where the map $p_* : B_r \to \Sigma_r$ is deduced from the action of $B_r = \pi_1(\mathsf{F}(\mathring{\mathbb{D}}^2, r)/\Sigma_r, *)$ on the fiber of the covering $q : \mathsf{F}(\mathring{\mathbb{D}}^2, r) \to \mathsf{F}(\mathring{\mathbb{D}}^2, r)/\Sigma_r$ at any base point $* \in \mathsf{F}(\mathring{\mathbb{D}}^2, r)/\Sigma_r$.* ☐

5.0.5. *Braids and braid diagrams.* The braids, which motivate the name given to the braid groups, occur in a representation of the paths in the configuration space $\mathsf{F}(M, r)$ associated to any manifold M. We review this representation of a braid before recalling the classical presentation of the braid groups by generators and relations. For our purpose, we focus on the case $M = \mathring{\mathbb{D}}^2$, and our braids are defined in the cylinder $\mathring{\mathbb{D}}^2 \times [0,1]$. In our study, we refer to works which deal with braids in the Euclidean plane $M = \mathbb{R}^2$ rather than in the open disc $M = \mathring{\mathbb{D}}^2$, but we have an obvious isomorphism between the braid groups associated to these spaces since the Euclidean plane $\mathbb{R}^2$ is homeomorphic to the open disc $\mathring{\mathbb{D}}^2$.

We precisely define a braid with r strands in $\mathring{\mathbb{D}}^2$ as a collection of r disjoint arcs $\alpha_i : [0,1] \to \mathring{\mathbb{D}}^2 \times [0,1]$, $i = 1, \dots, r$, of the form

$$\alpha_i(t) = (x_i(t), y_i(t), t), \quad t \in [0,1],$$

and whose origin $\alpha_i(0) = (x_i(0), y_i(0), 0)$ and end-point $\alpha_i(1) = (x_i(1), y_i(1), 1)$ lie in a set of fixed contact points $\{(x^0_k, 0, t^0) | k = 1, \dots, r\}$ on the axis $y = 0$ of the boundary discs of our cylinder $\mathring{\mathbb{D}}^2 \times \{t^0\}$, where $t^0 = 0, 1$ (see [11]). We can take the sets of equidistant points

$$(x^0_k, 0, 0), (x^0_k, 0, 1), \quad \text{with } x^0_k = -1 + (2k-1)/(r+1),\ k = 1, \dots, r,$$

as contact points for the moment.

The requirement that the arcs α_i are disjoint is equivalent to the relation

$$(x_i(t), y_i(t)) \neq (x_j(t), y_j(t))$$

for all $i \neq j$ and for every $t \in [0,1]$. In the case $t^0 = 0, 1$, this assumption implies that the r-tuple $(\alpha_1(t^0), \dots, \alpha_r(t^0)) = ((x_1(t^0), 0, t^0), \dots, (x_r(t^0), 0, t^0))$ forms a permutation of $((x^0_1, 0, t^0), \dots, (x^0_r, 0, t^0))$. The mapping $s : k \mapsto s(k)$ such that

$$x_i(0) = x^0_k, \quad x_i(1) = x^0_{s(k)}, \quad \text{for } i = 1, \dots, r,$$

defines a permutation $s \in \Sigma_r$, naturally associated to our braid α, and usually referred to as the underlying permutation of the braid α.

The set of pure braids consists of the braids which have the identity as underlying permutation.

The arcs α_i define the strands of the braid. For the moment, we take the convention that the collection of strands α_i, $i = 1, \dots, r$, which defines a braid $\alpha = (\alpha_1, \dots, \alpha_r)$, is equipped with the indexing such that $\alpha_i(0) = (x^0_i, 0, 0)$, for $i = 1, \dots, r$. We then have $(\alpha_1(1), \dots, \alpha_1(1)) = ((x^0_{s(1)}, 0, 1), \dots, (x^0_{s(r)}, 0, 1))$, where $s \in \Sigma_r$ is the permutation associated to our braid. (We will adopt another convention in §5.2 where we consider braids equipped with additional structures for which this ordering is not natural.)

We use a projection onto the plane (x, t) to get a convenient representation of our braids. We give an example of this representation in Figure 5.1 for a braid on 4 strands with

$$s = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$$

as underlying permutation. The projection picture works for braids such that the intersection between the projected arcs $(x_i(t), t)$ reduce to isolated points, and where each intersection $(x_i(t), t) = (x_j(t), t)$ involves no more than two arcs $(x_i(t), t)$, $(x_j(t), t)$. In this context, the usual convention is to insert a gap at each intersection point $(x_i(t), t) = (x_j(t), t)$, as in the example of Figure 5.1, in order to specify the strand which goes under the other with respect to the y-coordinate. Such a figure is called a braid diagram.

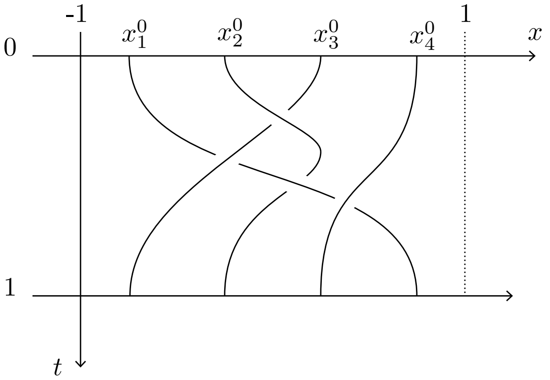


FIGURE 5.1. An instance of braid diagram. In the next pictures, we generally do not specify the abscissa x_i^0 of the contact points. We just specify the index of contact points when necessary.

In the next paragraph, we recall the definition of the isotopy relation between braids. The notion of isotopy can be formalized in terms of braid diagrams, and one can prove that braid diagrams are enough to give a faithful picture of braids up to isotopy. This observation is originally due to E. Artin, and we refer to his article [11], or to the subsequent textbook [101] by C. Kassel and V. Turaev, for more explanations about the relationship between braids and braid diagrams. In what follows, we just use braid diagrams informally in order to illustrate our constructions.

5.0.6. *Braid isotopies.* By definition, an isotopy from a braid α to another one β is a continuous family of braids h_s such that $h_0 = \alpha$ and $h_1 = \beta$. Two braids are isotopic if we have an isotopy between them, and in this case we write $\alpha \sim \beta$. The isotopy relation is clearly an equivalence relation on the set of braids.

Let us regard a braid as a single map $\alpha(t) = (\alpha_1(t), \dots, \alpha_r(t))$ rather than as a collection of maps. The assumption that the underlying braids of an isotopy h_s form a continuous family is equivalent to the requirement that the two parameter map $h : (s,t) \mapsto h_s(t)$ is continuous on $[0,1] \times [0,1]$. By continuity, the requirement that $h_s(1)$ belongs to the discrete space $\{((x^0_{w(1)}, 0, 1), \dots, (x^0_{w(k)}, 0, 1)) | w \in \Sigma_r\}$ implies that the map $s \mapsto h_s(1)$, given by the endpoints of the isotopy, is constant. Hence, we see that isotopic braids have the same underlying permutation.

By a standard abuse of language, we generally use the word braid to refer to an isotopy class of braids unless the distinction is made necessary by the context.

5.0.7. *Relationship with the fundamental groups.* We immediately see that a pure braid on r strands $\alpha_i(t) = (x_i(t), y_i(t), t)$ is equivalent to a based loop $\gamma(t) = ((x_1(t), y_1(t)), \dots, (x_r(t), y_r(t)))$ in the configuration space $F(\mathring{\mathbb{D}}^2, r)$, where we take the configuration of our contact points on the line $\underline{a}^0 = ((x^0_1, 0), \dots, (x^0_r, 0))$ as a base point. Similarly, an isotopy of pure braids is equivalent to a homotopy of based loops in $F(\mathring{\mathbb{D}}^2, r)$. Thus, the pure braid group P_r, which we define as the fundamental group of the space $F(\mathring{\mathbb{D}}^2, r)$, is identified with the set of isotopy classes of pure braids.

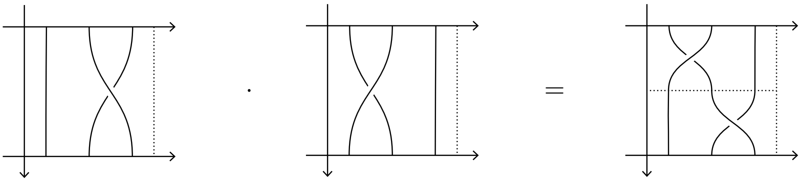


FIGURE 5.2. The concatenation of braids

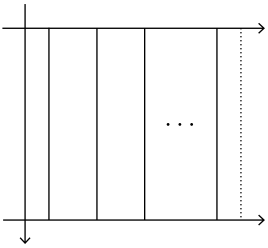


FIGURE 5.3. The identity braid

Let $\underline{b}^0 = q(\underline{a}^0)$ be the image of the element $\underline{a}^0 = ((x_1^0, 0), \dots, (x_r^0, 0))$ in the quotient space $F(\mathring{\mathbb{D}}^2, r)/\Sigma_r$. The fiber of this point $\underline{b}^0$ under the covering map $q : F(\mathring{\mathbb{D}}^2, r) \to F(\mathring{\mathbb{D}}^2, r)/\Sigma_r$ is $q^{-1}(\underline{b}^0) = \{((x_{w(1)}^0, 0), \dots, (x_{w(r)}^0, 0)),\ w \in \Sigma_r\}$. The set of all braids on r strands is identified with the set of paths connecting $\underline{a}^0$ to another point $w\underline{a}^0 = ((x_{w(1)}^0, 0), \dots, (x_{w(r)}^0, 0))$ in this fiber. Braid isotopies are also equivalent to path homotopies. By standard results of covering theory, any loop γ based at $\underline{b}^0$ in the quotient space $F(\mathring{\mathbb{D}}^2, r)/\Sigma_r$ lifts to a path of this form $\tilde{\gamma}$, with $\tilde{\gamma}(0) = \underline{a}^0$ and $\tilde{\gamma}(1) = w\underline{a}^0$ for some $w \in \Sigma_r$. Moreover, such a lifting is unique once we fix the starting point $\tilde{\gamma}(0) = \underline{a}^0$ and any homotopy of based loops lifts to a path homotopy. Hence, the full braid group B_r, which we define as the fundamental group of the quotient space $F(\mathring{\mathbb{D}}^2, r)/\Sigma_r$, is identified with the set of isotopy classes of all braids.

In both cases P_r and B_r, the group multiplication can readily be identified with a natural concatenation operation on braids, of which the Figure 5.2 gives an example. The unit element with respect to this group multiplication is given by the identity braid, represented in Figure 5.3. In what follows, we also use the notation id_r to refer to this braid in B_r. For short, we can still set $id = id_r$ when we do not need to specify the number of strands of our braid. Note that we perform compositions downwards, in the increasing direction of the t coordinates (as opposed to conventions adopted by other authors). Our choice is more natural when we regard braids as morphisms oriented from a source to a target object and we use this interpretation soon.

In Proposition 5.0.3, we refer to a general result of covering theory in order to define the morphism $p_* : B_r \to \Sigma_r$. By going back to the proof of this result, we immediately see that the morphism $p_* : B_r \to \Sigma_r$ is identified with the map that sends the isotopy class of a braid α to its underlying permutation s. The natural embedding of the subset of pure braids into the set of all braids gives the morphism

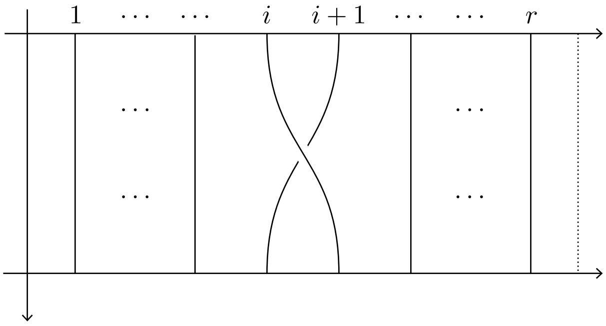


FIGURE 5.4. The generating braids

$q_* : P_r \to B_r$. Thus we have a full interpretation of the exact sequence of groups $1 \to P_r \to B_r \to \Sigma_r \to 1$ in terms of isotopy classes of braids.

5.0.8. *Generating elements.* For $i = 1, \ldots, r-1$, we consider the element $\tau_i \in B_r$ represented by the diagram of Figure 5.4

The mapping $p_* : B_r \to \Sigma_r$ assigns the elementary transposition $t_i = (i\, i+1) \in \Sigma_r$ to this braid $\tau_i \in B_r$. In §0.10, we recall that the symmetric group Σ_r admits a classical presentation, where we take these permutations t_i, $i = 1, \ldots, r-1$, as generating elements. For the braid group, we have the following statement:

THEOREM 5.0.9 (see [10]). *The braid group B_r admits a presentation where we take the braids τ_i, $i = 1, \ldots, r-1$, as generating elements together with the commutation relations*

$$\tau_i \tau_j = \tau_j \tau_i, \quad \text{for } i, j \in \{1, \ldots, r-1\} \text{ such that } |i-j| \geq 2,$$

and the braid relations

$$\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}, \quad \text{for } i \in \{1, \ldots, r-2\},$$

as generating relations (see also the picture of these relations in Figure 5.5*).* □

In other words, the braid group B_r is given by the same presentation as the symmetric group Σ_r except that we forget about the involution relation $t_i^2 = 1$ associated to transpositions. The idea of this result goes back to the work of E. Artin [10] cited in reference. We refer to [26], [63], and [101] for various proofs of the theorem.

The inverse of a generator τ_i in the braid group can actually be obtained by switching the disposition of the strands in the representation of Figure 5.4 (the $i+1$st strand comes in the foreground and the ith strand goes in the background).

5.0.10. *Change of contact points.* In the definition of §5.0.5, we assume that the origin points of a braid belong to the subset $\{(x_k^0, 0, 0) | k = 1, \ldots, r\}$, where $x_k^0 = -1 + (2k-1)/(r+1)$, and the end points belong to the subset $\{(x_k^0, 0, 1) | k = 1, \ldots, r\}$. Equivalently, our braids correspond to paths in the configuration space $F(\mathring{\mathbb{D}}^2, r)$ that starts at the element $((x_1^0, 0), \ldots, (x_r^0, 0))$ and ends at a permutation $((x_{w(1)}^0, 0), \ldots, (x_{w(r)}^0, 0))$ of this base point $((x_1^0, 0), \ldots, (x_r^0, 0))$.

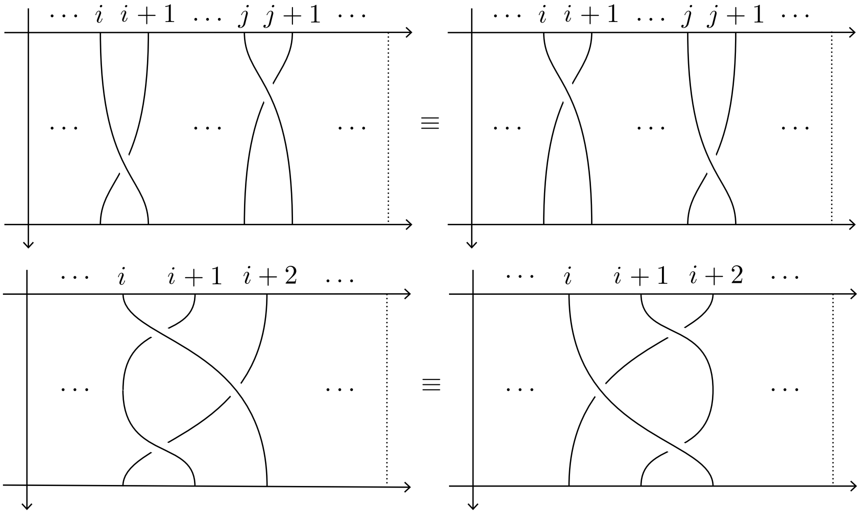


FIGURE 5.5. The commutation and braid relations in braid groups.

In principle, the fundamental groups associated to different choices of base points in a connected space are isomorphic. Hence, in our case, we obtain isomorphic groups if we replace our collection of contact points $\{(x_k^0, 0)|k = 1, \ldots, r\} \times \{0, 1\}$ in the definition of the braid group by another one with arbitrary ordinates $\{(a_k, b_k)|k = 1, \ldots, r\} \times \{0, 1\}$. But the definition of an isomorphism which compares the groups associated to these choices of base points involves the choice of a path $\gamma(t) = (\gamma_1(t), \ldots, \gamma_r(t))$ from one configuration $\gamma(0) = ((x_1^0, 0), \ldots, (x_r^0, 0))$ to the other one $\gamma(1) = ((a_1, b_1), \ldots, (a_r, b_r))$ in the space $\mathsf{F}(\mathring{\mathbb{D}}^2, r)$. In the braid picture, we represent this isomorphism by a concatenation of the strands of our braids with the arcs of the path γ. This isomorphism clearly depends on the homotopy class of the path γ, and hence, is not canonical in general.

In the sequel, we implicitly use changes of base points, but we also need a strict control of the isomorphisms which we use to compare our groups. For this aim, we restrict ourselves to base configurations of the form $((a_1, 0), \ldots, (a_r, 0))$, where all points lie on the line $y = 0$, and for which we assume $a_1 < \cdots < a_r$. Equivalently, we only consider base configurations $(a_1, \ldots, a_r)$ which lie in the equatorial 1-disc $\mathring{\mathbb{D}}^1 \subset \mathring{\mathbb{D}}^2$, and belong to the connected component associated to the permutation $(1, \ldots, r)$ in the configuration space $\mathsf{F}(\mathring{\mathbb{D}}^1, r)$. Since $\mathsf{F}(\mathring{\mathbb{D}}^1, r)$ has contractible connected components, all paths $\gamma(t) = (\gamma_1(t), \ldots, \gamma_r(t))$ which go from any such configuration to another one inside $\mathsf{F}(\mathring{\mathbb{D}}^1, r)$ are homotopic, and hence, induce the same isomorphism at the fundamental group level. Thus, all choices of contact points on the line $y = 0$ yield the same braid group up to a canonical and well determined isomorphism.

5.0.11. *Degenerate cases.* We should note that the definition of the braid group B_r makes sense for $r = 0$. We then deal with a degenerate situation where our braids have an empty set of strands. We therefore have $B_0 = *$ for formal reasons.

The braid group B_1 is also trivial (like the symmetric group Σ_1), with the isotopy class of a one-strand vertical braid as unique element.

5.1. Braided operads and E_2-operads

Let $\check{D}_2(r)$ be the universal coverings of the spaces of little 2-discs $D_2(r)$. The main purpose of this section is to check that the collection of these spaces $\check{D}_2(r)$ inherits the structure of an operad. To be precise, recall that we have to deal with a variant of the notion of an operad when we pass to these covering spaces $\check{D}_2(r)$. Namely, we have to consider an action of the braid groups instead of the action of the symmetric groups of the standard definition §1.1.1. In a preliminary step, we explain the general definition of this braided variant of the notion of an operad. Then we explain the statement of Fiedorowicz's recognition theorem [62] which asserts than we can define E_2-operads by taking the symmetrization of contractible braided operads.

5.1.1. *Braided operads.* We explicitly define a *braided operad* P in a base category $\mathcal{M}$ as a collection of objects $P(r) \in \mathcal{M}$, $r \in \mathbb{N}$, where $P(r)$ is now equipped with an action of the braid group B_r, together with:

(1) a unit morphism $\eta : \mathbb{1} \to P(1)$,

(2) and composition products $\mu : P(r) \otimes P(n_1) \otimes \cdots \otimes P(n_r) \to P(n_1 + \cdots + n_r)$, defined for every $r \geq 0$, and for all $n_1, \ldots, n_r \geq 0$,

such that natural equivariance, unit and associativity relations, modeled on the same commutative diagram as in the case of symmetric operads (Figure 1.1-1.3), hold. We just have to consider elements of the braid groups $\alpha \in B_r$ (respectively, $\beta_1 \in B_{n_1}, \ldots, \beta_r \in B_{n_r}$) instead of permutations $s \in \Sigma_r$ (respectively, $t_1 \in \Sigma_{n_1}, \ldots, t_r \in \Sigma_{n_r}$) in our equivariance relations. We therefore need an extension to the braid groups of the definition of block permutations and of the definition of the direct sum of permutations. We define these operations in the following proposition:

Proposition 5.1.2. *Let $r \in \mathbb{N}$. Let $n_1, \ldots, n_r \in \mathbb{N}$.*

(a) *The direct sum of permutations, regarded as a mapping $\Sigma_{n_1} \times \cdots \times \Sigma_{n_r} \to \Sigma_{n_1+\cdots+n_r}$, has a unique lifting to braid groups*

$$B_{n_1} \times \cdots \times B_{n_r} \to B_{n_1+\cdots+n_r},$$

which is given by the picture of Figure 5.6 when we consider the case of the direct sum $id_{n_1} \oplus \cdots \oplus \tau_k \oplus \cdots \oplus id_{n_r}$ of a generating braid $\tau_k \in B_{n_i}$ with identity braids $id_{n_j} \in B_{n_j}$, $j \neq i$, and which satisfies the following multiplicativity relation

$$(\alpha_1 \cdot \beta_1) \oplus \cdots \oplus (\alpha_r \cdot \beta_r) = (\alpha_1 \oplus \cdots \oplus \alpha_r) \cdot (\beta_1 \oplus \cdots \oplus \beta_r),$$

for all $(\alpha_1, \ldots, \alpha_r), (\beta_1, \ldots, \beta_r) \in B_{n_1} \times \cdots \times B_{n_r}$.

(b) *The block permutation construction, which we regard as a mapping $\Sigma_r \to \Sigma_{n_1+\cdots+n_r}$, has a unique lifting to braid groups*

$$B_r \to B_{n_1+\cdots+n_r},$$

which is given by the picture of Figure 5.7 when we consider the case of a generating element $\tau_i \in B_r$, and which satisfies the following multiplicativity relation

$$(\alpha \cdot \beta)_*(n_1, \ldots, n_r) = \alpha_*(n_1, \ldots, n_r) \cdot \beta_*(n_{s(1)}, \ldots, n_{s(r)}),$$

for all $\alpha, \beta \in B_r$, and where s denotes the underlying permutation of the braid α.

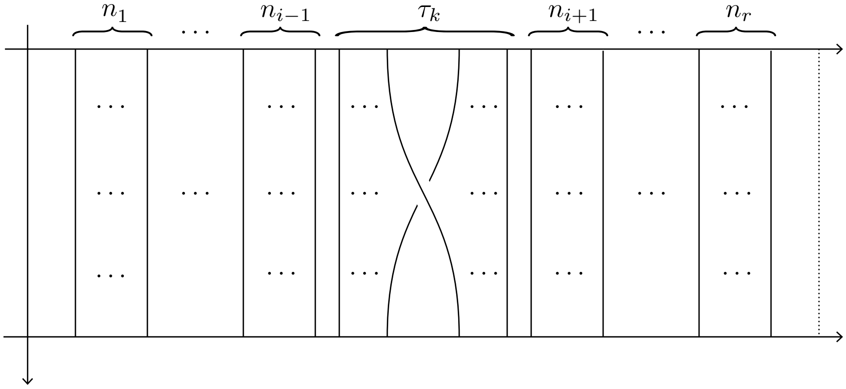


FIGURE 5.6. The direct sum $id_{n_1} \oplus \dots \oplus \tau_k \oplus \dots \oplus id_{n_r}$ in the braid group.

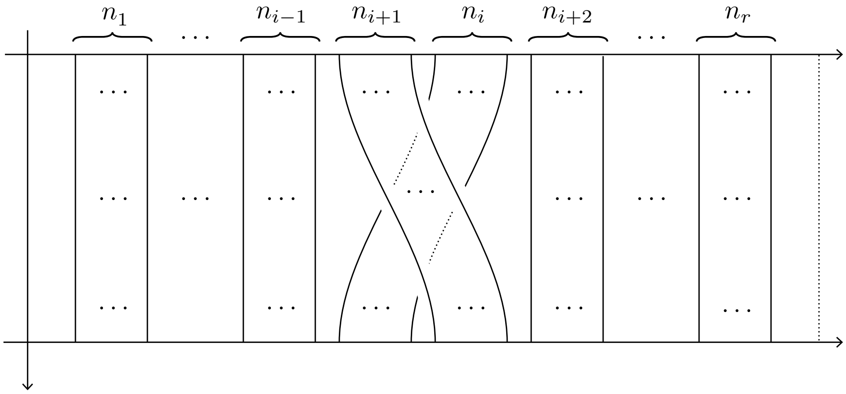


FIGURE 5.7. The block braid $(\tau_i)_*(n_1, \dots, n_r)$

(c) *In addition, we have the commutation relation*

$$\beta_1 \oplus \dots \oplus \beta_r \cdot \alpha_*(n_1, \dots, n_r) = \alpha_*(n_1, \dots, n_r) \cdot \beta_{s(1)} \oplus \dots \oplus \beta_{s(r)},$$

for all $\alpha \in B_r$, $(\beta_1, \dots, \beta_r) \in B_{n_1} \times \dots \times B_{n_r}$, *and where* s *denotes the underlying permutation of the braid* α *again.*

PROOF. The multiplicativity relations imply that these operations on braids are uniquely determined by fixing the image of generating elements. In each case, we just have to check that our mapping preserves the generating relations of the braid groups in order to establish the validity of our definition.

In the case of the direct sums (a), we have to check that our operations preserve the internal generating relations of braid groups which we form within each factor B_{n_i}, as well as the commutation relation

$$(id_{n_1} \oplus \dots \oplus \tau_k \oplus \dots \oplus id_{n_r}) \cdot (id_{n_1} \oplus \dots \oplus \tau_l \oplus \dots \oplus id_{n_r})$$
$$= (id_{n_1} \oplus \dots \oplus \tau_l \oplus \dots \oplus id_{n_r}) \cdot (id_{n_1} \oplus \dots \oplus \tau_k \oplus \dots \oplus id_{n_r})$$

when we take generating elements of disjoint factors B_{n_i} and B_{n_j}, $i \neq j$, of the cartesian product $B_{n_1} \times \cdots \times B_{n_r}$. These identities are visibly preserved by our mapping.

The preservation of the generating relations of the braid group B_r by the block braid construction of (b) is checked by a similar straightforward inspection.

The multiplicativity relations similarly imply that we are left to check the identity of assertion (c) in the case where one element among α and $\beta_1, \dots, \beta_r$ is a generating braid τ_k, and all the others are identity braids. The validity of the relation in this generating case is still immediate, and this verification completes the proof of our proposition. ☐

The braids of Figure 5.6 and Figure 5.7 can also be defined purely algebraically, in terms of the generating elements of the braid group $B_{n_1+\cdots+n_r}$. In the case of Figure 5.6, we have:

$$id_{n_1} \oplus \cdots \oplus \tau_k \oplus \cdots \oplus id_{n_r} = \tau_{k_i+k},$$

for all $\tau_k \in B_{n_i}$, where we set again $k_i = n_1 + \cdots + n_{i-1}$, for $i = 1, \dots, r$. In the case of Figure 5.7, we obtain:

$$\begin{aligned}(\tau_k)_*(n_1, \dots, n_r) = {} & \\ & (\tau_{k_i+n_i}\tau_{k_i+n_i-1} \cdots \tau_{k_i+1}) \cdot (\tau_{k_i+n_i+1}\tau_{k_i+n_i} \cdots \tau_{k_i+2}) \cdot \cdots \\ & \qquad \cdots \cdot (\tau_{k_i+n_i+n_{i+1}}\tau_{k_i+n_i+n_{i+1}-1} \cdots \tau_{k_i+n_{i+1}}).\end{aligned}$$

The definition of the permutation operad in Proposition 1.1.9 has the following braided analogue:

Proposition 5.1.3. *The collection of braid groups B_n, $n \in \mathbb{N}$, forms a braided operad in sets such that:*

(1) *the action of the braid group on each B_n is given by left translations;*
(2) *the identity braid on one strand $id_1 \in B_1$ defines the operadic unit;*
(3) *and the composition product $\mu : B_r \times (B_{n_1} \times \cdots \times B_{n_r}) \to B_{n_1+\cdots+n_r}$ maps a collection $\alpha \in B_r$, $(\beta_1, \dots, \beta_r) \in B_{n_1} \times \cdots \times B_{n_r}$, to the product element*

$$\alpha(\beta_1, \dots, \beta_r) = \beta_1 \oplus \cdots \oplus \beta_r \cdot \alpha_*(n_1, \dots, n_r)$$

in $B_{n_1+\cdots+n_r}$.

Proof. This statement easily follows from the relations of Proposition 5.1.2. ☐

In what follows, we use the notation B for the operad defined in this proposition which we also call the '*braid operad*'. To be more precise, when we use this notation B, we actually refer to a version of the braid operad where we forget about the term of arity zero of our object. We therefore assume $\mathsf{B}(r) = B_r$ for $r > 0$ and $\mathsf{B}(0) = \varnothing$. We use the notation B_+, with the extra subscript mark $+$, when we keep this term $\mathsf{B}_+(0) = B_0 = pt$ in our object. Thus, we adopt the same conventions for this operad as in the case of the permutation operad (see §1.1).

The result of §2.1, the equivalence between the plain definition of an operad and the definition in terms of partial composition operations has an obvious extension to braided operads. In the sequel, we use this definition, in terms of partial composites, rather than the definition of §5.1.1.

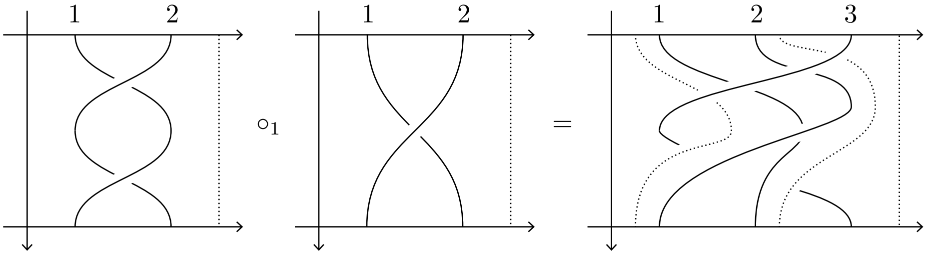


FIGURE 5.8. An operadic composition of braids

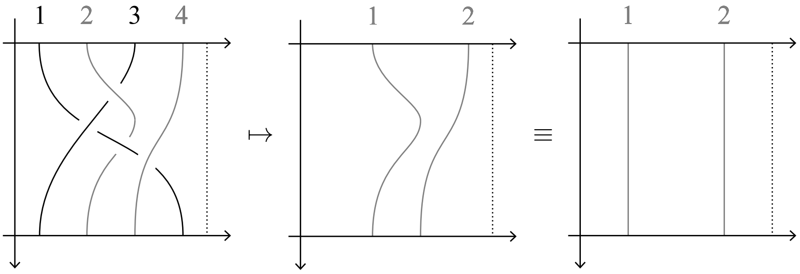


FIGURE 5.9. An instance of a restriction operator in the braid operad

Let $\alpha \in B_m$, $\beta \in B_n$. To illustrate the definition, we give an instance of an operadic composition of braids $\alpha \circ_k \beta = \alpha(id_1, \dots, \beta, \dots, id_1) \in B_{m+n-1}$ in Figure 5.8. Intuitively, the operadic composite $\alpha \circ_k \beta$ is obtained by inserting the braid β on the kth strand of the braid α. To ease the understanding of our picture, we have marked the array in which the braid β is inserted. In subsequent constructions, we will use that the strands which define the composite braid $\alpha \circ_k \beta$ in the outcome of this process are canonically in bijection with the strands of the braid α minus the kth one α_k plus the strands of the braid β.

5.1.4. *Unitary braided operads and restriction operators.* The notion of a unitary operad has an obvious analogue in the context of braided operads and so does the notion of a unitary extension of non-unitary operads. Furthermore, any unitary braided operad P_+ inherits restriction operators $u^* : \mathsf{P}_+(n) \to \mathsf{P}_+(m)$, associated to the increasing injections $u : \{1 < \cdots < m\} \to \{1 < \cdots < n\}$, and which we define as the composition operations $u^*(p) = p(*, \dots, 1, \dots, *, \dots, 1, \dots, *) \in \mathsf{P}_+(m)$ where we take an arity zero operation $* \in \mathsf{P}_+(0)$ at each place $j \notin \{u(1), \dots, u(m)\}$ as in the case of unitary symmetric operads (see §2.2). We can also define a braided analogue of the category of finite ordinals and injections of §2.2.2 in order to combine the action of braids $\alpha \in B_r$ with the restriction operators associated to these increasing injections $u \in \mathtt{Mor}_{\Lambda^+}(\underline{\mathtt{m}}, \underline{\mathtt{n}})$ in a single diagram structure.

The unitary braid operad B_+, for instance, defines a unitary extension of the non-unitary braid operad B since we have $\mathsf{B}_+(0) = B_0 = pt$ for this operad. The image of a braid $\alpha \in B_n$ under a restriction operator $u^* : B_n \to B_m$ is obtained by removing the strands α_k whose index k does not lie in the image of the map $u : \{1 < \cdots < m\} \to \{1 < \cdots < n\}$. Figure 5.9 gives an instance of application of this restriction process for the injection $u : \{1 < 2\} \to \{1 < 2 < 3 < 4\}$ such that $u(1) = 2$ and $u(2) = 4$.

The components of a symmetric operad naturally inherit an action of braid groups (by restriction through the canonical morphism $p_* : B_r \to \Sigma_r$) so that any symmetric operad naturally forms a braided operad. The next proposition gives a functor in the converse direction as this restriction operation:

Proposition 5.1.5.

(a) *Let* P *be any braided operad. We set* $\mathtt{Sym}\,\mathsf{P}(r) = \mathsf{P}(r)/P_r$, *for any* $r \in \mathbb{N}$. *The collection* $\mathtt{Sym}\,\mathsf{P} = \{\mathtt{Sym}\,\mathsf{P}(r), r \in \mathbb{N}\}$ *inherits a symmetric structure, a unit morphism, and operadic composition operations from the braided operad* P. *Hence, this collection forms a symmetric operad* $\mathtt{Sym}\,\mathsf{P}$ *naturally associated to* P.

(b) *The mapping* $\mathtt{Sym} : \mathsf{P} \mapsto \mathtt{Sym}\,\mathsf{P}$ *provides a left adjoint of the obvious functor which carries a braided operad to the symmetric operad defined by restricting the action of the braid groups on our objects to symmetric groups. The collection of quotient morphisms* $q : \mathsf{P}(r) \to \mathsf{P}(r)/P_r$ *defines a morphism of braided operads* $q : \mathsf{P} \to \mathtt{Sym}\,\mathsf{P}$ *which represents the augmentation of this adjunction.*

(c) *The mapping* $\mathtt{Sym} : \mathsf{P} \mapsto \mathtt{Sym}\,\mathsf{P}$ *preserves non-unitary operads and unitary extensions as well. To be explicit, for a unitary braided operad* P_+, *we have an obvious identity* $\mathtt{Sym}(\mathsf{P}_+) = \mathtt{Sym}(\mathsf{P})_+$ *in the category of symmetric operads.*

(d) *In the case of the braid operad* $\mathsf{B}(r) = B_r$, *we have* $\mathtt{Sym}\,\mathsf{B}(r) = B_r/P_r = \Sigma_r$ *and the symmetric operad* $\mathtt{Sym}\,\mathsf{B}$ *is identified with the permutation operad* $\mathsf{\Pi}$, *such as defined in Proposition* 1.1.9. *We have an analogous identity* $\mathtt{Sym}\,\mathsf{B}_+ = \mathsf{\Pi}_+$ *for the unitary version of the braid operad* B_+.

Proof. Since $\Sigma_r = B_r/P_r$, we immediately obtain that the action of B_r on $\mathsf{P}(r)$ induces an action of the symmetric group Σ_r on the quotient object $\mathsf{P}(r)/P_r$.

The operadic unit of P obviously defines a unit morphism $\mathbb{1} \xrightarrow{\eta} \mathtt{Sym}\,\mathsf{P}(1)$ at the level of the collection $\mathtt{Sym}\,\mathsf{P}$ since $\mathtt{Sym}\,\mathsf{P}(1) = \mathsf{P}(1)/P_1 = \mathsf{P}(1)$. Recall that the direct sums $\beta_1 \oplus \cdots \oplus \beta_r$ as well as the block braid construction $\alpha_*(n_1, \dots, n_r)$ of Proposition 5.1.2 lift the corresponding constructions on permutations. If $\beta_1, \dots, \beta_r$ are pure braids, then so is the direct sum $\beta_1 \oplus \cdots \oplus \beta_r$ because we have the identity $id_{n_1} \oplus \cdots \oplus id_{n_r} = id_{n_1+\cdots+n_r}$ at the level of permutations, and we have a similar assertion in the case of the block braid $\alpha_*(n_1, \dots, n_r)$. Thus, the permutations $\beta_1 \oplus \cdots \oplus \beta_r$ and $\alpha_*(n_1, \dots, n_r)$ that occur in the equivariance relations of braided operads are pure whenever α and $\beta_1, \dots, \beta_r$ are pure braids. From this observation, we immediately deduce that the composition products of the operad P induce composition products on the collection of quotient objects $\mathtt{Sym}\,\mathsf{P}(r) = \mathsf{P}(r)/P_r$ so that we have a commutative diagram

$$\begin{tikzcd}
\mathsf{P}(r) \otimes \mathsf{P}(n_1) \otimes \cdots \otimes \mathsf{P}(n_r) \arrow[r, "\mu"] \arrow[d] & \mathsf{P}(n_1 + \cdots + n_r) \arrow[d] \\
\mathsf{P}(r)/P_r \otimes \mathsf{P}(n_1)/P_{n_1} \otimes \cdots \otimes \mathsf{P}(n_r)/P_{n_r} \arrow[r, dotted, "\exists\mu"'] & \mathsf{P}(n_1 + \cdots + n_r)/P_{n_1+\cdots+n_r}
\end{tikzcd},$$

for every $r \geq 0$ and for all $n_1, \dots, n_r \geq 0$. The equivariance, unit and associativity relations of Figure 1.1-1.3 remain obviously satisfied in the quotient $\mathtt{Sym}\,\mathsf{P}$. This verification completes the construction of the symmetric operad $\mathtt{Sym}\,\mathsf{P}$ associated to P. The assertion about the adjunction relation follows from a straightforward inspection of our construction.

The identity between the symmetrization of the braid operad and the permutation operad follows from the observation that the operadic composition operations of braids $\alpha(\beta_1, \dots, \beta_r) = (\beta_1 \oplus \cdots \oplus \beta_r) \cdot \alpha_*(n_1, \dots, n_r)$ lift corresponding operadic composition operations on permutations.

The verification of assertion (c) of the proposition is immediate. □

We aim to prove that the topological operad of little 2-discs is given the symmetrization of a contractible braided operad in topological spaces. We actually have the following more explicit result:

THEOREM 5.1.6 (Z. Fiedorowicz [62]). *The universal coverings $\check{D}_2(r)$ of the little 2-disc spaces $D_2(r)$ form a braided operad in topological spaces $\check{D}_2$ with the operad of little 2-discs D_2 as associated symmetric operad.*

We have a similar result for the unitary extension of the operad of little 2-discs D_{2+}. We then get a unitary extension $\check{D}_{2+}$ of our braided operad of covering spaces $\check{D}_2$ such that $\mathtt{Sym}\,\check{D}_{2+} = D_{2+}$.

We address the proof of this theorem in a series of constructions and lemmas. We focus on the definition of the non-unitary operad structure on the collection of covering spaces $\check{D}_2(r)$, $r > 0$. The extension of our constructions to unitary operads is straightforward.

Recall that the definition of a universal covering depends on the choice of a base point in the base space. To be precise, the universal coverings associated to different base points are isomorphic, but we need to control the isomorphisms which relate our universal coverings in order to check that the axioms of operads hold at this level. We use a particular choice of base points in the little 2-disc spaces in order to work out this problem. We devote the next paragraph to this question.

5.1.7. *The choice of base points.* Recall that the operad of little 1-discs embeds into the little 2-discs operad by a topological inclusion $D_1 \hookrightarrow D_2$. In Proposition 4.1.6, we prove that each space $D_1(r)$ has contractible connected components $D_1(r)_w$ indexed by permutations $w \in \Sigma_r$. Recall that $\pi_0 D_1$ is also isomorphic to the permutation operad as an operad. Equivalently, the partial composition product $\circ_k : D_1(m) \times D_1(n) \to D_1(m+n-1)$ maps each cartesian product of connected components $D_1(m)_s \times D_1(n)_t$ into the connected component $D_1(m+n-1)_{s \circ_k t}$, associated to the composition product $s \circ_k t$ of the permutations $s \in \Sigma_m$ and $t \in \Sigma_n$ within the permutation operad.

We consider the contractible space $D_1(r)_{id}$ associated to the identity permutation $id = id_r \in \Sigma_r$ and the corresponding subspace of $D_2(r)$ which according to our definition (see §4.1.5) consists of configurations of little 2-discs of the form represented in Figure 5.10. We take a configuration of little 2-discs $\underline{c}^0$ inside the image of the space $D_1(r)_{id}$ as base point in the space of little 2-discs $D_2(r)$. We assume that $\check{D}_2(r)$ is the universal covering of the space $D_2(r)$ formed at this base point from now on.

Any disc configuration $\underline{c}$ coming from the subspace $D_1(r)_{id} \hookrightarrow D_2(r)$ can be connected to our base point $\underline{c}^0$ by a path γ^0 in that subspace $D_1(r)_{id} \hookrightarrow D_2(r)$. All paths of this form belong to the same homotopy class since $D_1(r)_{id}$ is contractible. Such a path gives a canonical isomorphism between the universal covering of $D_2(r)$ determined at the base point $\underline{c}$ and the universal covering $\check{D}_2(r)$ determined at our

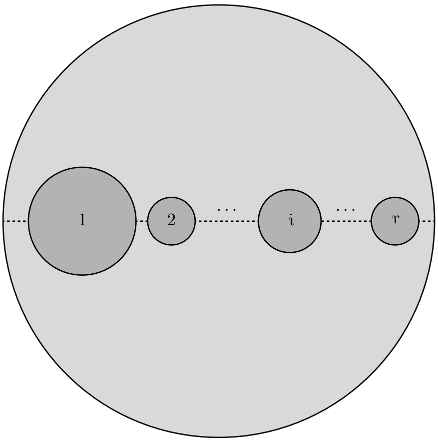


FIGURE 5.10. The form of a configuration of little 2-discs in the image of the contractible subspace $D_1(r)_{id} \hookrightarrow D_2(r)$ which we take as a base point in the space $D_2(r)$.

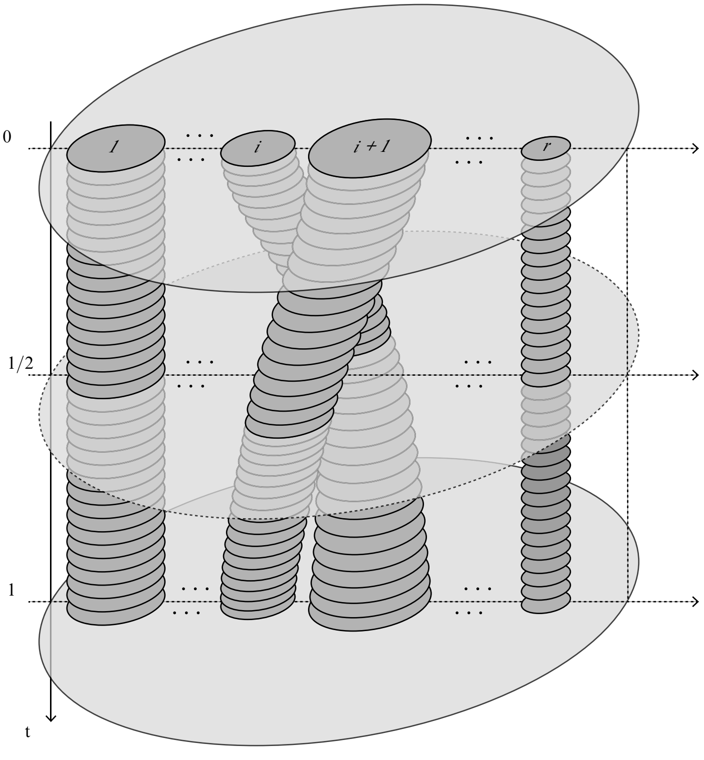


FIGURE 5.11. The path corresponding to the generating braid τ_i in the little 2-disc space $D_2(r)$.

chosen base point $\underline{c}^0$. We explain this process in the next paragraph. We use a classical construction of universal coverings, in terms of a space of homotopy classes of paths, in order to make our isomorphisms explicit.

5.1.8. *The construction of the universal coverings.* In short, we use that the covering space $\check{D}_2(r)$ can be defined by the set of homotopy classes of paths $\gamma : [0,1] \to D_2(r)$ with our base point $\underline{c}^0$ as origin:

$$\check{D}_2(r) = \{\gamma : [0,1] \to D_2(r) | \gamma(0) = \underline{c}^0\} / \simeq .$$

We briefly recall the definition of the topology of this space. We refer to standard textbooks (like [139, §V.10]) for further details on this general construction of the theory of coverings.

The homotopies considered in our definition of the space $\check{D}_2(r)$ are formed by continuous families of paths γ_s, $s \in [0,1]$, which have our base point as origin $\gamma_s(0) = \underline{c}^0$, and which are constant at the end-point $\gamma_s(1) \equiv \underline{c}^1$. In what follows, we use the notation $[\gamma]$ for the class of a path γ under this homotopy relation. We consider the map $q : \check{D}_2(r) \to D_2(r)$ which sends (the homotopy class of) a path $\gamma : [0,1] \to D_2(r)$ to its end-point $\gamma(1) \in D_2(r)$. We equip the set $\check{D}_2(r)$ with an appropriate topology in order to ensure that this map $q : \check{D}_2(r) \to D_2(r)$ defines a covering. We proceed as follows:

- each point $\underline{c} \in D_2(r)$ in the little 2-discs space $D_2(r)$ admits a basis of neighborhoods formed by contractible open sets $U_{\underline{c},\alpha}$, $\alpha \in \mathcal{I}$;
- to each path $\gamma : [0,1] \to D_2(r)$ such that $\gamma(0) = \underline{c}^0$ and $\gamma(1) = \underline{c}$, we associate the collection of sets $\check{U}_{\underline{c},\alpha} \subset \check{D}_2(r)$ which consist of homotopy classes of paths of the form $\alpha \cdot \gamma$ where α is any path such that $\alpha(0) = \underline{c}$ and $\alpha([0,1]) \subset U_{\underline{c},\alpha}$;
- we take this collection $\check{U}_{\underline{c},\alpha}$, $\alpha \in \mathcal{I}$, as a basis of open neighborhoods for the element $[\gamma]$ in the space $\check{D}_2(r)$.

This definition is actually forced by the requirement that the counterimage of the set $U_{\underline{c},\alpha}$ under the covering map $q : \check{D}_2(r) \to D_2(r)$ is a union of open sets.

In what follows, we omit to check the continuity of the maps which we define on covering spaces. These verifications generally reduce to straightforward inspections.

In the construction of this paragraph, the isomorphism which connects the space $\check{D}_2(r)$ with the universal covering taken at another base point $\underline{c}$ is given by the concatenation of the paths $\gamma : [0,1] \to D_2(r)$, which define the elements of the covering space $\check{D}_2(r)$, with a path $\gamma^0 : [0,1] \to D_2(r)$ such that $\gamma^0(0) = \underline{c}$ and $\gamma^0(1) = \underline{c}^0$. From this construction, we immediately see that this isomorphism is canonical as soon as the homotopy class of the path γ^0, which connects our base points, is uniquely determined and we can ensure that such a property hold when, as we set in §5.1.7, we restrict ourselves to base points $\underline{c}^0$ which belong to the image of the space $D_1(r)_{id}$ in $D_2(r)$.

5.1.9. *The action of braid groups.* The pure braid group P_r can immediately be identified with the group of automorphisms of the covering $\check{D}_2(r) \to D_2(r)$ because:

- the automorphism group of a universal covering is identified with the fundamental group of its base space,
- and the homotopy equivalence $\omega : D_2(r) \xrightarrow{\sim} F(\mathring{\mathbb{D}}^2, r)$, defined by the disc center mapping, induces a group isomorphism

$$\pi_1(D_2(r), *) \xrightarrow{\simeq} \pi_1(F(\mathring{\mathbb{D}}^2, r), *) = P_r.$$

One can adapt this approach in order to prove that the action of P_r on $\check{D}_2(r)$ extends to an action of the full braid group B_r. Indeed, we can also identify our covering space $\check{D}_2(r)$ with the universal covering of the quotient space $D_2(r)/\Sigma_r$, for which we have $\pi_1(D_2(r)/\Sigma_r, *) \xrightarrow{\simeq} \pi_1(F(\mathring{\mathbb{D}}^2, r)/\Sigma_r, *) = B_r$.

We give an explicit construction of this action in order to ease our subsequent verification of the equivariance relations of operadic composition products on our covering spaces $\check{D}_2(r)$. We rely on the explicit definition of the universal covering space $\check{D}_2(r)$ of §5.1.8. We consider a path in the little 2-disc space $\tau_i : [0,1] \to D_2(r)$ of the form represented in Figure 5.11. We immediately see from our picture that the image of this path under the disc center mapping $\omega : D_2(r) \to F(\mathring{\mathbb{D}}^2, r)$ is a representative of the generating braid of Figure 5.4.

Note that the endpoint of this path $\tau_i(1)$ is identified with the image of our base disc configuration $\underline{c}^0$ under the action of the transposition $t_i = (i\ i+1)$.

Let now $\gamma : [0,1] \to D_2(r)$ be a path in $D_2(r)$ with $\gamma(0) = \underline{c}^0$ as origin so that the homotopy class of this path $[\gamma]$ defines an element of the covering space $\check{D}_2(r)$. We apply the transposition t_i to this path in order to obtain a path $t_i\gamma$ with $t_i\gamma(0) = t_i\underline{c}^0$ as origin. We concatenate $t_i\gamma$ with the path represented in Figure 5.11 to obtain a new path $(t_i\gamma) \cdot \tau_i : [0,1] \to D_2(r)$ with $\underline{c}^0$ as origin and of which homotopy class $[(t_i\gamma) \cdot \tau_i]$ determines an element of the space $\check{D}_2(r)$.

By an immediate visual inspection of these constructions, we obtain that:

Lemma 5.1.10.
(a) *The mapping* $\tau_i : [\gamma] \mapsto [(t_i\gamma) \cdot \tau_i]$ *defines a lifting to the space* $\check{D}_2(r)$ *of the map* $t_i : D_2(r) \to D_2(r)$ *which gives the action of the transposition* $t_i = (i\ i+1)$ *on the space of little* 2*-discs* $D_2(r)$.
(b) *The maps* $\tau_i : \check{D}_2(r) \to \check{D}_2(r)$, $i = 1, \dots, r-1$, *which we deduce from this construction, satisfy the generating relations of braids groups, and hence, determine an action of the braid group* B_r *on the covering space* $\check{D}_2(r)$. ☐

This result completes the construction of the action of the braid groups on the collection of spaces $\check{D}_2 = \{\check{D}_2(r), r > 0\}$.

We can use a similar composition process $[\gamma] \mapsto [\gamma \cdot \omega]$ when $\omega : [0,1] \to D_2(r)$ is any loop based at $\omega(0) = \omega(1) = \underline{c}^0$ in order to determine the action of the fundamental group $\pi_1(D_2(r), \underline{c}^0)$ on the universal covering $\check{D}_2(r)$. We immediately see that this action corresponds to a restriction of the action considered in Lemma 5.1.10 when we apply the isomorphism $\pi_1(D_2(r), *) \xrightarrow{\simeq} \pi_1(F(\mathring{\mathbb{D}}^2, r), *)$ to identify $\pi_1(D_2(r), *)$ with the pure braid group P_r.

The following statement follows from this identification and from standard results of covering theory:

Lemma 5.1.11. *The covering map* $q : \check{D}_2(r) \to D_2(r)$ *defined in* §5.1.8 *induces a homeomorphism* $q_* : \check{D}_2(r)/P_r \xrightarrow{\simeq} D_2(r)$, *where the quotient space* $\check{D}_2(r)/P_r$ *is formed by considering the restriction of the action of Lemma* 5.1.10 *to the pure braid group* P_r. ☐

5.1.12. *The operadic composition structure.* We now aim to define operadic composition operations on the collection $\check{D}_2$. We can take the operadic unit of the little 2-disc operad $1 \in D_2(1)$ as base point in the space $D_2(1)$. We then take the homotopy class of the constant path $1(t) \equiv 1$ associated to this obvious element $1 \in D_2(1)$ to define the operadic unit of the object $\check{D}_2$.

We proceed as follows to define our composition products on $\check{D}_2$. Let $\alpha : [0,1] \to D_2(m)$ (respectively, $\beta : [0,1] \to D_2(n)$) be a path which represents an element of the covering space $\check{D}_2(m)$ (respectively, $\check{D}_2(n)$). Let $\underline{a}^0 = \alpha(0)$ (respectively, $\underline{b}^0 = \beta(0)$) be the base point in the little 2-discs space $D_2(m)$ (respectively, $D_2(n)$) which underlies this covering. We fix a composition index $k \in \{1,\dots,m\}$. By performing the operadic composition of little 2-discs pointwise, we obtain a path $\alpha \circ_k \beta : [0,1] \to D_2(m+n-1)$ with $\alpha \circ_k \beta(0) = \underline{a}^0 \circ_k \underline{b}^0$ as origin. This configuration of little 2-discs $\underline{a}^0 \circ_k \underline{b}^0$ is not necessarily equal to the base point $\underline{c}^0$ which we have chosen in the little 2-disc space $D_2(m+n-1)$, but the assumptions that $\underline{a}^0$ lies in the image of the space $D_1(m)_{id}$ inside $D_2(m)$ and that $\underline{b}^0$ lies in the image of the space $D_1(n)_{id}$ inside $D_2(n)$ imply that $\underline{a}^0 \circ_k \underline{b}^0$ comes from the space $D_1(m+n-1)_{id}$ too, because the operadic composition operations on the set of connected components of the little 1-discs operad correspond to the operadic composition of permutations and we have $id_m \circ_k id_n = id_{m+n-1}$ when we consider identity permutations (see Proposition 1.1.9). Thus, we can fix a path $\gamma^0 : [0,1] \to D_2(m+n-1)$ such that $\gamma^0(0) = \underline{c}^0$ and $\gamma^0(1) = \underline{a}^0 \circ_k \underline{b}^0$ which entirely lies in the image of the space $D_1(m+n-1)_{id}$ in $D_2(m+n-1)$ (see §5.1.7). We concatenate our composite $\alpha \circ_k \beta$ with such a path $\gamma^0 : [0,1] \to D_2(m+n-1)$. The homotopy class $[\alpha \circ_k \beta \cdot \gamma^0]$ defines an element of $\check{D}_2(m+n-1)$ naturally associated to $[\alpha] \in \check{D}_2(m+n-1)$ and $[\beta] \in \check{D}_2(m+n-1)$. This mapping gives a composition product

$$\circ_k : \check{D}_2(m) \times \check{D}_2(n) \to \check{D}_2(m+n-1)$$

which obviously lifts the corresponding composition product of the little 2-discs operad. We prove that:

LEMMA 5.1.13. *The composition products defined in the previous paragraph* $\circ_k : \check{D}_2(m) \times \check{D}_2(n) \to \check{D}_2(m+n-1)$ *fulfill the equivariance relations of braided operads, as well as the unit relations and the associativity relations of operads in the covering spaces* $\check{D}_2(r)$, $r > 0$.

PROOF. The proof of the unit and associativity relations of composition products follows from a straightforward verification. We use the explicit definition of the action of the generating braids τ_i in §§5.1.9-5.1.10 to check that our composition products are also equivariant with respect to the action of these elements τ_i in the braid group. The verification of this generating case suffices to prove the equivariance of our composition products in full generality. ☐

The covering maps $q : \check{D}_2(r) \to D_2(r)$ clearly define a morphism of braided operads $q : \check{D}_2 \to D_2$. The assertion of Lemma 5.1.11 also implies that this morphism induces an isomorphism between the symmetrized operad $\mathtt{Sym}\,\check{D}_2$ and D_2. This verification finishes the proof of Theorem 5.1.6. ☐

Theorem 5.1.6 has the following consequence:

THEOREM 5.1.14 (Z. Fiedorowicz [62]). *Let P be a non-unitary braided operad in topological spaces. Suppose that the action of B_r on $P(r)$ is free and proper, for all $r > 0$. If the spaces $P(r)$ are contractible for all $r > 0$, then the symmetric operad* $\mathtt{Sym}\,P$ *such that* $\mathtt{Sym}\,P(r) = P(r)/P_r$ *forms a (non-unitary) E_2-operad and we have an obvious extension of this result in the context of unitary operads.*

PROOF. We again focus on the case of non-unitary operads. The extension of our argument lines to general operads is straightforward. We form the aritywise product $\mathsf{Q}(r) = \mathsf{P}(r) \times \check{\mathsf{D}}_2(r)$ in the category of braided operads. The braid group B_r operates diagonally on the space $\mathsf{Q}(r) = \mathsf{P}(r) \times \check{\mathsf{D}}_2(r)$, for each $r > 0$, and we equip the collection $\mathsf{Q} = \{\mathsf{P}(r) \times \check{\mathsf{D}}_2(r), r > 0\}$ with a braided operad structure by using a straightforward extension, to braided operads, of the aritywise tensor product of symmetric operads (see §3.2.3). The canonical projections

$$\mathsf{P}(r) \leftarrow \mathsf{P}(r) \times \check{\mathsf{D}}_2(r) \rightarrow \check{\mathsf{D}}_2(r) \tag{1}$$

define morphisms of braided operads $\mathsf{P} \leftarrow \mathsf{Q} \rightarrow \check{\mathsf{D}}_2$.

Recall that the spaces $\mathsf{P}(r)$ are contractible by assumption and we have already observed that the spaces $\check{\mathsf{D}}_2(r)$ are contractible too. Thus, the considered projections are weak-equivalences between contractible spaces.

The braid group B_r operates freely and properly on $\mathsf{P}(r)$ by assumption, and on $\check{\mathsf{D}}_2(r)$ as well by definition of this space as a universal covering. The diagonal action of B_r on $\mathsf{P}(r) \times \check{\mathsf{D}}_2(r)$ is free and proper too. Hence, the maps (1) induce weak-equivalences

$$\mathsf{P}(r)/P_r \xleftarrow{\sim} (\mathsf{P}(r) \times \check{\mathsf{D}}_2(r))/P_r \xrightarrow{\sim} \check{\mathsf{D}}_2(r)/P_r = \mathsf{D}_2(r), \tag{2}$$

when we take the quotient of our spaces under the action of the group $P_r \subset B_r$. These maps (2) represent the components of the morphisms of symmetric operads $\mathtt{Sym}\,\mathsf{P} \leftarrow \mathtt{Sym}\,\mathsf{Q} \rightarrow \mathtt{Sym}\,\check{\mathsf{D}}_2$ associated to our morphisms of braided operads $\mathsf{P} \leftarrow \mathsf{Q} \rightarrow \check{\mathsf{D}}_2$. Hence, we conclude that these morphisms of symmetric operads define weak-equivalences $\mathtt{Sym}\,\mathsf{P} \xleftarrow{\sim} \mathtt{Sym}\,\mathsf{Q} \xrightarrow{\sim} \mathtt{Sym}\,\check{\mathsf{D}}_2 = \mathsf{D}_2$ and this observation completes the proof of our theorem. ☐

5.2. The classifying spaces of the colored braid operad

Recall that an Eilenberg-MacLane space of type $K(G, 1)$, where G is any group, is a connected space X such that $\pi_1(X) = G$ and $\pi_*(X) = 0$ for $* \neq 1$. These conditions determine the homotopy type of the space X (all Eilenberg-MacLane spaces of a given type $K(G, 1)$ are weakly-equivalent).

In §5.0, we mentioned that the underlying spaces of the little 2-discs operad D_2 are Eilenberg-MacLane spaces $K(P_r, 1)$ associated to the pure braid groups P_r. This result follows from the existence of the homotopy equivalence $\mathsf{D}_2(r) \xrightarrow{\sim} \mathsf{F}(\mathring{\mathbb{D}}^2, r)$, established in Proposition 4.2.2, and from the computation of the homotopy groups of the configuration spaces $\mathsf{F}(\mathring{\mathbb{D}}^2, r)$ in Proposition 5.0.1. We have a standard simplicial model BG for the Eilenberg-MacLane space $K(G, 1)$ which is usually called the *classifying space* of the group G, because this model BG represents the base space of a universal G-principal bundle.

The purpose of this section is to define a classifying space model of the little 2-disc operad D_2. We do not have a full operad structure on the collection of pure braid groups and we therefore have to consider an extension of the classifying space construction to groupoids and to operad in groupoids in order to define this model. We are precisely going to construct a collection of groupoids, the colored braid groupoids $\mathsf{CoB}(r)$, which contain the pure braid groups P_r as automorphism groups of objects, and which form an operad in the category of groupoids. We then prove that the collection of classifying spaces $\mathtt{B}(\mathsf{CoB}(r))$ which we associate to this operad in groupoids CoB defines a model of an E_2-operad.

To begin with, we make explicit the definition of an operad in the category of small categories and in the category of groupoids. Then we recall the definition of the classifying space of a category and we examine the application of this classifying space construction to operads in categories. We define the colored braid operad afterwards as an instance of an operad in groupoids.

5.2.1. *The category of small categories and groupoids.* We use the notation $\mathcal{C}at$ for the category of small categories. The cartesian product of categories $\times : \mathcal{C}at \times \mathcal{C}at \to \mathcal{C}at$ defines the tensor product operation of a symmetric monoidal structure on $\mathcal{C}at$. The one-point set pt, which is identified with the final object of the category of small categories, defines the unit object of this symmetric monoidal structure. (We generally assume that a set S is identified with a discrete category with S as object set and with no other morphism than identity morphisms.)

Recall that a *groupoid* is a small category in which all morphisms are invertible and that groups can be identified with groupoids with a single object. We use the notation $\mathcal{G}rd$ for the full subcategory of groupoids inside the category of small categories $\mathcal{C}at$. We immediately see that the embedding $\mathcal{G}rd \hookrightarrow \mathcal{C}at$ creates products and final objects. The category of groupoids $\mathcal{G}rd$ forms, therefore, a symmetric monoidal subcategory of the category of small categories $\mathcal{C}at$.

5.2.2. *Operads in small categories and in groupoids.* We use our general definition of an operad in a symmetric monoidal category in order to define the category of operads in the category of small categories and in the category of groupoids. Thus, an operad in the category of small categories P (we also speak about *operads in categories* for short) consists of a sequence of small categories $\mathsf{P}(r) \in \mathcal{C}at$, $r \in \mathbb{N}$, equipped with an action of the symmetric groups Σ_r, together with a unit morphism $\eta : pt \to \mathsf{P}(1)$, and composition products $\mu : \mathsf{P}(r) \times \mathsf{P}(n_1) \times \cdots \times \mathsf{P}(n_r) \to \mathsf{P}(n_1 + \cdots + n_r)$, all formed in the category of categories, and which satisfy our usual equivariance, unit and associativity relations in this category. Since the category of groupoids forms a symmetric monoidal subcategory of the category of small categories, we can also define an operad in the category of groupoids as an operad in the category of categories P of which components are groupoids $\mathsf{P}(r) \in \mathcal{G}rd$, for all $r \in \mathbb{N}$.

The equivalence between our first definition of an operad (see §1.1) and the definition in terms of partial composition operations (see §2.1) naturally holds in the context of the category of categories $\mathcal{M} = \mathcal{C}at$ (respectively, of groupoids $\mathcal{M} = \mathcal{G}rd$). Hence, the composition structure of an operad in categories (respectively, groupoids) can also be defined by giving a collection of functors $\circ_k : \mathsf{P}(m) \times \mathsf{P}(n) \to \mathsf{P}(m + n - 1)$, $k = 1, \dots, m$, which satisfies the equivariance, unit and associativity relations of §2.1.

The category of operads in categories is denoted by $\mathcal{C}at\,\mathcal{O}p$ (following our conventions). By definition, a morphism of operads in categories $\phi : \mathsf{P} \to \mathsf{Q}$ is a sequence of functors $\phi : \mathsf{P}(r) \to \mathsf{Q}(r)$ which preserve the structure operations of our operads. The category of operads in groupoids, also denoted by $\mathcal{G}rd\,\mathcal{O}p$, forms a full subcategory of the category of operads in categories $\mathcal{C}at\,\mathcal{O}p$.

In what follows, we consider operad morphisms $\phi : \mathsf{P} \to \mathsf{Q}$ of which all underlying functors $\phi : \mathsf{P}(r) \to \mathsf{Q}(r)$ are equivalences of categories. We then say that our operad morphism ϕ is a *categorical equivalence*, and we use the notation $\phi : \mathsf{P} \xrightarrow{\sim} \mathsf{Q}$, with the distinguishing mark $\sim$, for these morphisms. Note that the inverse equivalences of the functors $\phi : \mathsf{P}(r) \xrightarrow{\sim} \mathsf{Q}(r)$ do not necessarily define an operad morphism in general, and we do not assume that such a property holds in our definition of a categorical equivalence of operads.

5.2.3. *Recollections on classifying spaces.* The classifying space of a category $\mathcal{C}$ is the simplicial set $\mathtt{B}(\mathcal{C})$ defined in dimension n by the n-fold sequences of composable morphisms of $\mathcal{C}$

$$\underline{\alpha} = \{x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} x_n\}$$

together with the face operators such that:

$$d_i(\underline{\alpha}) = \begin{cases} x_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} x_n, & \text{for } i = 0, \\ x_0 \xrightarrow{\alpha_1} \cdots \to x_{i-1} \xrightarrow{\alpha_{i+1}\alpha_i} x_{i+1} \to \cdots \xrightarrow{\alpha_n} x_n, & \text{for } i = 1,\dots,n-1, \\ = x_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{n-1}} x_{n-1}, & \text{for } i = n, \end{cases}$$

and the degeneracy operators such that:

$$s_j(\underline{\alpha}) = x_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_j} x_j \xrightarrow{id} x_j \xrightarrow{\alpha_{j+1}} \cdots \xrightarrow{\alpha_n} x_n, \quad \text{for all } j = 0,\dots,n.$$

We define the classifying space of a groupoid $\mathtt{B}(\mathcal{G})$ by the same construction, by using the definition of a groupoid as a small category in which all morphisms are invertible. We can also apply this classifying space construction to a group G, which we identify with a groupoid with a single object $\mathtt{Ob}\, G = *$. We also use the notation $BG = \mathtt{B}(G)$ for the classifying space in this case. We soon explain that this space BG is identified with the quotient of a contractible space EG on which the group G operates freely (see §5.2.13). The space EG actually represents the total space of a universal principal G-bundle with the space BG as basis.

Let us mention that the simplicial set $\mathtt{B}(\mathcal{C})$ forms a Kan complex if and only if the category $\mathcal{C}$ is a groupoid (see for instance [79, §I.3]). This result can be used to check, by a direct and simple computation, that the geometric realization of the classifying space BG of a group G is an Eilenberg-MacLane space of type $K(G,1)$ (use the combinatorial definition of simplicial homotopy groups in [51, §2] or in [141, §1]).

The mapping $\mathtt{B} : \mathcal{C} \mapsto \mathtt{B}(\mathcal{C})$ defines a functor from the category of small categories $\mathcal{C}$ to the category of simplicial sets $s\mathcal{S}et$, and we have the following easy result:

Proposition 5.2.4. *The functor* $\mathtt{B} : \mathcal{C}at \to s\mathcal{S}et$ *is symmetric monoidal in the sense of* §3.3.1*:*

(1) *for the one-point set pt, viewed as the unit object of the category of small categories, we have an obvious identity* $\mathtt{B}(pt) = pt$*;*

(2) *for a cartesian product of categories $\mathcal{C} \times \mathcal{D}$, we have an isomorphism of simplicial sets* $\mathtt{B}(\mathcal{C} \times \mathcal{D}) \xrightarrow{\simeq} \mathtt{B}(\mathcal{C}) \times \mathtt{B}(\mathcal{D})$ *which is yielded by the maps of classifying spaces* $\mathtt{B}(\mathcal{C}) \xleftarrow{p_*} \mathtt{B}(\mathcal{C} \times \mathcal{D}) \xrightarrow{q_*} \mathtt{B}(\mathcal{D})$ *induced by the canonical projections* $\mathcal{C} \xleftarrow{p} \mathcal{C} \times \mathcal{D} \xrightarrow{q} \mathcal{D}$*;*

(3) *and these comparison isomorphisms (*1-2*) fulfill the unit, associativity and symmetry constraints of* §3.3.1.

Proof. The proof of assertions (1-2) reduces to a straightforward inspection of definitions. The definition of the isomorphism $\mathtt{B}(\mathcal{C} \times \mathcal{D}) \xrightarrow{\simeq} \mathtt{B}(\mathcal{C}) \times \mathtt{B}(\mathcal{D})$ from universal categorical constructions automatically ensures that the unit, associativity and symmetry constraints of §3.3.1 are fulfilled. ☐

Then the result of Proposition 3.1.1 implies:

PROPOSITION 5.2.5. *Let P be an operad in small categories. The collection of classifying spaces $\mathtt{B}(\mathsf{P})(r) = \mathtt{B}(\mathsf{P}(r))$, which we associate to the categories $\mathsf{P}(r)$, $r \in \mathbb{N}$, forms an operad in simplicial sets naturally associated to P.* □

We can also take a restriction of the functor defined in this proposition $\mathtt{B} : \mathsf{P} \mapsto \mathtt{B}(\mathsf{P})$ to the category non-unitary operads $\mathsf{P} \in \mathcal{C}at\,\mathcal{O}p_{\varnothing}$. We easily check that the object $\mathtt{B}(\mathsf{P})$ forms a non-unitary operad in this case (when we regard the category of non-unitary operads in simplicial sets as a full subcategory of the category of all operads). Recall that, in the situation of Proposition 5.2.4, the mapping $\mathtt{B} : \mathsf{P} \mapsto \mathtt{B}(\mathsf{P})$ also preserves unitary extensions, so that we have an identity $\mathtt{B}(\mathsf{P}_+) = \mathtt{B}(\mathsf{P})_+$ for any unitary operad in the category of small categories $\mathsf{P}_+ \in \mathcal{C}at\,\mathcal{O}p_*$ (see Proposition 3.1.1).

In §3.3.2, we observe that the geometric realization functor $|-| : s\mathcal{S}et \to \mathcal{T}op$ is also symmetric monoidal. We can apply this functor to the simplicial operad $\mathtt{B}(\mathsf{P})$ in order to form an operad in topological spaces naturally associated to P. In general, we abusively use the notation of the underlying simplicial operad $\mathtt{B}(\mathsf{P})$ for this operad in topological spaces. We only mark the application of the realization functor $|-|$ when the context requires to distinguish the topological object from its simplicial counterpart.

The mapping $\mathtt{B} : \mathsf{P} \mapsto \mathtt{B}(\mathsf{P})$ defines a functor from the category of operads in the category of small categories to the category of operads in simplicial sets. In §4, we introduced a notion of weak-equivalence for the category of operads in topological spaces. In the simplicial framework, we consider weak-equivalences of simplicial sets, which are maps $f : X \to Y$ of which geometric realization $|f| : |X| \to |Y|$ defines a weak-equivalence of topological spaces. We then say that a morphism of operads in simplicial sets $\phi : \mathsf{P} \to \mathsf{Q}$ is a weak-equivalence if each component of this morphism $\phi : \mathsf{P}(r) \to \mathsf{Q}(r)$ defines a weak-equivalence in the category of simplicial sets. From this definition, we immediately see that a morphism of operads in simplicial sets is a weak-equivalence $\phi : \mathsf{P} \xrightarrow{\sim} \mathsf{Q}$ if and only if the geometric realization of this morphism defines a weak-equivalence of operads in topological spaces $|\phi| : |\,\mathsf{P}\,| \xrightarrow{\sim} |\,\mathsf{Q}\,|$.

The following proposition, which is an immediate corollary of a standard result on classifying spaces, is worth recording:

PROPOSITION 5.2.6. *The classifying space functor $\mathtt{B} : \mathcal{C}at\,\mathcal{O}p \to s\mathcal{S}et\,\mathcal{O}p$ maps the categorical equivalences of operads in categories $\phi : \mathsf{P} \xrightarrow{\sim} \mathsf{Q}$ to weak-equivalences of operads in simplicial sets $\mathtt{B}(\phi) : \mathtt{B}(\mathsf{P}) \xrightarrow{\sim} \mathtt{B}(\mathsf{Q})$.* □

The rest of this section is devoted to the definition of the colored braid operad CoB and to the proof that the associated classifying space operad $\mathtt{B}(\mathsf{CoB})$ defines an instance of E_2-operad. We also establish a unitary extension of this result. In a first step, we define the underlying groupoids of this operad.

In general, we define a small category by giving an object set $\mathtt{Ob}\,\mathcal{C}$ together with morphism sets $\mathtt{Mor}_{\mathcal{C}}(x,y)$, for all pairs of objects $x, y \in \mathtt{Ob}\,\mathcal{C}$. But the information is carried by the morphisms in the case of the groupoids of colored braids $\mathsf{CoB}(r)$. Therefore, we use another approach for the definition of these groupoids $\mathsf{CoB}(r)$. We give an object set $\mathtt{Ob}\,\mathsf{CoB}(r)$ and a single morphism set $\mathtt{Mor}\,\mathsf{CoB}(r)$, which collects all morphisms of our groupoid, together with maps $s, t : \mathtt{Mor}\,\mathsf{CoB}(r) \to \mathtt{Ob}\,\mathsf{CoB}(r)$

which reflect the way to retrieve the source and the target object information from the definition of a morphism in $\mathtt{Mor}\,\mathsf{CoB}(r)$. We give a definition of the colored braid groupoid in these terms first. We make explicit an equivalent definition of the morphism set $\mathtt{Mor}_{\mathsf{CoB}(r)}(u,v)$ associated to each pair of objects $u,v\in\mathtt{Ob}\,\mathsf{CoB}(r)$ in terms of a coset decomposition of the braid groups B_r afterwards. We follow a similar plan when we explain the definition of an action of the symmetric groups Σ_r and of operadic composition operations on our groupoids $\mathsf{CoB}(r)$, $r\in\mathbb{N}$.

5.2.7. *Groupoids revisited.* In a preliminary step, we explain the general definition of a groupoid $\mathcal{G}$ when we use a single morphism set $\mathtt{Mor}\,\mathcal{G}$ to collect all morphisms of our object. In this context, we assume in that we have a pair of maps $s,t:\mathtt{Mor}\,\mathcal{G}\to\mathtt{Ob}\,\mathcal{G}$ such that $s(\alpha)\in\mathtt{Ob}\,\mathcal{G}$ represents the source object and $t(\alpha)\in\mathtt{Ob}\,\mathcal{G}$ represents the target object which we assign to any morphism $\alpha\in\mathtt{Mor}\,\mathcal{G}$. We also assume that we have a map $e:\mathtt{Ob}\,\mathcal{G}\to\mathtt{Mor}\,\mathcal{G}$, satisfying $se=te=id$, which gives the identity morphism $id_x=e(x)\in\mathtt{Mor}\,\mathcal{G}$ associated to each object $x\in\mathtt{Ob}\,\mathcal{G}$. The morphism set $\mathtt{Mor}_{\mathcal{G}}(x,y)$ which we associate to a pair of objects in our groupoid $x,y\in\mathtt{Ob}\,\mathcal{G}$ is defined by the subset of morphisms $\alpha\in\mathtt{Mor}\,\mathcal{G}$ such that $s(\alpha)=x$ and $t(\alpha)=y$. The fiber product

$$\begin{tikzcd}
\mathtt{Mor}\,\mathcal{G}\times_{st}\mathtt{Mor}\,\mathcal{G} \arrow[r, dotted, "q"] \arrow[d, dotted, "p"'] & \mathtt{Mor}\,\mathcal{G}\ , \arrow[d, "t"] \\
\mathtt{Mor}\,\mathcal{G} \arrow[r, "s"'] & \mathtt{Ob}\,\mathcal{G}
\end{tikzcd}$$

which we can more explicitly define as the set of pairs $(\alpha,\beta)\in\mathtt{Mor}\,\mathcal{G}\times\mathtt{Mor}\,\mathcal{G}$ such that $s(\alpha)=t(\beta)$, collects all pairs of composable morphisms in our groupoid. The composition of morphisms in $\mathcal{G}$ is given by a product operation $\mu:\mathtt{Mor}\,\mathcal{G}\times_{st}\mathtt{Mor}\,\mathcal{G}\to\mathtt{Mor}\,\mathcal{G}$, defined on this fiber product, and such that $s\mu=sq$, $t\mu=tp$. Thus, we have the formulas $s(\alpha\cdot\beta)=s(\beta)$ and $t(\alpha\cdot\beta)=t(\alpha)$, for all composable morphisms $(\alpha,\beta)\in\mathtt{Mor}\,\mathcal{G}\times_{st}\mathtt{Mor}\,\mathcal{G}$, where we set $\alpha\cdot\beta=\mu(\alpha,\beta)$.

To define the inverse of morphisms in a groupoid, we similarly consider a map $\iota:\mathtt{Mor}\,\mathcal{G}\to\mathtt{Mor}\,\mathcal{G}$ such that $s\iota=t$ and $t\iota=s$. The unit, associativity, and inverse relations of the composition structure of groupoids can be written in terms of commutative diagrams by using the product operation μ on the fiber product $\mathtt{Mor}\,\mathcal{G}\times_{st}\mathtt{Mor}\,\mathcal{G}$, but we prefer to use the standard pointwise expressions of these relations in what follows, because we define the product and inversion maps of our groupoids as maps between explicit point-sets.

5.2.8. *The groupoids of colored braids.* The object set $\mathtt{Ob}\,\mathsf{CoB}(r)$ of the *groupoid of colored braids on r strands* $\mathsf{CoB}(r)$ is the set of permutations $w\in\Sigma_r$ which we regard as orderings $(w(1),\dots,w(r))$ of the values $(1,\dots,r)$. The morphism set $\mathtt{Mor}\,\mathsf{CoB}(r)$ consists of isotopy classes of braids α together with a bijection $i\mapsto\alpha_i$ between the index set $i\in\{1,\dots,r\}$ and the strands $\alpha_i\in\{\alpha_1,\dots,\alpha_r\}$ of our braid α. Intuitively, this bijection assigns a *color* $i\in\{1,\dots,r\}$ to each strand α_i.

In §5.0.5, we assume that the strands of a braid form an r-tuple $(\alpha_1,\dots,\alpha_r)$ ordered according to the ordering of the points $(\alpha_1(0),\dots,\alpha_r(0))$ on the line $(y=0,t=0)$. In the colored braid case, we rather consider the ordering equivalent to the bijection $i\mapsto\alpha_i$ which we give with our braid α. Thus, we have $(\alpha_1(0),\dots,\alpha_r(0))=((x^0_{u(1)},0,0),\dots,(x^0_{u(r)},0,0)$ for some permutation $u\in\Sigma_r$ (not necessarily equal to the identity), where we again use the notation x^0_k, for $k=1,\dots,r$, for the

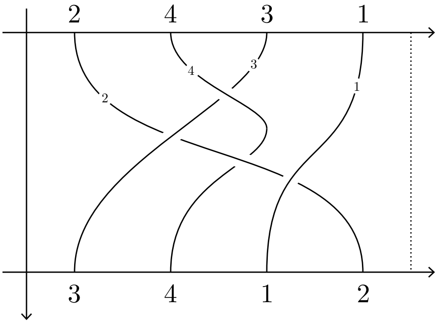


FIGURE 5.12. An instance of colored braid.

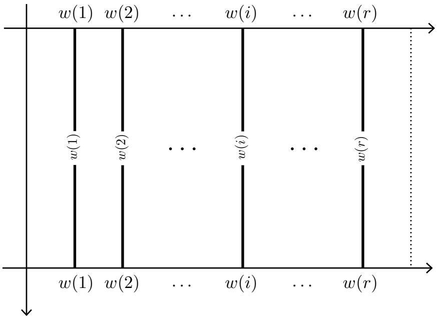


FIGURE 5.13. The representation of identity elements in the groupoid of colored braids.

abscissa of the contact points of our braids on the axis $y = 0$ (see §5.0.5). This permutation defines the source of our braid $u = s(\alpha)$ in the groupoid $\mathit{CoB}(r)$. The target of our braid $v = t(\alpha)$ is the permutation v such that $(\alpha_1(1), \dots, \alpha_r(1)) = ((x^0_{v(1)}, 0, 1), \dots, (x^0_{v(r)}, 0, 1)$. Intuitively, we simply take the ordering of the origin points of the strands on the line $(y = 0, t = 0)$ to determine an ordering of our set of colors which defines the source permutation u of the colored braid α. We similarly take the ordering of the end points of the strands on the line $(y = 0, t = 1)$ to determine the target permutation u of our colored braid α. To illustrate these definitions, we give an instance of a colored braid in Figure 5.12. The source and target permutations associated to this colored braid are given by the ordered sequences $u = (2, 4, 3, 1)$ and $v = (3, 4, 1, 2)$.

The identity morphism $id_w := e(w)$, which we assign to any permutation $w \in \Sigma_r$, is the identity braid together with the strand coloring such that $\alpha_{w(i)}(t) = (x^0_i, 0, t)$, for all $t \in [0, 1]$ and for any $i = 1, \dots, r$ (see Figure 5.13).

The composition of the groupoid is given by the standard concatenation operation on braids, inherited from the braid group, and represented in Figure 5.2. Note simply that the colors assigned to strands agree on contact points when our braids (α, β) satisfy the relation $s(\alpha) = t(\beta)$ and hence are composable in the sense of §5.2.7. In this situation, each composite strand inherits a single color from its components and we use this correspondence to define the coloring of our composite braid $\alpha \cdot \beta \in \mathtt{Mor}\,\mathsf{CoB}(r)$.

The inversion of colored braids can also be deduced from the inversion operation of the braid groups.

5.2.9. *Braid cosets and morphisms in the colored braid groupoids.* The coloring of the strands of a morphism α in the colored braid groupoid $\mathsf{CoB}(r)$ can be determined by giving the source object $u = s(\alpha)$ of this morphism α in $\mathtt{Ob}\,\mathsf{CoB}(r) = \Sigma_r$. Indeed, the values of this permutation $u = (u(1), \dots, u(r))$ represent the colors of our braid at the origin points of the strands $((x_1^0, 0, 0), \dots, (x_r^0, 0, 0))$ and fully determine the colors of the strands themselves. From this observation, we readily deduce that the morphism set $\mathtt{Mor}_{\mathsf{CoB}(r)}(u, v)$, which we associate to any pair of permutations $u, v \in \Sigma_r$ in the colored braid groupoid $\mathsf{CoB}(r)$, can be identified with the pre-image $p_*^{-1}(v^{-1}u) \subset B_r$ of the permutation $v^{-1}u \in \Sigma_r$ in the braid group B_r, where we consider the natural group morphism from braids to permutations $p_* : B_r \to \Sigma_r$. The composition operation of $\mathsf{CoB}(r)$ is also identified with the operation $p_*^{-1}(w^{-1}v) \times p_*^{-1}(v^{-1}u) \to p_*^{-1}(w^{-1}u)$ obtained by restriction of the natural multiplication of the braid group B_r. For a single permutation $w \in \Sigma_r$, we have an identity $\mathtt{Mor}_{\mathsf{CoB}(r)}(w, w) = p_*^{-1}(w^{-1}w) = P_r$ and the identity morphism associated to w in the groupoid $\mathsf{CoB}(r)$ corresponds to the neutral element of the pure braid group P_r.

5.2.10. *The symmetric structure of the colored braid groupoids.* Each groupoid of colored braids $\mathsf{CoB}(r)$ inherits a natural action of permutations. Therefore the collection $\mathsf{CoB} = \{\mathsf{CoB}(r), r > 0\}$ forms a symmetric sequence of groupoids. To be explicit, to each permutation $s \in \Sigma_r$, we associate a groupoid morphism $s_* : \mathsf{CoB}(r) \to \mathsf{CoB}(r)$ which is defined by the obvious left translation operation $s_* : \Sigma_r \to \Sigma_r$ on the object set $\mathtt{Ob}\,\mathsf{CoB}(r) = \Sigma_r$. We proceed as follows to define the action of our permutation $s \in \Sigma_r$ on the morphism set $\mathtt{Mor}\,\mathsf{CoB}(r)$. We assume that α is a braid equipped with a strand coloring $i \mapsto \alpha_i$ which represents an element of this morphism set $\mathtt{Mor}\,\mathsf{CoB}(r)$. We define $s_*(\alpha) \in \mathtt{Mor}\,\mathsf{CoB}(r)$ by taking the same underlying braid as α, but we equip this braid $s_*(\alpha)$ with the modified coloring $s(i) \mapsto \alpha_i$ which assigns the value $s(i) \in \{1, \dots, r\}$ to the strand α_i previously colored by the index $i \in \{1, \dots, r\}$ in $\alpha \in \mathtt{Mor}\,\mathsf{CoB}(r)$. These mappings $s_* : \mathtt{Mor}\,\mathsf{CoB}(r) \to \mathtt{Mor}\,\mathsf{CoB}(r)$ and $s_* : \mathtt{Ob}\,\mathsf{CoB}(r) \to \mathtt{Ob}\,\mathsf{CoB}(r)$ clearly preserve the structure operations of our groupoid. In the definition of §5.2.9, the mapping $s_* : \mathtt{Mor}_{\mathsf{CoB}(r)}(u, v) \to \mathtt{Mor}_{\mathsf{CoB}(r)}(su, sv)$ can also be identified with the identity map of the set $p_*^{-1}((sv)^{-1}(su)) = p_*^{-1}(v^{-1}u) \subset B_r$ when we use the definitions $\mathtt{Mor}_{\mathsf{CoB}(r)}(u, v) = p_*^{-1}(v^{-1}u)$ and $\mathtt{Mor}_{\mathsf{CoB}(r)}(su, sv) = p_*^{-1}((sv)^{-1}(su))$ of the morphism sets associated to the pairs (u, v) and (su, sv) in $\mathsf{CoB}(r)$.

5.2.11. *The operadic composition operations on colored braids.* We have an identity $\mathsf{CoB}(1) = pt$. We take the identity map of the one-point set pt to provide the collection of colored braid groupoids with an operadic unit morphism $\eta : pt \to \mathsf{CoB}(1)$. We define operadic composition operations $\circ_k : \mathsf{CoB}(m) \times \mathsf{CoB}(n) \to \mathsf{CoB}(m + n - 1)$ by the operadic composition of permutations at the object set level. (Hence, the collection of object-sets of our operad $\mathtt{Ob}\,\mathsf{CoB}$ is identified with the permutation operad in the category of sets.) We use the operadic composition operation for braids (see §§5.1.2-5.1.3) in order to define the value of this operadic composition operation on the morphism sets of our groupoids.

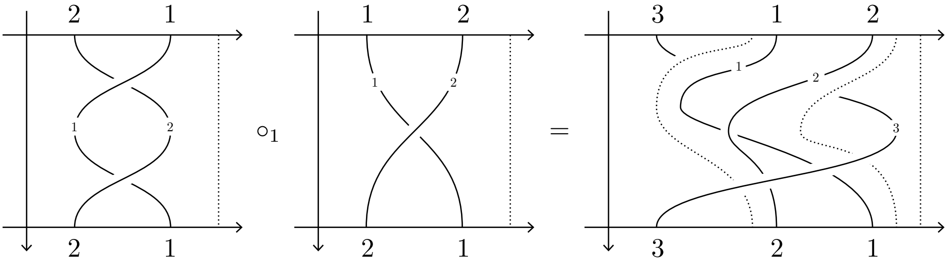


FIGURE 5.14. An operadic composition of colored braids

To be explicit, we fix such morphisms $\alpha \in \mathtt{Mor}\, \mathit{CoB}(m)$ and $\beta \in \mathtt{Mor}\, \mathit{CoB}(n)$. Intuitively, to define the composite $\alpha \circ_k \beta \in \mathtt{Mor}\, \mathit{CoB}(m+n-1)$, we insert the braid β in the strand of α colored by $k \in \{1, \dots, m\}$. We also apply the standard operadic shift $i \mapsto i+k-1$ to the index of the strands of β in the composite braid and the shift $i \mapsto i+n-1$ to the index of the strands of α when $k < i$. We use this correspondence to define the coloring of our composite braid $\alpha \circ_k \beta$. In comparison with the process of §§5.1.2-5.1.3, we simply use an ordering defined by the color indexing of the strands of α instead of the natural ordering of the source points on the line $y = t = 0$. Thus, the composition of braids in the colored braid groupoid is formally defined by the composition operation of §§5.1.2-5.1.3 up to an input reordering, which we determine from the source permutation of the braid α. To illustrate this process, we give an instance of partial composition operation $\alpha \circ_1 \beta \in \mathtt{Mor}\, \mathit{CoB}(3)$ in Figure 5.14. In order to ease the understanding of this picture, we have added dotted lines marking the array in which the braid β is inserted.

In the coset representation of morphism sets (see §5.2.9), the partial composition operation $\circ_k : \mathtt{Mor}_{\mathit{CoB}(m)}(s,t) \times \mathtt{Mor}_{\mathit{CoB}(n)}(u,v) \to \mathtt{Mor}_{\mathit{CoB}(m+n-1)}(s \circ_k u, t \circ_k v)$ maps any pair of elements $\alpha \in p_*^{-1}(t^{-1}s)$ and $\beta \in p_*^{-1}(v^{-1}u)$ to the composite braid $\alpha \circ_{s^{-1}(k)} \beta$ which has $p_*(\alpha \circ_{s^{-1}(k)} \beta) = (t \circ_k v)^{-1} \cdot (s \circ_k u)$ as associated permutation. This operation obviously preserves the groupoid structure, and hence, gives a morphism $\circ_k : \mathit{CoB}(m) \times \mathit{CoB}(n) \to \mathit{CoB}(m+n-1)$ in the category of groupoids. We easily check that these partial composition operations on the groupoids $\mathit{CoB}(r)$, $r > 0$, fulfill the equivariance, the unit and the associativity axioms of operads (we rely on the counterpart of this verification for the permutation and the braid operads).

These composition operations clearly extend to the degenerate case where β is a colored braid with an empty set of strands. This observation implies that the operad CoB has a unitary extension CoB_+. The restriction operator $u^* : \mathit{CoB}_+(n) \to \mathit{CoB}_+(m)$ which we define on this unitary extension CoB_+ can actually be identified with a natural extension to colored braids of the strand removal operations in the braid groups (see §5.1.4) just like the operadic composition of colored braids extends the operadic composition of braids.

The definition of the colored braid operad is now complete and we aim to prove:

THEOREM 5.2.12. *The operad* $\mathtt{B}(\mathsf{CoB})$ *associated to the (non-unitary) operad of colored braids* CoB *is a (non-unitary)* E_2*-operad and the unitary operad* $\mathtt{B}(\mathsf{CoB})_+$ *associated to the unitary version of this operad* CoB_+ *is a unitary* E_2*-operad.*

We focus on the case of non-unitary operads. The extension of our constructions to unitary operads is straightforward.

The idea is to identify $\mathtt{B}(\mathsf{CoB})$ with the symmetrization of a contractible braided operad and to deduce Theorem 5.2.12 from the recognition theorem of §5.1. This contractible braided operad is formed by a collection of contractible classifying spaces EB_r naturally associated to the braid groups B_r. In a preliminary step, we review the general definition of these contractible classifying spaces EG, which can be associated to any group G.

5.2.13. *Translation categories and their classifying spaces.* To a group G, we first associate a *translation category* E_G which has $\mathtt{Ob}\, E_G = G$ as object set and of which morphism sets are defined by the one-point sets $\mathtt{Mor}_{E_G}(\alpha,\beta) = \{\beta^{-1}\alpha\}$, for all $\alpha,\beta \in G$. The element $\beta^{-1}\alpha$ represents the right translation which connects β to α in G. This interpretation motivates the name 'translation category' which we give to this category E_G. The translation category E_G obviously forms a groupoid, for any group G.

The translation category E_G is also naturally equipped with a left G-action, which assigns a functor $g_* : E_G \to E_G$ to each $g \in G$. This functor is given by the left translation operation $g_*(\alpha) = g\alpha$ at the object set level, and by the identity of the translation factors $(g\beta)^{-1}(g\alpha) = \beta^{-1}\alpha$ at the morphism set level.

We then set $EG = \mathtt{B}(E_G)$, where we consider the classifying space of the translation category E_G. We can represent the n-simplices of this classifying space as chains

$$\underline{\alpha} = \{\alpha_0 \xrightarrow{\alpha_1^{-1}\alpha_0} \alpha_1 \xrightarrow{\alpha_2^{-1}\alpha_1} \cdots \xrightarrow{\alpha_n^{-1}\alpha_{n-1}} \alpha_n\},$$

where $(\alpha_0,\dots,\alpha_n)$ runs over G^{n+1}. The morphisms occurring in this simplex are determined by the sequence of vertices $(\alpha_0,\dots,\alpha_n)$, as we see in the above expression. The ith face of this simplex $d_i(\underline{\alpha})$ is obtained by omitting the vertex α_i in our expression, while the jth degeneracy $s_j(\underline{\alpha})$ is obtained by repeating the vertex α_j. By functoriality of the classifying space construction, the simplicial set EG inherits a left G-action from the translation category E_G. The image of a simplex $\underline{\alpha}$ under this action reads:

$$g_*(\underline{\alpha}) = \{g\alpha_0 \xrightarrow{(g\alpha_1)^{-1}(g\alpha_0)} g\alpha_1 \xrightarrow{(g\alpha_2)^{-1}(g\alpha_1)} \cdots \xrightarrow{(g\alpha_n)^{-1}(g\alpha_{n-1})} g\alpha_n\}.$$

This space EG is equipped with a natural map $p_* : EG \to BG$ with values in the classifying space of our group BG. If we regard the group G as a category with a single object $*$, then this map of classifying spaces is induced by the functor $p : E_G \to G$ defined by the obvious assignment $p(\alpha) = *$ on objects and by the mapping such that $p(\beta^{-1}\alpha) = \beta^{-1}\alpha$ on morphisms. We now have the following statement, which follows from an immediate inspection of our constructions:

OBSERVATION 5.2.14.

(a) *The groupoid* E_G *is equivalent to a point, and as a consequence, has a contractible classifying space* $EG = \mathtt{B}(E_G)$.

(b) *The action of* G *on* EG *is free.*

(c) *The mapping* $p_* : EG \to BG$ *induces an isomorphism* $p_* : EG/G \xrightarrow{\simeq} BG$ *when we take the quotient of the classifying space* EG *under the action of our group* G.

In the topological context, the contractibility of the simplicial set EG implies that the space $|EG|$ is contractible. The free action of G on EG gives rise to a free and proper action at the topological level. Furthermore, the mapping $p_* : EG \to BG$ induces a homeomorphism $|EG|/G = |EG/G| \xrightarrow{\simeq} |BG|$ since we have $|X/G| = |X|/G$ for any space X equipped with a G-action. The space $|EG|$ actually forms a principal G-bundle associated to G and is universal (as we already briefly explained in §5.2.3) in the sense that any principal G-bundle $q : E \to B$ can be obtained by taking the pullback of our projection map $p_* : |EG| \to |BG|$ along a classifying map $f : B \to |BG|$ which is uniquely determined (up to homotopy).

We now consider the translation categories E_{B_r} associated to the braid groups B_r. We immediately see that the collection formed by this sequence of groupoids $\mathsf{E}(r) = E_{B_r}$ inherits a natural braided operad structure from the braid groups: the braided structure of this collection is defined by the natural action of the braid groups B_r on the translation categories E_{B_r}; the unit morphism $\eta : pt \to \mathsf{E}(1)$ is given by the identity of the one-point set $E_{B_1} = pt$; the composition products $\circ_k : E_{B_m} \times E_{B_n} \to E_{B_{m+n-1}}$ are determined by the operadic composition of braids at the level of the object sets $\mathtt{Ob}\, E_{B_n} = B_n$. We easily check (by using the result obtained for braid groups) that these composition products fulfill the equivariance, unit and associativity axioms of braided operads.

We apply the symmetrization functor of Proposition 5.1.5 in the context of the category of categories $\mathcal{M} = \mathcal{C}at$ in order to get a symmetric operad $\mathtt{Sym}\, \mathsf{E}$ naturally associated to E. We have the following observation:

LEMMA 5.2.15. *The colored braid operad* CoB *is identified, as a symmetric operad in groupoids, with the symmetrization of the braided operad* E *such that* $\mathsf{E}(r) = E_{B_r}$ *for all* $r > 0$, *where we consider the translation categories of the braid groups* B_r.

PROOF. We use the definition of §5.2.9 where we identify the morphism sets of the groupoid $\mathsf{CoB}(r)$ with the cosets $\mathtt{Mor}_{\mathsf{CoB}(r)}(u, v) = p_*^{-1}(v^{-1}u)$ naturally associated to the morphism $p_* : B_r \to \Sigma_r$. We have an obvious functor $p_* : E_{B_r} \to \mathsf{CoB}(r)$ given by the map $p_* : B_r \to \Sigma_r$ on the object set $\mathtt{Ob}\, E_{B_r} = B_r$ and by the embedding $\{\beta^{-1}\alpha\} \hookrightarrow p_*^{-1}(p_*(\beta)^{-1}p_*(\alpha))$ on each morphism set $\mathtt{Mor}_{E_{B_r}}(\alpha, \beta) = \{\beta^{-1}\alpha\}$, for $\alpha, \beta \in B_r$. We immediately see that this functor carries the action of B_r on E_{B_r} to the natural action of Σ_r on $\mathsf{CoB}(r)$ and the action of $P_r \subset B_r$ on E_{B_r} to a trivial action. We can also readily check, by unraveling the definition of a quotient object in the category of categories, that $p_* : E_{B_r} \to \mathsf{CoB}(r)$ identifies $\mathsf{CoB}(r)$ with the quotient category E_{B_r}/P_r.

We have already observed that $p_* : E_{B_r} \to \mathsf{CoB}(r)$ carries the action of the braid group B_r on E_{B_r} to the natural action of the symmetric group Σ_r on the groupoid $\mathsf{CoB}(r)$. We readily obtain that p_* preserves the operadic composition products too (use the coset definition of these composition operations in §5.2.11). Hence, the collection of functors $p_* : E_{B_r} \to \mathsf{CoB}(r)$ defines a morphism in the category of braided operads $p_* : \mathsf{E} \to \mathsf{CoB}$ and the relation $E_{B_r}/P_r = \mathsf{CoB}(r)$ immediately implies that this morphism identifies CoB with the symmetric operad naturally associated to E. □

The conclusion of Proposition 5.2.5 remains obviously valid in the context of braided operads. In the particular case of the translation categories associated to braid groups $\mathsf{E}(r) = E_{B_r}$, we deduce from this assertion that:

FACT 5.2.16. *The collection of classifying spaces* $\mathtt{B}(\mathsf{E})(r) = \mathtt{B}(E_{B_r}) = EB_r$ *inherits a braided operad structure.*

The geometric realization and classifying space functors naturally commute with quotients under group actions. In the case of the symmetrization functor $\mathtt{Sym}$, which is essentially given by such a quotient process, this observation implies:

OBSERVATION 5.2.17. *We have operad identities* $\mathtt{Sym}\,|\,\mathtt{B}(\mathsf{E})| = |\,\mathtt{Sym}\,\mathtt{B}(\mathsf{E})| = |\,\mathtt{B}(\mathtt{Sym}\,\mathsf{E})|$.

Thus, from the identity $\mathtt{Sym}\,\mathsf{E} = \mathsf{CoB}$ established in Lemma 5.2.15, we conclude that $|\,\mathtt{B}(\mathsf{CoB})|$ is identified with the symmetrization of the contractible braided operad $|\,\mathtt{B}(\mathsf{E})|$. The braided operad $\mathtt{B}(\mathsf{E})$ is also contractible by observation 5.2.14 and the braid group B_r operates freely and properly at the level of the topological space $|\,\mathtt{B}(\mathsf{E}(r))| = EB_r$. By Theorem 5.1.14, these assertions imply that $|\,\mathtt{B}(\mathsf{CoB})| = \mathtt{Sym}\,|\,\mathtt{B}(\mathsf{E})|$ forms an E_2-operad, as claimed in Theorem 5.2.12. ☐

5.2.18. *Remark.* The category of algebras associated with the colored braid operad consists of braided categories equipped with a strictly associative tensor product. This statement is an operadic counterpart of a result of Joyal and Street [99] asserting that the disjoint union of the braids groups form a free braided category on one generating object. The correspondence between operads in groupoids and monoidal structures on categories is the subject of the next chapter. We go back to the connection between the colored braid operad and Joyal-Street's statement at this moment.

5.3. Fundamental groupoids and operads

In the previous section, we observed that the spaces underlying the little discs operad $\mathsf{D}_2(r)$ are identified with the Eilenberg-MacLane spaces $K(P_r, 1)$ associated to the pure braid groups P_r, and hence, have a well-determined isomorphism class in the homotopy category of spaces. Recall also that we can use the classifying space construction in order to make explicit a model of the objects $K(P_r, 1)$ in the category of topological spaces. But we have needed to replace the pure braid groups P_r by the groupoids of colored braids $\mathsf{CoB}(r)$ in the construction of the previous section in order to model the structure of the little 2-discs operad at the level of such classifying spaces.

The main purpose of this section is to explain the source of this problem and to give an explanation for the introduction of colored braids in §5.2.

The pure braid group P_r represents the fundamental group of the little 2-discs space $\mathsf{D}_2(r)$, and involves, by definition of the fundamental group, the choice of a base point in $\mathsf{D}_2(r)$. The problem comes from this choice: base points can not be chosen coherently with respect to the structure operations attached to our operad. The natural idea is to replace fundamental groups by fundamental groupoids in order to work out this difficulty. In the case of the little 2-discs operad D_2, we precisely prove that the fundamental groupoids of the spaces $\mathsf{D}_2(r)$ form an operad in groupoids which is equivalent to the colored braid operad of §5.2. The main goal of this section is to establish this result. Before, we quickly recall the definition of the fundamental groupoid and we check that the fundamental groupoids of the spaces underlying an operad in topological spaces form an operad in groupoids.

5.3.1. *Fundamental groupoids.* We denote the fundamental groupoid of a topological space X by πX. The object set of this groupoid πX is the underlying point-set of the space X. Let $x, y \in X$. The morphisms from x to y in πX are the homotopy classes of paths $\alpha : [0,1] \rightarrow X$ with $\alpha(0) = x$ as prescribed origin and $\alpha(1) = y$ as prescribed endpoint. The composition of morphisms in πX is given by the usual composition operation on paths and extends the composition of based loops considered in the definition of the fundamental group. The unit relation, the associativity relation and the existence of inverses in πX is proved by a straightforward extension of the arguments classically considered in the context of fundamental groups.

The fundamental group of X at a base point $x_0 \in X$ is clearly identified with the group of automorphisms of the point x_0 in the fundamental groupoid

$$\pi_1(X, x_0) = \mathtt{Mor}_{\pi X}(x_0, x_0)$$

and we have an isomorphism connecting $x_0 \in X$ to another point $x \in X$ in πX if an only if x_0 and x belong to the same path connected component of the space X.

Thus, if we regard a group as a groupoid with one object, then we can also identify the fundamental group $\pi_1(X, x_0)$ at a base point x_0 with the full subcategory of πX generated by the single object $\{x_0\} \subset X = \mathtt{Ob}\,\pi X$. If X is path connected, then the embedding $\pi_1(X, x_0) \hookrightarrow \pi X$, which we deduce from this categorical interpretation of the fundamental group, defines an equivalence of categories. In general, we get that the fundamental groupoid is equivalent (as a category) to the coproduct $\coprod_{[x_0] \in \pi_0(X)} \pi_1(X, x_0)$ formed by picking a representative $x_0 \in C$ in each path connected component $[x_0] = C \in \pi_0(X)$ of the space X.

In what follows, we also use the notation $\pi X_{|A} \subset \pi X$ for the full subcategory of the fundamental groupoid πX generated by a subset of our space $A \subset X$. We obviously have $\pi_1(X, x_0) = \pi X_{\{x_0\}}$ when we assume that our subset reduces to the one-point set $A = \{x_0\}$.

Even in the path connected case, we usually have no canonical choice for a single base point x_0 in a space X. We therefore consider full subcategories $\pi X_{|A} \subset \pi X$ associated to such base sets $A \subset X$.

The mapping $\pi : X \mapsto \pi X$ clearly gives a functor from spaces to groupoids and usual results on fundamental groups extend to fundamental groupoids. But we need to take care of the difference between the notion of an isomorphism and the notion of an equivalence in the groupoid context. For instance, a homeomorphism induces an isomorphism on fundamental groupoids, but a homotopy equivalence of spaces $f : X \xrightarrow{\sim} Y$ only induces an equivalence of fundamental groupoids in general $f_* : \pi X \xrightarrow{\sim} \pi Y$ (unless our map f defines a bijection at the point set level).

In order to study the image of topological operads under the fundamental groupoid functor $\pi : \mathcal{T}op \rightarrow \mathcal{G}rd$, we still establish that:

PROPOSITION 5.3.2. *The functor $\pi : \mathcal{T}op \rightarrow \mathcal{G}rd$ is symmetric monoidal:*

(1) *for the one-point set pt, viewed as the unit object of the category of spaces, we have an obvious identity $\pi\, pt = pt$;*

(2) *for a cartesian product of spaces $X \times Y$, we have an isomorphism $\pi(X \times Y) \xrightarrow{\simeq} \pi X \times \pi Y$ which is yielded by the morphisms of fundamental groupoids $\pi X \xleftarrow{p_*} \pi(X \times Y) \xrightarrow{q_*} \pi Y$ induced by the canonical projections $X \xleftarrow{p} X \times Y \xrightarrow{q} Y$;*

(3) *and these comparison isomorphisms (*1-2*) fulfill the unit, associativity and symmetry constraints of* §3.3.1*.*

Proof. The proof of assertion (1) is immediate. The proof of assertion (2) reduces to a straightforward extension of arguments classically used in the case of fundamental groups. The definition of the isomorphism $\pi(X \times Y) \xrightarrow{\simeq} \pi X \times \pi Y$ from universal categorical constructions automatically ensures, as usual, that the unit, associativity and symmetry constraints of symmetric monoidal functors are fulfilled. ☐

Proposition 5.3.3. *Let P be an operad in topological spaces. The collection $\pi\, P = \{\pi\, P(r), r \in \mathbb{N}\}$, where we consider the fundamental groupoids of the spaces $P(r)$, forms an operad in groupoids naturally associated to P.* ☐

We can also take a restriction of the functor defined in this proposition $\pi : P \mapsto \pi\, P$ to the category non-unitary operads $P \in \mathcal{T}op\,\mathcal{O}p_{\varnothing}$. We easily check that the object $\pi\, P$ forms a non-unitary operad in this case (when we regard the category of non-unitary operads in groupoids as a full subcategory of the category of all operads). From Proposition 3.1.1, we moreover deduce that the mapping $\pi : P \mapsto \pi\, P$ preserves unitary extensions, or more explicitly, that we have an identity $\pi(P_+) = (\pi\, P)_+$, for any unitary operad in topological spaces $P_+ \in \mathcal{T}op\,\mathcal{O}p_*$.

For the operad of little 2-discs $P = D_2$, we obtain the following result:

Theorem 5.3.4. *The fundamental groupoid operad of the little* 2*-discs operad $\pi\, D_2$ is connected to the colored braid operad CoB of* §5.2 *by a chain of categorical equivalences of operads in groupoids*

$$\pi\, D_2 \xleftarrow{\sim} \cdot \xrightarrow{\sim} CoB,$$

and similarly for the unitary extension of these operads $\pi\, D_{2+}$ and CoB_+.

Proof. We focus on the non-unitary operad case of this theorem. The extension of our construction to the setting of unitary operads is straightforward.

We still use the embedding $D_1 \hookrightarrow D_2$ to identify the operad of little 1-discs D_1 with a suboperad of D_2 (see §4.1.5). We consider, for each $r \in \mathbb{N}$, the full subcategory of the fundamental groupoid $\pi\, D_2(r)$ generated by the image of the set $D_1(r)$ in $D_2(r)$. We adopt the notation $\pi\, D_2(r)\,|_{D_1(r)}$ for this category, which obviously forms a groupoid. The collection of groupoids

$$\pi\, D_2\,|_{D_1} = \{\pi\, D_2(r)\,|_{D_1(r)}, r > 0\}$$

also defines a suboperad of $\pi\, D_2$ because the object sets $D_1(r)$ associated to these groupoids $\pi\, D_2(r)\,|_{D_1(r)}$ form themselves a suboperad of the little 2-discs operad D_2, regarded as an operad in sets. We use this operad $\pi\, D_2\,|_{D_1} \subset \pi\, D_2$ as an intermediate object between the fundamental groupoid operad $\pi\, D_2$ and the colored braid operad CoB.

The embeddings $\pi\, D_2(r)\,|_{D_1(r)} \hookrightarrow \pi\, D_2(r)$ are equivalences of categories since each space $D_2(r)$ is connected, and as a consequence, the embedding of operads in groupoids $\pi\, D_2\,|_{D_1} \hookrightarrow \pi\, D_2$ defined by the collection of these morphisms forms a categorical equivalence of operads. To complete our arguments, we define a second categorical equivalence of operads $\pi\, D_2\,|_{D_1} \xrightarrow{\sim} CoB$ connecting $\pi\, D_2\,|_{D_1}$ with the colored braid operad CoB. In a preliminary step, we construct the collection of groupoid equivalences $\pi\, D_2(r)\,|_{D_1(r)} \xrightarrow{\sim} CoB(r)$ underlying our operad morphism.

Let $\Pi(r)$ be the subset of the configuration space $F(\mathring{\mathbb{D}}^2, r)$ formed by the elements of the form $\underline{a}^0_w = ((x^0_{w(1)}, 0), \dots, (x^0_{w(r)}, 0))$, where $w \in \Sigma_r$. If we go back to the construction of §5.2.8, where we define the groupoids of colored braids, then we immediately see that the isotopy classes of braids defining the morphisms of this colored operad are nothing but homotopy classes of paths between elements of $\Pi(r)$. In other words, we have a formal identity $\mathit{CoB}(r) = \pi\, F(\mathring{\mathbb{D}}^2, r)\,|_{\Pi(r)}$, for each $r > 0$.

The homotopy equivalence $\omega : D_2(r) \xrightarrow{\sim} F(\mathring{\mathbb{D}}^2, r)$, defined by the disc center mapping (see §§4.2.1-4.2.2), induces an equivalence of fundamental groupoids $\omega_* : \pi\, D_2(r) \xrightarrow{\sim} \pi\, F(\mathring{\mathbb{D}}^2, r)$. In order to connect $\pi\, D_2(r)\,|_{D_1(r)} \subset \pi\, D_2(r)$ with the groupoid $\mathit{CoB}(r) = \pi\, F(\mathring{\mathbb{D}}^2, r)\,|_{\Pi(r)}$, we pick a collection of little 2-discs $\underline{c}^0$ in the image of our embedding $D_1(r) \to D_2(r)$ such that $\omega(\underline{c}^0) = \underline{a}^0_{id}$. Then we consider the subset $\Xi(r)$ formed by the elements $\underline{c}^0_w = w_*(\underline{c}^0)$ in $D_1(r) \hookrightarrow D_2(r)$, where we assume $w \in \Sigma_r$. The disc center mapping is clearly equivariant, so that $\omega(\underline{c}^0_w) = \underline{a}^0_w$, for all $w \in \Sigma_r$, and the equivalence $\omega_* : \pi\, D_2(r) \xrightarrow{\sim} \pi\, F(\mathring{\mathbb{D}}^2, r)$ induces, by restriction to $\Xi(r) \subset D_2(r)$, a groupoid isomorphism $\pi\, D_2(r)\,|_{\Xi(r)} \xrightarrow{\simeq} \pi\, F(\mathring{\mathbb{D}}^2, r)\,|_{\Pi(r)}$. To recap, we now have a groupoid diagram

(1)
$$\begin{tikzcd}
\pi\, D_2(r)\,|_{\Xi(r)} \arrow[r, dotted, "\simeq"] \arrow[d, hook] & \pi\, F(\mathring{\mathbb{D}}^2, r)\,|_{\Pi(r)} \arrow[r, "="] \arrow[dd, hook] & \mathit{CoB}(r) \\
\pi\, D_2(r)\,|_{D_1(r)} \arrow[d, hook] & & \\
\pi\, D_2(r) \arrow[r, "\sim"'] & \pi\, F(\mathring{\mathbb{D}}^2, r) &
\end{tikzcd}$$

where vertical morphisms are embeddings of full subgroupoids, the bottom horizontal morphism is a groupoid equivalence, and the upper horizontal morphism is a groupoid isomorphism. The connectedness of $D_2(r)$ implies that the first vertical embedding $\pi\, D_2(r)\,|_{\Xi(r)} \hookrightarrow \pi\, D_2(r)\,|_{D_1(r)}$ defines an equivalence of groupoids too, just like the second embedding $\pi\, D_2(r)\,|_{D_1(r)} \hookrightarrow \pi\, D_2(r)$.

The groupoid equivalence which we aim to define is obtained by picking an appropriate inverse equivalence of the embedding $\pi\, D_2(r)\,|_{\Xi(r)} \hookrightarrow \pi\, D_2(r)\,|_{D_1(r)}$.

Recall that the embedding of a configuration of little 1-discs $\underline{c} = (c_1, \dots, c_r)$ in the interval $\mathbb{D}^1 = [-1, 1]$ determines a linear ordering $i_1 < \dots < i_r$ of the indices of these 1-discs c_i, $i = 1, \dots, r$. In Proposition 4.1.6, we use this observation to assign a permutation $w = (i_1, \dots, i_r)$ to each element $\underline{c} \in D_1(r)$ and to establish the identity $\pi_0\, D_1(r) = \Sigma_r$. To an element $\underline{c}$ in the image of the space $D_1(r)$ in $D_2(r)$, we associate the base element $\underline{c}^0_w \in \Xi(r)$ formed by applying the permutation w associated to $\underline{c}$ to the initially chosen configuration of little 2-discs $\underline{c}^0$. Equivalently, we consider the base element $\underline{c}^0_w$ which lies in the same connected component $D_1(r)_w$ of the 1-disc space $D_1(r)$ as $\underline{c} \in D_1(r)$.

Recall that each space $D_1(r)_w$ is contractible. We pick a path γ which links $\gamma(0) = \underline{c}^0_w$ to $\gamma(1) = \underline{c}$ and lies in this contractible space. We perform such a choice of path for every element $\underline{c}$ in the image of the little 1-disc space $D_1(r)$ in $D_2(r)$. The homotopy class of our path γ represents an isomorphism between $\underline{c}$ and $\underline{c}^0_w$ in the fundamental groupoid $\pi\, D_2(r)$. We consider the groupoid morphism

$\pi\, D_2(r)\mid_{D_1(r)}\to \pi\, D_2(r)\mid_{\Xi(r)}$ which maps each object $\underline{c}$ to the associated configuration $\underline{c}^0_w$ in the set $\Xi(r)$, and which is given, at the morphism set level, by the composition with the isomorphism $[\gamma]\in \mathtt{Mor}_{\pi\, D_2(r)}(\underline{c}^0_w,\underline{c})$ determined by the homotopy class of our path connecting $\underline{c}^0_w$ and $\underline{c}$ inside $D_1(r)\hookrightarrow D_2(r)$. The contractibility of the space $D_1(r)_w$, where we define this path γ, implies that this isomorphism does not depend on choices.

Now we can take the composite of the just defined equivalence of groupoids with the obvious isomorphism $\pi\, D_2(r)\mid_{\Xi(r)}\xrightarrow{\simeq}\pi\, F(\mathring{\mathbb{D}}^2,r)\mid_{\Pi(r)}$ in order to get a morphism of groupoids

$$\pi\, D_2(r)\mid_{D_1(r)}\xrightarrow{\sim}\pi\, F(\mathring{\mathbb{D}}^2,r)\mid_{\Pi(r)}= CoB(r), \tag{2}$$

which is also an equivalence by construction. We see that our mapping, which associates an element $\underline{c}^0_w$ to any $\underline{c}$, is equivariant with respect to the action of permutations, and as a consequence, so is our groupoid morphism since we observed that our construction does not depend on any other choice.

We immediately see that our morphism sends the unit element of the operad $\pi\, D_2\mid_{D_1}$ to the unit element of the colored braid operad CoB too because we trivially have $CoB(1)=\pi\, F(\mathring{\mathbb{D}}^2,1)\mid_{\Pi(1)}= pt$. We also see that our groupoid morphisms commute with the operadic composition products at the object level, because we use the decomposition of the little 1-discs operad into connected components to determine our correspondence on objects and the operadic composition of permutations reflects the operadic composition associated to the connected components of the little 1-discs operad. The existence of a groupoid equivalence between $\pi\, D_2(r)\mid_{D_1(r)}$ and $CoB(r)$, for each $r>0$, implies that the morphisms of the groupoid $\pi\, D_2(r)\mid_{D_1(r)}$ are composites of elementary paths which correspond to the generating braids τ_i in the morphism sets of the groupoid $CoB(r)$. We easily see, by going back to our figures, that we retrieve the definition of the operadic composites of a generating braid with the identity braid in Proposition 5.1.2 when we form the operadic composites of the path of Figure 5.11 with a constant path in the fundamental groupoid of the little 2-discs operad. We readily conclude from the verification of these generating cases that our mappings preserve the operadic composites of all morphisms in our groupoids.

We conclude that our collection of groupoid equivalences (2) defines a morphism of operads in groupoids $\pi\, D_2\mid_{D_1}\to CoB$, which is also a categorical equivalence by construction. Hence, we finally have a chain of categorical equivalences of operads in groupoids

$$\pi\, D_2\xleftarrow{\sim}\pi\, D_2\mid_{D_1}\xrightarrow{\sim} CoB \tag{3}$$

that links the fundamental groupoid of the operad of little 2-discs $\pi\, D_2$ to the operad of colored braids CoB. We also readily see, by an immediate extension of our arguments, that these categorical equivalences preserve the restriction operators attached to our operads, and hence, extend to categorical equivalences of unitary operads. This observation completes the proof of Theorem 5.3.4. □

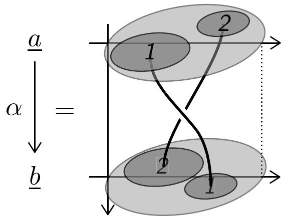
5.3.5. *Remark: The representation of morphisms in the fundamental groupoid of the little* 2*-discs operad.* The bijection

$$\omega_*:\mathtt{Mor}_{\pi\, D_2(r)}(\underline{a},\underline{b})\xrightarrow{\simeq}\mathtt{Mor}_{\pi\, F(\mathring{\mathbb{D}}^2,r)}(\omega(\underline{a}),\omega(\underline{b}))$$

induced by the disc center mapping $\omega : \mathsf{D}_2(r) \xrightarrow{\sim} \mathsf{F}(\mathring{\mathbb{D}}^2, r)$ implies that the morphisms of the fundamental groupoid of the little 2-discs space are specified by:

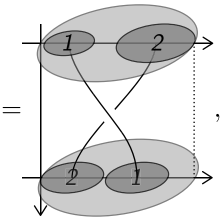
- a configuration of little 2-discs $\underline{a} = (a_1, \dots, a_r)$, which represents the source of our morphism,
- a configuration of little 2-discs $\underline{b} = (b_1, \dots, b_r)$, which defines the target,
- and a braid on r strands $\alpha = (\alpha_1, \dots, \alpha_r)$ such that α_i connects the center of the ith little 2-disc a_i in the source configuration $\underline{a}$ to the center of the ith little 2-disc b_i in the target configuration $\underline{b}$.

Thus, we get a picture of the following form:

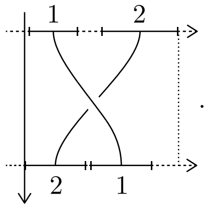


for a morphism $\alpha \in \mathtt{Mor}_{\pi\,\mathsf{D}_2(2)}(\underline{a}, \underline{b})$, with the configuration of little 2-discs $\underline{a}$ represented at the top of the picture, and $\underline{b}$ at the bottom.

In the special case of configurations of little 2-discs centered on the axis $y = 0$, like

$\alpha =$  ,

we may use a simplified picture where, as in the braid diagram representation, we only retain the trace of our little 2-discs configurations on the axis $y = 0$:

 .

This trace also represents the counterimage of our configurations of little 2-discs under the operad embedding $\mathsf{D}_1 \hookrightarrow \mathsf{D}_2$. The groupoid equivalence $\pi\,\mathsf{D}_2(r)\,|_{\mathsf{D}_1(r)} \xrightarrow{\sim} \mathsf{CoB}(r)$, considered in the proof of Theorem 5.3.4, is given at the morphism level by a simple path concatenation operation which recenters the contact points of such braid diagrams.

5.3.6. *Extra remarks.* The results of Theorem 5.2.12 and Theorem 5.3.4 are actually not independent though we give a direct proof of each statement. To explain the precise relationship between our results, we consider simplicial sets rather than topological spaces.

The classifying space construction of §5.2.3 is naturally given as a functor from categories to simplicial sets. The fundamental groupoid construction has a combinatorial analogue, defined on the category of simplicial sets, and which yields a left adjoint $\pi : s\mathcal{S}et \to \mathcal{G}rd$ of the restriction of the classifying space functor $\mathtt{B} : \mathcal{C}at \to s\mathcal{S}et$ to the category of groupoids $\mathcal{G}rd \subset \mathcal{C}at$. The augmentation $\pi\,\mathtt{B}(\mathcal{G}) \to \mathcal{G}$ of this adjunction $\pi : s\mathcal{S}et \rightleftarrows \mathcal{G}rd : \mathtt{B}$ defines an isomorphism of groupoids, for all $\mathcal{G} \in \mathcal{G}rd$. The adjunction unit $X \to \mathtt{B}(\pi X)$ defines a weak-equivalence of simplicial sets when X is a Kan complex with a trivial homotopy in degree $* > 1$.

The simplicial version of the fundamental groupoid $\pi : s\mathcal{S}et \to \mathcal{G}rd$ defines a symmetric monoidal functor like the fundamental groupoid of topological spaces (this result is a variation on the Eilenberg-Zilber correspondence). Therefore, the fundamental groupoid induces a functor $\pi : s\mathcal{S}et\,\mathcal{O}p \to \mathcal{G}rd\,\mathcal{O}p$ from the category of simplicial operads $s\mathcal{S}et\,\mathcal{O}p$ to the category of operads in groupoids $\mathcal{G}rd\,\mathcal{O}p$. This functor is still left adjoint of the functor on operads $\mathtt{B} : \mathcal{G}rd\,\mathcal{O}p \to s\mathcal{S}et\,\mathcal{O}p$ which we define by the aritywise application of the classifying space functor from groupoids to simplicial sets. By combining this adjunction with the realization and singular complex adjunction relation (see §3.1.4), we get a chain of adjunctions

$$\begin{tikzcd}
\mathcal{T}op\,\mathcal{O}p \arrow[r, shift right, "\mathtt{Sing}_\bullet(-)"'] \arrow[r, phantom, "(1)"] & s\mathcal{S}et\,\mathcal{O}p \arrow[l, shift right, "|-|"'] \arrow[r, shift left, "\pi"] \arrow[r, phantom, "(2)"] & \mathcal{G}rd\,\mathcal{O}p \arrow[l, shift left, "\mathtt{B}"]
\end{tikzcd}$$

which link the category of topological operads and the category of operads in groupoids.

The unit (respectively, the augmentation) of the adjunction between simplicial sets and topological spaces is a weak-equivalence, and so is the unit (respectively, the augmentation) of the corresponding adjunction (1) on operads.

The augmentation of the adjunction between simplicial sets and groupoids defines a groupoid isomorphism $\pi\,\mathtt{B}(\mathcal{G}) \xrightarrow{\simeq} \mathcal{G}$, for all $\mathcal{G} \in \mathcal{G}rd$, while the unit of this adjunction $X \to \mathtt{B}(\pi X)$ defines a weak-equivalence of simplicial sets as soon as X is a Kan complex with a trivial homotopy in degree $* > 1$. These assertions extend to the unit and the augmentation of the adjunction (2) on our categories of operads.

From these observations, we deduce that the existence of weak-equivalences of operads $\mathsf{D}_2 \xleftarrow{\sim} \cdot \xrightarrow{\sim} \mathtt{B}(\mathsf{CoB})$, asserted by Theorem 5.2.12, implies the existence of categorical equivalences of operads in groupoids between the fundamental groupoid operad $\pi\,\mathsf{D}_2$ and the operad of colored braids $\mathsf{CoB} \simeq \pi\,\mathtt{B}(\mathsf{CoB})$. On the other hand, since we observed that the underlying spaces of the little 2-discs operad D_2 are Eilenberg-MacLane spaces, we automatically have weak-equivalences of operads which connect the operads in topological spaces D_2 and $\mathtt{B}(\pi\,\mathsf{D}_2)$. Hence, the existence of categorical equivalences of operads in groupoids between $\pi\,\mathsf{D}_2$ and CoB, as asserted in Theorem 5.3.4, also implies the existence of weak-equivalences of simplicial operads connecting D_2 and $\mathtt{B}(\mathsf{CoB})$, as claimed in Theorem 5.2.12.

Our adjunctions (1-2) can also be used to give a necessary and sufficient recognition criterion for E_2-operads. Namely, an operad P is E_2 if and only if each space $\mathsf{P}(r)$ has a trivial homotopy in degree $* > 1$ and $\pi\,\mathsf{P}$ is equivalent to the colored braid operad CoB as an operad in groupoids.

5.4. Outlook: The recognition of E_n-operads for $n > 2$

The recognition of E_n-operads is more difficult in the case $n > 2$ than in the case $n = 2$, because the underlying spaces of the little n-discs operads are no longer Eilenberg-MacLane spaces when $n > 2$. On the other hand, we do have sufficient conditions asserting, as in Theorem 5.1.14, that certain operads $\mathtt{Sym}_n \mathsf{P}$ obtained by a quotient process from an appropriate contractible object P are E_n.

In the context of Theorem 5.1.14, we consider the category of braided operads, the obvious restriction functor from symmetric operads to braided operads, and the symmetrization functor which represents a left adjoint of this restriction functor. Nice analogues of these notions have been introduced by Michael Batanin with the aim of defining higher dimensional generalizations of fundamental groupoids (see [18] for this part of the program). In Batanin's approach [19, 20, 21], the category of braided operads is replaced by a category of n-operads which have an underlying collection $\mathsf{P}(\tau)$ indexed by n-level trees. These trees represent composition patterns that can be formed from the structure of an n-category. We again have an obvious functor $\mathcal{O}p \to {}_n\mathcal{O}p$ from the category of ordinary operads to the category of n-operads and we have an n-symmetrization functor which goes in the converse direction $\mathtt{Sym}_n : {}_n\mathcal{O}p \to \mathcal{O}p$. In [20], Batanin establishes that the symmetrization of a contractible n-operad (satisfying some suitable cofibration requirement) is an E_n-operad. In [19], he proves further that many usual models of E_n-operads, like the Fulton–MacPherson operads (see §4.3), can be defined by applying this symmetrization process.

Batanin's recognition criterions are used to define models of E_n-operads for each n independently. In [23], Clemens Berger explains that models of the little n-discs operads, regarded as a nested sequence of operads, can be obtained from contractible (symmetric) operads equipped with an appropriate cellular structure. The first application of this recognition method, given by Berger himself in [23], is the construction of simplicial models of E_n-operads from a basic simplicial operad, first considered by Barratt-Eccles in [17], and which is given by an application of the translation category construction of §§5.2.13-5.2.16 to the symmetric groups Σ_n. In short, one can check that the Barratt-Eccles operad E is an E_∞-operad in simplicial sets equipped with a filtration by suboperads $\mathsf{E}_1 \subset \cdots \subset \mathsf{E}_n \subset \cdots \subset \mathsf{E}$ such that the geometric realization of this sequence of operads in the category of simplicial sets defines a nested sequence of operads in topological spaces $|\,\mathsf{E}_1\,| \subset \cdots \subset |\,\mathsf{E}_n\,| \subset \cdots \subset |\,\mathsf{E}\,|$ which is connected to the nested sequence of the little discs operads $\mathsf{D}_1 \subset \cdots \subset \mathsf{D}_n \subset \cdots \subset \mathsf{D}$ by a chain of weak-equivalences of nested sequences of operads in topological spaces. The free algebras associated to these operads E_n, which we deduce from the Barratt-Eccles operad E, are related to simplicial models of n-fold spaces of suspensions $\Omega^n\Sigma^n X$ defined by Jeff Smith in [165]. This model is precisely given by a group completion of the reduced free algebra $\mathbb{S}_*(\mathsf{E}_{n+}, X)$ (see §2.2.23), for any pointed simplicial set X, and where we consider a unitary version of the operad E_n. Berger's method has also been applied successfully by Jim McClure and Jeff Smith in [143] to prove that a certain operad, defined by natural operations acting on Hochschild cochain complexes, is E_2. This result has lead to a new conceptual proof of the Deligne conjecture claiming the existence of a natural E_2-structure on the Hochschild cochain complex (see the preface of the book).

Other models of E_n-operads, related to the topics studied in the present chapter, arise from the iterated monoidal categories of [15], which generalize the classical braided monoidal categories of quantum algebra ($n = 2$) and yield higher intermediate structures between the standard (noncommutative) monoidal categories ($n = 1$) and symmetric monoidal categories ($n = \infty$).

CHAPTER 6

The Magma and Parenthesized Braid Operads

The operads of the category of small categories, like the operad of colored braids considered in the previous chapter, govern multiplicative structures on categories. In §5.2.18, we already mentioned that an action of the operad of colored braids on a category encodes a braided monoidal structure whose tensor product is associative in the strict sense. We give a detailed proof of this statement in this chapter. But our main purpose is to explain the definition of a variant of the operad of colored braids, the *operad of parenthesized braids*, whose actions encode general braided monoidal structures, where the tensor product is associative up to natural isomorphisms.

Recall that the colored braid operad is an operad in groupoids CoB of which object sets form an operad in sets isomorphic to the permutation operad Π. The morphisms of the rth component of this operad $\mathsf{CoB}(r)$ are isotopy classes of braids with r strands whose (fixed) contact points are labeled by indices $(i_1, \dots, i_r)$ (the colors) that form a permutation of the set $(1, \dots, r)$. These contact points, together with the associated colors, represent the objects of our operad. In the parenthesized braid operad, denoted by PaB, the object sets form an operad in sets isomorphic to a free operad $\Omega = \mathbb{O}(\mu(x_1, x_2), \mu(x_2, x_1))$ generated by a non-symmetric operation $\mu = \mu(x_1, x_2)$ in arity 2. We adopt the name '*magma*', which Bourbaki introduces for general non-associative structures (see [35, §I.1]), to refer to this free operad in sets.

In our geometrical picture, the morphisms of the rth component of the parenthesized braid operad $\mathsf{PaB}(r)$ are still defined by isotopy classes of braids with r strands, but we now consider contact points located on the center of diadic partitions of the interval $[-1, 1]$. These diadic partitions are in bijection with planar binary trees and this correspondence gives the isomorphism between the object sets of the parenthesized braid operad and the components of the magma operad.

The diadic partitions correspond to a suboperad of the little 2-disc operad defined by certain little 2-disc configurations centered on the horizontal axis. The components of the operad of parenthesized braids are actually identified with the full subgroupoids of the fundamental groupoid operad of little 2-discs defined by these particular subsets of base points. Recall that the connection between the colored braid operad CoB and the fundamental operad of little 2-discs $\pi\, \mathsf{D}_2$ involves a chain of categorical equivalences $\pi\, \mathsf{D}_2 \xleftarrow{\sim} \cdot \xrightarrow{\sim} \mathsf{CoB}$. The operad of parenthesized braids PaB is actually the minimal object which can be used to give the middle term in such a chain.

In a first step (§6.1), we explain the definition of an operad, which we call the *parenthesized permutation operad*, and which governs general monoidal category structures, where we have no symmetry constraint on the tensor product. By the way, we also give an operadic interpretation of the Mac Lane Coherence Theorem.

In a second step (§6.2), we address the definition of the parenthesized braid operad itself, and we give the proof that this operad governs the structure of a braided monoidal category with general associativity isomorphisms. To complete our account, we explain the definition of an analogous operad, the *operad of parenthesized symmetries,* which encodes the structure of a symmetric monoidal category with general associativity isomorphisms. We devote a third section §6.3 to this topic.

Let us mention that Bar-Natan uses the name 'parenthesized braid' and the notation PaB_r for objects that differ from the groupoids $\mathsf{PaB}(r)$ in our definition of the operad of parenthesized braids (see [16]). The objects considered by Bar-Natan PaB_r actually represent summands of a free enriched braided monoidal category on one generating object. We explain the connection between Bar-Natan formalism and our operadic approach with more details in §6.2.8. By the way, we also explain the relationship between the operad of colored braids of §5.2 and Joyal-Street's definition of the free braided monoidal category on one generating object.

6.1. Magmas and the parenthesized permutation operad

The ultimate objective of this chapter, as we just explained, is to define an operad in groupoids, the operad of parenthesized braids, with the same morphism sets as the operad of colored braids of §5.2, but where the object sets are changed into terms of the magma operad in order to encode general braided monoidal category structures. The rough idea is to perform a pullback operation in order to get this change of object sets. This pullback process can also be used to get an operad governing general non-symmetric monoidal categories, and we study this more basic example first. The relationship between monoidal structures and this pullback operation actually follows from an operadic interpretation of the Mac Lane Coherence Theorem which we explain in this section too.

The *magma operad*, as we explain in the introduction of this chapter, is a free operad (in sets) with a single (non-symmetric) generator μ in arity 2. We explicitly set:

$$\Omega = \Theta(\mu(x_1, x_2), \mu(x_2, x_1)),$$

where $\mu = \mu(x_1, x_2)$ denotes our generating operation, and $t\mu = \mu(x_2, x_1)$, with $t = (1\ 2)$, is the associated transposed element. The algebras associated to this operad are identified with Bourbaki's (non-commutative) magmas (see [35, §I.1]). To be explicit, by going back to the definition of free operads in §1.2, we see that an Ω-algebra in sets consists of an object $A \in \mathcal{S}et$ equipped with a (possibly non-commutative and non-associative) product $m : A \times A \to A$ which gives the action of the generating operation $\mu \in \Omega(2)$ on A. This is exactly Bourbaki's definition of a magma, and the name 'magma operad' is motivated by this correspondence.

To begin this section, we explain a representation of the elements of the magma operad in terms of non-commutative non-associative monomials and planar binary trees.

6.1.1. *The algebraic definition of the magma operad.* Recall that the elements of a free operad intuitively consists of formal operadic composites of generating operations with no relation between them apart from the general equivariance, unit and associativity relations of operads. In the case of the magma operad, we consider operadic composites of the product $\mu = \mu(x_1, x_2)$ and of the transposed operation $t\mu = \mu(x_2, x_1)$. If we take the usual product notation $\mu(x_1, x_2) = x_1 x_2$

for the generating operation $\mu = \mu(x_1, x_2)$, then these operadic composites are equivalent to *parenthesized words*

$$x_i x_j, \quad (x_i x_j) x_k, \quad x_i (x_j x_k), \qquad ((x_i x_j) x_k) x_l, \quad (x_i (x_j x_k)) x_l, \quad (x_i x_j)(x_k x_l), \quad \ldots$$

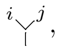
defined by providing any permutation of the variables $(x_1, \ldots, x_r)$ with a full binary bracketing (the *parenthesization*). These parenthesized words are the non-commutative non-associative monomials considered by Bourbaki.

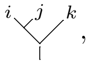
In this algebraic representation of the elements of the magma operad, the symmetric groups act by permuting variable indices, the unit is defined by the one-variable word $1 = 1(x_1) = x_1$, and the operadic composition operations $\circ_k : \Omega(m) \times \Omega(n) \to \Omega(m+n-1)$, $k = 1, \ldots, m$, are defined by the natural substitution of variables in non-commutative non-associative monomials. For instance, we have:

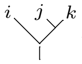
$$((x_3 x_1) x_2) \circ_1 ((x_2 x_1) x_3) = (x_5((x_2 x_1) x_3)) x_4.$$

To get this composite, we replace the variable x_1 (corresponding to our composition index $k = 1$) in the first monomial $p(x_1, x_2, x_3) = (x_3 x_1) x_2$ by the second monomial $q(x_1, x_2, x_3) = (x_2 x_1) x_3$, and we use the variable index shift of §1.1.4 to form a new monomial $p \circ_1 q = p(q(x_1, x_2, x_3), x_4, x_5)$ on the variables $(x_1, \ldots, x_5)$.

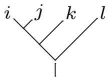
6.1.2. *The planar binary tree representation.* In our general construction of free operads, the elements are represented by trees whose vertices are labeled by generating operations. In the case of the magma operad, we can form a reduced version of this representation, where the elements of arity r consist of *planar binary trees* with r ingoing edges indexed by a permutation $(i_1, \ldots, i_r)$ of $(1, \ldots, r)$ as in the following pictures:

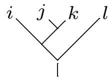
$x_i x_j =$ 

, $(x_i x_j) x_k =$ 

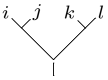
, $x_i (x_j x_k) =$ 

,

$((x_i x_j) x_k) x_l =$ 

, $(x_i (x_j x_k)) x_l =$ 

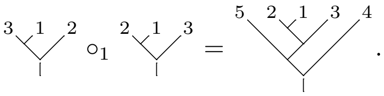
,

$(x_i x_j)(x_k x_l) =$ 

, $\ldots$

In the representation of §A, these planar binary trees correspond to treewise elements in which all vertices are labeled by the generating operation μ and where no vertex labeled by the transposed operation $t\mu$ occurs. The equivariance relation of §1.1.5 (see also §A.2) implies that the treewise elements considered in the appendix are, in the case of the magma operad, equivalent to treewise elements of this reduced form, and hence, that we can restrict ourselves to such planar binary trees in our construction.

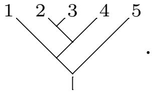
The symmetric action, the operadic unit and the operadic composition operations of the magma operad are given by the same operations as in §A.1 for planar binary trees. The symmetric group acts by permutation of the indices attached to the ingoing edges. To obtain the picture of an operadic composite of trees $\underline{\sigma} \circ_k \underline{\tau}$, we plug the second tree $\underline{\tau}$ in the ingoing edge of the tree $\underline{\sigma}$ marked by the index k and we perform the usual shift operation on the input indices of this composite object. For instance, in the case considered in §6.1.1, we get the following picture:



6.1.3. *The underlying permutation and parenthesization of the elements of the magma operad.* We readily see that each component of the magma operad $\Omega(r)$ forms a free Σ_r-set. We obtain, to be more precise, that each element $p \in \Omega(r)$ has a unique expression $p = s\pi = \pi(x_{s(1)}, \dots, x_{s(r)})$ such that π arises as a (multiple) composite of the generating operation $\mu = \mu(x_1, x_2)$ (with no occurrence of the transposed operation) and where s is a permutation acting on this monomial $\pi = \pi(x_1, \dots, x_r)$. We refer to the permutation s occurring in this expression $p = s\pi$ as the *underlying permutation* of the magma element $p \in \Omega(r)$. We also refer to the composite which defines the monomial π as the *underlying parenthesization* of the word $p = \pi(x_{s(1)}, \dots, x_{s(r)})$.

We can easily retrieve the permutation s and the composite π from the monomial expression of our element $p \in \Omega(r)$. Indeed, the permutation s represents the ordering of the variables in the word underlying p (where we forget about the parenthesization), while the monomial $\pi = \pi(x_1, \dots, x_r)$ is determined by the parenthesization itself (with the variables put in the canonical order). As an example, in the case $p = p(x_1, x_2, x_3, x_4, x_5) = (x_5((x_2x_1)x_3))x_4$, we obtain the permutation $s = (5, 2, 1, 3, 4)$ which corresponds to the ordering of variables $x_5x_2x_1x_3x_4$ and we have an identity $p = s\pi = \pi(x_5, x_2, x_1, x_3, x_4)$, where:

$$\pi(x_1, x_2, x_3, x_4, x_5) = (x_1((x_2x_3)x_4))x_5 =$$



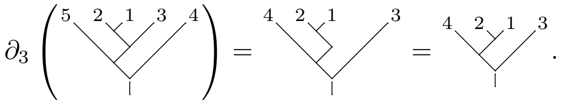
In the treewise representation, the permutation s can also be determined by the ordering (in the plane) of the indices attached to the ingoing edges of our tree (where we use the outgoing edge of the tree to fix the orientation of our figure).

6.1.4. *The unitary extension of the magma operad.* The magma operad has a unitary extension Ω_+ such that

$$\Omega_+(n) = \begin{cases} *, & \text{if } n = 0, \\ \Omega(n), & \text{otherwise,} \end{cases}$$

and of which composition structure extends the composition structure of the non-unitary operad Ω. In §2.2, we observe that the partial composition operations with the additional arity zero term of such an operad are equivalent to restriction operators $\partial_k : \Omega_+(n) \to \Omega_+(n-1)$ such that $\partial_k(p) = p \circ_k *$. Furthermore, for a connected free operad, such as the magma operad $\Omega = \Theta(\mu(x_1, x_2), \mu(x_2, x_1))$, the associativity of partial composites implies that our restriction operators are uniquely determined by their expression on generating operations $\mu = \mu(x_1, x_2)$ and $t\mu = \mu(x_2, x_1)$ (see §2.4). For these operations, we trivially have $\partial_1(\mu) = \partial_2(\mu) = 1$ and $\partial_1(t\mu) = \partial_2(t\mu) = 1$, since the component of arity one of our operad Ω is reduced to the one-point set formed by the unit element $1 \in \Omega(1)$.

If we use our correspondence between the restriction operators ∂_k and the composition operations $p \circ_k *$, then the formulas $\partial_1(\mu) = \partial_2(\mu) = 1$ are equivalent to the assumption that the operation $\mu(x_1, x_2) = x_1x_2$ satisfies the unit identities $\mu(*, x_1) = x_1 = \mu(x_1, *)$ in the operad Ω_+. In the algebraic approach of §6.1.1, we get the expression of a restriction operator on a non-commutative non-associative monomials by performing this substitution $x_k = *$ (and the standard index shift on the remaining variables). For instance, we have $\partial_3((x_5((x_2x_1)x_3))x_4) = (x_4((x_2x_1)*))x_3 = (x_4(x_2x_1))x_3$. In the treewise representation of §6.1.2, the operation $\partial_k(\underline{\sigma}) = \underline{\sigma} \circ_k *$ is identified with the removal of the ingoing edge indexed by k in the tree $\underline{\sigma}$. For instance, in the case $p(x_1, x_2, x_3, x_4, x_5) = (x_5((x_2x_1)x_3))x_4$, we get the following picture:



The algebras over the operad Ω_+ are identified with sets A equipped with a product $m : A \times A \to A$ (which determines the action of the non-unitary operad Ω on A) together with a distinguished element $e \in A$ such that $m(e, a) = a = m(a, e)$, for all $a \in A$. This element e represents the image of the arity zero operation $* \in \Omega_+(0)$ in A.

6.1.5. *Pullbacks of operads in groupoids.* By definition of free operads, giving a morphism $\omega : \Omega \to \mathsf{P}$ from the magma operad Ω to an operad in sets P amounts to fixing an operation of arity two $m \in \mathsf{P}(2)$ such that $m = \omega(\mu)$.

We consider such a morphism $\omega : \Omega \to \mathtt{Ob}\,\mathsf{Q}$ with values in the object-set operad $\mathsf{P} = \mathtt{Ob}\,\mathsf{Q}$ underlying an operad in groupoids Q. For each $r > 0$, we form a groupoid $\omega^*\,\mathsf{Q}(r)$ with $\mathtt{Ob}\,\omega^*\,\mathsf{Q}(r) = \Omega(r)$ as object set and with the morphism sets such that:

$$\mathtt{Mor}_{\omega^*\,\mathsf{Q}(r)}(p, q) = \mathtt{Mor}_{\mathsf{Q}(r)}(\omega(p), \omega(q)),$$

for all $p, q \in \Omega(r)$. These morphism sets are equipped with the identity morphisms and with the composition operations of the morphism sets of the groupoid $\mathsf{Q}(r)$.

The collection of groupoids $\omega^*\,\mathsf{Q}(r)$, $r > 0$, also inherits an operad structure:

- the action of a permutation $s \in \Sigma_r$ on the groupoid $\omega^*\,\mathsf{Q}(r)$ is the functor $s_* : \omega^*\,\mathsf{Q}(r) \to \omega^*\,\mathsf{Q}(r)$ given by the action of s on the magma operad at the object set level and by the maps
$$\underbrace{\mathtt{Mor}_{\mathsf{Q}(r)}(\omega(p), \omega(q))}_{=\mathtt{Mor}_{\omega^*\,\mathsf{Q}(r)}(p,q)} \xrightarrow{s_*} \underbrace{\mathtt{Mor}_{\mathsf{Q}(r)}(s\omega(p), s\omega(q))}_{=\mathtt{Mor}_{\omega^*\,\mathsf{Q}(r)}(sp,sq)}$$
inherited from the groupoid $\mathsf{Q}(r)$ at the morphism set level;
- the unit object $1 \in \mathtt{Ob}\,\omega^*\,\mathsf{Q}(1)$, equivalent to a functor $\eta : pt \to \omega^*\,\mathsf{Q}(1)$, is given by the unit element of the magma operad $1 \in \Omega(1)$;
- the composition operations $\circ_k : \omega^*\,\mathsf{Q}(m) \times \omega^*\,\mathsf{Q}(n) \to \omega^*\,\mathsf{Q}(m + n - 1)$, are the functors given by the partial composition operations of the magma operad at the object set level and by the mapping
$$\underbrace{\mathtt{Mor}_{\mathsf{Q}(m)}(\omega(p_0), \omega(p_1))}_{=\mathtt{Mor}_{\omega^*\,\mathsf{Q}(m)}(p_0,p_1)} \times \underbrace{\mathtt{Mor}_{\mathsf{Q}(n)}(\omega(q_0), \omega(q_1))}_{=\mathtt{Mor}_{\omega^*\,\mathsf{Q}(n)}(q_0,q_1)}$$
$$\xrightarrow{\circ_k} \underbrace{\mathtt{Mor}_{\mathsf{Q}(m+n-1)}(\omega(p_0) \circ_k \omega(q_0), \omega(p_1) \circ_k \omega(q_1))}_{=\mathtt{Mor}_{\omega^*\,\mathsf{Q}(m+n-1)}(p_0 \circ_k q_0, p_1 \circ_k q_1)}$$
inherited from the operad Q at the morphism set level, for all $m, n > 0$ and for each $k = 1, \dots, m$.

The collection $\omega^* \mathsf{Q}(r)$, $r > 0$, therefore forms an operad in groupoids $\omega^* \mathsf{Q}$. We refer to this operad as the *pullback* of the operad Q along the morphism $\omega : \Omega \to \mathtt{Ob}\,\mathsf{Q}$ determined by the object $m \in \mathtt{Ob}\,\mathsf{Q}(2)$.

In the case where our operad Q has a unitary extension Q_+, we immediately see that the morphism of non-unitary operads $\omega : \Omega \to \mathtt{Ob}\,\mathsf{Q}$ associated to an element $m \in \mathtt{Ob}\,\mathsf{Q}(2)$ has a unitary extension $\omega : \Omega_+ \to \mathtt{Ob}\,\mathsf{Q}_+$ as soon as the relations $m \circ_1 * = m \circ_2 * = 1$ hold in Q_+. In this situation, we also have a unitary version $\omega^* \mathsf{Q}_+$ of the operadic pullback $\omega^* \mathsf{Q}$. We define this object by an obvious extension of our construction.

6.1.6. *The pullback of the permutation operad.* We first examine the application of our pullback construction to the permutation operad Π which is basically defined in the category sets, but which we may also regard as formed of a collection of discrete groupoids. We adopt the notation CoP to distinguish this operad in the category of groupoids from the underlying operad in sets Π. We accordingly have

$$\mathtt{Ob}\,\mathsf{CoP}(r) = \Pi(r) = \Sigma_r,$$

for each $r > 0$, and the morphism sets of the groupoid $\mathsf{CoP}(r)$ are defined by:

$$\mathtt{Mor}_{\mathsf{CoP}(r)}(u,v) = \begin{cases} pt, & \text{if } u = v, \\ \varnothing, & \text{otherwise,} \end{cases}$$

for all $u, v \in \Sigma_r$. We also use the name '*colored permutation operad*' to refer to this operad in groupoids CoP.

Let $m \in \mathtt{Ob}\,\mathsf{CoP}(2)$ be the object defined by the identity permutation $(1,2) \in \Sigma_2$. The operad morphism $\omega : \Omega \to \Pi = \mathtt{Ob}\,\mathsf{CoP}$ associated to this element $m \in \mathtt{Ob}\,\mathsf{CoP}(2)$ is obviously identified with the map which sends a parenthesized words $p = p(x_{s(1)}, \dots, x_{s(r)})$ to the underlying permutation $(s(1), \dots, s(r))$ (in the sense of §6.1.3) and forgets about the bracketing (see §6.1.3). Let

$$\mathsf{PaP} := \omega^* \mathsf{CoP}$$

denote the pullback of the permutation operad under this morphism. We accordingly have

$$\mathtt{Ob}\,\mathsf{PaP}(r) = \Omega(r),$$

for every $r > 0$, by definition of our pullback operation. We call this operad in groupoids the '*operad of parenthesized permutations*'.

For parenthesized words $p = p(x_{u(1)}, \dots, x_{u(r)})$ and $q = q(x_{v(1)}, \dots, x_{v(r)})$, with $\omega(p) = u$ and $\omega(q) = v$ as underlying permutations, we have:

$$\mathtt{Mor}_{\mathsf{PaP}(r)}(p,q) = \begin{cases} pt, & \text{if } \omega(p) = \omega(q), \\ \varnothing, & \text{otherwise.} \end{cases}$$

The morphism $\omega : \Omega \to \mathtt{Ob}\,\mathsf{CoP}$ clearly admits a unitary extension, and we accordingly have a unitary version of the parenthesized permutation operad PaP_+ such that $\mathsf{PaP}_+(0) = *$.

The groupoid $\mathsf{PaP}(3)$ contains an *associativity isomorphism*

$$\alpha(x_1, x_2, x_3) \in \mathtt{Mor}_{\mathsf{PaP}(3)}((x_1x_2)x_3, x_1(x_2x_3))$$

which connects the objects $(x_1x_2)x_3, x_1(x_2x_3) \in \Omega(3)$.

$$\begin{tikzcd}
 & ((x_1 \Box x_2) \Box x_3) \Box x_4 \arrow[dl, "a(x_1,x_2,x_3) \Box x_4"'] \arrow[ddr, "a(x_1 \Box x_2,x_3,x_4)"] & \\
(x_1 \Box (x_2 \Box x_3)) \Box x_4 \arrow[dd, "a(x_1,x_2 \Box x_3,x_4)"'] & & \\
 & & (x_1 \Box x_2) \Box (x_3 \Box x_4) \arrow[ddl, "a(x_1,x_2,x_3 \Box x_4)"] \\
x_1 \Box ((x_2 \Box x_3) \Box x_4) \arrow[dr, "x_1 \Box a(x_2,x_3,x_4)"'] & & \\
 & x_1 \Box (x_2 \Box (x_3 \Box x_4)) &
\end{tikzcd}$$

FIGURE 6.1. The pentagon relation. We use the notation $m(x_1, x_2) = x_1 \Box x_2$ for an object of an operad $m = m(x_1, x_2) \in \mathtt{Ob}\, \mathsf{Q}(2)$. The expressions $((x_1 \Box x_2) \Box x_3) \Box x_4, \dots \in \mathtt{Ob}\, \mathsf{Q}(4)$ represent composites of this object in our operad. We have for instance $((m \circ_1 m) \circ_1 m)(x_1, x_2, x_3, x_4) = ((x_1 \Box x_2) \Box x_3) \Box x_4$. We similarly use algebraic expressions of the form $a(x_1, x_2, x_3) \Box x_4 \in \mathtt{Mor}\, \mathsf{Q}(4)$ to represent the operadic composites of morphisms in our operad. We have for instance $(id \circ_1 a)(x_1, x_2, x_3, x_4) = a(x_1, x_2, x_3) \Box x_4$, where we consider the image of the morphisms $id = id_{x_1 \Box x_2} \in \mathtt{Mor}\, \mathsf{Q}(2)$ and $a \in \mathtt{Mor}\, \mathsf{Q}(3)$ under the operadic composition functor $\circ_1 : \mathsf{Q}(2) \times \mathsf{Q}(3) \to \mathsf{Q}(4)$.

We have the following result, which gives an operadic interpretation of the Mac Lane Coherence Theorem:

THEOREM 6.1.7 (Operadic interpretation of the Mac Lane Coherence Theorem).

(a) *The morphisms of the groupoids $\mathsf{PaP}(r)$ can be obtained as (categorical) composites of morphisms which themselves decompose into operadic composition products of identity morphisms and of the associativity isomorphism $\alpha(x_1, x_2, x_3)$.*

(b) *Let Q be any operad in the category of categories. Let*

$$m = m(x_1, x_2) \in \mathtt{Ob}\, \mathsf{Q}(2)$$

be an object in the component of arity two of this operad. In what follows, we also set

$$m(x_1, x_2) = x_1 \Box x_2$$

and we use classical algebraic notation (rather than operadic notation) to represent the composites of this object in our operad Q. Let

$$a(x_1, x_2, x_3) \in \mathtt{Mor}_{\mathsf{Q}(3)}((x_1 \Box x_2) \Box x_3, x_1 \Box (x_2 \Box x_3))$$

be an isomorphism which connects the operadic composites

$$(m \circ_1 m)(x_1, x_2, x_3) = (x_1 \Box x_2) \Box x_3 \in \mathtt{Ob}\, \mathsf{Q}(3)$$
$$\text{and} \quad (m \circ_2 m)(x_1, x_2, x_3) = x_1 \Box (x_2 \Box x_3) \in \mathtt{Ob}\, \mathsf{Q}(3)$$

in the category $\mathsf{Q}(3)$.

If this associativity isomorphism $a = a(x_1, x_2, x_3)$ *makes the pentagon diagram of Figure* 6.1 *commute in* $\mathsf{Q}(4)$, *then all parallel morphisms of the category* $\mathsf{Q}(r)$ *which we obtain from this isomorphism* $a = a(x_1, x_2, x_3)$ *by combinations of the operations of (*a*) are equal, for any* $r > 0$. *In this situation, we also have a morphism of operads in groupoids*

$$\phi : \mathsf{PaP} \to \mathsf{Q}$$

uniquely determined by the assignment $\phi(\mu(x_1, x_2)) = x_1 \,\square\, x_2$ *at the object set level and by the assignment* $\phi(\alpha(x_1, x_2, x_3)) = a(x_1, x_2, x_3)$ *at the morphism set level.*

(c) *In the construction of (*b*), if we moreover assume the existence of an object*

$$e \in \mathsf{Ob}\, \mathsf{Q}(0)$$

which satisfies the relation

$$e \,\square\, x_1 = x_1 = x_1 \,\square\, e$$

at the object set level (where we again use algebraic notation to denote the composites with this object in our operad) and the relation

$$a(e, x_1, x_2) = a(x_1, e, x_2) = a(x_1, x_2, e) = id_{m(x_1,x_2)}$$

at the morphism set level (with the same notation conventions as above), then the morphism $\phi : \mathsf{PaP} \to \mathsf{Q}$ *has a unitary extension*

$$\phi : \mathsf{PaP}_+ \to \mathsf{Q}$$

which maps the distinguished arity 0 *element of the unitary operad of parenthesized permutations* $* \in \mathsf{PaP}_+(0)$ *to this object* $e \in \mathsf{Ob}\, \mathsf{Q}(0)$.

Explanations. The claims of this theorem follow from an operadic interpretation of the Mac Lane Coherence Theorem [130]. We use operadic notation in this proof rather than the algebraic notation which we adopt in the statement of our theorem.

To understand the claims of (a-b), we may look at the picture formed by the full subgroupoid of $\mathsf{PaP}(r)$ generated by the parenthesized words $p = p(x_{s(1)}, \dots, x_{s(r)})$ for a given underlying permutation $s \in \Sigma_r$. The morphisms of this subgroupoid are identified with the paths of a graph, represented in Figure 6.2 in the case $r = 4$, and in Figure 6.3 in the case $r = 5$. To simplify these pictures, we only represents the parenthesization (in the form of binary trees) underlying our objects $p \in \mathsf{Ob}\, \mathsf{PaP}(r)$ and we omit the permutation labeling $(s(1), \dots, (s(r))$ which is by assumption the same for all objects occurring in the figure. In the case $r = 4$, we just retrieve the pentagon diagram of the theorem.

The edges of our graph are operadic composition products of associativity isomorphisms and of identity morphisms. The claim of (a) is therefore equivalent to the connectedness of this graph, which visibly holds in the case $r = 4$ and in the case $r = 5$. The general case of this claim can be established by an easy induction.

Note that all possible combinations of categorical composites of operadic composites of associativity isomorphisms and identity morphisms which we may form to connect a pair of objects with the same underlying permutation in the groupoid $\mathsf{PaP}(r)$ are equal, since the morphism sets which we associated to any such pair of objects are reduced to a point by construction of this groupoid. The idea of the second claim of our statement (b) is that the validity of the pentagon equation and the universal relations satisfied by the composition operations of an operad in

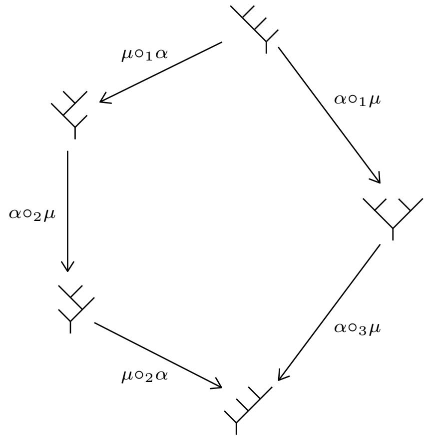


FIGURE 6.2. The associahedra in dimension 2. The edges are given by the image of the associativity isomorphism $\alpha \in \texttt{Mor}\, \mathsf{PaP}(3)$ under the functors $-\circ_k \mu : \mathsf{PaP}(3) \to \mathsf{PaP}(4)$, $k = 1,2,3$, and $\mu\circ_k - : \mathsf{PaP}(3) \to \mathsf{PaP}(4)$, $k = 1,2$, in the parenthesized permutation operad PaP.

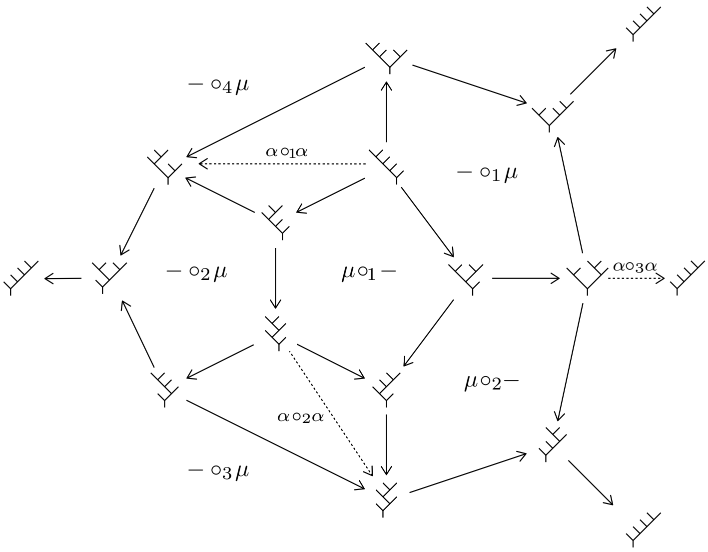


FIGURE 6.3. The picture of the 3 dimensional associahedra. The binary tree corresponding to the word $x_1(x_2(x_3(x_4x_5)))$ is put at the infinity of the figure. The pentagon cells of this picture are identified with the image of the pentagon of Figure 6.2 under the functors $-\circ_k \mu : \mathsf{PaP}(4) \to \mathsf{PaP}(5)$, $k = 1,2,3,4$, and $\mu\circ_k - : \mathsf{PaP}(4) \to \mathsf{PaP}(5)$, $k = 1,2$, marked in the figure. The square cells correspond to the factorization of morphisms $\alpha\circ_k\alpha$, marked by dotted arrows in the figure, and which arise from the bifunctoriality of the composition operations $-\circ_k - : \mathsf{PaP}(3)\times\mathsf{PaP}(3) \to \mathsf{PaP}(5)$, $k = 1,2,3$.

categories suffice to imply the identity of all these composites in our operad. To establish this result, we observe that the edges of our graphs form the 1-dimensional skeleton of a connected cell complex whose 2-dimensional cells are equivalent either to the pentagon of Figure 6.1, which correspond to the assumption of our claim, or to square diagrams, which reflect universal relations satisfied by the composition operations of an operad in categories. The picture of Figure 6.3 makes this observation clear in the case $r = 5$. The edges of our graph form, in general, the 1-dimensional skeleton of a polyhedron, the *Stasheff associahedron* $\mathsf{K}(r)$, whose boundary decomposes into cartesian products of associahedra of lower dimension:

$$\partial\,\mathsf{K}(r) = \bigcup_{\underline{\mathsf{m}} \circ_{i_k} \underline{\mathsf{n}} = \underline{\mathsf{r}}} \mathsf{K}(\underline{\mathsf{m}}) \times \mathsf{K}(\underline{\mathsf{n}}),$$

for a set of faces indexed by operadic composition schemes:

$$p = a(x_{i_1}, \dots, b(x_{j_1}, \dots, x_{j_n}), \dots, x_{i_m})$$

(see §2.1, §2.5). The form of the 2-dimensional cells of the associahedra can be obtained by induction from the shape of this decomposition. The associahedra actually define an operad in topological space which models homotopy associative monoids (see [167]). We refer to [73, 116, 125] for various constructions of the associahedra as convex polyhedra and to Stasheff's original article [167] for a realization as a cell complex. We can use all these constructions to get a geometrical proof of our statement (see [160]). We can also use a direct inductive argument, forgetting about the geometry of associahedra and focusing on the underlying combinatorics of trees, to establish that all relations between the paths of our graph reduce to composites of pentagon and square relations. We refer to Mac Lane's monograph [130] for further details on this purely combinatorial approach.

To establish our assertion (b), we consider the image of our graph in the groupoid $\mathsf{Q}(r)$, with the object $m \in \mathtt{Ob}\,\mathsf{Q}(2)$ and the associativity isomorphism $a \in \mathtt{Mor}\,\mathsf{Q}(3)$ substituted to the object μ and to the universal associativity isomorphism α of the parenthesized permutation operad PaP. The assumption on our associativity isomorphism $a(x_1, x_2, x_3)$ implies the commutativity of the pentagon diagrams of this graph in the groupoid $\mathsf{Q}(r)$. The bifunctoriality of operadic composition products implies the commutativity of the squares of our graph. The graph therefore commutes as a whole and this assertion gives the crux of the proof of our claim (b). Indeed, the commutativity of the graph implies that the image of our graph in the groupoid $\mathsf{Q}(r)$ gives a coherent definition of groupoid morphisms $\phi : \mathsf{PaP}(r) \to \mathsf{Q}(r)$, $r > 0$, that preserve the structure operations of our operads.

The requirements of (c) imply that our operad morphism $\phi : \mathsf{PaP} \to \mathsf{Q}$ makes the composite with the arity zero term $*$ in the unitary extension of the operad PaP correspond to the composite with the object e in the operad Q. Therefore, we immediately obtain that our operad morphism $\phi : \mathsf{PaP} \to \mathsf{Q}$ admits an extension $\phi : \mathsf{PaP}_+ \to \mathsf{Q}$ to the unitary operad PaP_+ as soon as we have an object e that satisfies these conditions in Q. □

To sum up, the result of Theorem 6.1.7 gives an equivalence between operad morphisms $\phi : \mathsf{PaP} \to \mathsf{Q}$ and pairs (m, a), where $m = m(x_1, x_2) \in \mathtt{Ob}\,\mathsf{Q}(2)$ and $a = a(x_1, x_2, x_3) \in \mathtt{Mor}\,\mathsf{Q}(3)$ is an isomorphism that makes this operation $m(x_1, x_2) = x_1 \,\square\, x_2$ associative in the operad Q. In the expression of this associativity relation, we assume the verification of coherence constraints, which can be reduced to the commutativity of the pentagon diagram of Figure 6.1. In the unitary case, we consider an additional object $e \in \mathsf{Q}(0)$ which satisfies strict unit relations $e \,\square\, x_1 = x_1 = x_1 \,\square\, e$ with respect to the product $m(x_1, x_2) = x_1 \,\square\, x_2$ and natural coherence constraints with respect to the associativity isomorphism $a = a(x_1, x_2, x_3)$.

For the sake of comparison, in the case of the discrete groupoid operad CoP we have the following statement:

Theorem 6.1.8.

(a) *Giving a morphism* $\phi : \mathsf{CoP} \to \mathsf{Q}$ *from the permutation operad* CoP *towards an operad in groupoids* Q *amounts to giving an object*

$$m(x_1, x_2) = x_1 \,\square\, x_2 \in \mathtt{Ob}\, \mathsf{Q}(2)$$

which satisfies a strict associativity relation

$$(x_1 \,\square\, x_2) \,\square\, x_3 = x_1 \,\square\, (x_2 \,\square\, x_3)$$

in the operad Q*.*

(b) *In the construction of (* a*), if we moreover assume the existence of an object*

$$e \in \mathtt{Ob}\, \mathsf{Q}(0) \quad \text{such that} \quad e \,\square\, x_1 = x_1 = x_1 \,\square\, e,$$

then the morphism $\phi : \mathsf{CoP} \to \mathsf{Q}$ *has a unitary extension* $\phi : \mathsf{CoP}_+ \to \mathsf{Q}$ *which maps the distinguished arity* 0 *element of the unitary operad of permutations* $* \in \mathsf{CoP}_+(0)$ *to this object* $e \in \mathtt{Ob}\, \mathsf{Q}(0)$*.*

Proof. If we regard an operad in sets P as a collection of discrete groupoids, then giving a morphism of operads in groupoids $\phi : \mathsf{P} \to \mathsf{Q}$ reduces to giving a morphism of operads in sets $\phi : \mathsf{P} \to \mathtt{Ob}\, \mathsf{Q}$ with values in the object-set operad underlying Q. In the case of the permutation operad $\mathsf{P} = \mathit{\Pi}$, which we identify with the associative operad in sets As (see Proposition 1.2.7), we deduce from the presentation by generators and relations of §1.2.6 that giving such a morphism amounts to giving an operation $m \in \mathsf{Q}(2)$ which satisfies the associativity relation in Q. The argument line is similar in the unitary case. ☐

6.1.9. *The operadic representation of monoidal structures on categories.* Recall that the action of an operad P on an object A in a base symmetric monoidal category $\mathcal{M}$ is equivalent to a morphism $\phi : \mathsf{P} \to \mathtt{End}_A$, where $\mathtt{End}_A$ is the endomorphism operad of A. In the case where we work within the category of (small) categories $\mathcal{M} = \mathcal{C}at$ and we deal with an object of this category $\mathcal{C} \in \mathcal{C}at$, the endomorphism operad $\mathtt{End}_{\mathcal{C}}$ is defined in arity r by the category which has the r-functors $f : \mathcal{C}^{\times r} \to \mathcal{C}$ as objects and the natural transformation between such functors as morphisms.

From the results established in this section, we obtain that giving a morphism $\phi : \mathsf{PaP}_+ \to \mathtt{End}_{\mathcal{C}}$ amounts to giving a monoidal structure on $\mathcal{C}$ with strict unit relations but general associativity isomorphisms (see [130]), while giving a morphism $\phi : \mathsf{CoP}_+ \to \mathtt{End}_{\mathcal{C}}$ is equivalent to giving a monoidal structure with both strict unit and strict associativity relations. In both cases, we take the image of the object $\mu = \mu(x_1, x_2)$ under ϕ to get the tensor product operation $m(X_1, X_2) = X_1 \otimes X_2 : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ of the monoidal structure on $\mathcal{C}$. In the unitary setting, we also take the image of the unitary element of the operad $* \in \mathsf{P}_+(0)$ to get a natural transformation $e : pt \to \mathcal{C}$ equivalent to a unit object $\mathbb{1} \in \mathcal{C}$ for this tensor product in $\mathcal{C}$. In the case of the parenthesized permutation operad, we take the image of the associativity isomorphism $\alpha \in \mathtt{Mor}\, \mathsf{P}(3)$ to get a natural isomorphism $a(X_1, X_2, X_3) : (X_1 \otimes X_2) \otimes X_3 \xrightarrow{\simeq} X_1 \otimes (X_2 \otimes X_3)$ that makes our tensor product associative. In the colored permutation case, we assume that this associativity relation holds strictly $(X_1 \otimes X_2) \otimes X_3 = X_1 \otimes (X_2 \otimes X_3)$ and we take the identity morphism as associativity isomorphism $a = id$. The pentagon relation of Theorem 6.1.7 is nothing but the usual coherence axiom of [130] for the associativity isomorphism of a monoidal category and we have a similar correspondence for the coherence constraints associated to the unit object. This identity gives the correspondence between the construction of Theorem 6.1.7–6.1.8 and the definition of a monoidal structure on a category $\mathcal{C}$.

To complete the account of this section, we record the following result which motivates the introduction of the pullback construction of §6.1.5 for operads in groupoids:

Proposition 6.1.10. *Let P be an operad in groupoids which has the magma operad as underlying object operad $\mathtt{Ob}\, P = \Omega$. For any lifting problem*

$$\begin{tikzcd}
 & R \arrow[d, "\phi"] \\
P \arrow[ur, dotted, "\exists ?\psi"] \arrow[r] & S
\end{tikzcd}$$

such that ϕ is a categorical equivalence of operads in groupoids (see §5.2.2*), we have a fill-in morphism ψ that makes our lifting diagram commute in the strict sense.*

Proof. Exercise. ☐

6.2. The parenthesized braid operad

Recall that the operad of colored braids CoB satisfies $\mathtt{Ob}\, CoB = \Pi$. We define the *operad of parenthesized braids PaB* by applying the pullback construction of §6.1.5 to this operad CoB. We explicitly set $PaB := \omega^* \, CoB$, where we again consider the morphism $\omega : \Omega \to \Pi$ which maps a parenthesized word $p = p(x_{s(1)}, \dots, x_{s(r)}) \in \Omega(r)$ to its underlying permutation $s \in \Sigma_r$ (as in §6.1.6). We then have $\mathtt{Ob}\, PaB(r) = \Omega(r)$ for each arity $r > 0$, and we have the relation $\mathtt{Mor}_{PaB(r)}(p, q) = \mathtt{Mor}_{CoB(r)}(\omega(p), \omega(q))$ for any pair of parenthesized words $p = p(x_{u(1)}, \dots, x_{u(r)})$ and $q = q(x_{v(1)}, \dots, x_{v(r)})$ with $u = \omega(p)$ and $v = \omega(q)$ as underlying permutations. The symmetric group actions, the unit, and the composition operations that define the operad structure of this collection of groupoids are inherited from the magma operad at the object set level and from the colored braid operad at the morphism set level (see §6.1.5). The parenthesized braid operad has a unitary version (like the parenthesized permutation operad) which is defined by an obvious unitary extension of our pullback construction.

Recall that we use the notation CoP for the permutation operad Π regarded as an operad in groupoids. We have an obvious morphism of operads in groupoids $\iota : CoP \to CoB$ which is given by the identity $\mathtt{Ob}\, CoB = \Pi$ at the object set level. This morphism admits a lifting

$$\begin{tikzcd}
PaP \arrow[r, dotted] \arrow[d] & PaB \arrow[d] \\
CoP \arrow[r, "\iota"'] & CoB
\end{tikzcd},$$

which identifies the operad of parenthesized permutations PaP with a suboperad of PaB such that $\mathtt{Ob}\,\mathit{PaB} = \mathtt{Ob}\,\mathit{PaP} = \varOmega$. This operad embedding has an obvious extension to unitary operads.

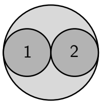
The main purpose of this section is to give an analogue of Theorem 6.1.7 for the parenthesized braid operad. To complete our account, we also state an analogue of the result of Theorem 6.1.8 for the colored braid operad. Before examining this question we give a topological interpretation of the operad PaB in terms of the fundamental groupoid of the little 2-discs operad $\pi\, D_2$.

6.2.1. *Parenthesized braids and fundamental groupoid elements.* In the definition of the operad of colored braids CoB, we make a choice of contact points $\underline{a}$ on the medium axis $y = 0$ of the open disc $\mathring{\mathbb{D}}^2$. The planar binary trees, which define the objects of the groupoids $\mathit{PaB}(r)$, actually have an interpretation in terms of particular choices of configurations of contact points on this line $y = 0$. To be explicit, instead of the equidistant contact points of §5.0, we consider the centers of diadic decompositions of the axis $y = 0$ of the open disc $\mathring{\mathbb{D}}^2$. In the next proposition, we establish that these configurations of points correspond to configurations of little 2-discs which generate a free operad (isomorphic to the magma operad) inside the little 2-disc operad D_2. The equivalence between these diadic decompositions, the free suboperad of little 2-discs, and the planar binary trees of the magma operad, is made explicit (for the low arity cases $r = 2, 3, 4$) in the picture of Figure 6.4.

In our proposition, we rely on this correspondence on objects in order to prove that the parenthesized braid operad PaB is identified with a suboperad of the operad $\pi\, D_2$ formed by the fundamental groupoids of the little 2-discs spaces $D_2(r)$, $r > 0$. In what follows, we use this relationship in order to represent the morphisms of the parenthesized braid operad PaB as braids whose origins and end-points are given by the centers of little 2-discs configurations corresponding to the elements of the object sets of our operad $\mathtt{Ob}\,\mathit{PaB} \subset \mathtt{Ob}\,\pi\, D_2$ (see Figure 6.5 for an example). In general, we use a simplified representation in terms of a braid diagram, where we replace the configurations of little 2-discs considered in our picture by their trace on the axis $y = 0$. This projection actually gives the diadic decompositions of the interval which we associate to our objects in the little 2-discs operad.

Proposition 6.2.2.

(a) *The configuration of little* 2*-discs*

$\mu =$  $\in D_2(2)$

generates a free operad, isomorphic to $\varOmega$*, within the little* 2*-disc operad* D_2*.*

(b) *The disc center mapping of* §4.2.2 *can be applied to paths in the spaces of little* 2*-discs to give an isomorphism*

$$\omega_* : \pi\, D_2\, |_{\varOmega} \xrightarrow{\simeq} \mathit{PaB}$$

between the operad of parenthesized braids PaB *and the restriction of the fundamental groupoid of the little* 2*-disc operad* D_2 *to the objects of this suboperad* $\varOmega \subset \mathtt{Ob}\,\pi\, D_2$*.*

Proof. Let $\phi : \varOmega \to D_2$ be the morphism that sends the generating element $\mu \in \varOmega(2)$ of the free operad $\varOmega$ to the little 2-disc configuration $\mu \in D_2(2)$ of assertion (a). The claim of assertion (a) is that this morphism defines an embedding

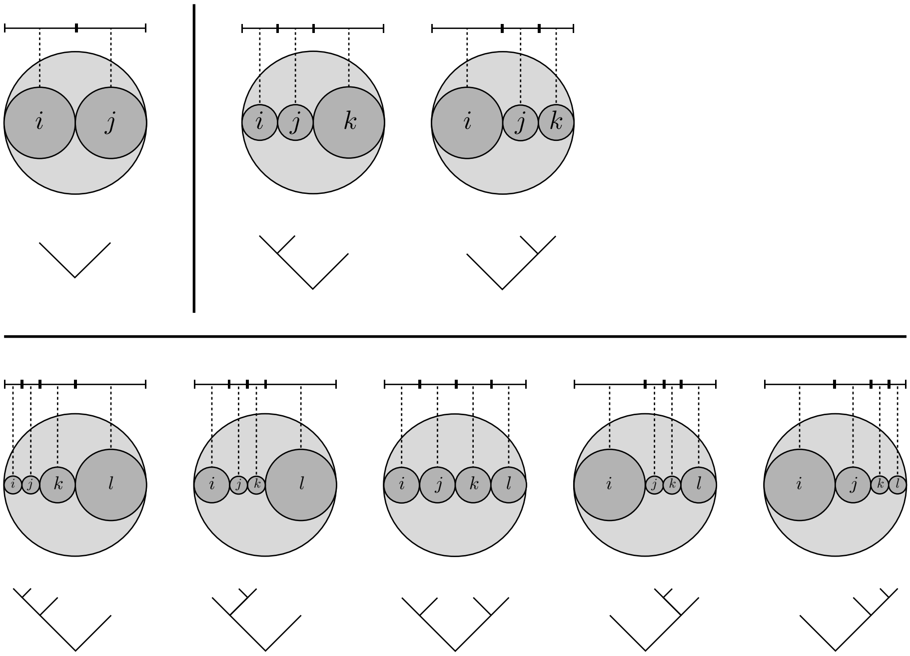


FIGURE 6.4. The correspondence between binary trees, operadic composites of a generating little 2-disc configurations and diadic decompositions of the interval in arity $r = 2, 3, 4$. The indices (i, j), (i, j, k), (i, j, k, l), ... run over permutations of $(1, 2)$, $(1, 2, 3)$, $(1, 2, 3, 4)$, ... The diadic decompositions of the interval represent a counterpart, in the little 1-disc operad, of the little 2-disc composites considered in this picture (see the proof of Proposition 6.2.2 for detailed explanations).

$\alpha \quad =$

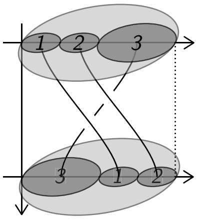


FIGURE 6.5. The picture, in the fundamental groupoid of the little 2-discs operad, of a morphism of the parenthesized braid operad.

of operads. In our verification, we use the symmetric collection equivalent to the magma operad Ω, and we consider monomials $p = p(x_{i_1}, \ldots, x_{i_r}) \in \Omega(\underline{\mathrm{r}})$ of which variables can be indexed by an arbitrary finite set $\underline{\mathrm{r}} = \{i_1, \ldots, i_r\}$. The element μ visibly comes from the operad of little 1-discs D_1 which we regard as a suboperad of D_2 by using the correspondence of §4.1.5 (see also §5.1.7). The morphism ϕ therefore admits a factorization through D_1 and we are left to prove that this factorization $\phi : \Omega \to D_1$ is an injection. Equivalently, we can look at the trace of any configuration of little 2-discs $\underline{c} \in D_2(r)$ on the axis $y = 0$ in the ambient disc $\mathbb{D}^2$ to determine the pre-image of this operation $\underline{c} \in D_2(r)$ in the magma operad Ω. (Recall that the image of D_1 in D_2 consists of configurations of little discs centered on this axis $y = 0$, and the trace, considered in our process, can be used to determine the counterimage in D_1 of an element of D_2.)

The configurations of little intervals that lie in the image of our map ϕ are associated to diadic decompositions of the interval $[-1, 1]$ (see Figure 6.4 for examples). To retrieve an element of Ω from the corresponding configuration of little intervals $\underline{c}$, we just observe that we have $\underline{c} = \mu(\underline{a}, \underline{b})$ where $\underline{a} \in D_1(\{i_1, \dots, i_m\})$ (respectively, $\underline{b} \in D_1(\{j_1, \dots, j_n\})$) is produced by applying the affine transformation $t \mapsto 2t + 1$ (respectively, $t \mapsto 2t - 1$) to the configuration of little intervals that lie in the subinterval $[-1, 0] \subset [-1, 1]$ (respectively, $[0, 1] \subset [-1, 1]$) of the collection $\underline{c}$. We carry on this process to obtain, by induction, the full decomposition of $\underline{c}$ in the magma operad Ω.

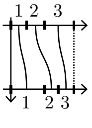
The second claim of the proposition is a variation on the result of Theorem 5.2.12. Simply note that we now have a direct isomorphism $\omega_* : \pi\, D_2 \mid_{\Omega} \xrightarrow{\simeq} PaB$ which lifts the chain of category equivalences considered in the proof of Theorem 5.2.12:

$$\begin{tikzcd}
& \pi\, D_2(r) \mid_{\Omega(r)} \arrow[r, dotted, "\omega_*", "\simeq"'] \arrow[d, hook] & PaB(r) \arrow[d] \\
\pi\, D_2(r) & \pi\, D_2(r) \mid_{D_1(r)} \arrow[l, hook'] \arrow[r] & CoB(r)
\end{tikzcd}.$$

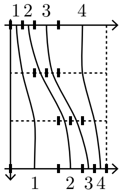
This verification completes the proof of Proposition 6.2.2.

We must note that the isomorphism of the proposition $\omega_* : \pi\, D_2 \mid_{\Omega} \xrightarrow{\simeq} PaB$ does not extend to a morphism of unitary operads. We therefore need to go back to the rectification process of Theorem 5.3.4 when we deal with the restriction operators $u^* : PaB_+(n) \to PaB_+(m)$ which define the composition operations with the additional term $PaB_+(0) = *$ of the unitary operad of parenthesized braids PaB_+. $\square$

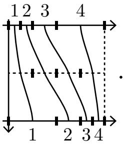
6.2.3. *The associativity isomorphism and the braiding of the parenthesized braid operad.* The morphism $\iota : PaP \to PaB$ considered in the introduction of this section is given by the identity $\mathtt{Ob}\, PaB = \mathtt{Ob}\, PaP = \Omega$ on object sets and is defined at the morphism set level by sending the associativity isomorphism $\alpha \in \mathtt{Mor}\, PaP(3)$ in the parenthesized permutation operad to the morphism

$\alpha =$ 

in the parenthesized braid operad. The pentagon relation of Figure 6.1 is equivalent to the identity of the following parenthesized braid diagrams:

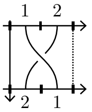


$=$

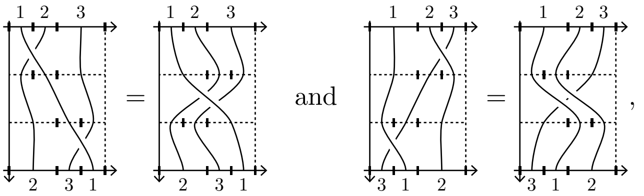


.

Besides the associativity isomorphism $\alpha \in \texttt{Mor}\,\mathit{PaB}(3)$, we consider the morphism

$\tau =$ 

which we call the *braiding isomorphism* (of the parenthesized braid operad). We readily see that the associativity isomorphism and the braiding isomorphism satisfy the following identities



which we express, in algebraic terms, by the commutativity of the hexagon diagrams of Figure 6.6. We moreover have the obvious identities $\alpha(*, x_1, x_2) = \alpha(x_1, *, x_2) = \alpha(x_1, x_2, *) = id_{\mu(x_1,x_2)}$ and $\tau(*, x_1) = \tau(x_1, *) = id_{x_1}$ (which we can also express by the restriction formulas $\partial_1\alpha = \partial_2\alpha = \partial_3\alpha = id_\mu$ and $\partial_1\tau = \partial_2\tau = id_1$) when we work in the unitary extension of our operad PaB_+.

We can now formulate the analogue of Theorem 6.1.7 for the parenthesized braid operad:

Theorem 6.2.4.

(a) *The morphisms of the groupoid $\mathit{PaB}(r)$ can be obtained as (categorical) composites of morphisms which themselves decompose into operadic composition products of identity morphisms, of the associativity isomorphism $\alpha \in \texttt{Mor}\,\mathit{PaB}(3)$, and of the braiding isomorphism $\tau \in \texttt{Mor}\,\mathit{PaB}(2)$.*

(b) *Let Q be any operad in the category of categories. Let*

$$m = m(x_1, x_2) \in \texttt{Ob}\,\mathit{Q}(2)$$

be an object in the component of arity two of this operad. In what follows, we also set

$$m(x_1, x_2) = x_1 \square x_2$$

and we use classical algebraic notation (rather than operadic notation) to represent the composites of this object in our operad Q. Let

$$a(x_1, x_2, x_3) \in \texttt{Mor}_{\mathit{Q}(3)}((x_1 \square x_2) \square x_3, x_1 \square (x_2 \square x_3))$$

be an isomorphism which connects the operadic composites

$$\begin{aligned} &(m \circ_1 m)(x_1, x_2, x_3) = (x_1 \square x_2) \square x_3 \in \texttt{Ob}\,\mathit{Q}(3) \\ \textit{and}\quad &(m \circ_2 m)(x_1, x_2, x_3) = x_1 \square (x_2 \square x_3) \in \texttt{Ob}\,\mathit{Q}(3) \end{aligned}$$

in the category $\mathit{Q}(3)$. Let

$$c = c(x_1, x_2) \in \texttt{Mor}_{\mathit{Q}(2)}(x_1 \square x_2, x_2 \square x_1)$$

be an isomorphism which connects the operation $m(x_1, x_2) = x_1 \square x_2 \in \texttt{Ob}\,\mathit{Q}(2)$ to its transposite $(1\ 2)m(x_1, x_2) = x_2 \square x_1 \in \texttt{Ob}\,\mathit{Q}(2)$ in the category $\mathit{Q}(2)$.

$$\begin{tikzcd}
 & (x_1 \square x_2) \square x_3 \arrow[dl, swap, "c(x_1,x_2)\square x_3"] \arrow[dr, "a(x_1,x_2,x_3)"] & \\
(x_2 \square x_1) \square x_3 \arrow[d, swap, "a(x_2,x_1,x_3)"] & & x_1 \square (x_2 \square x_3) \arrow[d, "c(x_1,x_2\square x_3)"] \\
x_2 \square (x_1 \square x_3) \arrow[dr, swap, "x_2\square c(x_1,x_3)"] & & (x_2 \square x_3) \square x_1 \arrow[dl, "a(x_2,x_3,x_1)"] \\
 & x_2 \square (x_3 \square x_1) &
\end{tikzcd}$$

$$\begin{tikzcd}
 & x_1 \square (x_2 \square x_3) \arrow[dl, swap, "x_1\square c(x_2,x_3)"] \arrow[dr, "a(x_1,x_2,x_3)^{-1}"] & \\
x_1 \square (x_3 \square x_2) \arrow[d, swap, "a(x_1,x_3,x_2)^{-1}"] & & (x_1 \square x_2) \square x_3 \arrow[d, "c(x_1\square x_2,x_3)"] \\
(x_1 \square x_3) \square x_2 \arrow[dr, swap, "c(x_1,x_3)\square x_2"] & & x_3 \square (x_1 \square x_2) \arrow[dl, "a(x_3,x_1,x_2)^{-1}"] \\
 & (x_3 \square x_1) \square x_2 &
\end{tikzcd}$$

FIGURE 6.6. The hexagon relations. In these diagrams, we again use the operator notation $m(x_1, x_2) = x_1 \square x_2$ for an object of an operad $m = m(x_1, x_2) \in \mathtt{Ob}\,\mathsf{Q}(2)$, and the expressions $(x_1 \square x_2) \square x_3, \dots \in \mathtt{Ob}\,\mathsf{Q}(4)$ in our diagram represent operadic composites of this object (as in the picture of the pentagon relation in Figure 6.1). We have for instance $(m \circ_1 m)(x_1, x_2, x_3) = (x_1 \square x_2) \square x_3$. We similarly use algebraic expressions of the form $c(x_1, x_2) \square x_3, \dots \in \mathtt{Mor}\,\mathsf{Q}(3)$ to represent operadic composites of morphisms in our operad Q. We have for instance $(id \circ_1 c)(x_1, x_2, x_3) = c(x_1, x_2) \square x_3$, where we consider the image of the morphisms $id = id_{x_1 \square x_2} \in \mathtt{Mor}\,\mathsf{Q}(2)$ and $c \in \mathtt{Mor}\,\mathsf{Q}(2)$ under the operadic composition functor $\circ_1 : \mathsf{Q}(2) \times \mathsf{Q}(2) \to \mathsf{Q}(2)$.

If these isomorphisms $a = a(x_1, x_2, x_3)$ *and* $c = c(x_1, x_2)$ *make the pentagon diagram of Figure* 6.1 *and the hexagon diagrams of Figure* 6.6 *commute, then we have a morphism of operads in groupoids*

$$\phi : \mathsf{PaB} \to \mathsf{Q}$$

uniquely determined by the assignments $\phi(\mu) = x_1 \square x_2$, $\phi(\alpha) = a(x_1, x_2, x_3)$ *and* $\phi(\tau) = c(x_1, x_2)$ *in the operad* Q.

(c) *In the construction of (*b*), if we moreover assume the existence of an object*

$$e \in \mathtt{Ob}\,\mathsf{Q}(0)$$

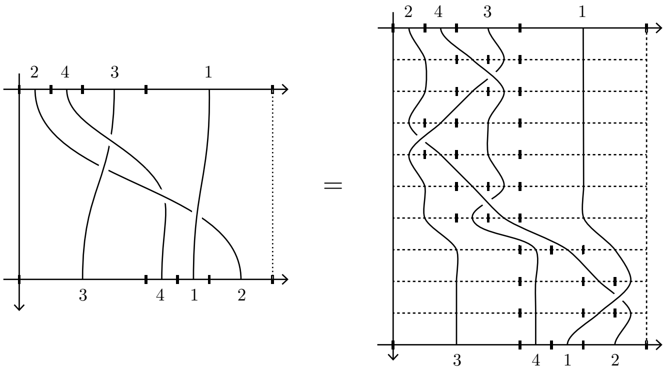


FIGURE 6.7. The decomposition of a parenthesized braid.

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which satisfies the relation

$$e \,\Box\, x_1 = x_1 = x_1 \,\Box\, e$$

at the object set level (where we again use algebraic notation to denote the composites with this object in our operad), together with the identities

$$a(e, x_1, x_2) = a(x_1, e, x_2) = a(x_1, x_2, e) = id_{m(x_1,x_2)},$$
$$c(e, x_1) = c(x_1, e) = id_{x_1}$$

at the morphism set level (with the same notation conventions as above), then the morphism $\phi : \mathsf{PaB} \to \mathsf{Q}$ *has a unitary extension*

$$\phi : \mathsf{PaB}_+ \to \mathsf{Q}$$

which maps the distinguished arity 0 *element of the unitary operad of parenthesized braids* $* \in \mathsf{PaB}_+(0)$ *to this object* $e \in \mathtt{Ob}\,\mathsf{Q}(0)$.

PROOF. We subdivide the proof of this theorem in several steps. We use operadic notation in this proof rather than the algebraic notation which we adopt in the statement of our theorem.

Step 1: The decomposition of morphisms in the parenthesized braid operad. We first prove that any given morphism $\beta \in \mathtt{Mor}_{\mathsf{PaB}(r)}(p, q)$ has a decomposition of the form specified in assertion (a). We suggest the reader to follow our argument lines on the example depicted in Figure 6.7.

We have $\mathtt{Mor}_{\mathsf{PaB}(r)}(p, q) = \mathtt{Mor}_{\mathsf{CoB}(r)}(\omega(p), \omega(q))) \subset B_r$ by definition of the groupoids of parenthesized braids. We immediately obtain, therefore, that our morphism $\beta \in \mathtt{Mor}_{\mathsf{PaB}(r)}(p, q)$ admits a decomposition

$$\beta = \beta_1 \cdot \ldots \cdot \beta_n,$$

where each factor $\beta_i \in \mathtt{Mor}_{\mathsf{PaB}(r)}(p_i, q_i)$ consists, after forgetting about parenthesizations, of a single generating element τ_k of the braid group B_r.

If $p_i = p_i(x_{s(1)}, \ldots, x_{s(r)})$ has $s = (s(1), \ldots, s(k), s(k+1), \ldots, s(r))$ as underlying permutation, then q_i has an underlying permutation of the form $st_k = (s(1), \ldots, s(k+1), s(k), \ldots, s(r))$, with the factors $(s(k), s(k+1))$ switched. We pick a parenthesization that gathers the factors $x_{s(k)}$ and $x_{s(k+1)}$ in the word $x_{s(1)} \cdot \dots \cdot x_{s(r)}$. We therefore consider a parenthesized word of the form

$$\kappa_i = \pi_i(x_{s(1)}, \dots, \mu(x_{s(k)}, x_{s(k+1)}), \dots, x_{s(r)}) \in \Omega(r),$$

where $\pi_i \in \Omega(r-1)$. By Theorem 6.1.7(a), we have a morphism

$$\rho = \rho(x_{s(1)}, \dots, x_{s(k)}, x_{s(k+1)}, \dots, x_{s(r)}) \in \mathtt{Mor}\, \mathit{PaP}(r)$$

which goes from p_i to κ_i in the operad of parenthesized permutations, and this morphism is given by a composition of associators. We similarly have a morphism

$$\sigma = \sigma(x_{s(1)}, \dots, x_{s(k+1)}, x_{s(k)}, \dots, x_{s(r)}) \in \mathtt{Mor}\, \mathit{PaP}(r)$$

which goes from q_i to $\lambda_i = \pi_i(x_{s(1)}, \dots, \mu(x_{s(k+1)}, x_{s(k)}), \dots, x_{s(r)})$ in $\mathit{PaP}(r)$. We therefore have a decomposition of each morphism β_i of the form

$$\begin{aligned}\beta_i = \sigma(x_{s(1)}, \dots, x_{s(k+1)}, x_{s(k)}, \dots, x_{s(r)})^{-1} \\ \cdot \pi_i(x_{s(1)}, \dots, \tau(x_{s(k)}, x_{s(k+1)}), \dots, x_{s(r)}) \\ \cdot \rho(x_{s(1)}, \dots, x_{s(k)}, x_{s(k+1)}, \dots, x_{s(r)})\end{aligned}$$

where ρ and σ are defined by composites of associators, and the medium factor, which represents the operadic composite $s \cdot id_{\pi_i} \circ_k \tau$ in the morphism set of our operad, reduces to the application of a braiding isomorphism τ within a fixed parenthesized word.

This observation completes the proof of assertion (a) of the theorem.

Step 2: The construction of the category morphisms $\phi(r) : \mathit{PaB}(r) \to \mathit{Q}(r)$. The result of Theorem 6.1.7(b) implies the existence (and the uniqueness) of a morphism $\phi : \mathit{PaP} \to \mathit{Q}$ satisfying $\phi(\mu) = m$ and $\phi(\alpha) = a$ whenever we have an object $m \in \mathtt{Ob}\, \mathit{Q}(2)$ and an associativity isomorphism $a \in \mathtt{Mor}\, \mathit{Q}(3)$ that satisfies the pentagon relation of Figure 6.1 in the operad Q. The aim of our next verifications is to establish that this morphism $\phi : \mathit{PaP} \to \mathit{Q}$ admits an extension to the parenthesized braid operad PaB which we determine by the additional assignment $\phi(\tau) = c$ when we have a braiding isomorphism $c \in \mathtt{Mor}\, \mathit{Q}(2)$ such that the hexagon diagrams of Figure 6.6 commute.

The definition of the morphism $\phi : \mathit{PaP} \to \mathit{Q}$ includes the definition of a map $\phi : \Omega(r) \to \mathit{Q}(r)$ at the object set level, for each $r \in \mathbb{N}$. In this second step, we precisely aim to define a map of morphism sets $\phi : \mathtt{Mor}\, \mathit{PaB}(r) \to \mathtt{Mor}\, \mathit{Q}(r)$ corresponding to this map of object sets $\phi : \Omega(r) \to \mathit{Q}(r)$ and to complete the construction of a groupoid morphism $\phi : \mathit{PaB}(r) \to \mathit{Q}(r)$, for each $r \in \mathbb{N}$.

The image of a morphism $\beta \in \mathtt{Mor}_{\mathit{PaB}(r)}(p, q)$ under an operad morphism $\phi : \mathit{PaB} \to \mathit{Q}$ is actually uniquely determined from the decomposition obtained in Step 1 and the assignments $\alpha \mapsto a = a(x_1, x_2, x_3)$, $\tau \mapsto c = c(x_1, x_2)$, since our operad morphism is supposed to commute with all structure operations involved in this decomposition. For instance, in the case of the braid of Figure 6.5, we obtain an expression of the form

$$\begin{aligned}\phi(\beta) = m(1, a^{-1}) \cdot m(1, m(1, c)) \cdot m(1, a) \cdot a(1, m, 1) \cdot m(m(1, c), 1) \\ \cdot m(a, 1) \cdot m(m(c, 1), 1) \cdot m(a^{-1}, 1) \cdot m(m(1, c), 1) \cdot m(a, 1)\end{aligned}$$

in $\mathtt{Mor}\, \mathit{Q}(4)$ (where we do not mark input permutations for simplicity).

The main purpose of our verifications is to establish that the map

$$\phi : \mathtt{Mor}_{\mathit{PaB}(r)}(p, q) \to \mathtt{Mor}_{\mathit{Q}}(\phi(p), \phi(q)),$$

which we determine from the decomposition process of Step 1, does not depend on the choices involved in this operation.

The Mac Lane Coherence Theorem implies that $\phi(\beta)$ does not depend on the choice of the decomposition of the isomorphisms of the parenthesized permutation operad that connect the parenthesized words of our factorization. We also see that the outcome of our construction does not depend on the parenthesizations $\pi \in \Omega(r-1)$, which we choose to gather the factors of the braiding isomorphisms in our words. Indeed, we can go from one parenthesization $\kappa_i = \kappa_i(x_1,\dots,x_{r-1})$ to another $\lambda_i = \lambda_i(x_1,\dots,x_{r-1})$ by a morphism $\rho = \rho(x_1,\dots,x_{r-1})$ defined within the parenthesized permutation operad (and hence formed by a composite of associators). The middle square of the commutative diagram

$$\begin{tikzcd}
& p_i(x_{s(1)},\dots,x_{s(r)}) \arrow[dl] \arrow[dr] & \\
\kappa_i(x_{s(1)},\dots,\mu(x_{s(k)},x_{s(k+1)}),\dots,x_{s(r)}) \arrow[rr, dotted, "\rho\circ_k\mu"] \arrow[d, "\kappa_i\circ_k\tau"'] & & \lambda_i(x_{s(1)},\dots,\mu(x_{s(k)},x_{s(k+1)}),\dots,x_{s(r)}) \arrow[d, "\lambda_i\circ_k\tau"] \\
\kappa_i(x_{s(1)},\dots,\mu(x_{s(k+1)},x_{s(k)}),\dots,x_{s(r)}) \arrow[rr, dotted, "\rho\circ_k\mu"] \arrow[dr] & & \lambda_i(x_{s(1)},\dots,\mu(x_{s(k+1)},x_{s(k)}),\dots,x_{s(r)}) \arrow[dl] \\
& q_i(x_{s(1)},\dots,x_{s(r)}) &
\end{tikzcd}$$

is carried to a commutative square by our morphism ϕ, for any choice of the morphism $c = \phi(\tau)$, because the composition products of an operad in the category of categories $\circ_k : Q(m)\times Q(n) \to Q(m+n-1)$ is a morphism of categories (and, hence, defines a bifunctor). The external triangles are carried to commutative triangles too (by the Mac Lane Coherence Theorem), and we conclude that both paths from $p_i = p_i(x_{s(1)},\dots,x_{s(r)})$ to $q_i = q_i(x_{s(1)},\dots,x_{s(r)})$ in the above diagram yield the same morphism in Q.

We still have to establish that the morphism $\phi(\beta)$ does not depend on the choice of the decomposition $\beta = \beta_1\cdot\ldots\cdot\beta_n$ which we form from the image of our morphism β in the braid group B_r after forgetting about parenthesizations. We are left to check that the application of the generating relations of braids does not change the result of our construction.

In the case of the commutation relation $\tau_k\tau_l = \tau_l\tau_k$, we can assume that we have chosen a parenthesization of the form

$$\lambda_i = \pi_i(x_{s(1)},\dots,\mu(x_{s(k)},x_{s(k+1)}),\dots,\mu(x_{s(l)},x_{s(l+1)}),\dots,x_{s(r)})$$

when we determine the image of morphisms β_i and β_{i+1} associated to the factors of this braid relation. The identity of the result associated to the decompositions

$$\beta = \beta_1\cdot\ldots\cdot\beta_i\cdot\beta_{i+1}\cdot\ldots\cdot\beta_n = \beta_1\cdot\ldots\cdot\beta_{i+1}\cdot\beta_i\cdot\ldots\cdot\beta_n$$

follows, in that case, from the associativity of the composition product of operads.

In the case of the braiding relation $\tau_k\tau_{k+1}\tau_k = \tau_{k+1}\tau_k\tau_{k+1}$, we assume that we have chosen a parenthesization of the form

$$\lambda_i = \pi_i(x_{s(1)},\dots,\mu(\mu(x_{s(k)},x_{s(k+1)}),x_{s(k+2)}),\dots,x_{s(r)})$$

$$\begin{tikzcd}
(x_1\Box x_2)\Box x_3 \arrow[r, "a(x_1{,}x_2{,}x_3)"] \arrow[d, "c(x_1{,}x_2)\Box x_3"'] & x_1\Box(x_2\Box x_3) \arrow[r, "x_1\Box c(x_2{,}x_3)"] \arrow[ddd, dotted, "c(x_1{,}x_2\Box x_3)"] & x_1\Box(x_3\Box x_2) \arrow[r, "a(x_1{,}x_3{,}x_2)^{-1}"] \arrow[ddd, dotted, "c(x_1{,}x_3\Box x_2)"] & (x_1\Box x_3)\Box x_2 \arrow[d, "c(x_1{,}x_3)\Box x_2"] \\
(x_2\Box x_1)\Box x_3 \arrow[d, "a(x_2{,}x_1{,}x_3)"'] & & & (x_3\Box x_1)\Box x_2 \arrow[d, "a(x_3{,}x_1{,}x_2)"] \\
x_2\Box(x_1\Box x_3) \arrow[d, "x_2\Box c(x_1{,}x_3)"'] & & & x_3\Box(x_1\Box x_2) \arrow[d, "x_3\Box c(x_1{,}x_2)"] \\
x_2\Box(x_3\Box x_1) \arrow[r, "a(x_2{,}x_3{,}x_1)^{-1}"] & (x_2\Box x_3)\Box x_1 \arrow[r, "c(x_2{,}x_3)\Box x_1"] & (x_3\Box x_2)\Box x_1 \arrow[r, "a(x_3{,}x_2{,}x_1)"] & x_3\Box(x_2\Box x_1)
\end{tikzcd}$$

FIGURE 6.8. The dodecagon relation. We adopt the same conventions as in the picture of the pentagon and hexagon relations in this diagram. In short, we again use the operator notation $m(x_1, x_2) = x_1 \Box x_2$ for an object of an operad $m = m(x_1, x_2) \in \mathtt{Ob}\, \mathcal{Q}(2)$, we use expressions of the form $(x_1\Box x_2)\Box x_3, \dots \in \mathtt{Ob}\, \mathcal{Q}(4)$ to represent operadic composites of this object, and we use the algebraic formulas $c(x_1, x_2) \Box x_3, \dots \in \mathtt{Mor}\, \mathcal{Q}(3)$ to represent operadic composites of morphisms in our operad $\mathcal{Q}$ (see Figure 6.1 and Figure 6.6 for further explanations on the correspondence between these expressions and standard operadic notation).

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when we determine the image of morphisms associated to the factors of this braid relation. The identity of our morphisms in $\mathcal{Q}$ reduces in that case to the commutation of the dodecagon diagram of Figure 6.8, which we establish next (in Lemma 6.2.5).

Step 3: The preservation of operadic composition structures. In the previous step, we checked that we have a coherent definition of the morphisms of small categories $\phi : \mathcal{PaB}(r) \to \mathcal{Q}(r)$ which extend an operad morphism $\phi : \mathcal{PaP} \to \mathcal{Q}$ on the parenthesized permutation operad $\mathcal{PaP}$. The purpose of this third step is to check that these morphisms $\phi : \mathcal{PaB}(r) \to \mathcal{Q}(r)$ preserve the structure operations of our operads, and hence, define a morphism in the category of operads. The equivariance and the preservation of operadic units are immediate, and the preservation of the operadic composition products of objects follows from the definition of our morphisms as an extension of the components of an operad morphism on the operad of parenthesized permutations. We are therefore left to check the preservation of the operadic composition products for the morphisms of the parenthesized braid operad.

The decomposition of morphisms, which we have used to determine our maps on morphism sets $\phi : \mathtt{Mor}\, \mathcal{PaB}(r) \to \mathtt{Mor}\, \mathcal{Q}(r)$ in Step 2, can be applied to reduce the verification of the relations $\phi(\beta \circ_k \gamma) = \phi(\beta) \circ_k \phi(\gamma)$ to generating cases. The preservation of operadic composites with associators is also included in the definition of our morphisms as an extension of the components of an operad morphism on the parenthesized permutation operad. We therefore reduce our verifications to the case where β (respectively, γ) is an identity morphism in PaB and γ (respectively, β) is given by the application of a braiding τ within a parenthesized word.

The verification of the relation $\phi(\beta \circ_k \gamma) = \phi(\beta) \circ_k \phi(\gamma)$ is immediate when β is the identity and the braiding occurs in the second factor γ. Thus we focus on the case where the braiding occurs in the first factor β.

We assume $\beta = \kappa(x_1, \dots, \tau(x_l, x_{l+1}), \dots, x_m)$ and $\gamma = id_\lambda$, for some $\kappa \in \Omega(m-1)$, $l \in \{1, \dots, m-1\}$, and $\lambda \in \Omega(n)$. We can still use the decomposition of the word λ within the magma operad to reduce our verification to the case where $\lambda = \mu$ and $n = 2$. If $\gamma = id_\mu$ is plugged in an input $k \neq l, l+1$ of the word $\beta = \kappa(x_1, \dots, \tau(x_l, x_{l+1}), \dots, x_m)$, then our relation follows from the associativity of the composition products in Q. If $\gamma = id_\mu$ is plugged in an input $k = l, l+1$ of the braiding $\tau = \tau(x_l, x_{l+1})$ within the composite $\beta = \kappa(x_1, \dots, \tau(x_l, x_{l+1}), \dots, x_m)$, then we see that the decomposition of the morphism $\tau(x_l, x_{l+1}) \circ_k id_\mu$, involved in the construction of our map $\phi : \beta \circ_k \gamma$, is equivalent to the application of the hexagon relations of Figure 6.6 within the parenthesized braid operad, and the commutation of these diagrams in Q implies the preservation of this operadic composition operation.

This verification completes the proof of assertion (b) of the theorem.

Step 4: The definition of the unitary extension of our morphism. To address the proof of assertion (c), we just observe that the relations of this assertion, which read $m \circ_1 e = m \circ_2 e = 1$, $a \circ_1 id_e = a \circ_2 id_e = a \circ_3 id_e = id_m$, and $c \circ_1 id_e = c \circ_1 id_e = id_1$, imply that the assignment $\phi : * \mapsto e$ gives a coherent extension of our morphism $\phi : \mathsf{PaB} \to \mathsf{Q}$ when we consider the image of the object $\mu \in \mathtt{Ob}\,\mathsf{PaB}(2)$, and of the morphisms $\alpha \in \mathtt{Mor}_{\mathsf{PaB}(3)}(\mu(\mu(x_1, x_2), x_3), \mu(x_1, \mu(x_2, x_3)))$, $\tau \in \mathtt{Mor}_{\mathsf{PaB}(2)}(\mu(x_1, x_2), \mu(x_2, x_1))$ under the restriction operators $\partial_k = - \circ_k *$ in PaB. From this verification, we readily deduce that ϕ carries any restriction operator in PaB to the corresponding composite with the object e in the operad Q, and our conclusion follows. $\square$

The next lemma, which we use in the proof of Theorem 6.2.4, is a standard statement of the theory of braided monoidal categories (see [99]):

Lemma 6.2.5. *If the morphisms $a(x_1, x_2, x_3)$ and $c(x_1, x_2)$ of Theorem 6.2.4 make the hexagon diagrams of Figure 6.6 commute, then the dodecagon of Figure 6.8, tiled with two hexagons and one square, commutes as well.*

We suggest the reader to make these relations explicit for the associativity isomorphism α and for the braiding τ of the parenthesized braid operad PaB.

Proof. The left hand side and right hand side hexagons in the dodecagon tiling of the lemma are identified with the hexagons of Figure 6.6 (with a factor $a^{\pm 1}$ inverted) and therefore, these hexagons commute. The medium square commutes as well. Indeed, for the morphism $c = c(x_1, x_2)$, going from $m = m(x_1, x_2)$ to $tm = m(x_2, x_1)$, where we set $t = (1\ 2)$, the functoriality of the composition product $\circ_2 : \mathsf{Q}(2) \times \mathsf{Q}(2) \to \mathsf{Q}(3)$ gives $c \circ_2 (tm) \cdot m \circ_2 c = c \circ_2 c = (tm) \circ_2 c \cdot c \circ_2 m$, which is the identity asserted by the commutation of that square. $\square$

To sum up, the result of Theorem 6.2.4 gives an equivalence between operad morphisms $\phi : \mathsf{PaB} \to \mathsf{Q}$ and triples (m, a, c) consisting of an operation $m(x_1, x_2) = x_1 \square x_2 \in \mathtt{Ob}\,\mathsf{Q}(2)$, an isomorphism $a = a(x_1, x_2, x_3) \in \mathtt{Mor}\,\mathsf{Q}(3)$ which makes this operation $m(x_1, x_2) = x_1 \square x_2$ associative in the operad Q, and an isomorphism $c = c(x_1, x_2) \in \mathtt{Mor}\,\mathsf{Q}(2)$ which makes m braided commutative in the sense that we have $c(x_1, x_2) : x_1 \square x_2 \xrightarrow{\simeq} x_2 \square x_1$, but we do not necessarily get the identity of the object $m(x_1, x_2) = x_1 \square x_2$ when we go back to $x_1 \square x_2$ by applying this commutation operator $c = c(x_1, x_2)$ twice. In both the expression of the associativity and braiding relations, we assume the verification of coherence constraints, which can be reduced to the commutativity of the pentagon diagram of Figure 6.1 in the case of the associativity relation and of the hexagon diagrams of Figure 6.6 in the case of the braiding relation. In the unitary case, we consider an additional object $e \in \mathsf{Q}(0)$ which satisfies strict unit relations $e \square x_1 = x_1 = x_1 \square e$ with respect to $m(x_1, x_2) = x_1 \square x_2$, and natural coherence constraints with respect to the associativity isomorphism and to the braiding isomorphism.

For the sake of comparison, in the case of the colored braid operad, we obtain the following statement:

THEOREM 6.2.6.

(a) *Giving a morphism $\phi : \mathsf{CoB} \to \mathsf{Q}$ from the colored braid operad CoB towards an operad in the category of categories Q amounts to giving an object*

$$m(x_1, x_2) = x_1 \square x_2 \in \mathtt{Ob}\,\mathsf{Q}(2)$$

and an isomorphism

$$c(x_1, x_2) \in \mathtt{Mor}_{\mathsf{Q}(2)}(x_1 \square x_2, x_2 \square x_1)$$

such that the strict associativity relation

$$(x_1 \square x_2) \square x_3 = x_1 \square (x_2 \square x_3)$$

holds in the operad Q and the hexagons of Figure 6.6*, where we take $a = id$, commute.*

(b) *In the construction of (*a*), if we moreover assume the existence of an object*

$$e \in \mathtt{Ob}\,\mathsf{Q}(0) \quad \text{such that} \quad e \square x_1 = x_1 = x_1 \square e,$$
$$\text{and} \quad c(e, x_1) = id = c(x_1, e),$$

then the morphism $\phi : \mathsf{CoB} \to \mathsf{Q}$ has a unitary extension $\phi : \mathsf{CoB}_+ \to \mathsf{Q}$ which maps the distinguished arity 0 *element of the unitary operad of colored braids $* \in \mathsf{CoB}_+(0)$ to this object $e \in \mathtt{Ob}\,\mathsf{Q}(0)$.*

PROOF. This result follows from the same argument lines as Theorem 6.2.4. We just forget about the associativity isomorphisms in our verifications. □

6.2.7. *The operadic representation of braided monoidal structures on categories.* We can extend the observations of §6.1.9 to get an interpretation of the action of the operads $\mathsf{P} = \mathsf{CoB}_+, \mathsf{PaB}_+$ on a category $\mathcal{C}$. We again use that such an action is encoded by a morphism $\phi : \mathsf{P} \to \mathtt{End}_{\mathcal{C}}$ which maps the objects of our operad $p \in \mathtt{Ob}\,\mathsf{P}(r)$ to multi-functors $f : \mathcal{C}^{\times r} \to \mathcal{C}$ and the morphisms of the operad to natural transformations.

In both cases, we can take the image of the object $\mu = \mu(x_1, x_2)$ under our morphism ϕ to get the tensor product operation $m(X_1, X_2) = X_1 \otimes X_2 : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ of a monoidal structure on $\mathcal{C}$ (as in §6.1.9). The image of the unitary element of the operad gives a natural transformation $e : pt \to \mathcal{C}$ equivalent to a unit object $\mathbb{1} \in \mathcal{C}$ for this tensor product in $\mathcal{C}$, and we can take the image of the braiding τ to get a natural isomorphism $c(X_1, X_2) : X_1 \otimes X_2 \xrightarrow{\simeq} X_2 \otimes X_1$ that makes the tensor product of our category braided commutative. In the case of the parenthesized braid operad, we take the image of the associativity isomorphism $\alpha \in \mathtt{Mor}\, \mathsf{PaB}(3)$ to get a natural isomorphism $a(X_1, X_2, X_3) : (X_1 \otimes X_2) \otimes X_3 \xrightarrow{\simeq} X_1 \otimes (X_2 \otimes X_3)$ that makes the tensor product associative. In the case of the colored braid operad, we assume that the tensor product satisfies the associativity relation in the strict sense $(X_1 \otimes X_2) \otimes X_3 = X_1 \otimes (X_2 \otimes X_3)$ and we take $a(X_1, X_2, X_3) = id$ as associativity isomorphism. Hence, we obtain that giving a morphism $\phi : \mathsf{PaB}_+ \to \mathtt{End}_{\mathcal{C}}$ is equivalent to giving a braided monoidal structure on $\mathcal{C}$, in the sense of [130], with strict unit relations but general associativity isomorphisms, while giving a morphism $\phi : \mathsf{CoB}_+ \to \mathtt{End}_{\mathcal{C}}$ is equivalent to giving a braided monoidal structure with both strict unit and strict associativity relations. The pentagon diagram of Figure 6.1 and the hexagon diagrams of Figure 6.6 are equivalent to the usual coherence axioms of braided monoidal categories (see [130]), and we have a similar correspondence for the coherence constraints associated to the unit object.

The constructions of this chapter can readily be adapted to get operads that govern symmetric monoidal category structures with strict or general associativity isomorphisms. We survey this case in the next section.

6.2.8. *Free braided monoidal categories.* The free algebra construction of §1.3.4 can be applied in the category of small categories to associate a free strict (respectively, general) braided monoidal category $\mathbb{S}(\mathsf{P}, \mathcal{X})$ to any small category $\mathcal{X} \in \mathcal{C}at$, when we take $\mathsf{P} = \mathsf{CoB}_+$ (respectively, $\mathsf{P} = \mathsf{PaB}_+$).

We focus on the case of a one-point set $\mathcal{X} = pt$. For the operad $\mathsf{P} = \mathsf{CoB}_+$, we obtain $\mathtt{Ob}\, \mathbb{S}(\mathsf{CoB}_+, pt) = \mathbb{N}$ and $\mathbb{S}(\mathsf{CoB}_+, pt) = \coprod_{r \in \mathbb{N}} (\mathsf{CoB}_+(r) \times pt^{\times r})_{\Sigma_r} = \coprod_{r \in \mathbb{N}} \mathsf{CoB}_+(r)_{\Sigma_r}$ is identified with the disjoint union of the braid groups B_r, regarded as categories with a single object. The tensor product $\otimes : \mathbb{S}(\mathsf{CoB}_+, pt) \times \mathbb{S}(\mathsf{CoB}_+, pt) \to \mathbb{S}(\mathsf{CoB}_+, pt)$ is given by the addition of non-negative integers at the object level, and by the direct sum of braids at the morphism level. We actually retrieve, with this operadic approach, the Joyal-Street construction of the free braided monoidal category (see [99]).

For the operad $\mathsf{P} = \mathsf{PaB}_+$, we have $\mathtt{Ob}\, \mathbb{S}(\mathsf{PaB}_+, pt) = \Omega(x)_+$, where we use the notation $\Omega(x)_+$ for a free magma on one variable x. The category $\mathbb{S}(\mathsf{PaB}_+, pt)$ admits a decomposition $\mathbb{S}(\mathsf{PaB}_+, pt) = \coprod_{r \in \mathbb{N}} (\mathsf{PaB}_+(r) \times pt^{\times r})_{\Sigma_r} = \coprod_{r \in \mathbb{N}} \mathsf{PaB}_+(r)_{\Sigma_r}$, whose rth summand $\mathsf{PaB}_+(r)_{\Sigma_r}$ is identified with the full subcategory generated by monomials of weight r in the object set defined by the free magma $\Omega(x)_+$. For any pair of such monomials $p, q \in \Omega(x)_+$, we moreover have $\mathtt{Mor}_{\mathbb{S}(\mathsf{PaB}_+, pt)}(p, q) = B_r$. The tensor product associated to this category $\otimes : \mathbb{S}(\mathsf{PaB}_+, pt) \times \mathbb{S}(\mathsf{PaB}_+, pt) \to \mathbb{S}(\mathsf{PaB}_+, pt)$ is given by the substitution operation $p(x, \dots, x) \otimes q(x, \dots, x) = \mu(p(x, \dots, x), q(x, \dots, x))$ at the object level and by the direct sum of braids at the morphism level again.

Bar-Natan's parenthesized braid categories (see [16]) are identified with Hopf groupoids $\Bbbk[\mathsf{PaB}_+(r)_{\Sigma_r}]$ associated to these summands of the free braided monoidal category (we formalize the definition of the notion of a Hopf groupoid in §9).

6.3. The parenthesized symmetry operad

We briefly mention in §6.2.7 that we can adapt the constructions of the previous section to get operads encoding symmetric monoidal category structures with strict or general associativity isomorphisms. We survey this construction in this section.

Recall that the notion of a symmetric monoidal category differs from the structure of a braided monoidal category by the requirement that the braiding isomorphism $c(X_1, X_2) : X_1 \otimes X_2 \xrightarrow{\simeq} X_2 \otimes X_1$, which we call the *symmetry isomorphism* in this context, satisfies the involution relation $c(X_1, X_2)c(X_2, X_1) = id$ for every pair of objects in our category $X_1, X_2 \in \mathcal{C}$.

We consider the operad in sets $\mathsf{\Gamma}$ such that $\mathsf{\Gamma}(r) = pt$, for all $r > 0$ (the operad of commutative monoids). We start with the operads in discrete groupoids P which has the components of this operad $\mathsf{\Gamma}$ as object sets. We explicitly have $\mathtt{Ob}\,\mathsf{P}(r) = \mathsf{\Gamma}(r) = pt$ and $\mathtt{Mor}_{\mathsf{P}(r)}(pt, pt) = pt$, for all $r > 0$. We then set $\mathsf{CoS} := \alpha^*\,\mathsf{P}$ where we consider the pullback of this operad P along the standard morphism $\alpha : \mathsf{\Pi} \to \mathsf{\Gamma}$ from the permutation operad $\mathsf{\Pi}$ to $\mathsf{\Gamma}$. We accordingly have $\mathtt{Ob}\,\mathsf{CoS}(r) = \mathsf{\Pi}(r) = \Sigma_r$ by construction and the morphism sets of the groupoids $\mathsf{CoS}(r)$ which form this operad CoS are given by $\mathtt{Mor}_{\mathsf{CoS}(r)}(u, v) = pt$, for all $u, v \in \Sigma_r$, $r > 0$. The structure operations of this operad are inherited from the permutation operad at the object set level and are given by trivial one-point set maps at the morphism set level. We are precisely going to see that this operad CoS, which we call the *colored symmetry operad*, is associated to the category of symmetric monoidal categories whose tensor product is associative in the strict sense.

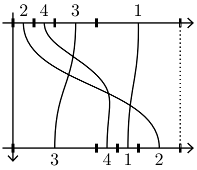
We then set $\mathsf{PaS} := \omega^*\,\mathsf{CoS}$, where we consider the pullback of the colored symmetry operad CoS along the standard morphism $\omega : \mathsf{\Omega} \to \mathsf{\Pi}$ from the magma operad $\mathsf{\Omega}$ to the permutation operad $\mathsf{\Pi}$, as in the construction of the parenthesized braid operad of the previous section. We accordingly have $\mathtt{Ob}\,\mathsf{PaS}(r) = \mathsf{\Omega}(r)$, and the morphism sets of the groupoids $\mathsf{PaS}(r)$ which form this operad PaS are given by $\mathtt{Mor}_{\mathsf{PaS}(r)}(p, q) = pt$, for all $p, q \in \mathsf{\Omega}(r)$, $r > 0$. The structure operations of this operad are inherited from the magma operad at the object set level and are again given by trivial one-point set maps at the morphism set level.

We call this operad PaS the *parenthesized symmetry operad*. We explain an analogue of the statement of Theorem 6.2.4 for this operad PaS.

We have an obvious unitary extension of both the colored symmetry operad and the parenthesized symmetry operad. We consider these unitary operads CoS_+ and PaS_+ in order to model the existence of strict units in symmetric monoidal categories.

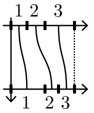
6.3.1. *The representation of morphisms in the parenthesized symmetry operad.* We can actually use a variant of the parenthesized braid diagram picture to represent the morphisms of the parenthesized symmetry operad PaS. We still represent the source and target objects of our morphisms $p, q \in \mathsf{\Omega}(r)$ by diadic decompositions of the intervals, which reflect the underlying parenthesization of these operations in the magma operad, together with a labeling of contact points, on the center of the components our diadic decompositions, which reflects the positions of the variables $(x_1, \dots, x_r)$ in our parenthesized words $p = p(x_1, \dots, x_r)$ and $q = q(x_1, \dots, x_r)$.

Recall that we have $\mathtt{Mor}_{\mathsf{PaS}(r)}(p, q) = pt$, for all $p, q \in \mathsf{\Omega}(r)$, by construction of the operad PaS. We represent our morphism by an r tuple of strands which connect the contact points with the same label in the source and target object of our morphisms. We simply forget about the relative positions of the strands which we consider in our representation of braid diagrams. We give an example of application of this representation in the following picture:

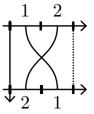


We can also use an obvious variant of the insertion of braids to represent the operadic composition of operations in *PaS*. We have an obvious operad morphism $q : \mathit{PaB} \to \mathit{PaS}$, which we form by forgetting the relative positions of the strands in our representation of the morphisms of the parenthesized braid operad as braid diagrams.

The operad of parenthesized symmetries inherits an associativity isomorphisms, which is given by the same picture as the associativity isomorphism of the parenthesized braid operad:

(1) $\alpha =$  $\in \mathtt{Mor}_{\mathit{PaS}(3)}((x_1 x_2)x_3, x_1(x_2 x_3)),$

and a symmetry homomorphism:

(2) $\tau =$  $\in \mathtt{Mor}_{\mathit{PaS}(3)}(x_1 x_2, x_2 x_1),$

which represents the underlying transposition of the braiding isomorphism of the parenthesized braid operad. We see that this symmetry isomorphism satisfies the involution relation

$$\tau(x_1, x_2)\tau(x_2, x_1) = id \tag{3}$$

(unlike the braiding isomorphism of the parenthesized braid operad), where $(1\ 2)\tau = \tau(x_2, x_1)$ represent the image of the operation $\tau = \tau(x_1, x_2)$ under the action of the transposition $(1\ 2) \in \Sigma_2$ on $\mathit{PaS}(2)$.

We immediately check that these isomorphisms satisfy the same pentagon and hexagon relations, expressed by the commutativity of the diagrams of Figure 6.1 and Figure 6.6, as the associativity isomorphism and the braiding of the parenthesized braid operad (see §6.2.3). We may simply observe that the hexagon diagrams of Figure 6.6 become equivalent to each other when our symmetry isomorphism $c = c(x_1, x_2)$ satisfies the involution relation $c(x_1, x_2)c(x_2, x_1) = id$.

We have the same obvious identities as in the case of the parenthesized braid operad when we pass to the unitary extension of our operad PaS_+ and we consider the composition operations with the element of arity zero $* \in \mathit{PaS}_+(0)$. We explicitly have the formulas $\alpha(*, x_1, x_2) = \alpha(x_1, *, x_2) = \alpha(x_1, x_2, *) = id_{\mu(x_1,x_2)}$ and $\tau(*, x_1) = \tau(x_1, *) = id_{x_1}$ We equivalently get $\partial_1\alpha = \partial_2\alpha = \partial_3\alpha = id_\mu$ and $\partial_1\tau = \partial_2\tau = id_1$ when we use restriction operators to represent these composition operations $\partial_k = - \circ_k *$.

We can now formulate the analogue of the statement of Theorem 6.2.4 for the parenthesized symmetry operad:

Theorem 6.3.2.

(a) *Let* Q *be any operad in the category of categories. Let*

$$m = m(x_1, x_2) \in \mathtt{Ob}\,\mathsf{Q}(2)$$

be an object in the component of arity two of this operad. In what follows, we also use the notation

$$m(x_1, x_2) = x_1 \square x_2$$

for this object and we again use classical algebraic notation (rather than operadic notation) to represent the composites of this object in our operad Q*. Let*

$$a(x_1, x_2, x_3) \in \mathtt{Mor}_{\mathsf{Q}(3)}((x_1 \square x_2) \square x_3, x_1 \square (x_2 \square x_3))$$

be an isomorphism which connects the operadic composites

$$\begin{aligned} &(m \circ_1 m)(x_1, x_2, x_3) = (x_1 \square x_2) \square x_3 \in \mathtt{Ob}\,\mathsf{Q}(3) \\ \textit{and} \quad &(m \circ_2 m)(x_1, x_2, x_3) = x_1 \square (x_2 \square x_3) \in \mathtt{Ob}\,\mathsf{Q}(3) \end{aligned}$$

in the category $\mathsf{Q}(3)$*. Let*

$$c = c(x_1, x_2) \in \mathtt{Mor}_{\mathsf{Q}(2)}(x_1 \square x_2, x_2 \square x_1)$$

be a symmetry isomorphism which connects the object $m(x_1, x_2) = x_1 \square x_2 \in \mathtt{Ob}\,\mathsf{Q}(2)$ *to its transposite* $(1\ 2)m(x_1, x_2) = x_2 \square x_1 \in \mathtt{Ob}\,\mathsf{Q}(2)$ *in the category* $\mathsf{Q}(2)$ *and which satisfies the involution relation* $c(x_1, x_2)c(x_2, x_1) = id$*, where* $(1\ 2)c = c(x_2, x_1)$ *represents the image of this isomorphism under the action of the transposition* $(1\ 2) \in \Sigma_2$ *on* $\mathsf{Q}(2)$*.*

If these isomorphisms $a = a(x_1, x_2, x_3)$ *and* $c = c(x_1, x_2)$ *make the pentagon diagram of Figure* 6.1 *commute, as well as (any one of) the hexagon diagrams of Figure* 6.6*, then we have a morphism of operads in groupoids*

$$\phi : \mathsf{PaS} \to \mathsf{Q}$$

uniquely determined by the assignments $\phi(\mu) = x_1 \square x_2$*,* $\phi(\alpha) = a(x_1, x_2, x_3)$ *and* $\phi(\tau) = c(x_1, x_2)$ *in the operad* Q*.*

(b) *In the construction of (*a*), if we moreover assume the existence of an object*

$$e \in \mathtt{Ob}\,\mathsf{Q}(0)$$

which satisfies the relation

$$e \square x_1 = x_1 = x_1 \square e$$

at the object set level (where we again use algebraic notation to denote the composites with this object in our operad), together with the identities

$$\begin{aligned} &a(e, x_1, x_2) = a(x_1, e, x_2) = a(x_1, x_2, e) = id_{m(x_1, x_2)}, \\ &c(e, x_1) = c(x_1, e) = id_{x_1} \end{aligned}$$

at the morphism set level (with the same notation conventions as above), then the morphism $\phi : \mathsf{PaB} \to \mathsf{Q}$ *has a unitary extension*

$$\phi : \mathsf{PaS}_+ \to \mathsf{Q}$$

which maps the distinguished arity 0 *element of the unitary operad of parenthesized symmetries* $* \in \mathsf{PaS}_+(0)$ *to this object* $e \in \mathtt{Ob}\,\mathsf{Q}(0)$*.*

Proof. The proof of this statement follows from the same argument lines as the result of Theorem 6.2.4 and we leave this verification as an exercise. ☐

In the case of the colored symmetry operad, we obtain the following statement:

THEOREM 6.3.3.

(a) *Giving a morphism* $\phi : \mathsf{CoS} \to \mathsf{Q}$ *from the colored symmetry operad* CoS *towards an operad in the category of categories* Q *amounts to giving an object*

$$m(x_1, x_2) = x_1 \square x_2 \in \mathtt{Ob}\,\mathsf{Q}(2)$$

which satisfies the strict associativity relation

$$(x_1 \square x_2) \square x_3 = x_1 \square (x_2 \square x_3)$$

in this operad Q, *together with a symmetry isomorphism*

$$c(x_1, x_2) \in \mathtt{Mor}_{\mathsf{Q}(2)}(x_1 \square x_2, x_2 \square x_1)$$

which satisfies the involution relation $c(x_1, x_2)c(x_2, x_1) = id$ *and makes commute (any one of) the hexagons of Figure* 6.6 *for the choice* $a = id$ *of associativity isomorphism.*

(b) *In the construction of (*a*), if we moreover assume the existence of an object*

$$e \in \mathtt{Ob}\,\mathsf{Q}(0) \quad \textit{such that} \quad e \square x_1 = x_1 = x_1 \square e, \qquad \textit{and} \quad c(e, x_1) = id = c(x_1, e),$$

then the morphism $\phi : \mathsf{CoS} \to \mathsf{Q}$ *has a unitary extension* $\phi : \mathsf{CoS}_+ \to \mathsf{Q}$ *which maps the distinguished arity* 0 *element of the unitary operad of colored symmetries* $* \in \mathsf{CoS}_+(0)$ *to this object* $e \in \mathtt{Ob}\,\mathsf{Q}(0)$.

PROOF. This statement parallels the result of Theorem 6.2.6, and follows again from a straightforward adaptation of the arguments of Theorem 6.2.4 (where we forget about the associativity isomorphism). □

We use the same ideas as in §6.2.7 to deduce from these statements that giving an action of the (unitary) operad of parenthesized symmetries PaS_+ on a category $\mathcal{C}$ amounts to providing this category $\mathcal{C}$ with a symmetric monoidal structure, with a strict unit but general associativity isomorphisms, while giving an action of the (unitary) operad of colored symmetries CoS_+ on $\mathcal{C}$ amounts to providing $\mathcal{C}$ with a symmetric monoidal structure where the tensor product satisfies both strict unit and strict associativity relations.

We observed in §6.3.1 that we have an obvious morphism from the operad of parenthesized braids PaB to the operad of parenthesized symmetries PaS. We actually have a whole commutative diagram of operad morphisms

$$\begin{tikzcd}
\mathsf{PaP}_+ \arrow[r] \arrow[d] & \mathsf{PaB}_+ \arrow[r] \arrow[d] & \mathsf{PaS}_+ \arrow[d] \\
\mathsf{CoP}_+ \arrow[r] & \mathsf{CoB}_+ \arrow[r] & \mathsf{CoS}_+
\end{tikzcd}$$

which summarizes the connections between the operads considered in this chapter and which obviously mirrors the diagram of functors between the categories of monoidal categories encoded by our operads.

Part I(c)

Hopf Algebras and the Malcev Completion

CHAPTER 7

Hopf Algebras

The purpose of this chapter is to review the definition of the notion of a *Hopf algebra* and to explain classical structure results on Hopf algebras.

Briefly recall for the moment that a Hopf algebra is an object which is equipped with both a counitary coalgebra structure, a unitary algebra structure, and with an operation, called the *antipode*, which is a generalization of the classical inversion operation for groups. Hopf algebras equipped with a commutative algebra structure naturally occur in the framework of algebraic geometry, as function rings of affine group schemes (see for instance [33, 55, 180]). Hopf algebras equipped with a cocommutative coalgebra structure notably occur in algebraic topology, as the homology of connected H-spaces (see for instance [182, §III.8] for an introduction to this subject), and as the natural structure of the Steenrod algebra (see [168]). Hopf algebras equipped with a cocommutative coalgebra structure also occur in representation theory, as the dual objects of the commutative Hopf algebras considered in the study of affine group schemes, and as the enveloping algebras of Lie algebras (we review the definition the enveloping algebra of a Lie algebra in the second section of this chapter). In the next chapter, we also use complete Hopf algebras in order to extend the rationalization of abelian groups to general groups. We refer to this construction as the *Malcev completion.* Further fields of applications of commutative and cocommutative Hopf algebra structures include algebraic combinatorics (see the monographs [4, 5]), the Grothendieck-Galois theory (see for instance [172, §6] for a nice introduction to this subject and [32] for a comprehensive account), and the Connes-Kreimer approach of the renormalization theory in mathematical physics (see [49]).

The notion of a Hopf algebra makes also sense without assuming any commutativity property, for both the coalgebraic and the algebraic part of the structure. Significant examples of Hopf algebras which are neither cocommutative nor commutative occur in the theory of quantum groups (see for instance [56] for an authoritative overview of this subject). In our applications however, we only deal with Hopf algebras which are cocommutative as coalgebras. Therefore, when we deal with a Hopf algebra, we generally assume that the coalgebra structure is cocommutative and we do not recall this convention, unless the precision is required by the context.

In the first section of the chapter (§7.1), we recall the precise definition of a Hopf algebra and we give a reminder of basic examples of Hopf algebras. To be more specific, we check that the free $\Bbbk$-module $\Bbbk[G]$ associated to a group G inherits a Hopf algebra structure.

The second section (§7.2) is devoted to the connection between Lie algebras and Hopf algebras: we recall the definition of the enveloping algebra of a Lie algebra and the statement of the theorems of Poincaré-Birkhoff-Witt and Milnor-Moore (the classical structure theorems of the theory of Hopf algebras). The main outcome of these theorems is that the enveloping algebra functor defines an equivalence of categories between the category of Lie algebras and a subcategory of the category of (cocommutative) Hopf algebras formed by objects which satisfy a certain conilpotence condition.

In the third section (§7.3), we study Hopf algebras in the category of complete filtered modules. We use these complete Hopf algebras in the next chapter (§8) in order to define the Malcev completion of groups.

7.1. The notion of a Hopf algebra

This first section is introductory. Our purpose is to recall the general definition of a Hopf algebra and the definition of the Hopf algebra structure on a group algebra $\Bbbk[G]$.

In short, the notion of a Hopf algebra is defined by replacing sets, underlying the usual group structures, by coalgebras, and by using tensor structures instead of cartesian structures in the definition of the unit, product, and inversion operations. In the case of a group algebra $\Bbbk[G]$, we consider the natural coalgebra structure of §3.0.6, with the coproduct defined by the diagonal $\Delta([g]) = [g] \otimes [g]$ on the elements of G, and the counit such that $\epsilon([g]) = 1$, for any $g \in G$. The Hopf structure of $\Bbbk[G]$ is yielded by the structure operations attached to our group G. In the sequel, we generally assume that the underlying coalgebra of a Hopf algebra is cocommutative, and we therefore take this convention in our definition.

The definition of a Hopf algebra makes sense in the general setting of symmetric monoidal categories. Throughout this section, we generally start with abstract definitions, which we formulate in this categorical framework, and we make explicit the applications of our concepts in the context of a category of modules over a ground ring $\Bbbk$. The purpose of this abstract approach is to give a conceptual introduction to the main ideas of the theory and to prepare the ground for applications of Hopf algebras in other contexts than the standard categories of modules over a ring.

In certain cases, we use pointwise formulas, directly transported from a module context, to specify morphisms in abstract categories. The idea is to interpret such formulas in terms of operations on abstract variables, so that our formulas actually represent combinations of morphisms which we produce by applying structure operations of the ambient category. For instance, we may use the expression $c(x \otimes y) = y \otimes x$ to refer to a symmetry isomorphism $c : M \otimes N \xrightarrow{\simeq} N \otimes M$.

To start with, we review the definition of the notion of a counitary (cocommutative) coalgebra, which we introduced in §3.0.4 in the context of symmetric monoidal categories. In a second step, we examine the definition of a bialgebra, which are monoid objects (algebras in the sense of §3.0) in the symmetric monoidal category of coalgebras. We explain the definition of a Hopf algebra afterwards.

7.1.1. *Counitary cocommutative coalgebras.* Briefly recall that a counitary cocommutative coalgebra (in a symmetric monoidal category) consists of an object C equipped with a counit $\epsilon : C \to \mathbb{1}$ (also referred to as the augmentation) and with a coproduct $\Delta : C \to C \otimes C$ which satisfies natural counit, coassociativity and cocommutativity relations (see §3.0.4).

In the module context, the augmentation $\epsilon : C \to \Bbbk$ assigns a scalar $\epsilon(x) \in \Bbbk$ to each element $x \in C$, and we represent the expansion of the coproduct of any such element $\Delta(x) \in C^{\otimes 2}$ by an expression of the form $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$, where $x_{(1)}, x_{(2)} \in C$ denote the factors of this tensor in $C^{\otimes 2}$.

The coassociativity relation implies that all $n-1$-fold iterations of the coproduct of our coalgebra C define the same morphism $\Delta^{(n)} : C \to C^{\otimes n}$, and we can naturally extend our notation of the coproduct to represent the expansion of the n-fold tensor $\Delta^{(n)}(x) \in C^{\otimes n}$ arising from any such iterated application of coproducts. Explicitly, we write $\Delta^{(n)}(x) = \sum_{(x)} x_{(1)} \otimes \cdots \otimes x_{(n)}$, for any $x \in C$. In this formalism the coassociativity relation reads:

$$\underbrace{\sum_{(x)} x_{(1)} \otimes x_{(2)} \otimes x_{(3)}}_{\Delta^{(3)}(x)} = \underbrace{\sum_{(x)} \Delta(x_{(1)}) \otimes x_{(2)}}_{\Delta \otimes id \cdot \Delta(x)} = \underbrace{\sum_{(x)} x_{(1)} \otimes \Delta(x_{(2)})}_{id \otimes \Delta \cdot \Delta(x)} .$$

The counit relations read:

$$\sum_{(x)} \epsilon(x_{(2)}) \cdot x_{(1)} = \sum_{(x)} \epsilon(x_{(1)}) \cdot x_{(2)} = x,$$

and the cocommutativity relation reads:

$$\sum_{(x)} x_{(2)} \otimes x_{(1)} = \sum_{(x)} x_{(1)} \otimes x_{(2)}.$$

The coassociativity implies that these relations have an obvious extension to multiple coproducts. In particular, we obtain from the cocommutativity relation that an n-fold coproduct $\Delta^{(n)}(x) \in C^{\otimes n}$ is invariant under the action of any permutation $s \in \Sigma_n$ on $C^{\otimes n}$.

Recall that we use the notation $\mathcal{C}om^c_+$ for the category formed by the counitary cocommutative coalgebras with the structure preserving morphisms of the base category as morphisms.

7.1.2. *Tensor product of counitary cocommutative coalgebras.* In §3.0.4, we observe that the tensor product of augmented cocommutative coalgebras inherits a counitary cocommutative coalgebra structure so that augmented cocommutative coalgebras form a symmetric monoidal category with unit, associativity and symmetry isomorphisms inherited from the base category.

In the module context, the definition of the augmentation on a tensor product of coalgebras $C, D \in \mathcal{C}om^c_+$ reads

$$\epsilon(x \otimes y) = \epsilon(x) \cdot \epsilon(y),$$

and the definition of the coproduct reads

$$\Delta(x \otimes y) = \sum_{(x),(y)} (x_{(1)} \otimes y_{(1)}) \otimes (x_{(2)} \otimes y_{(2)}),$$

for any $x \in C$, $y \in D$, and where we adopt the convention of §7.1.1 for the notation of the coproduct of x (respectively, y) in C (respectively, D). The ground ring $\Bbbk$, which represents the unit object of our module category $\mathcal{M}od$, is equipped with the augmented cocommutative coalgebra structure such that $\epsilon(1) = 1$ and $\Delta(1) = 1 \otimes 1$.

7.1.3. *Unitary associative algebras.* In §3.0, we focused on the study of commutative structures. However, we mentioned that most of our constructions can be handled for associative (non-commutative) algebras.

To get the definition of a unitary associative algebra, we just drop the commutativity requirement from our definition. Thus, a *unitary associative algebra* in a (symmetric) monoidal category $\mathcal{M}$ consists of an object $A \in \mathcal{M}$ equipped with a unit morphism $\eta : \mathbb{1} \to A$ and with a product $\mu : A \otimes A \to A$ which satisfies natural unit and associativity relations. In general, we express these relations by the commutativity of the following diagrams:

$$\begin{tikzcd}
A \otimes \mathbb{1} \arrow[r, "id \otimes \eta"] \arrow[dr, "\simeq"'] & A \otimes A \arrow[d, "\mu"] & \mathbb{1} \otimes A \arrow[l, "\eta \otimes id"'] \arrow[dl, "\simeq"] \\
& A &
\end{tikzcd}, \qquad
\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "id \otimes \mu"] \arrow[d, "\mu \otimes id"'] & A \otimes A \arrow[d, "\mu"] \\
A \otimes A \arrow[r, "\mu"] & A
\end{tikzcd}.$$

In the module context, we use the standard algebraic notation $1 = \eta(1) \in A$ and $a_1 a_2 = \mu(a_1 \otimes a_2)$, for the unit element and the product of an associativity algebra A. If necessary, then we just use a subscript in order to specify the associative algebra A which corresponds to a given unit morphism $\eta = \eta_A$ (respectively, to a given product morphism $\mu = \mu_A$).

The morphisms of unitary associative algebras consist, as in the commutative case, of the morphisms of the base category which preserve the unit and the product operation of our objects. We use the notation $\mathcal{A}s_+ = \mathcal{M}\mathcal{A}s_+$ for the category of unitary associative algebras in $\mathcal{M}$. We just forget the base category from this notation when this information is not necessary (as in the commutative algebra case again).

7.1.4. *Tensor product of unitary associative algebras.* The tensor product $A \otimes B$ of unitary associative algebras $A, B \in \mathcal{A}s_+$ inherits a unitary associative algebra structure whose definition is the same as in the unitary commutative algebra case:

- the unit morphism of the tensor product $A \otimes B$ is given by the composite $\mathbb{1} \xrightarrow{\simeq} \mathbb{1} \otimes \mathbb{1} \xrightarrow{\eta_A \otimes \eta_B} A \otimes B$, where we use the unit isomorphism of the base category before applying the unit morphisms of our algebras A and B;
- and the product morphism is given by the composite $A \otimes B \otimes A \otimes B \xrightarrow{(2\ 3)^*} A \otimes A \otimes B \otimes B \xrightarrow{\mu_A \otimes \mu_B} A \otimes B$, where we perform a tensor permutation $(2\ 3)^*$ (by using the symmetry isomorphism of the base category) before applying the product morphisms of our algebras A and B.

The unit object of the base symmetric monoidal category $\mathbb{1}$ inherits a canonical unitary associative algebra structure and represents a unit object for the tensor product of unitary associative algebras. The tensor product of unitary associative algebras inherits unit, associativity and symmetry isomorphisms from the base category too. The category of unitary associative algebras therefore inherits a full symmetric monoidal structure from the base category.

In the module context, the unit element of the tensor product $A \otimes B$ is given by the tensor product $1_{A \otimes B} = 1_A \otimes 1_B$ of the unit elements of our algebras $1_A \in A$ and $1_B \in B$, and the definition of the product reads $(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = (a_1 a_2) \otimes (b_1 b_2)$, for any $a_1 \otimes b_1, a_2 \otimes b_2 \in A \otimes B$.

This symmetric monoidal structure is similar to the symmetric monoidal structure of the category of unitary commutative algebras, such as defined in §3.0.3. However, we may observe that the tensor product of unitary associative algebras is not identified with the coproduct of the category (in contrast with the tensor product of unitary commutative algebras). To be specific, we have a commutation relation $(a \otimes 1) \cdot (1 \otimes b) = a \otimes b = (1_A \otimes b) \cdot (a \otimes 1_B)$, between the image of the elements of A and B in the tensor product $A \otimes B$, while such commutativity relations do not occur in coproducts when we work in the category of unitary associative algebras.

7.1.5. *Bialgebras.* We formally define a *(cocommutative) bialgebra* as a unitary associative algebra (in the sense of §7.1.3) in the symmetric monoidal category of counitary (cocommutative) coalgebras. Accordingly, a cocommutative bialgebra in a symmetric monoidal category $\mathcal{M}$ consists of an object $H \in \mathcal{M}$ equipped with:

(1) a counitary cocommutative coalgebra structure, determined by a counit $\epsilon : H \to \mathbb{1}$ and a coproduct $\Delta : H \to H \otimes H$ which satisfy the counit, coassociativity, and cocommutativity relations of §3.0.4,

(2) together with a unit morphism $\eta : \mathbb{1} \to H$ and a product morphism $\mu : H \otimes H \to H$, both formed in the category of counitary cocommutative coalgebras, and which satisfy the unit and associativity relations of §7.1.3 in that category.

Under our conventions for the notation of algebra categories, the category of bialgebras which we define in this paragraph is denoted by the expression $\mathcal{C}om^c_+\,\mathcal{A}s_+ = \mathcal{M}\,\mathcal{C}om^c_+\,\mathcal{A}s_+$. We drop the base category from this notation when we do not need to recall this information (as usual).

7.1.6. *The distribution relations underlying a bialgebra structure.* Since the category of counitary cocommutative coalgebras inherits its symmetric monoidal structure from the base category, the unit and product morphisms, which define the unitary associative algebra structure of our bialgebras, can be formed in the base category, and the requirement that these morphisms are morphisms of counitary cocommutative coalgebras is equivalent to the commutativity of the following diagrams:

(1)
$$\begin{tikzcd}
\mathbb{1} \arrow[r, "\eta"] \arrow[dr, "="'] & H \arrow[d, "\epsilon"] \\
& \mathbb{1}
\end{tikzcd}, \qquad
\begin{tikzcd}
\mathbb{1} \arrow[r, "\eta"] \arrow[d, "\simeq"'] & H \arrow[d, "\Delta"] \\
\mathbb{1} \otimes \mathbb{1} \arrow[r, "\eta\otimes\eta"] & H \otimes H
\end{tikzcd}$$

(as regards the unit morphism) and:

(2)
$$\begin{tikzcd}
H \otimes H \arrow[r, "\mu"] \arrow[d, "\epsilon\otimes\epsilon"'] & H \arrow[d, "\epsilon"] \\
\mathbb{1} \otimes \mathbb{1} \arrow[r, "\simeq"] & \mathbb{1}
\end{tikzcd}, \qquad
\begin{tikzcd}
H \otimes H \arrow[r, "\mu"] \arrow[d, "\Delta\otimes\Delta"'] & H \arrow[dd, "\Delta"] \\
H \otimes H \otimes H \otimes H \arrow[d, "(2\ 3)^*"'] & \\
H \otimes H \otimes H \otimes H \arrow[r, "\mu\otimes\mu"] & H \otimes H
\end{tikzcd}$$

(as regards the product). In the module context, the commutation of these diagrams are equivalent to identities:

$$\epsilon(1) = 1, \quad \Delta(1) = 1 \otimes 1,$$
$$\text{and} \quad \epsilon(a \cdot b) = \epsilon(a) \cdot \epsilon(b), \quad \Delta(a \cdot b) = \sum_{(a),(b)} a_{(1)} \cdot b_{(1)} \otimes a_{(2)} \cdot b_{(2)},$$

for any $a, b \in H$.

We just unravel the definition of the counitary cocommutative coalgebra structure on the unit object $\mathbb{1}$ ($\mathbb{1} = \Bbbk$ in the module context) and on the tensor product $H \otimes H$ to get these identities. We readily see that the distribution relation between the product and the coproduct is equivalent to the identity $\Delta(a \cdot b) = \Delta(a) \cdot \Delta(b)$ by definition of the product of tensors in unitary associative algebras. We more generally see that assuming the commutativity of the diagrams (1-2) amounts to requiring that the counit $\epsilon : H \to \mathbb{1}$ and the coproduct $\Delta : H \to H \otimes H$ define morphisms of unitary associative algebras, where we use the underlying unitary associative algebra structure of our bialgebra H, of the unit object $\mathbb{1}$ and of the tensor product $H \otimes H$. We therefore obtain that:

Proposition 7.1.7. *The structure of a bialgebra, initially defined as the structure of a unitary associative algebra in the category of counitary cocommutative coalgebras in* §7.1.5*, can equivalently be defined as the structure of a counitary cocommutative coalgebra in the category of unitary associative algebras (where we use the observations of* §7.1.4 *to provide the category of unitary associative algebras with a symmetric monoidal structure).* ☐

This proposition asserts that we have an identity of categories $\mathcal{C}om^c_+ \mathcal{A}s_+ = \mathcal{A}s_+ \mathcal{C}om^c_+$ when we apply our conventions for the notation of categories.

7.1.8. *Hopf algebras.* Recall that we generally use the phrase 'Hopf object' to refer to any class of structured object in the symmetric monoidal category of counitary cocommutative coalgebras (see §3). But we prefer to use the name '*bialgebra*' for the category unitary associative algebras in counitary cocommutative coalgebras and we reserve the name '*Hopf algebra*' for objects of this category which satisfy certain extra properties (in order to agree with the conventions of the literature). We explicitly define a *Hopf algebra H* as a bialgebra in the sense of §7.1.5 equipped with:

(1) morphisms $\sigma, \tau : H \to H$, both formed in the base category, and which fit in a commutative diagram

$$\begin{tikzcd}
H \arrow[r, "\epsilon"] \arrow[d, "\Delta"'] & \mathbb{1} \arrow[r, "\eta"] & H \\
H \otimes H \arrow[rr, dotted, shift left, "\sigma\otimes id"] \arrow[rr, dotted, shift right, "id \otimes \tau"'] & & H \otimes H \arrow[u, "\mu"']
\end{tikzcd} ,$$

where we consider the unit, the counit, the product and the coproduct operations of our bialgebra H.

This definition makes sense in any symmetric monoidal category, but we examine the particular case of Hopf algebras in modules with more details first. (We tackle more elaborate examples in the next sections.) In the module context, our relations, expressed by the commutativity of the diagram of (1), are equivalent to the equations

$$\sum_{(a)} \sigma(a_{(1)}) \cdot a_{(2)} \overset{(1)}{=} \epsilon(a) \cdot 1 \overset{(2)}{=} \sum_{(a)} a_{(1)} \cdot \tau(a_{(2)}),$$

which hold for any $a \in H$ and take values in H. In general, a morphism σ which fits in a relation of the form (1) is called a *left antipode*, and a morphism τ which fits in the symmetric relation (2) is called a *right antipode.*

We denote the category of Hopf algebras by the expression $\mathcal{H}opf\mathcal{A}lg$.

To complete the definition of a Hopf algebra, we check that:

Proposition 7.1.9. *In general, if we assume that a bialgebra H is equipped with a left antipode $\sigma : H \to H$, then we have at most one right antipode on H* *which is also necessarily equal to the left antipode as a morphism from H to H. If we symmetrically assume that a bialgebra H is equipped with a right antipode $\tau : H \to H$, then we have at most one left antipode on H which is also necessarily equal to the right antipode.*

Hence, in our definition of a Hopf algebra §7.1.8, *the left and the right antipodes are necessarily equal $\sigma = \tau$, and are also unique. Furthermore any morphism of bialgebras $\phi : G \to H$, where G and H are Hopf algebras, automatically preserves antipodes.*

In fact, we do not need the cocommutativity of the coalgebra structure in the proof of this statement, as well as in the proof of the next general statements on Hopf algebras up to Proposition 7.1.12.

Proof. The result of this proposition holds in any symmetric monoidal category, but we prefer to give a proof in the module setting in order to illustrate our coproduct notation. The reformulation of our arguments in a general setting is the matter of a straightforward transcription.

Let H be any bialgebra. Recall that we use the notation $\Delta^{(3)}$ for the 3-fold coproduct of any coalgebra (see §7.1.1). We adopt a similar notation $\mu^{(3)}$ for the 3-fold product of our bialgebra H. The proof of the first claims of the proposition reduce to the proof of an identity $\sigma(a) = \tau(a)$, for all $a \in H$, and for any given left and right antipodes $\sigma, \tau : H \to H$. To establish this relation, we perform different reductions of the expression

$$\mu^{(3)} \cdot (\sigma \otimes id \otimes \tau) \cdot \Delta^{(3)}(a) = \sum_{(a)} \sigma(a_{(1)}) \cdot a_{(2)} \cdot \tau(a_{(3)}),$$

leading to $\sigma(a)$ in one case, and to $\tau(a)$ in the other case. We just use that both the left and right antipode relations can be applied within our 3-fold coproduct $\Delta^{(3)}(a) = \Delta \otimes id \cdot \Delta(a) = id \otimes \Delta \cdot \Delta(a)$. We explicitly have:

$$\begin{aligned}\sum_{(a)} \sigma(a_{(1)}) \cdot a_{(2)} \cdot \tau(a_{(3)}) &= \sum_{(a)} \epsilon(a_{(1)}) 1 \cdot \tau(a_{(2)}) = \sum_{(a)} \tau(\epsilon(a_{(1)}) a_{(2)}) = \tau(a),\\ &= \sum_{(a)} \sigma(a_{(1)}) \cdot \epsilon(a_{(2)}) 1 = \sum_{(a)} \sigma(\epsilon(a_{(2)}) a_{(1)}) = \sigma(a),\end{aligned}$$

and these identities prove our claim $\sigma(a) = \tau(a)$.

The relation $\phi\sigma(a) = \tau\phi(a) = \sigma\phi(a)$, for a morphism $\phi : G \to H$, follows from the same argument line, by considering different reductions of the expression $(\phi\sigma \otimes \phi \otimes \tau\phi) \cdot \Delta^{(3)}(a) = \sum_{(a)} \phi\sigma(a_{(1)}) \cdot \phi(a_{(2)}) \cdot \tau\phi(a_{(3)})$. ☐

Thus, in what follows, we use the name '*antipode*' (without extra precision) to refer to the single morphism $\sigma = \tau$ which defines both the left and the right antipode of a Hopf algebra. The result of Proposition 7.1.9 also motivates us to regard Hopf algebras as bialgebras endowed with special properties (as alluded to before we introduce our definition) rather than bialgebras equipped with extra structures. In categorical terms, we regard the category of Hopf algebras $\mathcal{H}opf\mathcal{A}lg$ as a full subcategory of the category of bialgebras.

This interpretation diminishes the inconsistence between the definition of a Hopf algebra and our convention to use the name 'Hopf' as a qualifier for any category of structured object in the symmetric monoidal category of counitary cocommutative coalgebras. In fact, the terminology 'Hopf algebra' was originally used in algebraic topology for bialgebra structures, without any reference to antipodes, but in situations where antipodes automatically exist.

By elaborating on the arguments of Proposition 7.1.9, we also obtain that:

Proposition 7.1.10. *In a Hopf algebra H, the antipode $\sigma : H \to H$ defines:*

(1) *a morphism of counitary cocommutative coalgebras from H to H;*

(2) *and a morphism of unitary associative algebras from H to H^{op}, where we use the notation H^{op} for the unitary associative algebra obtained by changing the product of H into the transposite operation $\mu^{op}(x_1, x_2) = \mu(x_2, x_1)$ (we also say that σ defines an* anti-morphism of unitary associative algebras *from H to H).*

We check this proposition in the module context again. We reduce the proof of our proposition to the verification of the following statement:

Lemma 7.1.11.

(a) *The antipode $\sigma(a)$ of any element $a \in H$ in a Hopf algebra H satisfies the identities*

$$\epsilon(\sigma(a)) = \epsilon(a) \quad \textit{and} \quad \Delta\sigma(a) = \sum_{(a)} \sigma(a_{(2)}) \otimes \sigma(a_{(1)}),$$

with respect to the counit $\epsilon : H \to \Bbbk$ and the coproduct $\Delta : H \to H \otimes H$ of the counitary cocommutative coalgebra structure of our object H.

(b) *The antipode also preserves the unit element of our Hopf algebra $1 \in H$ in the sense that we have the identity $\sigma(1) = 1$, and we moreover have the formula*

$$\sigma(ab) = \sigma(b) \cdot \sigma(a),$$

for any product of elements $a, b \in H$.

Proof. We establish the product relation first. We use the identity $\sigma = \tau$ between the left and the right antipode of a Hopf algebra (see Proposition 7.1.9).

Let $a, b \in H$. We readily see (by using the same arguments as in the proof of Proposition 7.1.9) that the expression $\sum_{(a),(b)} \sigma(a_{(1)}b_{(1)}) \cdot a_{(2)}b_{(2)} \cdot \tau(b_{(3)}) \cdot \tau(a_{(3)})$ can either be reduced to $\sigma(ab)$ (if we start by applying the relations of the right antipode τ) or to $\tau(b) \cdot \tau(a)$ (if we use the relation for the left antipode σ first and the distribution relation between the product and the coproduct of our Hopf algebra H). We therefore obtain $\sigma(ab) = \tau(b) \cdot \tau(a) = \sigma(b) \cdot \sigma(a)$.

We establish the coproduct relation $\Delta\sigma(a) = \sum_{(a)} \sigma(a_{(2)}) \otimes \sigma(a_{(1)})$ by similar arguments, by considering different reductions of the expression $\sum_{(a)} \Delta\sigma(a_{(1)}) \cdot \Delta(a_{(2)}) \cdot (1\ 2)^*(\tau \otimes \tau)\Delta(a_{(3)})$ (we get $\Delta\sigma(a)$ in one case and $\sum_{(a)} \tau(a_{(2)}) \otimes \tau(a_{(1)})$ in the other case). We establish the augmentation relation $\epsilon\sigma(a) = \epsilon(a)$ by considering different reductions of the expression $\sum_{(a)} \epsilon\sigma(a_{(1)}) \cdot \epsilon(a_{(2)})$. The unit relation $\sigma(1) = 1$ is a direct consequence of the left antipode relation $\sigma(1) \cdot 1 = \epsilon(1) \cdot 1$ and of the identity $\epsilon(1) = 1$. ☐

For the sake of completeness, we also check that:

Proposition 7.1.12. *The antipode of a Hopf algebra satisfies the involution relation $\sigma^2 = id$ as soon as we assume that the coproduct of our Hopf algebra is cocommutative.*

Proof. We check this assertion in the module context again. The transcription of our arguments in a general categorical setting reduces to a straightforward exercise. We start with the expression $\sum_{(a)} \sigma(\sigma(a_{(1)})) \cdot \sigma(a_{(2)}) \cdot a_{(3)}$. By performing the left antipode relation on the second and third factors of this tensor product, we obtain that this expression reduces to $\sigma(\sigma(a))$. On the other hand, the already established coproduct identity $\Delta\sigma = \sigma \otimes \sigma \cdot \Delta$ (where we use the cocommutativity of the coproduct to drop the transposition) implies that we can also apply the left antipode relation on the first and second factors of our tensor product. If we perform this reduction first, then we retrieve the simple expression of our element $a \in A$. Hence, we have the relation $\sigma(\sigma(a)) = a$, for any $a \in A$, which is the claim of the proposition. ☐

7.1.13. *Monoid and group algebras.* Recall that we use the notation $\Bbbk[X]$ for the free $\Bbbk$-module associated to a set X, and the notation $[x]$ for the basis element of this $\Bbbk$-module $\Bbbk[X]$ associated to any element of our set $x \in X$. In §3.0.4, we observed that $\Bbbk[X]$ inherits a canonical counitary cocommutative coalgebra structure such that $\epsilon([x]) = 1$ and $\Delta([x]) = [x] \otimes [x]$, for any $x \in X$.

In the case of an associative monoid $X = M$, we readily see that $\Bbbk[M]$ inherits an additional unitary associative algebra structure, with a unit $1_{\Bbbk[M]} = [1]$ yielded by the unit of M, and a product induced by the product of M, so that we have $[a] \cdot [b] = [a \cdot b]$, for any $a, b \in M$. Furthermore, we easily check that these operations fulfill the relations of §7.1.6 so that our unit $\eta : \Bbbk \to \Bbbk[M]$ and product morphisms $\mu : \Bbbk[M] \otimes \Bbbk[M] \to \Bbbk[M]$ are morphisms of counitary cocommutative coalgebras. Hence, the free $\Bbbk$-module $\Bbbk[M]$ associated to a monoid M forms a bialgebra in the sense of the definition of §7.1.5.

In the case of a group $X = G$, we can check further that the mapping $\sigma : \Bbbk[G] \to \Bbbk[G]$ such that $\sigma([g]) = [g^{-1}]$, for any $g \in G$, satisfies the equation of a left and right antipode on $\Bbbk[G]$. Hence, the free $\Bbbk$-module $\Bbbk[G]$ associated to a group G (the *group algebra* of G) forms an instance of a Hopf algebra, in the sense of §7.1.8.

7.1.14. *Group like elements.* Recall that the subset of group-like elements of a counitary cocommutative coalgebra C, denoted by $\mathbb{G}(C)$, is defined by:

$$\mathbb{G}(C) = \{c \in C | \epsilon(c) = 1, \Delta(c) = c \otimes c\}.$$

In §3.0.6, we observed that the map $\mathbb{G} : C \mapsto \mathbb{G}(C)$ defines a right adjoint of the free $\Bbbk$-module map $\Bbbk[-] : X \mapsto \Bbbk[X]$, regarded as a functor $\Bbbk[-] : \mathcal{S}et \to \mathcal{C}om_{+}^{c}$ from the category of sets $\mathcal{S}et$ to the category of counitary cocommutative coalgebras in $\Bbbk$-modules $\mathcal{C}om_{+}^{c} = \mathcal{M}od\,\mathcal{C}om_{+}^{c}$. The unit of this adjunction is the morphism $\iota : X \to \mathbb{G}\,\Bbbk[X]$ yielded by the identity between the elements of X and the basis elements of the $\Bbbk$-module $\Bbbk[X]$ (which are group-like by definition of our counitary cocommutative coalgebra structure on $C = \Bbbk[X]$). The adjunction augmentation is the morphism $\rho : \Bbbk[\mathbb{G}(C)] \to C$ defined by the extension to the free $\Bbbk$-module $\Bbbk[\mathbb{G}(C)]$ of the set inclusion $\mathbb{G}(C) \subset C$.

In the context of groups and Hopf algebras, we obtain the following results:

PROPOSITION 7.1.15. *The set of group-like elements $\mathbb{G}(H)$ in a Hopf algebra H satisfies the following belonging relations:*

$$1 \in \mathbb{G}(H), \quad g, h \in \mathbb{G}(H) \Rightarrow gh \in \mathbb{G}(H), \quad \textit{and} \quad g \in \mathbb{G}(H) \Rightarrow \sigma(g) \in \mathbb{G}(H).$$

Furthermore, for a group-like element $g \in \mathbb{G}(H)$, the antipode relations imply:

$$g \cdot \sigma(g) = \sigma(g) \cdot g = 1.$$

The set of group-like elements $\mathbb{G}(H)$ *of a Hopf algebra* H *consequently forms a group with the multiplication* $\mu : \mathbb{G}(H) \times \mathbb{G}(H) \to \mathbb{G}(H)$ *induced by the product of our Hopf algebra.*

Proof. The axioms of bialgebras include the relations $\epsilon(1) = 1$, $\Delta(1) = 1 \otimes 1$, which are equivalent to the requirement that the unit element of H is group-like in the sense of our definition. Hence, we have $1 \in \mathbb{G}(H)$.

For a product of group-like elements $g, h \in H$, the axioms of §7.1.6 imply the relations $\epsilon(gh) = \epsilon(g) \cdot \epsilon(h) = 1 \cdot 1 = 1$ and $\Delta(gh) = \Delta(g) \cdot \Delta(h) = (g \otimes g) \cdot (h \otimes h) = (gh) \otimes (gh)$. Hence, we have $g, h \in \mathbb{G}(H) \Rightarrow gh \in \mathbb{G}(H)$.

For the antipode $\sigma(g) \in H$ of a group-like element $g \in \mathbb{G}(H)$, the identities established in Lemma 7.1.11 imply $\epsilon\sigma(g) = \epsilon(g) = 1$ and $\Delta\sigma(g) = (1\ 2)^*(\sigma \otimes \sigma)\Delta(g) = \sigma(g) \otimes \sigma(g)$. Hence, we have $g \in \mathbb{G}(H) \Rightarrow \sigma(g) \in \mathbb{G}(H)$.

The identities $g \cdot \sigma(g) = \sigma(g) \cdot g = 1$ are formal consequences of the application of the antipode relations when we assume $\epsilon(g) = 1 \Rightarrow \eta\epsilon(g) = 1$ and $\Delta(g) = g \otimes g$. This observation completes our verifications. ☐

Proposition 7.1.16. *The functor* $\mathbb{G} : \mathcal{H}opf\mathcal{A}lg \to \mathcal{G}rp$ *obtained by the construction of Proposition* 7.1.15 *is also right adjoint to the group algebra functor* $\Bbbk[-] : \mathcal{G}rp \to \mathcal{H}opf\mathcal{A}lg$*, from groups to Hopf algebras.*

Proof. We easily see that the unit and augmentation of the adjunction $\Bbbk[-] : \mathcal{S}et \rightleftarrows \mathcal{C}om_+^c : \mathbb{G}$ between the category of sets $\mathcal{S}et$ and the category of counitary cocommutative coalgebras $\mathcal{C}om_+^c$ (see our reminder on the definition of these morphisms in §7.1.14) preserve the additional unit and product structures of our objects when we deal with groups and Hopf algebras. Therefore our functors $\Bbbk[-] : \mathcal{G}rp \rightleftarrows \mathcal{H}opf\mathcal{A}lg : \mathbb{G}$ still form an adjoint pair between the category of groups $\mathcal{G}rp$ and the category of Hopf algebras $\mathcal{H}opf\mathcal{A}lg$. ☐

7.2. Lie algebras and Hopf algebras

We survey the relationship between Lie algebras and Hopf algebras in this second section. Lie algebras arose in the mathematical literature as infinitesimal versions of group structures. The tangent space of a Lie group (a manifold equipped with a group structure) is a fundamental instance of Lie algebra. The classical Lie's third theorem asserts that any finite dimensional real Lie algebra can be integrated into a Lie group, and hence occurs as such a tangent space.

The relationship between Lie algebras and Hopf algebras which we aim to review in this section is an algebraic counterpart of this correspondence. The main device for this study is the enveloping algebra functor, of which we recall the formal definition. To be explicit, we will check that the enveloping algebra functor induces an equivalence of categories between the category of Lie algebras and a subcategory of Hopf algebras which satisfy a local conilpotence condition (see §7.2.15). This assertion is a consequence of the Milnor-Moore Theorem. In this section, we also recall the statement of the Poincaré-Birkhoff-Witt Theorem and the statement of a general structure theorem for locally conilpotent Hopf algebras which we use in our proof of the Milnor-Moore Theorem.

Throughout this section, we assume that we work in an additive base symmetric monoidal category $\mathcal{M}$ whose morphism sets are uniquely divisible as abelian groups and hence form $\mathbb{Q}$-modules. To coin this situation, we say that $\mathcal{M}$ forms a $\mathbb{Q}$-*additive* *symmetric monoidal category.* In the case of a module category $\mathcal{M} = \mathcal{M}od$, this requirement is equivalent to the assumption that the ground ring $\Bbbk$ satisfies $\mathbb{Q} \subset \Bbbk$. The notion of a Lie algebra makes sense in other contexts. Notably, we will consider Lie algebras in module categories defined over more general rings later on, but we have to distinguish several variants of the notion of a Lie algebra in this context.

We need colimits in order to define the enveloping algebra of Lie algebras. We therefore also assume that colimits exist in $\mathcal{M}$ and that the tensor product of our category $\mathcal{M}$ distributes over colimits (see §0.9).

The primitive element functor in §7.2.11 is defined by a kernel in the base category. We need the existence of all kernels to define the subobjects of primitive elements of arbitrary coalgebras. We assume that this is so all through this section to simplify our account. We may observe, nonetheless, that our proof of the Poincaré-Birkhoff-Witt Theorem and of the Milnor-Moore Theorem implies that we can realize the subobject of primitive elements of a Hopf algebra in a $\mathbb{Q}$-additive symmetric monoidal category as the kernel of an idempotent morphism. Moreover, the proof of the Poincaré-Birkhoff-Witt Theorem and of the Milnor-Moore Theorem which we give in this book is valid as soon as such kernels exist in our base category $\mathcal{M}$.

The $\mathbb{Q}$-additive category requirement implies that our category $\mathcal{M}$ is canonically enriched over the category of $\mathbb{Q}$-modules $\mathcal{M}od_{\mathbb{Q}}$, with the natural morphism sets of our category as hom-objects $\mathtt{Hom}_{\mathcal{M}}(-,-) = \mathtt{Mor}_{\mathcal{M}}(-,-)$. In good cases, the existence of this $\mathbb{Q}$-additive structure implies that the category is equipped with an external tensor product $\otimes : \mathcal{M}od_{\mathbb{Q}} \times \mathcal{M} \to \mathcal{M}$ such that $\mathtt{Mor}_{\mathcal{M}}(K \otimes M, N) = \mathtt{Mor}_{\mathcal{M}od_{\mathbb{Q}}}(K, \mathtt{Hom}_{\mathcal{M}}(M, N))$, for all $K \in \mathcal{M}od_{\mathbb{Q}}$ and $M, N \in \mathcal{M}$. By adjunction, we readily see that the classical unit, associativity and symmetry relations of tensor products hold for any combination of this external tensor product $\otimes : \mathcal{M}od_{\mathbb{Q}} \times \mathcal{M} \to \mathcal{M}$ with the internal tensor product of the category of $\mathbb{Q}$-modules $\mathcal{M}od_{\mathbb{Q}}$ and of the our category $\mathcal{M}$. This external tensor product can be associated to a symmetric monoidal functor $\eta : \mathcal{M}od_{\mathbb{Q}} \to \mathcal{M}$, which maps any $\mathbb{Q}$-module $K \in \mathcal{M}od_{\mathbb{Q}}$ to the tensor product $K \otimes \mathbb{1}$ with the unit object $\mathbb{1}$ in our symmetric monoidal category $\mathcal{M}$ (see [66]). In the case where $\mathcal{M}$ is a category of modules $\mathcal{M}od = \mathcal{M}od_{\Bbbk}$ over a ground ring such that $\mathbb{Q} \subset \Bbbk$, this symmetric monoidal functor $\eta : \mathcal{M}od_{\mathbb{Q}} \to \mathcal{M}od_{\Bbbk}$ is the standard functor of extension of scalars. In §7.2.3, we use the external tensor product $\otimes : \mathcal{M}od_{\mathbb{Q}} \times \mathcal{M} \to \mathcal{M}$ to extend the operadic expansion of the free Lie algebra (see §1.3.5) to the category of Lie algebras in our symmetric monoidal category $\mathcal{M}$.

In this section, we still use the idea to specify general morphisms by pointwise formulas, as in the module context, since we can generally interpret these formulas as combinations of the structure operations of our ambient category (as we explain in the introduction of the previous section). In our survey, we formulate all definitions in the setting of a general symmetric monoidal category first, and we make explicit the applications in the module context afterwards. To begin with, we review the definition of a Lie algebra.

7.2.1. *Lie algebras.* In §1.3.1, we recall the definition of a *Lie algebra* as an instance of a category of algebras associated to an operad Lie.

In the context of a $\mathbb{Q}$-additive symmetric monoidal category $\mathcal{M}$, we define a Lie algebra as an object $\mathfrak{g} \in \mathcal{M}$ equipped with a morphism $\lambda : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, called the *Lie bracket*, which satisfies the antisymmetry relation $\lambda \cdot (id + (1\ 2)^*) = 0$ and a

3-fold tensor relation $\lambda(\lambda, 1) \cdot (id + (1\ 2\ 3)^* + (1\ 3\ 2)^*) = 0$ which corresponds to the classical *Jacobi relation*. In these formulas, we use the notation σ^* for the action of a permutation $\sigma \in \Sigma_r$ on a tensor power $\mathfrak{g}^{\otimes r}$.

In the module context, we write $\lambda(x_1, x_2) = [x_1, x_2] \in \mathfrak{g}$ for the image of elements $x_1, x_2 \in \mathfrak{g}$ under the Lie bracket $\lambda : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$. The antisymmetry relation reads $[x_1, x_2] = -[x_2, x_1]$, and the Jacobi relation reads

$$[[x_1, x_2], x_3] + [[x_2, x_3], x_1] + [[x_3, x_1], x_2] = 0,$$

for $x_1, x_2, x_3 \in \mathfrak{g}$. The Jacobi relation is equivalent to the formula $[[x, y], a] = [[x, a], y] + [x, [y, a]]$, for all $x, y, a \in \mathfrak{g}$, which asserts that the operation $\theta_a = [-, a]$ forms a derivation with respect to the Lie bracket. By antisymmetry, we have also have the relation $[x, [a, b]] = [[x, a], b] - [[x, b], a]$, for $a, b, x \in \mathfrak{g}$, which asserts that these derivation operations $\theta_a = [-, a]$ satisfy the identity $[\theta_a, \theta_b] = \theta_{[a,b]}$, where we set $[\theta_a, \theta_b] = \theta_a\theta_b - \theta_b\theta_a$, for any $a, b \in \mathfrak{g}$. (We are mainly going to use these variations of the Jacobi relation in the definition of semi-direct products of Lie algebras in §8.5.4.) We will also see that the relation $[x, [a, b]] = [[x, a], b] - [[x, b], a]$, for $a, b, x \in \mathfrak{g}$, has an interpretation in terms of representations of Lie algebras (see §7.2.9).

We denote the category of Lie algebras by $\mathcal{L}ie$. We obviously define a morphism of Lie algebras as a morphism of the base category which preserves the structure operation (the Lie bracket) of our Lie algebras. As usual, we just specify the ambient symmetric monoidal category $\mathcal{M}$ in our notation $\mathcal{L}ie = \mathcal{M}\mathcal{L}ie$ when this information is necessary.

7.2.2. *Remarks.* In the standard definition of a Lie algebra, we assume that we have the vanishing relation $[x, x] = 0$, for all $x \in \mathfrak{g}$, instead of the antisymmetry relation. In §8.2.2, where we give a short introduction to Lie algebras over the integers, we will take this vanishing relation $[x, x] = 0$ in our definition. Let us observe that $[x, x] = 0$ is equivalent to the antisymmetry relation $[x_1, x_2] = -[x_2, x_1]$ when we work in a category of modules over a ring $\Bbbk$ such that 2 is invertible (and hence, when we assume $\mathbb{Q} \subset \Bbbk$), but this is no longer the case when 2-torsion phenomena may occur in our category of modules.

Further subtleties occur in other examples of symmetric monoidal categories. In the context of graded modules, where we use the symmetric monoidal structure of §4.4, the antisymmetry relation reads $[x_1, x_2] = - \pm [x_2, x_1]$, with an extra sign that arises from the permutation of the elements $x_1, x_2 \in \mathfrak{g}$ (see §0.2, §4.4.1). In this case, we assume that we have the vanishing relation $[x, x] = 0$ for the homogeneous elements of even degree of our graded Lie algebra and the relation $[[x, x], x] = 0$ for the homogeneous elements of odd degree. These requirements follow from the (graded) antisymmetry and Jacobi relations when we work in a category of graded modules over a ring $\Bbbk$ where 2 and 3 are invertible (and hence, when we assume $\mathbb{Q} \subset \Bbbk$), but are not automatically satisfied otherwise.

7.2.3. *Relationship with the operadic definition and free Lie algebras.* We see that the definition of the structure of a Lie algebra which we give in §7.2.1 is identical to the description of the structure of an algebra over the Lie operad which we give in §1.3.1. We more precisely get, by the observations of §1.3.1, that giving an operation $\lambda : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ such that the antisymmetry and the Jacobi relations hold (as in §7.2.1) amounts to giving an operad morphism $\phi : \mathsf{Lie} \to \mathtt{End}_{\mathfrak{g}}$, where Lie is our notation for the Lie operad and $\mathtt{End}_{\mathfrak{g}}$ is the endomorphism operad associated to the object $\mathfrak{g}$.

We only consider the case of a base category of modules $\mathcal{M} = \mathcal{M}od$ in §1.3.1, but we can extend this correspondence in the setting of a $\mathbb{Q}$-additive symmetric monoidal category. We just consider the external hom-objects of our base category $\mathtt{Hom}_{\mathcal{M}}(-,-)$ in the definition of the components of the endomorphism operad $\mathtt{End}_{\mathfrak{g}}(r) = \mathtt{Hom}_{\mathcal{M}}(\mathfrak{g}^{\otimes r}, \mathfrak{g})$, for $r \in \mathbb{N}$.

The interpretation of Lie algebras in terms of algebras over operads implies that the category of Lie algebras inherits free objects, which admit an expansion of the form $\mathbb{L}(M) = \bigoplus_{r=0}^{\infty}(\mathit{Lie}(r) \otimes M^{\otimes r})_{\Sigma_r}$. We just use the external tensor product of our base category with the category of $\mathbb{Q}$-modules, where the Lie operad is defined, in order to form the tensor products $\mathit{Lie}(r) \otimes M^{\otimes r}$ which occur in this formula. We can equivalently use the symmetric monoidal functor $\eta : \mathcal{M}od_{\mathbb{Q}} \to \mathcal{M}$ equivalent to this external tensor product in order to map the Lie operad Lie in our symmetric monoidal category $\mathcal{M}$.

Recall also that the expression $(-)_{\Sigma_r}$ in this expansion refers to the application of a coinvariant functor which we use to identify the right action of permutations on the tensor powers $M^{\otimes r}$ with their left action on the components of the Lie operad $\mathit{Lie}(r)$ in our object (see §§1.3.2-1.3.5). In the sequel, we refer to the summands of this expansion $\mathbb{L}_r(M) = (\mathit{Lie}(r) \otimes M^{\otimes r})_{\Sigma_r}$ as the *components of homogeneous weight* of the free Lie algebra.

In the literature, the free Lie algebra is usually defined as a quotient of a free magma (see for instance [34, II.2.2], or [155, §0.2]). This construction parallels the definition of the Lie operad by generators and relations. (Magmas, as we observed in §6.1, are identified with structures associated to free operads.) In this approach, the free Lie algebra $\mathbb{L}(M)$ intuitively consists of Lie monomials on the elements of our generating object $x \in M$ (when we work in a concrete symmetric monoidal category), where a Lie monomial refers to a formal operadic composite of Lie brackets quotiented by the antisymmetry and Jacobi relations. The Lie bracket on $\mathbb{L}(M)$ is intuitively defined by the obvious substitution operation on Lie monomials. The homogeneous component of weight r of the free Lie algebra $\mathbb{L}_r(M)$ is linearly generated by the Lie monomials in r variables $[\cdots[[x_1,x_2],\dots],x_r]$, where $x_1,\dots,x_r \in M$. The Lie bracket preserves the weight grading in the sense that we have the relation $[\mathbb{L}_s(M), \mathbb{L}_t(M)] \subset \mathbb{L}_{s+t}(M)$, for all $s,t \geq 0$. (We go back to this observation in the next section.)

In §1.2.11, we mention that the Lie operad has an intricate symmetric structure. The structure theorems of Hopf algebras imply that the free Lie algebra functor has a more effective realization in terms of a retract of the tensor algebras, and we rather use this approach when we have to deal with free Lie algebras. We review the definition of the tensor algebra and of the symmetric algebras before tackling this subject.

7.2.4. *The tensor algebra and the symmetric algebra.* The (unitary) *tensor algebra* $\mathbb{T}(M)$ associated to an object $M \in \mathcal{M}$ in our base category $\mathcal{M}$ is explicitly defined by the sum $\mathbb{T}(M) = \bigoplus_{r=0}^{\infty} M^{\otimes r}$, where we form the tensor powers of our object $M^{\otimes r}$ by using the tensor product operation of the base category $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$. In the sequel, we refer to the summands of this expansion $\mathbb{T}_r(M) = M^{\otimes r}$ as the *components of homogeneous weight* of the tensor algebra.

The (unitary) *symmetric algebra* $\mathbb{S}(M)$ is explicitly defined by the sum $\mathbb{S}(M) = \bigoplus_{r=0}^{\infty}(M^{\otimes r})_{\Sigma_r}$, where we apply the coinvariant functor $(-)_{\Sigma_r}$ in order to make the action of permutations $\sigma \in \Sigma_r$ on the tensor powers $M^{\otimes r}$ equal to identity morphisms. In the module context, we get the relation $x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(r)} \equiv x_1 \otimes \cdots \otimes x_r$ for every $x_1 \otimes \cdots \otimes x_r \in M^{\otimes r}$, and for each $\sigma \in \Sigma_r$. The summands $\mathbb{S}_r(M) = (M^{\otimes r})_{\Sigma_r}$ define the components of homogeneous weight of the symmetric algebra.

The tensor algebra inherits a unit $\eta : \mathbb{1} \to \mathbb{T}(M)$ given by the identity between the unit object $\mathbb{1}$ and the summand $M^{\otimes 0} = \mathbb{1}$ of weight $r = 0$ of our expansion $\mathbb{T}(M) = \bigoplus_{r=0}^{\infty} M^{\otimes r}$, as well as a product $\mu : \mathbb{T}(M) \otimes \mathbb{T}(M) \to \mathbb{T}(M)$ which is defined by the concatenation operations $M^{\otimes s} \otimes M^{\otimes t} \xrightarrow{=} M^{\otimes s+t}$ termwise, so that $\mathbb{T}(M)$ forms a unitary associative algebra. The symmetric algebra inherits a similarly defined unit $\eta : \mathbb{1} \to \mathbb{S}(M)$, as well as a product $\mu : \mathbb{S}(M) \otimes \mathbb{S}(M) \to \mathbb{S}(M)$ which is given by the morphisms $(M^{\otimes s})_{\Sigma_s} \otimes (M^{\otimes t})_{\Sigma_t} \to (M^{\otimes s+t})_{\Sigma_{s+t}}$ induced by the concatenation operations of the tensor algebra termwise. This product operation becomes commutative on the symmetric algebra. The object $\mathbb{S}(M)$ forms a unitary commutative algebra therefore.

In the tensor algebra case, we have a canonical embedding $\iota : M \to \mathbb{T}(M)$, given by the identity between M and the object $M^{\otimes 1} = M$. In the symmetric algebra case, we have a similarly defined embedding $\iota : M \to \mathbb{S}(M)$, given by the identity between M and the object $(M^{\otimes 1})_{\Sigma_1} = M$.

In §1.3.5, we already briefly recalled the definition of the (non-unitary) tensor algebra and of the (non-unitary) symmetric algebra to illustrate the construction of free algebras over operads. The (unitary) tensor algebras which we consider in this paragraph represent the free objects of the category of (unitary) associative algebras in our base symmetric monoidal category, while the (unitary) symmetric algebras represent the free objects of the category of (unitary) commutative algebras. The universal properties of free objects are equivalent to the following adjunction statements:

PROPOSITION 7.2.5.

(a) *The tensor algebra functor* $\mathbb{T} : \mathcal{M} \to \mathcal{A}s_+$ *is left adjoint to the forgetful functor* $\omega : \mathcal{A}s_+ \to \mathcal{M}$ *from the category of unitary associative algebras* $\mathcal{A}s_+$ *to the base category* $\mathcal{M}$. *The embedding* $\iota : M \to \mathbb{T}(M)$ *represents the unit of this adjunction relation.*

(b) *The symmetric algebra functor* $\mathbb{S} : \mathcal{M} \to \mathcal{C}om_+$ *is left adjoint to the forgetful functor* $\omega : \mathcal{C}om_+ \to \mathcal{M}$ *from the category of unitary commutative algebras* $\mathcal{C}om_+$ *to the base category* $\mathcal{M}$. *The embedding* $\iota : M \to \mathbb{S}(M)$ *also represents the unit of this adjunction relation.*

EXPLANATIONS. In §§1.3.3-1.3.4, we explained that the assertions of this proposition have an equivalent formulation in terms of universal properties. In the case of the tensor algebra $R = \mathbb{T}(M)$ (respectively, in the case of the symmetric algebra $R = \mathbb{S}(M)$), we explicitly obtain that any morphism $f : M \to A$ with values in a unitary associative (respectively, commutative) algebra A admits a unique factorization

$$\begin{tikzcd}
M \arrow[rr, "f"] \arrow[dr, "\iota"'] & & A \\
& R \arrow[ur, dotted, "\phi_f"'] &
\end{tikzcd}$$

such that ϕ_f is a morphism of unitary associative (respectively, commutative) algebras.

The image of a tensor $x_1 \otimes \cdots \otimes x_r \in M^{\otimes r}$ in the tensor algebra $\mathbb{T}(M)$ is denoted by $x_1 \cdot \ldots \cdot x_r \in \mathbb{T}(M)$ (when we work in a concrete base symmetric monoidal category), because by identifying the object M with a summand of $\mathbb{T}(M)$, we obtain that this tensor represents the product of the elements $x_1, \ldots, x_r \in M$ in $\mathbb{T}(M)$. We adopt similar conventions for the symmetric algebra. In this case, we have the identity $x_{\sigma(1)} \cdot \ldots \cdot x_{\sigma(r)} = x_1 \cdot \ldots \cdot x_r$ which reflects the commutativity of the product in $\mathbb{S}(M)$. The product of the tensor algebra is given by the concatenation operation $(x_1 \cdot \ldots \cdot x_s) \cdot (y_1 \cdot \ldots \cdot y_t) = x_1 \cdot \ldots \cdot x_s \cdot y_1 \cdot \ldots \cdot y_t$, and we have a similar correspondence in the symmetric algebra case.

The extension of a module morphism $f : M \to A$ to the tensor (respectively, symmetric) algebra is explicitly defined by the formula $\phi_f(x_1 \cdot \ldots \cdot x_r) = f(x_1) \cdot \ldots \cdot f(x_r)$ for any monomial $x_1 \cdot \ldots \cdot x_r \in R$ where we take the product of the image of the elements $x_1, \ldots, x_r \in M$ in the algebra A. □

We use the statement of Proposition 7.2.5 to establish the following structure result:

Proposition 7.2.6. *The tensor algebra $R = \mathbb{T}(M)$ (respectively, the symmetric algebra $R = \mathbb{S}(M)$) inherits a Hopf algebra structure such that:*

- *the augmentation $\epsilon : R \to \mathbb{1}$ is the morphism of unitary associative (respectively, commutative) algebras associated to the zero morphism $\epsilon(x) \equiv 0$ from M to the unit object $\mathbb{1}$;*
- *the coproduct $\Delta : R \to R \otimes R$ is the morphism of unitary associative (respectively, commutative) algebras given by the formula $\Delta(x) = x \otimes 1 + 1 \otimes x$ on $M \subset R$;*
- *the antipode $\sigma : R \to R$ is the anti-morphism of unitary associative (respectively, commutative) algebras given by the opposite of the identity map $\sigma(x) = -x$ on $M \subset R$.*

In an abstract categorical setting, we regard our pointwise formulas as an algebraic combination of morphisms involving the structure operations of the ambient category (as we explained in the introduction of this section).

Explanations. In the module context, we can apply the formula given in the proof of Proposition 7.2.5 to determine the image of any monomial $x_1 \cdot \ldots \cdot x_r \in R$ under our structure morphisms. For the augmentation, we obtain $\epsilon(x_1 \cdot \ldots \cdot x_r) = \epsilon(x_1) \cdot \ldots \cdot \epsilon(x_r) = 0$ as soon as $r > 0$. For the coproduct, we get the expression:

$$\begin{aligned}\Delta(x_1 \cdot \ldots \cdot x_r) &= \underbrace{(x_1 \otimes 1 + 1 \otimes x_1)}_{\Delta(x_1)} \cdot \ldots \cdot \underbrace{(x_r \otimes 1 + 1 \otimes x_r)}_{\Delta(x_r)} \\ &= \sum_{\substack{\{i_1 < \cdots < i_s\} \amalg \{j_1 < \cdots < j_t\} \\ = \{1 < \cdots < r\}}} (x_{i_1} \cdot \ldots \cdot x_{i_s}) \otimes (x_{j_1} \cdot \ldots \cdot x_{j_t}).\end{aligned}$$

For the antipode, we get $\sigma(x_1 \cdot \ldots \cdot x_r) = \sigma(x_r) \cdot \ldots \cdot \sigma(x_1) = (-1)^r \cdot (x_r \cdot \ldots \cdot x_1)$.

The proof of the structure relations of Hopf algebras reduces to straightforward verifications, which are also immediate because the uniqueness claim in the definition of morphisms on tensor (respectively, symmetric) algebras enables us to reduce these verifications to the case of generating elements. □

7.2.7. *The adjunction between Lie and associative algebras.* Let A be any (unitary) associative algebra. One can readily check that the commutator $[a_1, a_2] = a_1a_2 - a_2a_1$ satisfies the antisymmetry and Jacobi relation of a Lie bracket, and hence provides A with a natural Lie algebra structure.

In §1.3.9, we interpret (a non-unitary version of) this correspondence as an instance of a restriction functor $\iota^* : A \mapsto \iota^* A$, associated to an operad morphism from the Lie operad to the (non-unitary) associative operad. This interpretation works same in the unitary context. In the case of the tensor algebra, the existence of this structure implies that, for any object $M \in \mathcal{M}$, we have a natural morphism of Lie algebras $\iota : \mathbb{L}(M) \to \mathbb{T}(M)$ which fits in a factorization

$$\begin{tikzcd}
M \arrow[rr] \arrow[dr] & & \mathbb{T}(M) \\
& \mathbb{L}(M) \arrow[ur, dotted, "\iota"'] &
\end{tikzcd}$$

of the canonical embedding $M \hookrightarrow \mathbb{T}(M)$. In the operadic approach, we have $\mathbb{L}(M) = \bigoplus_{r=0}^{\infty}(\mathsf{Lie}(r) \otimes M^{\otimes r})_{\Sigma_r}$ (see §1.3.5, §7.2.3), $\mathbb{T}(M) = \bigoplus_{r=0}^{\infty}(\mathsf{As}_+(r) \otimes M^{\otimes r})_{\Sigma_r}$ (see §1.3.5), and our free algebra morphism is the natural transformation induced by the morphisms $\iota : \mathsf{Lie}(r) \to \mathsf{As}_+(r)$ at the operad level.

Intuitively, the morphism $\iota : \mathbb{L}(M) \to \mathbb{T}(M)$ maps the Lie monomials, which represent the elements of the free Lie algebra, into commutators in the tensor algebra. From this representation, we retrieve that the morphism $\iota : \mathbb{L}(M) \to \mathbb{T}(M)$ preserves the weight grading of our free algebras and splits as a sum of homogeneous components $\iota : \mathbb{L}_r(M) \to \mathbb{T}_r(M)$.

In Proposition 1.3.8, we give a general construction of extension functors on categories of algebras associated to operads. These extension functors are left adjoint to the restriction functors associated to operad morphisms. In the case of the Lie operad and the associative operad, the application of our construction returns a functor $\iota_! : \mathcal{L}ie \to \mathcal{A}s_+$ which is left adjoint to our explicitly defined restriction functor $\iota^* : \mathcal{A}s_+ \to \mathcal{L}ie$. The image of a Lie algebra $\mathfrak{g}$ under this extension functor $\iota_! : \mathcal{L}ie \to \mathcal{A}s_+$ is usually called the *enveloping algebra* of $\mathfrak{g}$, and is denoted by $\iota_! \mathfrak{g} = \mathbb{U}(\mathfrak{g})$. The enveloping algebra of a Lie algebra is endowed with a Lie algebra morphism $\iota : \mathfrak{g} \to \mathbb{U}(\mathfrak{g})$ which represents the unit of our adjunction.

In the approach of Proposition 1.3.8, the image of a Lie algebra under the extension functor $\iota_! \mathfrak{g} = \mathbb{U}(\mathfrak{g})$, is defined by a reflexive coequalizer of free algebras of the form:

$$\begin{tikzcd}
\mathbb{T}(\mathbb{L}(\mathfrak{g})) \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & \mathbb{T}(\mathfrak{g}) \arrow[l, bend right=50, "s_0"'] \arrow[r, dotted, "\epsilon"] & \mathbb{U}(\mathfrak{g})
\end{tikzcd} .$$

The morphism $\epsilon : \mathbb{T}(\mathfrak{g}) \to \mathbb{U}(\mathfrak{g})$ which occurs in this coequalizer is identified with the morphism of associative algebras induced by the canonical morphism $\iota : \mathfrak{g} \to \mathbb{U}(\mathfrak{g})$ underlying the enveloping algebra. This coequalizer construction differs from the classical definition of the enveloping algebra. We review this classical approach in §7.2.9. We just examine some applications of our extension functor construction $\mathbb{U} = \iota_! : \mathcal{L}ie \to \mathcal{A}s_+$ when we take a free Lie algebra $\mathfrak{g} = \mathbb{L}(M)$ before. We then have $\iota_! \mathbb{L}(M) = \mathbb{U}\,\mathbb{L}(M) = \mathbb{T}(M)$ by composition of adjunction relations. Furthermore, we readily see that the previously considered morphism $\iota : \mathbb{L}(M) \to \mathbb{T}(M)$, which we defined by using the definition of free Lie algebras, is identified with the canonical

Lie algebra morphism $\iota : \mathbb{L}(M) \to \mathbb{U}\,\mathbb{L}(M)$ attached to the enveloping algebra. We have the following observation:

PROPOSITION 7.2.8. *The morphism* $\iota : \mathbb{L}(M) \to \mathbb{T}(M)$ *admits a retraction* $\rho : \mathbb{T}(M) \to \mathbb{L}(M)$, *which is formed in the base category* $\mathcal{M}$, *and which is defined by the following formula:*

$$\rho(x_1 \cdot \ldots \cdot x_r) = \frac{1}{r} \cdot [\cdots [[x_1, x_2], x_3], \ldots, x_r],$$

for any monomial $x_1 \cdot \ldots \cdot x_r \in \mathbb{T}_r(M)$, *and for each* $r > 0$.

PROOF. We borrow our argument from [151, §B.2, Lemma 2.2]. We establish the proposition within a base category of modules for simplicity. We use that the mapping $\Delta : \mathbb{L}(M) \to \mathbb{L}(M)$ such that $\Delta(p) = rp$ for any homogeneous monomial $p \in \mathbb{L}_r(M)$ defines a *derivation of the free Lie algebra.* We explicitly have the derivation relation $\Delta([p, q]) = [\Delta(p), q] + [p, \Delta(q)]$ for all $p, q \in \mathbb{L}(M)$. We then equip the sum $\Bbbk \oplus \mathbb{L}(M)$ with the Lie bracket such that $[(\lambda, p), (\mu, q)] = (0, \lambda\Delta(q) - \mu\Delta(p) + [p, q])$, for any $(\lambda, p), (\mu, q) \in \Bbbk \oplus \mathbb{L}(M)$. We consider the morphism of associative algebras $ad : \mathbb{U}\,\mathbb{L}(M) \to \mathtt{End}(\Bbbk \oplus \mathbb{L}(M))^{op}$ which is defined by the mapping $ad(q) : (\lambda, p) \mapsto [(\lambda, p), (0, q)]$ when $q \in \mathbb{L}(M)$.

For a monomial of the tensor algebra $x_1 \cdot \ldots \cdot x_r \in \mathbb{T}(M)$, which we identify with a product of generating elements $x_1, \ldots, x_r \in M$ in the enveloping algebra $\mathbb{T}(M) = \mathbb{U}\,\mathbb{L}(M)$, we get the formula:

$$ad(x_1 \cdot \ldots \cdot x_r)(1, 0) = ad(x_r) \cdot \ldots \cdot ad(x_1)(1, 0) = (0, [\cdots [[x_1, x_2], x_3], \ldots, x_r]).$$

We therefore have $ad(u)(1, 0) = (0, r\rho(u))$, for any homogeneous element of weight r of the tensor algebra $u \in \mathbb{T}_r(M)$. For a homogeneous Lie polynomial $p \in \mathbb{L}_r(M)$, we obtain on the other hand:

$$ad(p)(1, 0) = [(1, 0), (0, p)] = (0, \Delta(p)) = (0, rp).$$

We therefore have the identity $rp = r\rho(p) \Rightarrow p = \rho(p)$ and our proposition follows. □

The structure theorems of Hopf algebras, which we explain soon, give a characterization of the object $\mathbb{L}(M)$ within the tensor algebra $\mathbb{T}(M)$, and in the sequel, we actually use this representation when we need to handle free Lie algebra structures.

7.2.9. *The classical definition of enveloping algebras.* In the module context, the enveloping algebra $\mathbb{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is classically defined as a quotient

$$\mathbb{U}(\mathfrak{g}) = \mathbb{T}(\mathfrak{g})/\langle x \cdot y - y \cdot x - [x, y],\ x, y \in \mathfrak{g}\rangle,$$

where we divide the tensor algebra $\mathbb{T}(\mathfrak{g})$ by the ideal generated by the relations $x \cdot y - y \cdot x - [x, y] \equiv 0$, for $x, y \in \mathfrak{g}$. The morphism $\iota : \mathfrak{g} \to \mathbb{U}(\mathfrak{g})$ associated to the enveloping algebra $\mathbb{U}(\mathfrak{g})$ is defined as the composite of the morphism $\iota : \mathfrak{g} \to \mathbb{T}(\mathfrak{g})$ with the canonical quotient morphism $q : \mathbb{T}(\mathfrak{g}) \to \mathbb{U}(\mathfrak{g})$. In general, we use the same notation for the elements of the tensor algebra and their image in the enveloping algebra. Intuitively, the quotient process makes the commutator of Lie algebra elements $x, y \in \mathfrak{g}$ equal to the image of the Lie bracket $[x, y] \in \mathfrak{g}$ in the enveloping algebra $\mathbb{U}(\mathfrak{g})$.

From this quotient definition, we easily retrieve that the enveloping algebra $\mathbb{U}(\mathfrak{g})$ fits in an adjunction relation $\mathtt{Mor}_{\mathcal{A}s_+}(\mathbb{U}(\mathfrak{g}), A) = \mathtt{Mor}_{\mathcal{L}ie}(\mathfrak{g}, A)$, for $A \in \mathcal{A}s_+$, and the canonical morphism $\iota : \mathfrak{g} \to \mathbb{U}(\mathfrak{g})$ represents the unit of this adjunction. The morphism of unitary associative algebras $\phi_f : \mathbb{U}(\mathfrak{g}) \to A$ associated to a Lie algebra morphism $f : \mathfrak{g} \to A$ is given by the same formula as in the tensor algebra case $\phi_f(x_1 \cdot \ldots \cdot x_r) = f(x_1) \cdot \ldots \cdot f(x_r)$ except that we now assume that the monomial $x_1 \cdot \ldots \cdot x_r$ represents an element of the enveloping algebra $\mathbb{U}(\mathfrak{g})$.

The enveloping algebra of a Lie algebra is also used to study the category of representations of a Lie algebra. In short, we define a representation of a Lie algebra $\mathfrak{g}$ as an object of the base category $M \in \mathcal{M}$ equipped with an operation $[-,-] : M \otimes \mathfrak{g} \to M$ which satisfies the relation of a Lie bracket $[\xi,[x,y]] = [[\xi,x],y] - [[\xi,y],x]$ for $\xi \in M$ and $x, y \in \mathfrak{g}$. The category of representations of $\mathfrak{g}$ is isomorphic to the category of right modules over the enveloping algebra $\mathbb{U}(\mathfrak{g})$, where we define a right module over an associative algebra A as an object of the base category $M \in \mathcal{M}$ equipped with a morphism $\rho : M \otimes A \to M$ that satisfies the usual unit and associativity relations of the structure of a right module in $\mathcal{M}$. We explicitly set $\xi \cdot (x_1 \cdot \ldots \cdot x_n) = [\cdots[[\xi,x_1],x_2],\ldots,x_n]$ to define the action $\xi \cdot u = \rho(\xi \otimes u)$ of a monomial $u = x_1 \cdot \ldots \cdot x_n \in \mathbb{U}(\mathfrak{g})$, where $x_1,\ldots,x_n \in \mathfrak{g}$, on an element $\xi \in M$. We use the relation of Lie brackets $[\xi,[x,y]] = [[\xi,x],y] - [[\xi,y],x]$ to check that this action remains well-defined when we pass to the quotient of the tensor algebra by the ideal of defining relations of the enveloping algebra.

We use the adjunction relation of enveloping algebras to establish the following structure result:

PROPOSITION 7.2.10. *The enveloping algebra of a Lie algebra* $\mathbb{U}(\mathfrak{g})$ *inherits a Hopf algebra structure such that:*

- *the augmentation* $\epsilon : \mathbb{U}(\mathfrak{g}) \to \mathbb{1}$ *is the morphism of unitary associative algebras induced by the zero morphism* $\epsilon(x) = 0$ *from the Lie algebra* $\mathfrak{g}$ *to the unit object* $\mathbb{1}$*;*
- *the coproduct* $\Delta : \mathbb{U}(\mathfrak{g}) \to \mathbb{U}(\mathfrak{g}) \otimes \mathbb{U}(\mathfrak{g})$ *is the morphism of unitary associative algebras whose restriction to the Lie algebra* $\mathfrak{g}$ *is given by the formula* $\Delta(x) = x \otimes 1 + 1 \otimes x$*;*
- *the antipode* $\sigma : \mathbb{U}(\mathfrak{g}) \to \mathbb{U}(\mathfrak{g})$ *is the anti-morphism of unitary associative algebras whose restriction to the Lie algebra* $\mathfrak{g}$ *is given by the opposite of the identity map* $\sigma(x) = -x$*.*

EXPLANATIONS. This proposition follows from the same argument line as the result of Proposition 7.2.6 (where we define the Hopf algebra structure of the tensor and symmetric algebras). For our purpose, we only have to check that the formulas of the proposition correspond to the definition of Lie algebra morphisms on $\mathfrak{g}$. (We then use our adjunction relation to extend these morphisms to well-defined structure morphisms on the enveloping algebra.)

This condition is obvious for the augmentation. In the case of the coproduct, we readily obtain the relation $[\Delta(x),\Delta(y)] = [x,y] \otimes 1 + 1 \otimes [x,y]$ in the tensor product $\mathbb{U}(\mathfrak{g}) \otimes \mathbb{U}(\mathfrak{g})$, and therefore, we have $[\Delta(x),\Delta(y)] = \Delta([x,y])$, for every $x, y \in \mathfrak{g}$. In the case of the antipode, we have $\sigma([x,y]) = -[x,y] = yx - xy = \sigma(y)\sigma(x) - \sigma(x)\sigma(y)$, and this result agrees with the expression of the commutator of $\sigma(x)$ and $\sigma(y)$ (in this order) in the opposite algebra $\mathbb{U}(\mathfrak{g})^{op}$.

The explicit formulas for the augmentation, the coproduct, and the antipode of monomials are also the same as in the tensor algebra case. □

The first objective of the Lie theory of Hopf algebras is to identify the image of a Lie algebra $\mathfrak{g}$ inside the associated enveloping algebra $\mathbb{U}(\mathfrak{g})$. For this aim, we explain the definition of a primitive element functor on coalgebras. Intuitively, the primitive element functor represents an infinitesimal version of the functor of group-like elements of the previous section from Hopf algebras to groups.

7.2.11. *Primitive elements in coaugmented counitary cocommutative coalgebras.* We consider a (counitary cocommutative) coalgebra C equipped with a morphism of coalgebras $\eta : \mathbb{1} \to C$, where we regard the tensor unit $\mathbb{1}$ as the terminal object of the category of (counitary cocommutative) coalgebras. We refer to this morphism as a *coaugmentation* associated to the coalgebra C. We define the *primitive part* of a coaugmented coalgebra C as an object $\mathbb{P}(C) \subset C$ in the ambient category $\mathcal{M}$ such that:

$$\mathbb{P}(C) = \{x \in C | \epsilon(x) = 0, \Delta(x) = x \otimes 1 + 1 \otimes x\}.$$

We then write $x \otimes 1 \in C \otimes C$ (respectively, $1 \otimes x \in C \otimes C$) for the morphism $id \otimes \eta : C \to C \otimes C$ (respectively, $\eta \otimes id : C \to C \otimes C$) yielded by the tensor product of the coaugmentation $\eta : \mathbb{1} \to C$ with the identity morphism of the object C (as in the construction of Proposition 7.2.6).

In principle, we have to define this object $\mathbb{P}(C)$ by an appropriate kernel in $\mathcal{M}$. To simplify our presentation, we assume from now on that kernels automatically exist in $\mathcal{M}$ (and, more generally, that the base category is equipped with limits in addition to biproducts and colimits). Nevertheless, in each statement where we explicitly determine the primitive part of a coalgebra structure, we proceed by a direct approach, without assuming the existence of a subobject of primitive elements as a preliminary result. In general, we only need the existence of split kernels for idempotent morphisms. This assumption is actually sufficient for the Poincaré-Birkhoff-Witt Theorem (in our formulation) and for the Milnor-Moore Theorem (which we establish soon).

The following observations parallel the assertions of Proposition 7.1.15 (about the definition of a group structure on the set of group-like elements in a Hopf algebra):

PROPOSITION 7.2.12. *Let H be a Hopf algebra. Let $[x, y] = xy - yx$ denote the commutator operation in H. We have $[\mathbb{P}(H), \mathbb{P}(H)] \subset \mathbb{P}(H)$. The object $\mathbb{P}(H) \subset H$ consequently inherits a Lie algebra structure with the morphism $[-,-] : \mathbb{P}(H) \otimes \mathbb{P}(H) \to \mathbb{P}(H)$ induced by the commutator of H as Lie bracket.*

PROOF. In the proof of Proposition 7.2.10, we already used an identity of the form $[x \otimes 1 + 1 \otimes x, y \otimes 1 + 1 \otimes y] = [x, y] \otimes 1 + 1 \otimes [x, y]$. If we assume $\Delta(x) = x \otimes 1 + 1 \otimes x$ and $\Delta(y) = y \otimes 1 + 1 \otimes y$, then we deduce from this relation that we have the identity $\Delta([x, y]) = [\Delta(x), \Delta(y)] = [x, y] \otimes 1 + 1 \otimes [x, y]$. We clearly have the implication $\epsilon(x) = \epsilon(y) = 0 \Rightarrow \epsilon([x, y]) = [\epsilon(x), \epsilon(y)] = 0$ too, and these verifications establish that the object $\mathbb{P}(H)$ is stable under commutators, which is the claim of the proposition. □

PROPOSITION 7.2.13. *The functor of primitive elements $\mathbb{P} : \mathcal{H}opf\mathcal{A}lg \to \mathcal{L}ie$ is right adjoint to the enveloping algebra functor $\mathbb{U} : \mathcal{L}ie \to \mathcal{H}opf\mathcal{A}lg$ (which we regard as a functor with values in the category of Hopf algebras by using the result of Proposition 7.2.10).*

PROOF. Let $\mathfrak{g}$ be a Lie algebra. Let H be a Hopf algebra. We elaborate on the adjunction relation of §7.2.7, between Lie algebra morphisms $f : \mathfrak{g} \to H$ and unitary associative algebra morphisms $\phi = \phi_f : \mathbb{U}(\mathfrak{g}) \to H$. We use pointwise formulas to make our argument more explicit, as usual.

We have $f = \phi_f|_{\mathfrak{g}}$ (by definition of this adjunction), and as a consequence, we have $f(\mathfrak{g}) \subset \mathbb{P}(H)$ (equivalently, the map f comes from a Lie algebra morphism with values in $\mathbb{P}(H)$) if and only if the associated morphism of unitary associative algebras $\phi : \mathbb{U}(\mathfrak{g}) \to H$ satisfies $\epsilon\phi(x) = 0 = \epsilon(x)$ and $\Delta\phi(x) = f(x)\otimes 1 + 1\otimes f(x) = \phi\otimes\phi\cdot\Delta(x)$ for any $x \in \mathfrak{g}$. We deduce from the injectivity of the adjunction correspondence that the verification of these relations on $\mathfrak{g}$ implies that the identities $\epsilon\phi = \epsilon$ and $\Delta\phi = \phi\otimes\phi\cdot\Delta$ hold on the whole $\mathbb{U}(\mathfrak{g})$. We therefore conclude that the adjunction relation of §7.2.7 restricts to an adjunction relation between Lie algebra morphisms $f : \mathfrak{g} \to \mathbb{P}(H)$ and Hopf algebra morphisms $\phi : \mathbb{U}(\mathfrak{g}) \to H$, and this result proves the claim of our proposition. □

The Milnor-Moore Theorem, which we state soon, implies that this adjunction defines an equivalence of categories when we restrict ourselves to a subcategory of Hopf algebras satisfying an appropriate conilpotence condition. Before addressing this general statement, we determine the primitive elements of the symmetric algebra and of the tensor algebra. The result reads as follows:

PROPOSITION 7.2.14.

(a) *For the symmetric algebra* $\mathbb{S}(M)$*, which comes equipped with the Hopf algebra structure of Proposition* 7.2.6*, we have* $\mathbb{P}\,\mathbb{S}(M) = M$*.*

(b) *For the tensor algebra* $\mathbb{T}(M)$*, which comes equipped with the Hopf algebra structure of Proposition* 7.2.6*, the morphism* $\iota : \mathbb{L}(M) \to \mathbb{T}(M)$ *of* §§7.2.7-7.2.8 *defines an isomorphism between the free Lie algebra* $\mathbb{L}(M)$ *and the Lie algebra of primitive elements* $\mathbb{P}\,\mathbb{T}(M) \subset \mathbb{T}(M)$*.*

PROOF. We again use pointwise formulas in order to make our argument more explicit.

The definition of the coproduct in the symmetric algebra $\mathbb{S}(M)$ immediately implies $M \subset \mathbb{P}\,\mathbb{S}(M)$. To check the converse inclusion, we consider the morphism $\phi : \mathbb{S}(M) \to \mathbb{S}(M)$ defined by the projection onto the summand $\mathbb{S}_1(M) = M$ in the symmetric algebra $\mathbb{S}(M)$. For a homogeneous element $u \in \mathbb{S}_r(M)$ of weight $r > 0$, we have $u = (1/r)\cdot\sum_{(u)} u_{(1)}\cdot\phi(u_{(2)})$. (The proof of this identity follows from a straightforward verification, by using the explicit formula of the coproduct of monomials $u = x_1\cdot\ldots\cdot x_r$ in the proof of Proposition 7.2.6.) This equation implies that we have the following relations $u \in \mathbb{P}\,\mathbb{S}(M) \Rightarrow u = (1/r)\cdot(u\cdot\phi(1) + 1\cdot\phi(u)) = (1/r)\cdot\phi(u) \Rightarrow u \in M$, for any homogeneous element $u \in \mathbb{S}_r(M)$, from which we conclude that $\mathbb{P}\,\mathbb{S}(M) = M$.

In the case of the tensor algebra, we again immediately have $M \subset \mathbb{P}\,\mathbb{T}(M)$, and this inclusion implies $\mathbb{L}(M) \subset \mathbb{P}\,\mathbb{T}(M)$ since primitive elements are preserved by commutators (see Proposition 7.2.12). To check the converse inclusion, we consider the morphism $\psi : \mathbb{T}(M) \to \mathbb{T}(M)$ (closely related to the morphism of Proposition 7.2.8) such that $\psi(1) = 0$, $\psi(x_1) = x_1$, and

$$\psi(x_1\cdot\ldots\cdot x_r) = [\cdots[[x_1,x_2],x_3],\ldots,x_r],$$

for a monomial $u = x_1\cdot\ldots\cdot x_r$ of order $r > 1$.

For a homogeneous element $u \in \mathbb{T}_r(M)$ of weight $r > 0$, we have again the relation:

$$(*) \qquad u = \frac{1}{r}\cdot\sum_{(u)} u_{(1)}\cdot\psi(u_{(2)})$$

which we establish as follows (we borrow our argument from [183]). We can assume $u = x_1 \cdot \ldots \cdot x_r$ and we argue by induction on the order of this tensor $r > 0$. The identity clearly holds when $r = 1$. In the case $r > 1$, we write $u = v \cdot x_r$, where we set $v = x_1 \cdot \ldots \cdot x_{r-1}$. The formula given in the proof of Proposition 7.2.6 implies that the coproduct of this monomial $v \in \mathbb{T}(M)$ can be written $\sum_{(v)} v_{(1)} \otimes v_{(2)} = v \otimes 1 + \sum'_{(v)} v_{(1)} \otimes v_{(2)} + 1 \otimes v$, where the middle sum consists of tensors $v_{(1)} \otimes v_{(2)}$ whose factors $v_{(1)}, v_{(2)} \in \mathbb{T}(M)$ are monomials of order > 0 in the tensor algebra. We assume by induction that we have the relation $\sum_{(v)} v_{(1)} \cdot \psi(v_{(2)}) = (r-1)v \Leftrightarrow \sum'_{(v)} v_{(1)} \cdot \psi(v_{(2)}) + \psi(v) = (r-1)v$. We then have the formula:

$$\begin{aligned}\Delta(u) = \Delta(v) \cdot \Delta(x_r) = \Delta(v) \cdot (x_r \otimes 1 + 1 \otimes x_r)&\\ \Rightarrow \sum_{(u)} u_{(1)} \cdot \psi(u_{(2)}) = v\psi(x_r) + {\sum_{(v)}}' v_{(1)}\psi(v_{(2)}x_r) + \psi(vx_r)&\\ + {\sum_{(v)}}' v_{(1)}x_r\psi(v_{(2)}) + x_r\psi(v).&\end{aligned}$$

We use the identities $\psi(x_r) = x_r$, $\psi(v_{(2)}x_r) = [\psi(v_{(2)}), x_r] = \psi(v_{(2)}) \cdot x_r - x_r \cdot \psi(v_{(2)})$ and $\psi(vx_r) = [\psi(v), x_r] = \psi(v) \cdot x_r - x_r \cdot \psi(v)$ to reduce this expression to the formula:

$$\sum_{(u)} u_{(1)} \cdot \psi(u_{(2)}) = v \cdot x_r + {\sum_{(v)}}' v_{(1)} \cdot \psi(v_{(2)}) \cdot x_r + \psi(v) \cdot x_r,$$

and we use the induction hypothesis to get our relation $(*)$ for the element $u = v \cdot x_r = x_1 \cdot \ldots \cdot x_r$.

This equation $(*)$ readily implies, as in the symmetric algebra case, that we have $u \in \mathbb{P}\,\mathbb{T}(M) \Rightarrow u \in \mathbb{L}(M)$ and the proof of this relation completes the verification of our identity $\mathbb{P}\,\mathbb{T}(M) = \mathbb{L}(M)$. □

We now explain the concept of a locally conilpotent Hopf algebra. We use this notion in our formulation of the Poincaré-Birkhoff-Witt Theorem and of the Milnor-Moore Theorem in the setting of $\mathbb{Q}$-additive symmetric monoidal categories.

7.2.15. *Locally conilpotent Hopf algebras.* In any Hopf algebra H the relation $\epsilon\eta = id$, between the unit $\eta : \mathbb{1} \to H$ and the counit $\epsilon : H \to \mathbb{1}$, implies that we have a decomposition $H = \mathbb{1} \oplus \mathbb{I}(H)$, where we set $\mathbb{I}(H) = \ker(\epsilon : H \to \mathbb{1})$. We call this subobject $\mathbb{I}(H)$ the *augmentation ideal* of the Hopf algebra H. We consider the morphism $\pi = id - \epsilon\eta$, which defines the projector associated to this summand $\mathbb{I}(H)$ in the Hopf algebra H.

Let $\Delta^{(n)} : H \to H^{\otimes n}$ denote the n-fold coproduct associated to our Hopf algebra (see §7.1.1). Let $\pi^{(n)} : H^{\otimes n} \to H^{\otimes n}$ denote the n-fold tensor power of our projector π. The composite $\pi^{(n)}\Delta^{(n)}$ represents the components of the n-fold coproduct $\Delta^{(n)}$ on the summand $\mathbb{I}(H)^{\otimes n}$ of the tensor product $H^{\otimes n}$ (we remove all terms involving a unit factor).

We say that H is *locally conilpotent* when H admits a colimit decomposition $K^0 \to \cdots \to K^m \to \cdots \to \operatorname{colim}_m K^m = H$ such that:

(1) we have $\pi^{(n)}\Delta^{(n)}{}_{|K^m} = 0$ as soon as $n > m$;

(2) and the coproduct $H \to H \otimes H$ admits a factorization

$$\begin{tikzcd} H \arrow[r, "\Delta"] & H \otimes H \\ K^m \arrow[u] \arrow[r, dotted, "\exists"] & \operatorname{colim}_{p+q \leq m} K^p \otimes K^q \arrow[u] \end{tikzcd} ,$$

for each $m \in \mathbb{N}$.

We use the notation $\mathcal{H}opf\mathcal{A}lg_c$ for the full subcategory of the category of Hopf algebras $\mathcal{H}opf\mathcal{A}lg$ formed by the locally conilpotent Hopf algebras.

We can retrieve the definition of [151, §B.3] (where our locally conilpotent Hopf algebras are called connected Hopf algebras) by taking $K^m = \ker(\pi^{(m+1)}\Delta^{(m+1)})$. We easily check (by using the coassociativity of the coproduct) that these kernels form a nested sequence $\ker(\pi^{(1)}\Delta^{(1)}) \subset \cdots \subset \ker(\pi^{(m+1)}\Delta^{(m+1)}) \subset \cdots \subset H$. We automatically have the vanishing condition (1). When we work in a category of modules over a field (so that the tensor product preserves kernels), we easily check (by using the coassociativity of the coproduct again) that the coproduct condition (2) is automatically fulfilled too. We accordingly get that our local conilpotence condition is equivalent to the relation $\operatorname{colim}_m \ker(\pi^{(m+1)}\Delta^{(m+1)}) = H$ when we fix $K^m = \ker(\pi^{(m+1)}\Delta^{(m+1)})$.

The tensor algebra (and the symmetric algebra similarly) is an instance of a locally conilpotent Hopf algebra. Indeed, we see that the objects $K^m = \bigoplus_{r \leq m} \mathbb{T}_r(M)$ fulfill our coproduct condition (by using the explicit expression of this coproduct in Proposition 7.2.6) and we trivially have $\mathbb{T}(M) = \operatorname{colim}_m(\bigoplus_{r \leq m} \mathbb{T}_r(M))$. The enveloping algebra $\mathbb{U}(\mathfrak{g})$ of a Lie algebra is locally conilpotent too. In the context of modules over a field, we take $K^m = \operatorname{im}(\bigoplus_{r \leq m} \mathbb{T}_r(\mathfrak{g}) \to \mathbb{T}(\mathfrak{g}) \to \mathbb{U}(\mathfrak{g}))$ and our requirements follow from the observation that the coproduct of $\mathbb{U}(\mathfrak{g})$ is identified with a quotient of the coproduct of the tensor algebra $\mathbb{T}(\mathfrak{g})$. In the general context, we can arrange this construction by extending the colimit decomposition $\mathbb{T}(\mathfrak{g}) = \operatorname{colim}_m\{\bigoplus_{r \leq m} \mathbb{T}_r(\mathfrak{g})\}$ to the coequalizer of §7.2.7 which serves to define the enveloping algebra.

We can now state the first main structure theorem of the theory of Hopf algebras:

Theorem 7.2.16 (Structure Theorem). *Let H be a Hopf algebra. The morphism $e : \mathbb{S}\,\mathbb{P}(H) \to H$ defined by the symmetrized sum*

$$e(x_1 \cdot \ldots \cdot x_r) = \frac{1}{r!} \cdot \sum_{\sigma \in \Sigma_r} x_{\sigma(1)} \cdot \ldots \cdot x_{\sigma(r)}$$

on the monomials $x_1 \cdot \ldots \cdot x_r \in \mathbb{S}(\mathbb{P}(H))$ is an isomorphism of counitary cocommutative coalgebras as soon as H is locally conilpotent (see §7.2.15*).*

Proof. The proof of this theorem forms the technical heart of this section. We adapt ideas of [150] (see also [39, 149]). We divide our argument line in several steps.

Preliminaries: Convolution algebras. Let $\mathtt{End}(H)$ be the module formed by the endomorphisms $f : H \to H$ of the object H in the base category. The composition of endomorphisms gives a product $\circ$ that provides this module $\mathtt{End}(H)$ with a natural unitary associative algebra structure. To prove our theorem, we use that $\mathtt{End}(H)$ is equipped with an additional associative product, called the *convolution* *product*, and which we define by the composition operation $f * g = \mu \circ (f \otimes g) \circ \Delta$, for any $f, g \in \mathtt{End}(H)$, where $\Delta : H \to H \otimes H$ denotes the coproduct of our Hopf algebra while $\mu : H \otimes H \to H$ refers to the product as usual. We equivalently have:

$$(f * g)(u) = \sum_{(u)} f(u_{(1)}) \cdot g(u_{(2)})$$

for any $u \in H$. The morphism $\eta\epsilon$, defined by the composite of the unit and of the counit of the Hopf algebra, is a unit with respect to the convolution product since we have $(\eta\epsilon * f)(u) = \sum_{(u)} \epsilon(u_{(1)}) \cdot f(u_{(2)}) = f(\sum_{(u)} \epsilon(u_{(1)}) \cdot u_{(2)}) = f(u)$, for any $u \in H$, and similarly $(f * \eta\epsilon)(u) = f(u)$.

Let us observe that the definition of this convolution product makes sense in the more general case of a hom-object $\mathtt{Hom}(C, A)$ such that C is a counitary cocommutative coalgebra (or just a counitary coassociative coalgebra) and A is a unitary associative algebra. Furthermore, our construction is natural in C and in A. In the course of our verifications, we notably consider the convolution algebras defined by the hom-objects $\mathtt{Hom}(H, H \otimes H)$ and $\mathtt{Hom}(H \otimes H, H \otimes H)$. In these cases, we use the natural counitary cocommutative coalgebra structure (respectively, the natural unitary associative algebra structure) of the tensor product $H \otimes H$ to define our convolution product.

For our purpose, we still consider the morphism $\pi : H \to H$ such that $\pi = id - \eta\epsilon \Leftrightarrow id = \eta\epsilon + \pi$. Recall that this morphism represents the projection onto the summand $\mathbb{I}(H) = \ker(\epsilon : H \to \mathbb{1})$ of our Hopf algebra H (see §7.2.15).

Step 1: A subalgebra of the convolution algebra. We set $\pi^n := \pi^{*n}$, for all $n \in \mathbb{N}$, and we consider formal sums $f = \sum_{n=0}^{\infty} \lambda_n \pi^n$, $\lambda_n \in \mathbb{Q}$, which we aim to identify with elements of the endomorphism algebra $\mathtt{End}(H)$. For this purpose, we use the colimit decomposition $H = \operatorname{colim}_m K^m$ which we consider in our local conilpotence condition in §7.2.15 and which gives the relation $\mathtt{End}(H) = \lim_m \mathtt{Hom}(K^m, H)$ at the hom-object level. Each morphism π^n can be written as a composite $\pi^n = \nabla^{(n)} \pi^{(n)} \Delta^{(n)}$, where $\Delta^{(n)} : H \to H^{\otimes n}$ denotes the n-fold coproduct of our Hopf algebra (as in §7.2.15), the morphism $\pi^{(n)} : H^{\otimes n} \to H^{\otimes n}$ is the n-fold tensor power of our projector π, and $\nabla^{(n)} : H^{\otimes n} \to H$ denotes the n-fold product. By definition of the local conilpotence condition (see §7.2.15), we have $\pi^{(n)} \Delta^{(n)}(K^m) = 0 \Rightarrow \pi^n(K^m) = 0$ when $n > m$. This vanishing relation enables us to regard our formal sum as an element of the limit of hom-objects $\mathtt{End}(H) = \lim_m \mathtt{Hom}(K^m, H)$ defined by the collection of partial sums $f_m = \sum_{n=0}^{m} \lambda_n \pi^n \in \mathtt{Hom}(K^m, H)$, for $m \geq 0$.

Let $\mathtt{S} = \{\sum_{n=0}^{\infty} \lambda_n \pi^n \,|\, \lambda_n \in \mathbb{Q}\, (\forall n)\}$ be the submodule of the endomorphism algebra $\mathtt{End}(H)$ formed by the endomorphisms $f \in \mathtt{End}(H)$ which admit an expansion of this form $f = \sum_{n=0}^{\infty} \lambda_n \pi^n$. We have $\pi^m * \pi^n = \pi^{m+n}$ and the coproduct condition §7.2.15(2) in our definition of the local conilpotence implies that the convolution product $f * g \in \mathtt{End}(H)$ of endomorphisms of the form $f = \sum_{n=0}^{\infty} \lambda_n \pi^n$ and $g = \sum_{n=0}^{\infty} \mu_n \pi^n$ satisfies $f * g = \sum_{n=0}^{\infty} (\sum_{p+q=n} \lambda_p \mu_q) \pi^n$. Hence, our module $\mathtt{S}$ is preserved by the convolution product of the endomorphism algebra $\mathtt{End}(H)$.

In the next steps of this proof, we consider a new collection of elements $e^s \in \mathtt{End}(H)$, which we define by the formulas:

$$e^1 = \log_*(id) = \log_*(\eta\epsilon + \pi) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\pi^n}{n} \quad \text{and} \quad e^s = \frac{(e^1)^{*s}}{s!} \quad \text{for } s \in \mathbb{N}.$$

We see that each e^s has an expansion of the form $e^s = \sum_{n\geq s}\lambda_n\pi^n$ with a leading term of order $n = s$. We can therefore give a sense to infinite sums $\sum_{s=0}^{\infty} c_s e^s$ in $\mathtt{End}(H)$. We moreover have the relation

$$\mathtt{S} = \Big\{\sum_{n=0}^{\infty}\lambda_n\pi^n \,|\, \lambda_n\in\mathbb{Q}\,(\forall n)\Big\} = \Big\{\sum_{s=0}^{\infty} c_s e^s \,|\, c_s\in\mathbb{Q}\,(\forall s)\Big\}$$

inside the endomorphism algebra $\mathtt{End}(H)$.

Step 2: The coproduct relations. We determine a distribution relation between the action of the endomorphisms e^s on H and the coproduct of the Hopf algebra H. For this purpose, we use the convolution structure associated to the hom-objects $\mathtt{Hom}(H, H\otimes H)$ and $\mathtt{Hom}(H\otimes H, H\otimes H)$. We have an obvious extension of the formal sum representation of Step 1 to $\mathtt{Hom}(H, H\otimes H)$ since we also have a limit decomposition $\mathtt{Hom}(H, H\otimes H) = \lim_n \mathtt{Hom}(K^n, H\otimes H)$ in this case. We have a similar observation for the hom-object $\mathtt{Hom}(H\otimes H, H\otimes H)$. We then use the identity $\mathtt{Hom}(H\otimes H, H\otimes H) = \lim_{pq}\mathtt{Hom}(K^p\otimes K^q, H\otimes H)$.

We easily check that the distribution relation between the coproduct and the product of H implies that we have the distribution relation $\Delta\circ(f*g) = (\Delta\circ f)*(\Delta\circ g)$ in $\mathtt{Hom}(H, H\otimes H)$ when we form the composite of endomorphisms $f, g\in\mathtt{End}(H) = \mathtt{Hom}(H, H)$ with the coproduct $\Delta\in\mathtt{Hom}(H, H\otimes H)$. Hence, the morphism $\Delta_* : \mathtt{Hom}(H, H)\to\mathtt{Hom}(H, H\otimes H)$ such that $\Delta_*(f) = \Delta\circ f$ preserves the convolution product. We have a similar result for the morphism $\Delta^* : \mathtt{Hom}(H\otimes H, H\otimes H)\to\mathtt{Hom}(H, H\otimes H)$ such that $\Delta^*(f) = f\circ\Delta$. We then use that the coproduct $\Delta : H\to H\otimes H$ defines a morphism of coalgebras (and hence, the assumption that this coproduct is cocommutative) in order to get our distribution relation.

We study the composite of the endomorphism $e^1 = \log_*(id)$ with the coproduct of H. We have $\Delta\circ id = (id\otimes id)\circ\Delta$, and we deduce from the usual logarithm addition formula that we have the identity:

$$\begin{aligned}\Delta\circ\log_*(id) &= \log_*(id\otimes id)\circ\Delta\\ &= \log_*((id\otimes\eta\epsilon)*(\eta\epsilon\otimes id))\circ\Delta\\ &= (\log_*(id)\otimes\eta\epsilon + \eta\epsilon\otimes\log_*(id))\circ\Delta,\end{aligned}$$

where we use the coproduct condition §7.2.15(2) again in order to give a sense to the terms of this relation in our limit of hom-objects.

We have $e^0 = (e^1)^{*0} = \eta\epsilon$ (the convolution unit), and accordingly, we can rewrite the above relation as $\Delta\circ e^1 = (e^1\otimes e^0 + e^0\otimes e^1)\circ\Delta$. We then have $\Delta\circ e^r = (\Delta\circ e^1)^{*r}/r! = (e^1\otimes e^0 + e^0\otimes e^1)^{*r}\circ\Delta/r!$ for each endomorphism $e^r = (e^1)^{*r}/r!$, and the binomial identity implies that we have the following distribution relation:

$$(*)\qquad \Delta\circ e^r = \sum_{s+t=r}(e^s\otimes e^t)\circ\Delta,$$

for all $r\in\mathbb{N}$.

Step 3: The Eulerian idempotents. We now prove that the endomorphisms e^s, $s\in\mathbb{N}$, form a complete collection of orthogonal idempotents in the endomorphism algebra $\mathtt{End}(H)$. For this purpose, we consider a third collection of elements, which we define by the simple formula $\psi^n = id^{*n}$, where $id : H\to H$ is the identity morphism. For each $n \in \mathbb{N}$, we readily get:

$$\psi^n = \exp_*(n \log_*(id)) = \exp_*(ne^1) = \sum_{s=0}^{\infty} n^s e^s$$

when we define the exponential $\exp_*(x)$ by the usual power series expansion in the convolution algebra.

Recall that we use the notation $\Delta^{(n)}$ for the n-fold coproduct of the Hopf algebra H and the notation $\nabla^{(n)}$ for the n-fold product. We have already observed that the elements $\pi^n \in \mathtt{End}(H)$ introduced in Step 1 satisfy the relation $\pi^n = \nabla^{(n)}\pi^{(n)}\Delta^{(n)}$, where we set $\pi^{(n)} = \pi^{\otimes n}$. We immediately deduce, from the definition of the convolution product, that our new elements $\psi^n = id^{*n}$ are also identified with the composites $\psi^n = \nabla^{(n)}\Delta^{(n)}$, for all $n \in \mathbb{N}$.

The distribution relation $(*)$ implies, by an immediate induction, that we have the formula

$$(*') \qquad \Delta^{(n)} \circ e^r = \sum_{r_1+\cdots+r_n=r} (e^{r_1} \otimes \cdots \otimes e^{r_n}) \circ \Delta^{(n)}$$

for all $n \in \mathbb{N}$. From this formula, we get the identity:

$$(\nabla^{(n)}\pi^{(n)}\Delta^{(n)}) \circ e^r = \nabla^{(n)} \circ \Big(\sum_{r_1+\cdots+r_n=r} (\pi \circ e^{r_1}) \otimes \cdots \otimes (\pi \circ e^{r_n}) \Big) \circ \Delta^{(n)},$$

from which we deduce the vanishing relation:

$$\pi^n \circ e^r = 0 \quad \text{for } n > r,$$

because if we assume $n > r$, then we have $r_k = 0$ for some factor in each term of the above formula and $\pi e^0 = \pi\eta\epsilon = 0$. The relation $(*')$ similarly implies that we have the identity:

$$\begin{aligned}\psi^n \circ e^r = (\nabla^{(n)}\Delta^{(n)}) \circ e^r &= \nabla^{(n)} \circ \Big(\sum_{r_1+\cdots+r_n=r} (e^{r_1} \otimes \cdots \otimes e^{r_n}) \Big) \circ \Delta^{(n)} \\ &= \sum_{r_1+\cdots+r_n=r} e^{r_1} * \cdots * e^{r_n} = \sum_{r_1+\cdots+r_n=r} \frac{r!}{r_1! \cdots r_n!} \cdot e^r,\end{aligned}$$

which gives:

$$\psi^n \circ e^r = n^r e^r,$$

for each $n \in \mathbb{N}$, $r \in \mathbb{N}$, when we consider the composite of the morphism $\psi^n = id^{*n}$ with any $e^r \in \mathtt{End}(H)$.

We consider the variants $\mathtt{S}_r = \{\sum_{s=r}^{\infty} c_s e^s \,|\, c_s \in \mathbb{Q}\,(\forall s)\} = \{\sum_{n=r}^{\infty} \lambda_n \pi^n \,|\, \lambda_n \in \mathbb{Q}\,(\forall n)\}$ of the module $\mathtt{S} = \mathtt{S}_0$ introduced in Step 1. We form the quotient object $\mathtt{S}/\mathtt{S}_{r+1}$ and we write $\bar{p} \in \mathtt{S}/\mathtt{S}_{r+1}$ for the image of an element $p \in \mathtt{S}$ in this quotient. From the expansion $\psi^n = \sum_{s=0}^{\infty} n^s e^s$, we deduce the relation:

$$\bar{\psi}^n = \sum_{s=0}^{r} n^s \bar{e}^s \Rightarrow \bar{e}^s = \sum_{n=0}^{r} \theta_{sn} \bar{\psi}^n,$$

where $(\theta_{kl})_{kl}$ denotes the inverse of the Vandermonde matrix $(n^s)_{ns}$.

We deduce from the vanishing of the product $\pi^n \circ e^r$ for $n > r$ that we have the relation $\mathtt{S}_{r+1} \circ e^r = 0$ in $\mathtt{End}(H)$ and that the mapping $\rho : f \mapsto f \circ e^r$ induces a well-defined module morphism $\rho : \mathtt{S}/\mathtt{S}_{r+1} \to \mathtt{End}(H)$. We also deduce from our computation of the product $\psi^n \circ e^r$ that we have the formula

$\rho(\bar{\psi}^n) = n^r e^r$, for all $n \in \mathbb{N}$. By using the Vandermonde matrix again, we obtain $\rho(\bar{e}^s) = \sum_{n=0}^r \theta_{sn}\rho(\bar{\psi}^n) = \sum_{n=0}^r \theta_{sn} n^r e^r = \delta_s^r e^r$, where δ_s^r is the Kronecker delta (compare this argument with [124]). We conclude from this computation that our endomorphisms e^r, $r \in \mathbb{N}$, satisfy the relations

$$e^s \circ e^t = \begin{cases} e^s, & \text{if } s = t, \\ 0, & \text{otherwise.} \end{cases}$$

We moreover have $\sum_s e^s = \exp_* \log_*(id) = id$. We therefore obtain that the endomorphisms e^r, $r \in \mathbb{N}$, form a complete set of orthogonal idempotents in the endomorphism algebra of the object H.

From now on, we use the name '*Eulerian idempotents*' (following the convention of the article [150]) to refer to these endomorphisms $e^s \in \mathtt{End}(H)$, $s \in \mathbb{N}$. The original Eulerian idempotents, such as defined in [154], are collections of idempotent elements e_n^s defined in the group algebra of the symmetric groups $\mathbb{Q}[\Sigma_n]$. These idempotents correspond to the endomorphisms $e^s \in \mathtt{End}(H)$ which we associate to the tensor algebra $H = \mathbb{T}(M)$ (see [124, 150] and [155, §9]).

Step 4: The Eulerian splitting. We consider the splitting

$$H = \bigoplus_{r=0}^{\infty} e^r(H),$$

which we deduce from the action of the Eulerian idempotents $e^r \in \mathtt{End}(H)$ on our Hopf algebra H and where $e^r(H) \subset H$ denotes the image of the endomorphism $e^r : H \to H$, for each $r \in \mathbb{N}$. Recall that we have $e^0 = \eta\epsilon$, the unit of the convolution product. We readily see that we have the inclusion relation $e^1(H) \subset \mathbb{P}(H)$, because we have $e^0 e^1(u) = 0 \Rightarrow \epsilon(e^1(u)) = 0$, and for $r = 1$, the distribution relation $(*)$ implies:

$$\begin{aligned}\Delta(e^1(u)) &= \sum_{(u)} \big[e^1(u_{(1)}) \otimes e^0(u_{(2)}) + e^0(u_{(1)}) \otimes e^1(u_{(2)})\big] \\ &= \sum_{(u)} \big[\epsilon(u_{(2)}) \cdot e^1(u_{(1)}) \otimes 1 + 1 \otimes \epsilon(u_{(1)}) \cdot e^1(u_{(2)})\big] \\ &= e^1(u) \otimes 1 + 1 \otimes e^1(u).\end{aligned}$$

The goal of this step is to prove that we have an isomorphism $e^r(H) \simeq \mathbb{S}_r(e^1(H))$, for every $r \in \mathbb{N}$. Later on (in the concluding step of the proof of this theorem), we will prove that the inclusion $e^1(H) \subset \mathbb{P}(H)$ is actually an equality.

Recall that we set $\mathbb{I}(H) = \ker(\epsilon : H \to \mathbb{1})$ and the morphism $\pi : H \to H$ such that $\pi = id - \eta\epsilon = id - e^0$ is identified with the projector associated to this summand of our Hopf algebra. Recall besides that we use the notation $\nabla^{(r)} : H^{\otimes r} \to H$ for the r-fold product in our Hopf algebra H and the notation $\Delta^{(r)} : H \to H^{\otimes r}$ for the r-fold coproduct. We also consider the morphism $\pi^{(r)} : H^{\otimes r} \to H^{\otimes r}$ defined by the tensor power of our projector $\pi : H \to H$.

In the case $n = r$, the distribution formula $(*')$ together with the identities $\pi e^0 = 0$ and $\pi e^1 = e^1$ imply that we have the relation:

$$(**) \qquad \pi^{(r)}\Delta^{(r)}(e^r(u)) = \sum_{(u)} e^1(u_{(1)}) \otimes \cdots \otimes e^1(u_{(r)}),$$

for all $u \in H$. We then consider the morphism $\eta : \mathbb{S}_r(e^1(H)) \hookrightarrow e^1(H)^{\otimes r} \hookrightarrow H^{\otimes r}$ such that $\eta(x_1 \cdot \ldots \cdot x_r) = (1/r!) \cdot \sum_{\sigma \in \Sigma_r} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(r)}$ for any monomial

$x_1 \cdot \ldots \cdot x_r \in \mathbb{S}_r(e^1(H))$. The above formula $(**)$, where we also use the symmetry of the r-fold coproduct $\Delta^{(r)}$, implies that $\pi^{(r)}\Delta^{(r)}(e^r(u))$ lies in the image of this morphism. Hence, we have a morphism $S_r : e^r(H) \to \mathbb{S}_r(e^1(H))$ which fits in a commutative diagram:

$$\begin{tikzcd}
e^r(H) \arrow[rr, dotted, "S_r"] \arrow[dr, "\pi^{(r)}\Delta^{(r)}"'] & & \mathbb{S}_r(e^1(H)) \arrow[dl, hook', "\eta"] \\
& H^{\otimes r} &
\end{tikzcd} ,$$

for any $r \in \mathbb{N}$. Let $\Pi_r = e^r\nabla^{(r)}\eta$ be the morphism $\Pi_r : \mathbb{S}_r(e^1(H)) \to e^r(H)$ defined by taking the composite

$$\mathbb{S}_r(e^1(H)) \overset{\eta}{\hookrightarrow} H^{\otimes r} \xrightarrow{\nabla^{(r)}} H \xrightarrow{e^r} e^r(H) ,$$

for any $r \in \mathbb{N}$.

The previous distribution formula $(**)$ implies that we have the relation:

$$\nabla^{(r)}\pi^{(r)}\Delta^{(r)}(e^r(u)) = \sum_{(u)} e^1(u_{(1)}) \cdot \ldots \cdot e^1(u_{(r)}) = (e^1 * \cdots * e^1)(u),$$

and hence that we have the identity:

$$\nabla^{(r)}\pi^{(r)}\Delta^{(r)}(e^r(u)) = r! \cdot e^r(u),$$

for all $u \in H$, from which we readily deduce that we have the identity $\Pi_r S_r(e^r(u)) = r! \cdot e^r \circ e^r(u) = r! \cdot e^r(u)$. We aim to establish a converse relation $S_r\Pi_r(\varpi) = r! \cdot \varpi$ for any $\varpi = e^1(u_1) \cdot \ldots \cdot e^1(u_r) \in \mathbb{S}_r(e^1(H))$. We have by definition:

$$\Pi_r(\varpi) = \frac{1}{r!} \cdot \sum_{\sigma\in\Sigma_r} e^r(e^1(u_{\sigma(1)}) \cdot \ldots \cdot e^1(u_{\sigma(r)})).$$

We now consider a general element of the form $v = e^1(v_1) \cdot \ldots \cdot e^1(v_r)$ in the Hopf algebra H. We have:

$$\Delta^{(r)}(e^1(v_i)) = \sum_{k=1}^{r} 1 \otimes \cdots \otimes \underbrace{e^1(v_i)}_{k} \otimes \cdots \otimes 1, \quad \text{for each } i = 1, \ldots, r,$$

since we observed that $e^1(H)$ consists of primitive elements. We use the distribution relation between the product and the coproduct of a Hopf algebra to obtain that the r-fold coproduct of our element $v = e^1(v_1) \cdot \ldots \cdot e^1(v_r)$ satisfies:

$$\Delta^{(r)}(v) = \Delta^{(r)}(e^1(v_1)) \cdot \ldots \cdot \Delta^{(r)}(e^1(v_r)),$$

and admits an expansion of the form:

$$\Delta^{(r)}(v) = \sum_{\tau\in\Sigma_r} e^1(v_{\tau(1)}) \otimes \cdots \otimes e^1(v_{\tau(r)}) + \text{tensors with a unit factor},$$

where the remainder consists of terms $\rho = \rho_1 \otimes \cdots \otimes \rho_r \in H^{\otimes r}$ which contain at least one unit factor $\rho_i = 1 \in H$. We apply the distribution relation $(**)$ to our element $u = v = e^1(v_1) \cdot \ldots \cdot e^1(v_r)$ and we use the relations $e^1(1) = 0$ and $e^1 \circ e^1 = e^1$ to obtain the identity:

$$\pi^{(r)}\Delta^{(r)}(e^r(v)) = \sum_{\tau\in\Sigma_r} e^1(v_{\tau(1)}) \otimes \cdots \otimes e^1(v_{\tau(r)}).$$

We apply this formula to the terms $v = e^1(u_{\sigma(1)}) \cdot \ldots \cdot e^1(u_{\sigma(r)})$ in the expansion of $\Pi_r(\varpi)$. We then get:

$$\begin{aligned}\pi^{(r)}\Delta^{(r)}(\Pi_r(\varpi)) &= \frac{1}{r!} \cdot \sum_{\sigma,\tau\in\Sigma_r} e^1(u_{\tau\sigma(1)}) \otimes \cdots \otimes e^1(u_{\tau\sigma(r)})\\ &= \sum_{\sigma\in\Sigma_r} e^1(u_{\sigma(1)}) \otimes \cdots \otimes e^1(u_{\sigma(r)})\end{aligned}$$

after an obvious change of summation variables and a reduction of identical terms. We consequently have the relation $\eta S_r \Pi_r(\varpi) = \pi^{(r)}\Delta^{(r)}\Pi_r(\varpi) = r! \cdot \eta(\varpi)$, from which we conclude that we have the identity $S_r\Pi_r(\varpi) = r! \cdot \varpi$, for all $\varpi \in \mathbb{S}_r(e^1(H))$. We therefore obtain that our maps S_r and Π_r do define converse isomorphisms between $\mathbb{S}_r(e^1(H))$ and $e^r(H)$ (up to the multiplicative scalar $r! \in \mathbb{Q}$).

We go back to the consideration of a general element of the form $v = e^1(v_1) \cdot \ldots \cdot e^1(v_r)$ in the Hopf algebra H. We can also use our distribution relation $(**)$ to compute the expression $\pi^{(n)}\Delta^{(n)}(e^n(v))$ for $n > r$. Indeed, we see that all tensors in the expansion of the n-fold coproduct $\Delta^{(n)}(v)$ contain at least one unit factor $1 \in H$ in this case and the application of our distribution relation $(**)$ therefore returns a trivial result $\pi^{(n)}\Delta^{(n)}(e^n(v)) = 0$.

Conclusions. We now consider the symmetrization map $e : \mathbb{S}(e^1(H)) \to H$, which we define by the same formula as in the statement of our theorem:

$$e(e^1(u_1) \cdot \ldots \cdot e^1(u_r)) = \frac{1}{r!} \cdot \sum_{\sigma\in\Sigma_r} e^1(u_{\sigma(1)}) \cdot \ldots \cdot e^1(u_{\sigma(r)}),$$

for $\varpi = e^1(u_1) \cdot \ldots \cdot e^1(u_r) \in \mathbb{S}_r(e^1(H))$. We trivially have $e^r(e(\varpi)) = \Pi_r(\varpi)$ when we take the projection of $e(\varpi) \in H$ onto the summand $e^r(H) \subset H$. We already observed, on the other hand, that we have the relation $\pi^{(n)}\Delta^{(n)}(e^n(v)) = 0$ for any element of the form $v = e^1(v_1) \cdot \ldots \cdot e^1(v_r)$ when $n > r$. We accordingly have $\eta S_n(e^n(v)) = 0 \Rightarrow S_n(e^n(v)) = 0 \Rightarrow e^n(v) = 0$ for any such $v = e^1(v_1) \cdot \ldots \cdot e^1(v_r)$. This vanishing relation implies that $e : \mathbb{S}(e^1(H)) \to H$ carries $\mathbb{S}_r(e^1(H))$ into $\bigoplus_{n\leq r} e^n(H)$ and satisfies $e(\varpi) \equiv \Pi_r(\varpi) (\operatorname{mod} \bigoplus_{n<r} e^n(H))$, for any $\varpi = e^1(u_1) \cdot \ldots \cdot e^1(u_r) \in \mathbb{S}_r(e^1(H))$. Then we can use that our morphisms $\Pi_r : \mathbb{S}_r(e^1(H)) \to e^r(H)$ are invertible in the base category to obtain that the symmetrization map $e : \mathbb{S}(e^1(H)) \to H$ is invertible too.

To complete our verifications, we just check, by using the inclusion $e^1(H) \subset \mathbb{P}(H)$ and a straightforward computation, that e defines a coproduct preserving morphism from the symmetric algebra $\mathbb{S}(e^1(H))$, which we equip with the coalgebra structure of Proposition 7.2.6, towards our Hopf algebra H. Moreover, from the relation $e^1(H) \subset \mathbb{P}(H)$, we conclude that e preserves the augmentation attached to our objects too. Hence, our symmetrization map $e : \mathbb{S}(e^1(H)) \to H$ defines an isomorphism in the category of counitary cocommutative coalgebras. Then we can use the result of Proposition 7.2.14(a), where we determine the primitive elements of a symmetric coalgebra, to deduce the additional relation $e^1(H) = \mathbb{P}\,\mathbb{S}(e^1(H)) = \mathbb{P}(H)$ from the existence of this isomorphism $e : \mathbb{S}(e^1(H)) \xrightarrow{\simeq} H$. The statement of our theorem follows. □

Proposition 7.2.14(b) and Theorem 7.2.16 give the free Lie algebra case of the Poincaré-Birkhoff-Witt Theorem:

THEOREM 7.2.17 (Poincaré-Birkhoff-Witt Theorem). *The morphism* $e : \mathbb{S}(\mathfrak{g}) \to \mathbb{U}(\mathfrak{g})$ *defined by the symmetrized sum*

$$e(x_1 \cdot \ldots \cdot x_r) = \frac{1}{r!} \cdot \sum_{\sigma \in \Sigma_r} x_{\sigma(1)} \cdot \ldots \cdot x_{\sigma(r)}$$

on the symmetric algebra monomials $x_1 \cdot \ldots \cdot x_r \in \mathbb{S}(\mathfrak{g})$ *yields an isomorphism of counitary cocommutative coalgebras from the symmetric algebra* $\mathbb{S}(\mathfrak{g})$ *to the enveloping algebra* $\mathbb{U}(\mathfrak{g})$*, for all Lie algebras in our base category* $\mathfrak{g} \in \mathcal{L}ie$*.*

PROOF. The equivalence between the claim of Theorem 7.2.17 and the combined results of Proposition 7.2.14(b) and Theorem 7.2.16 in the case of a free Lie algebra $\mathfrak{g} = \mathbb{L}(M)$ follows from the identity $\mathbb{U}\,\mathbb{L}(M) = \mathbb{T}(M)$ (see §7.2.7). The assertion that our morphism $e : \mathbb{S}(\mathfrak{g}) \to \mathbb{U}(\mathfrak{g})$ is a morphism of counitary cocommutative coalgebras in the theorem follows from a straightforward verification (similar to the verification of the parallel claim of Theorem 7.2.16), and we do not come back to this claim.

To establish the general case of our theorem, we use that any Lie algebra $\mathfrak{g}$ fits in a reflexive coequalizer of free Lie algebras

$$\begin{tikzcd}
\mathbb{L}(M_1) \arrow[r, shift left, "d^0"] \arrow[r, shift right, "d^1"'] & \mathbb{L}(M_0) \arrow[l, bend right=50, "s^0"'] \arrow[r, dotted] & \mathfrak{g} ,
\end{tikzcd}$$

of which construction can be deduced from the free Lie algebra adjunction (see [130, §VI.7] for details). The symmetric algebra functor preserves reflexive coequalizers (by the general statements of §1.4) as well as the enveloping algebra functor (by adjunction and because reflexive coequalizers of algebras are created in the underlying category). Thus, our natural transformation fits in a diagram of coequalizers of the form:

$$\begin{tikzcd}
\mathbb{S}\,\mathbb{L}(M_1) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\simeq"'] & \mathbb{S}\,\mathbb{L}(M_0) \arrow[l, bend right=50] \arrow[r, dotted] \arrow[d, "\simeq"'] & \mathbb{S}(\mathfrak{g}) \arrow[d, dotted, "e"] , \\
\mathbb{T}(M_1) \arrow[r, shift left] \arrow[r, shift right] & \mathbb{T}(M_0) \arrow[l, bend right=50] \arrow[r, dotted] & \mathbb{U}(\mathfrak{g})
\end{tikzcd}$$

where we use the identity $\mathbb{U}\,\mathbb{L}(M) = \mathbb{T}(M)$ of §7.2.7 and the combined results of Proposition 7.2.14(b) and Theorem 7.2.16 to get that this diagram involves isomorphisms between the terms of our coequalizers. The existence of these isomorphisms implies that we get an isomorphism at the level of the coequalizers themselves, and this assertion finishes the proof of our theorem.

We borrow this argument for the proof of the general case of the Poincaré-Birkhoff-Witt Theorem from [151, Theorem B.2.3]. We however do not use the same approach as in this reference for the verification of the case of free Lie algebras. We also refer to [1, §3.3] for another approach of the Poincaré-Birkhoff-Witt Theorem (in a different setting) and to [81] for a historical overview of the subject. Let us mention that our proof works as soon as we have kernels for idempotent morphisms. □

Theorem 7.2.17 admits the following immediate consequence which extends the results of Proposition 7.2.14(b) to arbitrary Lie algebras:

THEOREM 7.2.18. *The canonical morphism* $\iota : \mathfrak{g} \to \mathbb{U}(\mathfrak{g})$*, which we associate to any enveloping algebra* $\mathbb{U}(\mathfrak{g})$*, admits a natural retraction* $\rho : \mathbb{U}(\mathfrak{g}) \to \mathfrak{g}$ *and induces an isomorphism between the Lie algebra* $\mathfrak{g}$ *and the Lie algebra of primitive elements* $\mathbb{P}\,\mathbb{U}(\mathfrak{g}) \subset \mathbb{U}(\mathfrak{g})$. ☐

This result gives one part of the Milnor-Moore Theorem:

THEOREM 7.2.19 (Milnor-Moore Theorem). *The enveloping algebra and primitive element functors* $\mathbb{U} : \mathcal{L}ie \rightleftarrows \mathcal{H}opf\mathcal{A}lg : \mathbb{P}$ *induce adjoint equivalences of categories between the category of Lie algebras* $\mathcal{L}ie$ *and the subcategory of locally conilpotent Hopf algebras* $\mathcal{H}opf\mathcal{A}lg_c$.

The original Milnor-Moore Theorem [145] deals with Hopf algebras in (weight) graded modules that satisfy a stronger conilpotence condition as the one considered in this theorem. (We actually use this conilpotence condition for weight graded Hopf algebras in §§7.3.15-7.3.16.) The reference [151, Theorem B.4.5] provides a generalization of Milnor-Moore's result in the setting of locally conilpotent (connected) Hopf algebras in $\mathbb{Q}$-modules (respectively, graded modules, differential graded modules). We give a direct proof of the general case of our theorem by relying on the result of Theorem 7.2.16 (the Structure Theorem), on the result of Theorem 7.2.17 (the Poincaré-Birkhoff-Witt Theorem), and on the result of Theorem 7.2.18.

PROOF. The claim of Theorem 7.2.18 actually gives one of the inversion relations of our category equivalence $\mathfrak{g} \xrightarrow{\simeq} \mathbb{P}\,\mathbb{U}(\mathfrak{g})$. We use the statements of Theorem 7.2.16 and Theorem 7.2.17 to get the converse relation $\mathbb{U}\,\mathbb{P}(H) \xrightarrow{\simeq} H$. For this purpose, we simply observe that our morphism $\mathbb{U}\,\mathbb{P}(H) \to H$ fits in a commutative triangle

$$\begin{tikzcd} & \mathbb{U}\,\mathbb{P}(H) \arrow[dd] \\ \mathbb{S}\,\mathbb{P}(H) \arrow[ur, "\simeq"] \arrow[dr, "\simeq"'] & \\ & H \end{tikzcd},$$

where the diagonal morphisms are the isomorphisms of Theorem 7.2.16 and Theorem 7.2.17. ☐

To complete the survey of this section, and as a preparation for subsequent applications of enveloping algebras to the study of operads, we examine the definition of a symmetric monoidal structure on Lie algebras.

7.2.20. *Direct sums of Lie algebras.* The category of Lie algebras inherits limits and colimits, like any category of algebras over an operad. Furthermore, the limits of Lie algebras, as well as the filtered colimits and the reflexive coequalizers, are created in the ambient symmetric monoidal category. In our setting, we also have an identity between the direct sum $\mathfrak{g} \oplus \mathfrak{h}$ and the product of $\mathfrak{g}$ and $\mathfrak{h}$ in the category of Lie algebras since we assume that the ambient category is additive. We precisely take this direct sum operation $\oplus : \mathcal{L}ie \times \mathcal{L}ie \to \mathcal{L}ie$ to provide the category of Lie algebras $\mathcal{L}ie$ with a symmetric monoidal structure. The zero object 0, which also represents the initial object of the category of Lie algebras, defines the monoidal unit. The direct sum satisfies the unit, associativity, and symmetry axioms of symmetric monoidal categories, but it clearly does not satisfy the distribution relation of §0.9 with respect to colimits.

The Lie bracket of the Lie algebra $\mathfrak{g} \oplus \mathfrak{h}$ is defined by $[(x_1, y_1), (x_2, y_2)] = ([x_1, x_2], [y_1, y_2])$, for all $x_1, x_2 \in \mathfrak{g}$, and $y_1, y_2 \in \mathfrak{h}$.

The canonical embeddings $i : \mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{h}$ and $j : \mathfrak{h} \to \mathfrak{g} \oplus \mathfrak{h}$ define morphisms of Lie algebras. We moreover have $[i(\mathfrak{g}), j(\mathfrak{h})] = 0$ in $\mathfrak{g} \oplus \mathfrak{h}$. We readily see that the Lie algebra $\mathfrak{g} \oplus \mathfrak{h}$ is universal with this property in the sense that giving a morphism from $\mathfrak{g} \oplus \mathfrak{h}$ towards a Lie algebra $\mathfrak{m}$ amounts to giving a pair of Lie algebra morphisms $(f : \mathfrak{g} \to \mathfrak{m}, g : \mathfrak{h} \to \mathfrak{m})$ such that we have the relation $[f(\mathfrak{g}), g(\mathfrak{h})] = 0$ in $\mathfrak{m}$. This result holds in the general setting of $\mathbb{Q}$-additive symmetric monoidal categories.

The Lie algebra embeddings $\mathfrak{g} \xrightarrow{i} \mathfrak{g} \oplus \mathfrak{h} \xleftarrow{j} \mathfrak{h}$ induce morphisms $\mathbb{U}(\mathfrak{g}) \xrightarrow{i_*} \mathbb{U}(\mathfrak{g} \oplus \mathfrak{h}) \xleftarrow{j_*} \mathbb{U}(\mathfrak{h})$ at the enveloping algebra level, and we can use the product of the enveloping algebra in order to get a morphism $\mu(i_*, j_*) : \mathbb{U}(\mathfrak{g}) \otimes \mathbb{U}(\mathfrak{h}) \to \mathbb{U}(\mathfrak{g} \oplus \mathfrak{h})$ such that $\mu(i_*, j_*)(u \otimes v) = i_*(u) \cdot j_*(v)$, for $u \otimes v \in \mathbb{U}(\mathfrak{g}) \otimes \mathbb{U}(\mathfrak{h})$. We claim that:

LEMMA 7.2.21. *The just defined morphism $\mu(i_*, j_*) : \mathbb{U}(\mathfrak{g}) \otimes \mathbb{U}(\mathfrak{h}) \to \mathbb{U}(\mathfrak{g} \oplus \mathfrak{h})$ is an isomorphism.*

PROOF. In general, we have a bijection between the morphisms of unitary associative algebras on a tensor product $\phi : U \otimes V \to T$ and the pairs of unitary associative algebra morphisms $(\phi_f : U \to T, \phi_g : V \to T)$ that satisfy the relation $[\phi_f(U), \phi_g(V)] = 0$ in T because we can set $\phi(u \otimes v) = \phi_f(u) \cdot \phi_g(v)$ to get a morphism on $U \otimes V$ when this condition is satisfied. In the case of the enveloping algebras $U = \mathbb{U}(\mathfrak{g})$ and $V = \mathbb{U}(\mathfrak{h})$, the verification of the relation $[f(\mathfrak{g}), g(\mathfrak{h})] = 0$ for the Lie algebra morphisms $(f : \mathfrak{g} \to T, g : \mathfrak{h} \to T)$ associated to $(\phi_f : \mathbb{U}(\mathfrak{g}) \to T, \phi_g : \mathbb{U}(\mathfrak{h}) \to T)$, readily implies that we have the commutation relation $[\phi_f(u), \phi_g(v)] = 0$ on the whole tensor product $\mathbb{U}(\mathfrak{g}) \otimes \mathbb{U}(\mathfrak{h})$.

Hence, giving a morphism of unitary associative algebras $\phi : \mathbb{U}(\mathfrak{g}) \otimes \mathbb{U}(\mathfrak{h}) \to T$ amounts to giving a pair of Lie algebra morphisms $(f : \mathfrak{g} \to T, g : \mathfrak{h} \to T)$ such that $[f(\mathfrak{g}), g(\mathfrak{h})] = 0$, and according to the analysis of §7.2.20, this data amounts to defining a Lie algebra morphism on the direct sum $\mathfrak{g} \oplus \mathfrak{h}$.

From this result, we conclude that the tensor product $\mathbb{U}(\mathfrak{g}) \otimes \mathbb{U}(\mathfrak{h})$ satisfies the same adjunction relation as the enveloping algebra of the Lie algebra $\mathfrak{g} \oplus \mathfrak{h}$, and hence, is isomorphic to this enveloping algebra $\mathbb{U}(\mathfrak{g} \oplus \mathfrak{h})$. The morphism considered in the lemma can readily be identified with the comparison isomorphism which we define in this proof. ☐

7.2.22. *The symmetric monoidal category of Hopf algebras.* We have already observed that the category of counitary cocommutative coalgebras in a symmetric monoidal category inherits a symmetric monoidal structure and we have a similar result for the category of unitary associative algebras. As we define bialgebras in terms of a combination of these structures, we deduce from our general primary results that the category of bialgebras inherits a symmetric monoidal structure too. When we deal with Hopf algebras $G, H \in \mathcal{H}opf\mathcal{A}lg$, we have an obvious antipode on the tensor product $G \otimes H$, which is defined factorwise by the tensor product of the antipodes associated to G and H. We conclude that the category of Hopf algebras $\mathcal{H}opf\mathcal{A}lg$ forms a symmetric monoidal subcategory of the category of bialgebras.

We implicitly check, in the proof of Lemma 7.2.21, that our pairing $\mu(i_*, j_*) : \mathbb{U}(\mathfrak{g}) \otimes \mathbb{U}(\mathfrak{h}) \to \mathbb{U}(\mathfrak{g} \oplus \mathfrak{h})$ defines a morphism of unitary associative algebras. We readily see that our morphism preserves counits and coproducts too (since this is so on the Lie algebras which generate our unitary algebra tensor product). Accordingly, our pairing defines an isomorphism of Hopf algebras.

In the formalism of symmetric monoidal categories, the result of Lemma 7.2.21 implies:

PROPOSITION 7.2.23. *The enveloping algebra functor* $\mathbb{U} : \mathcal{L}ie \to \mathcal{H}opf\mathcal{A}lg$ *is symmetric monoidal in the sense that:*

(1) *in the case of the zero object* 0*, viewed as the unit object of the category of Lie algebras, we have an obvious identity* $\mathbb{U}(0) = \mathbb{1}$*;*
(2) *in the case of a direct sum of Lie algebras* $\mathfrak{g} \oplus \mathfrak{h}$*, the pairing of* §7.2.20 *defines a Hopf algebra isomorphism* $\mathbb{U}(\mathfrak{g}) \otimes \mathbb{U}(\mathfrak{h}) \xrightarrow{\simeq} \mathbb{U}(\mathfrak{g} \oplus \mathfrak{h})$*;*
(3) *and these comparison isomorphisms fulfill the unit, associativity and symmetry constraints of* §3.3.1.

PROOF. The statement of assertion (1) is obvious and we have already checked the result of assertion (2). The proof of the unit, associativity and symmetry constraints, claimed by assertion (3), follows from a straightforward inspection of definitions. □

7.3. Lie algebras and Hopf algebras in complete filtered modules

In this section, we examine the definition of Hopf algebras and the applications of the concepts of §§7.1-7.2 in the case where the ambient category consists of modules M equipped with a filtration $M = \mathtt{F}_0 M \supset \cdots \supset \mathtt{F}_s M \supset \cdots$ such that $M = \lim_s M/\mathtt{F}_s M$. We use the phrase '*complete filtered module*' to refer to such objects. We also use the name '*complete Hopf algebra*' to refer to a subcategory of Hopf algebras in complete filtered modules which satisfy an appropriate connectedness condition.

Our main purpose is to check that the main results of §7.2, about the relationship between Lie algebras and Hopf algebras, do work for complete Hopf algebras. We first check that the category of complete filtered modules forms an example of symmetric monoidal category which fulfills the requirements of the previous section §7.2. We revisit the definition of the adjunction between Lie algebras and Hopf algebras in the context of complete filtered modules afterwards. We are notably going to check that the universal algebra structures of §7.2, namely the free Lie algebra, the symmetric algebra, the tensor algebra and the enveloping algebras of Lie algebras, can be realized as completions of ordinary universal algebra structures when we pass from the category of plain modules to the category of complete filtered modules.

Throughout this section, we assume that our ground ring $\mathbb{k}$ is a field of characteristic zero. The assumption that $\mathbb{k}$ is a field ensures us that the tensor product of $\mathbb{k}$-modules preserves monomorphisms, kernels and finite limits. Let us mention that our constructions have generalizations in the context of complete modules over a complete local ring R, like power series ring $\mathbb{k}[[t]]$, which are naturally considered in problems of deformation theory (see for instance [137] for a general reference on this subject).

We explain the precise definition of the categories of filtered modules and of complete filtered modules which we use in this section first. We also explain the definition of a category of weight graded modules which we use as an auxiliary category in our constructions. We check that the category of filtered modules and the category of complete filtered modules form examples of $\mathbb{Q}$-additive symmetric monoidal categories in the sense considered in §7.2. We have a similar result for the category of weight graded modules.

7.3.1. *The category of filtered modules.* We call *filtered module* the structure defined by a module M equipped with a decreasing filtration of the form:

$$M = \mathtt{F}_0\, M \supset \cdots \supset \mathtt{F}_s\, M \supset \mathtt{F}_{s+1}\, M \supset \cdots .$$

We also say that a module morphism $f : M \to N$ is *filtration preserving* when we have $f(\mathtt{F}_s\, M) \subset \mathtt{F}_s\, N$, for all $s \in \mathbb{N}$. We use the notation $f\,\mathcal{M}od$ for the category formed by the filtered modules as objects together with the filtration preserving morphisms as morphisms.

We actually use this category $f\,\mathcal{M}od$ as an intermediate object in our definition of the category of complete filtered modules. We are also going to use the following observations in our constructions:

(a) The direct sum $\bigoplus_{\alpha\in\mathcal{I}} M_\alpha$ of filtered modules M_α, $\alpha \in \mathcal{I}$, inherits a canonical filtration, defined by the obvious formula $\mathtt{F}_s(\bigoplus_\alpha M_\alpha) = \bigoplus_\alpha \mathtt{F}_s(M_\alpha)$, and represents the coproduct of the objects M_α, $\alpha \in \mathcal{I}$, in the category of filtered modules. The category of filtered modules is obviously additive so that we have an identity between finite direct sums and cartesian products of filtered modules.

(b) A submodule $K \subset M$ of a filtered module M inherits a canonical filtration, defined by $\mathtt{F}_s\, K = K \cap \mathtt{F}_s\, M$. We call this filtration the induced filtration on K. We easily see that the kernel $K = \ker(f)$ of a filtration preserving morphism $f : M \to N$, where we assume $M, N \in f\,\mathcal{M}od$ and we equip this object K with the induced filtration, represents the kernel of the morphism f in the category of filtered modules.

(c) A quotient N/M of a filtered module N by a submodule M is also equipped with a canonical filtration, defined by $\mathtt{F}_s(N/M) = \mathtt{F}_s(N)/M \cap \mathtt{F}_s(N)$. We easily see that this quotient filtered module N/M represents the cokernel of the canonical embedding $i : M \hookrightarrow N$ in the category of filtered modules. In general, the cokernel of a morphism $f : M \to N$ in the category of filtered modules can be realized as the quotient filtered module $N/f(M)$, where we regard the image of our morphism $f(M) \subset N$ as a submodule of the codomain N.

The existence of coproducts and cokernels implies that the category of filtered modules has all colimits. Recall that, in an additive category, the coequalizer of a parallel pair (d_0, d_1) is identified with the cokernel of the difference $d_0 - d_1$.

The observations of (b) imply that the monomorphisms of the category of filtered modules are the filtration preserving morphisms $i : M \to N$ which are injective as module morphisms. In general, the preservation of filtrations by a morphism $i : M \to N$ is equivalent to the relation $i(\mathtt{F}_s\, M) \subset \mathtt{F}_s\, N \Leftrightarrow \mathtt{F}_s\, M \subset i^{-1}(\mathtt{F}_s\, N)$. We say that a monomorphism of filtered modules $i : M \to N$ is a *filtered module inclusion* and we write $i : M \hookrightarrow N$ when we have an equality $\mathtt{F}_s\, M = i^{-1}(\mathtt{F}_s\, N)$ so that we can identify the subobject M with a submodule of N equipped with the induced filtration of (b).

Note that a monomorphism of filtered modules is not necessarily a filtered module inclusion, and hence, is not necessarily a kernel in the category of filtered modules. This observation immediately proves that the category of filtered modules, though additive, fails to be abelian. Note also that a morphism of filtered modules which is bijective as a module morphism is not an isomorphism in the category of filtered modules in general. The isomorphisms of the category of filtered modules precisely consist of the filtration preserving morphisms $f : M \to N$ which are bijective as module morphisms and of which inverse bijection $f^{-1} : N \to M$ is also filtration preserving.

7.3.2. *Towers.* We immediately see that giving a filtration as in §7.3.1 amounts to giving a coaugmented tower of surjections:

$$M \twoheadrightarrow \cdots \twoheadrightarrow M/\,\mathtt{F}_{s+1}\,M \twoheadrightarrow M/\,\mathtt{F}_{s}\,M \twoheadrightarrow \cdots \twoheadrightarrow M/\,\mathtt{F}_{0}\,M = 0$$

since we have the identity $\mathtt{F}_s\,M = \ker(M \to q_{s-1}M)$ when we set $q_sM = M/\,\mathtt{F}_{s+1}\,M$ for all $s \geq -1$. We refer to the quotient object $q_sM = M/\,\mathtt{F}_{s+1}\,M$ as the *sth level* of the tower associated to the filtered module M. We also have an equivalence between the morphisms of filtered modules and the morphisms of coaugmented towers, which formally consist of the module morphisms $f : M \to N$ such that we have factorizations:

$$\begin{tikzcd}
M \arrow[r, two heads] \arrow[d, "f"'] & \cdots \arrow[r, two heads] & M/\,\mathtt{F}_{s+1}\,M \arrow[r, two heads] \arrow[d, dotted, "f_*"'] & M/\,\mathtt{F}_{s}\,M \arrow[r, two heads] \arrow[d, dotted, "f_*"'] & \cdots \arrow[r, two heads] & M/\,\mathtt{F}_{0}\,M \arrow[d, dotted, "f_*"'] \\
N \arrow[r, two heads] & \cdots \arrow[r, two heads] & N/\,\mathtt{F}_{s+1}\,N \arrow[r, two heads] & N/\,\mathtt{F}_{s}\,N \arrow[r, two heads] & \cdots \arrow[r, two heads] & N/\,\mathtt{F}_{0}\,N
\end{tikzcd}$$

when we consider the towers associated of our objects $M, N \in f\,\mathcal{M}od$. In this equivalence, the isomorphisms of the category of filtered modules correspond to the morphisms of coaugmented towers which form an isomorphism levelwise.

We mainly use the tower representation when we define the completion of filtered modules (in the next paragraph). In the tower representation, we can easily realize colimits by an obvious levelwise construction, and we can see that the filtration constructions of §7.3.1 match this process. For our purpose, we record the following observations:

(a) For a direct sum $\bigoplus_{\alpha\in\mathcal{I}} M_\alpha$ of filtered modules M_α, $\alpha \in \mathcal{I}$, we have an obvious identity:

$$\Big(\bigoplus_{\alpha\in\mathcal{I}} M_\alpha\Big)/\,\mathtt{F}_s\Big(\bigoplus_{\alpha\in\mathcal{I}} M_\alpha\Big) = \bigoplus_{\alpha\in\mathcal{I}}\big(M_\alpha/\,\mathtt{F}_s\,M_\alpha\big),$$

for each $s \in \mathbb{N}$.

(b) For a submodule $K \subset M$ of a filtered module M, which we equip with the induced filtration of §7.3.1(b), we have a tower identity $K/\,\mathtt{F}_s\,K = K/K \cap \mathtt{F}_s\,M$, and the inclusion $K \subset M$ induces an embedding

$$K/K \cap \mathtt{F}_s\,M \hookrightarrow K/\,\mathtt{F}_s\,M,$$

for each $s \in \mathbb{N}$. For the kernel $K = \ker(f)$ of a filtration preserving morphism $f : M \to N$, we have the relation

$$\ker(f)/\ker(f) \cap \mathtt{F}_s\,M = \ker(f_* : M/\,\mathtt{F}_s\,M \to N/\,\mathtt{F}_s\,N),$$

at each level $s \in \mathbb{N}$.

(c) For the quotient N/M of a filtered module N by a submodule M, where we consider the quotient filtration of §7.3.1(c), we have a short exact sequence

$$0 \to M/M \cap \mathtt{F}_s\,N \to N/\,\mathtt{F}_s\,N \to (N/M)/\,\mathtt{F}_s(N/M) \to 0,$$

for each $s \in \mathbb{N}$.
The proof of these observations reduces to easy verifications.

7.3.3. *Completions.* The *completion* of a filtered module M is the module $\hat{M}$ such that $\hat{M} = \lim_s M/\mathtt{F}_s M$. The quotient morphisms $q : M \to M/\mathtt{F}_s M$ lift to a canonical morphism $q : M \to \hat{M}$ with values in this limit $\hat{M} = \lim_s M/\mathtt{F}_s M$. The module $\hat{M}$ inherits a canonical filtration, defined by the kernels

$$\mathtt{F}_s \hat{M} = \ker(\hat{M} \to M/\mathtt{F}_s M),$$

where we consider the canonical morphisms $\hat{M} \to M/\mathtt{F}_s M$ associated to the limit $\hat{M} = \lim_s M/\mathtt{F}_s M$. The morphism $q : M \to \hat{M}$ is clearly filtration preserving. The mapping $M \mapsto \hat{M}$ obviously defines a functor on the category of filtered modules $f\,\mathcal{M}od$ and the morphism $q : M \to \hat{M}$ is obviously natural in $M \in f\,\mathcal{M}od$ too.

In the language of §7.3.2, setting $\mathtt{F}_s \hat{M} = \ker(\hat{M} \to M/\mathtt{F}_s M)$ amounts to providing the completed module $\hat{M}$ with the filtration associated to the tower

$$\hat{M} \twoheadrightarrow \cdots \twoheadrightarrow M/\mathtt{F}_{s+1} M \twoheadrightarrow M/\mathtt{F}_s M \twoheadrightarrow \cdots \twoheadrightarrow M/\mathtt{F}_0 M = 0$$

which we use in our definition of the object $\hat{M}$. We accordingly have an identity:

$$\hat{M}/\mathtt{F}_{s+1} \hat{M} = M/\mathtt{F}_{s+1} M,$$

for every $s \in \mathbb{N}$. This tower identity implies that the completion functor is idempotent in the sense that the canonical morphism $q : N \to \hat{N}$ associated to a completed module $N = \hat{M}$ is an isomorphism.

In general, we say that a filtered module M is *complete* when the associated morphism $q : M \to \hat{M}$ is an isomorphism (equivalently, when we have $M = \lim_s M/\mathtt{F}_s M$). The idempotence of the completion functor implies that the completion of a filtered module M gives a complete filtered module $\hat{M}$ naturally associated to M. This complete filtered module $\hat{M}$ is also universal in the sense that any filtration preserving morphism $f : M \to N$, where N is complete, admits a unique factorization

$$\begin{tikzcd}
M \arrow[rr, "f"] \arrow[dr, "q"'] & & N \\
& \hat{M} \arrow[ur, dotted, "\hat{f}"'] &
\end{tikzcd}$$

in the category of filtered modules (we take the image of the morphism f under the completion functor and we use the identity $N = \hat{N}$ to get this factorization).

7.3.4. *The category of complete filtered modules.* The category formed by the complete filtered modules as objects together with the filtration preserving morphisms as morphisms is denoted by $\hat{f}\,\mathcal{M}od$. The completion functor can be interpreted as a left adjoint of the obvious category embedding $i : \hat{f}\,\mathcal{M}od \hookrightarrow f\,\mathcal{M}od$.

In what follows, we use notation of the form $\widehat{\mathrm{colim}}_\alpha M_\alpha$, with a hat mark, to distinguish the colimit of a diagram in the category of complete filtered modules $M_\alpha \in \hat{f}\,\mathcal{M}od$, $\alpha \in \mathcal{J}$, from the colimit of this diagram $\mathrm{colim}_\alpha M_\alpha$ in the category of filtered modules $f\,\mathcal{M}od$. We also use the phrase '*complete colimit*' to distinguish the colimit $\widehat{\mathrm{colim}}_\alpha M_\alpha$ in the category of complete filtered modules $\hat{f}\,\mathcal{M}od$ from the '*ordinary colimit*' $\mathrm{colim}_\alpha M_\alpha$, which we form in the category of filtered modules $f\,\mathcal{M}od$.

The idempotence of the completion functor actually implies that the complete colimit $\widehat{\operatorname{colim}}_\alpha M_\alpha$ can be realized as a completion of the ordinary colimit $(\operatorname{colim}_\alpha M_\alpha)^\wedge$. This observation implies that the category of complete filtered modules has all colimits too. We more generally have the identity of complete filtered modules:

$$(\operatorname*{colim}_\alpha M_\alpha)^\wedge = \widehat{\operatorname{colim}}_\alpha \hat{M}_\alpha$$

for any diagram of filtered modules $M_\alpha \in f\mathcal{M}od$, $\alpha \in \mathcal{I}$, where we consider the complete colimit of the completion of our objects on the right-hand side.

For our purpose, we record the following assertions about the definition of particular instances of colimits and limits in the category of complete filtered modules (we rely on the observations of §7.3.2 and on general properties of limits for the verification of these claims):

(a) For the direct sum of a finite collection of filtered modules M_{α_i}, $i = 1, \dots, n$, we have an obvious relation

$$\big(M_{\alpha_1} \oplus \dots \oplus M_{\alpha_n}\big)^\wedge = \hat{M}_{\alpha_1} \oplus \dots \oplus \hat{M}_{\alpha_n}.$$

In the case of complete filtered modules $M_{\alpha_i} = \hat{M}_{\alpha_i}$ we deduce from this identity that the direct sum $M_{\alpha_1} \oplus \dots \oplus M_{\alpha_n}$ is complete, and we obtain, besides, that this direct sum represents the coproduct of the objects M_{α_i}, $i = 1, \dots, n$, in the category of complete filtered modules. The category of complete filtered modules is therefore additive (like the category of filtered modules). For the direct sum $\bigoplus_{\alpha\in\mathcal{I}} M_\alpha$ of a (possibly infinite) collection of filtered modules M_α, $\alpha \in \mathcal{I}$, the completion returns a complete filtered module $(\bigoplus_{\alpha\in\mathcal{I}} M_\alpha)^\wedge$ which represents the coproduct of the objects $\hat{M}_\alpha$ in the category of complete filtered modules. We can also use the notation $\widehat{\bigoplus}_{\alpha\in\mathcal{I}} M_\alpha$ (with the hat mark) to refer to these coproducts in the complete sense.

(b) Let $K \subset M$ be a submodule of a filtered module M, which we equip with the induced filtration of §7.3.1(b). The morphism of complete filtered modules $\hat{K} \to \hat{M}$ which extends the inclusion $i : K \hookrightarrow M$ is an inclusion of filtered modules. The kernel $\ker(f)$ of a filtration preserving morphism $f : M \to N$, equipped with the induced filtration, is automatically complete as soon as M and N are complete, and represents the kernel of the morphism f in the category of complete filtered modules. In general, we have the relation

$$\ker(f)^\wedge = \ker(\hat{f} : \hat{M} \to \hat{N}),$$

where $\hat{f} : \hat{M} \to \hat{N}$ is the morphism of complete filtered modules induced by our morphism $f : M \to N$.

(c) For the quotient N/M of a filtered module N by a submodule M, which we equip with the quotient filtration of §7.3.1(c), we have the relation

$$(N/M)^\wedge = \hat{N}/\hat{M},$$

where we use the observation of assertion (b) to identify the completion $\hat{M}$ with a submodule of the complete filtered module $\hat{N}$. In particular, the quotient of a complete filtered module $N = \hat{N}$ by a complete submodule $M = \hat{M}$ is automatically complete. The object $(N/M)^\wedge$ represents the cokernel of the inclusion $\hat{M} \hookrightarrow \hat{N}$ in the category of complete filtered modules. In general, the cokernel of a morphism $\hat{f} : \hat{M} \to \hat{N}$ in the category of complete filtered modules can be identified with the completion $(N/f(M))\hat{}$, where we regard the image of our morphism $f(M) \subset N$ as a submodule of the object N.

We easily see that the category of complete filtered modules, though additive, fails to be abelian (for the same reasons as the category of plain filtered modules). Nevertheless, we have an abelian category structure at the level of a category of weight graded modules, which we use as approximations of our complete filtered modules. We explain the definition of a weight graded modules in the next paragraph and we explain the definition of a weight graded module associated to a (complete) filtered module afterwards.

7.3.5. *The category of weight graded modules.* The category of *weight graded modules*, denoted by $w\,\mathcal{M}od$, consists of modules M which are equipped with a decomposition into a sum $M = \bigoplus_{s\in\mathbb{N}} M_s$ of *components of homogeneous weight* $M_s \in \mathcal{M}od$, $s \in \mathbb{N}$. The morphisms of the category of weight graded modules are the module morphisms $f : M \to N$ which preserve the weight decomposition of our objects in the sense that we have the relation $f(M_s) \subset N_s$, for each weight $s \in \mathbb{N}$. This definition of a weight graded module is obviously the same as the definition of a graded module in §4.4 (except that our weight grading decompositions are supposed to range over the set of non-negative integers). Nevertheless, the weight gradings which we consider in this section have a different nature than the gradings of §4.4, which we generally associate to homological constructions. This difference motivates us to introduce another category for these objects. We also use a symmetric monoidal structure for weight graded modules that differs from the symmetric monoidal structure of §4.4 on the category of graded modules (see §7.3.13).

The category of weight graded modules inherits both limits and colimits (realized componentwise) and is also clearly abelian (unlike the category of filtered modules and the category of filtered modules).

7.3.6. *The weight graded module associated to a filtered module.* To a filtered module $M \in f\,\mathcal{M}od$ we associate a weight graded module $\mathtt{E}^0\,M \in w\,\mathcal{M}od$ with the subquotients

$$\mathtt{E}^0_s\,M = \mathtt{F}_s\,M/\,\mathtt{F}_{s+1}\,M, \quad s \in \mathbb{N},$$

as homogeneous components. The mapping $\mathtt{E}^0 : M \mapsto \mathtt{E}^0\,M$ defines a functor from the category of filtered modules $f\,\mathcal{M}od$ to the category of weight graded modules $w\,\mathcal{M}od$. In what follows, we also consider the obvious restriction of this functor to the category of complete filtered modules $\hat{f}\,\mathcal{M}od$.

The subquotients $\mathtt{E}^0_s\,M$ can also be defined in terms of the tower associated to M. We explicitly have:

$$\mathtt{E}^0_s\,M = \ker(M/\,\mathtt{F}_{s+1}\,M \to M/\,\mathtt{F}_s\,M), \quad \text{for any } s \in \mathbb{N}.$$

For the completion of a filtered module $\hat{M}$, we immediately deduce from this representation that we have the relation:

$$\mathtt{E}^0_s\,\hat{M} = \mathtt{E}^0_s\,M,$$

for every $s \in \mathbb{N}$.

The following easy statement motivates the introduction of weight graded modules for the study of complete filtered modules:

PROPOSITION 7.3.7. *A morphism of complete filtered modules defines an isomorphism $f : M \xrightarrow{\simeq} N$ (in the category of complete filtered modules) if and only if* *the morphism of weight graded modules which we associate to this morphism forms an isomorphism* $\mathtt{E}^0 f : \mathtt{E}^0 M \xrightarrow{\simeq} \mathtt{E}^0 N$ *(in the category of weight graded modules).*

Explanations and proof. Recall that we consider the categorical notion of isomorphism in the category of (complete) filtered modules (see §7.3.1), and a morphism of complete filtered modules $f : M \to N$ is an isomorphism in this sense if and only if this morphism induces an isomorphism levelwise on the tower decomposition of the objects $M, N \in \hat{f}\,\mathcal{M}od$ (see §7.3.2). The morphism of weight graded modules $\mathtt{E}^0 f : \mathtt{E}^0 M \to \mathtt{E}^0 N$, on the other hand, forms an isomorphism in the category of weight graded modules if and only if this morphism defines an isomorphism componentwise.

The "only if" part of the proposition follows from the functoriality of the map $\mathtt{E}^0 : f \mapsto \mathtt{E}^0 f$. We therefore focus on the proof of the "if" part. The definition $\mathtt{E}^0_s M = \ker(M/\mathtt{F}_{s+1} M \to M/\mathtt{F}_s M)$ implies that the modules of homogeneous weight $\mathtt{E}^0_s M$ fit in short exact sequences

$$0 \to \mathtt{E}^0_s M \to M/\mathtt{F}_{s+1} M \to M/\mathtt{F}_s M \to 0,$$

for all $s \in \mathbb{N}$. From these exact sequences, we obtain by induction that a morphism of filtered modules $f : M \to N$ induces an isomorphism at each level s of the towers associated to our modules as soon as the morphism of weight graded modules $\mathtt{E}^0 f : \mathtt{E}^0 M \to \mathtt{E}^0 N$ is an isomorphism. The proposition follows. □

In subsequent applications, we combine the result of this proposition with the following observations:

Proposition 7.3.8. *The mapping* $\mathtt{E}^0 : M \mapsto \mathtt{E}^0 M$ *preserves the categorical operations considered in* §7.3.1*(*a-c*). To be explicit, we have the following assertions:*

(a) *For a direct sum* $\bigoplus_\alpha M_\alpha$ *of (complete) filtered modules* M_α*, we have the obvious relation*

$$\mathtt{E}^0(\bigoplus_\alpha M_\alpha) = \bigoplus_\alpha \mathtt{E}^0 M_\alpha.$$

(b) *For the kernel* $K = \ker(f : M \to N)$ *of a filtration preserving morphism* $f : M \to N$*, where* M *and* N *are (complete) filtered modules, we have the relation*

$$\mathtt{E}^0 \ker(f : M \to N) = \ker(\mathtt{E}^0 f : \mathtt{E}^0 M \to \mathtt{E}^0 N).$$

(c) *For a submodule* $M \subset N$ *of a filtered module* N*, which we equip with the induced filtration of* §7.3.1*(*b*), the weight graded module* $\mathtt{E}^0 M$ *associated to* M *embeds into* $\mathtt{E}^0 N$*, and we have a short exact sequence*

$$0 \to \mathtt{E}^0 M \to \mathtt{E}^0 N \to \mathtt{E}^0(N/M) \to 0$$

which identifies $\mathtt{E}^0 N/\mathtt{E}^0 M$ *with the weight graded module* $\mathtt{E}^0(N/M)$*, where the module* N/M *is equipped with the quotient filtration of* §7.3.1*(*c*).*

Proof. The proof of this proposition reduces to easy verifications by using the observations of §7.3.2 on the definition of direct sums, kernels and quotient objects in the category of complete filtered modules. □

7.3.9. *The tensor product of filtered modules.* The tensor product $M \otimes N$ of filtered modules $M, N \in f\,\mathcal{M}od$ inherits a canonical filtration, which we define by:

$$\mathtt{F}_r(M \otimes N) = \sum_{s+t=r} \mathtt{F}_s(M) \otimes \mathtt{F}_t(N) \subset M \otimes N, \quad \text{for each } r \in \mathbb{N}.$$

The category of filtered modules is therefore equipped with a natural tensor product operation.

The ground field $\Bbbk$, equipped with the filtration such that $\mathtt{F}_0\,\Bbbk = \Bbbk$ and $\mathtt{F}_s\,\Bbbk = 0$ for $s > 0$, forms a unit for this tensor product. We readily check that the associativity isomorphism of the tensor product of $\Bbbk$-modules $(K \otimes L) \otimes M \simeq K \otimes (L \otimes M)$ preserves filtrations, and hence, defines an associativity isomorphism in the category of filtered modules. We similarly have a symmetry isomorphism $M \otimes N \simeq N \otimes M$ inherited from the base category of $\Bbbk$-modules. Thus we have a full symmetric monoidal structure on the category of filtered modules. We readily see, moreover, that the tensor product of filtered modules distributes over the direct sum operation of §7.3.1(a), over the cokernel operation of §7.3.1(c), and as a consequence over all colimits, as we require in §0.9(a).

The tensor product of complete filtered modules is not complete in general, but we can lift the symmetric monoidal structure of the category of filtered modules to the category of complete filtered modules by using our completion functor. We explicitly set:

$$M \hat{\otimes} N = \lim_r (M \otimes N)/\,\mathtt{F}_r(M \otimes N),$$

for any $M, N \in \hat{f}\,\mathcal{M}od$, in order to get a tensor product operation $\hat{\otimes}$ on the category of complete filtered modules $\hat{f}\,\mathcal{M}od$. We aim to establish that the category of complete filtered modules, equipped with this completed tensor product, is symmetric monoidal. We rely on the following observation:

Lemma 7.3.10. *The natural morphism*

$$\bigoplus_{s+t=r} \underbrace{\mathtt{F}_s\,M/\,\mathtt{F}_{s+1}\,M}_{=\mathtt{E}^0_s\,M} \otimes \underbrace{\mathtt{F}_t\,N/\,\mathtt{F}_{t+1}\,N}_{=\mathtt{E}^0_t\,N} \\ \to \underbrace{\sum_{s+t=r} \mathtt{F}_s(M) \otimes \mathtt{F}_t(N)/ \sum_{s+t=r+1} \mathtt{F}_s(M) \otimes \mathtt{F}_t(N)}_{=\mathtt{E}^0_r(M \otimes N)}$$

is an isomorphism.

Proof. The proof of this lemma reduces to an elementary exercise of linear algebra. □

This lemma gives our main argument in the proof of the following proposition:

Proposition 7.3.11. *The canonical morphism $M \otimes N \to \hat{M} \otimes \hat{N} \to \hat{M} \hat{\otimes} \hat{N}$, defined for any pair of filtered modules $M, N \in f\,\mathcal{M}od$, extends to an isomorphism*

$$(M \otimes N)^{\wedge} \xrightarrow{\simeq} \hat{M} \hat{\otimes} \hat{N}$$

in the category of complete filtered modules.

Proof. Lemma 7.3.10 implies that we have the identity

$$\mathtt{E}^0_r(M \otimes N)^{\wedge} = \mathtt{E}^0_r(M \otimes N) = \textstyle\bigoplus_{s+t=r} \mathtt{E}^0_s\,M \otimes \mathtt{E}^0_t\,N$$

as well as

$$\mathtt{E}^0_r(\hat{M}\hat{\otimes}\hat{N}) = \mathtt{E}^0_r(\hat{M}\otimes\hat{N}) = \bigoplus\nolimits_{s+t=r}\mathtt{E}^0_s\hat{M}\otimes\mathtt{E}^0_t\hat{N} = \bigoplus\nolimits_{s+t=r}\mathtt{E}^0_s M\otimes\mathtt{E}^0_t N,$$

for every $r \in \mathbb{N}$. Besides, we immediately see that the morphism of the proposition $(M\otimes N)\hat{} \to \hat{M}\hat{\otimes}\hat{N}$ induces the identity morphism at the level of these weight graded modules. Proposition 7.3.7 immediately implies, therefore, that this morphism is an isomorphism. ☐

In the next paragraphs §§7.3.12-7.3.13, we reinterpret these intermediate results as the definition of symmetric monoidal functors between the symmetric monoidal categories formed by the filtered modules, the complete filtered modules and the weight graded modules.

7.3.12. *The symmetric monoidal structure of the category of complete filtered modules.* We equip the category of complete filtered modules with the completed tensor product

$$M\hat{\otimes}N = \lim_r M\otimes N/\,\mathtt{F}_r(M\otimes N)$$

of §7.3.9. We see that the ground field, for which we have $\hat{\mathbb{k}} = \mathbb{k}$, also defines a unit for this tensor structure. Furthermore, we easily deduce from the result of Proposition 7.3.11 that the completed tensor product inherits an associativity isomorphism from the ordinary tensor product of filtered modules:

$$((K\hat{\otimes}L)\hat{\otimes}M) \simeq ((K\otimes L)\otimes M)\hat{}\simeq (K\otimes(L\otimes M))\hat{}\simeq (K\hat{\otimes}(L\hat{\otimes}M)).$$

We also have an obvious symmetry isomorphism $M\hat{\otimes}N \simeq N\hat{\otimes}M$ which is induced by the symmetry isomorphism of the category of filtered modules (we just use the functoriality of completions in this case). Thus the category of complete filtered modules $\hat{f}\,\mathcal{M}od$, equipped with our completed tensor product $\hat{\otimes}$, has a full symmetric monoidal category structure. We immediately check, moreover, that the tensor product of complete filtered modules distributes over colimits, as we require in §0.9(a), since this is so in the category of filtered modules (see §7.3.9), the colimits of diagrams of complete filtered modules are given by the completions of their counterparts in the category of filtered modules (see §7.3.4), and our completed tensor product commutes with the completion operation (by Proposition 7.3.11).

The result of Proposition 7.3.11 and the definition of our symmetric monoidal structure on complete filtered modules implies that the completion functor $(-)\hat{}$: $f\,\mathcal{M}od \to \hat{f}\,\mathcal{M}od$ is symmetric monoidal.

7.3.13. *The symmetric monoidal structure of the category of weight graded modules.* We already briefly mentioned that the category of weight graded modules inherits a symmetric monoidal structure as well. We make the definition of this symmetric monoidal structure explicit in this paragraph. We use that the tensor product of weight graded modules $M, N \in w\,\mathcal{M}od$ inherits a canonical weight decomposition $M\otimes N = \bigoplus_{r=0}^{\infty}(M\otimes N)_r$ where we set:

$$(M\otimes N)_r = \bigoplus_{s+t=r} M_s\otimes N_t,$$

for any $r \in \mathbb{N}$ (as in the case of the tensor product of graded modules in §4.4.1). We equip the category of weight graded modules with this tensor product bifunctor $\otimes : w\,\mathcal{M}od \times w\,\mathcal{M}od \to w\,\mathcal{M}od$. The ground ring $\mathbb{k}$, regarded as a weight graded module of rank 1 concentrated in weight $r = 0$, represents a unit object for this tensor product, and the associativity isomorphism of the tensor product of $\mathbb{k}$-modules defines an associativity isomorphism for this tensor product of weight graded modules yet. We also consider the obvious symmetry isomorphism $M \otimes N \simeq N \otimes M$, defined by the plain symmetry isomorphism of the category of $\Bbbk$-modules (with no sign involved), when $M, N \in w\,\mathcal{M}od$ (while we follow the sign rule of differential graded algebra to define the symmetry isomorphism of the symmetric monoidal category of graded modules). The tensor product of weight graded modules clearly distributes over colimits too.

By Lemma 7.3.10, we have an isomorphism of weight-graded module

$$\mathtt{E}^0(M \otimes N) \simeq \mathtt{E}^0(M) \otimes \mathtt{E}^0(N),$$

for any $M, N \in f\,\mathcal{M}od$. We have an analogous result in the case of the completed tensor product since the definition of the tensor product $M \hat{\otimes} N$ as a completion implies $\mathtt{E}^0(M \hat{\otimes} N) = \mathtt{E}^0(M \otimes N)$ (see §7.3.6). The mapping $\mathtt{E}^0 : M \mapsto \mathtt{E}^0(M)$ clearly preserves unit objects. Furthermore, our isomorphism $\mathtt{E}^0(M \otimes N) \simeq \mathtt{E}^0(M) \otimes \mathtt{E}^0(N)$ satisfies the coherence constraints of a symmetric monoidal transformation (see §3.3.1). In particular, we easily see that the isomorphism $\mathtt{E}^0(M \otimes N) \simeq \mathtt{E}^0(N \otimes M)$ induced by the symmetry isomorphism of filtered modules is carried to the symmetry isomorphism of the category of weight graded modules $\mathtt{E}^0(M) \otimes \mathtt{E}^0(N) \simeq \mathtt{E}^0(N) \otimes \mathtt{E}^0(M)$ (of which definition was actually motivated by this correspondence) when we apply our symmetric monoidal transformation. Thus, the mapping $\mathtt{E}^0 : M \mapsto \mathtt{E}^0(M)$ defines a symmetric monoidal functor from the category of filtered modules (respectively, the category of complete filtered modules) to the category of weight graded modules.

7.3.14. *Hopf algebras in filtered, complete and weight graded modules.* The definition of the symmetric monoidal structures of the previous paragraphs enables us to apply the concepts of the previous sections §§7.1-7.2 to the category of filtered modules, to the category of complete filtered modules, and to the category of weight graded modules. In particular, we can define Hopf algebras in any of these categories.

We readily see that defining a Hopf algebra in the category of weight graded modules amounts to giving a weight graded module H equipped with a Hopf algebra structure (in the ordinary sense) such that the unit $\eta : \Bbbk \to H$, the product $\mu : H \otimes H \to H$, the counit $\epsilon : H \to \Bbbk$, the coproduct $\Delta : H \to H \otimes H$, and the antipode map $\sigma : H \to H$ are weight preserving morphisms. We similarly see that defining a Hopf algebra in the category of filtered modules H amounts to giving a filtered module H equipped with a Hopf algebra structure (in the ordinary sense) such that the unit $\eta : \Bbbk \to H$, the product $\mu : H \otimes H \to H$, the counit $\epsilon : H \to \Bbbk$, the coproduct $\Delta : H \to H \otimes H$, and the antipode map $\sigma : H \to H$ are filtration preserving morphisms. In the context of complete filtered modules, we have to replace the plain tensor product by the completed tensor product $\hat{\otimes}$ in the definition of a Hopf algebra. The product can still be composed with the canonical morphism $H \otimes H \to H \hat{\otimes} H$ (associated to our completion) to give an ordinary product on the Hopf algebra H (we go back to this observation in §7.3.21), but the coproduct $\Delta : H \to H \hat{\otimes} H$ does not factor through the ordinary tensor product in general, and hence, is not equivalent to an ordinary coproduct.

The preservation of symmetric monoidal structures implies that the filtration subquotient functor $\mathtt{E}^0 : M \mapsto \mathtt{E}^0 M$ maps a Hopf algebra in filtered modules (respectively, in complete filtered modules) to a Hopf algebra in weight graded modules. The completion functor $(-)\hat{}\colon M \mapsto \hat{M}$ similarly maps a Hopf algebra in filtered modules to a Hopf algebra in complete filtered modules.

7.3.15. *Connected weight graded Hopf algebras and complete Hopf algebras.* In order to follow usual conventions (see [145]), we say that a *weight graded Hopf algebra* (for a Hopf algebra in weight graded modules) is *connected* when we have $H_0 = \Bbbk$. In this case, the unit morphism of our Hopf algebra $\eta : \Bbbk \to H$ (respectively, the counit morphism $\epsilon : H \to \Bbbk$) is necessarily given by the identity morphism between the ground field $\Bbbk$ and the module $H_0 = \Bbbk$. We use the notation $w\,\mathcal{H}opf\mathcal{A}lg$ for the category formed by the connected weight graded Hopf algebras.

In the filtered module context, we analogously say that a *filtered Hopf algebra* (for a Hopf algebra in filtered modules) is *connected* when we have $\mathtt{E}^0_0 H = H/\mathtt{F}_1 H = \Bbbk$, and we use the notation $f\,\mathcal{H}opf\mathcal{A}lg$ for this subcategory of the category of Hopf algebras in filtered modules. In the case of Hopf algebras in complete filtered modules, we suppose that the requirement $\mathtt{E}^0_0 H = \Bbbk$ is satisfied in all applications. Therefore we only use the name '*complete Hopf algebra*' to refer to Hopf algebras in complete filtered modules which satisfy our connectedness condition $\mathtt{E}^0_0 H = \Bbbk$. We also use the notation $\hat{f}\,\mathcal{H}opf\mathcal{A}lg$ (with no further precision) for this subcategory of complete Hopf algebras.

The requirement $H/\mathtt{F}_1 H = \Bbbk$ implies that our category of connected filtered (respectively, complete) Hopf algebras represents the preimage of the category of connected weight graded Hopf algebras under the mapping $\mathtt{E}^0 : H \mapsto \mathtt{E}^0 H$. We accordingly have a diagram of functors

$$\begin{tikzcd}
f\,\mathcal{H}opf\mathcal{A}lg \arrow[rr, "(-)\hat{}"] \arrow[dr, "\mathtt{E}^0"'] & & \hat{f}\,\mathcal{H}opf\mathcal{A}lg \arrow[dl, "\mathtt{E}^0"] \\
& w\,\mathcal{H}opf\mathcal{A}lg &
\end{tikzcd}$$

which summarizes the connections between our categories of Hopf algebras.

The following proposition motivates the introduction of the connectedness condition for weight graded Hopf algebras:

Proposition 7.3.16. *The connected weight graded Hopf algebras are locally conilpotent in the sense of the definition of* §7.2.15.

Proof. Let H be a connected weight graded Hopf algebra. We check that the conditions of local conilpotence hold for the objects $K^m = H_0 \oplus \cdots \oplus H_m$. We obviously have $\operatorname{colim}_m K^m = H$ and the homogeneity of the coproduct implies the inclusion relation $\Delta(K^m) \subset \sum_{p+q=m} K^p \otimes K^q$. We are therefore left to checking the vanishing condition $n > m \Rightarrow \pi^{(n)}\Delta^{(n)}(K^m) = 0$.

Recall that the morphism $\pi^{(n)}\Delta^{(n)}$ represents the components of the n-fold coproduct $\Delta^{(n)} : H \to H^{\otimes n}$ on the summand $\mathbb{I}(H)^{\otimes n} \subset H^{\otimes n}$. In the case of a connected weight graded Hopf algebra, for which we have $H_0 = \Bbbk$, the augmentation ideal $\mathbb{I}(H) = \ker(\epsilon : H \to \Bbbk)$ is identified with the sum $\mathbb{I}(H) = \bigoplus_{r>0} H_r$. The reduced n-fold coproduct $\pi^{(n)}\Delta^{(n)}$ is equivalently defined by dropping all terms which involve at least one unit factor $1 \in H_0$ in the expansion of the n-fold coproduct of an element $u \in H$. The preservation of the grading by the coproduct implies

$$\pi^{(n)}\Delta^{(n)}(H_r) \subset \bigoplus_{\substack{r_1+\dots+r_n=r\\ r_1,\dots,r_n>0}} H_{r_1}\otimes\dots\otimes H_{r_n},$$

and we accordingly have $n > r \Rightarrow \pi^{(n)}\Delta^{(n)}(H_r) = 0$. This observation finishes the proof of our statement. ☐

7.3.17. *Weight graded Lie algebras and Hopf algebras.* We can formally apply the definitions and the constructions of §7.2 to the category of weight graded modules $\mathcal{M} = w\,\mathcal{M}od$ since this symmetric monoidal category fulfills all our requirements, including the distribution relation of §0.9(a) with respect to colimits. We accordingly have a category of *weight graded Lie algebras* (for Lie algebras in weight graded modules), an enveloping algebra functor which assigns a weight graded Hopf algebra $\mathbb{U}(\mathfrak{g})$ to any weight graded Lie algebra $\mathfrak{g}$, as well as a primitive element functor $\mathbb{P}(H)$ which forms a right adjoint of this enveloping algebra functor on weight graded Lie algebras. We basically get all constructions by applying the definitions of §7.2 to the symmetric monoidal category of weight graded modules, where we consider the tensor product operation of §7.3.13. We also have, by the way, a weight graded version of the symmetric algebra $\mathbb{S}(M)$, of the tensor algebra $\mathbb{T}(M)$, and of the free Lie algebra $\mathbb{L}(M)$ of §§7.2.3-7.2.4.

We immediately get, from the general definition of a Lie algebra in a symmetric monoidal category, that a weight graded Lie algebra consists of a weight graded module $\mathfrak{g}$ together with a weight preserving morphism $\lambda : \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{g}$ which defines a Lie structure (in the ordinary sense) on the module $\mathfrak{g}$. The preservation of the weight grading is equivalent to the relation $[\mathfrak{g}_s, \mathfrak{g}_t] \subset \mathfrak{g}_{s+t}$, for all $s, t \in \mathbb{N}$, where we use the Lie bracket notation $[x, y] = \lambda(x \otimes y)$. We say that a weight graded Lie algebra is *connected* when we have $\mathfrak{g}_0 = 0$ and we use the notation $w\,\mathcal{L}ie$ for this subcategory of the category of weight graded Lie algebras. We readily see that the enveloping algebra functor maps a connected weight graded Lie algebra to a connected weight graded Hopf algebra (in the sense of §7.3.15) and we have a converse relation for the functor of primitive elements. We accordingly have adjoint functors between our subcategories of connected objects $\mathbb{U} : w\,\mathcal{L}ie \rightleftarrows w\,\mathcal{H}opf\mathcal{A}lg : \mathbb{P}$ and the result of Proposition 7.2.23, which asserts that the enveloping algebra functor is symmetric monoidal, holds in the weight graded setting. We also immediately see, by the way, that the symmetric algebra $R = \mathbb{S}(M)$ associated to a weight graded module $M \in w\,\mathcal{M}od$ is connected in the sense that we have the relation $R_0 = \Bbbk$ when $M_0 = 0$. We have a similar result $M_0 = 0 \Rightarrow \mathbb{T}(M)_0 = \Bbbk$ for the tensor algebra $R = \mathbb{T}(M)$, while we obtain $M_0 = 0 \Rightarrow \mathbb{L}(M)_0 = 0$ for the free Lie algebra $R = \mathbb{L}(M)$.

We record the following weight graded version of the main theorems of §7.2:

Theorem 7.3.18.

(a) *The result of Theorem* 7.2.16 *(the Structure Theorem of Hopf algebras) implies that we have an isomorphism of weight graded counitary cocommutative coalgebras*

$$e : \mathbb{S}\,\mathbb{P}(H) \xrightarrow{\simeq} H,$$

for any connected weight graded Hopf algebra $H \in w\,\mathcal{H}opf\mathcal{A}lg$.

(b) *The result of Theorem* 7.2.17 *(the Poincaré-Birkhoff-Witt Theorem) implies that we have an isomorphism of weight graded counitary cocommutative coalgebras*

$$e : \mathbb{S}(\mathfrak{g}) \xrightarrow{\simeq} \mathbb{U}(\mathfrak{g}),$$

for any connected weight graded Lie algebra $\mathfrak{g} \in w\,\mathcal{L}ie$.

(c) *The result of Theorem* 7.2.19 *(the Milnor-Moore Theorem) implies that the weight graded versions of the enveloping algebra functor* $\mathbb{U} : \mathfrak{g} \mapsto \mathbb{U}(\mathfrak{g})$ *and of the primitive element functor* $\mathbb{P} : H \mapsto \mathbb{P}(H)$ *induce adjoint equivalences of categories*

$$\mathbb{U} : w\,\mathcal{L}ie \rightleftarrows w\,\mathcal{H}opf\mathcal{A}lg : \mathbb{P}$$

between the category of connected weight graded Lie algebras $w\,\mathcal{L}ie$ *and the category of connected weight graded Hopf algebras* $w\,\mathcal{H}opf\mathcal{A}lg$.

The third assertion of this theorem is actually the original version of the Milnor-Moore theorem (see [145]).

Proof. These assertions are applications of the results of Theorem 7.2.16, Theorem 7.2.17 and Theorem 7.2.19 since we established in Proposition 7.3.16 that the connected weight graded Hopf algebras are locally conilpotent in the sense of §7.2.15. ☐

We now review the applications of the concepts of §7.2 to the category of complete filtered modules. We can formally apply the definitions and constructions of §7.2 in this context since we observed in §7.3.12 that the tensor product of complete filtered modules (and the tensor product of plain filtered modules similarly) fulfill all our requirements, including the distribution relation of §0.9(a) with respect to colimits. We use another approach in order to make these constructions more explicit. In the next paragraphs, we precisely check that the complete versions of the free Lie algebra, of the symmetric algebra, of the tensor algebra and of the enveloping algebra of a Lie algebra are identified with completions of their ordinary counterpart. We examine the definition of the structure of a Lie algebra first.

7.3.19. *Lie algebras in filtered modules and in complete filtered modules.* We immediately get, from the definition of §7.2.1, that the structure a filtered Lie algebra (for a Lie algebra in filtered modules) consists of a filtered module $\mathfrak{g}$ equipped with a Lie bracket $\lambda : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ which defines a Lie structure (in the ordinary sense) on $\mathfrak{g}$ and preserves filtration. We explicitly assume $[\mathtt{F}_s \mathfrak{g}, \mathtt{F}_t \mathfrak{g}] \subset \mathtt{F}_{s+t} \mathfrak{g}$, for all $s, t \in \mathbb{N}$, where we use the bracket notation $[x, y] = \lambda(x \otimes y)$. When we deal with Lie algebras in complete filtered modules, we assume that $\mathfrak{g}$ is a complete filtered module $\hat{\mathfrak{g}} = \mathfrak{g}$ and we replace the plain tensor product in the definition of the Lie bracket by the completed one. We immediately see that any such Lie bracket on the completion $\mathfrak{g} \hat{\otimes} \mathfrak{g} = (\mathfrak{g} \otimes \mathfrak{g})^{\wedge}$ arises as the extension of an ordinary filtration preserving Lie bracket on $\mathfrak{g}$:

$$\begin{tikzcd}
\mathfrak{g} \otimes \mathfrak{g} \arrow[rr, "\lambda"] \arrow[dr] & & \mathfrak{g} . \\
& \mathfrak{g} \hat{\otimes} \mathfrak{g} \arrow[ur, dotted, "\hat{\lambda}"'] &
\end{tikzcd}$$

Hence, a Lie algebra in complete filtered modules is equivalent to a filtered Lie algebra $\mathfrak{g}$ whose underlying filtered module is complete $\hat{\mathfrak{g}} = \mathfrak{g}$. This observation implies that we have an embedding of the category of Lie algebras in complete filtered modules into the category of Lie algebras in filtered modules. We see that this embedding is a right adjoint of the functor induced by the completion $(-)^\wedge\colon \mathfrak{g} \mapsto \hat{\mathfrak{g}}$. We simply use the preservation of symmetric monoidal structures by the completion functor (see §7.3.12) in order to check that the completion $\hat{\mathfrak{g}}$ of a Lie algebra in filtered modules $\mathfrak{g}$ inherits a Lie algebra structure.

7.3.20. *Connected filtered Lie algebras and complete Lie algebras.* We say that a filtered Lie algebra is *connected* when we have $\mathtt{E}_0^0\,\mathfrak{g} = \mathfrak{g}/\mathtt{F}_1\,\mathfrak{g} = 0$, so that the filtration of our Lie algebra has the form $\mathfrak{g} = \mathtt{F}_1\,\mathfrak{g} \supset \cdots \supset \mathtt{F}_s\,\mathfrak{g} \supset \cdots$. We use the notation $f\,\mathcal{L}ie$ for the subcategory of connected filtered algebras. In the case of Lie algebras in complete filtered modules, we suppose that the requirement $\mathtt{E}_0^0\,\mathfrak{g} = 0$ (equivalently, $\mathfrak{g} = \mathtt{F}_1\,\mathfrak{g}$) is satisfied in all applications. Therefore we only use the name '*complete Lie algebra*' to refer to Lie algebras in complete filtered modules which satisfy our connectedness condition $\mathtt{E}_0^0\,\mathfrak{g} = 0$. We also use the notation $\hat{f}\,\mathcal{L}ie$ (with no further precision) for this subcategory of complete Lie algebras.

The functor $\mathtt{E}^0 : M \mapsto \mathtt{E}^0\,M$ also preserves Lie algebra structures since we observed in §7.3.13 that this functor is symmetric monoidal in the sense of §3.3. The requirement $\mathtt{E}^0\,\mathfrak{g} = 0$ implies that our category of connected filtered (respectively, complete) Lie algebras represents the preimage of the category of connected weight graded Lie algebras of §7.3.17 under the functor $\mathtt{E}^0 : \mathfrak{g} \mapsto \mathtt{E}^0\,\mathfrak{g}$. We accordingly have a diagram of functors

$$\begin{tikzcd}
f\,\mathcal{L}ie \arrow[rr, shift left, "(-)^\wedge"] \arrow[dr, "\mathtt{E}^0"'] & & \hat{f}\,\mathcal{L}ie \arrow[ll, shift left] \arrow[dl, "\mathtt{E}^0"] \\
& w\,\mathcal{L}ie &
\end{tikzcd},$$

which summarizes the connections between our categories of lie algebras, and where the horizontal arrows are the embedding and completion functors of §7.3.19.

We now revisit the construction of the symmetric algebras, tensor algebras, and enveloping algebras of §7.2 in the setting of complete filtered modules. We then deal with unitary associative algebras and unitary commutative algebras in complete filtered modules. We examine the definition of these structures in a preliminary step. We also deal, for our purpose, with auxiliary categories of unitary associative algebras and of unitary commutative algebras in filtered modules.

7.3.21. *Unitary associative and unitary commutative algebras in complete filtered modules.* We immediately see (as in the Lie algebra case) that a unitary associative (respectively, commutative) algebra in filtered modules is equivalent to a filtered module A equipped with a unit morphism $\eta : \Bbbk \to A$ and with a product morphism $\mu : A \otimes A \to A$ which provide A with a unitary associative (respectively, commutative) structure (in the ordinary sense), and preserve filtrations (this condition is void for the unit since we assume $\mathtt{F}_1\,\Bbbk = 0$). We also readily see that a unitary associative (respectively, commutative) algebra in complete filtered modules is equivalent to a unitary associative (respectively, commutative) algebra in filtered modules A which is complete as a filtered module $A = \hat{A}$, because we have $\hat{\Bbbk} = \Bbbk$ and any product in the symmetric monoidal category of complete filtered modules $\hat{\mu} : A \hat{\otimes} A \to A$ arises as the extension of an ordinary product of filtered modules $\mu : A \otimes A \to A$.

We say that a unitary associative (respectively, commutative) algebra in filtered modules is *connected* when we have $\mathtt{E}^0_0 A = A/\mathtt{F}_1 A = \Bbbk$. When we work in the complete setting, we only use the phrase '*complete unitary associative (respectively, commutative) algebra* for the unitary associative (respectively, commutative) algebras in complete filtered modules which fulfill this connectedness requirement. We adopt the notation $f\,\mathcal{A}s_+$ (respectively, $f\,\mathcal{C}om_+$) for the subcategory of connected filtered unitary associative (respectively, commutative) algebras, and the notation $\hat{f}\,\mathcal{A}s_+$ (respectively, $\hat{f}\,\mathcal{C}om_+$) for the subcategory of complete unitary associative (respectively, commutative) algebras. We use similar conventions for unitary associative (respectively, commutative) algebras in weight graded modules and we have a diagram, similar to the functor diagram of §7.3.20, which summarizes the connections between these categories of algebras.

The observations of this paragraph imply that the completion functor $(-)\hat{}\colon A \mapsto \hat{A}$ induces a functor on filtered unitary associative (respectively, commutative) algebras. We readily see that this functor on filtered unitary associative (respectively, commutative) algebras $(-)\hat{}\colon A \mapsto \hat{A}$ defines a left adjoint of the category embedding $i : \hat{f}\,\mathcal{A}s_+ \hookrightarrow f\,\mathcal{A}s_+$ (respectively, $i : \hat{f}\,\mathcal{C}om_+ \hookrightarrow f\,\mathcal{C}om_+$).

7.3.22. *The completion of free algebras.* We can apply the general construction of §7.2.4 to get the definition of tensor (respectively, symmetric) algebras in the category of filtered modules (and in the category of complete filtered modules similarly). We can also use the general construction of §7.2.3 to get the definition of free Lie algebras. In all case, we simply replace the generic direct sums and tensor products of §7.2 by the coproduct and tensor product of our categories (the complete direct sum $\hat{\oplus}$ and the complete tensor product $\hat{\otimes}$ in the complete filtered module setting).

When we deal with plain filtered modules, we can identify the tensor algebra $\mathbb{T}(M)$ with the standard tensor algebra associated to the module M, which we equip with the filtration $\mathbb{T}(M) = \mathtt{F}_0\,\mathbb{T}(M) \supset \cdots \supset \mathtt{F}_s\,\mathbb{T}(M) \supset \cdots$ such that $\mathtt{F}_s\,\mathbb{T}(M) = \bigoplus_{r=0}^{\infty} \mathtt{F}_s(M^{\otimes r})$ and $\mathtt{F}_s(M^{\otimes r}) = \sum_{s_1+\cdots+s_r=s} \mathtt{F}_{s_1} M \otimes \cdots \otimes \mathtt{F}_{s_r} M$. We have a similar observation in the symmetric algebra and in the free Lie algebra case.

For a complete filtered module $M = \hat{M}$, we use the notation $\hat{\mathbb{T}}(M)$ to refer to the complete tensor algebra associated to M, and $\mathbb{T}(M)$ for the ordinary tensor algebra (formed in the category of filtered modules). We adopt an analogous notation $\hat{\mathbb{S}}(M)$ for the complete symmetric algebra associated to M, which we oppose to the ordinary symmetric algebra $\mathbb{S}(M)$. We similarly set $\hat{\mathbb{L}}(M)$ for the complete free Lie algebra associated to M, and $\mathbb{L}(M)$ for the ordinary free Lie algebra in the category of filtered modules.

In fact, by using the adjunction between unitary associative (respectively, commutative) algebras in filtered modules and in complete filtered modules, we immediately get that the complete tensor (respectively, symmetric) algebra can be realized as the completion of the ordinary tensor (respectively, symmetric) algebra. We more generally have $\hat{\mathbb{T}}(\hat{M}) = \mathbb{T}(M)\hat{}$ for any filtered module $M \in f\,\mathcal{M}od$. We similarly have $\hat{\mathbb{S}}(\hat{M}) = \mathbb{S}(M)\hat{}$ in the case of the symmetric algebra, and $\hat{\mathbb{L}}(\hat{M}) = \mathbb{L}(M)\hat{}$ in the case of the free Lie algebra. We can use this relationship to get an explicit representation of the complete tensor algebra (respectively, of the complete symmetric algebra, of the complete free Lie algebra).

Let $R = \mathbb{S}(M)$ (respectively, $R = \mathbb{T}(M)$) denote the symmetric (respectively, tensor) algebra associated to a filtered module M. We immediately see that the counit $\epsilon : R \to \Bbbk$, the coproduct $\Delta : R \to R \otimes R$, and the antipode $\sigma : R \to R$ in the construction of Proposition 7.2.6 are filtration preserving morphisms and are identified with the canonical structure morphisms of our Hopf algebras $R = \mathbb{S}(M), \mathbb{T}(M)$ when we carry out the construction of Proposition 7.2.6 in the symmetric monoidal category of filtered modules. In the complete case, the counit $\hat{\epsilon} : \hat{R} \to \Bbbk$, the coproduct $\hat{\Delta} : \hat{R} \to \hat{R} \hat{\otimes} \hat{R}$ and the antipode $\hat{\sigma} : \hat{R} \to \hat{R}$, which define the Hopf algebra structure of the complete algebras $\hat{R} = \hat{\mathbb{S}}(M), \hat{\mathbb{T}}(M)$, are identified with the completion of these structure morphisms $\epsilon : R \to \Bbbk$, $\Delta : R \to R \otimes R$ and $\sigma : R \to R$ on the ordinary symmetric and tensor algebras $R = \mathbb{S}(M), \mathbb{T}(M)$.

7.3.23. *Connectedness assumptions and complete free algebras.* We usually apply our complete tensor algebra construction to complete filtered modules M such that $\mathtt{E}_0^0 M = 0 \Leftrightarrow M = \mathtt{F}_1 M$. We readily see that we have the implication $M = \mathtt{F}_1 M \Rightarrow \mathtt{E}_0^0 \hat{\mathbb{T}}(M) = \mathtt{E}_0^0 \mathbb{T}(M) = \Bbbk$ so that the complete tensor algebra $\hat{\mathbb{T}}(M)$ associated to a complete filtered modules M which satisfies this connectedness requirement $\mathtt{E}_0^0 M = 0$ forms a complete unitary associative algebra in the sense of §7.3.21. We have similar results in the symmetric algebra and free Lie algebra case.

In the case $\mathtt{E}_0^0 M = 0$, we moreover have

$$\hat{\mathbb{S}}(M) = \prod_{r=0}^{\infty} (M^{\hat{\otimes} r})_{\Sigma_r} \quad \text{and} \quad \hat{\mathbb{T}}(M) = \prod_{r=0}^{\infty} M^{\hat{\otimes} r},$$

because this condition $\mathtt{E}_0^0 M = 0 \Leftrightarrow M = \mathtt{F}_1 M$ implies that we have the inclusion relation $M^{\otimes r} = \mathtt{F}_r(M^{\otimes r}) \subset \mathtt{F}_r \mathbb{T}(M)$ for each $r \in \mathbb{N}$, and similarly in the symmetric algebra case. In other terms, the complete direct sums $\hat{\oplus}$, which we usually have in the expansion of the tensor and symmetric algebra, reduce to an ordinary product when the module M is connected. We also have

$$\hat{\mathbb{L}}(M) = \prod_{r=0}^{\infty} \hat{\mathbb{L}}_r(M)$$

in the free Lie algebra case, where we take the completion of the homogeneous summands of the ordinary free Lie algebra $\hat{\mathbb{L}}_r(M) = \mathbb{L}_r(M)\hat{} = (\mathsf{Lie}(r) \otimes M^{\hat{\otimes} r})_{\Sigma_r}$ in the expansion of §7.2.3.

7.3.24. *The completed enveloping algebras of Lie algebras, primitive elements and adjunctions.* We can readily extend our analysis of the construction of symmetric and tensor algebras to enveloping algebras.

When we deal with a Lie algebra in filtered modules $\mathfrak{g}$, we can provide the usual enveloping algebra associated to $\mathfrak{g}$ (as explicitly defined §7.2.9) with a canonical filtration so that this algebra $\mathbb{U}(\mathfrak{g})$ naturally forms a unitary associative algebra in the category of filtered modules and satisfies the adjunction relation of enveloping algebras (see §7.2.7) in this category. When the Lie algebra is complete $\hat{\mathfrak{g}} = \mathfrak{g}$, we adopt the notation $\hat{\mathbb{U}}(\mathfrak{g})$ for the enveloping algebra in complete filtered modules associated to $\mathfrak{g}$, as opposed to the ordinary enveloping algebra in filtered modules $\mathbb{U}(\mathfrak{g})$. We can actually identify the complete enveloping algebra $\hat{\mathbb{U}}(\mathfrak{g})$ with the completion of the ordinary enveloping algebra $\mathbb{U}(\mathfrak{g})$. We more generally have $\hat{\mathbb{U}}(\hat{\mathfrak{g}}) = \mathbb{U}(\mathfrak{g})\hat{}$ for any Lie algebra in filtered modules $\mathfrak{g}$.

We moreover easily see that the counit $\hat{\epsilon} : \hat{\mathbb{U}}(\mathfrak{g}) \to \Bbbk$, the coproduct $\hat{\Delta} : \hat{\mathbb{U}}(\mathfrak{g}) \to \hat{\mathbb{U}}(\mathfrak{g}) \hat{\otimes} \hat{\mathbb{U}}(\mathfrak{g})$ and the antipode $\hat{\sigma} : \hat{\mathbb{U}}(\mathfrak{g}) \to \hat{\mathbb{U}}(\mathfrak{g})$, which define the Hopf algebra structure of the enveloping algebra in the complete case, are identified with the morphisms induced by the counit, coproduct and antipode of the ordinary enveloping algebra $\mathbb{U}(\mathfrak{g})$ on the completion. We have besides $\mathtt{E}^0_0\,\hat{\mathbb{U}}(\mathfrak{g}) = \Bbbk$ as soon as our Lie algebra satisfies $\mathtt{E}^0\,\mathfrak{g} = 0 \Leftrightarrow \mathfrak{g} = \mathtt{F}_1\,\mathfrak{g}$. Hence, we obtain that the complete enveloping algebra functor induces a functor from the category of complete Lie algebras $\hat{f}\,\mathcal{L}ie$ towards the category of complete Hopf algebras $\hat{f}\,\mathcal{H}opf\mathcal{A}lg$:

$$\hat{\mathbb{U}} : \hat{f}\,\mathcal{L}ie \to \hat{f}\,\mathcal{H}opf\mathcal{A}lg\,.$$

In the converse direction, the image of a Hopf algebra H under the primitive element functor $\mathbb{P} : H \mapsto \mathbb{P}(H)$ is defined, in the complete case, as the submodule such that:

$$\mathbb{P}(H) = \{x \in H | \epsilon(x) = 0, \Delta(x) = x \hat{\otimes} 1 + 1 \hat{\otimes} x\},$$

and which we equip with the induced filtration of §7.3.1(b). Recall that this object is complete and is identified with the appropriate kernel in the category of complete filtered modules (see §7.3.4). In the case of a complete Hopf algebra, we moreover have $\mathtt{E}^0_0\,H = \Bbbk \Rightarrow \mathtt{E}^0_0\,\mathbb{P}(H) = 0$ so that the mapping $\mathbb{P} : H \mapsto \mathbb{P}(H)$ yields a functor from the category of complete Hopf algebras $\hat{f}\,\mathcal{H}opf\mathcal{A}lg$ towards the category of complete Lie algebras $\hat{f}\,\mathcal{L}ie$:

$$\mathbb{P} : \hat{f}\,\mathcal{H}opf\mathcal{A}lg \to \hat{f}\,\mathcal{L}ie\,.$$

The adjunction of Proposition 7.2.13 between the enveloping algebra functor and the primitive element functor also holds in the complete context.

The results of Proposition 7.2.14 also hold in the category of complete filtered modules since this category fits the assumptions of §7.2. Thus, we have:

$$\mathbb{P}\,\hat{\mathbb{T}}(M) = \hat{\mathbb{L}}(M).$$

In the rest of this section, we check the analogue of the structure theorems of §7.2 for complete Hopf algebras. To start with, we observe that:

Theorem 7.3.25. *In the complete setting, the symmetrization morphism of Theorem* 7.2.16 *(the Structure Theorem of Hopf algebras) gives an isomorphism of counitary cocommutative coalgebras*

$$e : \hat{\mathbb{S}}\,\mathbb{P}(H) \xrightarrow{\simeq} H,$$

for any complete Hopf algebra $H \in \hat{f}\,\mathcal{H}opf\mathcal{A}lg$ *(which satisfies the connectedness requirement* $H/\,\mathtt{F}_1\,H = \Bbbk$ *of our definition).*

Explanation and proof. In §7.2, we use a local conilpotence condition to establish this statement in a general context. Recall that this assumption is essentially used to have a limit decomposition of the endomorphism algebra $\mathtt{End}(H)$ and to give a sense to formal sums in this object.

In the case of a complete Hopf algebra H, we rather use the relation $H = \lim_s H/\,\mathtt{F}_s\,H$ to get a limit decomposition at the level of hom-objects $\mathtt{Hom}(-,H) = \lim_s \mathtt{Hom}(-,H/\,\mathtt{F}_s\,H)$, and we can similarly take the decomposition $\mathtt{Hom}(-,H\hat{\otimes}H) = \lim_s \mathtt{Hom}(-,H\otimes H/\,\mathtt{F}_s(H\otimes H))$ when we have to deal with hom-objects with a tensor product as target object. We easily check that the proof of Theorem 7.2.16 works same when we take this limit decomposition instead of the one considered in §7.2. We precisely use the connectedness condition $H/\,\mathtt{F}_1\,H = \Bbbk$ to give a sense to our formal sums $\sum_n \lambda_n \pi^n$. We therefore get a version of the result of Theorem 7.2.16 for complete Hopf algebras, as claimed in the present theorem. (Just note that we have to take the complete direct sums of §7.3.4(a) instead of the ordinary direct sums when we work in the category of complete filtered modules.) ☐

This structure theorem is completed by the following analogues of the Poincaré-Birkhoff-Witt and Milnor-Moore theorems:

Theorem 7.3.26.

(a) *In the complete setting, the symmetrization morphism of Theorem* 7.2.17 *(the Poincaré-Birkhoff-Witt Theorem) gives an isomorphism of counitary cocommutative coalgebras*

$$e : \hat{\mathbb{S}}(\mathfrak{g}) \xrightarrow{\simeq} \hat{\mathbb{U}}(\mathfrak{g}),$$

for any complete Lie algebra $\mathfrak{g} \in \hat{f}\,\mathcal{L}ie$.

(b) *In the complete setting, the result of Theorem* 7.2.19 *(the Milnor-Moore Theorem) implies that the complete enveloping algebra functor* $\hat{\mathbb{U}} : \mathfrak{g} \mapsto \hat{\mathbb{U}}(\mathfrak{g})$ *and the primitive element functor* $\mathbb{P} : H \mapsto \mathbb{P}(H)$ *define adjoint equivalences of categories*

$$\hat{\mathbb{U}} : \hat{f}\,\mathcal{L}ie \rightleftarrows \hat{f}\,\mathcal{H}opf\mathcal{A}lg : \mathbb{P}$$

between the category of complete Lie algebras $\hat{f}\,\mathcal{L}ie$ *and the category of complete Hopf algebras* $\hat{f}\,\mathcal{H}opf\mathcal{A}lg$.

Proof. The claim of assertion (a) follows from the application of the statement of Theorem 7.2.17 in the symmetric monoidal category of complete filtered modules (just recall that our result remains valid in the general setting $\mathbb{Q}$-additive symmetric monoidal categories). The argument line of Theorem 7.2.19 does work and yields a proof of assertion (b) as soon as we have the results of the Structure Theorem and of the Poincaré-Birkhoff-Witt Theorem. In the case of complete filtered modules, these results are provided by the statements of Theorem 7.3.25 and Theorem 7.3.26(a). □

For the sake of completeness, we record the following relationship between the complete version and the weight graded version of our functors:

Proposition 7.3.27.

(a) *For any complete filtered module* $M \in \hat{f}\,\mathcal{M}od$, *we have identities* $\mathtt{E}^0\,\hat{\mathbb{S}}(M) = \mathbb{S}(\mathtt{E}^0\,M)$, $\mathtt{E}^0\,\hat{\mathbb{T}}(M) = \mathbb{T}(\mathtt{E}^0\,M)$, *and* $\mathtt{E}^0\,\hat{\mathbb{L}}(M) = \mathbb{L}(\mathtt{E}^0\,M)$.

(b) *For a complete Hopf algebra* $H \in \hat{f}\,\mathcal{H}opf\mathcal{A}lg$, *we have* $\mathtt{E}^0\,\mathbb{P}(H) = \mathbb{P}(\mathtt{E}^0\,H)$.

(c) *For a complete Lie algebra* $\mathfrak{g} \in \hat{f}\,\mathcal{L}ie$, *we have* $\mathtt{E}^0\,\hat{\mathbb{U}}(\mathfrak{g}) = \mathbb{U}(\mathtt{E}^0\,\mathfrak{g})$.

Proof. In fact, we prove that we have the relations $\mathtt{E}^0\,\mathbb{S}(M) = \mathbb{S}(\mathtt{E}^0\,M)$ and $\mathtt{E}^0\,\mathbb{T}(M) = \mathbb{T}(\mathtt{E}^0\,M)$ at the level of plain filtered modules, and we use the general identity $\mathtt{E}^0(-)\hat{}= \mathtt{E}^0(-)$ to get the complete case of these identities, as asserted in our proposition.

The identity $\mathtt{E}^0\,\mathbb{T}(M) = \mathbb{T}(\mathtt{E}^0\,M)$ follows from the preservation of the tensor product (see §7.3.13) and from the preservation of direct sums (see Proposition 7.3.8) by the filtration subquotient functor $\mathtt{E}^0 : f\,\mathcal{M}od \to w\,\mathcal{M}od$. In any $\mathbb{Q}$-additive symmetric monoidal category, the quotient map $\mathbb{T}_r(M) \to \mathbb{S}_r(M)$ from the tensor product $\mathbb{T}_r(M) = M^{\otimes r}$ to the symmetric tensor product $\mathbb{S}_r(M) = (M^{\otimes r})_{\Sigma_r}$ admits a natural section, for every $r \in \mathbb{N}$, which is defined by the symmetrization map $e(x_1 \cdot \dots \cdot x_r) = \sum_{\sigma \in \Sigma_r} x_{\sigma(1)} \otimes \dots \otimes x_{\sigma(r)}$. The functor $\mathtt{E}^0 : M \mapsto \mathtt{E}^0\,M$ preserves this retraction diagram (because $\mathtt{E}^0$ preserves the symmetry isomorphism of our symmetric monoidal structure). Therefore, we also have $\mathtt{E}^0\,\mathbb{S}_r(M) = \mathbb{S}_r(\mathtt{E}^0\,M)$ for the symmetric tensor functor $\mathbb{S}_r(-)$, and we conclude that we have $\mathtt{E}^0\,\mathbb{S}(M) = \mathbb{S}(\mathtt{E}^0\,M)$. The case of the free Lie algebra follows from the same argument line by using the observation, established in Proposition 7.2.8, that the free Lie algebra forms a natural retract of the tensor algebra.

The second and third assertions of the proposition readily follow from the identity $\mathtt{E}^0\,\hat{\mathbb{S}}(M) = \mathbb{S}(\mathtt{E}^0\,M)$ and from the result of the Structure Theorem of Hopf algebras in weight graded modules (Theorem 7.3.18) and in complete filtered modules (Theorem 7.3.25-7.3.26). $\square$

CHAPTER 8

The Malcev Completion for Groups

In this chapter, we explain the definition of a category of *Malcev complete groups*, which are groups endowed with power operations g^a whose exponents a can take values in any given ground field of characteristic zero $\Bbbk$.

The main idea of our approach is to consider a complete version of the group algebra functor of §7.1.13 and of the group-like element functor of §7.1.14. These functors fit in an adjunction relation:

$$\Bbbk[-]\hat{}\colon \mathcal{G}rp \rightleftarrows \hat{f}\,\mathcal{H}opf\mathcal{A}lg : \mathbb{G},$$

like the ordinary group algebra and group-like element functors. We define our category of Malcev complete groups $\hat{f}\,\mathcal{G}rp$ as the image of the category of complete Hopf algebras under the group-like element functor $\mathbb{G} : \hat{f}\,\mathcal{H}opf\mathcal{A}lg \to \mathcal{G}rp$ (see §8.2). We have an obvious Malcev complete group $\hat{G}$, associated to any group G, which is defined by the formula

$$\hat{G} = \mathbb{G}\,\Bbbk[G]\hat{},$$

where we take the composite of the group-like element functor and of the complete Hopf algebra functor of our adjunction relation.

The structure theorems of complete Hopf algebras imply that we have an equivalence between the category of Malcev complete groups $G = \mathbb{G}(H)$ and the category of complete Lie algebras $\mathfrak{g} = \mathbb{P}(H)$. We use this correspondence to get insights into the structure of Malcev complete groups. To be specific, we will see that we can use logarithm and exponential functions to get inverse bijections between the module of primitive elements and the set of group-like elements in any complete Hopf algebra. This observation implies that every element in a Malcev complete group G can be represented by an exponential $g = e^x$, where x belongs to the Lie algebra $\mathfrak{g}$ associated to G. The definition of general power operations g^a in G, where $a \in \Bbbk$, follows from this exponential representation. Indeed, for an element $g = e^x$, we can simply set $g^a = e^{ax}$.

We explain the definition of the complete group algebra associated to a group and the definition of our group-like element functor for complete Hopf algebras in the first section of the chapter §8.1. We also define our exponential correspondence between primitive and group-like elements in this first section. We devote the second section of the chapter §8.2 to the definition of the category of Malcev complete groups and we make explicit the definition of our Malcev completion functor in the third section §8.3.

We study the Malcev completion of free groups and of groups defined by generators and relations in the fourth section of the chapter §8.4. We then use the correspondence between Hopf algebras and Lie algebras to give an explicit description, in terms of commutator expansions, of elements in the Malcev completion of a free group. We devote a fifth section §8.5 to a study of the Malcev completion of semi-direct products of groups.

We assume throughout this chapter that the ground ring $\Bbbk$ is a field of characteristic 0. We consider the category of modules associated to this field $\mathcal{M}od$, and the corresponding category of complete filtered modules $\hat{f}\,\mathcal{M}od$ (see §7.3).

8.1. The adjunction between groups and complete Hopf algebras

The main purpose of this section is to define the complete versions of the group algebra functor and of the group-like element functor of §7.1. We use that any ordinary algebra H inherits a canonical filtration, in the sense of §7.3.1. We apply the completion process of §7.3.3 to the ordinary group algebra $\Bbbk[G]$, which we equip with this canonical filtration, in order to get the complete Hopf algebra $\Bbbk[G]^{\wedge}$associated to any group G. We explain the definition of our filtration on a Hopf algebra first.

8.1.1. *The canonical completion of a Hopf algebra.* Let H be any Hopf algebra in the category of $\Bbbk$-modules. Let $\mathbb{I}(H) = \ker(\epsilon : H \to \Bbbk)$ be the augmentation ideal of H. Recall that H admits a splitting $H = \Bbbk\,1 \oplus \mathbb{I}(H)$, where $1 \in H$ denotes the unit of this algebra H. We consider the nested sequence of ideals

$$H = \mathbb{I}^0(H) \supset \mathbb{I}^1(H) \supset \cdots \supset \mathbb{I}^n(H) \supset \cdots\,, \tag{1}$$

where $\mathbb{I}^n(H)$ denotes the nth power of $\mathbb{I}(H)$ in H. We provide H with the filtration such that $\mathtt{F}_n\,H = \mathbb{I}^n(H)$, for any $n \in \mathbb{N}$.

The counit $\epsilon : H \to \Bbbk$ satisfies $\epsilon(\mathbb{I}(H)) = 0$ by definition, and hence, defines a filtration preserving morphism with values in the ground field $\Bbbk$, which we equip with the filtration such that $\mathtt{F}_0\,\Bbbk = \Bbbk$ and $\mathtt{F}_s\,\Bbbk = 0$ for $s > 0$ (see §7.3.9). The counit identities $\epsilon \otimes id \cdot \Delta(u) = id \otimes \epsilon \cdot \Delta(u) = u$ imply that the coproduct of any element $u \in \mathbb{I}(H)$ has an expansion of the form:

$$\Delta(u) = \underbrace{u \otimes 1 + 1 \otimes u}_{\in\,\mathbb{I}(H)\otimes 1 + 1\otimes\mathbb{I}(H)} + \underbrace{\sum_{(u)}{}' u_{(1)} \otimes u_{(2)}}_{\in\,\mathbb{I}(H)\otimes\mathbb{I}(H)}, \tag{2}$$

where we use the expression $\sum'_{(u)} u_{(1)} \otimes u_{(2)}$ to denote the terms of this coproduct $\Delta(u) = \sum_{(u)} u_{(1)} \otimes u_{(2)}$ which lie in module $\mathbb{I}(H) \otimes \mathbb{I}(H) \subset H \otimes H$. From this expansion, we deduce that we have $\Delta(u) \in \mathbb{I}^1(H) \otimes \mathbb{I}^0(H) + \mathbb{I}^0(H) \otimes \mathbb{I}^1(H)$ when $u \in \mathbb{I}(H)$. For an n-fold product $u = u_1 \cdot \ldots \cdot u_n$, we get the implication:

$$u = u_1 \cdot \ldots \cdot u_n \in \mathbb{I}^n(H) \Rightarrow \Delta(u) = \Delta(u_1) \cdot \ldots \cdot \Delta(u_n) \in \sum_{p+q=n} \mathbb{I}^p(H) \otimes \mathbb{I}^q(H).$$

This relation proves that the coproduct of our Hopf algebra $\Delta : H \to H \otimes H$ is a filtration preserving morphism.

Recall that the preservation of filtrations is a void condition for the unit morphism (see §7.3.21). The product $\mu : H \otimes H \to H$ of our Hopf algebra obviously defines a filtration preserving morphism too since we have $\mathbb{I}^p(H) \cdot \mathbb{I}^q(H) = \mathbb{I}^{p+q}(H)$ by definition of the powers of an ideal. For the antipode, we have the implications $u \in \mathbb{I}(H) \Rightarrow \sigma(u) \in \mathbb{I}(H)$ and $u = u_1 \cdot \ldots \cdot u_n \in \mathbb{I}^n(H) \Rightarrow \sigma(u) = \sigma(u_n) \cdot \ldots \cdot \sigma(u_1) \in \mathbb{I}^n(H)$ by Proposition 7.1.10. Hence, the antipode $\sigma : H \to H$ preserves our filtration as well.

We also trivially have $\mathbb{I}(H) = \ker(\epsilon : H \to \Bbbk) \Leftrightarrow H/\mathbb{I}(H) = \Bbbk 1$. We conclude from these observations that our filtration by the powers of the augmentation ideal (1) provides H with the structure of a connected filtered Hopf algebra in the sense of §7.3.15. By observations of §§7.3.14-7.3.15, we can take the completion of this filtered object (1) to get a complete Hopf algebra

$$\hat{H} = \lim_n H/\mathbb{I}^n(H) \tag{3}$$

canonically associated to H. We still have the relation:

$$\mathtt{E}^0\hat{H} = \mathtt{E}^0 H = \bigoplus_{n=0}^{\infty} \mathbb{I}^n(H)/\mathbb{I}^{n+1}(H) \tag{4}$$

by general properties of the completion of filtered modules (see §7.3.6).

8.1.2. *The complete group algebra and group-like element functors.* We associate a *complete group algebra* $\Bbbk[G]\hat{}$ to any group G by taking the completion §8.1.1(3) of the ordinary group algebra of §7.1.13. We explicitly set

$$\Bbbk[G]\hat{} = \lim_n \Bbbk[G]/\mathbb{I}^n\Bbbk[G]$$

to get a functor $\Bbbk[-]\hat{}\colon \mathcal{G}rp \to \hat{f}\,\mathcal{H}opf\mathcal{A}lg$ from the category of groups $\mathcal{G}rp$ towards the category of complete Hopf algebras $\hat{f}\,\mathcal{H}opf\mathcal{A}lg$.

We use a complete analogue of the group-like element functor of §7.1.14 to define a functor in the converse direction. To be explicit, we define the set of group-like elements of a counitary cocommutative coalgebra in the category of complete filtered modules by:

$$\mathbb{G}(C) = \{c \in C | \epsilon(c) = 1, \Delta(c) = c \hat{\otimes} c\},$$

where the tensor $c\hat{\otimes}c \in C\hat{\otimes}C$, associated to any $c \in C$, represents the image of the ordinary tensor product $c \otimes c \in C \otimes C$ under the completion morphism $C \otimes C \to C\hat{\otimes}C$. We also have $\mathbb{G}(C) = \mathtt{Mor}_{\hat{f}\,\mathcal{C}om_+^c}(\Bbbk, C)$, where we use the notation $\hat{f}\,\mathcal{C}om_+^c$ for the category of counitary cocommutative coalgebras in complete filtered modules.

In the case of a complete Hopf algebra $C = H$, we have the relations:

$$\begin{aligned}
&1 \in \mathbb{G}(H),\\
&g, h \in \mathbb{G}(H) \Rightarrow gh \in \mathbb{G}(H),\\
&g \in \mathbb{G}(H) \Rightarrow \sigma(g) \in \mathbb{G}(H) \quad \text{and} \quad g\,\sigma(g) = \sigma(g)\,g = 1
\end{aligned}$$

(compare with Proposition 7.1.15), from which we deduce that the set of group-like elements of a complete Hopf algebra $\mathbb{G}(H)$ forms a group naturally associated to H (like the set of group-like elements of an ordinary Hopf algebra).

Proposition 7.1.16 has the following analogue in the context of complete Hopf algebras:

Proposition 8.1.3. *The complete group algebra functor $\Bbbk[-]\hat{}\colon G \mapsto \Bbbk[G]\hat{}$ and the complete group-like element functor $\mathbb{G} : H \to \mathbb{G}(H)$ define a pair of adjoint functors $\Bbbk[-]\hat{}\colon \mathcal{G}rp \rightleftarrows \hat{f}\,\mathcal{H}opf\mathcal{A}lg : \mathbb{G}$ between the category of groups $\mathcal{G}rp$ and the category of complete Hopf algebras $\hat{f}\,\mathcal{H}opf\mathcal{A}lg$.*

Proof. Let $G \in \mathcal{G}rp$. Let $H \in \hat{f}\,\mathcal{H}opf\mathcal{A}lg$. In §7.3.21, we observe that the completion functor defines a left adjoint of the obvious functor from the category of unitary associative algebras in complete filtered modules to the category of unitary associative algebras in filtered modules. Thus, we have an equivalence between the morphisms of complete unitary associative algebras $\hat{\phi} : \Bbbk[G]^{\wedge} \to H$ and the morphisms of ordinary unitary associative algebras $\phi : \Bbbk[G] \to H$ which satisfy the relation $\phi(\mathbb{I}^n \Bbbk[G]) \subset \mathtt{F}_n H$ for all $n \in \mathbb{N}$. Moreover, such a morphism $\phi : \Bbbk[G] \to H$ is uniquely determined by fixing the images $\phi([g]) \in H$ of the basis elements $[g] \in \Bbbk[G]$ of our group algebra $\Bbbk[G]$, for $g \in G$. The preservation of the unit and of the product reduces to the usual equations $\phi(1) = 1$ and $\phi([gh]) = \phi([g])\phi([h])$ in H. The adjunction relation also implies that the preservation of augmentation and coproducts by our morphism is equivalent to the verification of the identities $\epsilon\phi([g]) = 1$ and $\Delta\phi([g]) = \phi([g])\hat{\otimes}\phi([g])$ in H, and hence, is equivalent to the relation $\phi([g]) \in \mathbb{G}(H)$. Finally, we readily see that for a complete Hopf algebra, such that $\mathtt{E}^0_0 H = \Bbbk \Leftrightarrow \mathbb{I}(H) = \mathtt{F}_1 H$, the relation $\epsilon\phi([g]) = 1 \Rightarrow \phi([g] - 1) \in \mathtt{F}_1 H$ automatically implies $\phi(\mathbb{I}^n \Bbbk[G]) \subset (\mathtt{F}_1 H)^n \subset \mathtt{F}_n H$. Hence, we get that our morphism automatically preserves the filtration of our Hopf algebras. ☐

8.1.4. *Logarithms and exponentials.* Recall that we assume the relation $\mathtt{E}^0_0 H = H/\mathtt{F}_1 H = \Bbbk$ for any complete Hopf algebra H. In the proof of Proposition 8.1.3, we already observed that this condition implies $\mathtt{F}_1 H = \mathbb{I}(H)$ and $\mathbb{I}(H)^n = (\mathtt{F}_1 H)^n \subset \mathtt{F}_n H$, for any $n \in \mathbb{N}$. We can use this observation to give a sense to a logarithm map

$$\log(\underbrace{1+h}_{=g}) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{h^n}{n}$$

for $h \in \mathbb{I}(H) \Leftrightarrow \epsilon(g) = \epsilon(1+h) = 1$ and to an exponential map

$$\exp(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

for $x \in \mathbb{I}(H) \Leftrightarrow \epsilon(x) = 0$. Formally, we consider the sequence of truncated power series $\log_{\langle r\rangle}(1+h) = \sum_{n=1}^{r} (-1)^{n-1} \cdot (1/n) \cdot h^n$, for $r \geq 0$. We use the relation $h \in \mathbb{I}(H) \Rightarrow h^{r+1} \in \mathtt{F}_{r+1} H \Rightarrow \log_{\langle r+1\rangle}(1+h) \equiv \log_{\langle r\rangle}(1+h) (\operatorname{mod} \mathtt{F}_{r+1} H)$, to establish that these truncated power series correspond to each others in the quotient algebras $H/\mathtt{F}_{r+1} H$ and hence determine a well-defined element $\log(1+h) \in H$ such that we have the relation $\log(1+h) \equiv \log_{\langle r\rangle}(1+h) (\operatorname{mod} \mathtt{F}_{r+1} H)$ in the quotient module $H/\mathtt{F}_{r+1} H$, for any $r \geq 0$. We use a similar construction to define the exponential map $\exp(x)$ for any element such that $x \in \mathbb{I}(H) \Leftrightarrow \epsilon(x) = 0$.

We have the following general observation:

Proposition 8.1.5. *In a complete Hopf algebra H, we have the relations*

$$g \in \mathbb{G}(H) \Rightarrow \log(g) \in \mathbb{P}(H) \quad \textit{and} \quad x \in \mathbb{P}(H) \Rightarrow \exp(x) \in \mathbb{G}(H)$$

so that the logarithm map $\log : g \mapsto \log(g)$ *and the exponential map* $\exp : x \mapsto \exp(x)$ *define inverse bijections between the set of primitive elements and the set of group-like elements of H.*

Proof. The identity $\epsilon(\log(g)) = 0$ is obvious, for any $g \in 1 + \mathbb{I}(H)$, as well as the identity $\epsilon(\exp(x)) = 1$, for any $x \in \mathbb{I}(H)$.

The definition of the coproduct as an algebra morphism $\Delta : H \to H\hat{\otimes}H$ implies that we have the relation $\Delta(\log(g)) = \log(\Delta(g))$, where on the right hand side we consider the logarithm of the element $\Delta(g)$ in the tensor product of Hopf algebras

$H\hat{\otimes}H$. For a group-like element, we have $\Delta(\log(g)) = \log(g\hat{\otimes}g)$, and according to the usual logarithm addition formula, which we apply to the commutative product $(g\hat{\otimes}1)\cdot(1\hat{\otimes}g) = g\hat{\otimes}g = (1\hat{\otimes}g)\cdot(g\hat{\otimes}1)$, we have the relation:

$$\begin{aligned}\Delta(\log(g)) = \log(g\hat{\otimes}g) &= \log((g\hat{\otimes}1)\cdot(1\hat{\otimes}g))\\ &= \log(g\hat{\otimes}1) + \log(1\hat{\otimes}g) = \log(g)\hat{\otimes}1 + 1\hat{\otimes}\log(g).\end{aligned}$$

Hence, we have $g \in \mathbb{G}(H) \Rightarrow \log(g) \in \mathbb{P}(H)$ as stated in the proposition.

In the case of the exponential of a primitive element $x \in \mathbb{P}(H)$, we argue similarly to get:

$$\begin{aligned}\Delta(\exp(x)) = \exp(x\hat{\otimes}1 + 1\hat{\otimes}x) &= \exp(x\hat{\otimes}1)\cdot\exp(1\hat{\otimes}x)\\ &= (\exp(x)\hat{\otimes}1)\cdot(1\hat{\otimes}\exp(x)) = \exp(x)\hat{\otimes}\exp(x).\end{aligned}$$

Thus, we have $x \in \mathbb{P}(H) \Rightarrow \exp(x) \in \mathbb{G}(H)$.

The conclusion of the proposition follows from the fact that the logarithm and the exponential are inverse to each other with respect to the composition of power series. ☐

We use the result of this proposition to establish the following statement:

Proposition 8.1.6. *The functor $\mathbb{G} : \hat{f}\,\mathcal{H}opf\mathcal{A}lg \to \mathcal{G}rp$ induces an injective map on morphism sets*

$$\mathtt{Mor}_{\hat{f}\,\mathcal{H}opf\mathcal{A}lg}(A, B) \hookrightarrow \mathtt{Mor}_{\mathcal{G}rp}(\mathbb{G}(A), \mathbb{G}(B)),$$

for all $A, B \in \hat{f}\,\mathcal{H}opf\mathcal{A}lg$, and hence, is faithful.

Proof. The group morphism $\mathbb{G}(f) : \mathbb{G}(A) \to \mathbb{G}(B)$ associated to a morphism of complete Hopf algebras $f : A \to B$ fits in a commutative diagram

$$\begin{tikzcd}\mathbb{P}(A) \arrow[r, "\mathbb{P}(f)"] \arrow[d, "\exp"', "\simeq"] & \mathbb{P}(B) \arrow[d, "\exp"', "\simeq"] \\ \mathbb{G}(A) \arrow[r, "\mathbb{G}(f)"] & \mathbb{G}(B)\end{tikzcd},$$

where we consider the exponential correspondence of Proposition 8.1.5. Hence, if we have $\mathbb{G}(f) = \mathbb{G}(g)$ for morphisms of complete Hopf algebras $f, g : A \to B$, then we also have $\mathbb{P}(f) = \mathbb{P}(g)$ and Theorem 7.3.26(b) (the Milnor-Moore Theorem) implies that we have $f = g$ as soon as this relation holds. ☐

We also use the exponential correspondence in the following proposition, where we examine the first examples of applications of our adjunction between groups and complete Hopf algebras:

Proposition 8.1.7.

(a) *We consider the complete symmetric algebra $H = \hat{\mathbb{S}}(M)$ associated to a plain $\Bbbk$-module M which we identify with a (complete) filtered module such that $\mathtt{F}_1 M = M$ and $\mathtt{F}_n M = 0$ for $n \geq 2$. We equip this complete symmetric algebra with the (complete version of) the Hopf algebra structure of Proposition* 7.2.5. *We then have an isomorphism:*

$$\exp : M \xrightarrow{\simeq} \mathbb{G}\,\hat{\mathbb{S}}(M)$$

between the underlying abelian group of our module and the group of group-like elements of this complete Hopf algebra $H = \hat{\mathbb{S}}(M)$.

(b) *We now assume that* A *is an abelian. We use additive notation to identify this abelian group with a* $\mathbb{Z}$*-module. We then have an isomorphism of complete Hopf algebras*

$$\rho : \Bbbk[A]^{\wedge} \xrightarrow{\simeq} \hat{\mathbb{S}}(A \otimes_{\mathbb{Z}} \Bbbk),$$

where we consider the complete symmetric algebra generated by the $\Bbbk$*-module* $M = A \otimes_{\mathbb{Z}} \Bbbk$ *on the left-hand side, and we again take the filtration such that* $\mathtt{F}_1 M = M$ *and* $\mathtt{F}_n M = 0$ *for* $n \geq 2$ *in order to identify this* $\Bbbk$*-module* $M = A \otimes_{\mathbb{Z}} \Bbbk$ *with an object of the category of complete filtered modules.*

Proof. We have $M = \mathbb{P}\,\hat{\mathbb{S}}(M)$ by the result of Proposition 7.2.14 (which we apply to the category of complete filtered modules). We therefore obtain the first assertion of this proposition as the particular case $H = \hat{\mathbb{S}}(M)$ of the result of Proposition 8.1.5.

We use additive notation for the abelian group A which we consider in the second assertion of the proposition, but we keep multiplicative notation when we work in the complete group algebra $\Bbbk[A]^{\wedge}$. We accordingly have the identity $[a]\cdot[b] = [a+b]$ when we consider the class of elements $a, b \in A$ in $\Bbbk[A]^{\wedge}$. We use that the exponential $\exp : a \mapsto e^{a\otimes 1}$ defines a group morphism $\exp : A \to \mathbb{G}(\hat{\mathbb{S}}(A \otimes_{\mathbb{Z}} \Bbbk))$ (since we have the relation $e^{a\otimes 1 + b\otimes 1} = e^{a\otimes 1} \cdot e^{b\otimes 1}$ as soon as we assume that our group is abelian), and we use the adjunction relation of Proposition 8.1.3 to define the morphism of our assertion $\rho : \Bbbk[A]^{\wedge} \to \hat{\mathbb{S}}(A \otimes_{\mathbb{Z}} \Bbbk)$. We accordingly have $\rho([a]) = e^{a\otimes 1}$, for any element $a \in A$.

We have an obvious morphism which goes the other way round $\psi : \hat{\mathbb{S}}(A \otimes_{\mathbb{Z}} \Bbbk) \to \Bbbk[A]^{\wedge}$ and which is induced by the map $\psi : A \to \Bbbk[A]^{\wedge}$ such that $\psi(a) = \log([a])$. We just use the addition formula of logarithms to check that this map does define a morphism of $\mathbb{Z}$-modules. We explicitly have $\log([a+b]) = \log([a]\cdot[b]) = \log([a]) + \log([b]) \Rightarrow \psi(a+b) = \psi(a) + \psi(b)$, for all $a, b \in A$ (since we assume that our group is abelian again). We easily see that the map $\psi : A \to \Bbbk[A]^{\wedge}$ induces a morphisms of complete Hopf algebras on $\hat{\mathbb{S}}(A \otimes_{\mathbb{Z}} \Bbbk)$, because each $a \in A$ defines a primitive element in the symmetric algebra $\hat{\mathbb{S}}(A \otimes_{\mathbb{Z}} \Bbbk)$ (by construction of our Hopf algebra structure in Proposition 7.2.5) and we have the relation $[a] \in \mathbb{G}(\Bbbk[A]^{\wedge}) \Rightarrow \psi(a) = \log([a]) \in \mathbb{P}(\Bbbk[A]^{\wedge})$ for any such $a \in A$ (by the observations of Proposition 8.1.5).

We clearly have $\rho\psi(a) = \log(e^{a\otimes 1}) = a \otimes 1$ when we consider the image of such generating elements $a \in A$ under the composite of our morphisms, and we conversely have the relation $\psi\rho([a]) = \exp(\log([a])) = [a]$ in $\Bbbk[A]^{\wedge}$, for all $a \in A$. We conclude from these computations (and an obvious application of adjunction relations) that our morphisms are converse to each other, and hence, define isomorphisms between the complete group algebra $\Bbbk[A]^{\wedge}$ and the symmetric algebra $\hat{\mathbb{S}}(A \otimes_{\mathbb{Z}} \Bbbk)$. We therefore get the claim of the second assertion of our proposition. □

Recall that the category of Hopf algebras in a symmetric monoidal category inherits a symmetric monoidal structure (see §7.2.22). We can apply this observation to the category of complete Hopf algebras which consists of Hopf algebras in the category of complete filtered modules by definition. The category of groups is also equipped with a symmetric monoidal structure (with the cartesian product as tensor product operation). To complete the results of this section, we observe that:

Proposition 8.1.8. *The functors* $\Bbbk[-]^{\wedge}\colon \mathcal{G}rp \rightleftarrows \hat{f}\,\mathcal{H}opf\mathcal{A}lg : \mathbb{G}$ *are symmetric monoidal, as well as the adjunction relation between them.*

Proof. For the trivial group $G = 1$, we immediately obtain $\Bbbk[1] = \Bbbk$. For a cartesian product $G \times H$, we have a Hopf algebra identity $\Bbbk[G \times H] = \Bbbk[G] \otimes \Bbbk[H]$, and we can also readily check that the filtration by the powers of the augmentation ideal of the group algebra $\Bbbk[G \times H]$ agrees with the filtration of the tensor product $\Bbbk[G] \otimes \Bbbk[H]$. We explicitly have $\mathbb{I}^n \Bbbk[G \times H] = \sum_{p+q=n} \mathbb{I}^p \Bbbk[G] \otimes \mathbb{I}^q \Bbbk[H]$ for every n. We immediately deduce from this relation that we have an isomorphism $\Bbbk[G \times H]^{\wedge} \simeq (\Bbbk[G] \otimes \Bbbk[H])^{\wedge} \simeq \Bbbk[G]^{\wedge} \hat{\otimes} \Bbbk[H]^{\wedge}$ at the level of our completed group algebras. We easily check that the isomorphisms which give these relations satisfy the coherence constraints of §3.3.1.

We immediately see that the ground field $\Bbbk$ is identified with the terminal object of the category of complete Hopf algebras. We easily check that a tensor product of complete Hopf algebras is identified with a cartesian product in the category of complete Hopf algebras. (We actually have these relations for any category of Hopf algebras which we may form in an ambient symmetric monoidal category.) We deduce from these observations that the group-like element functor is symmetric monoidal, because this functor preserves final objects and cartesian products by adjunction.

We easily check that the unit and augmentation of our adjunction commute with the isomorphisms which make our functors symmetric monoidal. We therefore conclude that our functors $\Bbbk[-]^{\wedge}\colon \mathcal{G}rp \rightleftarrows \hat{f}\,\mathcal{H}opf\mathcal{A}lg : \mathbb{G}$ define a symmetric monoidal adjunction. ☐

8.2. The category of Malcev complete groups

We observed in Proposition 8.1.6 that the group-like element functor from complete Hopf algebras to groups $\mathbb{G} : \hat{f}\,\mathcal{H}opf\mathcal{A}lg \to \mathcal{G}rp$ is faithful. We therefore define our category of Malcev complete groups, which we denote by $\hat{f}\,\mathcal{G}rp$, as the faithful image of the category of complete Hopf algebras $\hat{f}\,\mathcal{H}opf\mathcal{A}lg$ in the category of groups:

$$\hat{f}\,\mathcal{G}rp = \mathbb{G}(\hat{f}\,\mathcal{H}opf\mathcal{A}lg).$$

We also say that a group G is *Malcev complete* when we have $G = \mathbb{G}(H)$ for some $H \in \hat{f}\,\mathcal{H}opf\mathcal{A}lg$.

To give a first class of examples, the result of Proposition 8.1.7(a) implies that a $\Bbbk$-module admits a Malcev complete structure, and we easily deduce from the functoriality of the construction of this proposition and the equivalence of the Milnor-Moore Theorem (Theorem 7.3.26) that the morphisms of Malcev complete groups between $\Bbbk$-modules are exactly the $\Bbbk$-module morphisms. Thus, our category of Malcev complete groups can be interpreted as a non-abelian generalization of the category of $\Bbbk$-modules.

Most of this section is devoted to the study of natural structures which we associate to Malcev complete groups $G = \mathbb{G}(H)$. Notably, we already briefly explained, in the introduction of this chapter, that we can use the exponential correspondence to define power operations g^a with exponents in our ground field $a \in \Bbbk$ when g represents the element of a Malcev complete group $G = \mathbb{G}(H)$. We explicitly have $g = e^x$, for some primitive element in the underlying complete Hopf algebra of our group $x \in \mathbb{P}(H)$, and we just set $g^a = e^{ax}$, for any $a \in \Bbbk$. The morphisms of Malcev complete groups clearly preserve these extra power operations, and in the case of $\Bbbk$-modules (where we use additive notation for our group structure) we obviously retrieve the action of the scalars on our object.

We first explain the definition of a general notion of filtration which we naturally associate to Malcev complete groups.

8.2.1. *Filtrations on groups.* We use the notation (a, b) for the commutator operation $(a, b) = a^{-1}b^{-1}ab$ in a group G. When we have subgroups $A, B \subset G$, we also use the notation (A, B) for the subgroup of G generated by the commutators $(a, b) \in G$ such that $a \in A$ and $b \in B$. We consider general groups G equipped with a filtration

$$G = \mathtt{F}_1 G \supset \cdots \supset \mathtt{F}_n G \supset \mathtt{F}_{n+1} G \supset \cdots \tag{1}$$

by subgroups $\mathtt{F}_n G \subset G$ such that

$$(\mathtt{F}_m G, \mathtt{F}_n G) \subset \mathtt{F}_{m+n} G \quad \text{for all } m, n > 0. \tag{2}$$

We are going to see that a Malcev complete group inherits a filtration of this form. We may also readily check that the lower series filtration of a group, inductively defined by $\Gamma_1 G = G$ and $\Gamma_n G = (G, \Gamma_{n-1}(G))$ for $n > 1$, gives a universal example of a filtration which meets our requirement. To be more precise, our condition (2) implies that any filtration (1) satisfies $\Gamma_n G \subset \mathtt{F}_n G$ for $n > 0$.

8.2.2. *The weight graded Lie algebra associated to a group.* Suppose we have a group G equipped with a filtration of the form considered in the previous paragraph §8.2.1(1) and that satisfies the commutator condition §8.2.1(2). This condition (2) implies that our filtration §8.2.1(1) consists of a nested sequence of normal subgroups and that each subquotient $\mathtt{F}_n G/\mathtt{F}_{n+1} G$ is abelian. We more precisely see that the conjugation operation $x^g = g^{-1}xg$ preserves each subgroup $\mathtt{F}_n G$ and reduces to the identity morphism on the subquotient $\mathtt{F}_n G/\mathtt{F}_{n+1} G$, for all $g \in G$, because we have the trivial relation $x^g = x \cdot (x, g)$ and, under our assumptions, we have $x \in \mathtt{F}_n G \Rightarrow (x, g) \in \mathtt{F}_{n+1} G$, for all $g \in G = \mathtt{F}_1 G$.

We use additive notation for the abelian group structure of the subquotients $\mathtt{F}_n G/\mathtt{F}_{n+1} G$. We explicitly set $\overline{u} + \overline{v} = \overline{u \cdot v}$, for any pair $u, v \in \mathtt{F}_n G$, where we use the notation $\overline{g}$ for the class of any element $g \in \mathtt{F}_n G$ in $\mathtt{F}_n G/\mathtt{F}_{n+1} G$. We also adopt the notation $\mathtt{E}^0 G$ for the connected weight graded $\mathbb{Z}$-module such that:

$$\mathtt{E}^0 G = \bigoplus_{n=1}^{\infty} \underbrace{\mathtt{F}_n G/\mathtt{F}_{n+1} G}_{=\mathtt{E}^0_n G},$$

where we consider the obvious extension, in the context of modules over a ring, of the notion of weight graded module which we define in §7.3.5.

The inclusion relation $(\mathtt{F}_m G, \mathtt{F}_n G) \subset \mathtt{F}_{m+n} G$ of §8.2.1(2) implies that we can associate a well-defined element $[\overline{u}, \overline{v}] = \overline{(u, v)} \in \mathtt{F}_{m+n} G/\mathtt{F}_{m+n+1} G$ to any pair $\overline{u} \in \mathtt{F}_m G/\mathtt{F}_{m+1} G$ and $\overline{v} \in \mathtt{F}_n G/\mathtt{F}_{n+1} G$. The Philip Hall identities

$$\begin{aligned} &(a, b) \cdot (b, a) = 1, \\ &(a, b \cdot c) = (a, c) \cdot (a, b) \cdot ((a, b), c), \\ &((a, b), c^a) \cdot ((c, a), b^c) \cdot ((b, c), a^b) = 1, \end{aligned}$$

where $(-,-)$ is our commutator operation and we set $g^h = h^{-1}gh$ for any $g, h \in G$ (see [84, 113]), imply that the mapping $[\overline{x}, \overline{y}] = \overline{(x, y)}$ induces a biadditive operation

$$\underbrace{\mathtt{F}_m G/\mathtt{F}_{m+1} G}_{=\mathtt{E}^0_m G} \times \underbrace{\mathtt{F}_n G/\mathtt{F}_{n+1} G}_{=\mathtt{E}^0_n G} \xrightarrow{[-,-]} \underbrace{\mathtt{F}_{m+n} G/\mathtt{F}_{m+n+1} G}_{=\mathtt{E}^0_{m+n} G}$$

for every $m, n > 0$. This operation satisfies the vanishing relation $[x, x] = 0$, for all $x \in \mathtt{E}^0 G$, and the Jacobi relation $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$, for $x, y, z \in \mathtt{E}^0 G$. (The vanishing relation is obvious and the Jacobi relation follows from the Philip Hall identities.) Hence, the weight graded $\mathbb{Z}$-module $\mathtt{E}^0 G$ inherits the structure of a connected weight graded Lie algebra, where we again consider an obvious analogue, in the category of $\mathbb{Z}$-modules, of the notion of weight graded Lie algebra which we define §7.3.17 (since our subquotients $\mathtt{E}^0_n G = \mathtt{F}_n G/\mathtt{F}_{n+1} G$ are just abelian groups in general).

In the case of a Malcev complete group, which is given by the group of group-like element of a complete Hopf algebra $G = \mathbb{G}(H)$, we have the following result:

Proposition 8.2.3. *Let H be any complete Hopf algebra with $G = \mathbb{G}(H)$ as associated Malcev complete group.*

(a) *The sets*

$$\mathtt{F}_n \mathbb{G}(H) = \{g \in \mathbb{G}(H) | g - 1 \in \mathtt{F}_n H\}, \quad n > 0,$$

define a filtration of the group $G = \mathbb{G}(H)$ by a series of subgroups $\mathtt{F}_n \mathbb{G}(H) \subset \mathbb{G}(H)$ which fulfills the requirements of §8.2.1*(*1-2*). We moreover have the relation $\mathbb{G}(H) = \lim_n \mathbb{G}(H)/\mathtt{F}_n \mathbb{G}(H)$.*

(b) *The exponential map* $\exp : \mathbb{P}(H) \to \mathbb{G}(H)$ *induces an isomorphism of weight graded Lie algebras*

$$\exp : \mathtt{E}^0 \mathbb{P}(H) \xrightarrow{\simeq} \mathtt{E}^0 \mathbb{G}(H),$$

where:

- *on the one hand, we consider the weight graded Lie algebra $\mathtt{E}^0 \mathbb{P}(H)$ associated to the complete Lie algebra $\mathbb{P}(H) \subset H$, which we equip with the filtration induced by the natural filtration of the complete Hopf algebra H (see* §§7.3.19-7.3.20*);*
- *and, on the other hand, we consider the weight graded Lie algebra $\mathtt{E}^0 \mathbb{G}(H)$ which we define by using the subquotients of our filtration $\mathbb{G}(H) = \mathtt{F}_1 \mathbb{G}(H) \supset \cdots \supset \mathtt{F}_n \mathbb{G}(H) \supset \cdots$ of the group $G = \mathbb{G}(H)$ (see* §8.2.2*).*

In the case of a $\Bbbk$-module $M = \mathbb{G}(\hat{\mathbb{S}}(M))$ (see Proposition 8.1.7), we trivially get $M = \mathtt{F}_1 M$ and $\mathtt{F}_n M = 0$ for $n \geq 2$.

The second assertion of this proposition implies that the weight graded Lie algebra $\mathtt{E}^0 G = \mathtt{E}^0 \mathbb{G}(H)$ which we associate to a Malcev complete group $G = \mathbb{G}(H)$ naturally forms a weight graded Lie algebra in the category of $\Bbbk$-modules (and not only in the category of $\mathbb{Z}$-modules). The arguments which we give in our proof of this proposition actually imply that the power operations of our Malcev complete group g^a, where we assume $a \in \Bbbk$, preserve the filtration of the proposition, and when we pass to the subquotients of our filtration $\mathtt{E}^0_m G = \mathtt{E}^0_m \mathbb{G}(H)$ these operations trivially correspond to the action of the scalars on the Lie algebra $\mathtt{E}^0 \mathbb{P}(H)$ in the second assertion of the proposition.

Recall that we also have the commutation relation $\mathtt{E}^0 \mathbb{P}(H) = \mathbb{P}(\mathtt{E}^0 H)$ for every complete Hopf algebra $H \in \hat{f}\,\mathcal{H}opf\mathcal{A}lg$ (see Proposition 7.3.27). By combining this relation with the result of the above proposition we get the identity $\mathbb{P}(\mathtt{E}^0 H) = \mathtt{E}^0 \mathbb{G}(H)$. In the next section, we use this relation in order to give a representation of elements of the Malcev completion of free groups in terms of (infinite) products of iterated commutators.

PROOF. We use the exponential correspondence $g = e^x \Leftrightarrow x = \log(g)$. We readily deduce from the multiplicativity of the filtration of Hopf algebras that the elements $g \in \mathtt{F}_n \mathbb{G}(H)$ of the sets $\mathtt{F}_n \mathbb{G}(H)$ of assertion (a) are given by the exponentials $g = e^x$ of primitive elements $x \in \mathbb{P}(H)$ such that $x \in \mathbb{P}(H) \cap \mathtt{F}_n H$. Moreover, we clearly have

$$x, y \in \mathbb{P}(H) \cap \mathtt{F}_n H \Rightarrow e^x, e^y \equiv 1 (\operatorname{mod} \mathtt{F}_n H) \Rightarrow e^x e^y \equiv 1 (\operatorname{mod} \mathtt{F}_n H),$$

so that each $\mathtt{F}_n \mathbb{G}(H)$ forms a subgroup of $\mathbb{G}(H)$.

Recall that we have $\mathtt{F}_n \mathbb{P}(H) = \mathbb{P}(H) \cap \mathtt{F}_n H$ for each weight $n > 0$ by definition of the filtration of the submodule of a complete module. Let $\sigma : H \to H$ denote the antipode of our Hopf algebra. For $x \in \mathtt{F}_m \mathbb{P}(H)$ and $y \in \mathtt{F}_n \mathbb{P}(H)$, we have:

$$e^x = 1 + u \quad \text{and} \quad e^y = 1 + v,$$

where $u = e^x - 1 \in \mathtt{F}_m H$ and $v = e^y - 1 \in \mathtt{F}_n H$. If we use the implication $\xi \in \mathbb{P}(H) \Rightarrow e^\xi \in \mathbb{G}(H) \Rightarrow \sigma(e^\xi) = (e^\xi)^{-1} = e^{-\xi}$, then we moreover get:

$$e^{-x} = \sigma(e^x) = 1 + \sigma(u), \quad e^{-y} = \sigma(e^y) = 1 + \sigma(v).$$

The antipode relation now implies

$$1 + u + \sigma(u) + u\sigma(u) = 1 \quad \text{and} \quad 1 + v + \sigma(v) + v\sigma(v) = 1$$

and we use the multiplicativity of the filtration of the Hopf algebra to obtain

$$(e^x, e^y) \equiv 1 + uv - vu \equiv 1 + xy - yx (\operatorname{mod} \mathtt{F}_{m+n+1} H) \equiv 1 (\operatorname{mod} \mathtt{F}_{m+n} H).$$

This computation proves that we have the inclusion relation $(\mathtt{F}_m \mathbb{G}(H), \mathtt{F}_n \mathbb{G}(H)) \subset \mathtt{F}_{m+n} \mathbb{G}(H)$, for all $m, n > 0$, and hence that our sequence of subgroups $\mathtt{F}_n \mathbb{G}(H) \subset \mathbb{G}(H)$ satisfies the requirements of §8.2.2. Note also that our requirement $H/\mathtt{F}_1 H = \Bbbk \Leftrightarrow \mathbb{I}(H) = \mathtt{F}_1 H$ for a complete Hopf algebra $H \in \hat{f}\,\mathcal{H}opf\mathcal{A}lg$ implies $\mathbb{G}(H) = \mathtt{F}_1 \mathbb{G}(H)$.

We now consider a collection of group elements $g_n \in \mathbb{G}(H)$, $n > 0$, such that we have the relation $g_{n+1} \equiv g_n$ in the quotient group $\mathbb{G}(H)/\mathtt{F}_n \mathbb{G}(H)$, for each $n > 0$ (thus, this collection represents an element of the limit of the tower of groups $\mathbb{G}(H)/\mathtt{F}_n \mathbb{G}(H)$, $n > 0$). We have $g_n = e^{x_n}$ for some $x_n \in \mathbb{P}(H)$. We use that the relation $g_{n+1} \equiv g_n$ holds in $\mathbb{G}(H)/\mathtt{F}_n \mathbb{G}(H)$ if and only if we have this relation in the quotient $H/\mathtt{F}_n H$ of our Hopf algebra, and equivalently, if and only if we have the relation $e^{x_{n+1}} = e^{x_n} (\operatorname{mod} \mathtt{F}_n H)$, which gives $x_{n+1} = x_n (\operatorname{mod} \mathtt{F}_n H)$ when we take the logarithm of these exponential elements in H. We then have $x_n \equiv x (\operatorname{mod} \mathtt{F}_n H)$ for some $x \in H$ by completeness of our Hopf algebra H. We also have $x \in \mathbb{P}(H)$ since we have $x_n \in \mathbb{P}(H)$ for every $n > 0$ and $\mathbb{P}(H)$ forms a subobject of H in the category of complete filtered modules. We accordingly get that $g = e^x$ defines a group-like element of H. We moreover have $g \equiv g_n$ in $\mathbb{G}(H)/\mathtt{F}_n \mathbb{G}(H)$, for each $n > 0$. This verification proves that the map $\mathbb{G}(H) \to \lim_n \mathbb{G}(H)/\mathtt{F}_n \mathbb{G}(H)$ is surjective. We use similar computations to establish that this map is also injective. We therefore have an identity $\mathbb{G}(H) = \lim_n \mathbb{G}(H)/\mathtt{F}_n \mathbb{G}(H)$ and this observation finishes the verification of the first part of the proposition.

We now examine the second assertion of the proposition. We first assume $x, y \in \mathtt{F}_n \mathbb{P}(H)$, for some $n > 0$. We easily check that we have the relation $e^{-x}e^{-y}e^{x+y} \equiv 1 (\operatorname{mod} \mathtt{F}_{n+1} H)$ in the quotient $H / \mathtt{F}_{n+1} H$ of our Hopf algebra. We deduce from this observation that we have the identity $e^{x+y} \equiv e^x e^y$ in the quotient group $\mathtt{E}^0_n \mathbb{G}(H) = \mathtt{F}_n \mathbb{G}(H) / \mathtt{F}_{n+1} \mathbb{G}(H)$. If we assume $x \in \mathtt{F}_n \mathbb{P}(H)$, $y \in \mathtt{F}_{n+1} \mathbb{P}(H)$, then we readily get $e^{-x}e^{x+y} \equiv 1 (\operatorname{mod} \mathtt{F}_n H)$, and this relation implies that we have the identity $e^{x+y} \equiv e^x e^y$ in $\mathtt{E}^0_n \mathbb{G}(H)$. We deduce from these verifications that the exponential map induces a well-defined group morphism from $\mathtt{E}^0_n \mathbb{P}(H)$ to $\mathtt{E}^0_n \mathbb{G}(H)$, for each weight $n > 0$. We use that the equivalence $g = e^x \in \mathtt{F}_n \mathbb{G}(H) \Leftrightarrow x \in \mathtt{F}_n \mathbb{P}(H)$ holds for every weight $n > 0$ in order to check that this group morphism is bijective.

We then assume $x \in \mathtt{F}_m \mathbb{P}(H)$ and $y \in \mathtt{F}_n \mathbb{P}(H)$. We already observed that we have the relation $(e^x, e^y) \equiv 1 + xy - yx (\operatorname{mod} \mathtt{F}_{m+n+1} H)$ in the Hopf algebra H. We also have $e^{[x,y]} \equiv 1 + [x, y] \equiv 1 + xy - yx (\operatorname{mod} \mathtt{F}_{m+n+1} H)$. We readily deduce from these formulas that we have the relation $(e^x, e^y)e^{-[x,y]} \equiv 1 (\operatorname{mod} \mathtt{F}_{m+n+1} H)$ in the Hopf algebra H and the identity $(e^x, e^y) \equiv e^{[x,y]}$ in the quotient group $\mathtt{E}^0_{m+n+1} \mathbb{G}(H) = \mathtt{F}_{m+n} \mathbb{G}(H) / \mathtt{F}_{m+n+1} \mathbb{G}(H)$, from which we conclude that our map $\exp : \mathtt{E}^0 \mathbb{P}(H) \to \mathtt{E}^0 \mathbb{G}(H)$ defines a morphism of weight graded Lie algebras. This verification finishes the proof of the second assertion of the proposition. □

The following proposition gives a generalization for Malcev complete groups of the claim of Proposition 7.3.7 about the definition of isomorphisms in a category of complete filtered modules:

Proposition 8.2.4. *A morphism of Malcev complete groups defines an isomorphism*

$$\psi : G \xrightarrow{\cong} H$$

(in the category of Malcev complete groups) if and only if the morphism of weight graded Lie algebras which we associate to this morphism forms an isomorphism

$$\mathtt{E}^0 \psi : \mathtt{E}^0 G \xrightarrow{\cong} \mathtt{E}^0 H$$

(in the category of weight graded Lie algebras).

Explanations and proof. Recall that we define the category of Malcev complete groups as the image of the category of complete Hopf algebras under the group-like element functor $\mathbb{G} : \hat{f}\,\mathcal{H}opf\mathcal{A}lg \to \mathcal{G}rp$. Consequently, when we define an isomorphism of Malcev complete groups, we implicitly assume that our morphism $\psi : G \to H$ is invertible in this category, and hence, that we have an inverse of our morphism $\psi^{-1} : H \to G$ which comes from a morphism of complete Hopf algebras as well. The morphism of weight graded Lie algebras $\mathtt{E}^0 \psi : \mathtt{E}^0 G \to \mathtt{E}^0 H$, on the other hand, defines an isomorphism if and only if this morphism defines an isomorphism componentwise. Thus, our statement actually implies that a morphism of Malcev complete groups defines an isomorphism in the category of Malcev complete groups $\psi : G \xrightarrow{\cong} H$ if and only if this morphism induces an isomorphism of groups componentwise $\mathtt{E}^0_m \psi : \mathtt{E}^0_m G \xrightarrow{\cong} \mathtt{E}^0_m H$, $m \geq 1$, when we consider the subquotients of the filtration of Proposition 8.2.3.

The "only if" part of the proposition trivially follows from the functoriality of the map $\mathtt{E}^0 : \psi \mapsto \mathtt{E}^0 \psi$. We therefore focus on the proof of the "if" part, and we assume that $\mathtt{E}^0 \psi : \mathtt{E}^0 G \to \mathtt{E}^0 H$ defines an isomorphism in the category of weight graded Lie algebras. Let $\phi : A \to B$ be the morphism of complete Hopf algebras which underlies our group morphism $\psi = \mathbb{G}(\phi)$. We deduce from the functoriality of the correspondence of Proposition 8.2.3(b) that ϕ induces an isomorphism of weight graded Lie algebra $\mathtt{E}^0\,\mathbb{P}(\phi) : \mathtt{E}^0\,\mathbb{P}(A) \to \mathtt{E}^0\,\mathbb{P}(B)$ when we take the primitive part of our Hopf algebras. By Proposition 7.3.7 this statement implies that $\mathbb{P}(\phi)$ forms an isomorphism of complete filtered modules, and hence, forms an isomorphism of complete Lie algebras itself. We can then use the complete version of the Milnor-Moore Theorem (see Theorem 7.3.26) to conclude that ϕ also forms an isomorphism of complete Hopf algebras, and hence, that ψ defines an isomorphism of Malcev complete groups as asserted in our proposition. □

Recall that every complete Hopf algebra is identified with the enveloping algebra of a complete Lie algebra by the (complete version of) the Milnor-Moore Theorem (see §7.3). We can also use this result to give an expression, in terms of Lie algebras, of the tower of quotient groups which we consider in the statement of Proposition 8.2.3:

Proposition 8.2.5. *Let $H = \hat{\mathbb{U}}(\mathfrak{g})$ be the complete enveloping algebra of a complete Lie algebra $\mathfrak{g} \in \hat{f}\,\mathcal{L}ie$. Let $G = \mathbb{G}(\hat{\mathbb{U}}(\mathfrak{g}))$. We have an isomorphism:*

$$G/\,\mathtt{F}_{m+1}\,G \xrightarrow{\simeq} \mathbb{G}\,\hat{\mathbb{U}}(\mathfrak{g}\,/\,\mathtt{F}_{m+1}\,\mathfrak{g}),$$

for each $m \geq 0$, where we consider the tower of quotient groups $q_m G = G/\,\mathtt{F}_{m+1}\,G$ associated to the filtration of Proposition 8.2.3.

Proof. We naturally provide the quotient Lie algebra $\mathfrak{g}\,/\,\mathtt{F}_{m+1}\,\mathfrak{g}$ with the filtration such that:

$$\mathtt{F}_n(\mathfrak{g}\,/\,\mathtt{F}_{m+1}\,\mathfrak{g}) = \begin{cases} \mathtt{F}_n\,\mathfrak{g}\,/\,\mathtt{F}_{m+1}\,\mathfrak{g}, & \text{for } n \leq m, \\ 0, & \text{otherwise,} \end{cases}$$

in order to identify this Lie algebra with an object of the category of complete Lie algebras. We adapt the argument lines of the proof of Proposition 8.2.3 to check our claims, and we apply some of the relations established in the verification of this previous statement to the complete Hopf algebra such that $H = \hat{\mathbb{U}}(\mathfrak{g}) \Leftrightarrow \mathfrak{g} = \mathbb{P}(H)$. We consider the obvious map $G = \mathbb{G}\,\hat{\mathbb{U}}(\mathfrak{g}) \to \mathbb{G}\,\hat{\mathbb{U}}(\mathfrak{g}\,/\,\mathtt{F}_{m+1}\,\mathfrak{g})$ induced by the quotient map of Lie algebras $\mathfrak{g} \to \mathfrak{g}\,/\,\mathtt{F}_{m+1}\,\mathfrak{g}$. We use the commutative diagram:

$$\begin{tikzcd}
\mathfrak{g} \arrow[r] \arrow[d, "\exp"', "\simeq"] & \mathfrak{g}\,/\,\mathtt{F}_{m+1}\,\mathfrak{g} \arrow[d, "\exp"', "\simeq"] \\
\mathbb{G}\,\hat{\mathbb{U}}(\mathfrak{g}) \arrow[r] & \mathbb{G}\,\hat{\mathbb{U}}(\mathfrak{g}\,/\,\mathtt{F}_{m+1}\,\mathfrak{g})
\end{tikzcd},$$

where the vertical maps are the bijections of the exponential correspondence.

We observed in the proof of Proposition 8.2.3 that the elements $g \in \mathtt{F}_{m+1}\,G$ are given by exponentials $g = e^x$, where we assume $x \in \mathfrak{g} \cap \mathtt{F}_{m+1}\,H$, and such elements become obviously trivial in $\mathbb{G}\,\hat{\mathbb{U}}(\mathfrak{g}\,/\,\mathtt{F}_{m+1}\,\mathfrak{g})$ (since we have $\mathtt{F}_{m+1}\,\mathfrak{g} = \mathfrak{g} \cap \mathtt{F}_{m+1}\,H$). We deduce from this observation that our map induces a well-defined morphism from the quotient group $G/\,\mathtt{F}_{m+1}\,G$ towards the group of group-like elements $\mathbb{G}\,\hat{\mathbb{U}}(\mathfrak{g}\,/\,\mathtt{F}_{m+1}\,\mathfrak{g})$. We immediately deduce from the above diagram that this group morphism is surjective too. We moreover see that an element $g \in G$ such that $g = e^x$ for some $x \in \mathfrak{g}$ is carried to the unit in the group $\mathbb{G}\,\hat{\mathbb{U}}(\mathfrak{g}\,/\,\mathtt{F}_{m+1}\,\mathfrak{g})$ if and only if x is carried to 0 in the quotient module $\mathfrak{g}\,/\,\mathtt{F}_{m+1}\,\mathfrak{g}$, and hence if and only if we have $x \in \mathtt{F}_{m+1}\,\mathfrak{g} \Leftrightarrow g \in \mathtt{F}_{m+1}\,G$. Thus, our map is also injective on $G/\,\mathtt{F}_{m+1}\,G$ and induces a group isomorphism $G/\,\mathtt{F}_{m+1}\,G \xrightarrow{\simeq} \mathbb{G}\,\hat{\mathbb{U}}(\mathfrak{g}\,/\,\mathtt{F}_{m+1}\,\mathfrak{g})$. □

This statement implies that the quotient groups $q_m G = G/\,\mathtt{F}_{m+1}\,G$ of a Malcev complete group $G = \mathbb{G}(H)$ naturally form Malcev complete groups. The morphisms $p_m : G/\,\mathtt{F}_{m+1}\,G \to G/\,\mathtt{F}_m\,G$ in the tower decomposition of our group $G = \lim_m G/\,\mathtt{F}_{m+1}\,G$ define morphisms of Malcev complete groups too, and the identity $G = \lim_m G/\,\mathtt{F}_{m+1}\,G$ also holds in the category of Malcev complete groups. Indeed, if we set $H = \hat{\mathbb{U}}(\mathfrak{g})$ as in the statement our proposition, then we can obviously identify these morphisms $p_m : G/\,\mathtt{F}_{m+1}\,G \to G/\,\mathtt{F}_m\,G$ with the image of the canonical morphisms of complete Hopf algebras $p_m : \hat{\mathbb{U}}(\mathfrak{g}\,/\,\mathtt{F}_{m+1}\,\mathfrak{g}) \to \hat{\mathbb{U}}(\mathfrak{g}\,/\,\mathtt{F}_m\,\mathfrak{g})$ under the group-like element functor. The result of Proposition 8.2.5 also admits the following straightforward consequence:

Proposition 8.2.6. *Any morphism of Malcev complete groups $\phi : G \to q_m H$, where we consider the mth quotient group of a Malcev complete group as target object $q_m H = H/\,\mathtt{F}_{m+1}\,H$, admits a unique factorization:*

$$\begin{tikzcd}
G \arrow[rr, "\phi"] \arrow[dr] & & H/\,\mathtt{F}_{m+1}\,H \\
& G/\,\mathtt{F}_{m+1}\,G \arrow[ur, dotted, "\exists!\bar{\phi}"'] &
\end{tikzcd}$$

such that $\bar{\phi}$ is a morphism in the category of Malcev complete groups.

Proof. We assume $G = \mathbb{G}(\hat{\mathbb{U}}(\mathfrak{g}))$ and $H = \mathbb{G}(\hat{\mathbb{U}}(\mathfrak{h}))$ for some complete Lie algebras $\mathfrak{g}, \mathfrak{h} \in \hat{f\,\mathcal{L}ie}$ as in the construction of Proposition 8.2.5. We use the correspondence of this proposition and the complete version of the Milnor-Moore Theorem (Theorem 7.3.26) to get that our diagram is equivalent to the diagram of Lie algebras

$$\begin{tikzcd}
\mathfrak{g} \arrow[rr, "\phi"] \arrow[dr] & & \mathfrak{h}\,/\,\mathtt{F}_{m+1}\,\mathfrak{h} \\
& \mathfrak{g}\,/\,\mathtt{F}_{m+1}\,\mathfrak{g} \arrow[ur, dotted, "\exists!\bar{\phi}"'] &
\end{tikzcd} ,$$

and the existence of the fill-in morphism is immediate in this case (go back to the definition of the filtration on a quotient Lie algebra in the proof of Proposition 8.2.5). □

8.2.7. *Remarks: Truncated filtrations and group-like elements.* The group of group-like elements of a complete Hopf algebra H also admits a natural tower decomposition $\mathbb{G}(H) = \lim_m \mathbb{G}_{\langle m\rangle}(H)$, where $\mathbb{G}_{\langle m\rangle}(H)$ is the group formed by the classes $\bar{g} \in H/\,\mathtt{F}_{m+1}\,H$ of elements $g \in H$ which fulfill the equations of a group-like element modulo terms of filtration $\geq m+1$. We explicitly set:

$$(1) \qquad \mathbb{G}_{\langle m\rangle}(H) = \{\bar{g} \in H/\,\mathtt{F}_{m+1}\,H \,|\, \epsilon(g) = 1, \Delta(g) \equiv g\hat{\otimes}g(\operatorname{mod}\mathtt{F}_{m+1}(H \otimes H))\},$$

for each $m \geq 0$, and we readily check, by relying on the same observations as in the case of the group of group-like elements (see §8.1.2), that this set inherits a group structure. We have an obvious group morphism $\mathbb{G}_{\langle m\rangle}(H) \to \mathbb{G}_{\langle m-1\rangle}(H)$ induced by the quotient map $p_m : H/\,\mathtt{F}_{m+1}\,H \to H/\,\mathtt{F}_m\,H$ for each $m \geq 1$, and we immediately see that a sequence of classes $\bar{g}_m \in \mathbb{G}_{\langle m\rangle}(H)$ such that $g_m \equiv g_{m-1}(\operatorname{mod} \mathtt{F}_m H)$ determines an element $g \in H$ which fulfill the equations of a group-like element in H. We therefore have the isomorphism $\mathbb{G}(H) = \lim_m \mathbb{G}_{\langle m\rangle}(H)$ specified at the beginning of this remark. We immediately see that the morphism $p : \mathbb{G}(H) \rightarrow \mathbb{G}_{\langle m\rangle}(H)$ cancels the subgroup $\mathtt{F}_{m+1}\,\mathbb{G}(H)$ in the filtration of Proposition 8.2.3. We actually have an isomorphism

$$(2) \qquad \mathbb{G}(H)/\,\mathtt{F}_{m+1}\,\mathbb{G}(H) \xrightarrow{\simeq} \mathbb{G}_{\langle m\rangle}(H),$$

for each $m \geq 0$. We use a truncated version of our constructions to establish this claim.

We can consider the category $\hat{f}_{\langle m\rangle}\,\mathcal{M}od$ formed by the filtered modules M whose filtration satisfies $\mathtt{F}_{m+1} M = 0$. We trivially have $M = M/\,\mathtt{F}_{m+1} M = \lim_n M/\,\mathtt{F}_{n+1} M$ when this vanishing condition is satisfied. We can therefore regard this category $\hat{f}_{\langle m\rangle}\,\mathcal{M}od$ as a full subcategory of the category of the category of complete filtered modules $\hat{f}\,\mathcal{M}od$. We can moreover identify the quotient operation $q_m M = M/\,\mathtt{F}_{m+1} M$ with a functor $q_m : \hat{f}\,\mathcal{M}od \rightarrow \hat{f}_{\langle m\rangle}\,\mathcal{M}od$ which defines a left adjoint of the category embedding $i_m : \hat{f}_{\langle m\rangle}\,\mathcal{M}od \rightarrow \hat{f}\,\mathcal{M}od$. We trivially have $q_m i_m M = M$, for any $M \in \hat{f}_{\langle m\rangle}\,\mathcal{M}od$. We equip the category $\hat{f}_{\langle m\rangle}\,\mathcal{M}od$ with the tensor product such that $M \hat{\otimes}_{\langle m\rangle} N = M \hat{\otimes} N/\,\mathtt{F}_{m+1}(M \hat{\otimes} N)$, for any pair of objects $M, N \in \hat{f}_{\langle m\rangle}\,\mathcal{M}od$. We readily check that the functor $q_m : \hat{f}\,\mathcal{M}od \rightarrow \hat{f}_{\langle m\rangle}\,\mathcal{M}od$ is (strongly) symmetric monoidal, while the category embedding $i_m : \hat{f}_{\langle m\rangle}\,\mathcal{M}od \hookrightarrow \hat{f}\,\mathcal{M}od$ is just unit-preserving and equipped with a monoidal transformation, which is given by the obvious quotient map $M \hat{\otimes} N \rightarrow M \hat{\otimes}_{\langle m\rangle} N$, for any pair of objects $M, N \in \hat{f}_{\langle m\rangle}\,\mathcal{M}od$.

We can apply the constructions of the previous chapter §7 to this symmetric monoidal category $\mathcal{M} = \hat{f}_{\langle m\rangle}\,\mathcal{M}od$. We then use the notation $\hat{f}_{\langle m\rangle}\,\mathcal{L}ie$ for the category of Lie algebras in $\mathcal{M} = \hat{f}_{\langle m\rangle}\,\mathcal{M}od$ which satisfy the same connectedness condition $\mathtt{E}^0_0\,\mathfrak{g} = 0 \Leftrightarrow \mathfrak{g} = \mathtt{F}_1\,\mathfrak{g}$ as in §7.3.20. This category $\hat{f}_{\langle m\rangle}\,\mathcal{L}ie$ also forms a full subcategory of the category of complete Lie algebras $\hat{f}\,\mathcal{L}ie$. We easily check that the functor $q_m : \hat{f}\,\mathcal{M}od \rightarrow \hat{f}_{\langle m\rangle}\,\mathcal{M}od$ preserves Lie algebra structures (since we observed that this functor is symmetric monoidal) and induces a left adjoint of the canonical embedding $i_m : \hat{f}_{\langle m\rangle}\,\mathcal{L}ie \hookrightarrow \hat{f}\,\mathcal{L}ie$ on the category of complete Lie algebras $\hat{f}\,\mathcal{L}ie$.

We also use the notation $\hat{f}_{\langle m\rangle}\,\mathcal{H}opf\mathcal{A}lg$ for the category of Hopf algebras in $\hat{f}_{\langle m\rangle}\,\mathcal{M}od$ such that $\mathtt{E}^0_0 H = \Bbbk$. We again readily check that our functor $q_m : \hat{f}\,\mathcal{M}od \rightarrow \hat{f}_{\langle m\rangle}\,\mathcal{M}od$ preserves Hopf algebra structures, and hence, induces a functor on the category of complete Hopf algebras $q_m : \hat{f}\,\mathcal{H}opf\mathcal{A}lg \rightarrow \hat{f}_{\langle m\rangle}\,\mathcal{H}opf\mathcal{A}lg$, but we do not have such statements for the category embedding $i_m : \hat{f}_{\langle m\rangle}\,\mathcal{M}od \hookrightarrow \hat{f}\,\mathcal{M}od$ (because this functor is only lax symmetric monoidal). We could check, nevertheless, that the functor $q_m : \hat{f}\,\mathcal{H}opf\mathcal{A}lg \rightarrow \hat{f}_{\langle m\rangle}\,\mathcal{H}opf\mathcal{A}lg$ has a left adjoint $i^{\sharp}_m : \hat{f}_{\langle m\rangle}\,\mathcal{H}opf\mathcal{A}lg \rightarrow \hat{f}\,\mathcal{H}opf\mathcal{A}lg$.

We can easily define an analogue of the complete enveloping algebra functor

$$(3) \qquad \mathbb{U}_{\langle m\rangle} : \hat{f}_{\langle m\rangle}\,\mathcal{L}ie \rightarrow \hat{f}_{\langle m\rangle}\,\mathcal{H}opf\mathcal{A}lg$$

on this category of truncated complete Lie algebras $\hat{f}_{\langle m\rangle}\,\mathcal{L}ie$, as well as analogues of the primitive element functor $\mathbb{P}_{\langle m\rangle} : \hat{f}_{\langle m\rangle}\,\mathcal{H}opf\mathcal{A}lg \to \hat{f}_{\langle m\rangle}\,\mathcal{L}ie$ and of the functor of group-like elements $\mathbb{G}_{\langle m\rangle} : \hat{f}_{\langle m\rangle}\,\mathcal{H}opf\mathcal{A}lg \to \mathcal{G}rp$. We use the same notation for the obvious composite of these functors with the truncation operation $q_m : M \mapsto q_m M$ (on the source). We actually have

$$\mathbb{U}_{\langle m\rangle}(\mathfrak{g}) = q_m\,\mathbb{U}(\mathfrak{g}), \tag{4}$$

for any $\mathfrak{g} \in \hat{f}\,\mathcal{L}ie$, where we consider the image of the standard complete enveloping algebra of this Lie algebra $\mathbb{U}(\mathfrak{g}) \in \hat{f}\,\mathcal{H}opf\mathcal{A}lg$ under the truncation functor $q_m : \hat{f}\,\mathcal{H}opf\mathcal{A}lg \to \hat{f}_{\langle m\rangle}\,\mathcal{H}opf\mathcal{A}lg$, while the group of group-like elements $\mathbb{G}_{\langle m\rangle}(H)$ is given by the formula introduced at the beginning of this paragraph (1), and the Lie algebra $\mathbb{P}_{\langle m\rangle}(H)$ consists of elements $\xi \in H$ which have a trivial augmentation $\epsilon(\xi) = 0$ and are primitive modulo terms of filtration $\geq m+1$.

The complete versions of the Structure Theorem of Hopf algebras, of the Poincaré-Birkhoff-Witt Theorem, and of the Milnor-Moore Theorem (see Theorem 7.3.26) also hold in this category of Hopf algebras $\hat{f}_{\langle m\rangle}\,\mathcal{H}opf\mathcal{A}lg$. The exponential and logarithm maps still induce inverse bijections between the Lie algebra $\mathbb{P}_{\langle m\rangle}(H)$ and the group $\mathbb{G}_{\langle m\rangle}(H)$. In this context, we can actually consider truncated series $\exp_{\langle m\rangle}(u) = \sum_{n=0}^{m} u^n/n!$ and $\log_{\langle m\rangle}(1+u) = \sum_{n=1}^{m}(-1)^{n-1}u^n/n$ since we have $u \in \mathtt{F}_1\,H \Rightarrow u^n \equiv 0(\operatorname{mod}\mathtt{F}_{m+1}\,H)$ for $n \geq m+1$.

Let H be a complete Hopf algebra. We have $H = \hat{\mathbb{U}}(\mathfrak{g})$ for some Lie algebra $\mathfrak{g} \in \hat{f}\,\mathcal{L}ie$ by the Milnor-Moore Theorem for the category of complete Hopf algebras $\hat{f}\,\mathcal{H}opf\mathcal{A}lg$. We already explained that $q_m\hat{\mathbb{U}}(\mathfrak{g}) = \hat{\mathbb{U}}(\mathfrak{g})/\mathtt{F}_{m+1}\,\hat{\mathbb{U}}(\mathfrak{g})$ is identified with the truncated complete enveloping algebra $\mathbb{U}_{\langle m\rangle}(\mathfrak{g}) = \mathbb{U}_{\langle m\rangle}(q_m\,\mathfrak{g})$ which we associate to the Lie algebra $q_m\,\mathfrak{g} = \mathfrak{g}\,/\,\mathtt{F}_{m+1}\,\mathfrak{g}$ in the category $\hat{f}_{\langle m\rangle}\,\mathcal{M}od$. We consequently have $\mathbb{P}_{\langle m\rangle}\,q_m\hat{\mathbb{U}}(\mathfrak{g}) = q_m\,\mathfrak{g}$ by the Milnor-Moore Theorem for the Hopf algebras of the category $\hat{f}_{\langle m\rangle}\,\mathcal{H}opf\mathcal{A}lg$, and the truncated exponential map $\exp_{\langle m\rangle} : u \mapsto \exp_{\langle m\rangle}(u)$ induces a bijection from this Lie algebra $q_m\,\mathfrak{g} = \mathfrak{g}\,/\,\mathtt{F}_{m+1}\,\mathfrak{g}$ to the group $\mathbb{G}_{\langle m\rangle}(\hat{\mathbb{U}}(\mathfrak{g})) = \mathbb{G}_{\langle m\rangle}(q_m\hat{\mathbb{U}}(\mathfrak{g}))$ which we consider in (2). By Proposition 8.2.5, we have on the other hand $G/\,\mathtt{F}_{m+1}\,G = \mathbb{G}\,\hat{\mathbb{U}}(\mathfrak{g}\,/\,\mathtt{F}_{m+1}\,\mathfrak{g}) = \mathbb{G}\,\hat{\mathbb{U}}(q_m\,\mathfrak{g})$ where we set $G = \mathbb{G}\,\hat{\mathbb{U}}(\mathfrak{g})$ for short. We use the commutative diagram

$$\begin{tikzcd}
 & q_m\,\mathfrak{g} \arrow[dl, "\exp"', "\simeq"] \arrow[dr, "\exp_{\langle m\rangle}", "\simeq"'] & \\
\mathbb{G}\,\hat{\mathbb{U}}(q_m\,\mathfrak{g}) \arrow[rr] & & \mathbb{G}_{\langle m\rangle}(\hat{\mathbb{U}}(\mathfrak{g}))
\end{tikzcd},$$

where the diagonal arrows are the bijections given the exponential correspondence, to establish that our map in (2) does form a group isomorphism, as stated.

8.2.8. *Remarks: The equivalence with the category of complete Lie algebras and the Baker-Campbell-Hausdorff formula.* The category of Malcev complete groups is, according to our definition, equivalent to the category of complete Hopf algebras, with an equivalence of categories yielded by the group-like element functor from complete Hopf algebras to groups.

We can obviously compose this equivalence with the equivalence of categories of the Milnor-Moore Theorem to get an equivalence between the category of Malcev complete groups and the category of complete Lie algebras. We then consider the functor which assigns the group $G = \mathbb{G}\,\hat{\mathbb{U}}(\mathfrak{g})$, to any complete Lie algebra $\mathfrak{g} \in \hat{f}\,\mathcal{L}ie$. Since we have $\mathfrak{g} = \mathbb{P}\,\hat{\mathbb{U}}(\mathfrak{g})$, the result of Proposition 8.1.5 implies that the exponential map induces a natural bijection $\exp : \mathfrak{g} \xrightarrow{\simeq} G$ between the Lie algebra $\mathfrak{g}$ and the associated Malcev complete group $G = \mathbb{G}\,\hat{\mathbb{U}}(\mathfrak{g})$. In particular, for $a, b \in \mathfrak{g}$, we have an identity $e^a e^b = e^c$ between the product of the elements e^a and e^b in the group G and the exponential of a certain element c in the Lie algebra $\mathfrak{g}$.

We can use the functoriality of the exponential correspondence to get a universal formula, usually referred to as the *Baker-Campbell-Hausdorff formula* in the literature, for this Lie algebra element (see for instance [34, §II.6]). We proceed as follows. We first work within a free complete Lie algebra $\mathfrak{g} = \hat{\mathbb{L}}(\Bbbk\, x \oplus \Bbbk\, y)$, where x and y now represent abstract variables, and we use the exponential correspondence to get a Lie power series $\phi(x, y) \in \hat{\mathbb{L}}(\Bbbk\, x \oplus \Bbbk\, y)$ that satisfies our relation

$$e^x e^y = e^{\phi(x,y)}$$

in the tensor algebra $\hat{\mathbb{T}}(\Bbbk\, x \oplus \Bbbk\, y) = \hat{\mathbb{U}}\hat{\mathbb{L}}(\Bbbk\, x \oplus \Bbbk\, y)$. We also have $\phi(x, y) = \log(e^x e^y)$ and we can use the retraction of Proposition 7.2.8 to get an explicit definition of this Lie power series. We obtain for the first terms:

$$\phi(x, y) = x + y + \frac{1}{2}[x, y] + \frac{1}{12}([[x, y], y] + [[y, x], x]) + \cdots .$$

We then consider the Lie algebra morphism $\mathbb{L}(\Bbbk\, x \oplus \Bbbk\, y) \to \mathfrak{g}$ which maps our variables (x, y) to given elements (a, b) of the Lie algebra $\mathfrak{g}$. We have the relation $e^a e^b = e^{\phi(a,b)}$, where we consider the Lie algebra element $c = \phi(a, b)$ defined by performing the substitution $(x, y) = (a, b)$ in our Lie power series $\phi(x, y) \in \hat{\mathbb{L}}(\Bbbk\, x \oplus \Bbbk\, y)$.

The Baker-Campbell-Hausdorff formula can be used to give a direct definition of the Malcev complete group G associated to a Lie algebra $\mathfrak{g}$ without referring to Hopf algebras. This approach is used by Bourbaki [34] for instance. We then define G as the set of formal exponential elements e^ξ, where $\xi \in \mathfrak{g}$, and we merely set $e^a e^b = e^{\phi(a,b)}$ to provide this set $G = \exp \mathfrak{g}$ with a group structure.

8.2.9. *Remarks: The relationship with the notion of a unipotent algebraic group.* In the case where the Lie algebra $\mathfrak{g}$ in §8.2.8 is finite dimensional, the exponential map gives, for any choice of a basis on $\mathfrak{g}$, an identity between the underlying set of the group G associated to $\mathfrak{g}$ and the affine space $\Bbbk^N$, where we set $N = \dim \mathfrak{g}$. The identity $\mathfrak{g} = \lim_n \mathfrak{g} / \mathrm{F}_n\, \mathfrak{g}$ also implies that we have the vanishing relation $\mathrm{F}_{m+1}\, \mathfrak{g} = 0$ for some $m \geq 0$. Hence, our Lie algebra $\mathfrak{g}$ is nilpotent in the sense that all Lie monomials of weight $> m$ vanish in $\mathfrak{g}$. From this observation, we deduce that the Baker-Campbell-Hausdorff formula reduces to a finite sum in $\mathfrak{g}$, and as a consequence, is given by a polynomial expression in our choice of coordinates $G \simeq \Bbbk^N$. Thus, when we have $\dim \mathfrak{g} = N$, we obtain that the group G associated to $\mathfrak{g}$ forms an algebraic group in the classical sense of algebraic geometry (see for instance [33]).

Recall that an algebraic group is unipotent if and only if this group admits an embedding into a group of upper triangular matrices with unit entries on the diagonal. One can see that the algebraic group G which we associate to a finite dimensional nilpotent Lie algebra $\mathfrak{g}$ is unipotent. The embedding is provided by the Ado Theorem (see for instance [151, Appendix A, Proposition 3.6(b)]). One can conversely see that the Lie algebra of a unipotent group is nilpotent, and that a unipotent group is identified with the exponential group of this Lie algebra (see [55, Chapter IV, Proposition 4.1]).

8.3. The Malcev completion functor on groups

We define the Malcev completion of a group $G \in \mathcal{G}rp$ by the formula $\hat{G} = \mathbb{G}\,\Bbbk[G]^\wedge$, where we consider the complete group algebra of G. This group is automatically Malcev complete in our sense. We moreover have a natural morphism $\eta : G \to \hat{G}$ given by the unit of the adjunction between the complete group algebra functor and the functor of group-like elements. In fact, we formally get that this map $G \mapsto \hat{G}$ represents the left adjoint of the obvious forgetful functor from our category of Malcev complete groups to the category of groups, and the morphism $\eta : G \to \hat{G}$ defines the unit of this adjunction relation. Equivalently, we obtain that the object $\hat{G}$ is characterized by the following universal property:

Proposition 8.3.1. *Any group morphism $\phi : G \to H$ where $H = \mathbb{G}(A)$ is Malcev complete admits a unique factorization*

$$\begin{tikzcd}
G \arrow[rr, "\phi"] \arrow[dr] & & H \\
& \hat{G} \arrow[ur, dotted, "\hat{\phi}"'] &
\end{tikzcd}$$

such that $\hat{\phi}$ is a morphism of Malcev complete groups.

Proof. This proposition is an immediate consequence of the adjunction relation between the complete group algebra functor and the functor of group-like elements. ☐

To give a first example, we can easily determine the Malcev completion of an abelian group from the results of Proposition 8.1.7:

Proposition 8.3.2. *We assume that A is an abelian group. We use additive notation to identify this abelian group with a $\mathbb{Z}$-module (as in Proposition 8.1.7). We then have an identity:*

$$\hat{A} = A \otimes_{\mathbb{Z}} \Bbbk$$

where we consider the Malcev completion of our group on the left-hand side and the $\Bbbk$-module $M = A \otimes_{\mathbb{Z}} \Bbbk$ on the right-hand side.

Proof. The assertions of Proposition 8.1.7 give the identities $\Bbbk[A]^\wedge = \hat{\mathbb{S}}(A \otimes_{\mathbb{Z}} \Bbbk) \Rightarrow \hat{A} = \mathbb{G}(\hat{\mathbb{S}}(A \otimes_{\mathbb{Z}} \Bbbk)) = A \otimes_{\mathbb{Z}} \Bbbk$ from which we deduce the claim of this proposition. ☐

In the case of the Malcev completion $\hat{G} = \mathbb{G}\,\Bbbk[G]^\wedge$ of a group $G \in \mathcal{G}rp$, the weight graded Lie algebra $\mathtt{E}^0\,\hat{G} = \mathtt{E}^0\,\mathbb{P}\,\Bbbk[G]^\wedge$ which we determine from the filtration of the Malcev complete group $\hat{G} = \mathbb{G}\,\Bbbk[G]^\wedge$ in Proposition 8.2.3 has the following additional feature:

Proposition 8.3.3. *The weight graded Lie algebra $\mathtt{E}^0\,\hat{G} = \mathtt{E}^0\,\mathbb{P}\,\Bbbk[G]^\wedge$ is generated by its homogeneous component of weight one in the sense that every homogeneous element of this Lie algebra $\pi \in \mathtt{E}^0_m\,\hat{G}$, $m \geq 1$, can be expressed as a linear combination of iterated Lie brackets of homogeneous elements $\alpha \in \mathtt{E}^0_1\,\hat{G}$.*

Proof. For short, we set $H = \Bbbk[G]^\wedge$all along this proof. Recall that we have the relation $\mathtt{E}^0\,\mathbb{P}(H) = \mathbb{P}(\mathtt{E}^0\,H)$ (see Proposition 7.3.27). We identify the object $M = \mathtt{E}_0^1\,H$ with a weight graded module concentrated in weight one. We have $\mathtt{E}_0^0\,H = \Bbbk \Rightarrow \mathtt{E}_1^0\,H \subset \mathbb{P}(\mathtt{E}^0\,H)$ for homogeneity reasons, and we accordingly have the identity $\mathtt{E}_1^0\,\mathbb{P}(H) = \mathtt{E}_1^0\,H$ for the component of weight one of the weight graded Lie algebra $\mathtt{E}^0\,\mathbb{P}(H) = \mathbb{P}(\mathtt{E}^0\,H)$. We consider the morphism of weight graded Lie algebras $\rho : \mathbb{L}(\mathtt{E}_0^1\,H) \to \mathbb{P}(\mathtt{E}^0\,H)$ induced by this inclusion $\mathtt{E}_0^1\,H \subset \mathbb{P}\,\mathtt{E}^0\,H$. The claim of the proposition is equivalent to the assertion that this morphism is surjective.

The identity $\mathtt{E}^0\,\Bbbk[G]^\wedge = \bigoplus_{n=0}^{\infty} \mathbb{I}^n\,\Bbbk[G]/\,\mathbb{I}^{n+1}\,\Bbbk[G]$ implies that the weight graded Hopf algebra $\mathtt{E}^0\,H = \mathtt{E}^0\,\Bbbk[G]^\wedge$is generated by its homogeneous component of weight one $\mathtt{E}_1^0\,\Bbbk[G]^\wedge = \mathbb{I}\,\Bbbk[G]/\,\mathbb{I}^2\,\Bbbk[G]$ as an associative algebra. Equivalently, the morphism of weight graded associative algebras $\psi : \mathbb{T}(\mathtt{E}_0^1\,H) \to \mathtt{E}^0\,H$ induced by the inclusion $\mathtt{E}_0^1\,H \subset \mathtt{E}^0\,H$, where we still identify this object $M = \mathtt{E}_0^1\,H$ with a weight graded module concentrated in weight one, is surjective. We deduce our proposition from this observation and from the weight graded versions of the Poincaré-Birkhoff-Witt Theorem and of the Milnor-Moore Theorem which we establish in §7.3. We then provide the tensor algebra $L = \mathbb{T}(\mathtt{E}_1^0\,H)$ with the Hopf algebra structure of Proposition 7.2.6, where the coproduct is determined by the formula $\Delta(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha$ on generators $\alpha \in \mathtt{E}_1^0\,H$. We immediately see that our morphism $\psi : \mathbb{T}(\mathtt{E}_0^1\,H) \to \mathtt{E}^0\,H$ is a morphism of Hopf algebras.

Theorem 7.3.18(c) (the weight graded version of the Milnor-Moore Theorem) implies that we have the identity $\mathtt{E}^0\,H = \mathbb{U}\,\mathbb{P}(\mathtt{E}^0\,H)$ for the weight graded Hopf algebra $\mathtt{E}^0\,H$. We have on the other hand $\mathbb{T}(\mathtt{E}_1^0\,H) = \mathbb{U}\,\mathbb{L}(\mathtt{E}_1^0\,H)$ (see §7.2.7) and, by composing adjunction relations, we can readily identify the morphism $\psi : \mathbb{T}(\mathtt{E}_0^1\,H) \to \mathtt{E}^0\,H$ with the morphism of enveloping algebras $\psi : \mathbb{U}\,\mathbb{L}(\mathtt{E}_1^0\,H) \to \mathbb{U}\,\mathbb{P}(\mathtt{E}^0\,H)$ associated to our morphism of Lie algebras $\rho : \mathbb{L}(\mathtt{E}_1^0\,H) \to \mathbb{P}(\mathtt{E}^0\,H)$. Theorem 7.3.18(b) (the weight graded version of the Poincaré-Birkhoff-Witt Theorem) now implies that this morphism of Lie algebras $\rho : \mathbb{L}(\mathtt{E}_1^0\,H) \to \mathbb{P}(\mathtt{E}^0\,H)$ forms a retract of $\psi : \mathbb{T}(\mathtt{E}_0^1\,H) \to \mathtt{E}^0\,H$, and, hence, we get that this morphism $\rho : \mathbb{L}(\mathtt{E}_1^0\,H) \to \mathbb{P}(\mathtt{E}^0\,H)$ is surjective (as we require) as soon as the morphism of associative algebras $\psi : \mathbb{T}(\mathtt{E}_0^1\,H) \to \mathtt{E}^0\,H$ is so. ☐

Let us mention that the assertion of the previous proposition is part of the conventions of [151, Appendix A, Definition 3.1] for the definition of the category of Malcev groups (in the context of rational coefficients $\Bbbk = \mathbb{Q}$). To be more precise, in the approach of this reference, the category of (rational) Malcev groups is defined as the category of groups G equipped with a filtration $G = \mathtt{F}_1\,G \supset \cdots \supset \mathtt{F}_s\,G \supset \cdots$ such that the conditions of §8.2.1 hold and where the weight graded Lie algebra $\mathtt{E}^0\,G$, which we determine from this filtration, forms a $\mathbb{Q}$-module and is generated by its homogeneous component of weight one $\mathtt{E}_1^0\,G \subset \mathtt{E}^0\,G$. For short, we also say that the Lie $\mathtt{E}^0\,G$ is *generated in weight one* when this property holds. The main result of [151, Appendix A, §3] implies that this category of Malcev groups is equivalent to the subcategory of our category of Malcev complete groups $G = \mathbb{G}(H)$ whose associated weight graded Lie algebra $\mathtt{E}^0\,G = \mathtt{E}^0\,\mathbb{P}(H)$ is generated in weight one.

8.3.4. *Remarks: complements on the weight graded Lie algebra and on the filtration of the Malcev completion of a group.* We explained in §8.2.2 that we can associate a weight graded Lie algebra (defined over $\mathbb{Z}$) to any filtration of a group G that satisfies the commutator conditions of §8.2.1. We consider the weight graded Lie algebra $\mathtt{E}^0\,G = \bigoplus_m \Gamma_m\,G/\,\Gamma_{m+1}\,G$ defined by the subquotients of the lower central series filtration of any group G (see §8.2.1) and the weight graded Lie algebra $\mathtt{E}^0\hat{G} = \bigoplus_m \mathtt{F}_m\hat{G}/\mathtt{F}_{m+1}\hat{G}$ associated to the Malcev completion of our group $\hat{G} = \mathbb{G}(\Bbbk[G]\hat{\ })$ (see Proposition 8.2.3). We have a natural morphism of Lie algebras $\upsilon : \bigoplus_m \Gamma_m G/\Gamma_{m+1} G \to \bigoplus_m \mathtt{F}_m\hat{G}/\mathtt{F}_{m+1}\hat{G}$ because the canonical morphism $\eta : G \to \hat{G}$ with values in the Malcev completion of our group $\hat{G} = \mathbb{G}(\Bbbk[G]\hat{\ })$ carries $\Gamma_m G \subset G$ into $\Gamma_m\hat{G} \subset \mathtt{F}_m\hat{G}$ for each $m \geq 1$. The main theorem of [153] asserts that the extension of this morphism to our coefficient ring $\Bbbk$ defines an isomorphism of Lie algebras:

$$\upsilon : \underbrace{(\bigoplus_m \Gamma_m G/\Gamma_{m+1} G)}_{=\mathtt{E}^0 G} \otimes_{\mathbb{Z}} \Bbbk \xrightarrow{\simeq} \underbrace{(\bigoplus_m \mathtt{F}_m\hat{G}/\mathtt{F}_{m+1}\hat{G})}_{=\mathtt{E}^0\hat{G}}, \tag{1}$$

for any group G.

By using this relation and the assertions of Proposition 8.2.4 and Proposition 8.2.5, we readily get that the natural morphism $G/\Gamma_{m+1} G \to \hat{G}/\mathtt{F}_{m+1}\hat{G}$ factors through an isomorphism of Malcev complete groups:

$$(G/\Gamma_{m+1} G)\hat{\ } \xrightarrow{\simeq} \hat{G}/\mathtt{F}_{m+1}\hat{G}, \tag{2}$$

where we consider the Malcev completion of the quotient group $G/\Gamma_{m+1} G$ on the one hand and the quotient of the Malcev group $\hat{G}$ by the $m+1$st layer of its natural filtration on the other hand. If we take these isomorphisms (1-2) together, then we conclude that our Malcev completion process carries the central extensions $1 \to \Gamma_m G/\Gamma_{m+1} G \to G/\Gamma_{m+1} G \to G/\Gamma_m G \to 1$ to the short exact sequences $1 \to \mathtt{E}^0_m\hat{G} \to \hat{G}/\mathtt{F}_{m+1}\hat{G} \to \hat{G}/\mathtt{F}_m\hat{G} \to 1$ where we consider the subquotients $\mathtt{E}^0_m\hat{G}$ of this filtration of the group $\hat{G}$. This observation also implies that our Malcev completion functor, of which we borrow the definition from [151, Appendix A], returns the same result as the initial definition of the classical Malcev completion process in [132].

We use the observations of this paragraph in side remarks, but not really in our main applications. We refer to [153] for the proof of the above isomorphism relation (1), which we mainly recall for the sake of completeness.

8.3.5. *Remark: The Malcev completion of nilpotent groups.* Recall that a group is G is nilpotent when we have $\Gamma_s G = 1$ for some $s > 0$. The main examples of groups which we consider in this monograph (free groups, pure braid groups) are not nilpotent, but we will take the case of nilpotent groups as a model for the other examples of applications of the Malcev completion which we consider in this work. We therefore give a short survey of the main results of the literature on the Malcev completion of nilpotent groups. We refer to [151, Appendix A.3] for the proof of the general statements which we recall in this overview.

First of all, we may check that the Malcev completion of a nilpotent group G is still nilpotent. To be more precise, the identity of §8.3.4(2) implies that we have the relation $\Gamma_s G = 1 \Rightarrow \mathtt{F}_s\hat{G} = 1$ for any group G, where we consider the natural filtration of this Malcev complete group $\hat{G}$. By [151, Appendix A, Proposition 3.5], the vanishing relation $\mathtt{F}_s\hat{G} = 1$ also implies that we have an identity:

$$\mathtt{F}_m\hat{G} = \Gamma_m\hat{G}$$

for every $m \geq 1$, where we now consider the lower central series filtration of the group $\hat{G}$.

If we take $\Bbbk = \mathbb{Q}$ as coefficient ring for our Malcev completion process, then we can also check that the Malcev completion functor is idempotent on nilpotent groups. To be more explicit, in this case $\Bbbk = \mathbb{Q}$, we get that the Malcev completion functor carries the universal morphism $\eta : G \to \hat{G}$ associated to a nilpotent group G to an isomorphism of Malcev complete groups:

$$\hat{\eta} : \hat{G} \xrightarrow{\simeq} \hat{\hat{G}}.$$

This observation implies that any group morphism $\psi : \hat{G} \to H$, where $H = \mathbb{G}(A)$ is a Malcev complete group in our sense, defines a morphism of Malcev complete groups and, hence, arises from a morphism of complete Hopf algebras $\phi : \mathbb{Q}[G]^\frown \to A$ (see Proposition 8.4.6 for the proof of an analogous result in the case of free groups with a finite number of generators).

Recall that a group G is uniquely divisible when the equation $g^n = h$ has a unique solution $g \in G$, for any $h \in G$ and $n \in \mathbb{Z}\backslash\{0\}$. By [151, Appendix A, Corollary 3.7] (see also [113, Theorem 4.15]), we actually have an equivalence of categories between the category of uniquely divisible nilpotent groups and the category formed by the Malcev complete groups $G = \mathbb{G}(H)$ whose filtration satisfies $\mathtt{F}_s\, G = 1$ for some $s > 0$ and whose associated weight graded Lie algebra $\mathtt{E}^0\, G$ is generated by $\mathtt{E}^0_1\, G$ as a Lie algebra. (We still assume that we take $\Bbbk = \mathbb{Q}$ as ground ring in this case.) The group returned by our Malcev completion functor $\hat{G}$ actually represents a universal uniquely divisible group associated to G (see [151, Appendix A, Corollary 3.8]).

8.4. The Malcev completion of free groups

We now study the Malcev completion of free groups. We also briefly explain the Malcev completion of groups defined by a presentation by generators and relations in the concluding paragraph of this section.

We focus on the case of finitely generated groups and we use the notation $\mathbb{F}(x_1, \dots, x_n)$ for the free group generated by an n-tuple of variables $(x_1, \dots, x_n)$. We consider the Malcev completion with coefficients in an arbitrary field of characteristic zero $\Bbbk$ for the moment. We have the following result:

Proposition 8.4.1. *For a free group $F = \mathbb{F}(x_1, \dots, x_n)$, we have an isomorphism of complete Hopf algebras*

$$\Bbbk[\mathbb{F}(x_1, \dots, x_n)]^\frown \xrightarrow{\simeq} \hat{\mathbb{T}}(\xi_1, \dots, \xi_n),$$

where $\hat{\mathbb{T}}(\xi_1, \dots, \xi_n)$ is a short notation for the complete tensor algebra associated to the free $\Bbbk$-module $M = \Bbbk\,\xi_1 \oplus \dots \oplus \Bbbk\,\xi_n$ equipped with the filtration such that $\mathtt{F}_1\, M = M$ and $\mathtt{F}_s\, M = 0$ for $s > 1$. This complete tensor algebra is equipped with the canonical Hopf algebra structure of §7.2.6 *(see also* §7.3.22*) so that each generator ξ_i defines a primitive element in $\hat{\mathbb{T}}(\xi_1, \dots, \xi_n)$. We also have an identity:*

$$\mathbb{P}\,\Bbbk[\mathbb{F}(x_1, \dots, x_n)]^\frown = \hat{\mathbb{L}}(\xi_1, \dots, \xi_n)$$

when we pass to the primitive part, where $\hat{\mathbb{L}}(\xi_1, \dots, \xi_n)$ is again a short notation for the free complete Lie algebra associated to the free $\Bbbk$-module $M = \Bbbk\,\xi_1 \oplus \dots \oplus \Bbbk\,\xi_n$.

Proof. The elements ξ_i are primitive by definition of the Hopf algebra structure of the tensor algebra, and the associated exponential elements e^{ξ_i} are group-like by Proposition 8.1.5.

We consider the group morphism $\phi : \mathbb{F}(x_1,\dots,x_n) \to \mathbb{G}(\hat{\mathbb{T}}(\xi_1,\dots,\xi_n))$ such that $\phi(x_i) = e^{\xi_i}$, for $i = 1,\dots,n$, and the associated morphism of complete Hopf algebras $\phi_\sharp : \Bbbk[\mathbb{F}(x_1,\dots,x_n)]^\wedge \to \hat{\mathbb{T}}(\xi_1,\dots,\xi_n)$. We have a Hopf algebra morphism in the conversion direction $\psi : \hat{\mathbb{T}}(\xi_1,\dots,\xi_n) \to \Bbbk[\mathbb{F}(x_1,\dots,x_n)]^\wedge$ which assigns the logarithm $\log[x_i]$ of the group-like elements $[x_i]$ in $\Bbbk[\mathbb{F}(x_1,\dots,x_n)]^\wedge$ to the generating elements of the tensor algebra ξ_i, $i = 1,\dots,n$.

We have $\phi_\sharp\psi(\xi_i) = \log\exp(\xi_i) = \xi_i$ for each i so that $\phi_\sharp\psi = id$. We also have $\mathbb{G}(\psi)(\phi(x_i)) = x_i$ for each generator of the free group x_i, where we consider the morphism induced by our complete Hopf algebra morphism ψ on the sets of group-like elements $\mathbb{G}(-)$. We have as a consequence $\mathbb{G}(\psi)\phi = \iota$, where ι refers to the standard morphism $\iota : \mathbb{F}(x_1,\dots,x_n) \to \mathbb{G}\,\Bbbk[\mathbb{F}(x_1,\dots,x_n)]^\wedge$ that defines the unit morphism of the adjunction between the complete group algebra functor and the functor of group-like elements. We have $\mathbb{G}(\psi)\phi = \iota \Rightarrow \psi\phi_\sharp = id$ by adjunction. We conclude from this result that $\phi_\sharp$ and ψ define inverse isomorphisms between the complete group algebra of the free group and the complete tensor algebra, while the second assertion of the proposition follows from the result of Proposition §7.2.14 (see also §7.3.24). ☐

8.4.2. *Commutator expansions and the Malcev completion of the free group.* In what follows, we use the notation $\hat{\mathbb{F}}(x_1,\dots,x_n)$ for the Malcev completion of the free group $F = \mathbb{F}(x_1,\dots,x_n)$. The idea is that an element of this group $g \in \hat{\mathbb{F}}(x_1,\dots,x_n)$ can be regarded as a group-like power series $g = g(x_1,\dots,x_n)$ on the variables $(x_1,\dots,x_n)$. In a first step, we can use the result of Proposition 8.4.1 to get that any such $g \in \hat{\mathbb{F}}(x_1,\dots,x_n)$ is represented by the exponential of a Lie power series $h = h(\xi_1,\dots,\xi_n)$ on variables $(\xi_1,\dots,\xi_n)$ such that $\xi_i = \log(x_i) \Leftrightarrow x_i = e^{\xi_i}$, for $i = 1,\dots,n$.

We can also give an explicit representation of these group-like power series in terms of iterated commutators on the variables $(x_1,\dots,x_n)$ in the free group. We review the statement of an analogous observation for the plain free group $F = \mathbb{F}(x_1,\dots,x_n)$ before tackling the case of the Malcev complete group $\hat{F} = \hat{\mathbb{F}}(x_1,\dots,x_n)$. We use that the free Lie algebra $\mathbb{L}(\xi_1,\dots,\xi_n)$ has a definition over $\mathbb{Z}$ and forms a free $\mathbb{Z}$-module (see [34, II.2.9] or [155, §0.3]). We can actually provide each homogeneous component of the free Lie algebra $\mathbb{L}_r(\xi_1,\dots,\xi_n)$, $r > 0$, with a basis $H(r) = \{h(\xi_1,\dots,\xi_n)\}$, whose elements $h = h(\xi_1,\dots,\xi_n) \in H(r)$ are Lie monomials of weight $r > 0$ on the variables $\xi_1,\dots,\xi_n$ (see [155] for a general reference on this subject).

By a fundamental result of combinatorial group theory, the subquotients of the lower central series filtration of a free group (see §8.2.1) form a free weight graded Lie algebra (over $\mathbb{Z}$):

$$(1)\qquad \bigoplus_{s>0} \Gamma_s\,\mathbb{F}(x_1,\dots,x_n)/\Gamma_{s+1}\,\mathbb{F}(x_1,\dots,x_n) = \mathbb{L}(\xi_1,\dots,\xi_n),$$

where we assume that each generator ξ_i is homogeneous of weight 1 (see for instance [34, §II.5.4] or [131, Theorem 5.12]). This statement implies that any element of the quotient $F/\Gamma_{s+1}F = \mathbb{F}(x_1,\dots,x_n)/\Gamma_{s+1}\,\mathbb{F}(x_1,\dots,x_n)$ of the free group $F = \mathbb{F}(x_1,\dots,x_n)$ has a unique representative

$$(2)\qquad g_s(x_1,\dots,x_n) = x_1^{a_1}\cdot\ldots\cdot x_n^{a_n}\cdot \prod_{\substack{h\in H(r)\\ 2\leq r\leq s}} h(x_1,\dots,x_n)^{a_h},$$

where we assume $a_1, \dots, a_n \in \mathbb{Z}$ and $a_h \in \mathbb{Z}$ for $h \in H(r)$. We replace the iterated Lie brackets $[-,-]$ of our Lie monomials $h(\xi_1, \dots, \xi_n)$ by iterated commutators of the elements x_i, $i = 1, \dots, n$, in the free group to form the factors of this expansion $h(x_1, \dots, x_n) \in \mathbb{F}(x_1, \dots, x_r)$ (see [84] for the original statement of this result, see [34, §II.5.4] or [131, Theorem 5.13A] for an account of this result in our reference books on this subject). We perform this product in the increasing direction of the weight index $r > 0$, and with respect to a fixed order within each indexing set $H(r)$.

Recall that the pro-nilpotent completion of a group G is defined by the limit $\hat{G} = \lim_s G/\Gamma_{s+1} G$, where we consider the lower central series filtration of our group. In our expansion (2), we obviously have $h(\xi_1, \dots, \xi_n) \in \mathbb{L}_r(\xi_1, \dots, \xi_n) \Rightarrow h(x_1, \dots, x_n) \in \Gamma_r \mathbb{F}(x_1, \dots, x_n)$ and we can therefore take $s \to \infty$ in order to get a representation of the elements of the pro-nilpotent completion of the free group $F = \mathbb{F}(x_1, \dots, x_n)$, and not only of the quotient groups $F/\Gamma_{s+1} F$.

We can now use the result of Proposition 8.4.1 together with the general results of Proposition 7.3.27 and Proposition 8.2.3 in order to establish parallel statements for the Malcev completion of the free group $\hat{F} = \hat{\mathbb{F}}(x_1, \dots, x_n)$. We consider the isomorphisms

$$\mathbb{L}(\xi_1, \dots, \xi_n) \xrightarrow{\simeq} \mathtt{E}^0 \hat{\mathbb{L}}(\xi_1, \dots, \xi_n) \xrightarrow{\simeq} \mathtt{E}^0 \hat{\mathbb{F}}(x_1, \dots, x_n) \tag{3}$$

given by the results of these propositions. We deduce from these isomorphisms that the weight graded object $\mathtt{E}^0 \hat{F}$, which we construct by using the natural filtration $\hat{F} = \mathtt{F}_1 \hat{F} \supset \cdots \supset \mathtt{F}_s \hat{F} \supset \cdots$ of the Malcev complete group $\hat{F} = \hat{\mathbb{F}}(x_1, \dots, x_n)$, is identified with a free weight graded Lie algebra, as in (1), but we now work over our ground field $\Bbbk$ instead of the ring of integers $\mathbb{Z}$.

We use this correspondence to check that each element $g = g(x_1, \dots, x_n)$ of the Malcev completion $\hat{F} = \hat{\mathbb{F}}(x_1, \dots, x_n)$ has an expansion of the form (2) in the quotient group $\hat{F}/\mathtt{F}_{s+1} \hat{F}$, where we consider the natural filtration of our Malcev complete group again, but we now take our exponents in our coefficient field $a_1, \dots, a_n \in \Bbbk$, $a_h \in \Bbbk$ (and not only in $\mathbb{Z}$). When we form this expansion, we also set $x_i = e^{\xi_i}$ to identify our variables x_i, $i = 1, \dots, n$, with the image of elements of the complete free Lie algebra $\hat{\mathbb{L}}(\xi_1, \dots, \xi_n)$ under the map of Proposition 8.4.1. We can still take $s \to \infty$ in our expansion (2) to extend this representation to the elements of the Malcev complete group itself $\hat{F} = \hat{\mathbb{F}}(x_1, \dots, x_n)$. We then use the identity $\hat{F} = \lim_s \hat{F}/\mathtt{F}_{s+1} \hat{F}$. We can also identify $\mathtt{F}_s \hat{F} = \mathtt{F}_s \hat{\mathbb{F}}(x_1, \dots, x_n)$ with the subgroup of $\hat{F}$ formed by the elements whose infinite expansion have a leading term in weight $r = s$.

We also use the relationship between the weight graded Lie algebra $\mathtt{E}^0 \hat{F} = \mathtt{E}^0 \hat{\mathbb{F}}(x_1, \dots, x_n)$ and the free Lie algebra $L = \mathbb{L}(\xi_1, \dots, \xi_n)$ in the proof of the following proposition:

PROPOSITION 8.4.3. *For a free group $F = \mathbb{F}(x_1, \dots, x_n)$ with a finite number of generators $n \in \mathbb{N}$, we have an identity $\mathtt{F}_s \hat{F} = \Gamma_s \hat{F}$, for every $s > 0$, where we consider the natural filtration of the Malcev complete group $\hat{F} = \hat{\mathbb{F}}(x_1, \dots, x_n)$ on the left-hand side, and the lower central series filtration of the plain (abstract) group underlying $\hat{F}$ on the right-hand side.*

PROOF. We set $\hat{F} = \hat{\mathbb{F}}(x_1, \dots, x_n)$ for short all along this proof. Recall that our filtration automatically satisfies $\Gamma_s \hat{F} \subset \mathtt{F}_s \hat{F}$ (see §8.2.1). We aim to check that a converse inclusion relation holds. We prove that the image of any element $g = g(x_1, \dots, x_n) \in \mathtt{F}_s \hat{F}$ in the quotient group $\hat{F}/\mathtt{F}_t \hat{F}$ has a representative of the form:

$$g_t(x_1, \dots, x_n) = \prod_{i_2,\dots,i_s} ((\cdots((h^{\underline{i}}_t, x_{i_2}), x_{i_3}), \dots), x_{i_s}), \tag{1}$$

where the product ranges over all collections $\underline{i} = (i_1, \dots, i_{s-1})$ and where we have the identity $h^{\underline{i}}_{t+1} \equiv h^{\underline{i}}_t$ in $\hat{F}/\mathtt{F}_{t-s} \hat{F}$ for all $t \geq s$. We then have $h^{\underline{i}}_t \equiv h_{\underline{i}}$ for an element of the Malcev complete group $h_{\underline{i}} \in \hat{F}$, and we can take $t \to \infty$ in this formula (1) to get the required expression of $g = g(x_1, \dots, x_n) \in \mathtt{F}_s \hat{F}$ in terms of s-fold commutators in $\hat{F}$.

We use that the components $\mathbb{L}_t(\xi_1, \dots, \xi_n)$, $t > 0$, of the free weight graded Lie algebra $\mathbb{L}(\xi_1, \dots, \xi_n)$ are spanned by Lie monomials $[[\cdots[[\xi_{i_1}, \xi_{i_2}], \xi_{i_3}], \dots], \xi_{i_t}]$, where we consider iterated Lie brackets of the same shape as the iterated commutators of our expansions (1). We assume that we have established our formula (1) up to some level $t > 0$. We then have $g \equiv g_t r_t$ in $\hat{F}/\mathtt{F}_{t+1} \hat{F}$, for some element $r_t \in \mathtt{F}_t \hat{F}$. We moreover have $r_t \equiv e^{\rho_t}$, where $\rho_t = \rho_t(\xi_1, \dots, \xi_n)$ is a Lie polynomial of weight t, which also admits an expansion of the form:

$$\rho_t(\xi_1, \dots, \xi_n) = \sum_{i_2,\dots,i_s} [[\cdots[[\sigma^{\underline{i}}_t, \xi_{i_2}], \xi_{i_3}], \dots], \xi_{i_s}], \tag{2}$$

for some Lie polynomials $\sigma^{\underline{i}}_t \in \mathbb{L}_{t-s}(\xi_1, \dots, \xi_n)$. We use that the exponential map carries Lie brackets to commutators when we pass to the subquotient $\mathtt{E}^0_t \hat{F} = \mathtt{F}_t \hat{F}/\mathtt{F}_{t+1} \hat{F}$ of the Malcev complete group $\hat{F}$ (see Proposition 8.2.3) and that we have $\theta^{\underline{i}}_t = \exp(\sigma^{\underline{i}}_t) \in \mathtt{F}_{t-s} \hat{F}$. We accordingly have the identity:

$$\begin{aligned} g \equiv g_t r_t \equiv & \prod_{i_2,\dots,i_s} ((\cdots((h^{\underline{i}}_t, x_{i_2}), x_{i_3}), \dots), x_{i_s}) \\ & \cdot \prod_{i_2,\dots,i_s} ((\cdots((\theta^{\underline{i}}_t, x_{i_2}), x_{i_3}), \dots), x_{i_s}) \end{aligned} \tag{3}$$

in $\hat{F}/\mathtt{F}_{t+1} \hat{F}$. We can move the factors $\pi = ((\cdots((\theta^{\underline{i}}_t, x_{i_2}), x_{i_3}), \dots), x_{i_s})$ in this expansion without changing the value of our expression in $\hat{F}/\mathtt{F}_{t+1} \hat{F}$ since we have the relation $\pi \in \mathtt{F}_t \hat{F} \Rightarrow (h, \pi) \in \mathtt{F}_{t+1} \hat{F} \Rightarrow h\pi = \pi h(h, \pi) \equiv \pi h (\operatorname{mod} \mathtt{F}_{t+1} \hat{F})$, for any $h \in \hat{F}$. We moreover have the relation

$$\begin{aligned} ((\cdots((h^{\underline{i}}_t, x_{i_2}), x_{i_3}), \dots), x_{i_s}) \cdot ((\cdots((\theta^{\underline{i}}_t, x_{i_2}), x_{i_3}), \dots), x_{i_s}) & \\ \equiv ((\cdots((\theta^{\underline{i}}_t \cdot h^{\underline{i}}_t, x_{i_2}), x_{i_3}), \dots), x_{i_s}) & \end{aligned} \tag{4}$$

by the Philip Hall identities (see §8.2.2). We can therefore set $h^{\underline{i}}_{t+1} = h^{\underline{i}}_t \cdot \theta^{\underline{i}}_t$ to carry on our process. □

We use the result of this proposition in the proof of the following lemma:

LEMMA 8.4.4. *If we take $\Bbbk = \mathbb{Q}$ as a ground ring, and we consider a free group $F = \mathbb{F}(x_1, \dots, x_n)$ with a finite number of generators again $n \in \mathbb{N}$, then we have an isomorphism of complete Hopf algebras*

$$\rho : \mathbb{Q}[\hat{\mathbb{F}}(x_1, \dots, x_n)]^{\wedge} \xrightarrow{\simeq} \hat{\mathbb{T}}(\xi_1, \dots, \xi_n),$$

where we consider the complete group algebra of the Malcev completion of our group $F = \mathbb{F}(x_1,\dots,x_n)$ *on the left hand side, and the completed tensor algebra of Proposition* 8.4.1 *on the right hand side.*

Proof. For short, we still set $\hat{F} = \hat{\mathbb{F}}(x_1,\dots,x_n)$ all along this proof. We use the identity $\hat{\mathbb{F}}(x_1,\dots,x_n) = \mathbb{G}\,\hat{\mathbb{T}}(\xi_1,\dots,\xi_n)$ obtained in Proposition 8.4.4. The morphism of complete Hopf algebras $\rho : \mathbb{Q}[\hat{F}]\hat{\ } \to \hat{\mathbb{T}}(\xi_1,\dots,\xi_n)$ which we consider in our statement is induced by the inclusion $\hat{F} = \mathbb{G}\,\hat{\mathbb{T}}(\xi_1,\dots,\xi_n) \subset \hat{\mathbb{T}}(\xi_1,\dots,\xi_n)$ and represents the augmentation of the adjunction $\mathbb{Q}[-]\hat{\ } : \mathcal{G}rp \rightleftarrows \hat{f}\,\mathcal{H}opf\mathcal{A}lg : \mathbb{G}$ for the complete tensor algebra $H = \hat{\mathbb{T}}(\xi_1,\dots,\xi_n)$.

We have an obvious morphism of complete Hopf algebras which goes the other way round $\psi : \hat{\mathbb{T}}(\xi_1,\dots,\xi_n) \to \mathbb{Q}[\hat{F}]\hat{\ }$ and which we explicitly define by the formula $\psi(\xi_i) = \log[e^{\xi_i}]$, for any generating element ξ_i, $i = 1,\dots,n$. We immediately see that we have the identity $\rho\psi(\xi_i) = \log(e^{\xi_i}) = \xi_i$ in $\hat{\mathbb{T}}(\xi_1,\dots,\xi_n)$, for any $i = 1,\dots,n$, and we are left to check the validity of a converse relation $\psi\rho([g]) = [g]$, for any element $g \in \hat{F}$.

We consider the class of such an element g in the quotient group $\hat{F}/\,\mathtt{F}_{s+1}\,\hat{F}$. We use the notation $g_s = g_s(x_1,\dots,x_n)$ for the representative of this class defined by the expansion of §8.4.2(2). Recall that we set $x_i = e^{\xi_i}$ to identify the variables x_i, $i = 1,\dots,n$, with group-like elements of the complete tensor algebra $\hat{\mathbb{T}}(\xi_1,\dots,\xi_n)$. We have $g = g_s h_s$, where $h_s \in \mathtt{F}_{s+1}\,\hat{F}$, and the result of Proposition 8.4.3 implies that this remainder h_s belongs to $\Gamma_{s+1}\,\hat{F}$.

We clearly have the identity $\psi\rho([h(e^{\xi_1},\dots,e^{\xi_n})]) = [h(e^{\xi_1},\dots,e^{\xi_n})]$, for any factor of our expansion §8.4.2(2) since each of these factors consist of iterated commutators of the generating variables $x_i = e^{\xi_i}$, $i = 1,\dots,n$. We moreover have $\psi\rho([h]) = [h] \Leftrightarrow \psi\rho([h^a]) = [h^a]$ for any rational exponent $a = p/q$ and any $h \in \mathbb{G}\,\hat{\mathbb{T}}(\xi_1,\dots,\xi_n)$, because the element $[h^{p/q}]$ is characterized by the relation $[h^{p/q}]^q = [h^p] = [h]^p$ in $\mathbb{G}\,\hat{\mathbb{T}}(\xi_1,\dots,\xi_n) \subset \hat{\mathbb{T}}(\xi_1,\dots,\xi_n)$.

We deduce from these observations that we have the relation $\psi\rho([g_s]) = [g_s]$, for any $s > 0$. Recall that $h_s \in \mathtt{F}_{s+1}\,\hat{F}$ can be expressed as a product of $(s+1)$-fold commutators in $\hat{F}$ by the result of Proposition 8.4.3. Now, we have the general formula $[(u,v)] = [u^{-1}v^{-1}](([u]-1)([v]-1)-([v]-1)([u]-1))$ in the complete group algebra $\mathbb{Q}[\hat{F}]\hat{\ }$, for any commutator $(u,v) = u^{-1}v^{-1}uv$. We also have $[u]-1, [v]-1 \in \mathbb{I}\,\mathbb{Q}[\hat{F}]$. For a product of $(s+1)$-fold commutators such as $h_s \in \Gamma_{s+1}\,\hat{F}$, we obtain by a straightforward induction that we have $[h_s] \in \mathbb{I}^{s+1}\,\mathbb{Q}[\hat{F}]\hat{\ }$, where we consider the $(s+1)$th power of the augmentation ideal of the complete algebra $\mathbb{Q}[\hat{F}]\hat{\ }$. We accordingly have $g \equiv g_s (\operatorname{mod} \mathbb{I}^{s+1}\,\mathbb{Q}[\hat{F}]\hat{\ })$ and hence we have $\psi\rho([g_s]) = [g_s] \Rightarrow \psi\rho([g]) \equiv [g] (\operatorname{mod} \mathbb{I}^{s+1}\,\mathbb{Q}[\hat{F}]\hat{\ })$, for any $s > 0$. We conclude from these relations that we have the identity $\psi\rho([g]) = [g]$ for our element $g \in \hat{F}$ in the complete group algebra $\mathbb{Q}[\hat{F}]\hat{\ }$, and this result finishes the proof of the lemma. □

We use this lemma in the proof of the following statement:

Proposition 8.4.5. *The Malcev completion with rational coefficients is idempotent on free groups with a finite number of generators. To be explicit, if we take* $\Bbbk = \mathbb{Q}$ *as coefficient ring for our Malcev completion process, then the Malcev completion functor carries the universal morphism* $\eta : F \to \hat{F}$ *associated to such a* *group $F = \mathbb{F}(x_1, \dots, x_n)$ to an isomorphism of Malcev complete groups:*

$$\hat{\eta} : \hat{F} \xrightarrow{\simeq} \hat{\hat{F}}.$$

Proof. Recall that we have $\mathbb{Q}[F]^\wedge = \hat{\mathbb{T}}(\xi_1, \dots, \xi_n)$ for a free group $F = \mathbb{F}(x_1, \dots, x_n)$ and the isomorphism of the previous lemma, which reads $\rho : \mathbb{Q}[\hat{F}]^\wedge \xrightarrow{\simeq} \mathbb{Q}[F]^\wedge$, represents the augmentation of the adjunction $\mathbb{Q}[-]^\wedge : \mathcal{G}rp \rightleftarrows \hat{f}\,\mathcal{H}opf\mathcal{A}lg : \mathbb{G}$ for this complete group algebra $\mathbb{Q}[F]^\wedge = \hat{\mathbb{T}}(\xi_1, \dots, \xi_n)$. We have $\mathbb{G}(\rho)\hat{\eta} = id$ by general properties of adjunction relations, and we conclude from this relation that $\hat{\eta}$ is an isomorphism, as claimed in our proposition. ☐

Let us emphasize that we forget about the Malcev complete group structure of our object $\hat{F}$ when we perform the completion a second time. In a sense, the idea of this proposition is that the structures which we attach to the Malcev completion of the group $\hat{F}$ are determined by the group structure of our object. We have the following observation which relies on the same idea:

Proposition 8.4.6. *Let $F = \mathbb{F}(x_1, \dots, x_n)$ be a free group with a finite number of generators as in Proposition* 8.4.3. *If we take $\Bbbk = \mathbb{Q}$ as coefficient ring for our Malcev completion process, then every group morphism $\psi : \hat{F} \to H$, where H is Malcev complete, automatically defines a morphism of Malcev complete groups in our sense.*

Proof. We consider the universal morphism $\eta : \hat{F} \to \hat{\hat{F}}$ associated to the group $G = \hat{F}$ and the morphism of Malcev complete groups $\hat{id} : \hat{\hat{F}} \to \hat{F}$ which extends the identity morphism of $\hat{F}$. We have the relations $\hat{id}\,\hat{\eta} = id$ and $\hat{id}\,\eta = id$ by adjunction. We use that $\hat{\eta}$ is an isomorphism of Malcev complete groups (Proposition 8.4.5) and the first of these relations $\hat{id}\,\hat{\eta} = id$ to get that $\hat{id}$ forms an isomorphism as well. We explicitly have $\hat{id} = \hat{\eta}^{-1}$. We then have $\hat{id}\,\eta = id \Rightarrow \eta = \hat{id}^{-1}$ and we deduce from this identity that η defines morphism of Malcev complete groups itself. We now form the following diagram:

$$\begin{tikzcd}
\hat{F} \arrow[rr, "\psi"] \arrow[dr, "\eta"'] & & H \\
& \hat{\hat{F}} \arrow[ur, dotted, "\exists\hat{\psi}"'] &
\end{tikzcd},$$

where we consider the morphism of Malcev complete groups extending ψ. We eventually obtain that our morphism $\psi = \hat{\psi}\eta$ defines a morphism of Malcev complete groups by composition. ☐

8.4.7. *The Malcev completion of a group defined by a presentation by generators and relations.* The result of Proposition 8.4.1 can be used to determine the primitive Lie algebra $\mathbb{P}\,\Bbbk[G]^\wedge$ for a group given by a presentation by generators and relations $G = \langle x_1, \dots, x_n : w_0^1 \equiv w_1^1, \dots, w_0^r \equiv w_1^r \rangle$. The definition of a group by such a presentation can be formulated in terms of a reflexive coequalizer of free groups

$$\begin{tikzcd}
\underbrace{\mathbb{F}(x_1, \dots, x_n, z_1, \dots, z_r)}_{=F_1} \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & \underbrace{\mathbb{F}(x_1, \dots, x_n)}_{=F_0} \arrow[l, bend right=40, "s_0"'] \arrow[r, dotted, "\epsilon"] & \underbrace{\langle x_1, \dots, x_n : w_0^j \equiv w_1^j \rangle}_{=G}
\end{tikzcd},$$

where d_0, d_1 and s_0 are both the identity on the generating elements x_i, and we set $d_0(z_j) = w_0^j(x_1, \dots, x_n)$, $d_1(z_j) = w_1^j(x_1, \dots, x_n)$ for the remaining variables.

The category of complete Lie algebras inherits a colimits from the base category of complete filtered modules like any category of algebras over an operad in a symmetric monoidal category whose tensor product distributes over colimits (see Proposition 1.3.6). Moreover, the forgetful functor from complete Lie algebras to complete filtered modules preserves the coequalizers which are reflexive in the base category (see again Proposition 1.3.6). The category of complete Hopf algebras inherits colimits too, since this category is equivalent to the category of complete Lie algebras (by the Milnor-Moore Theorem for complete Hopf algebras, Theorem 7.3.26). The primitive element functor, which we use to define this equivalence of categories, obviously preserves colimits.

The functor $\Bbbk[-]^\wedge$ now preserves coequalizers by adjunction, and as a consequence, we get a coequalizer in the category of complete Lie algebras

$$\begin{tikzcd}
\underbrace{\hat{\mathbb{L}}(\xi_1, \dots, \xi_n, \zeta_1, \dots, \zeta_r)}_{=\mathbb{P}\,\Bbbk[F_1]^\wedge} \arrow[r, shift left, "(d_0)_*"] \arrow[r, shift right, "(d_1)_*"'] & \underbrace{\hat{\mathbb{L}}(\xi_1, \dots, \xi_n)}_{=\mathbb{P}\,\Bbbk[F_0]^\wedge} \arrow[l, bend right, "(s_0)_*"'] \arrow[r, dotted, "\epsilon"] & \mathbb{P}\,\Bbbk[G]^\wedge ,
\end{tikzcd}$$

when we compose this functor with the primitive element functor (we also use the result of Proposition 8.4.1 to replace the primitive Lie algebras $\mathbb{P}\,\Bbbk[F]^\wedge$ of the complete Hopf algebras associated to free groups $F = F_0, F_1$ by complete free Lie algebras). The morphisms $(d_0)_*$, $(d_1)_*$ and $(s_0)_*$ occurring in this coequalizer are given by the identity on the generating elements ξ_i, and we deduce from the exponential correspondence between group-like and primitive elements that we have the formulas $(d_0)_*(\zeta_j) = \log(w_0^j(e^{\xi_1}, \dots, e^{\xi_n}))$ and $(d_1)_*(\zeta_j) = \log(w_1^j(e^{\xi_1}, \dots, e^{\xi_n}))$ for the remaining generators ζ_j. We therefore have a presentation (in the complete sense) of the Lie algebra associated to our group.

This result implies that the Malcev completion $\hat{G}$ of the group $G = \langle x_1, \dots, x_n : w_0^j \equiv w_1^j \rangle$ fits in a reflexive coequalizer in the category of Malcev complete groups

$$\begin{tikzcd}
\underbrace{\hat{\mathbb{F}}(x_1, \dots, x_n, z_1, \dots, z_r)}_{=\hat{F}_1} \arrow[r, shift left, "(d_0)_*"] \arrow[r, shift right, "(d_1)_*"'] & \underbrace{\hat{\mathbb{F}}(x_1, \dots, x_n)}_{=\hat{F}_0} \arrow[l, bend right, "(s_0)_*"'] \arrow[r, dotted, "\epsilon"] & \hat{G} ,
\end{tikzcd}$$

but this observation does not imply that we get a reflexive coequalizer in the category of plain (abstract) groups when we perform this construction.

8.5. The Malcev completion of semi-direct products of groups

The purpose of this section is to study the Malcev completion of semi-direct products of groups. For this purpose, we use that the semi-direct operation has an analogue in the category of complete Hopf algebras and we check that, under mild assumptions, the complete group algebra of a semi-direct product is given by this corresponding semi-direct product operation in the category of complete Hopf algebras. We also check that our equivalence of categories between complete Hopf algebras and complete Lie algebras (the Milnor-Moore Theorem) makes the semi-direct product of complete Hopf algebras correspond to the semi-direct product operation in the category complete Lie algebras. We address this correspondence in the second part of the section. We will moreover check that the correspondence between the semi-direct product of complete Hopf algebras and the semi-direct product of Lie algebras passes to weight graded objects.

We explain the definition of semi-products in the category of Hopf algebras first, and we check that the complete group algebra of a semi-direct product of groups is identified with a semi-product of complete Hopf algebras afterwards.

8.5.1. *Semi-direct products of Hopf algebras.* The *semi-direct product of Hopf algebras*, denoted by $H \sharp K$ (and also called the *smash product* in the literature), is defined for any pair (H, K), where H is a Hopf algebra and K is another Hopf algebra equipped with a right action of H. We explain the definition of this notion of right action of Hopf algebras before tackling the definition of the semi-direct product itself.

The construction works in any symmetric monoidal category $\mathcal{M}$. We use the symmetric monoidal structure of the category of counitary cocommutative coalgebras in $\mathcal{M}$ and that a Hopf algebra H is identified with a unitary associative algebra (equipped with an extra antipode operation) in this category $\mathcal{M}\,\mathcal{C}om^c_+$ (see §7.1). We then consider the category of right modules over H in the category $\mathcal{M}\,\mathcal{C}om^c_+$. We define the objects of this category as counitary cocommutative coalgebras $K \in \mathcal{M}\,\mathcal{C}om^c_+$ equipped with a morphism $\rho : K \otimes H \to K$, formed in the category of counitary cocommutative coalgebras $\mathcal{M}\,\mathcal{C}om^c_+$ as well, and which satisfies the usual unit and associativity constraints of the action of an algebra on a right module in this category. We follow our general convention to mark the structure morphism of a right module K by the corresponding object $\rho = \rho_K$ whenever necessary. In the context of a concrete symmetric monoidal category, we generally use the product notation $x \cdot a = \rho(x \otimes a)$ for this action of H on K, for any $a \in H$ and $x \in K$. In this setting, the unit and associativity relations of the action are equivalent to the standard relations $x \cdot 1 = x$ and $x \cdot (ab) = (x \cdot a) \cdot b$, while the preservation of coalgebra structures is equivalent to the same equations $\epsilon(x \cdot a) = \epsilon(x)\epsilon(a)$ and $\Delta(x \cdot a) = \sum_{(a),(x)} (x_{(1)} \cdot a_{(1)}) \otimes (x_{(2)} \cdot a_{(2)})$ as in the definition of the product of a bialgebra (see §7.1.6).

We easily check that the tensor product $K \otimes L$, where K and L are right modules over a Hopf algebra H, forms a right module over H for the action determined by the composite morphism

$$(1)\quad K \otimes L \otimes H \xrightarrow{id \otimes id \otimes \Delta_H} K \otimes L \otimes H \otimes H \xrightarrow{(2\ 3)^*} K \otimes H \otimes L \otimes H \xrightarrow{\rho_K \otimes \rho_L} K \otimes L.$$

We just use an obvious restriction of structure, by using the coproduct of our Hopf algebra H to provide the tensor product $K \otimes L$ with the structure of a right module over this Hopf algebra H. We explicitly have the formula $(x \otimes y) \cdot a = \sum_{(a)} (x \cdot a_{(1)}) \otimes (y \cdot a_{(2)})$, for any $a \in H$ and $x \in K$, $y \in L$, when we work in a concrete symmetric monoidal category. We accordingly get that the category of right modules over H in counitary cocommutative coalgebras forms a symmetric monoidal category.

In the definition of a semi-direct product of Hopf algebras $H \sharp K$, we precisely assume that the Hopf algebra K forms an algebra in the symmetric monoidal category of right modules over H in the category of counitary cocommutative coalgebras, for a unit morphism $\eta : \mathbb{1} \to K$ and a product morphism $\mu : K \otimes K \to K$ given by the plain Hopf algebra structure of this object K. Thus, we assume that these structure morphisms of the Hopf algebra K define morphisms of right modules over

H. We readily check that this requirement is equivalent to the following equations in K (when our base category forms a concrete symmetric monoidal category):

$$1 \cdot a = \epsilon(a) 1 \quad \text{and} \quad (xy) \cdot a = \sum_{(a)} (x \cdot a_{(1)})(y \cdot a_{(2)}), \tag{2}$$

for all $a \in H$ and $x, y \in K$. We moreover easily see that the antipode of the Hopf algebra K satisfies the relation $\sigma(x \cdot a) = \sigma(x) \cdot a$ with respect to the action of H on K, for all $a \in H$ and $x \in K$ (adapt the arguments of the proof of Lemma 7.1.11).

We now set $H \sharp K = H \otimes K$. We provide this object $H \sharp K$ with its natural counitary cocommutative coalgebra structure (given by the symmetric monoidal structure of the category of counitary cocommutative coalgebras), with the obvious unit morphism $\eta : \mathbb{1} \to H \sharp K$ (given by the tensor product of the unit morphisms of K and H), and with the product operation $\mu : H \sharp K \otimes H \sharp K \to H \sharp K$ such that:

$$(a \otimes x)(b \otimes y) = \sum_{(b)} ab_{(1)} \otimes (x \cdot b_{(2)}) y, \tag{3}$$

for any $a, b \in H$ and $x, y \in K$. We easily check that this construction provides $H \sharp K$ with a well-defined bialgebra structure, and that we moreover have a (left and right) antipode on $H \sharp K$ which is given by the formula

$$\sigma(a \otimes x) = \sum_{(a)} \sigma(a_{(1)}) \otimes (\sigma(x) \cdot \sigma(a_{(2)})), \tag{4}$$

for any $a \in H$ and $x \in K$.

We immediately see that we have an identity $\Bbbk[H \ltimes K] = \Bbbk[H] \sharp \Bbbk[K]$ for the group algebra of a semi-direct product of groups $G = H \ltimes K$, where we extend the right action of H on K by linearity to get a right action of the Hopf algebra $\Bbbk[H]$ on $\Bbbk[K]$. We can also form semi-direct products of Hopf algebras in the category of complete filtered modules and in the category of weight graded modules. We then have the following statement:

Proposition 8.5.2. *We consider a semi-direct product of groups $G = H \ltimes K$ such that the action of H on K reduces to the identity on the abelianization $K / \Gamma_2 K = K/[K, K]$, where we take the quotient of K by the subgroup of commutators $\Gamma_2 K = [K, K]$. We then have an identity*

$$\Bbbk[H \ltimes K]\hat{} = \Bbbk[H]\hat{} \sharp \Bbbk[K]\hat{}$$

in the category of complete Hopf algebras, for an action of $\Bbbk[H]\hat{}$ on $\Bbbk[K]\hat{}$ obtained by completion of the natural action of $\Bbbk[H]$ on $\Bbbk[K]$.

Proof. We identify the elements of the group $G = H \ltimes K$ with formal products $ax \in G$, where $a \in H$ and $x \in K$. We aim to check that the obvious coalgebra isomorphism

$$\Bbbk[H] \otimes \Bbbk[K] \xrightarrow{\simeq} \Bbbk[H \ltimes K], \tag{1}$$

which carries any tensor $[a] \otimes [x]$, where $a \in K$, $x \in H$, to the basis element $[g] = [ax]$ of the group algebra $\Bbbk[H \ltimes K]$, preserves filtrations and extends to an isomorphism on the completion of our objects.

We provide the tensor product $\Bbbk[H] \otimes \Bbbk[K]$ with the filtration such that:

$$\mathtt{F}_s(\Bbbk[H] \otimes \Bbbk[K]) = \sum_{m+n=s} \mathbb{I}^m \Bbbk[H] \otimes \mathbb{I}^n \Bbbk[K] \subset \Bbbk[H] \otimes \Bbbk[K], \tag{2}$$

for all $s \geq 1$, where we consider the filtration of the group algebras $\Bbbk[H]$ and $\Bbbk[K]$ by the powers of their augmentation ideals. We immediately see that our isomorphism maps this module (2) to the module spanned by products of the form $\varpi = ([a_1]-1)\cdots([a_m]-1)([x_1]-1)\cdots([x_n]-1)$ inside $\mathbb{I}^s\,\Bbbk[H \ltimes K]$, where we assume $a_1,\dots,a_m \in H$ and $x_1,\dots,x_n \in K$, for each $s = m+n$. We check that any element of the submodule $\mathbb{I}^s\,\Bbbk[H \ltimes K]$ can conversely be expressed as a sum of products of this form in the group algebra $\Bbbk[H \ltimes K]$. In the case $s = 1$, where we have a single factor $[g]-1 \in \mathbb{I}\,\Bbbk[H \ltimes K]$, with $g = ax \in H \ltimes K$, we have the identity $[g]-1 = ([a]-1)([x]-1) + ([a]-1) + ([x]-1)$ which gives the desired result. We assume by induction that our claim holds for the elements of the sth power of the augmentation ideal $\mathbb{I}^s\,\Bbbk[H \ltimes K]$, for some $s > 0$, We accordingly have $\varpi = ([a_1]-1)\cdots([a_m]-1)([x_1]-1)\cdots([x_n]-1)$ for any such $\varpi \in \mathbb{I}^s\,\Bbbk[H \ltimes K]$. We form a product $\varpi\cdot([g]-1) \in \mathbb{I}^{s+1}\,\Bbbk[H \ltimes K]$ with $[g]-1 \in \mathbb{I}\,\Bbbk[H \ltimes K]$. We check that we can permute the factor $[a]-1 \in \mathbb{I}\,\Bbbk[H]$ in the just obtained expression $[g]-1 = ([a]-1)([x]-1) + ([a]-1) + ([x]-1)$ of the element $[g]-1 \in \mathbb{I}\,\Bbbk[H \ltimes K]$ with the last factor $[x_n]-1 \in \mathbb{I}\,\Bbbk[H \ltimes K]$, $x_n \in K$, of our reduced expression of $\varpi \in \mathbb{I}^s\,\Bbbk[H \ltimes K]$.

We have the formula $([x_n]-1)([a]-1) = ([a]-1)([x_n^a]-1) + ([x_n^a]-[x_n])$ in $\Bbbk[H \ltimes K]$, where we use the notation x_n^a for the image of $x_n \in K$ under the action of $a \in H$ in the group K. We use the assumption on this action to get an identity $x_n^a = x_n(u_1,v_1)\cdots(u_l,v_l)$, where we consider a product of commutators $(u_i,v_i) = u_i^{-1}v_i^{-1}u_i v_i$ such that $u_i,v_i \in K$, for every $i = 1,\dots,l$. We now have the general relation $[(u,v)]-1 = [u^{-1}v^{-1}uv]-1 = [u^{-1}v^{-1}](([u]-1)([v]-1)-([v]-1)([u]-1))$ in any group algebra. This formula implies that we can identify the difference $[(u,v)]-1$ with an element of the ideal $\mathbb{I}^2\,\Bbbk[K]$. We deduce from this general observation that the difference $[x_n^a]-[x_n] = [x_n]([(u_1,v_1)\cdots(u_l,v_l)]-1)$ in our formula is also identified with an element of the ideal $\mathbb{I}^2\,\Bbbk[K]$. We accordingly have $[x_n^a]-[x_n] = \sum_j c_j([\xi_j]-1)([\eta_j]-1)$, where we assume $\xi_j,\eta_j \in K$ and $c_j \in \Bbbk$, for all j. We therefore have an identity $\varpi\cdot([a]-1) = \varpi'\cdot([a]-1)([x_n^a]-1)+\sum_j c_j\varpi'([\xi_j]-1)([\eta_j]-1)$, where we set $\varpi' = ([a_1]-1)\cdots([a_m]-1)([x_1]-1)\cdots([x_{n-1}]-1)$. We can use our induction hypothesis to express $\varpi'\cdot([a]-1) \in \mathbb{I}^s\,\Bbbk[H \ltimes K]$ in the desired form of a product of factors $[b_i]-1, [y_j]-1 \in \mathbb{I}\,\Bbbk[H \ltimes K]$, where $b_i \in H$, $y_j \in K$. This result completes our rewriting process.

We therefore conclude that our isomorphism (1) carries the module (2) isomorphically onto $\mathbb{I}^s\,\Bbbk[H \ltimes K]$, and hence, induces an isomorphism

$$\Bbbk[H]^{\wedge}\hat{\otimes}\,\Bbbk[K]^{\wedge} \xrightarrow{\simeq} \Bbbk[H \ltimes K]^{\wedge} \tag{3}$$

when we pass to completions. We immediately get that this isomorphism preserves coalgebra structures since this is so for our initial isomorphism (1).

We similarly check that the morphism $\rho : \Bbbk[K] \otimes \Bbbk[H] \to \Bbbk[K]$ such that $\rho([x]\otimes[a]) = [x^a]$ which gives the right action of the Hopf algebra $\Bbbk[H]$ on $\Bbbk[K]$ preserves filtrations and extends to the completion of our objects:

$$\rho : \Bbbk[K]^{\wedge}\hat{\otimes}\,\Bbbk[H]^{\wedge} \to \Bbbk[K]^{\wedge}. \tag{4}$$

We immediately get that the product operation of the complete group algebra $\Bbbk[H \ltimes K]^{\wedge}$ is identified with the product operation of the semi-direct product $\Bbbk[H]^{\wedge}\sharp\,\Bbbk[K]^{\wedge}$ which we get by considering this action of Hopf algebras (4) because this is so for the product operation of the ordinary group algebra $\Bbbk[H \ltimes K]$ which determines this product operation by completion. We therefore have the identity of complete Hopf algebras $\Bbbk[H \ltimes K]^{\wedge} = \Bbbk[H]^{\wedge}\hat{\sharp}\,\Bbbk[K]^{\wedge}$ asserted in the proposition. ☐

We then get the following result for the Malcev completion of a semi-direct product of groups:

Proposition 8.5.3. *We consider a semi-direct product of groups $G = H \ltimes K$ such that the action of H on K reduces to the identity when we take the quotient of K by the subgroup of commutators $\Gamma_2 K = [K, K]$ (as in Proposition* 8.5.2*). We then get an identity*

$$\hat{G} = \hat{H} \ltimes \hat{K}$$

when we take the Malcev completion of the group $G = H \ltimes K$. The action of the group $\hat{H}$ on $\hat{K}$ which we consider in this semi-direct product expression preserves the natural filtration of the Malcev complete group $\hat{K}$ and reduces to the identity when we take the quotient of $\hat{K}$ by the second subgroup of this filtration $\mathtt{F}_2 \hat{K} \subset \hat{K}$.

We moreover have the identity $\mathtt{F}_s \hat{G} = \mathtt{F}_s \hat{H} \ltimes \mathtt{F}_s \hat{K}$, for every $s > 0$, where we consider the natural filtration of the Malcev complete group $\hat{G}$. We just take the induced action of the group $\mathtt{F}_s \hat{H} \subset \hat{H}$ on the subgroup $\mathtt{F}_s \hat{K} \subset \hat{K}$ to form the semi-direct products of this formula.

Proof. We use the isomorphism $\Bbbk[H \ltimes K]^{\wedge} \simeq \Bbbk[H]^{\wedge}\hat{\sharp}\,\Bbbk[K]^{\wedge}$ of Proposition 8.5.2. We easily check, by using the arguments as in Proposition 7.1.15, that the group-like element functor $\mathbb{G}(-)$ from counitary cocommutative coalgebras in complete filtered modules to sets carries the completed tensor products of complete counitary cocommutative coalgebras to cartesian products of sets. We use this general observation to check that the isomorphism of counitary cocommutative coalgebras $\Bbbk[H \ltimes K]^{\wedge} \simeq \Bbbk[H]^{\wedge}\hat{\otimes}\,\Bbbk[K]^{\wedge}$ underlying our complete group algebras induces a bijection of sets $\mathbb{G}\,\Bbbk[H \ltimes K]^{\wedge} \simeq \mathbb{G}\,\Bbbk[H]^{\wedge} \times \mathbb{G}\,\Bbbk[K]^{\wedge}$ when we apply the functor of group-like elements $\mathbb{G}(-)$. We similarly see that the morphism $\rho : \Bbbk[K]^{\wedge}\hat{\otimes}\,\Bbbk[H]^{\wedge} \rightarrow \Bbbk[K]^{\wedge}$ which gives the right action of the complete Hopf algebra $\Bbbk[H]^{\wedge}$ on $\Bbbk[K]^{\wedge}$ induces a map $\rho : \mathbb{G}\,\Bbbk[K]^{\wedge} \times \mathbb{G}\,\Bbbk[H]^{\wedge} \rightarrow \mathbb{G}\,\Bbbk[K]^{\wedge}$, which gives a right action of the group $\hat{H} = \mathbb{G}\,\Bbbk[H]^{\wedge}$ on $\hat{K} = \mathbb{G}\,\Bbbk[K]^{\wedge}$ when we pass to the sets of group-like elements. We readily check that the multiplication of the group $\hat{G} = \mathbb{G}\,\Bbbk[H \ltimes K]^{\wedge}$ is identified with the semi-direct product operation yielded by this action too. We therefore have an identity $\hat{G} = \hat{H} \ltimes \hat{K}$ in the category of groups.

Recall that the morphism $\rho : \Bbbk[K]^{\wedge}\hat{\otimes}\,\Bbbk[H]^{\wedge} \rightarrow \Bbbk[K]^{\wedge}$ is yielded by the map such that $\rho([x] \otimes [a]) = [x^a]$ on plain group algebras, where we use the notation $x^a \in K$ for the image of an element $x \in K$ under the action of an element $a \in H$ in the group K. In the proof of Proposition 8.5.2 we used that we have a relation $[x^a] - [x] = \sum_j ([\xi_j] - 1)([\eta_j] - 1) \in \mathbb{I}^2\,\Bbbk[K]$ for any such $x^a \in K$. We get the same relation for our morphism on complete group algebras, and we accordingly have the relation

$$g^h \equiv g (\operatorname{mod} \mathbb{I}^2\,\Bbbk[K]^{\wedge})$$

in $\Bbbk[K]^{\wedge}$ for any pair of group-like elements $g \in \mathbb{G}\,\Bbbk[K]^{\wedge}$, $h \in \mathbb{G}\,\Bbbk[H]^{\wedge}$, where we use the notation $g^h = \rho(g \otimes h)$ for the action in this case. This relation implies that we have the identity $g^h \equiv g$ in the quotient group $\hat{K}/\mathtt{F}_2 \hat{K}$. The rest of the proposition follows from straightforward verifications from this result. ☐

We also have a classical notion of semi-direct product in the category of Lie algebras. We already mentioned that the equivalence of categories provided by the

Milnor-Moore Theorem makes semi-direct products of Hopf algebras correspond to semi-direct products of Lie algebras. We check this assertion in the next proposition after recalling the definition of a semi-direct product of Lie algebras.

8.5.4. *Recollections on semi-direct products of Lie algebras.* We go back to the general setting of a $\mathbb{Q}$-additive symmetric monoidal category $\mathcal{M}$ and we consider the category of Lie algebras in $\mathcal{M}$ (see §7.2). We assume that the tensor product of $\mathcal{M}$ distributes over colimits. We consider a pair of Lie algebras $(\mathfrak{h},\mathfrak{k})$, where $\mathfrak{h}$ acts on $\mathfrak{k}$ in the sense that we have an operation $[-,-]:\mathfrak{k}\otimes\mathfrak{h}\to\mathfrak{k}$ which satisfies the relations of Lie brackets $[x,[a,b]]=[[x,a],b]-[[x,b],a]$, for all $x\in\mathfrak{k}$, $a,b\in\mathfrak{h}$, and $[[x,y],a]=[[x,a],y]+[x,[y,a]]$, for all $x,y\in\mathfrak{k}$, $a\in\mathfrak{h}$. (We refer to §7.2.1 for the correspondence between these relations and the classical Jacobi relation.) The first relation $[x,[a,b]]=[[x,a],b]-[[x,b],a]$ implies that our operation $[-,-]:\mathfrak{k}\otimes\mathfrak{h}\to\mathfrak{k}$ provides the object $\mathfrak{k}$ with the structure of a representation of the Lie algebra $\mathfrak{h}$ (in the sense of §7.2.9), while the second relation $[[x,y],a]=[[x,a],y]+[x,[y,a]]$ implies that $\mathfrak{k}$ is equipped with a Lie bracket $[-,-]:\mathfrak{k}\otimes\mathfrak{k}\to\mathfrak{k}$ which defines a morphism of representations of $\mathfrak{h}$, where we equip the tensor product $\mathfrak{k}\otimes\mathfrak{k}$ with the action of $\mathfrak{h}$ such that $[x\otimes y,a]=[x,a]\otimes y+x\otimes[y,a]$, for $x,y\in\mathfrak{k}$, $a\in\mathfrak{h}$. This formula $[[x,y],a]=[[x,a],y]+[x,[y,a]]$ is also equivalent to the assertion that the operation $\theta_a=[-,a]$ forms a derivation with respect to the Lie bracket of $\mathfrak{k}$.

We then set $\mathfrak{h}\ltimes\mathfrak{k}=\mathfrak{h}\oplus\mathfrak{k}$, and we provide this object with the Lie bracket such that $[(a,x),(b,y)]=([a,b],[x,y]+[x,b]-[y,a])$, for all $a,b\in\mathfrak{h}$ and $x,y\in\mathfrak{k}$, in order to define the semi-direct product of the Lie algebras $\mathfrak{h}$ and $\mathfrak{k}$. We have the following statement:

Proposition 8.5.5.

(a) *If we have a semi-direct product of Lie algebras $\mathfrak{g}=\mathfrak{h}\ltimes\mathfrak{k}$, then the Hopf algebra $K=\mathbb{U}(\mathfrak{k})$ inherits an action of $H=\mathbb{U}(\mathfrak{h})$, and we have the identity $\mathbb{U}(\mathfrak{h}\ltimes\mathfrak{k})=\mathbb{U}(\mathfrak{h})\sharp\,\mathbb{U}(\mathfrak{k})$ in the category of Hopf algebras.*

(b) *If we have a semi-direct product of Hopf algebras $L=H\sharp K$, then the Lie algebra $\mathfrak{k}=\mathbb{P}(K)$ inherits an action of $\mathfrak{h}=\mathbb{P}(H)$ and we have the identity $\mathbb{P}(H\sharp K)=\mathbb{P}(H)\ltimes\mathbb{P}(K)$ in the category of Lie algebras.*

Proof. We use the derivation formula $[x_1\cdot\ldots\cdot x_n,a]:=x_1\cdot\ldots\cdot[x_i,a]\cdot\ldots\cdot x_n$, for $x_1,\ldots,x_n\in\mathfrak{k}$ and $a\in\mathfrak{h}$, in order to extend the action of the Lie algebra $\mathfrak{h}$ to the enveloping algebra $\mathbb{U}(\mathfrak{k})$ and in order to provide this object $\mathbb{U}(\mathfrak{k})$ with the structure of a representation of $\mathfrak{h}$ (see §7.2.9). We then use the derivation relation $[[x,y],a]=[[x,a],y]+[x,[y,a]]$, for $x,y\in\mathfrak{k}$ and $a\in\mathfrak{h}$, in order to check that this morphism $[-,-]:\mathbb{U}(\mathfrak{k})\otimes\mathfrak{h}\to\mathbb{U}(\mathfrak{k})$ preserves the defining relations of the enveloping algebra $\mathbb{U}(\mathfrak{k})$, while we easily check that the structure relation of a representation $[x,[a,b]]=[[x,a],b]-[[x,b],a]$ on the Lie algebra $\mathfrak{k}$ implies the validity of this relation for our extension of the action to the enveloping algebra $\mathbb{U}(\mathfrak{k})$.

We use the isomorphism between the category of representations of a Lie algebra and the category of right modules over the associated enveloping algebra to get a morphism $\rho:\mathbb{U}(\mathfrak{k})\otimes\mathbb{U}(\mathfrak{h})\to\mathbb{U}(\mathfrak{k})$ which extends this action of $\mathfrak{h}$ on $\mathbb{U}(\mathfrak{k})$ and which provides $K=\mathbb{U}(\mathfrak{k})$ with the structure of a right module over the associative algebra $H=\mathbb{U}(\mathfrak{h})$. We easily check that this right module structure satisfies the coherence constraint of §8.5.1 with respect to the product of $K=\mathbb{U}(\mathfrak{k})$, because we have the derivation relation $[uv,a]=[u,a]v+u[v,a]$, for all $u,v\in\mathbb{U}(\mathfrak{k})$ and $a\in\mathfrak{h}$, by construction of our action, and this relation is equivalent to our coherence constraint $(uv)\cdot a=\sum_{(a)}(u\cdot a_{(1)})(v\cdot a_{(2)})$ for the Lie algebra elements $a\in\mathfrak{h}$ which we identify with primitive generators of the enveloping algebra $\mathbb{U}(\mathfrak{h})$. We easily check that our morphism $\rho : \mathbb{U}(\mathfrak{k}) \otimes \mathbb{U}(\mathfrak{h}) \to \mathbb{U}(\mathfrak{k})$ preserves coproducts too, and hence, forms a morphism in the category of counitary cocommutative coalgebras, as we require in §8.5.1. We can therefore form the semi-direct $H \sharp K = \mathbb{U}(\mathfrak{h}) \sharp \mathbb{U}(\mathfrak{k})$ of the Hopf algebras $H = \mathbb{U}(\mathfrak{h})$ and $K = \mathbb{U}(\mathfrak{k})$.

We have a morphism of Lie algebra $\mathfrak{h} \ltimes \mathfrak{k} \to \mathbb{U}(\mathfrak{h}) \sharp \mathbb{U}(\mathfrak{k})$ given by the canonical maps $\mathfrak{h} \subset \mathbb{U}(\mathfrak{h}) = \mathbb{U}(\mathfrak{h}) \otimes 1$ and $\mathfrak{k} \subset \mathbb{U}(\mathfrak{k}) = 1 \otimes \mathbb{U}(\mathfrak{k})$ on the factors of the semi-direct product $\mathfrak{h} \ltimes \mathfrak{k} = \mathfrak{h} \oplus \mathfrak{k}$. We immediately see that this morphism carries $\mathfrak{h} \ltimes \mathfrak{k}$ into the primitive part of $\mathbb{U}(\mathfrak{h}) \sharp \mathbb{U}(\mathfrak{k})$, and hence, extends to a morphism of Hopf algebras $\mathbb{U}(\mathfrak{h} \ltimes \mathfrak{k}) \to \mathbb{U}(\mathfrak{h}) \sharp \mathbb{U}(\mathfrak{k})$ (see Proposition 7.2.13). We have an obvious map in the converse direction $\mathbb{U}(\mathfrak{h}) \sharp \mathbb{U}(\mathfrak{k}) \to \mathbb{U}(\mathfrak{h} \ltimes \mathfrak{k})$, which is given by the morphism of enveloping algebras induced by the obvious inclusions of Lie algebras $\mathfrak{h} \subset \mathfrak{h} \ltimes \mathfrak{k}$ and $\mathfrak{k} \subset \mathfrak{h} \ltimes \mathfrak{k}$ on each factor of the tensor product $\mathbb{U}(\mathfrak{h}) \sharp \mathbb{U}(\mathfrak{k}) = \mathbb{U}(\mathfrak{h}) \otimes \mathbb{U}(\mathfrak{k})$. We easily check that this map defines an inverse morphism of our morphism of Hopf algebras $\mathbb{U}(\mathfrak{h} \ltimes \mathfrak{k}) \to \mathbb{U}(\mathfrak{h}) \sharp \mathbb{U}(\mathfrak{k})$, which is an isomorphism therefore, and this verification finishes the proof of the first assertion of the proposition.

We now assume that we have a semi-direct product of Hopf algebras $H \sharp K$ as in the second assertion of the proposition. We immediately see that the morphism $\rho : K \otimes H \to K$ which defines the right action of H on K preserves primitives elements since we assume that this morphism preserves coalgebra structures in our definition (see §8.5.1). We accordingly have a morphism $[-,-] : \mathbb{P}(K) \otimes \mathbb{P}(H) \to \mathbb{P}(K)$ induced by ρ. We easily check that this morphism satisfies the relations of §8.5.4, and we can therefore use this action to form a semi-direct product $\mathbb{P}(H) \ltimes \mathbb{P}(K)$ in the category of Lie algebras

We have $H \sharp K = H \otimes K$ if we forget about the product operation of the semi-direct product $H \sharp K$. We easily check that the primitive element functor on the category of (coaugmented) counitary cocommutative coalgebras in §7.2.11 is right adjoint to the obvious functor $(-)_+ : \mathcal{M} \to \mathcal{M}\,\mathcal{C}om_+^c$ which carries any object $M \in \mathcal{M}$ to the coalgebra such that $M_+ = \mathbb{1} \oplus M$ and where we take a trivial coproduct on M. Recall that the tensor product is identified with the cartesian product in the category of counitary cocommutative coalgebras (see §3.0.4). Hence, we have an isomorphism in the base category $\mathbb{P}(H \sharp K) \simeq \mathbb{P}(H) \oplus \mathbb{P}(K)$ by adjunction.

We easily check that the Lie bracket of $\mathbb{P}(H \sharp K)$ corresponds to the Lie bracket of the semi-direct product $\mathbb{P}(H) \ltimes \mathbb{P}(K) = \mathbb{P}(H) \oplus \mathbb{P}(K)$ when we keep track of the image of commutators under this isomorphism, and this verification finishes the proof of the second assertion of the proposition. ☐

We can apply this proposition in the complete filtered module setting to get an identity of complete Hopf algebras $\hat{\mathbb{U}}(\mathfrak{h} \ltimes \mathfrak{k}) = \hat{\mathbb{U}}(\mathfrak{h}) \sharp \hat{\mathbb{U}}(\mathfrak{h})$, for any semi-direct product of complete Lie algebras $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{k}$. We conversely have an identity of complete Lie algebras $\mathbb{P}(H \sharp K) = \mathbb{P}(H) \ltimes \mathbb{P}(K)$ when H and K are complete Hopf algebras and we form the semi-direct product $H \sharp K$ in the category of complete Hopf algebras.

8.5.6. *Semi-direct products of weight graded Hopf algebras and Lie algebras.* We also easily check that the mapping $\mathtt{E}^0 : M \mapsto \mathtt{E}^0\, M$ which assigns a weight graded object to any complete filtered module carries a semi-direct product of complete Hopf algebras to a semi-direct product of weight graded Hopf algebras, and a semi-direct product of complete Lie algebras to a semi-direct product of weight graded

Lie algebras. We explicitly have an identity:

$$\mathtt{E}^0(H\sharp K) = \mathtt{E}^0(H)\sharp\,\mathtt{E}^0(K), \tag{1}$$

for any semi-direct product of complete Hopf algebras $L = H\sharp K$, and an identity:

$$\mathtt{E}^0(\mathfrak{h}\ltimes\mathfrak{k}) = \mathtt{E}^0(\mathfrak{h})\ltimes\mathtt{E}^0(\mathfrak{k}) \tag{2}$$

for any semi-direct product of complete Lie algebras $\mathfrak{g} = \mathfrak{h}\ltimes\mathfrak{k}$. We mainly use that the functor $\mathtt{E}^0 : M \mapsto \mathtt{E}^0\, M$ is symmetric monoidal (and preserves direct sums) to get these identities.

We use these observations in the proof of the following proposition:

Proposition 8.5.7. *We consider a semi-direct product of groups $G = H \ltimes K$ such that the action of H on K reduces to the identity when we take the quotient of K by the subgroup of commutators $\Gamma_2\, K = [K, K]$ (as in Proposition* 8.5.2 *and in Proposition* 8.5.3*). We then have an identity*

$$\mathtt{E}^0\,\hat{G} = \mathtt{E}^0\,\hat{H}\ltimes\mathtt{E}^0\,\hat{K}$$

in the category of weight graded Lie algebras, where we consider the weight graded Lie algebras of the Malcev complete groups associated to our groups (see §8.2.2*).*

Proof. We use the general relation $\hat{G} = \mathbb{G}\,\Bbbk[G]^{\wedge} \Rightarrow \mathtt{E}^0\,\hat{G} \simeq \mathtt{E}^0\,\mathbb{P}\,\Bbbk[G]^{\wedge}$ of Proposition 8.2.3(b) and the identity $\Bbbk[H\ltimes K]^{\wedge} = \Bbbk[H]^{\wedge}\sharp\,\Bbbk[K]^{\wedge}$ of Proposition 8.5.2.

We apply the result of Proposition 8.5.5(b) to this semi-direct tensor product of complete Hopf algebras. We then get the relation $\mathbb{P}\,\Bbbk[H\ltimes K]^{\wedge} = \mathbb{P}\,\Bbbk[H]^{\wedge}\ltimes\mathbb{P}\,\Bbbk[K]^{\wedge}$ which implies $\mathtt{E}^0\,\mathbb{P}\,\Bbbk[H\ltimes K]^{\wedge} = \mathtt{E}^0\,\mathbb{P}\,\Bbbk[H]^{\wedge}\ltimes\mathtt{E}^0\,\mathbb{P}\,\Bbbk[K]^{\wedge}$, since we just observed that the functor $\mathtt{E}^0 : M \mapsto \mathtt{E}^0\, M$ carries a semi-direct product of complete Lie algebras to a semi-direct product of weight graded Lie algebras. We therefore get the relation asserted in our proposition. □

CHAPTER 9

The Malcev Completion for Groupoids and Operads

In the previous chapter §7, we reviewed the applications of the adjunction between groups and Hopf algebras to the definition of a rationalization functor, the Malcev completion, on the category of groups.

To be more explicit, recall that the free $\Bbbk$-module $\Bbbk[G]$ associated to a group G inherits a Hopf algebra structure such that the mapping $\Bbbk[-] : G \mapsto \Bbbk[G]$ defines a functor from the category of groups $\mathcal{G}rp$ to the category of Hopf algebras $\mathcal{H}opf\mathcal{A}lg$. The other way round, we have a functor from Hopf algebras to groups which is defined by observing that the set of group-like elements $\mathbb{G}(H)$ in a Hopf algebra H inherits a group structure. We checked in §7.1 that this functor $\mathbb{G} : \mathcal{H}opf\mathcal{A}lg \to \mathcal{G}rp$ forms a right adjoint of the group algebra functor $\Bbbk[-] : \mathcal{G}rp \to \mathcal{H}opf\mathcal{A}lg$. To define our Malcev completion functor in §8, we consider an extension of this adjunction relation, where the category of plain Hopf algebras is replaced by a category of complete Hopf algebras. The complete Hopf algebra $\Bbbk[G]^{\wedge}$associated to a group G is precisely defined as the completion $\Bbbk[G]^{\wedge}= \lim_n \Bbbk[G]/\,\mathbb{I}^n \Bbbk[G]$ of the Hopf algebra associated to G with respect to the powers of the augmentation ideal $\mathbb{I}\,\Bbbk[G] = \ker(\epsilon : \Bbbk[G] \to \Bbbk)$ and the completion of the group G is defined by the set of group-like elements $\hat{G} = \mathbb{G}\,\Bbbk[G]^{\wedge}$associated to this completed Hopf algebra $\Bbbk[G]^{\wedge}$.

In §8, we also crucially assume that the ground ring $\Bbbk$ is a field of characteristic 0. The elements of the group $\hat{G}$ are then identified with exponentials $g = e^x$ such that x belongs to the Lie algebra of primitive elements of $\Bbbk[G]^{\wedge}$. This representation enables us to define powers g^{α} for arbitrary exponents $\alpha \in \Bbbk$ in $\hat{G}$. In the case $\Bbbk = \mathbb{Q}$, our construction therefore returns a rationalization of the group G. The case $\Bbbk = \mathbb{C}$ of our construction will be used in the next chapter, for the definition of the Knizhnik–Zamolodchikov associator.

We still take an arbitrary field of characteristic zero as ground ring $\Bbbk$ throughout this chapter. Our first purpose is to check that the Malcev completion process for groups extends to groupoids. Then we prove that the obtained completion functor on groupoids preserves symmetric monoidal structures, and hence can be applied to operads aritywise in order to yield a Malcev completion functor on the category of operads in groupoids. Some care is necessary when we deal with groupoids, and not all arguments are generalizable, because the morphism sets of groupoids, as opposed to the underlying set of a group, are not naturally pointed.

In a preliminary section (§9.0), we explain the definition of the notion of a Hopf groupoid as an analogue for groupoids of the notion of a Hopf algebra. We explain the definition of our Malcev completion functor for groupoids in §9.1 and we tackle the applications to operads in §9.2. We devote an appendix section §9.3 to the statement of a local connectedness condition for complete Hopf groupoids.

9.0. The notion of a Hopf groupoid

The goal of this section is to make the definition of a Hopf groupoid explicit. In short, we define Hopf groupoids by replacing the morphism sets of ordinary groupoids by coalgebras, and the cartesian products of morphism sets in the definition of the composition operation of a groupoid by tensor products.

We follow the same plan as in §7.1. To begin with, we make explicit the definition of the notion of a *Hopf category*, which is a generalization of the notion of a bialgebra. We give the definition of a Hopf groupoid afterwards and we explain the definition of the extension of the group algebra functor of §7.1 to the category of groupoids in order to complete the account of this section.

9.0.1. *Hopf categories.* In §7.1, we define (cocommutative) bialgebras as unitary associative algebras (or monoids) in the category of counitary cocommutative coalgebras. The *Hopf categories* which we consider in this chapter are small categories enriched in counitary cocommutative coalgebras, and are defined by applying the general concepts of §0.13 to this instance of symmetric monoidal category $\mathcal{C}om_+^c$. This definition makes sense within any base symmetric monoidal category in which we can form our category of counitary cocommutative coalgebras. For the moment we focus on the case of categories enriched in counitary cocommutative coalgebras in $\Bbbk$-modules $\mathcal{C}om_+^c = \mathcal{M}od\,\mathcal{C}om_+^c$.

We then define a Hopf category $\mathcal{H}$ as a set of objects $\mathtt{Ob}\,\mathcal{H}$ together with a collection of counitary cocommutative coalgebras $\mathtt{Hom}_{\mathcal{H}}(x,y) \in \mathcal{C}om_+^c$, associated to each pair of objects $x, y \in \mathtt{Ob}\,\mathcal{H}$, and equipped with unit morphisms

$$(1) \qquad \eta : \Bbbk \to \mathtt{Hom}_{\mathcal{H}}(x,x), \quad \text{for } x \in \mathtt{Ob}\,\mathcal{H},$$

and composition morphisms

$$(2) \qquad \mu : \mathtt{Hom}_{\mathcal{H}}(y,z) \otimes \mathtt{Hom}_{\mathcal{H}}(x,y) \to \mathtt{Hom}_{\mathcal{H}}(x,z), \quad \text{for } x,y,z \in \mathtt{Ob}\,\mathcal{H},$$

that satisfy the usual unit and associativity axioms of categories within the symmetric monoidal category of counitary cocommutative coalgebras (see §0.12).

We also refer to these hom-objects $\mathtt{Hom}_{\mathcal{H}}(x,y)$ as the *hom-coalgebras* of our Hopf category $\mathcal{H}$, and we call '*homomorphisms*' the elements of these hom-coalgebras (see §0.12). In applications, we also use the classical notation $id_x \in \mathtt{Hom}_{\mathcal{H}}(x,x)$ for the homomorphisms $id_x = \eta(1)$, which represent the identity morphisms of our hom-objects. We similarly use the notation $f \circ g$ (or just fg) for the image of homomorphisms under the composition operation (2). Let us mention that the identity homomorphisms $id_x \in \mathtt{Hom}_{\mathcal{H}}(x,x)$ are automatically group-like. We explicitly have the relations $\epsilon(1) = 1 \Rightarrow \epsilon(id_x) = 1$ and $\Delta(1) = 1 \otimes 1 \Rightarrow \Delta(id_x) = id_x \otimes id_x$, for all $x \in \mathtt{Ob}\,\mathcal{H}$.

We consider the obvious notion of morphism of Hopf categories. In brief, a morphism of Hopf categories $\phi : \mathcal{H} \to \mathcal{K}$ consists of a map on the underlying object-sets of our Hopf categories $\phi : \mathtt{Ob}\,\mathcal{H} \to \mathtt{Ob}\,\mathcal{K}$ together with a collection of coalgebra morphisms

$$(3) \qquad \phi : \mathtt{Hom}_{\mathcal{H}}(x,y) \to \mathtt{Hom}_{\mathcal{K}}(\phi(x),\phi(y)), \quad \text{for } x,y \in \mathtt{Ob}\,\mathcal{H},$$

which preserve the unit and the composition operations of our hom-coalgebras.

9.0.2. *Hopf groupoids.* We define a *Hopf groupoid* as a Hopf category $\mathcal{H}$ where we have extra operations on hom-objects

$$(1) \qquad \sigma : \mathtt{Hom}_{\mathcal{H}}(x,y) \to \mathtt{Hom}_{\mathcal{H}}(y,x),$$

which make the following diagrams commute:

(2)
$$\begin{tikzcd}
\mathtt{Hom}_{\mathcal{H}}(x,y) \arrow[r, "\epsilon"] \arrow[d, "\Delta"'] & \Bbbk \arrow[r, "\eta"] & \mathtt{Hom}_{\mathcal{H}}(x,x) \\
\mathtt{Hom}_{\mathcal{H}}(x,y)\otimes\mathtt{Hom}_{\mathcal{H}}(x,y) \arrow[rr, "\sigma\otimes id"] & & \mathtt{Hom}_{\mathcal{H}}(y,x)\otimes\mathtt{Hom}_{\mathcal{H}}(x,y) \arrow[u, "\mu"']
\end{tikzcd},$$

(3)
$$\begin{tikzcd}
\mathtt{Hom}_{\mathcal{H}}(x,y) \arrow[r, "\epsilon"] \arrow[d, "\Delta"'] & \Bbbk \arrow[r, "\eta"] & \mathtt{Hom}_{\mathcal{H}}(y,y) \\
\mathtt{Hom}_{\mathcal{H}}(x,y)\otimes\mathtt{Hom}_{\mathcal{H}}(x,y) \arrow[rr, "id\otimes\sigma"] & & \mathtt{Hom}_{\mathcal{H}}(x,y)\otimes\mathtt{Hom}_{\mathcal{H}}(y,x) \arrow[u, "\mu"']
\end{tikzcd},$$

for all $x, y \in \mathtt{Ob}\,\mathcal{H}$. We obviously assume in this definition that our operation σ is defined by a morphism of counitary cocommutative coalgebras. We immediately see that a Hopf algebra is identified with a Hopf groupoid with one object, and our extra operations are obvious generalizations of the antipode of a Hopf algebra. We therefore keep the same name, '*antipode*', to refer to these morphisms (1). We readily see that the endomorphism coalgebra $\mathtt{End}_{\mathcal{H}}(x) = \mathtt{Hom}_{\mathcal{H}}(x,x)$ which we associate to an object $x \in \mathcal{H}$ in a Hopf groupoid $\mathcal{H}$ inherits a Hopf algebra structure, with the unit given by the identity homomorphism of our object $id_x \in \mathtt{End}_{\mathcal{H}}(x)$, the product yielded by the composition product of our Hopf groupoid $\mathcal{H}$, and the antipode yielded by our antipode operation in $\mathcal{H}$.

The relations which we express by the commutativity of the above diagrams are obvious coalgebra analogues of the inversion relation of morphisms in groupoids. For a group-like element $f \in \mathtt{Hom}_{\mathcal{H}}(x,y)$, which has $\epsilon(f) = 1$ and $\Delta(f) = f\otimes f$, these relations read $f\cdot\sigma(f) = id$, $\sigma(f)\cdot f = id$, and hence are equivalent to the requirement that f is invertible with $\sigma(f) = f^{-1}$ as inverse. We easily check, besides, that the antipode operation of a Hopf groupoid is unique and satisfies the relation $\sigma(id_x) = id_x$, for any $x \in \mathtt{Ob}\,\mathcal{H}$, as well as the relation $\sigma(f\circ g) = \sigma(g)\circ\sigma(f)$, for any pair of composable homomorphisms $f \in \mathtt{Hom}_{\mathcal{H}}(y,z)$, $g \in \mathtt{Hom}_{\mathcal{H}}(x,y)$ (the proof of these assertions follows from a straightforward generalization of the arguments which we use in the context of Hopf algebras).

We define the category of Hopf groupoids as the full subcategory of the category of Hopf categories generated by the Hopf groupoids. We use the notation $\mathcal{H}opf\,\mathcal{G}rd$ for this category. We should simply observe that a morphism of Hopf categories $\phi : \mathcal{G} \to \mathcal{H}$ automatically preserves antipodes when $\mathcal{G}$ and $\mathcal{H}$ are Hopf groupoids (this assertion is a variation of the uniqueness of antipodes in Hopf algebras). Recall that we use the notation $\mathcal{G}rd$ for the category of groupoids.

9.0.3. *The Hopf groupoid associated to a groupoid.* We can easily extend the definition of the group algebra functor in the groupoid context. To be explicit, to any groupoid $\mathcal{G}$, we associate the Hopf groupoid $\Bbbk[\mathcal{G}]$ with the same set of objects as our original groupoid $\mathtt{Ob}\,\Bbbk[\mathcal{G}] := \mathtt{Ob}\,\mathcal{G}$, and where we take the coalgebras associated to the morphism sets $S = \mathtt{Mor}_{\mathcal{G}}(x,y)$ as hom-objects:

$$\mathtt{Hom}_{\Bbbk[\mathcal{G}]}(x,y) := \Bbbk[\mathtt{Mor}_{\mathcal{G}}(x,y)],$$

for all $x, y \in \mathtt{Ob}\,\mathcal{G}$. Recall that $\Bbbk[S]$ denotes the free $\Bbbk$-module associated to any set S. We use the notation $[f]$ for the elements of this $\Bbbk$-module which we associate to the elements of our set $f \in S$, and we equip $\Bbbk[S]$ with the coalgebra structure such that $\epsilon([f]) = 1$ and $\Delta([f]) = [f]\otimes[f]$, for any $f \in S$ (see §3.0.6).

The unit morphism of the Hopf groupoid $\Bbbk[\mathcal{G}]$ is defined by $\eta(1) = [id_x]$, for each $x \in \mathtt{Ob}\,\mathcal{G}$, where we consider the element of the hom-coalgebra $[id_x] \in \Bbbk[\mathtt{Mor}_{\mathcal{G}}(x,x)]$ associated the identity morphisms of our object x in $\mathcal{G}$. The composition is defined by the obvious linear extension of the composition operation of the groupoid $\mathcal{G}$, and the antipode is given by $\sigma([f]) = [f^{-1}]$, for any $f \in \mathtt{Mor}_{\mathcal{G}}(x,y)$.

9.0.4. *The group-like element functor on Hopf groupoids.* In the converse direction, we can use the group-like functor $\mathbb{G} : C \mapsto \mathbb{G}(C)$ on the category of counitary cocommutative coalgebras $C \in \mathcal{C}om_{+}^{c}$ (see §3.0.6) to construct a functor from the category of Hopf groupoids to the category of groupoids. To be explicit, to any Hopf groupoid $\mathcal{H}$, we associate the groupoid $\mathbb{G}(\mathcal{H})$ with the same object set as our original Hopf groupoid $\mathtt{Ob}\,\mathbb{G}(\mathcal{H}) := \mathtt{Ob}\,\mathcal{H}$, and where we take the sets of group-like elements of the coalgebras $C = \mathtt{Hom}_{\mathcal{H}}(x,y)$ as morphism sets:

$$\mathtt{Mor}_{\mathbb{G}(\mathcal{H})}(x,y) := \mathbb{G}(\mathtt{Hom}_{\mathcal{H}}(x,y)),$$

for all $x, y \in \mathtt{Ob}\,\mathcal{H}$. We already observed that the identity homomorphisms of a Hopf category are automatically group-like (see §9.0.1), and we readily check, as in the case of Hopf algebras, that a composite of group-like homomorphisms remains group-like. Recall also that for a group-like homomorphism $f \in \mathbb{G}(\mathtt{Hom}_{\mathcal{H}}(x,y))$ the antipode relations are equivalent to the identities $\sigma(f) \cdot f = id_x$ and $f \cdot \sigma(f) = id_y$, and hence, imply that our homomorphism f is invertible with respect to the composition operation of our Hopf groupoid (see 9.0.2). Thus, the morphisms of our category $\mathbb{G}(\mathcal{H})$ are all invertible.

9.0.5. *The adjunction between groupoids and Hopf groupoids.* We easily check that the adjunction of §3.0.6 (between the free module functor from sets to counitary cocommutative coalgebras and the group-like element functor) extends to an adjunction between our Hopf groupoid functor on groupoids and the group-like element functor on Hopf groupoids:

$$\Bbbk[-] : \mathcal{G}rd \rightleftarrows \mathcal{H}opf\,\mathcal{G}rd : \mathbb{G}\,.$$

In short, for an object of the category of groupoids $\mathcal{G} \in \mathcal{G}rd$, we have an obvious morphism $\iota : \mathcal{G} \to \mathbb{G}(\Bbbk[\mathcal{G}])$ defined by the identity map at the object set level and by the obvious inclusions $\mathtt{Mor}_{\mathcal{G}}(x,y) \subset \mathbb{G}(\Bbbk[\mathtt{Mor}_{\mathcal{G}}(x,y)])$ at the morphism set level. For a Hopf groupoid $\mathcal{H} \in \mathcal{H}opf\,\mathcal{G}rd$, we consider the morphism $\rho : \Bbbk[\mathbb{G}(\mathcal{H})] \to \mathcal{H}$ which is still defined by the identity map at the object set level and by the morphisms of $\Bbbk$-modules $\rho : \Bbbk[\mathbb{G}(\mathtt{Hom}_{\mathcal{G}}(x,y))] \to \mathtt{Hom}_{\mathcal{G}}(x,y)$ induced by the inclusion maps $\mathbb{G}(\mathtt{Hom}_{\mathcal{G}}(x,y)) \subset \mathtt{Hom}_{\mathcal{G}}(x,y)$ at the hom-object level. (Thus, we just apply the unit and the augmentation of the adjunction of §3.0.6 to the hom-coalgebras and to the morphism sets of our objects.) We simply check that these natural transformations preserve the structure operations of groupoids and of Hopf groupoids to conclude that they still give the unit and the augmentation of an adjunction between the category of groupoids and the category Hopf groupoids.

9.0.6. *Geometrical, local and global connectedness of Hopf groupoids.* We usually say that a groupoid $\mathcal{G}$ is *connected* when we have $\mathtt{Mor}_{\mathcal{G}}(x,y) \neq \varnothing$, for all $x, y \in \mathtt{Ob}\,\mathcal{G}$. We then have the relation $\mathtt{Hom}_{\Bbbk[\mathcal{G}]}(x,y) \neq 0$, for all $x, y \in \mathtt{Ob}\,\mathcal{G}$, in the Hopf groupoid $\Bbbk[\mathcal{G}]$ associated to $\mathcal{G}$. We also say that a Hopf groupoid $\mathcal{H}$ is *geometrically connected* when we have this non-vanishing relation $\mathtt{Hom}_{\mathcal{H}}(x,y) \neq 0$, for all $x, y \in \mathtt{Ob}\,\mathcal{H}$. By general structure results on coalgebras (see for instance [1, Theorem 2.3.3], or [171, §8.0]), we then have $\mathbb{G}(\mathtt{Hom}_{\mathcal{H}}(x,y) \otimes_{\Bbbk} \mathbb{l}) \neq \varnothing$ for an algebraic extension $\mathbb{l}$ of our ground field $\Bbbk$. Hence, if we assume $\Bbbk = \bar{\Bbbk}$, then the groupoid of group-like elements $\mathcal{G} = \mathbb{G}(\mathcal{H})$ which we associate to a geometrically connected Hopf groupoid $\mathcal{H}$ is connected in the ordinary sense.

We have examples of Hopf groupoids such that $\mathtt{Hom}_{\mathcal{H}}(x,y) \neq 0$ but where we have $\mathbb{G}(\mathtt{Hom}_{\mathcal{H}}(x,y)) = \varnothing$ for some pairs of objects $x, y \in \mathtt{Ob}\,\mathcal{H}$ in the case where the ground field is not algebraically closed $\Bbbk \neq \bar{\Bbbk}$. We therefore introduce new notions of connectedness for Hopf groupoids. Namely, we say that a Hopf groupoid $\mathcal{H}$ is *locally connected* (over $\Bbbk$) when every non-zero hom-coalgebra $\mathtt{Hom}_{\mathcal{H}}(x,y) \neq 0$ contains at least one group-like element (defined over $\Bbbk$), and we say that $\mathcal{H}$ is *globally connected* when we have $\mathbb{G}(\mathtt{Hom}_{\mathcal{H}}(x,y)) \neq \varnothing$ for all $x, y \in \mathtt{Ob}\,\mathcal{H}$, so that the groupoid of group-like elements associated to $\mathcal{H}$ is connected. Thus, the local connectedness condition is void when $\Bbbk = \bar{\Bbbk}$, and the geometrical connectedness implies the global connectedness in this case.

9.1. The Malcev completion for groupoids

The goal of this section is to explain the definition of the Malcev completion of a groupoid. The idea, as in the group context, is to perform a completion of the Hopf groupoid which we associate to any groupoid in §9.0.5, and to go back to groupoids by using a complete version of the group-like element functor from Hopf groupoids to groupoids. To achieve this program, we need to introduce a suitable notion of complete Hopf groupoid.

In §9.0, we mention that our definition of the notion of a Hopf groupoid makes sense in any ambient symmetric monoidal category. We actually define our category of complete Hopf groupoids as a subcategory of the category of Hopf groupoids in complete filtered modules formed by objects which satisfy extra local connectedness conditions. We make the definition of a Hopf groupoid in complete filtered modules explicit first and we explain our local connectedness requirements for the definition of the category of complete Hopf groupoids afterwards.

9.1.1. *Hopf groupoids in complete filtered modules.* We still define our category of Hopf groupoids in complete filtered modules as a subcategory of a category of Hopf categories in complete filtered modules. We just replace the category of ordinary counitary cocommutative coalgebras $\mathcal{C}om_{+}^{c} = \mathcal{M}od\,\mathcal{C}om_{+}^{c}$ in the definition of §§9.0.1-9.0.2 by the category of counitary cocommutative coalgebras in complete filtered modules $\hat{f}\,\mathcal{C}om_{+}^{c} = \hat{f}\,\mathcal{M}od\,\mathcal{C}om_{+}^{c}$ in order to get our definition of these complete analogues of the concepts of §9.0. Thus, a Hopf category in complete filtered modules $\mathcal{H}$ consists of an object set $\mathtt{Ob}\,\mathcal{H}$ together with a collection of hom-objects with values in the category of complete counitary cocommutative coalgebras:

$$\mathtt{Hom}_{\mathcal{H}}(x,y) \in \hat{f}\,\mathcal{C}om_{+}^{c},$$

for $x, y \in \mathtt{Ob}\,\mathcal{H}$. We also consider the symmetric monoidal structure of the category of complete counitary cocommutative coalgebras when we define the unit morphisms §9.0.1(1) and the composition morphisms §9.0.1(2) associated to these complete hom-coalgebras in a complete Hopf category $\mathcal{H}$. We therefore replace the ordinary tensor product by the completed tensor product of §7.3.12 in the definition of these structure morphisms.

Then we define a Hopf groupoid in complete filtered modules as a Hopf category in complete filtered modules $\mathcal{H}$ equipped with antipode morphisms, formed within the category of complete counitary cocommutative coalgebras, and which fulfill the relations of §9.0.2 in this category. We just have to replace the ordinary tensor products of our diagrams §9.0.2(2-3) by completed tensor products again in order to get this complete version of the antipode relations. We observed in §9.0.2 that the endomorphism coalgebra of an object $\mathtt{End}_{\mathcal{H}}(x) = \mathtt{Hom}_{\mathcal{H}}(x, x)$ in a Hopf groupoid inherits a Hopf algebra structure. We similarly get that this endomorphism coalgebra $\mathtt{End}_{\mathcal{H}}(x)$ forms a Hopf algebra in complete filtered modules when we work in the complete setting.

Recall that the ground field $\Bbbk$ is identified with a complete filtered module equipped with a trivial filtration and forms a unit for the completed tensor product. The unit morphisms of a Hopf category in complete filtered modules are therefore equivalent to ordinary unit morphisms of counitary cocommutative coalgebras §9.0.1(1). The preservation of filtration, which we require for morphisms of complete filtered modules in general, is automatically fulfilled for these unit morphisms. The preservation of counitary cocommutative coalgebra structures is equivalent to the requirement that the element $id_x = \eta(1)$ associated to each unit morphism $\eta : \Bbbk \to \mathtt{Hom}_{\mathcal{H}}(x, x)$ is group-like as an element of the complete counitary cocommutative coalgebra $\mathtt{Hom}_{\mathcal{H}}(x, x)$. We explicitly have $\epsilon(id_x) = 1$ and $\Delta(id_x) = id_x \hat{\otimes} id_x$, where we use our definition of the notion of a group-like element in the complete sense (see §8.1.2).

The composition operations of a Hopf category in complete filtered modules can also be identified with extensions of ordinary filtration preserving composition products $\mu : \mathtt{Hom}_{\mathcal{H}}(y, z) \otimes \mathtt{Hom}_{\mathcal{H}}(x, y) \to \mathtt{Hom}_{\mathcal{H}}(x, z)$ as in the Hopf algebra case (see §7.3.14), but we have to consider the completed tensor product when we deal with the coalgebra structure of our hom-objects.

We also define a morphism of Hopf groupoids in complete filtered modules $\phi : \mathcal{G} \to \mathcal{H}$ as a map of object sets $\phi : \mathtt{Ob}\,\mathcal{G} \to \mathtt{Ob}\,\mathcal{H}$ together with a collection of morphisms of complete counitary cocommutative coalgebras $\phi : \mathtt{Hom}_{\mathcal{G}}(x, y) \to \mathtt{Hom}_{\mathcal{H}}(\phi(x), \phi(y))$ which preserve the units and the composition operations attached to our objects. We still get that such a morphism automatically preserves the antipode operations.

9.1.2. *Local connectedness assumptions for complete Hopf groupoids.* In §7.3.15, we introduce a connectedness condition in the definition of the category of complete Hopf algebras. Explicitly, we require that the augmentation of a complete Hopf algebra H induces an isomorphism between the weight zero subquotient $\mathtt{E}^0_0 H = H/\mathtt{F}_1 H$ of our filtration on H and the ground ring $\Bbbk$. We equivalently have an identity between the first layer of our filtration $\mathtt{F}_1 H \subset H$ and the augmentation ideal of our Hopf algebra $\mathbb{I}(H) = \ker(\epsilon : H \to \Bbbk)$.

In the Hopf groupoid context, we similarly assume that the augmentation $\epsilon : \mathtt{Hom}_{\mathcal{H}}(x, y) \to \Bbbk$ of the hom-coalgebras such that $\mathtt{Hom}_{\mathcal{H}}(x, y) \neq 0$ induces an isomorphism from the weight zero subquotient of our filtration to the ground field:

$$\mathtt{E}^0_0 \mathtt{Hom}_{\mathcal{H}}(x, y) = \mathtt{Hom}_{\mathcal{H}}(x, y)/\mathtt{F}_1 \mathtt{Hom}_{\mathcal{H}}(x, y) \xrightarrow{\simeq} \Bbbk\,.$$

This is exactly the generalization of our connectedness condition for complete Hopf algebras. Let us mention that for a (complete or ordinary) coalgebra the non-vanishing relation $C \neq 0$ implies $\epsilon \neq 0$ (because we have $\epsilon = 0 \Rightarrow id_C = 0 \Rightarrow C = 0$ by the counit relation $\epsilon \otimes id \cdot \Delta = id_C$). Thus the augmentation $\epsilon : \mathtt{Hom}_{\mathcal{H}}(x, y) \to \Bbbk$ is automatically surjective when $\mathtt{Hom}_{\mathcal{H}}(x, y) \neq 0$ and our requirement is equivalent to the relation

$$\mathtt{F}_1 \mathtt{Hom}_{\mathcal{H}}(x, y) = \ker(\epsilon : \mathtt{Hom}_{\mathcal{H}}(x, y) \to \Bbbk),$$

for any $x, y \in \mathtt{Ob}\,\mathcal{H}$, where we consider the first filtration layer of the coalgebra $\mathtt{Hom}_{\mathcal{H}}(x, y)$.

Besides this requirement on the filtration of our hom-coalgebras, we assume that any non-zero complete hom-coalgebra in our Hopf groupoid $\mathtt{Hom}_{\mathcal{H}}(x, y) \neq 0$ contains at least one group-like element (in the complete sense) $g \in \mathbb{G}(\mathtt{Hom}_{\mathcal{H}}(x, y))$. Explicitly, we assume that we have an element $g \in \mathtt{Hom}_{\mathcal{H}}(x, y)$ such that $\epsilon(g) = 1$ and $\Delta(g) = g \hat{\otimes} g$. This assumption is obviously an analogue for Hopf groupoids in complete filtered modules of our local connectedness condition for ordinary Hopf groupoids (see §9.0.6).

We call *complete Hopf groupoids* the Hopf groupoids in complete filtered modules whose non-zero hom-coalgebras $\mathtt{Hom}_{\mathcal{H}}(x, y) \neq 0$ satisfy the above connectedness conditions concerning the filtration $\mathtt{E}^0_0\,\mathtt{Hom}_{\mathcal{H}}(x, y) = \Bbbk \Leftrightarrow \mathtt{F}_1\,\mathtt{Hom}_{\mathcal{H}}(x, y) = \ker(\epsilon : \mathtt{Hom}_{\mathcal{H}}(x, y) \to \Bbbk)$ (the *filtration connectedness condition*) as well as our local connectedness conditions concerning the existence of group-like elements $g \in \mathbb{G}(\mathtt{Hom}_{\mathcal{H}}(x, y))$ (the *local connectedness condition*). We use the notation $\hat{f}\,\mathcal{H}opf\,\mathcal{G}rd$ for the subcategory of the category of Hopf groupoids in complete filtered modules generated by the complete Hopf groupoids.

We immediately see that our filtration connectedness condition implies that the endomorphism coalgebra $\mathtt{End}_{\mathcal{H}}(x) = \mathtt{Hom}_{\mathcal{H}}(x, x)$ of any object $x \in \mathtt{Ob}\,\mathcal{H}$ in a complete Hopf groupoid $\mathcal{H}$ satisfies the connectedness requirement of our definition of a complete Hopf algebra (see §7.3.15) and hence forms a complete Hopf algebra in our sense. We can conversely identify a complete Hopf algebra with a complete Hopf groupoid with a single object.

Note that we allow the existence of null hom-coalgebras $\mathtt{Hom}_{\mathcal{H}}(x, y) = 0$ in this definition. We just say that a complete Hopf groupoid is globally connected when we have $\mathtt{Hom}_{\mathcal{H}}(x, y) \neq 0$ for all $x, y \in \mathtt{Ob}\,\mathcal{H}$. We have the following straightforward proposition:

PROPOSITION 9.1.3. *Any complete Hopf groupoid* $\mathcal{H}$ *decomposes as a coproduct* $\mathcal{H} = \coprod_{\alpha} \mathcal{H}_{\alpha}$ *(in the category of complete Hopf groupoids) such that each term* $\mathcal{H}_{\alpha}$ *is globally connected.*

PROOF. We define these objects $\mathcal{H}_{\alpha}$ as the complete Hopf groupoids on the maximal subsets of objects $\mathtt{Ob}\,\mathcal{H}_{\alpha} = S_{\alpha} \subset \mathtt{Ob}\,\mathcal{H}$ such that we have $x, y \in S_{\alpha} \Rightarrow \mathtt{Hom}_{\mathcal{H}}(x, y) \neq 0$ in the complete Hopf groupoid $\mathcal{H}$, and we set $\mathtt{Hom}_{\mathcal{H}_{\alpha}}(x, y) = \mathtt{Hom}_{\mathcal{H}}(x, y)$ for any such pair $x, y \in S_{\alpha}$. ☐

9.1.4. *Connected components and categorical equivalences of complete Hopf groupoids.* We obviously refer to the complete Hopf groupoids $\mathcal{H}_{\alpha}$ in the decomposition of the previous proposition as the connected components of $\mathcal{H}$. We also use the notation $\pi_0\,\mathcal{H}$ for the set of these maximal subsets of objects $\mathtt{Ob}\,\mathcal{H}_{\alpha} = S_{\alpha} \subset \mathtt{Ob}\,\mathcal{H}$ satisfying $x, y \in S_{\alpha} \Rightarrow \mathtt{Hom}_{\mathcal{H}}(x, y) \neq 0$ in the proof of the previous proposition. We can abusively identify an element of this set $S_{\alpha} \in \pi_0\,\mathcal{H}$ with the groupoid $\mathcal{H}_{\alpha}$ which this set determines. We therefore also refer to this set $\pi_0\,\mathcal{H}$ as the set of connected components of the complete Hopf groupoid $\mathcal{H}$.

The map $\pi_0 : \mathcal{H} \to \pi_0\,\mathcal{H}$ obviously defines a functor on the category of complete Hopf groupoids. We can use this functor $\pi_0 : \mathcal{H} \to \pi_0\,\mathcal{H}$ to define an analogue of the classical notion of an equivalence of groupoids in the complete Hopf groupoid context. We explicitly say that a morphism $\phi : \mathcal{A} \to \mathcal{B}$ is an *equivalence of complete*

Hopf groupoids if this morphism induces a bijection on the sets of connected components $\pi_0\phi : \pi_0 \mathcal{A} \xrightarrow{\simeq} \pi_0 \mathcal{B}$ and defines an isomorphism of complete counitary cocommutative coalgebras at the hom-object level $\phi : \mathtt{Hom}_{\mathcal{A}}(x,y) \xrightarrow{\simeq} \mathtt{Hom}_{\mathcal{B}}(\phi(x),\phi(y))$, for each pair of objects $x,y \in \mathtt{Ob}\,\mathcal{A}$. We use the notation $\phi : \mathcal{A} \xrightarrow{\sim} \mathcal{B}$ with the extra mark $\sim$ to distinguish the class of equivalences among the morphisms of the category of complete Hopf groupoids. We can obviously identify the isomorphisms of complete Hopf groupoids with the equivalences which are defined by a bijection at the object set level.

9.1.5. *The determination of complete Hopf groupoids from a complete endomorphism Hopf algebra.* Recall that an element $g \in C$ in a complete counitary cocommutative coalgebra C is group-like (in the complete sense) when we have $\epsilon(g) = 1$ and $\Delta(g) = g \hat{\otimes} g$. For a group-like element in the hom-coalgebra of a complete Hopf groupoid $g \in \mathtt{Hom}_{\mathcal{H}}(x,y)$, the antipode relations read $\sigma(g) \cdot g = id_x$, $g \cdot \sigma(g) = id_y$, and therefore imply that the antipode represents an inverse of our homomorphism $\sigma(g) = g^{-1}$ with respect to the composition operation of our complete Hopf groupoid.

We immediately see that the composition operation $g_* : u \mapsto g \circ u$ determines an isomorphism of complete counitary coalgebras $g_* : \mathtt{End}_{\mathcal{H}}(x) \xrightarrow{\simeq} \mathtt{Hom}_{\mathcal{H}}(x,y)$ when $g \in \mathtt{Hom}_{\mathcal{H}}(x,y)$ is group-like. We similarly have an isomorphism of complete counitary coalgebras $g^* : \mathtt{End}_{\mathcal{H}}(y) \xrightarrow{\simeq} \mathtt{Hom}_{\mathcal{H}}(x,y)$ when we consider the composition operation on the right $g^* : v \mapsto v \circ g$. We also write $\mathtt{End}_{\mathcal{H}}(x,y) = g\,\mathtt{End}_{\mathcal{H}}(x)$ and $\mathtt{End}_{\mathcal{H}}(x,y) = \mathtt{End}_{\mathcal{H}}(y) g$ to express the identity of our hom-coalgebra with the image of these translation isomorphisms.

We can use this observation to determine the structure of any complete Hopf groupoid $\mathcal{H}$ from the collection of complete endomorphism Hopf algebras $\mathtt{End}_{\mathcal{H}}(x_\alpha)$ associated to the choice of an object $x_\alpha \in \mathtt{Ob}\,\mathcal{H}_\alpha$ in each connected component of our complete Hopf groupoid $\mathcal{H}_\alpha \subset \mathcal{H}$, together with group-like homomorphisms $g_x \in \mathbb{G}(\mathtt{Hom}_{\mathcal{H}}(x_\alpha,x))$ which connect any object $x \in \mathtt{Ob}\,\mathcal{H}$ with one of our base objects $x_\alpha \in \mathtt{Ob}\,\mathcal{H}_\alpha$. We explicitly use the relation $\mathtt{Hom}_{\mathcal{H}}(x,y) = g_y\,\mathtt{End}_{\mathcal{H}}(x_\alpha) g_x^{-1}$ to determine the hom-coalgebra associated to any pair of objects in $\mathcal{H}$. We use the formula $(g_z u g_y^{-1})(g_y v g_x^{-1}) = g_z(uv)g_x^{-1}$ to determine the composite of homomorphisms $f = g_z u g_y^{-1} \in \mathtt{Hom}_{\mathcal{H}}(y,z)$ and $g = g_y v g_x^{-1} \in \mathtt{Hom}_{\mathcal{H}}(x,y)$ from the composition of the factors $u,v \in \mathtt{End}_{\mathcal{H}}(x_\alpha)$ in the complete endomorphism coalgebra $\mathtt{End}_{\mathcal{H}}(x_\alpha)$.

We can formalize the result of these observations as the following statement:

PROPOSITION 9.1.6. *Any complete Hopf groupoid $\mathcal{H}$ is isomorphic to a complete Hopf groupoid $\underline{H}$ with $\mathtt{Ob}\,\underline{H} = \mathtt{Ob}\,\mathcal{H}$ and which is determined by a partition of this object set $\mathtt{Ob}\,\mathcal{H} = \coprod_{\alpha\in\mathcal{I}} S_\alpha$ together with a collection of complete Hopf algebras H_α, $\alpha \in \mathcal{I}$, such that:*

$$\mathtt{Hom}_{\underline{H}}(x,y) = \begin{cases} H_\alpha, & \textit{if } x,y \in S_\alpha \textit{ for some } \alpha \in \mathcal{I}, \\ 0, & \textit{otherwise.} \end{cases}$$

PROOF. We just take $H_\alpha = \mathtt{End}_{\mathcal{H}}(x_\alpha)$ in the construction of §9.1.5 and we take the compositions $\phi : u \mapsto g_y u g_x^{-1}$ to define the morphisms of hom-coalgebras $\phi : H_\alpha \to \mathtt{Hom}_{\mathcal{H}}(x,y)$ which determine our isomorphism of complete Hopf groupoids $\phi : \underline{H} \xrightarrow{\simeq} \mathcal{H}$. □

We coin the phrase '*locally constant complete Hopf groupoid*' to refer to the complete Hopf groupoids $\underline{H}$ of the form considered in this proposition. We can use this proposition to deduce the proof of the validity of some constructions on complete Hopf groupoids from corresponding statements on complete Hopf algebras, but the isomorphism which we define in this proposition is by no way canonical in general, and we crucially need functorial constructions when we tackle the applications of complete Hopf groupoids to operads. We therefore avoid to apply this proposition in what follows.

9.1.7. *The completion of Hopf groupoids.* The natural filtration of Hopf algebras, which is given by the tensor powers of the augmentation ideal, has a natural generalization in the context of Hopf groupoids.

To be explicit, let $\mathcal{H} \in \mathcal{H}opf\mathcal{G}rd$ be a Hopf groupoid. We equip the hom-coalgebra $\mathtt{Hom}_{\mathcal{H}}(x,y)$, associated to each pair of objects $x, y \in \mathtt{Ob}\,\mathcal{H}$, with a filtration

$$\mathtt{Hom}_{\mathcal{H}}(x,y) = \mathbb{I}^0\,\mathtt{Hom}_{\mathcal{H}}(x,y) \supset \mathbb{I}^1\,\mathtt{Hom}_{\mathcal{H}}(x,y) \supset \cdots \supset \mathbb{I}^n\,\mathtt{Hom}_{\mathcal{H}}(x,y) \supset \cdots$$

such that $\mathbb{I}^n\,\mathtt{Hom}_{\mathcal{H}}(x,y)$ is the submodule of $\mathtt{Hom}_{\mathcal{H}}(x,y)$ spanned by the n-fold composites of homomorphisms $f_1 \cdot \ldots \cdot f_n$ satisfying $\epsilon(f_i) = 0$, for $i = 1, \ldots, n$. Thus, we assume that each factor of these composites f_i, $i = 1, \ldots, n$, lies in the kernel of the augmentation of our hom-coalgebras $\epsilon : \mathtt{Hom}_{\mathcal{H}}(-,-) \to \mathbb{k}$. For the first layer of this filtration $\mathbb{I}\,\mathtt{Hom}_{\mathcal{H}}(x,y) \subset \mathtt{Hom}_{\mathcal{H}}(x,y)$, we have the trivial identity $\mathbb{I}\,\mathtt{Hom}_{\mathcal{H}}(x,y) = \ker(\mathtt{Hom}_{\mathcal{H}}(x,y) \xrightarrow{\epsilon} \mathbb{k})$, for each pair of objects $x, y \in \mathtt{Ob}\,\mathcal{H}$.

The preservation of the counitary cocommutative coalgebra structure by the composition operations of $\mathcal{H}$ implies that we have the inclusion relation

$$\Delta(\mathbb{I}^n\,\mathtt{Hom}_{\mathcal{H}}(x,y)) \subset \sum_{p+q=n} \mathbb{I}^p\,\mathtt{Hom}_{\mathcal{H}}(x,y) \otimes \mathbb{I}^q\,\mathtt{Hom}_{\mathcal{H}}(x,y)$$

for each $n \in \mathbb{N}$, as in the Hopf algebra case (see §8.1.1). Thus, the coproduct of our hom-coalgebra $\Delta : \mathtt{Hom}_{\mathcal{H}}(x,y) \to \mathtt{Hom}_{\mathcal{H}}(x,y) \otimes \mathtt{Hom}_{\mathcal{H}}(x,y)$ defines a filtration preserving morphism for this canonical choice of filtration on our object $\mathtt{Hom}_{\mathcal{H}}(x,y)$. The counit morphism $\epsilon : \mathtt{Hom}_{\mathcal{H}}(x,y) \to \mathbb{k}$ trivially defines a filtration preserving morphism too.

Hence, each hom-object $\mathtt{Hom}_{\mathcal{H}}(x,y)$ of our Hopf groupoid $\mathcal{H}$ canonically inherits the structure of a counitary cocommutative coalgebra in filtered modules, of which we perform the completion

$$\mathtt{Hom}_{\mathcal{H}}(x,y)\widehat{\ } = \lim_n \mathtt{Hom}_{\mathcal{H}}(x,y) / \mathbb{I}^n\,\mathtt{Hom}_{\mathcal{H}}(x,y)$$

to get a complete counitary cocommutative coalgebra $\mathtt{Hom}_{\mathcal{H}}(x,y)\widehat{\ }$, for each pair of objects $x, y \in \mathtt{Ob}\,\mathcal{H}$. Recall that the filtration associated to such a completed module satisfies $\mathtt{Hom}_{\mathcal{H}}(x,y)\widehat{\ }/ \mathrm{F}_n\,\mathtt{Hom}_{\mathcal{H}}(x,y)\widehat{\ } = \mathtt{Hom}_{\mathcal{H}}(x,y) / \mathbb{I}^n\,\mathtt{Hom}_{\mathcal{H}}(x,y)$, for every $n \in \mathbb{N}$. In the case $n = 1$, we get $\mathbb{I}\,\mathtt{Hom}_{\mathcal{H}}(x,y) = \ker(\epsilon : \mathtt{Hom}_{\mathcal{H}}(x,y) \to \mathbb{k})$ and we have $\mathtt{Hom}_{\mathcal{H}}(x,y) / \mathbb{I}\,\mathtt{Hom}_{\mathcal{H}}(x,y) = \mathbb{k}$ as soon as $\mathtt{Hom}_{\mathcal{H}}(x,y) \neq 0$. We deduce from this relation that our completed hom-coalgebras $\mathtt{Hom}_{\mathcal{H}}(x,y)\widehat{\ }$ fulfill the filtration connectedness condition of §9.1.2.

In §9.0.6, we briefly mention that any hom-coalgebra $C = \mathtt{Hom}_{\mathcal{H}}(x,y)$ in a Hopf groupoid $\mathcal{H}$ admits an extension of scalars $C_{\mathbb{I}} = \mathtt{Hom}_{\mathcal{H}}(x,y) \otimes_{\mathbb{k}} \mathbb{I}$ such that $\mathbb{G}(C_{\mathbb{I}}) \neq \varnothing$. This observation follows from general structure theorems on coalgebras (see the references cited in §9.0.6). The group-like element $g \in \mathbb{G}(C_{\mathbb{I}})$ which we may form in such an extension $C_{\mathbb{I}} = C \otimes_{\mathbb{k}} \mathbb{I}$ of the ordinary coalgebra $C = \mathtt{Hom}_{\mathcal{H}}(x,y)$ determines a group-like element in a scalar extension $C_{\mathbb{I}}\widehat{\ } = \mathtt{Hom}_{\mathcal{H}}(x,y)_{\mathbb{I}}\widehat{\ }$ of the completed coalgebra $C^{\wedge} = \mathtt{Hom}_{\mathcal{H}}(x,y)^{\wedge}$. In the appendix section §9.3, we prove that these completed hom-coalgebras $\mathtt{Hom}_{\mathcal{H}}(x,y)^{\wedge}$ contain group-like elements defined over our ground field $g \in \mathbb{G}(\mathtt{Hom}_{\mathcal{H}}(x,y)^{\wedge})$ (and not only over an extension) as soon as we can ensure the existence of group-like elements in a extension of scalars for these completed hom-coalgebras. Thus, our completed hom-coalgebras $\mathtt{Hom}_{\mathcal{H}}(x,y)^{\wedge}$ automatically fulfill the local connectedness condition of §9.1.2 besides the filtration connectedness condition. (The verification of this condition is actually immediate for the main examples of Hopf groupoids, like $\mathcal{H} = \Bbbk[\mathcal{G}]$, to which we apply our completion process.)

The unit morphisms of the Hopf groupoid $\mathcal{H}$ have an obvious prolongment $\eta : \Bbbk \to \mathtt{Hom}_{\mathcal{H}}(x,x)^{\wedge}$ for all $x \in \mathtt{Ob}\,\mathcal{H}$. The composition operations of $\mathcal{H}$ trivially preserve the filtration of our counitary cocommutative coalgebras, and hence induce composition operations at the level of our completed hom-objects $\mathtt{Hom}_{\mathcal{H}}(-,-)^{\wedge}$. The relations satisfied by antipodes imply, as in the Hopf algebra case, that the antipodes of our Hopf groupoid preserve filtrations too and hence admit an extension to the completed hom-objects as well.

We define the completion of the Hopf groupoid $\mathcal{H}$ as the Hopf groupoid in complete filtered modules $\hat{\mathcal{H}}$ which has the same object set as our original Hopf groupoid $\mathtt{Ob}\,\hat{\mathcal{H}} := \mathtt{Ob}\,\mathcal{H}$ and the complete counitary cocommutative coalgebras defined in this paragraph as hom-objects:

$$\mathtt{Hom}_{\hat{\mathcal{H}}}(x,y) := \mathtt{Hom}_{\mathcal{H}}(x,y)^{\wedge},$$

for all pairs of objects $x, y \in \mathtt{Ob}\,\mathcal{H}$. We use the previous observations to determine the identity homomorphisms and the composition operations of this Hopf groupoid in complete filtered modules $\hat{\mathcal{H}}$. We just observed that this Hopf groupoid $\hat{\mathcal{H}}$ fulfills the connectedness conditions of §9.1.2 too and hence does form a complete Hopf groupoid according to our conventions §9.1.2.

Recall that the endomorphism coalgebra $\mathtt{End}_{\mathcal{H}}(x) = \mathtt{Hom}_{\mathcal{H}}(x,x)$ which we associate to any object $x \in \mathcal{H}$ in a Hopf groupoid $\mathcal{H}$ inherits a Hopf algebra structure, with the identity homomorphism of our object $id_x \in \mathtt{End}_{\mathcal{H}}(x)$ as unit element, the product yielded by the composition of homomorphisms in the Hopf groupoid $\mathcal{H}$, and the antipode yielded by the antipode operation of $\mathcal{H}$ (see §9.0.2). We have a similar result in the case of a complete Hopf groupoid (see §§9.1.1-9.1.2). We have the following observation:

Proposition 9.1.8. *The complete endomorphism Hopf algebra* $\mathtt{End}_{\hat{\mathcal{H}}}(x) := \mathtt{End}_{\mathcal{H}}(x)^{\wedge}$ *which we associate to any object* $x \in \mathtt{Ob}\,\mathcal{H}$ *in the completion* $\hat{\mathcal{H}}$ *of a Hopf groupoid* $\mathcal{H}$ *(in the sense of* §9.1.7*) is isomorphic to the completion of the endomorphism Hopf algebra* $\mathtt{End}_{\mathcal{H}}(x)$ *which we associate to this object* $x \in \mathtt{Ob}\,\mathcal{H}$ *in the Hopf groupoid* $\mathcal{H}$ *and which we treat as an isolated object (to perform the completion of* §8.1.1*).*

Proof. We fix an object $x \in \mathtt{Ob}\,\mathcal{H}$ in our Hopf groupoid $\mathcal{H}$. We check that the filtration of §9.1.7, where we consider all composites of composable homomorphisms (with a null augmentation) in the Hopf groupoid $\mathcal{H}$, agrees with the filtration of §8.1.1 for the Hopf algebra $\mathtt{End}_{\mathcal{H}}(x)$, where we only consider composites of endomorphisms of the objects x (with a null augmentation yet). The latter is obviously included in the former. To check the converse inclusion, we consider a chain of composable homomorphisms $x_0 \xleftarrow{f_1} x_1 \xleftarrow{f_2} \cdots \xleftarrow{f_n} x_n$, which goes from $x_0 = x$ to $x_n = x$ in the Hopf groupoid $\mathcal{H}$, and where we have $\epsilon(f_1) = \cdots = \epsilon(f_n) = 0$.

We assume $f = f_1 \cdot f_2 \cdot \dots \cdot f_n \neq 0$. We accordingly have $\mathtt{Hom}_{\mathcal{H}}(x_i, x) \neq 0$ and $\mathtt{Hom}_{\mathcal{H}}(x, x_i) \neq 0$, for all intermediate objects in our chain $x_i \in \mathtt{Ob}\,\mathcal{H}$, $i = 1, \dots, n-1$. We pick a homomorphism $g_i \in \mathtt{Hom}_{\mathcal{H}}(x_i, x)$ such that $\epsilon(g_i) = 1$ for each $i = 1, \dots, n-1$. We form the coproduct of these homomorphisms $\Delta(g_i) = \sum_{(g_i)} g_{i(1)} \otimes g_{i(2)}$ in the coalgebras $\mathtt{Hom}_{\mathcal{H}}(x_i, x)$, for $i = 1, \dots, n-1$. We insert the antipode relations $\sum_{(g_i)} \sigma(g_{i(1)}) \cdot g_{i(2)} = id_{x_i}$ in our composite to express the result of this composition operation as an n-fold composite of endomorphisms of the object x in $\mathcal{H}$:

$$f_1 \cdot f_2 \cdot \dots \cdot f_n = \sum_{(g_1),\dots,(g_{n-1})} (f_1 \cdot \sigma(g_{1(1)})) \cdot (g_{1(2)} \cdot f_2 \cdot \sigma(g_{2(1)})) \cdot \dots (g_{n-1(2)} \cdot f_n).$$

Note that these endomorphisms trivially have a null augmentation and hence belong to the augmentation ideal of the Hopf algebra $\mathtt{End}_{\mathcal{H}}(x)$.

This verification completes the proof that the filtration of the hom-coalgebra $\mathtt{End}_{\mathcal{H}}(x)$ in the Hopf groupoid $\mathcal{H}$, such as defined in §9.1.7, reduces to the filtration of §8.1.1 when we treat the Hopf algebra $\mathtt{End}_{\mathcal{H}}(x)$ as an isolated object. ☐

9.1.9. *The complete Hopf groupoid and group-like element functors.* We associate a complete Hopf groupoid $\Bbbk[\mathcal{G}]^\wedge$ to any groupoid $\mathcal{G}$ by taking the completion §9.1.7 of the Hopf groupoid $\Bbbk[\mathcal{G}]$ of §9.0.3. This complete Hopf groupoid $\Bbbk[\mathcal{G}]^\wedge$ has the same object set as our original groupoid $\mathtt{Ob}\,\Bbbk[\mathcal{G}]^\wedge = \mathtt{Ob}\,\mathcal{G}$, and the completed counitary cocommutative coalgebras

$$\mathtt{Hom}_{\mathcal{G}}(x,y)^\wedge = \Bbbk[\mathtt{Mor}_{\mathcal{G}}(x,y)]^\wedge$$

as hom-objects. The map $\Bbbk[-]^\wedge : \mathcal{G} \mapsto \Bbbk[\mathcal{G}]^\wedge$ defines a functor $\Bbbk[-]^\wedge : \mathcal{G}rd \to \hat{f}\,\mathcal{H}opf\mathcal{G}rd$ from the category of groupoids $\mathcal{G}rd$ to the category of complete Hopf groupoids $\hat{f}\,\mathcal{H}opf\mathcal{G}rd$.

In the converse direction, we see that the group-like element functor of §9.0.4 has an obvious complete analogue which enables us to associate a groupoid of group-like elements $\mathbb{G}(\mathcal{H})$ to any complete Hopf groupoid $\mathcal{H}$. This groupoid has the same object set as our original complete groupoid $\mathtt{Ob}\,\mathbb{G}(\mathcal{H}) = \mathtt{Ob}\,\mathcal{H}$ and the sets of group-like elements

$$\mathbb{G}(\mathtt{Hom}_{\mathcal{H}}(x,y)) = \{f \in \mathtt{Hom}_{\mathcal{H}}(x,y) \,|\, \epsilon(f) = 1 \text{ and } \Delta(f) = f \hat{\otimes} f\}$$

as morphism sets. The identity morphisms and the composition operation of this groupoid are yielded by the identity elements and by the composition of the complete Hopf groupoid $\mathcal{H}$, as in the construction of §9.0.4. The inverse of morphisms in $\mathbb{G}(\mathcal{H})$ is also given by the antipode operation on $\mathcal{H}$.

The adjunction relation of §9.0.5 has the following analogue in the context of complete Hopf groupoids:

PROPOSITION 9.1.10. *The complete Hopf groupoid functor* $\Bbbk[-]^\wedge : \mathcal{G} \mapsto \Bbbk[\mathcal{G}]^\wedge$ *and the group-like functor* $\mathbb{G} : \mathcal{H} \to \mathbb{G}(\mathcal{H})$, *given by the construction of the previous paragraph (*§9.1.9*), define a pair of adjoint functors* $\Bbbk[-]^\wedge : \mathcal{G}rd \rightleftarrows \hat{f}\,\mathcal{H}opf\mathcal{G}rd : \mathbb{G}$ *between the category of groupoids* $\mathcal{G}rd$ *and the category of complete Hopf groupoids* $\hat{f}\,\mathcal{H}opf\mathcal{G}rd$.

Proof. This proposition follows from a straightforward extension of the arguments of Proposition 8.1.3 where we define the adjunction between groups and complete Hopf algebras. ☐

Furthermore, we have the following generalization of the result of Proposition 8.1.6:

Proposition 9.1.11. *The functor $\mathbb{G} : \hat{f}\,\mathcal{H}opf\mathcal{G}rd \to \mathcal{G}rd$ induces an injective map on morphism sets:*

$$\mathtt{Mor}_{\hat{f}\,\mathcal{H}opf\mathcal{G}rd}(\mathcal{A},\mathcal{B}) \hookrightarrow \mathtt{Mor}_{\mathcal{G}rd}(\mathbb{G}(\mathcal{A}),\mathbb{G}(\mathcal{B})),$$

for all $\mathcal{A},\mathcal{B} \in \hat{f}\,\mathcal{H}opf\mathcal{G}rd$, and hence, is faithful.

Proof. Let $\phi,\psi : \mathcal{A} \to \mathcal{B}$ be a pair of parallel morphisms such that $\mathbb{G}(\phi) = \mathbb{G}(\psi)$. We implicitly assume that these morphisms $\phi,\psi : \mathcal{A} \to \mathcal{B}$ agree on the object sets of our Hopf groupoids when we make this assumption $\mathbb{G}(\phi) = \mathbb{G}(\psi)$. We check that our morphisms agree on hom-objects too. We use the ideas of §9.1.5. Proposition 8.1.6 implies that our morphisms agree on the complete endomorphism Hopf algebra of each object in $\mathcal{A}$. In the case of an arbitrary pair of objects $x,y \in \mathcal{A}$ such that $\mathtt{Hom}_{\mathcal{A}}(x,y) \neq 0$, we pick a group-like homomorphism $f \in \mathbb{G}(\mathtt{Hom}_{\mathcal{A}}(x,y))$ by using our local connectedness requirement in the definition of a complete Hopf groupoid. We form the commutative diagram

$$\begin{tikzcd}
\mathtt{End}_{\mathcal{A}}(x) \arrow[r, "\phi=\psi"] \arrow[d, "f_*"'] & \mathtt{End}_{\mathcal{B}}(u) \arrow[d, "\phi(f)_*=\psi(f)_*"] \\
\mathtt{Hom}_{\mathcal{A}}(x,y) \arrow[r, shift left, "\phi"] \arrow[r, shift right, "\psi"'] & \mathtt{Hom}_{\mathcal{B}}(u,v)
\end{tikzcd},$$

where we set $u = \phi(x) = \psi(x)$, $v = \phi(y) = \psi(y)$ and we use the obvious composition as vertical maps, to conclude that our morphisms also agree on the hom-coalgebra $\mathtt{Hom}_{\mathcal{A}}(x,y)$. We therefore have $\phi = \psi$. ☐

9.1.12. *The category of Malcev complete groupoids.* We now define the category of *Malcev complete groupoids* $\hat{f}\,\mathcal{G}rd$ as the faithful image of the category of complete Hopf groupoids $\hat{f}\,\mathcal{H}opf\mathcal{G}rd$ in the category of groupoids $\mathcal{G}rd$:

$$\hat{f}\,\mathcal{G}rd = \mathbb{G}(\hat{f}\,\mathcal{H}opf\mathcal{G}rd).$$

We also say that a groupoid $\mathcal{G}$ is *Malcev complete* when we have $\mathcal{G} = \mathbb{G}(\mathcal{H})$, for some $\mathcal{H} \in \hat{f}\,\mathcal{H}opf\mathcal{G}rd$. We moreover say that a morphism $\phi : \mathcal{G} \to \mathcal{H}$ in this category $\hat{f}\,\mathcal{G}rd$ is an *equivalence of Malcev complete groupoids* when this morphism is the image of an equivalence of complete Hopf groupoids (see §9.1.4) under the group-like element functor $\mathbb{G} : \hat{f}\,\mathcal{H}opf\mathcal{G}rd \to \hat{f}\,\mathcal{G}rd$. We again use the notation $\phi : \mathcal{G} \xrightarrow{\sim} \mathcal{H}$ with the extra mark $\sim$ to distinguish the class of equivalences among the morphisms of the category of Malcev complete groupoids. We can still identify the isomorphisms of Malcev complete groupoids with the equivalences which are given by a bijection at the object set level.

In the next paragraph, we explain a generalization, for our category of Malcev complete groupoids, of the structures which we attach to the objects of the category of Malcev complete groups in §8.2. Let us observe that a Malcev complete group in the sense of §8.2 is identified with a Malcev complete groupoid with a single object.

In the converse direction, we immediately see that the automorphism group of an object $\mathtt{Aut}_{\mathcal{G}}(x) = \mathtt{Mor}_{\mathcal{G}}(x,x)$ in a Malcev complete groupoid $\mathcal{G} = \mathbb{G}(\mathcal{H})$ forms a Malcev complete group since we have $\mathtt{Aut}_{\mathcal{G}}(x) = \mathbb{G}(\mathtt{End}_{\mathcal{H}}(x))$ by definition of our group-like functor on complete Hopf groupoids.

9.1.13. *The tower decomposition of Malcev complete groupoids.* We explained in §8.2 that each Malcev complete group G is equipped with a natural filtration $G = \mathtt{F}_1 G \supset \cdots \supset \mathtt{F}_m G \supset \cdots$ by normal subgroups $\mathtt{F}_m G \subset G$ such that $G = \lim_m G/\mathtt{F}_m G$.

In the case of a Malcev complete groupoid $\mathcal{G}$, we have a tower decomposition $\mathcal{G} = \lim_m q_m \mathcal{G}$ which we use to extend this construction. We take $\mathtt{Ob}\, q_m \mathcal{G} := \mathtt{Ob}\, \mathcal{G}$ for each $m \geq 0$, and we define the morphism sets of this groupoid as the quotients $\mathtt{Mor}_{q_m \mathcal{G}}(x,y) = \mathtt{Mor}_{\mathcal{G}}(x,y)/ \equiv$ of the morphism sets $\mathtt{Mor}_{\mathcal{G}}(x,y)$ under the equivalence relation such that $f \equiv g$ if we have $g = f \cdot \gamma$ for an element $\gamma \in \mathtt{F}_{m+1} \mathtt{Aut}_{\mathcal{G}}(x)$ in the $m+1$st layer of the canonical filtration of the Malcev complete group $G = \mathtt{Aut}_{\mathcal{G}}(x)$. We also use the notation

$$\mathtt{Mor}_{q_m \mathcal{G}}(x,y) = \mathtt{Mor}_{\mathcal{G}}(x,y)/\mathtt{F}_{m+1} \mathtt{Aut}_{\mathcal{G}}(x)$$

for the result of this quotient construction. We trivially have the group identity $\mathtt{Aut}_{q_m \mathcal{G}}(x) = \mathtt{Aut}_{\mathcal{G}}(x)/\mathtt{F}_{m+1} \mathtt{Aut}_{\mathcal{G}}(x)$ for the automorphism group of an object $x \in \mathtt{Ob}\, \mathcal{G}$ in $q_m \mathcal{G}$ and the relation $\mathtt{Aut}_{\mathcal{G}}(x) = \lim_m \mathtt{Aut}_{\mathcal{G}}(x)/\mathtt{F}_{m+1} \mathtt{Aut}_{\mathcal{G}}(x)$, which holds at the level of these Malcev complete groups $G = \mathtt{Aut}_{\mathcal{G}}(x)$, immediately implies that we have the relation $\mathtt{Mor}_{\mathcal{G}}(x,y) = \lim_m \mathtt{Mor}_{q_m \mathcal{G}}(x,y)$ for all morphisms sets $\mathtt{Mor}_{\mathcal{G}}(x,y)$, $x,y \in \mathcal{G}$, since the composition with any morphism $f \in \mathtt{Mor}_{\mathcal{G}}(x,y)$ induces a bijection $f_* : \mathtt{Aut}_{\mathcal{G}}(x) \to \mathtt{Mor}_{\mathcal{G}}(x,y)$ in our groupoid.

Furthermore, we have $\mathtt{Mor}_{q_0 \mathcal{G}}(x,y) = pt$ if x and y belong to the same connected component of the groupoid $\mathcal{G}$ and $\mathtt{Mor}_{q_0 \mathcal{G}}(x,y) = \varnothing$ otherwise since our Malcev complete groups $G = \mathtt{Aut}_{\mathcal{G}}(x)$ satisfy $G = \mathtt{F}_1 G \Leftrightarrow G/\mathtt{F}_1 G = pt$. Equivalently, this groupoid $q_0 \mathcal{G}$ is obtained by collapsing all non-trivial morphism sets in $\mathcal{G}$ to a point. To express this relationship, we also abusively write $q_0 \mathcal{G} = \pi_0 \mathcal{G}$ where $\pi_0 \mathcal{G}$ denotes the discrete groupoid (the set) of the connected components of the groupoid $\mathcal{G}$ (actually, we have to take the pullback of this discrete groupoid along the natural map of object sets $q : \mathtt{Ob}\, \mathcal{G} \to \pi_0 \mathcal{G}$ in order to give a sense to this identity).

Let us observe that the conjugation operation $c_f(\gamma) = f \cdot \gamma \cdot f^{-1}$ with a morphism $f \in \mathtt{Mor}_{\mathcal{G}}(x,y)$ in our groupoid $\mathcal{G}$ comes from a morphism of complete Hopf algebras $c_f : \mathtt{End}_{\mathcal{H}}(x) \to \mathtt{End}_{\mathcal{H}}(y)$, where $\mathcal{H}$ denotes the complete Hopf groupoid such that $\mathcal{G} = \mathbb{G}(\mathcal{H})$. This conjugation operation accordingly defines a morphism of Malcev complete groups $c_f : \mathtt{Aut}_{\mathcal{G}}(x) \to \mathtt{Aut}_{\mathcal{G}}(y)$, and, as a consequence, we have $\gamma \in \mathtt{F}_{m+1} \mathtt{Aut}_{\mathcal{G}}(x) \Rightarrow f \cdot \gamma \cdot f^{-1} \in \mathtt{F}_{m+1} \mathtt{Aut}_{\mathcal{G}}(y)$. From this observation, we deduce that we can equivalently take the equivalence relation such that $f \equiv g$ when we have $g = \gamma' \cdot f$ for an element $\gamma' \in \mathtt{F}_{m+1} \mathtt{Aut}_{\mathcal{G}}(y)$ in the definition of the morphism sets of our groupoid $q_m \mathcal{G}$. We also get that the composition of morphisms in $\mathtt{Mor}_{\mathcal{G}}(x,y)$ is compatible with this equivalence relation and, hence, does induce a composition operation on the morphism sets of the groupoid $q_m \mathcal{G}$.

This construction is clearly functorial. Furthermore, we have the following statement:

PROPOSITION 9.1.14. *The objects $q_m \mathcal{G}$ in the tower decomposition of a Malcev complete groupoid $\mathcal{G}$ form Malcev complete groupoids themselves and the identity*

$\mathcal{G} = \lim_m q_m \mathcal{G}$ *holds in the category Malcev complete groupoids. Besides, the morphisms* $p : \mathcal{G} \to q_n \mathcal{G}$*, which we consider in this tower decomposition, induce isomorphisms of Malcev complete groupoids* $p_* : q_m \mathcal{G} \xrightarrow{\simeq} q_m(q_n \mathcal{G})$ *for all* $m \leq n$ *when we apply our tower decomposition twice.*

Proof. We assume $\mathcal{G} = \mathbb{G}(\mathcal{H})$ for some complete Hopf groupoid $\mathcal{H}$. We fix $m \geq 0$. We aim to define a complete Hopf groupoid $q_m \mathcal{H}$ such that $q_m \mathcal{G} = \mathbb{G}(q_m \mathcal{H})$. We trivially take $\mathtt{Ob}\, q_m \mathcal{H} = \mathtt{Ob}\, \mathcal{H} = \mathtt{Ob}\, \mathcal{G}$ and our main purpose is to define the hom-objects of this Hopf groupoid $q_m \mathcal{H}$.

In a first step, we consider the case of the complete endomorphism Hopf algebra of an object $x \in \mathtt{Ob}\, q_m \mathcal{H}$. Let:

$$\mathfrak{g}_x := \mathbb{P}(\mathtt{End}_{\mathcal{H}}(x)). \tag{1}$$

By Proposition 8.2.5, we have an identity

$$\mathtt{Aut}_{\mathcal{G}}(x)/\,\mathtt{F}_{m+1}\,\mathtt{Aut}_{\mathcal{G}}(x) = \mathbb{G}(\hat{\mathbb{U}}(\mathfrak{g}_x\,/\,\mathtt{F}_{m+1}\,\mathfrak{g}_x)), \tag{2}$$

where we consider the quotient of this Lie algebra by its $m+1$st filtration layer $\mathtt{F}_{m+1}\,\mathfrak{g}_x \subset \mathfrak{g}_x$. We therefore set

$$\mathtt{End}_{q_m \mathcal{H}}(x) := \hat{\mathbb{U}}(\mathfrak{g}_x\,/\,\mathtt{F}_{m+1}\,\mathfrak{g}_x) \tag{3}$$

to get a complete Hopf algebra such that $\mathtt{Aut}_{q_m \mathcal{G}}(x) = \mathbb{G}(\mathtt{End}_{q_m \mathcal{H}}(x))$.

The idea is to define the underlying endomorphism coalgebra of the morphism set $\mathtt{Mor}_{q_m \mathcal{G}}(x,y) = \mathtt{Mor}_{\mathcal{G}}(x,y)/\,\mathtt{F}_{m+1}\,\mathtt{Aut}_{\mathcal{G}}(x)$, for any pair $x, y \in \mathtt{Ob}\,\mathcal{G}$, by using a counterpart, in the complete Hopf groupoid $\mathcal{H}$, of the quotient construction of §9.1.13. For this purpose, we use the identity of complete modules:

$$\hat{\mathbb{U}}(\mathfrak{g}_x\,/\,\mathtt{F}_{m+1}\,\mathfrak{g}_x) = \hat{\mathbb{U}}(\mathfrak{g}_x)/\hat{\mathbb{U}}(\mathfrak{g}_x)\cdot \mathtt{F}_{m+1}\,\mathfrak{g}_x, \tag{4}$$

where

$$\hat{\mathbb{U}}(\mathfrak{g}_x)\cdot \mathtt{F}_{m+1}\,\mathfrak{g}_x := \mathrm{im}\big(\hat{\mathbb{U}}(\mathfrak{g}_x)\hat{\otimes}\,\mathtt{F}_{m+1}\,\mathfrak{g}_x \xrightarrow{\mu} \hat{\mathbb{U}}(\mathfrak{g}_x)\big) \tag{5}$$

represents the left ideal (in the complete sense) generated by $\mathtt{F}_{m+1}\,\mathfrak{g}_x \subset \mathfrak{g}_x$ in the complete enveloping algebra $\hat{\mathbb{U}}(\mathfrak{g}_x)$, for any object $x \in \mathtt{Ob}\,\mathcal{H}$. To establish this relation, we use that the commutator relation $[\mathfrak{g}_x, \mathtt{F}_{m+1}\,\mathfrak{g}_x] \subset \mathtt{F}_{m+1}\,\mathfrak{g}_x$ implies that we have the identity $\hat{\mathbb{U}}(\mathfrak{g}_x)\cdot \mathtt{F}_{m+1}\,\mathfrak{g}_x = \mathtt{F}_{m+1}\,\mathfrak{g}_x \cdot \hat{\mathbb{U}}(\mathfrak{g}_x)$, where we consider a symmetrically defined right ideal generated by $\mathfrak{g}_x$ in $\hat{\mathbb{U}}(\mathfrak{g}_x)$. This identity implies that our quotient complete module in (4) inherits a well-defined complete algebra structure, and then we easily check that this algebra fulfills the universal property of the complete enveloping algebra of the Lie algebra $\mathfrak{g}_x\,/\,\mathtt{F}_{m+1}\,\mathfrak{g}_x$.

We now set:

$$\mathtt{Hom}_{q_m \mathcal{H}}(x,y) := \mathtt{Hom}_{\mathcal{H}}(x,y)/\,\mathtt{Hom}_{\mathcal{H}}(x,y)\cdot \mathtt{F}_{m+1}\,\mathfrak{g}_x, \tag{6}$$

for each pair of objects $x, y \in \mathtt{Ob}\,\mathcal{G}$, where

$$\mathtt{Hom}_{\mathcal{H}}(x,y)\cdot \mathtt{F}_{m+1}\,\mathfrak{g}_x = \mathrm{im}\big(\mathtt{Hom}_{\mathcal{H}}(x,y)\hat{\otimes}\,\mathtt{F}_{m+1}\,\mathfrak{g}_x \xrightarrow{\mu} \mathtt{Hom}_{\mathcal{H}}(x,y)\big) \tag{7}$$

represents an analogue, in this complete filtered module $\mathtt{Hom}_{\mathcal{H}}(x,y)$, of the left ideal generated by $\mathtt{F}_{m+1}\,\mathfrak{g}_x \subset \mathfrak{g}_x$ (in the complete sense yet). We easily deduce from the primitive coproduct formula $\Delta(\mathtt{F}_{m+1}\,\mathfrak{g}_x) \subset \mathtt{F}_{m+1}\,\mathfrak{g}_x \hat{\otimes}\, id_x + id_x \hat{\otimes}\, \mathtt{F}_{m+1}\,\mathfrak{g}_x$ that this module $I = \mathtt{Hom}_{\mathcal{H}}(x,y)\cdot \mathtt{F}_{m+1}\,\mathfrak{g}_x$ satisfies the relation $\Delta(I) \subset I\hat{\otimes} C + C\hat{\otimes} I$ in the coalgebra $C = \mathtt{Hom}_{\mathcal{H}}(x,y)$ and this result implies that our quotient object (6) inherits a well-defined counitary cocommutative coalgebra structure.

We now prove that these quotient hom-coalgebras (6) inherit coherently defined composition operations and antipodes. Let $u \in \mathtt{Hom}_{\mathcal{H}}(x,y)$. Let $\gamma \in \mathtt{F}_{m+1}\,\mathfrak{g}_x$. For this purpose, we first consider the expression:

$$\gamma' = \sum_{(u)} u_{(1)} \cdot \gamma \cdot \sigma(u_{(2)}), \tag{8}$$

where we use the Sweedler notation to express the coproduct of our homomorphism $\Delta(u) = \sum_{(u)} u_{(1)} \hat{\otimes} u_{(2)}$ in the complete coalgebra $\mathtt{Hom}_{\mathcal{H}}(x,y)$ and we apply the antipode of the complete Hopf groupoid $\mathcal{H}$ to the second factor of this coproduct. We easily check that we have the implication $\gamma \in \mathbb{P}(\mathtt{End}_{\mathcal{H}}(x)) \Rightarrow \gamma' \in \mathbb{P}(\mathtt{End}_{\mathcal{H}}(y))$. We clearly have the relation $\gamma \in \mathtt{F}_{m+1}\,\mathtt{End}_{\mathcal{H}}(x) \Rightarrow \gamma' \in \mathtt{F}_{m+1}\,\mathtt{End}_{\mathcal{H}}(y)$ too. Hence our expression (8) returns an element in the $m+1$st layer $\mathtt{F}_{m+1}\,\mathfrak{g}_y = \mathfrak{g}_y \cap \mathtt{F}_{m+1}\,\mathtt{End}_{\mathcal{H}}(y)$ of the filtration of the complete Lie algebra $\mathfrak{g}_y = \mathbb{P}(\mathtt{End}_{\mathcal{H}}(y))$ which we associate to the object $y \in \mathtt{Ob}\,\mathcal{H}$. We then have the identity:

$$u \cdot \gamma = \sum_{(u)} \underbrace{u_{(1)} \cdot \gamma \cdot \sigma(u_{(2)})}_{\in \mathtt{F}_{m+1}\,\mathfrak{g}_y} \cdot u_{(3)} \tag{9}$$

which implies that any element in our right translated module (7) is identified with an element of the symmetrically defined left translated module $\mathtt{F}_{m+1}\,\mathfrak{g}_y \cdot \mathtt{Hom}_{\mathcal{H}}(x,y)$, and since this argument can be symmetrized, we conclude that we have an identity:

$$\mathtt{Hom}_{\mathcal{H}}(x,y) \cdot \mathtt{F}_{m+1}\,\mathfrak{g}_x = \mathtt{F}_{m+1}\,\mathfrak{g}_y \cdot \mathtt{Hom}_{\mathcal{H}}(x,y), \tag{10}$$

for every pair of objects $x, y \in \mathtt{Ob}\,\mathcal{H}$. The definition of our quotient composition morphisms

$$\mu : \mathtt{Hom}_{q_m\,\mathcal{H}}(y,z) \hat{\otimes} \mathtt{Hom}_{q_m\,\mathcal{H}}(x,y) \to \mathtt{Hom}_{q_m\,\mathcal{H}}(x,z) \tag{11}$$

readily follows. We check the definition of antipode operations similarly. We also have an obvious morphism of complete Hopf groupoids $\mathcal{H} \to q_m\,\mathcal{H}$ by construction of our object.

We now immediately see that we have the relation

$$\mathtt{Mor}_{q_m\,\mathcal{G}}(x,y) = \mathbb{G}(\mathtt{Hom}_{q_m\,\mathcal{H}}(x,y)) \tag{12}$$

for this quotient complete Hopf groupoid $q_m\,\mathcal{H}$, for every pair of objects $x, y \in \mathtt{Ob}\,\mathcal{H}$, because this is so for $x = y$ by the result of Proposition 8.2.5 and we can pick group-like homomorphisms $g \in \mathbb{G}(\mathtt{Hom}_{\mathcal{H}}(x,y))$ to relate our coalgebras. We immediately see that the structure operations of the groupoid $q_m\,\mathcal{G}$ also correspond to the operations induced by the structure operations of the complete Hopf groupoid $q_m\,\mathcal{H}$ on group-like elements, because these operations are obtained by quotient of the natural structure operations of the groupoid $\mathcal{G}$.

Furthermore, our quotient complete Hopf groupoids $q_m\,\mathcal{H}$ clearly fit in a tower:

$$\mathcal{H} \to \cdots \to q_m\,\mathcal{H} \to q_{m-1}\,\mathcal{H} \to \cdots \to q_0\,\mathcal{H}, \tag{13}$$

which forms a counterpart, in the category of complete Hopf groupoids, of our tower decomposition $\mathcal{G} = \lim_m q_m\,\mathcal{G}$ of the Malcev complete groupoid $\mathcal{G}$.

The second assertion of the proposition $p_* : q_m\,\mathcal{G} \xrightarrow{\simeq} q_m(q_n\,\mathcal{G})$ is a straightforward consequence of the obvious identities

$$(\mathfrak{g}_x \,/\, \mathtt{F}_{n+1}\,\mathfrak{g}_x)/(\mathtt{F}_{m+1}\,\mathfrak{g}_x \,/\, \mathtt{F}_{n+1}\,\mathfrak{g}_x) = \mathfrak{g}_x \,/\, \mathtt{F}_{m+1}\,\mathfrak{g}_x \tag{14}$$

which hold for the complete Lie algebras associated to the complete endomorphism Hopf algebras of the objects $x \in \mathcal{G}$ in our complete Hopf groupoids. $\square$

We already observed that the construction of §9.1.13 is functorial. We also have the following result:

PROPOSITION 9.1.15. *Any morphism $\phi : \mathcal{G} \to q_m \mathcal{H}$, where $\mathcal{G}$ and $\mathcal{H}$ are Malcev complete groupoids, admits a unique factorization:*

$$\begin{tikzcd} \mathcal{G} \arrow[rr, "\phi"] \arrow[dr] & & q_m \mathcal{H} \\ & q_m \mathcal{G} \arrow[ur, dotted, "\exists ! \bar{\phi}"'] & \end{tikzcd} ,$$

where $\bar{\phi}$ is a morphism of Malcev complete groupoids.

PROOF. The uniqueness of the factorization is clear, because the morphism sets of the groupoid $q_m \mathcal{G}$ are defined as quotients of the morphism sets of $\mathcal{G}$. To produce the factorization, we just form the diagram

$$\begin{tikzcd} \mathcal{G} \arrow[r, "\phi"] \arrow[d] & q_m \mathcal{H} \arrow[d, "\simeq"] \\ q_m \mathcal{G} \arrow[r, "q_m \phi"] & q_m(q_m \mathcal{H}) \end{tikzcd} ,$$

where we use the functoriality of our tower decomposition and that the applications of the tower decomposition twice returns isomorphic objects (Proposition 9.1.14). □

9.1.16. *The principal fibers of the tower decomposition of a Malcev complete groupoid and local coefficient systems.* In the case of a Malcev complete group G, we observed in §8.2.2 that the subquotients of the filtration of our object $\mathtt{E}^0_m G = \mathtt{F}_m G / \mathtt{F}_{m+1} G$ naturally inherit a $\Bbbk$-module structure. We checked, besides, that the conjugation operation $c_\gamma(x) = \gamma \cdot x \cdot \gamma^{-1}$, which we associate to any element $\gamma \in G$, defines a filtration morphism on G which reduces to the identity map on these filtration subquotients $\mathtt{E}^0_m G$ (see §8.2.2).

In the case of a Malcev complete groupoid $\mathcal{G}$, we consider the collections of subquotients $\mathtt{E}^0_m \mathtt{Aut}_{\mathcal{G}}(x)$ which we associate the automorphism group of each object in our groupoid $x \in \mathtt{Ob}\,\mathcal{G}$. We immediately see that the action of the automorphism group $\mathtt{Aut}_{\mathcal{G}}(x)$ on a morphism set $\mathtt{Mor}_{\mathcal{G}}(x,y)$ in $\mathcal{G}$ descends to an action of the subquotient module $\mathtt{E}^0_m \mathtt{Aut}_{\mathcal{G}}(x)$ on the morphism set $\mathtt{Mor}_{q_m \mathcal{G}}(x,y)$ of the groupoid $q_m \mathcal{G}$ in the tower decomposition of §9.1.13. Furthermore, a pair of morphisms in this morphism set $f, g \in \mathtt{Mor}_{q_m \mathcal{G}}(x,y)$ have the same image in $\mathtt{Mor}_{q_{m-1} \mathcal{G}}(x,y)$ (when we go down by one level in our tower) if and only if we have $f = g \cdot \bar{\gamma}$ for a class $\bar{\gamma} \in \mathtt{E}^0_m \mathtt{Aut}_{\mathcal{G}}(x)$. To express this relationship, we can regard the map $p_m : \mathtt{Mor}_{q_m \mathcal{G}}(x,y) \to \mathtt{Mor}_{q_{m-1} \mathcal{G}}(x,y)$ as a principal fibration (of discrete spaces), and the module $\mathtt{E}^0_m \mathtt{Aut}_{\mathcal{G}}(x)$ as the structure group of this principal fibration. In what follows, we also say for short that the collection $\mathtt{E}^0_m \mathcal{G} = \{\mathtt{E}^0_m \mathtt{Aut}_{\mathcal{G}}(x), x \in \mathtt{Ob}\,\mathcal{G}\}$ represents the *(principal) fiber* of the groupoid morphism $p_m : q_m \mathcal{G} \to q_{m-1} \mathcal{G}$ in the tower decomposition $\mathcal{G} = \lim_m q_m \mathcal{G}$ of the Malcev complete groupoid $\mathcal{G}$.

We already observed that the conjugation operation $c_f(\gamma) = f \cdot \gamma \cdot f^{-1}$ with a morphism $f \in \mathtt{Mor}_{\mathcal{G}}(x,y)$ in our groupoid $\mathcal{G}$ comes from a morphism of complete Hopf algebras $c_f : \mathtt{End}_{\mathcal{H}}(x) \to \mathtt{End}_{\mathcal{H}}(y)$, for any pair of objects $x, y \in \mathtt{Ob}\,\mathcal{G}$, where $\mathcal{H}$ denotes the complete Hopf groupoid such that $\mathcal{G} = \mathbb{G}(\mathcal{H})$. We deduce from this representation that this conjugation operation preserves the filtration of our automorphism groups and induces a morphism on the subquotients of our filtration $c_f : \mathtt{E}^0_m \mathtt{Aut}_{\mathcal{G}}(x) \to \mathtt{E}^0_m \mathtt{Aut}_{\mathcal{G}}(y)$, for all $m \geq 0$. These conjugation operations are, in an obvious sense, compatible with the action of the modules $\mathtt{E}^0_m \mathtt{Aut}_{\mathcal{G}}(x)$, $x \in \mathtt{Ob}\,\mathcal{G}$, on the morphism sets of the groupoid $q_m\,\mathcal{G}$.

We moreover have $c_f = c_g$, for all pairs of parallel morphisms $f, g \in \mathtt{Mor}_{\mathcal{G}}(x, y)$, since we have $f = g \cdot \gamma$ for some $\gamma \in \mathtt{Aut}_{\mathcal{G}}(x)$ in this case, and we recalled at the beginning of this paragraph that the inner automorphisms of a Malcev complete group induce the identity map on our filtration subquotients. We conclude that the collection $\mathtt{E}^0_m\,\mathcal{G} = \{\mathtt{E}^0_m \mathtt{Aut}_{\mathcal{G}}(x), x \in \mathtt{Ob}\,\mathcal{G}\}$ forms a diagram over the groupoid $q_0\,\mathcal{G}$. We also use the name '*local coefficient system*', borrowed from algebraic topology, for this structure.

We record the following statement:

Proposition 9.1.17. *If a morphism of Malcev complete groupoids $\psi : \mathcal{G} \to \mathcal{H}$ induces a bijection on the sets of connected components $\pi_0\psi : \pi_0\,\mathcal{G} \xrightarrow{\simeq} \pi_0\,\mathcal{H}$ and an isomorphism on the fibers of the tower decomposition of our objects $\mathtt{E}^0_m\,\psi : \mathtt{E}^0_m\,\mathcal{G} \xrightarrow{\simeq} \mathtt{E}^0_m\,\mathcal{H}$, for all $m \geq 1$, then this morphism defines an equivalence of Malcev complete groupoids $\psi : \mathcal{G} \xrightarrow{\sim} \mathcal{H}$ (see* §9.1.12*).*

Proof. We consider the morphism of complete Hopf groupoids $\phi : \mathcal{A} \to \mathcal{B}$ such that $\mathbb{G}(\phi) = \psi$. We fix an object $x \in \mathtt{Ob}\,\mathcal{A}$ first. By assumption, our morphism $\phi : \mathcal{A} \to \mathcal{B}$ induces an isomorphism on the subquotients of the groups of group-like elements associated to the complete endomorphism Hopf algebras of the objects $x \in \mathtt{Ob}\,\mathcal{A}$ and $\phi(x) \in \mathtt{Ob}\,\mathcal{B}$ in our complete Hopf groupoids:

$$\mathtt{E}^0_m\,\mathbb{G}(\phi) : \mathtt{E}^0_m\,\mathbb{G}(\mathtt{End}_{\mathcal{A}}(x)) \xrightarrow{\simeq} \mathtt{E}^0_m\,\mathbb{G}(\mathtt{End}_{\mathcal{A}}(\phi(x))),$$

for all $m \geq 1$. By Proposition 8.2.3, this statement implies that $\phi : \mathcal{A} \to \mathcal{B}$ induces an isomorphism of weight graded Lie algebras

$$\mathtt{E}^0_m\,\mathbb{P}(\phi) : \mathtt{E}^0\,\mathbb{P}(\mathtt{End}_{\mathcal{A}}(x)) \xrightarrow{\simeq} \mathtt{E}^0\,\mathbb{P}(\mathtt{End}_{\mathcal{A}}(\phi(x))),$$

when we consider the primitive part of these complete endomorphism Hopf algebras $\mathtt{End}_{\mathcal{A}}(x)$ and $\mathtt{End}_{\mathcal{B}}(\phi(x))$. By Proposition 7.3.7, this result implies that $\mathbb{P}(\phi)$ is an isomorphism too. Then we can use the complete version of the Milnor-Moore Theorem (Theorem 7.3.26) to conclude that our morphism defines an isomorphism at the level of the complete endomorphism Hopf algebras of our object:

$$\phi : \mathtt{End}_{\mathcal{A}}(x) \xrightarrow{\simeq} \mathtt{End}_{\mathcal{A}}(\phi(x)).$$

We now consider the case of a pair of objects $x, y \in \mathtt{Ob}\,\mathcal{A}$. We use our assumption that our morphism induces a bijection on the sets of connected components to check x and y belong to the same connected component of the complete Hopf groupoid $\mathcal{A}$ if and only if their images belong to the same connected component of the complete Hopf groupoid $\mathcal{B}$. We then pick a group-like homomorphism $g \in \mathbb{G}(\mathtt{Hom}_{\mathcal{A}}(x, y))$ which connects x to y in our complete Hopf groupoid $\mathcal{A}$, and we form the commutative diagram

$$\begin{tikzcd}
\mathtt{End}_{\mathcal{A}}(x) \arrow[r, "\phi", "\simeq"'] \arrow[d, "g_*"'] & \mathtt{End}_{\mathcal{B}}(\phi(x)) \arrow[d, "\phi(g)_*"] \\
\mathtt{Hom}_{\mathcal{A}}(x, y) \arrow[r, "\phi"] & \mathtt{Hom}_{\mathcal{B}}(\phi(x), \phi(y))
\end{tikzcd} ,$$

where the vertical maps are given by the obvious composition operations, to conclude that our morphism ϕ also defines an isomorphism of counitary cocommutative coalgebras on the hom-object associated to each pair $x, y \in \mathtt{Ob}\,\mathcal{A}$. Recall simply that the composition with a group-like homomorphism in a complete Hopf groupoid defines an isomorphism of counitary cocommutative coalgebras (see §9.1.5). ☐

9.1.18. *The Malcev completion of groupoids.* We define the *Malcev completion of a groupoid* $\mathcal{G} \in \mathcal{G}rd$ by the formula $\hat{\mathcal{G}} = \mathbb{G}\,\Bbbk[\mathcal{G}]^{\wedge}$, where we consider the complete Hopf groupoid which we associate to $\mathcal{G}$ in §9.1.9. We accordingly have $\mathtt{Ob}\,\hat{\mathcal{G}} = \mathtt{Ob}\,\mathcal{G}$ by construction and $\mathtt{Mor}_{\hat{\mathcal{G}}}(x, y) = \mathbb{G}\,\Bbbk[\mathtt{Mor}_{\mathcal{G}}(x, y)]^{\wedge}$for any pair of objects $x, y \in \mathtt{Ob}\,\mathcal{G}$. This groupoid $\hat{\mathcal{G}}$ is automatically Malcev complete in our sense. We moreover have a natural morphism $\eta : \mathcal{G} \to \hat{\mathcal{G}}$ given by the unit of the adjunction of Proposition 9.1.10 and we can see that $\hat{\mathcal{G}}$ is characterized by the following universal property:

PROPOSITION 9.1.19. *Any groupoid morphism* $\phi : \mathcal{G} \to \mathcal{H}$, *where* $\mathcal{H} = \mathbb{G}(\mathcal{A})$ *is Malcev complete, admits a unique factorization*

$$\begin{tikzcd}
\mathcal{G} \arrow[rr, "\phi"] \arrow[dr] & & \mathcal{H} \\
& \hat{\mathcal{G}} \arrow[ur, dotted, "\exists!\hat{\phi}"'] &
\end{tikzcd}$$

such that $\hat{\phi}$ *is a morphism in the category of Malcev complete groupoids.*

PROOF. This proposition is an immediate consequence of the adjunction relation of Proposition 9.1.10. ☐

Recall that we use the notation $\mathtt{Aut}_{\mathcal{G}}(x) = \mathtt{Mor}_{\mathcal{G}}(x, x)$ when we consider the automorphism group of an object x in a groupoid $\mathcal{G}$. We have the following observation:

PROPOSITION 9.1.20. *We have the relation* $\mathtt{Aut}_{\hat{\mathcal{G}}}(x) = \mathtt{Aut}_{\mathcal{G}}(x)^{\wedge}$*for any object* $x \in \mathtt{Ob}\,\mathcal{G}$ *in a groupoid* $\mathcal{G}$, *where we consider the automorphism group of this object* x *in the Malcev complete groupoid* $\hat{\mathcal{G}}$ *on the left-hand side and the Malcev completion (in the sense of* §8*) of the group of automorphisms of* x *in the groupoid* $\mathcal{G}$ *on the right-hand side.*

PROOF. This statement is an immediate consequence of the result of Proposition 9.1.8 which, for the endomorphism Hopf algebra of our object $\mathtt{End}_{\Bbbk[\mathcal{G}]}(x) = \Bbbk[\mathtt{Aut}_{\mathcal{G}}(x)]$ in the Hopf groupoid $\Bbbk[\mathcal{G}]$, gives the relation $\mathtt{End}_{\Bbbk[\mathcal{G}]^{\wedge}}(x) = \Bbbk[\mathtt{Aut}_{\mathcal{G}}(x)]^{\wedge}$, where we consider the completed group algebra of the group $\mathtt{Aut}_{\mathcal{G}}(x)$ on the right-hand side. We just pass to group-like elements to get the identity of Malcev complete groups stated in the proposition. ☐

9.2. The Malcev completion of operads in groupoids

We now check that the Malcev completion process of the previous section defines a symmetric monoidal functor on the category of groupoids and, as a consequence, induces a functor from the category operads in groupoids to itself. We actually prove that the adjunction $\Bbbk[-]^{\wedge} : \mathcal{G}rd \rightleftarrows \hat{f}\,\mathcal{H}opf\,\mathcal{G}rd : \mathbb{G}$, which we use in our construction of the Malcev completion, is symmetric monoidal in the sense of §3.3.3.

In a preliminary step, we explain the definition of a symmetric monoidal structure on (complete) Hopf groupoids. The idea is to combine the (cartesian) symmetric monoidal structures of categories with the symmetric monoidal structure of (complete) counitary cocommutative coalgebras.

9.2.1. *Symmetric monoidal structures on Hopf categories and Hopf groupoids.* In §5.2.1, we equip the category of categories with the symmetric monoidal structure defined by the cartesian product of categories. In §3.0.4, we observe that the tensor product defines the cartesian product in the category of counitary cocommutative coalgebras.

To Hopf categories $\mathcal{G}$ and $\mathcal{H}$, we associate the Hopf category $\mathcal{G}\otimes\mathcal{H}$ with the cartesian product $\mathtt{Ob}(\mathcal{G}\otimes\mathcal{H}) = \mathtt{Ob}\,\mathcal{G}\times\mathtt{Ob}\,\mathcal{H}$ as object set, and the tensor products of coalgebras $\mathtt{Hom}_{\mathcal{G}\otimes\mathcal{H}}((u,x),(v,y)) = \mathtt{Hom}_{\mathcal{G}}(u,v)\otimes\mathtt{Hom}_{\mathcal{H}}(x,y)$ as hom-coalgebras. These tensor products inherit identity morphisms and composition products from the hom-coalgebras of $\mathcal{G}$ and $\mathcal{H}$ so that $\mathcal{G}\otimes\mathcal{H}$ forms a Hopf category. We moreover have natural functors $\mathcal{G}\xleftarrow{p}\mathcal{G}\otimes\mathcal{H}\xrightarrow{q}\mathcal{H}$ given by the natural projections $\mathtt{Ob}\,\mathcal{G}\xleftarrow{p}\mathtt{Ob}\,\mathcal{G}\times\mathtt{Ob}\,\mathcal{H}\xrightarrow{q}\mathtt{Ob}\,\mathcal{H}$ on object sets, and yielded by the tensor products with augmentation morphisms $\mathtt{Hom}_{\mathcal{G}}(u,v)\xleftarrow{\epsilon\otimes id}\mathtt{Hom}_{\mathcal{G}}(u,v)\otimes\mathtt{Hom}_{\mathcal{H}}(x,y)\xrightarrow{id\otimes\epsilon}\mathtt{Hom}_{\mathcal{H}}(x,y)$ on hom-coalgebras (see §3.0.4). This Hopf category $\mathcal{G}\otimes\mathcal{H}$ actually represents the cartesian product of $\mathcal{G}$ and $\mathcal{H}$ in the category of Hopf categories (this assertion follows from our analogous interpretation of the tensor product of counitary cocommutative coalgebras in §3.0.4).

We can replace the plain tensor product by the completed one in order to define an analogous tensor product construction $\mathcal{G}\hat{\otimes}\mathcal{H}$ for Hopf categories in complete filtered modules. We readily see that the Hopf category in complete filtered modules $\mathcal{G}\hat{\otimes}\mathcal{H}$ obtained by this operation represents the cartesian product of $\mathcal{G}$ and $\mathcal{H}$ in the category of Hopf categories in complete filtered modules too.

In §5.2.1, we observed that the cartesian product of groupoids $\mathcal{G}\times\mathcal{H}$, formed in the category of small categories, defines a groupoid and represents the cartesian product of $\mathcal{G}$ and $\mathcal{H}$ in the category of groupoids as well. In the context of Hopf categories, we can similarly prove that the tensor product of Hopf groupoids $\mathcal{G}\otimes\mathcal{H}$ forms a Hopf groupoid and represents the cartesian product of $\mathcal{G}$ and $\mathcal{H}$ in the category of Hopf groupoids. We have the same statement for the completed tensor product of complete Hopf groupoids.

The existence of these symmetric monoidal structures enables us to give a sense to the notion of an operad in Hopf groupoids and in complete Hopf groupoids. We also have the following result:

PROPOSITION 9.2.2.

(a) *The functors* $\Bbbk[-]^{\wedge}\colon \mathcal{G}rd\to\hat{f}\,\mathcal{H}opf\mathcal{G}rd$ *and* $\mathbb{G}:\hat{f}\,\mathcal{H}opf\mathcal{G}rd\to\mathcal{G}rd$ *are symmetric monoidal and define a symmetric adjunction between the category of groupoids and the category of complete Hopf groupoids.*

(b) *These functors can also be applied to operads aritywise in order to yield functors between the category of operads in groupoids and the category of operads in complete Hopf groupoids. Furthermore we still have an adjunction relation at the level of these categories of operads:*

$$\Bbbk[-]^{\wedge}\colon \mathcal{G}rd\,\mathcal{O}p \rightleftarrows \hat{f}\,\mathcal{H}opf\mathcal{G}rd\,\mathcal{O}p : \mathbb{G}\,.$$

Proof. The functor $\mathbb{G} : \hat{f}\,\mathcal{H}opf\,\mathcal{G}rd \to \mathcal{G}rd$, which defines a right-adjoint of $\Bbbk[-]^{\wedge} : \mathcal{G}rd \to \hat{f}\,\mathcal{H}opf\,\mathcal{G}rd$, preserves terminal objects and cartesian products and is therefore symmetric monoidal since we observed that the (complete) tensor product of (complete) Hopf groupoids represent the cartesian product in this category.

The proof that the functor $\Bbbk[-]^{\wedge} : \mathcal{G}rd \to \hat{f}\,\mathcal{H}opf\,\mathcal{G}rd$ is symmetric monoidal follows from easy verifications. For the trivial one-point set groupoid pt, we obviously have $\Bbbk[pt]^{\wedge} = \Bbbk$. For a cartesian product of groupoids $\mathcal{G} \times \mathcal{H}$, we can easily check that the filtration of §9.1.7 satisfies

$$\mathbb{I}^n\,\Bbbk[\mathtt{Mor}_{\mathcal{G}\times\mathcal{H}}((u,x),(v,y))] = \sum_{p+q=n} \mathbb{I}^p\,\Bbbk[\mathtt{Mor}_{\mathcal{G}}(u,v)] \otimes \mathbb{I}^q\,\Bbbk[\mathtt{Mor}_{\mathcal{H}}(x,y)]$$

inside the module

$$\begin{aligned}\Bbbk[\mathtt{Mor}_{\mathcal{G}\times\mathcal{H}}((u,x),(v,y))] &= \Bbbk[\mathtt{Mor}_{\mathcal{G}}(u,v) \times \mathtt{Mor}_{\mathcal{H}}(x,y)]\\ &= \Bbbk[\mathtt{Mor}_{\mathcal{G}}(u,v)] \otimes \Bbbk[\mathtt{Mor}_{\mathcal{H}}(x,y)].\end{aligned}$$

The isomorphism $\Bbbk[\mathtt{Mor}_{\mathcal{G}\times\mathcal{H}}((u,x),(v,y))] \simeq \Bbbk[\mathtt{Mor}_{\mathcal{G}}(u,v)] \otimes \Bbbk[\mathtt{Mor}_{\mathcal{H}}(x,y)]$ is therefore an identity of filtered modules which, as such, induces an isomorphism at the level of completions

$$\Bbbk[\mathtt{Mor}_{\mathcal{G}\times\mathcal{H}}((u,x),(v,y))]^{\wedge} \simeq \Bbbk[\mathtt{Mor}_{\mathcal{G}}(u,v)]^{\wedge} \hat{\otimes}\, \Bbbk[\mathtt{Mor}_{\mathcal{H}}(x,y)]^{\wedge}$$

(compare with the proof of Proposition 8.1.8, where we establish a similar result for the Malcev completion of a cartesian product of groups). This verification proves that the natural morphism $\Bbbk[\mathcal{G} \times \mathcal{H}]^{\wedge} \to \Bbbk[\mathcal{G}]^{\wedge} \hat{\otimes}\, \Bbbk[\mathcal{H}]^{\wedge}$ induced by the canonical projections $\mathcal{G} \xleftarrow{p} \mathcal{G} \times \mathcal{H} \xrightarrow{q} \mathcal{H}$ (where we use the interpretation of the complete tensor product as a categorical cartesian product) is an isomorphism. The definition of this comparison isomorphism from categorical constructions immediately implies the fulfillment of the unit, associativity and symmetry constraints of §3.3.1 (as usual). We conclude that the functor $\Bbbk[-]^{\wedge} : \mathcal{G}rd \to \hat{f}\,\mathcal{H}opf\,\mathcal{G}rd$ is symmetric monoidal as asserted.

The proof that the adjunction unit (respectively, augmentation) associated to our functors preserve symmetric monoidal structure reduces to straightforward verifications. The second assertion of the proposition is a consequence of the general observations of Proposition 3.1.1 and Proposition 3.1.3. □

9.2.3. *The category of operads in Malcev complete groupoids.* The result of Proposition 9.2.2 implies that the category of Malcev complete groupoids of §9.1.12 inherits a symmetric monoidal structure. We can therefore form operads in Malcev complete groupoids by applying our general definition of the notion of an operad in this symmetric monoidal category $\hat{f}\,\mathcal{G}rd$. We can equivalently define the category of operads in Malcev complete groupoids as the image of the category of operads in complete Hopf groupoids under the aritywise application of the group-like functor $\mathbb{G} : \hat{f}\,\mathcal{H}opf\,\mathcal{G}rd \to \hat{f}\,\mathcal{G}rd$. We use that this functor is faithful (see Proposition 9.1.11) to check this category identity, which under our notation conventions reads:

$$(\hat{f}\,\mathcal{G}rd)\,\mathcal{O}p = \mathbb{G}(\hat{f}\,\mathcal{H}opf\,\mathcal{G}rd\,\mathcal{O}p).$$

We can use the concepts of the previous section to define an analogue of the notion of a categorical equivalence of operads in groupoids in the context of operads in complete Hopf groupoids (respectively, in Malcev complete groupoids). We explicitly say that a morphism $\phi : \mathcal{G} \to \mathcal{H}$ defines a *categorical equivalence of operads in complete Hopf groupoids* (respectively, in Malcev complete groupoids) when this morphism defines an equivalence of complete Hopf groupoids (respectively, of Malcev complete groupoids) $\phi : \mathcal{G}(r) \xrightarrow{\sim} \mathcal{H}(r)$ in each arity $r \in \mathbb{N}$. We still use the notation $\phi : \mathcal{G} \xrightarrow{\sim} \mathcal{H}$ with the extra mark $\sim$ to distinguish this class of categorical equivalences in the category of operads.

We study the applications of the other constructions of §9.1 to operads. We first check the existence of operad structures on the tower decompositions of §9.1.13:

Proposition 9.2.4. *Let $\mathcal{G}$ be an operad in the category of Malcev complete groupoids. The collection $q_m \mathcal{G} = \{q_m \mathcal{G}(r), r \in \mathbb{N}\}$ where we apply the tower decomposition construction of* §9.1.13 *to the components of our operad aritywise, forms an operad in the category of Malcev complete groupoids, for each $m \geq 0$, and we have an identity $\mathcal{G} = \lim_m q_m \mathcal{G}$ in the category of operads in Malcev complete groupoids. This construction preserves non-unitary operad structures too, and we have an obvious identity $(q_m \mathcal{G})_+ = q_m(\mathcal{G}_+)$ when we consider the unitary extension of an operad.*

Proof. Let $\mathcal{H}$ be the operad in complete Hopf groupoids such that $\mathcal{G} = \mathbb{G}(\mathcal{H})$. To warm up, we can easily check that we have a well-defined operad structure on the collection of groupoids $q_m \mathcal{G}(r)$, $r \in \mathbb{N}$, which forms our object, when we forget about Malcev complete structures. But we need to check that this operad structure is defined at the level of the complete Hopf groupoids $q_m \mathcal{H}(r)$, $r \in \mathbb{N}$, which underlie our objects $q_m \mathcal{G}(r) = \mathbb{G}(q_m \mathcal{H}(r))$, and for this purpose, we have to go back to the construction of these complete Hopf groupoids in the proof of Proposition 9.1.14.

The definition of the action of symmetric groups is immediate from the functoriality of our construction. The definition of the operadic unit is obvious too. We therefore focus on the definition of the operadic composition operations. Recall that we have $\mathtt{End}_{q_m \mathcal{H}(r)}(x) = \hat{\mathbb{U}}(\mathfrak{g}_x / \mathtt{F}_{m+1} \mathfrak{g}_x)$, for any object $x \in \mathtt{Ob}\,\mathcal{H}(r)$, where we consider the complete Lie algebra such that $\mathfrak{g}_x = \mathbb{P}(\mathtt{End}_{q_m \mathcal{H}(r)}(x))$. The composition operations $\circ_i : \mathtt{End}_{\mathcal{H}(k)}(x) \hat{\otimes} \mathtt{End}_{\mathcal{H}(l)}(y) \to \mathtt{End}_{\mathcal{H}(k+l-1)}(x \circ_i y)$ of the complete endomorphism Hopf algebras of our operad in complete Hopf groupoids $\mathcal{H}$ are associated to composition operations defined on these complete Lie algebras $\circ_i : \mathfrak{g}_x \oplus \mathfrak{g}_y \to \mathfrak{g}_{x \circ_i y}$, because the category equivalence of the Milnor-Moore Theorem is symmetric monoidal (we apply the result of Proposition 7.2.23 in the category of complete filtered modules). These composition operations of complete Lie algebras preserve filtrations (by definition of the morphisms of complete Hopf algebras) and hence, carry the module $\mathtt{F}_{m+1} \mathfrak{g}_x \oplus \mathtt{F}_{m+1} \mathfrak{g}_y \subset \mathfrak{g}_x \oplus \mathfrak{g}_y$ into $\mathtt{F}_{m+1} \mathfrak{g}_{x \circ_i y} \subset \mathfrak{g}_{x \circ_i y}$. Recall that we also have $\mathtt{Hom}_{q_m \mathcal{H}(r)}(x, y) = \mathtt{Hom}_{\mathcal{H}(r)}(x, y) / \mathtt{Hom}_{\mathcal{H}(r)}(x, y) \cdot \mathtt{F}_{m+1} \mathfrak{g}_x$, for any pair of objects $x, y \in \mathtt{Ob}\,\mathcal{H}(r)$, where $\mathtt{Hom}_{\mathcal{H}(r)}(x, y) \cdot \mathtt{F}_{m+1} \mathfrak{g}_x \subset \mathtt{Hom}_{\mathcal{H}(r)}(x, y)$ denotes the image (in the complete sense) of the module $\mathtt{Hom}_{\mathcal{H}(r)}(x, y) \otimes \mathtt{F}_{m+1} \mathfrak{g}_x$ under the composition operation $\mu : \mathtt{Hom}_{\mathcal{H}(r)}(x, y) \hat{\otimes} \mathtt{End}_{\mathcal{H}(r)}(x) \to \mathtt{Hom}_{\mathcal{H}(r)}(x, y)$. The preservation of the filtration by the composition products of our Lie algebras $\circ_i : \mathfrak{g}_x \oplus \mathfrak{g}_y \to \mathfrak{g}_{x \circ_i y}$, readily implies that the composition products of the hom-coalgebras of our complete Hopf groupoids $\mathcal{H}$ preserve this filtration of our hom-coalgebras and do induce composition products on the corresponding quotient objects.

Hence, we obtain that the collection of complete Hopf groupoids $q_m\,\mathcal{H} = \{q_m\,\mathcal{H}(r), r\in\mathbb{N}\}$ inherits a full operad structure from our complete Hopf groupoid $\mathcal{H}$ such that we have the identity $q_m\,\mathcal{G} = \mathbb{G}(q_m\,\mathcal{H})$ in the category of operads in groupoids. ☐

We moreover have the following operadic counterpart of the claim of Proposition 9.1.15:

Proposition 9.2.5. *Any morphism $\phi : \mathcal{G}\to q_m\,\mathcal{H}$, where $\mathcal{G}$ and $\mathcal{H}$ are operads in Malcev complete groupoids, admits a unique factorization:*

$$\begin{tikzcd}
\mathcal{G} \arrow[rr, "\phi"] \arrow[dr] & & q_m\,\mathcal{H} \ , \\
& q_m\,\mathcal{G} \arrow[ur, dotted, "\exists!\bar{\phi}"'] &
\end{tikzcd}$$

where $\bar{\phi}$ is a morphism of operads in Malcev complete groupoids.

Proof. We use the same argument as in the proof of Proposition 9.1.15. We just observe that the morphism of operads in Malcev complete groupoids $p : \mathcal{H}\to q_n\,\mathcal{H}$ induces an isomorphism when we apply our decomposition twice $p_* : q_m\,\mathcal{H}\to q_m(q_n\,\mathcal{H})$, for all levels $m\leq n$, since this is so aritywise in the category of Malcev complete groupoids (see Proposition 9.1.14). ☐

9.2.6. *Local coefficient system operads.* We now study the structures which we associate to the local coefficient systems $\mathtt{E}^0_m\,\mathcal{G}(r) = \{\mathtt{E}^0_m\,\mathtt{Aut}_{\mathcal{G}}(x), x\in\mathtt{Ob}\,\mathcal{G}(r)\}$ which define the structure groups of the principal fibers of the morphisms $p_m : q_m\,\mathcal{G}(r)\to q_{m-1}\,\mathcal{G}(r)$ in the tower decomposition of the Malcev complete groupoids $\mathcal{G}(r) = \lim_m q_m\,\mathcal{G}(r)$ when $\mathcal{G} = \{\mathcal{G}(r), r\in\mathbb{N}\}$ is an operad. We already get, by the general observations of §9.1.16, that any morphism $f\in\mathtt{Mor}_{q_0\,\mathcal{G}(r)}(x,y)$ in the groupoid $q_0\,\mathcal{G}(r)$ determines a conjugation operation:

$$c_f : \mathtt{E}^0_m\,\mathtt{Aut}_{\mathcal{G}(r)}(x)\to\mathtt{E}^0_m\,\mathtt{Aut}_{\mathcal{G}(r)}(y), \tag{1}$$

and the local coefficient system $\mathtt{E}^0_m\,\mathcal{G}(r)$ therefore forms a diagram over the groupoid $q_0\,\mathcal{G}(r)$, for each $r\in\mathbb{N}$.

By functoriality of our general construction in §9.1.16, the morphisms of Malcev complete groupoids $s_* : \mathcal{G}(r)\to\mathcal{G}(r)$, which define the action of permutations $s\in\Sigma_r$ on our operad, induce morphisms of $\mathbb{k}$-modules:

$$s_* : \mathtt{E}^0_m\,\mathtt{Aut}_{\mathcal{G}(r)}(x)\to\mathtt{E}^0_m\,\mathtt{Aut}_{\mathcal{G}(r)}(sx), \tag{2}$$

for each $x\in\mathtt{Ob}\,\mathcal{G}(r)$, where we consider the module $\mathtt{E}^0_m\,\mathtt{Aut}_{\mathcal{G}(r)}(sx)$ associated to the image of our object x under the map $s_* : \mathtt{Ob}\,\mathcal{G}(r)\to\mathtt{Ob}\,\mathcal{G}(r)$ on the target. These morphisms moreover preserve the conjugation operations (1) on our objects. To be explicit, for each $f\in\mathtt{Mor}_{q_0\,\mathcal{G}(r)}(x,y)$, we have a commutative diagram:

$$\begin{tikzcd}
\mathtt{E}^0_m\,\mathtt{Aut}_{\mathcal{G}(r)}(x) \arrow[r, "s_*"] \arrow[d, "c_f"'] & \mathtt{E}^0_m\,\mathtt{Aut}_{\mathcal{G}(r)}(sx) \ , \arrow[d, "c_{sf}"] \\
\mathtt{E}^0_m\,\mathtt{Aut}_{\mathcal{G}(r)}(y) \arrow[r, "s_*"] & \mathtt{E}^0_m\,\mathtt{Aut}_{\mathcal{G}(r)}(sy)
\end{tikzcd} \tag{3}$$

where, on the right-hand side, we consider the conjugation operation c_{sf} associated to the image of our morphism sf under the action of the permutation s on the groupoid $q_0\,\mathcal{G}(r)$. In what follows, we also use the notation $s_* : \gamma \mapsto s\gamma$ for the map defined by this operation on our subquotient modules (2).

The morphisms $\circ_i : \mathcal{G}(k) \times \mathcal{G}(l) \to \mathcal{G}(k+l-1)$, $i = 1, \dots, k$, which define the composition operations of our operad similarly induce morphisms of $\Bbbk$-modules:

$$\circ_i : \mathtt{E}^0_m \,\mathtt{Aut}_{\mathcal{G}(k)}(x) \oplus \mathtt{E}^0_m \,\mathtt{Aut}_{\mathcal{G}(l)}(y) \to \mathtt{E}^0_m \,\mathtt{Aut}_{\mathcal{G}(k+l-1)}(x \circ_i y), \tag{4}$$

for each pair of objects $x \in \mathtt{Ob}\,\mathcal{G}(k)$, $y \in \mathtt{Ob}\,\mathcal{G}(k)$, where we consider the module $\mathtt{E}^0_m \,\mathtt{Aut}_{\mathcal{G}(k+l-1)}(x \circ_i y)$ associated to the image of our objects (x, y) under the map $\circ_i : \mathtt{Ob}\,\mathcal{G}(k) \times \mathtt{Ob}\,\mathcal{G}(l) \to \mathtt{Ob}\,\mathcal{G}(k+l-1)$. To check this assertion, we also use that our subquotient construction satisfies the relation $\mathtt{E}^0_m(G \times H) \xrightarrow{\simeq} \mathtt{E}^0_m(G) \oplus \mathtt{E}^0_m(H)$, for any pair of Malcev complete groups $G, H \in \hat{f}\,\mathcal{G}rp$, and defines a symmetric monoidal functor in this sense. The verification of this assertion is an easy exercise. These morphisms preserve the conjugation operations (1) too. To be explicit, for each pair of morphisms $f \in \mathtt{Mor}_{q_0\,\mathcal{G}(k)}(a, x)$, $g \in \mathtt{Mor}_{q_0\,\mathcal{G}(l)}(b, y)$, we have a commutative diagram:

$$\begin{tikzcd}
\mathtt{E}^0_m \,\mathtt{Aut}_{\mathcal{G}(k)}(a) \oplus \mathtt{E}^0_m \,\mathtt{Aut}_{\mathcal{G}(l)}(b) \arrow[r, "\circ_i"] \arrow[d, "(c_f, c_g)"'] & \mathtt{E}^0_m \,\mathtt{Aut}_{\mathcal{G}(k+l-1)}(a \circ_i b) \arrow[d, "c_{f \circ_i g}"] \\
\mathtt{E}^0_m \,\mathtt{Aut}_{\mathcal{G}(k)}(x) \oplus \mathtt{E}^0_m \,\mathtt{Aut}_{\mathcal{G}(l)}(y) \arrow[r, "\circ_i"] & \mathtt{E}^0_m \,\mathtt{Aut}_{\mathcal{G}(k+l-1)}(x \circ_i y)
\end{tikzcd} , \tag{5}$$

where, on the right-hand side, we consider the conjugation operation $c_{f \circ_i g}$ associated to the image of our morphisms (f, g) under the composition operation $\circ_i : q_0\,\mathcal{G}(k) \times q_0\,\mathcal{G}(l) \to q_0\,\mathcal{G}(k+l-1)$.

These operations (2-4) satisfy an obvious generalization of the equivariance, unit and associativity axioms of operads. We use the phrase '*local coefficient system operad*' to refer to this form of structure which we get on the collection $\mathtt{E}^0_m \,\mathtt{Aut}_{\mathcal{G}(r)}(x)$, $x \in \mathtt{Ob}\,\mathcal{G}(r)$, $r \in \mathbb{N}$. We should mention, to be precise, that the operadic unit of our object is represented by the null morphism $\eta : 0 \to \mathtt{E}^0_m \,\mathtt{Aut}_{\mathcal{G}(1)}(1)$, which corresponds to the unit morphism of our operad in Malcev complete groupoids $\eta : pt \to \mathcal{G}(1)$, and where we consider the module $\mathtt{E}^0_m \,\mathtt{Aut}_{\mathcal{G}(1)}(1)$ associated to the unit of our object-set operad $1 \in \mathtt{Ob}\,\mathcal{G}(1)$.

By the construction of §9.1.16, the morphism set $\mathtt{Mor}_{q_m\,\mathcal{G}(r)}(x, y)$ in the category $q_m\,\mathcal{G}(r)$ inherits an action of the module $\mathtt{E}^0_m \,\mathtt{Aut}_{\mathcal{G}(r)}(x)$, for each pair of objects $x, y \in \mathtt{Ob}\,\mathcal{G}(r)$. Furthermore, elements of this morphism set $f, g \in \mathtt{Mor}_{q_m\,\mathcal{G}(r)}(x, y)$ become equal in $q_{m-1}\,\mathcal{G}(r)$ if and only if they differ by the action $g = f \cdot \gamma$ of the class of a morphism in this subquotient $\gamma \in \mathtt{E}^0_m \,\mathtt{Aut}_{\mathcal{G}(r)}(x)$. We readily see that this action is preserved by the structure operations of our operads. We explicitly have $s \cdot (f \cdot \gamma) = (sf) \cdot (s\gamma)$, when we consider the action of a permutation and $(f \cdot \alpha) \circ_i (g \cdot \beta) = (f \circ_i g) \cdot (\alpha \circ_i \beta)$ for the composition products.

We can obviously adapt the constructions of this paragraph in the context of non-unitary operads. We just drop arity zero components in this case. We also see that, for the unitary extension of an operad $\mathcal{G}_+$, our subquotient construction satisfies $\mathtt{E}^0_m(\mathcal{G}_+) = (\mathtt{E}^0_m\,\mathcal{G})_+$, where $(\mathtt{E}^0_m\,\mathcal{G})_+$ is the object which we obtain by keeping track of an extra arity zero component $(\mathtt{E}^0_m\,\mathcal{G})_+(0) = 0$ in the definition of our structure. We then have composition operations (4) with a second term of arity $l = 0$, and which can be non-trivial though this object is null $(\mathtt{E}^0_m\,\mathcal{G})_+(0) = 0$.

In the second volume, we use the tower decompositions of operads in Malcev complete groupoids to compute the homotopy of mapping spaces of operads in simplicial sets. For the moment, we only use the elementary concept of a unitary extension when we deal with unitary operads, but in these subsequent applications we will rather use the concept of a Λ-operad (see §2.2) to encode unitary operad structures.

We now study the analogue of the Malcev completion process for operads in groupoids. We use the result of Proposition 9.2.2 to get the following statement:

PROPOSITION 9.2.7. *The Malcev completion functor on groupoids* $(-)^{\wedge}\colon \mathcal{G}rd \to \hat{f}\,\mathcal{G}rd$ *is symmetric monoidal (as a composite of symmetric monoidal functors) and can be applied aritywise to operads in groupoids in order to yield a Malcev completion functor on operads* $(-)^{\wedge}\colon \mathcal{G}rd\,\mathcal{O}p \to \hat{f}\,\mathcal{G}rd\,\mathcal{O}p$.

EXPLANATIONS. To recap the construction of this proposition, we define the *Malcev completion of an operad in groupoids* $P \in \mathcal{G}rd\,\mathcal{O}p$ as the operad $\hat{P}$ formed by the collection $\hat{P}(r) = P(r)^{\wedge}$, where we consider the Malcev completion of each groupoid $P(r)$, $r \in \mathbb{N}$. We also have $P = \mathbb{G}\,\Bbbk[P]^{\wedge}$, where $\Bbbk[P]^{\wedge}$ is the operad in complete Hopf groupoids defined by the completion of the Hopf groupoid $\Bbbk[P(r)]$ associated to each $P(r) \in \mathcal{G}rd$ and $\mathbb{G}(-)$ refers to the aritywise application of the group-like element functor on complete Hopf groupoids. Recall that the functors $\Bbbk[-]^{\wedge}\colon \mathcal{G}rd\,\mathcal{O}p \to \hat{f}\,\mathcal{H}opf\mathcal{G}rd\,\mathcal{O}p$ and $\mathbb{G} : \hat{f}\,\mathcal{H}opf\mathcal{G}rd\,\mathcal{O}p \to \mathcal{G}rd\,\mathcal{O}p$ automatically preserve unitary extensions of operads (see Proposition 3.1.1), and as a byproduct, so does the composite functor $(-)^{\wedge} = \mathbb{G}\,\Bbbk[-]^{\wedge}$. In the notation of §2.2, we have the identity $(P_{+})^{\wedge} = (\hat{P})_{+}$ for any unitary operad in groupoids P_{+}. ☐

We also have the following operadic analogue of the result of Proposition 9.1.19:

PROPOSITION 9.2.8. *Any morphism of operads in groupoids* $\phi : \mathcal{G} \to \mathcal{H}$, *where* $\mathcal{H} = \mathbb{G}(\mathcal{A})$ *admits the structure of an operad in Malcev complete groupoids, has a unique factorization*

$$\begin{tikzcd}
\mathcal{G} \arrow[rr, "\phi"] \arrow[dr] & & \mathcal{H} \\
& \hat{\mathcal{G}} \arrow[ur, dotted, "\exists!\hat{\phi}"'] &
\end{tikzcd}$$

such that $\hat{\phi}$ *is a morphism of operads in the category of Malcev complete groupoids.*

PROOF. This proposition is again an immediate consequence of our adjunction relations (see Proposition 9.2.2). ☐

9.3. Appendix: The local connectedness of complete Hopf groupoids

We check in this appendix section that group-like elements exist in any (nontrivial) hom-coalgebra $C = \mathtt{Hom}_{\mathcal{H}}(x,y)$ of a complete Hopf groupoid $\mathcal{H}$ as soon as we can ensure the existence of such group-like elements in an extension of scalars of this hom-coalgebra $C_{\mathbb{l}} = \mathtt{Hom}_{\mathcal{H}}(x,y)_{\mathbb{l}}$.

We have not explained the definition of the extension of scalars for a counitary cocommutative coalgebra in complete filtered modules yet. We use subscripts in order to mark the changes of ground fields which we consider in this extension process. We notably use the notation $\mathcal{M}od_{\Bbbk}$ to distinguish our base module category $\mathcal{M}od = \mathcal{M}od_{\Bbbk}$ from for the category of modules $\mathcal{M}od_{\mathbb{l}}$ defined over the extension $\mathbb{l}$ of our base field $\Bbbk$. We similarly use a subscript in the notation of tensor products in order to specify the field which we consider when we form our operations. We define the extension of scalars of a complete filtered module M by the limit $M_{\mathbb{l}} = \lim_n (M/\mathtt{F}_{n+1} M) \otimes_{\mathbb{k}} \mathbb{l}$, where we take the ordinary extension of scalars $(M/\mathtt{F}_{n+1} M)_{\mathbb{l}} = (M/\mathtt{F}_{n+1} M) \otimes_{\mathbb{k}} \mathbb{l}$ of the tower of modules $M/\mathtt{F}_{n+1} M \in \mathcal{M}od_{\mathbb{k}}$ which we associate to our object $M \in \hat{f}\,\mathcal{M}od_{\mathbb{k}}$. We clearly have a natural isomorphism $(M \hat{\otimes}_{\mathbb{k}} N)_{\mathbb{l}} \xrightarrow{\simeq} M_{\mathbb{l}} \hat{\otimes}_{\mathbb{l}} N_{\mathbb{l}}$ in the category $\hat{f}\,\mathcal{M}od_{\mathbb{l}}$ when we take the extension of scalars of the completed tensor product of objects $M, N \in \hat{f}\,\mathcal{M}od_{\mathbb{k}}$. We deduce from this observation that the extension of scalars $C_{\mathbb{l}}$ of a counitary cocommutative coalgebra $C \in \hat{f}\,\mathcal{C}om_{+}^{c}$ forms a counitary cocommutative coalgebra in the category of complete filtered modules over $\mathbb{l}$.

For short, we also use the phrase '*complete counitary cocommutative coalgebra*' to refer to counitary cocommutative coalgebras in the category of complete filtered modules in what follows.

We can now make explicit our statement concerning the existence of group-like elements in the hom-coalgebras of complete Hopf groupoids:

Proposition 9.3.1. *Let $\mathcal{H}$ be a Hopf groupoid in the category of complete filtered modules such that $\mathtt{Hom}_{\mathcal{H}}(x,y) \neq 0 \Rightarrow \mathtt{E}_0^0 \mathtt{Hom}_{\mathcal{H}}(x,y) = \mathbb{k}$ for all $x, y \in \mathtt{Ob}\,\mathcal{H}$ (see §9.1.2).*

Let $x, y \in \mathtt{Ob}\,\mathcal{H}$. If the hom-coalgebra $C = \mathtt{Hom}_{\mathcal{H}}(x,y)$ admits an extension of scalars $C_{\mathbb{l}} = \mathtt{Hom}_{\mathcal{H}}(x,y)_{\mathbb{l}}$ such that $\mathbb{G}(\mathtt{Hom}_{\mathcal{H}}(x,y)_{\mathbb{l}}) \neq \varnothing$, then this hom-coalgebra $C = \mathtt{Hom}_{\mathcal{H}}(x,y)$ automatically contains a group-like element $g \in \mathbb{G}(\mathtt{Hom}_{\mathcal{H}}(x,y))$ defined over our ground field $\mathbb{k}$ (and not only over the extension $\mathbb{l}$).

Thus, if we can ensure that the assumption of this proposition $\mathbb{G}(\mathtt{Hom}_{\mathcal{H}}(x,y)_{\mathbb{l}}) \neq \varnothing$ is valid for all hom-coalgebras such that $\mathtt{Hom}_{\mathcal{H}}(x,y) \neq 0$, then our Hopf groupoid $\mathcal{H}$ fully satisfies the connectedness requirements of §9.1.2, and hence forms a complete Hopf groupoid in our sense. The result of this proposition notably applies to the completion $\hat{\mathcal{H}}$ of a Hopf groupoid $\mathcal{H}$ and implies that this object $\hat{\mathcal{H}}$ does fulfill our local connectedness condition in the definition of a complete Hopf groupoid (see §9.1.7).

Proof. We use that the set of group-like elements in any complete counitary cocommutative coalgebra C has a decomposition $\mathbb{G}(C) = \lim_m \mathbb{G}_{\langle m\rangle}(C)$, where we set

$$\mathbb{G}_{\langle m\rangle}(C) = \{\bar{g} \in C/\mathtt{F}_{m+1} C \mid \epsilon(g) = 1, \Delta(g) \equiv g \hat{\otimes} g (\operatorname{mod} \mathtt{F}_{m+1}(C \hat{\otimes} C))\}$$

as in §8.2.7.

We fix a pair of objects $x, y \in \mathtt{Ob}\,\mathcal{H}$. We set $H = \mathtt{End}_{\mathcal{H}}(x)$ and $C = \mathtt{Hom}_{\mathcal{H}}(x,y)$. Each set $\mathbb{G}_{\langle m\rangle}(H)$ inherits a group structure, as we already observed in §8.2.7. Furthermore, we easily check that the product $\mu : H \otimes C \to C$, which we determine by the composition operation of our Hopf groupoid, induces a free and transitive action of this group $\mathbb{G}_{\langle m\rangle}(H)$ on the set $\mathbb{G}_{\langle m\rangle}(C)$ as soon as we have $\mathbb{G}_{\langle m\rangle}(C) \neq \varnothing$, for any $m \geq 0$. The identity $\mathbb{G}_{\langle m\rangle}(H) = \mathbb{G}(H)/\mathtt{F}_{m+1}\mathbb{G}(H)$, which we establish in §8.2.7, implies that the morphisms $p_m : \mathbb{G}_{\langle m\rangle}(H) \to \mathbb{G}_{\langle m-1\rangle}(H)$ in our decomposition of the group $\mathbb{G}(H)$ are surjective. We use the existence of a group-like element g_0 in a scalar extension $C_{\mathbb{l}}$ of our complete coalgebra C to prove that a similar result holds for the tower decomposition of the set of group-like elements in C.

We fix $\bar{g} \in \mathbb{G}_{\langle m-1\rangle}(C)$ for some $m \geq 1$. We have $\bar{g} = \bar{g}_0 \cdot \bar{h}$ for some class $\bar{h} \in \mathbb{G}_{\langle m-1\rangle}(H_{\mathbb{l}})$, where we also consider the scalar extension $H_{\mathbb{l}}$ of the complete

Hopf algebra H. We pick a class $\bar{h}' \in \mathbb{G}_{\langle m\rangle}(H_{\mathbb{l}})$ such that $h' \equiv h (\operatorname{mod} \mathtt{F}_m H)$ by using the surjectivity of the group morphism $p_m : \mathbb{G}_{\langle m\rangle}(H_{\mathbb{l}}) \to \mathbb{G}_{\langle m-1\rangle}(H_{\mathbb{l}})$, and we set $g' = g_0 h'$ to establish the existence of an element $\bar{g}' \in \mathbb{G}_{\langle m-1\rangle}(C_{\mathbb{l}})$, defined over $\mathbb{l}$, which satisfies the equation of a group-like element $\Delta(g') \equiv g' \hat{\otimes}_{\mathbb{l}} g'$ in the quotient module $(C_{\mathbb{l}} \hat{\otimes}_{\mathbb{l}} C_{\mathbb{l}}) / \mathtt{F}_{m+1}(C_{\mathbb{l}} \hat{\otimes}_{\mathbb{l}} C_{\mathbb{l}}) = (C \hat{\otimes} C / \mathtt{F}_{m+1}(C \hat{\otimes} C)) \otimes_{\mathbb{k}} \mathbb{l}$ and the congruence relation $g' \equiv g$ in $C_{\mathbb{l}} / \mathtt{F}_m C_{\mathbb{l}} = (C / \mathtt{F}_m C) \otimes_{\mathbb{k}} \mathbb{l}$.

If we now consider a general element of the form $\bar{g}' = \bar{g} + \bar{u}$ in the quotient module $C / \mathtt{F}_{m+1} C$, for some class $\bar{u} \in \mathtt{F}_m C / \mathtt{F}_{m+1} C$ so that we have the relation $g' \equiv g$ in $C / \mathtt{F}_m C$, then we see that the equation $\Delta(g') \equiv g' \hat{\otimes} g'$ in $C \hat{\otimes} C / \mathtt{F}_{m+1}(C \hat{\otimes} C)$ is equivalent to the affine equation $\Delta(g) + \Delta(u) \equiv (u \hat{\otimes} g + g \hat{\otimes} u) + (g \hat{\otimes} g)$ since $u \in \mathtt{F}_m C \Rightarrow u \hat{\otimes} u \equiv 0 (\operatorname{mod} \mathtt{F}_{m+1}(C \hat{\otimes} C))$ when $m \geq 1$. The existence of a solution of this equation over $\mathbb{l}$ guarantees that the solution exists over $\mathbb{k}$, and hence that our map $p_m : \mathbb{G}_{\langle m\rangle}(C) \to \mathbb{G}_{\langle m-1\rangle}(C)$ is surjective over the ground field yet.

We can now start with the relation $C / \mathtt{F}_1 C = \mathtt{E}_1^0 C = \mathbb{k} \Rightarrow \mathbb{G}_{\langle 0\rangle}(C) = pt$ for our hom-coalgebra $C = \mathtt{Hom}_{\mathcal{H}}(x, y)$ to produce a sequence of elements $g_m \in \mathbb{G}_{\langle m\rangle}(C)$ such that $p_m(g_m) = g_{m-1}$ for each level $m \geq 1$. We eventually get a group-like element $g \in \mathbb{G}(C)$ when we pass to the limit $\mathbb{G}(C) = \lim_m \mathbb{G}_{\langle m\rangle}(C)$, and this construction completes the proof of our proposition. ☐

Part I(d)

The Operadic Definition of the Grothendieck–Teichmüller Group

CHAPTER 10

The Malcev Completion of the Braid Operads and Drinfeld's Associators

We can apply our Malcev completion process to the operad of parenthesized braids PaB and to the operad of colored braids CoB of §§5-6. We then get operads in Malcev complete groupoids $\mathit{PaB}^{\wedge}$ and $\mathit{CoB}^{\wedge}$ whose morphism sets are given by the Malcev completion of the morphism sets of these operads PaB and CoB. We devote this chapter to the study of these operads $\mathit{PaB}^{\wedge}$ and $\mathit{CoB}^{\wedge}$.

Recall that the operad of parenthesized braids, defined in §6.2, is an operad in groupoids PaB which has the magma operad $\mathit{\Omega}$ as object set operad $\mathtt{Ob}\,\mathit{PaB} = \mathit{\Omega}$, and where the morphisms $\alpha \in \mathtt{Mor}_{\mathit{PaB}(r)}(p,q)$ consist of braids $\alpha \in B_r$ with contact points centered on a diadic decomposition of the horizontal axis. The colored braid operad CoB has the permutation operad $\mathit{\Pi}$, which governs the category of associative monoids in the category of sets, as underlying object set operad $\mathtt{Ob}\,\mathit{CoB} = \mathit{\Pi}$, and has the same morphism sets as the parenthesized braid operad. We just consider braids with equidistant contact points in this case, and we forget about the diadic decomposition which we consider in the case of parenthesized braid operad, because we only use this decomposition to reflect the composition structure of the underlying object set operad of parenthesized braids.

We have a morphism $\omega : \mathit{PaB} \to \mathit{CoB}$ given by the obvious morphism $\omega : \mathit{\Omega} \to \mathit{\Pi}$ on the underlying object set operads of our operads in groupoids $\mathit{PaB}, \mathit{CoB} \in \mathcal{G}rd\,\mathcal{O}p$, and by the identities $\mathtt{Mor}_{\mathit{PaB}(r)}(p,q) = \mathtt{Mor}_{\mathit{CoB}(r)}(\omega(p),\omega(q))$ at the morphism set level, where we assume $p,q \in \mathtt{Ob}\,\mathit{PaB}(r)$, $r > 0$. This morphism trivially defines an equivalence of categories aritywise. We obviously have the same relationship when we pass to the Malcev completion and we consider the morphisms of operads in Malcev complete groupoids $\omega : \mathit{PaB}^{\wedge} \to \mathit{CoB}^{\wedge}$ induced by our morphism $\omega : \mathit{PaB} \to \mathit{CoB}$. This morphism is still given by the morphism of set operads $\omega : \mathit{\Omega} \to \mathit{\Pi}$ at the object set level and reduces to the identities $\mathtt{Mor}_{\mathit{PaB}(r)^{\wedge}}(p,q) = \mathtt{Mor}_{\mathit{CoB}(r)^{\wedge}}(\omega(p),\omega(q))$ at the morphism set level, for $p,q \in \mathit{\Omega}(r)$.

The group of automorphisms $\mathtt{Aut}_{B(r)}(p)$ of an object $p \in \mathtt{Ob}\,B(r)$ in any of the operads $B = \mathit{CoB}, \mathit{PaB}$ is identified with the pure braid group on r strands P_r, where r is the arity. We therefore have the identity $\mathtt{Aut}_{B(r)^{\wedge}}(p) = \hat{P}_r$, where $\hat{P}_r$ denotes the Malcev completion of the pure braid group $\hat{P}_r$, when we pass to the operad in Malcev complete groupoids $B^{\wedge} = \mathit{CoB}^{\wedge}, \mathit{PaB}^{\wedge}$.

We work with a fixed coefficient field of characteristic zero $\Bbbk$ all through this chapter. We devote a preliminary section §10.0 to the study of the Malcev completion of the pure braid groups $\hat{P}_r$. We make explicit weight graded Lie algebras, the *Drinfeld–Kohno Lie algebras* $\mathfrak{p}(r)$, such that we have an identity $\mathtt{E}^0\,\hat{P}_r = \mathfrak{p}(r)$, where we consider the weight graded Lie algebra naturally associated the Malcev complete group $\hat{P}_r$. We also have an obvious complete analogue of the Drinfeld–Kohno Lie algebras $\hat{\mathfrak{p}}(r)$ such that we have the relation $\mathtt{E}^0\,\hat{p}(r) = \mathfrak{p}(r)$. We are going to see that we actually have an isomorphism of Malcev complete groups $\hat{P}_r \simeq \mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r))$, between the Malcev completion of the pure braid group $\hat{P}_r$ and the group of group-like elements in the complete enveloping algebra of these complete Lie algebras $\hat{\mathfrak{p}}(r)$.

We study the Malcev completion of the operads $\mathsf{B} = \mathsf{PaB}, \mathsf{CoB}$ afterwards. We check that the Drinfeld–Kohno Lie algebras $\mathfrak{p}(r)$ form an operad in the category of weight graded Lie algebras. The components of homogeneous weight $m \geq 1$ of these weight graded Lie algebras $\mathfrak{p}(r)_m$ inherit an additive operad structure in the category of $\mathbb{k}$-modules. We are going to see that this additive operad determine the principal fibers of the natural tower decomposition $\mathsf{B}^{\widehat{\ }} = \lim_m q_m\, \mathsf{B}^{\widehat{\ }}$ of the operads in Malcev complete groupoids $\mathsf{B}^{\widehat{\ }} = \mathsf{PaB}^{\widehat{\ }}, \mathsf{CoB}^{\widehat{\ }}$. We retrieve the identities $\mathtt{E}^0_m\,\hat{P}_r = \mathfrak{p}(r)_m$ established in the preliminary section when we apply this relationship to the automorphism sets of our operads. We explain this construction in §10.1.

We can obviously extend the operad structure of the Drinfeld–Kohno Lie algebras $\mathfrak{p}(r)$ to the completion of these Lie algebras $\hat{\mathfrak{p}}(r)$. The groups of group-like elements $\mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r))$, which we associate to the enveloping algebras of these complete Lie algebras, inherit an operad structure as well. We refer to this operad, which we identify with an operad in Malcev complete groupoids with a single object, as the *operad of chord diagrams*. We also use the notation $\mathsf{CD}^{\widehat{\ }}$ for this object. This name 'chord diagram' comes from the applications of the enveloping algebras $\hat{\mathbb{U}}\hat{\mathfrak{p}}(r)$ in the theory of Vassiliev invariants.

We check that we can define a morphism of operads $\phi : \mathsf{PaB}^{\widehat{\ }} \to \mathsf{CD}^{\widehat{\ }}$ which gives the identity of the preliminary section of the chapter $\hat{P}_r = \mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r))$ when we consider the automorphism set group of an object in these operads. We use the interpretation of the operad of parenthesized braids as the operad governing braided monoidal categories to reduce the definition of such a morphism $\phi : \mathsf{PaB}^{\widehat{\ }} \to \mathsf{CD}^{\widehat{\ }}$ to the definition of a braiding and of an associativity isomorphism in the operad $\mathsf{CD}^{\widehat{\ }}$. We are actually going to see that we retrieve the notion of associator introduced by Drinfeld in quantum group theory when we perform this construction. We can therefore rely on the existence of Drinfeld's associators to check the existence of a morphism which meets our requirements $\phi : \mathsf{PaB}^{\widehat{\ }} \to \mathsf{CD}^{\widehat{\ }}$. We review some significant constructions, which are used to give a proof of this existence result, in §§10.2-10.4. By the way, we explain the definition of the graded Grothendieck–Teichmüller group, which acts on the sets of associators and which is used as an auxiliary device to handle the solutions of our associator construction problem.

Recall that we use the notation $\mathsf{B} = \mathsf{PaB}, \mathsf{CoB}$ for non-unitary operads (which have no term in arity 0). In what follows, we also consider a unitary extension of these operads $\mathsf{B}_+ = \mathsf{PaB}_+, \mathsf{CoB}_+$, which are given by $\mathsf{B}_+(0) = pt$ and $\mathsf{B}_+(r) = \mathsf{B}(r)$ for $r > 0$. The operad of chord diagram $\mathsf{CD}^{\widehat{\ }}$ also admits a unitary extension $\mathsf{CD}^{\widehat{\ }}_+$ and the results, which we mention in this introduction, have an extension in the unitary context. In fact, we have to deal with the unitary extension of our operads when we study the correspondence between operad morphisms $\phi : \mathsf{PaB}^{\widehat{\ }} \to \mathsf{CD}^{\widehat{\ }}$ and Drinfeld's associators.

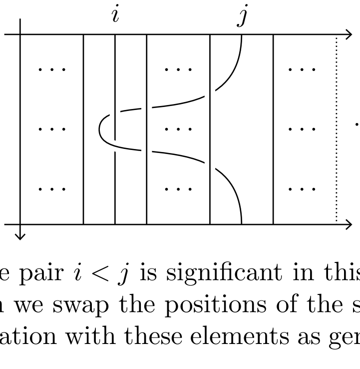
Recall that we introduce the notion of an (augmented) Λ-operad to encode the composition structure of a unitary operad P_+ from the underlying non-unitary operad P (see §2.2). We mainly use this concept in the next volume, when we apply methods of homotopy theory to operads. We express our constructions in the language of unitary operads and unitary extensions for the moment. We just consider the restriction operators $\partial_k : \mathsf{P}(r) \to \mathsf{P}(r-1)$, $k = 1, \dots, r$, which are part of the structure of an (augmented) Λ-operad, and which encode the composition operations $\partial_k(p) = p \circ_k *$ of any element $p \in \mathsf{P}(r)$ with the extra arity zero operation $* \in \mathsf{P}_+(0)$ in the unitary extension of our operad P_+. The composition structure of the unitary extension P_+ is fully determined by these restriction operators $\partial_k : \mathsf{P}(r) \to \mathsf{P}(r-1)$, $k = 1, \dots, r$, and by augmentation morphisms $\epsilon : \mathsf{P}(r) \to \mathbb{1}$ with values in the unit object of our symmetric monoidal category $\mathbb{1} \in \mathcal{M}$. In the context of (Malcev complete) groupoids, we have $\mathbb{1} = pt$, and we therefore forget about these augmentations which are trivially given by the constant map with values in the terminal object of this base category $\mathcal{M} = \mathcal{G}rd$.

10.0. The Malcev completion of the pure braid groups and the Drinfeld–Kohno Lie algebras

Recall that the pure braid group on r strands P_r is the kernel of the natural map $p_* : B_r \to \Sigma_r$ from the Artin braid group B_r towards the symmetric group Σ_r (see §5.0). We devote this section to the study of the Malcev completion of the pure braid groups P_r. We survey the definition of a presentation of the pure braid group by generators and relations first. We then make explicit the weight graded Lie algebras $\mathfrak{p}(r)$ such that $\mathfrak{p}(r) = \mathtt{E}^0 \hat{P}_r$, where we consider the weight graded Lie algebra of the Malcev complete group associated to P_r. We aim to check that we have an isomorphism of Malcev complete groups $\hat{P}_r \simeq \mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r))$, where we consider a completion $\hat{\mathfrak{p}}(r)$ of this weight graded Lie algebra $\mathfrak{p}(r)$ and the group of group-like elements in the complete enveloping algebra $\hat{\mathbb{U}}\hat{\mathfrak{p}}(r)$ (as we explain in the introduction of the chapter).

The Lie algebra $\mathfrak{p}(r)$ is usually called the *Drinfeld–Kohno Lie algebra*. The name '*Lie algebra of infinitesimal braids*' is also used for these Lie algebras in the literature. In what follows, we still refer to the complete Hopf algebra $\hat{\mathbb{U}}\hat{\mathfrak{p}}(r)$ as the *algebra of chord diagrams on r strands*. This terminology is motivated by a correspondence between the elements of this enveloping algebra and certain chord diagrams that occur in the definition of universal Vassiliev invariants.

10.0.1. *The presentation and the semi-direct product decomposition of the pure braid groups.* For each pair $\{i < j\} \subset \{1 < \cdots < r\}$, we consider the element $x_{ij} \in P_r$ such that:

(1) $x_{ij} =$ 

.

Note that the ordering of the pair $i < j$ is significant in this definition since we do not get the same braid when we swap the positions of the strands (i, j). The pure braid group P_r has a presentation with these elements as generators, and where the generating relations read:

$$(2) \quad x_{kl}x_{ij}x_{kl}^{-1} = \begin{cases} x_{ij}, & \text{for } k < l < i < j \text{ or } i < k < l < j, \\ x_{kj}^{-1}x_{ij}x_{kj}, & \text{for } k < l = i < j, \\ (x_{lj}x_{kj})^{-1}x_{ij}(x_{lj}x_{kj}), & \text{for } k = i < l < j, \\ (x_{lj}^{-1}x_{kj}^{-1}x_{lj}x_{kj})^{-1}x_{ij}(x_{lj}^{-1}x_{kj}^{-1}x_{lj}x_{kj}), & \text{for } k < i < l < j. \end{cases}$$

Recall that the pure braid group P_r can also be defined as the fundamental group of the configuration space of r points in the open disc $\mathsf{F}(\mathring{\mathbb{D}}^2, r)$. The above presentation can be established by induction, by using the homotopy exact sequence associated to the fibration $f : \mathsf{F}(\mathring{\mathbb{D}}^2, r) \to \mathsf{F}(\mathring{\mathbb{D}}^2, r-1)$ which forgets about the last point of a configuration. In the proof of Proposition 5.0.1, we already observed that the fiber of this map is a disc with $r-1$ punctures $\mathring{\mathbb{D}}^2 \setminus \{z_1^0, \dots, z_{r-1}^0\}$, and that our homotopy exact sequence reduces to a short exact sequence of fundamental groups

$$(3) \qquad 1 \to \pi_1(\mathring{\mathbb{D}}^2 \setminus \{z_1^0, \dots, z_{r-1}^0\}) \to \underbrace{\pi_1 \mathsf{F}(\mathring{\mathbb{D}}^2, r)}_{=P_r} \to \underbrace{\pi_1 \mathsf{F}(\mathring{\mathbb{D}}^2, r-1)}_{=P_{r-1}} \to 1.$$

The elements x_{ir}, $i = 1, \dots, r-1$, in our presentation (1) actually represent generating elements of the fundamental group of the punctured disc $\mathring{\mathbb{D}}^2 \setminus \{z_1^0, \dots, z_{r-1}^0\}$ when we use the classical identity between this group $\pi_1(\mathring{\mathbb{D}}^2 \setminus \{z_1^0, \dots, z_{r-1}^0\})$ and the free group on $r-1$ generators F_{r-1}. The map $p : P_r \to P_{r-1}$ in our short exact sequence (3) has an obvious section $s : P_{r-1} \to P_r$ which is given by the insertion of an extra vertical strand on the side of the diagram of any pure braid with $r-1$ strands $\alpha = (\alpha_1, \dots, \alpha_{r-1}) \in P_{r-1}$. The existence of this section implies that the group P_r decomposes as a semi-direct product:

$$(4) \qquad P_r = P_{r-1} \ltimes F_{r-1}.$$

We just compute the conjugates of the elements $x_{ir} \in F_{r-1}$, $i = 1, \dots, r-1$ by the other generators $x_{ij} \in P_{r-1}$ in the group P_r to determine our set of generating relations (2) by induction on r. We refer to [26] for details on this computation.

We apply the completion construction of §8 to define the Malcev complete version of the pure braid groups $\hat{P}_r$. We immediately see from the expression of our relations (2) that the semi-direct products $P_r = P_{r-1} \ltimes F_{r-1}$ fulfill the assumptions of Proposition 8.5.2, Proposition 8.5.3 and Proposition 8.5.7 in §8.5, where we study the Malcev completion of semi-direct products. By the results of these propositions, we have the relation:

$$(5) \qquad \hat{P}_r = \hat{P}_{r-1} \ltimes \hat{F}_{r-1}$$

in the category of Malcev complete groups.

We now explain the definition of the Drinfeld–Kohno Lie algebras alluded to in the introduction of this section.

10.0.2. *The Drinfeld–Kohno Lie algebras.* The *rth Drinfeld–Kohno Lie algebra* $\mathfrak{p}(r)$ is given by the presentation

$$\mathfrak{p}(r) = \mathbb{L}(t_{ij}, 1 \leq i \neq j \leq r)/\langle [t_{ij}, t_{kl}], [t_{ij}, t_{ik} + t_{jk}] \rangle$$

where:

(1) we consider the free Lie algebra with a generator t_{ij} such that $t_{ij} = t_{ji}$ associated to each pair $\{i \neq j\} \subset \{1, \dots, r\}$,

(2) and we take the quotient of this free Lie algebra $\mathbb{L}(t_{ij}, 1 \leq i \neq j \leq r)$ by the ideal generated by the relations

$$[t_{ij}, t_{kl}] = 0, \quad [t_{ij}, t_{ik} + t_{jk}] = 0,$$

which hold for all quadruples of (pairwise) distinct indices $i, j, k, l \in \{1, \dots, r\}$ in the first case, for all triples of (pairwise) distinct indices $i, j, k \subset \{1, \dots, r\}$ in the second case. We also use the notation $i \neq j \neq k \neq l$ and $i \neq j \neq k$ to specify these situations in which our relations holds. We refer to our first set of relations $[t_{ij}, t_{kl}] = 0$ as the *commutation relations*, while the relations $[t_{ij}, t_{ik} + t_{jk}] = 0$ are usually called the *Yang–Baxter relations* in the literature.

These generating relations (2) are homogeneous with respect to the natural weight grading of free Lie algebras. The Lie algebra $\mathfrak{p}(r)$ accordingly inherits a weight grading where we assume that the generating elements t_{ij} are homogeneous of weight 1. The homogeneous component of weight m of the Lie algebra $\mathfrak{p}(r)$ is given by:

$$\mathfrak{p}(r)_m = \mathbb{L}_m \,/\, \mathbb{L}_m \cap \langle [t_{ij}, t_{kl}], [t_{ij}, t_{ik} + t_{kj}] \rangle,$$

where we use the notation $\mathbb{L}_m$ for the homogeneous component of weight m of the free Lie algebra $\mathbb{L} = \mathbb{L}(t_{ij}, 1 \leq i \neq j \leq r)$.

We have the following proposition, which gives a counterpart, for the Drinfeld–Kohno Lie algebras, of the semi-direct product decompositions of the pure braid groups §10.0.1(4):

Proposition 10.0.3. *We have an isomorphism* $\mathfrak{p}(r) \simeq \mathfrak{p}(r-1) \ltimes \mathbb{L}(t_{ir}, i = 1, \dots, r-1)$*, where we consider a semi-direct product of the* $r-1$*th Drinfeld–Kohno Lie algebra* $\mathfrak{p}(r-1)$ *with a free Lie algebra generated by the elements* t_{ir}*,* $i = 1, \dots, r-1$*, inside* $\mathfrak{p}(r)$*.*

Explanations and proof. We easily check that we have a well-defined morphism of Lie algebras $p_* : \mathfrak{p}(r) \to \mathfrak{p}(r-1)$ such that:

$$p_*(t_{ij}) = \begin{cases} t_{ij}, & \text{if } 1 \leq i \neq j \leq r-1, \\ 0, & \text{otherwise.} \end{cases}$$

This morphism has an obvious section $s_* : \mathfrak{p}(r-1) \to \mathfrak{p}(r)$ which identifies $\mathfrak{p}(r-1)$ with the Lie subalgebra of $\mathfrak{p}(r)$ generated by the elements $t_{ij} \in \mathfrak{p}(r)$ such that $1 \leq i \neq j \leq r-1$. We are going to see that this morphism $p_* : \mathfrak{p}(r) \to \mathfrak{p}(r-1)$ represents a counterpart, for the Drinfeld–Kohno Lie algebras, of the morphism $p_* : P_r \to P_{r-1}$ which forgets the last strand in the pure braid group P_r while our section $s_* : \mathfrak{p}(r-1) \to \mathfrak{p}(r)$ represents a counterpart of the group inclusion $P_{r-1} \subset P_r$ which we consider in our semi-direct product decomposition §10.0.1(4).

We also have an obvious morphism of Lie algebras $i_* : \mathbb{L}(t_{ir}, i = 1, \dots, r-1) \to \mathfrak{p}(r)$ which carries the free Lie algebra $\mathbb{L}(t_{ir}, i = 1, \dots, r-1)$ to the Lie subalgebra generated by the elements t_{ir} inside $\mathfrak{p}(r)$. We are going to see that this morphism $i_* : \mathbb{L}(t_{ir}, i = 1, \dots, r-1) \to \mathfrak{p}(r)$ represents a counterpart of the group morphism $i_* : F_r \to P_r$ which carries the free group on r generators F_r to the subgroup generated by the elements x_{ir}, $i = 1, \dots, r-1$, inside the pure braid group P_r.

We use the relations of our presentation §10.0.2(2) to determine an action of the Lie algebra $\mathfrak{p}(r-1)$ on $\mathfrak{k} = \mathbb{L}(t_{ir}, i = 1, \dots, r-1)$ from an action of the generating elements of this Lie algebra $t_{kl} \in \mathfrak{p}(r-1)$, $1 \leq k \neq l \leq r-1$, on the generating elements t_{ir} of the free Lie algebra $\mathbb{L}(t_{ir}, i = 1, \dots, r-1)$. We explicitly set:

$$[t_{ir}, t_{kl}] = \begin{cases} -[t_{ir}, t_{lr}], & \text{in the case } 1 \leq k = i \neq l \leq r-1, \\ -[t_{ir}, t_{kr}], & \text{in the case } 1 \leq k \neq i = l \leq r-1, \\ 0, & \text{otherwise.} \end{cases}$$

We use the derivation relations of §8.5.4 to extend this action to our Lie algebras. We just check that this construction carries the defining relations of the Lie algebra $\mathfrak{p}(r-1)$ to zero.

We have an obvious morphism $\mathfrak{p}(r-1) \ltimes \mathbb{L}(t_{ir}, i = 1, \dots, r-1) \to \mathfrak{p}(r)$ when we consider the semi-direct product $\mathfrak{p}(r-1) \ltimes \mathbb{L}(t_{ir}, i = 1, \dots, r-1)$ associated to this action $\mathfrak{p}(r-1) \curvearrowright \mathbb{L}(t_{ir}, i = 1, \dots, r-1)$, since we construct this action from the defining relations of the Lie algebra $\mathfrak{p}(r)$. We readily check that this morphism has an inverse (and hence, defines an isomorphism) by considering the morphism of Lie algebras $\mathfrak{p}(r) \to \mathfrak{p}(r-1) \ltimes \mathbb{L}(t_{ir}, i = 1, \dots, r-1)$ which carries the generators t_{ij} of $\mathfrak{p}(r)$ such that $1 \leq i \neq j \leq r-1$ to the corresponding elements of $\mathfrak{p}(r-1)$, while the other generators t_{ir} are mapped to the corresponding generators of the free Lie algebra $\mathbb{L}(t_{ir}, i = 1, \dots, r-1)$. ☐

We use the previous proposition in the proof of the following theorem:

THEOREM 10.0.4 (T. Kohno [106], M. Xicoténcatl [185]). *We have an isomorphism of weight graded Lie algebras*

$$\upsilon : \mathfrak{p}(r) \xrightarrow{\simeq} \mathtt{E}^0 \hat{P}_r$$

defined on the generating elements of the Lie algebra $\mathfrak{p}(r)$ by the mapping such that:

$$\upsilon(t_{ij}) = \bar{x}_{ij}, \quad \text{for } 1 \leq i < j \leq r,$$

where we use the notation $\bar{x}_{ij}$ for the image of the generators of the pure braid group $x_{ij} \in P_r$, $1 \leq i < j \leq r$, in the subquotient $\mathtt{E}^0_1 \hat{P}_r = \mathtt{F}_1 \hat{P}_r / \mathtt{F}_2 \hat{P}_r$ of the Malcev completion of P_r.

PROOF. This result follows from the main theorem of [106], where methods of rational homotopy theory are used to give a description of the Malcev completion of the nilpotent quotients $P_r / \Gamma_{s+1} P_r$ of the pure braid groups P_r. We give a more elementary proof of this statement, which we borrow from [185], and which relies on the semi-direct product decomposition of the pure braid groups.

Recall that we equip the weight graded module $\mathtt{E}^0 \hat{P}_r = \bigoplus_{s=1}^{\infty} \mathtt{F}_s \hat{P}_r / \mathtt{F}_{s+1} \hat{P}_r$ with the Lie bracket induced by the commutator in $\hat{P}_r$. By using the relations (2) of §10.0.1, we easily check that the classes of the elements $x_{ij} \in P_r$, $1 \leq i < j \leq r$, in this weight graded Lie algebra $\mathtt{E}^0 \hat{P}_r$ satisfy the defining relations of the Drinfeld–Kohno Lie algebra $\mathfrak{p}(r)$. We therefore have a well-defined Lie algebra morphism $\upsilon : \mathfrak{p}(r) \to \mathtt{E}^0 \hat{P}_r$ given by the mapping of our statement.

By Proposition 8.5.7, the semi-direct product decomposition of the pure braid group $P_r = P_{r-1} \ltimes F_{r-1}$ implies that we have a semi-direct product decomposition $\mathtt{E}^0 \hat{P}_r = \mathtt{E}^0 \hat{P}_{r-1} \ltimes \mathtt{E}^0 \hat{F}_{r-1}$ for the weight graded Lie algebra $\mathtt{E}^0 \hat{P}_r$ which we associate to the Malcev completion of this group $\hat{P}_r$. By the result of Proposition 8.4.1, we have the identity $\mathtt{E}^0 \hat{F}_{r-1} = \mathbb{L}(t_{ir}, i = 1, \dots, r-1)$ for the Malcev completion of a free group $\hat{F}_{r-1} = \hat{\mathbb{F}}(x_{ir}, i = 1, \dots, r-1)$. We use these observations and the semi-direct product decomposition of the Drinfeld–Kohno Lie algebra $\mathfrak{p}(r) \simeq$

$\mathfrak{p}(r-1) \ltimes \mathbb{L}(t_{ir}, i = 1, \dots, r-1)$ of Proposition 10.0.3 to establish by induction that our morphism defines an isomorphism, for each $r > 0$. ☐

Recall that the pure braid group P_r trivially forms a normal subgroup of the Artin braid group B_r since this group is identified with the kernel of the morphism $p_* : B_r \to \Sigma_r$ which maps a braid $\alpha \in B_r$ to its underlying permutation $w \in \Sigma_r$. Let $c_\alpha : P_r \to P_r$ be the conjugation morphism $c_\alpha(x) = \alpha x \alpha^{-1}$ which we associate to any element of the Artin braid group $\alpha \in B_r$. This morphism induces a morphism of Malcev complete groups when we pass to the Malcev completion $c_\alpha : \hat{P}_r \to \hat{P}_r$, and we have a morphism of weight graded Lie algebras $c_\alpha : \mathtt{E}^0 \hat{P}_r \to \mathtt{E}^0 \hat{P}_r$ which we define by using that this morphism of Malcev complete groups preserves the natural filtration associated to our objects. We have the following result:

Proposition 10.0.5. *Let $\alpha \in B_r$ by any element of the pure braid group. Let $w = p_*(\alpha) \in \Sigma_r$ be the underlying permutation of this braid. The morphism of weight graded Lie algebras $c_\alpha : \mathtt{E}^0 \hat{P}_r \to \mathtt{E}^0 \hat{P}_r$ induced by the conjugation with $\alpha \in B_r$ on the pure braid group P_r is determined by the formula*

$$c_\alpha(t_{ij}) = t_{w(i)w(j)}$$

when we use the identity $\mathtt{E}^0 \hat{P}_r = \mathfrak{p}(r)$ of Theorem 10.0.4 *and the presentation of this Lie algebra $\mathfrak{p}(r)$ by generators and relations in* §10.0.2.

Proof. We can reduce the verification of this proposition to the case of the standard generators of the Artin braid group $\alpha = \tau_k$ (see §5.0.8) since the mapping $c : \alpha \mapsto c_\alpha$ which associates the conjugation morphism $c_\alpha(x) = \alpha x \alpha^{-1}$ to any element in a group defines a group morphism in general.

We fix a pair $1 \leq i < j \leq r$. We easily check that we have the following formulas in P_r:

$$c_{\tau_k}(x_{ij}) = \begin{cases} x_{i-1j}, & \text{if } k+1 = i, \\ x_{ii+1} x_{i+1j} x_{ii+1}^{-1}, & \text{if } i = k < k+1 < j, \\ x_{ij-1}, & \text{if } i < k < k+1 = j, \\ x_{ij}, & \text{if } i = k < k+1 = j, \\ x_{jj+1} x_{ij+1} x_{jj+1}^{-1}, & \text{if } i < j = k, \end{cases}$$

and we trivially have $c_{\tau_k}(x_{ij}) = x_{ij}$ when $\{k, k+1\} \cap \{i, j\} = \varnothing$. We also have $c_{\tau_k}(x_{ij}) = x_{i+1j} \cdot (x_{i+1j}, x_{ii+1}^{-1}) \Rightarrow c_{\tau_k}(x_{ij}) = x_{i+1j} (\operatorname{mod} \Gamma_2 P_r)$ in the case $i = k < k+1 < j$, and we similarly get $c_{\tau_k}(x_{ij}) = x_{ij+1} (\operatorname{mod} \Gamma_2 P_r)$ in the case $i < j = k$. We immediately deduce from these formulas that we have the relation $c_{\tau_k}(\bar{x}_{ij}) = \bar{x}_{t_k(i) t_k(j)}$, for any pair $1 \leq i \neq j \leq r$, when we consider the class of the generating elements $x_{ij} \in P_r$ in the quotient group $\hat{P}_r = \hat{P}_r / \mathtt{F}_2 \hat{P}_r$. We therefore have $c_{\tau_k}(t_{ij}) = t_{t_k(i) t_k(j)}$ since these classes $\bar{x}_{ij} \in \hat{P}_r / \mathtt{F}_2 \hat{P}_r$ represent the generating elements of the Drinfeld–Kohno Lie algebras $t_{ij} \in \mathfrak{p}(r)$ in the correspondence of Theorem 10.0.4, and this result finishes the proof of the proposition. ☐

10.0.6. *The complete Drinfeld–Kohno Lie algebras.* We now consider a complete version of the Drinfeld–Kohno Lie algebras $\hat{\mathfrak{p}}(r)$, which we define by the completion $\hat{\mathfrak{p}}(r) = \lim_s \mathfrak{p}(r) / \mathtt{F}_s \mathfrak{p}(r)$ of the ordinary Drinfeld–Kohno Lie algebras $\mathfrak{p}(r)$ with respect to the filtration such that $\mathtt{F}_s \mathfrak{p}(r) = \bigoplus_{m \geq s} \mathfrak{p}(r)_m$, for $s \geq 1$. We accordingly have $\mathtt{E}^0 \hat{\mathfrak{p}}(r) = \mathfrak{p}(r)$ by construction. We actually have $\mathfrak{p}(r) = \bigoplus_{m=1}^{\infty} \mathfrak{p}(r)_m \Rightarrow$

$\hat{\mathfrak{p}}(r) = \prod_{m=1}^{\infty} \mathfrak{p}(r)_m$, where we consider the product of the homogeneous components of the weight graded Drinfeld–Kohno Lie algebra $\mathfrak{p}(r)_m$ (instead of the direct sum which we consider in the natural expression of a weight graded object).

We can also define the complete Drinfeld–Kohno Lie algebra by the presentation

$$\hat{\mathfrak{p}}(r) = \hat{\mathbb{L}}(t_{ij}, 1 \leq i \neq j \leq r)/\langle [t_{ij}, t_{kl}], [t_{ij}, t_{ik} + t_{jk}] \rangle,$$

where we take the same conventions as in Proposition 8.4.1 for the definition of the free complete Lie algebra $\hat{\mathbb{L}}(t_{ij}, 1 \leq i \neq j \leq r)$.

We use these complete Lie algebras $\hat{\mathfrak{p}}(r)$, $r > 0$, in the following improvement of the result of Theorem 10.0.4:

THEOREM 10.0.7 (T. Kohno [106]). *We have an isomorphism of Malcev complete groups*

$$\hat{\phi} : \hat{P}_r \xrightarrow{\simeq} \mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r))$$

which induces the isomorphism of Theorem 10.0.4 *when we pass to the weight graded Lie algebras associated to these groups.*

PROOF (OUTLINE). This theorem is again a byproduct of the results of [106] which are obtained by applying general statements of rational homotopy theory. We outline a direct definition of a complex version of the isomorphism of the theorem which involves the same crucial ingredient as the arguments of this reference. We then take $\Bbbk = \mathbb{C}$ as ground field. To construct our isomorphism, we use the monodromy of a certain flat connection, which is called the *Knizhnik–Zamolodchikov connection* after the work of these authors in conformal field theory [104] (see also [60] for a reference book on this subject), and which takes values in the complex coefficient version of the Drinfeld–Kohno Lie algebra $\hat{\mathfrak{p}}(r) = \hat{\mathfrak{p}}(r)_{\mathbb{C}}$.

In the context of rational homotopy theory, the existence of this flat connection is equivalent to the statement that the configuration space of r points in the complex plane is formal (see [41, 107]). By general results of the rational homotopy theory, the formality property is independent from the ground field (see [85], and [170, Theorem 12.1]), and this observation implies the existence of rational analogues of the Knizhnik–Zamolodchikov connection. We may use this statement to define a rational analogue of the isomorphism which we define in this proof, but we will give another proof of the existence of such a rational isomorphism of Malcev complete groups in §§10.2-10.4 when we explain the definition of Drinfeld's associators. We therefore focus on the complex coefficient case for the moment.

Preliminaries: The holonomy of the Knizhnik–Zamolodchikov connection. In our process, we more precisely consider connections defined on trivial fiber bundles $X \times G$ with $G = \mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r))$ as structure group. We use that such connections are determined by connection forms $\omega \in \Omega^1(X, \mathfrak{g})$ with values in the Lie algebra $\mathfrak{g} = \hat{\mathfrak{p}}(r)$. We refer to [146] for a modern introduction to the theory of connections in general fiber bundles. The Knizhnik-Zamolodchikov connection is defined on the configuration space of r points in the complex plane $\mathsf{F}(\mathbb{C}, r) = \{(z_1, \dots, z_r) | z_i \neq z_j (\forall i \neq j)\}$ by the complex 1-form such that:

$$\omega_{KZ} = \sum_{1 \leq i < j \leq r} t_{ij} \otimes d\log(z_i - z_j) \in \Omega^1(\mathsf{F}(\mathbb{C}, r), \mathfrak{p}(r)), \tag{1}$$

where we set $d\log(u) = du/u$.

We review the definition of holonomy transformations for this example of connection. We assume that $\gamma(s) = (z_1(s), \dots, z_r(s))$ is a smooth loop $\gamma : [0, 1] \to$

$\mathsf{F}(\mathbb{C},r)$ based at a fixed point $\gamma(0)=\gamma(1)=\underline{a}^0$ of the configuration space $\mathsf{F}(\mathbb{C},r)$. Let

$$h_\gamma : s\mapsto h_\gamma(s)\in\hat{\mathbb{U}}(\hat{\mathfrak{p}}(r)) \tag{2}$$

be the solution of the differential equation

$$\frac{dh_\gamma}{ds}=\sum_{1\leq i<j\leq r}\frac{d\log(z_i-z_j)}{ds}\cdot t_{ij}\cdot h_\gamma \tag{3}$$

with values in the complete enveloping algebra $\hat{\mathbb{U}}(\hat{\mathfrak{p}}(r))$ and such that $h_\gamma(0)=1$. We see that the maps $s\mapsto h_\gamma(s)\otimes h_\gamma(s)$ and $s\mapsto\Delta h_\gamma(s)$ with values in $\hat{\mathbb{U}}\hat{\mathfrak{p}}(r)\hat{\otimes}\hat{\mathbb{U}}\hat{\mathfrak{p}}(r)$ satisfy an identical differential equation, have the same initial value $\Delta h_\gamma(0)=h_\gamma(0)\otimes h_\gamma(0)=1\otimes 1$, and as a consequence, agree on all $s\in[0,1]$. We deduce from this observation that the element $h_\gamma(1)$ is group-like and hence defines an element of our structure group $G=\mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r))$. The holonomy of our connection around the loop γ is precisely defined by this group-like element $h_\gamma(1)\in\mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r))$ associated to γ. We can also interpret the isomorphism $h_\gamma(1):\{\gamma(0)\}\times G\to\{\gamma(1)\}\times G$ determined by the action of $h_\gamma(1)\in G$ on the fibers of a trivial bundle $\mathsf{F}(\mathbb{C},r)\times G$ with structure group $G=\mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r))$ as a parallel displacement along the loop γ in this fiber bundle (see [146, §6.3]).

Main construction: The definition of the isomorphism from the monodromy of the Knizhnik–Zamolodchikov connection. The Knizhnik–Zamolodchikov connections are flat (see for instance [40, §16.2] or [100, §XIX.2] for the details of this verification) and as a consequence (see [146, §6.6]), we have an identity $h_\alpha(1)=h_\beta(1)$ for all homotopic loops $\alpha,\beta:[0,1]\to\mathsf{F}(\mathbb{C},r)$ in the configuration space $\mathsf{F}(\mathbb{C},r)$. We then set:

$$\phi([\alpha])=h_\alpha(1), \tag{4}$$

for any homotopy class of loop $[\alpha]\in\pi_1(\mathsf{F}(\mathbb{C},r),\underline{a}^0)$, in order to get a morphism

$$P_r=\pi_1(\mathsf{F}(\mathbb{C},r),\underline{a}^0)\xrightarrow{\phi}\mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r)), \tag{5}$$

the *monodromy morphism*, from the pure braid group $P_r=\pi_1(\mathsf{F}(\mathbb{C},r),\underline{a}^0)$ to $G=\mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r))$.

We easily check that we have the identity $\phi(x_{ij})\equiv t_{ij}$ in the abelian quotient $G/\mathtt{F}_2\,G=\mathtt{E}_1^0\,\hat{\mathfrak{p}}(r)$ of our structure group $G=\mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r))$, where we take a representative of the generating element x_{ij} of the pure braid group P_r in $\pi_1(\mathsf{F}(\mathbb{C},r),\underline{a}^0)$. We can therefore take the extension of this morphism to the Malcev completion

$$\begin{tikzcd} P_r \arrow[rr, "\phi"] \arrow[dr] & & \mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r)) \\ & \hat{P}_r \arrow[ur, dotted, "\exists\hat{\phi}"'] & \end{tikzcd} \tag{6}$$

in order to get the isomorphism of the theorem when we work over the field of complex numbers $\Bbbk=\mathbb{C}$. ☐

We revisit the proof of this theorem in the next sections. We use that the pure braid group P_r represents the group of automorphisms of objects in the component $\mathsf{PaB}(r)$ of the parenthesized braid operad PaB. We explain that the isomorphism of Theorem 10.0.7 is induced by an isomorphism of operads in groupoids.

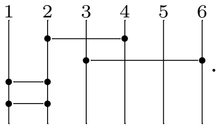
10.0.8. *The algebras of chord diagrams.* The enveloping algebra of the Drinfeld–Kohno Lie algebra $\mathfrak{p}(r)$ is identified with the associative algebra defined by the presentation:

$$\mathbb{U}\,\mathfrak{p}(r) = \mathbb{T}(t_{ij}, 1 \leq i \neq j \leq r)/\langle [t_{ij}, t_{kl}], [t_{ij}, t_{ik} + t_{kj}] \rangle,$$

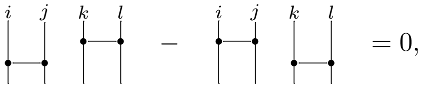
where we consider the same relations as in the Lie algebra case, but the bracket now refers to the commutator $[a, b] = ab - ba$. We have a similar presentation for the complete algebra $\hat{\mathbb{U}}\hat{\mathfrak{p}}(r)$ which we associate to our completion of the Drinfeld–Kohno Lie algebra $\hat{\mathfrak{p}}(r)$. We just take the completion of the tensor algebra in this case.

This associative algebra $\mathbb{U}\,\mathfrak{p}(r)$ is also called the *algebra of chord diagrams* in the literature (as we briefly explain in the introduction of this section). This phrase refers to a representation of the monomials $t_{i_1 j_1} \cdot \ldots \cdot t_{i_m j_m}$ by chord diagrams on r strands. In short, the diagrams corresponding to such a monomial is obtained by drawing a chord between the strand i_k and the strand j_k, for each factor $t_{i_k j_k}$, so that the composition ordering of the monomial, read from right to left, corresponds to a downwards orientation of the diagram. For instance, we have:

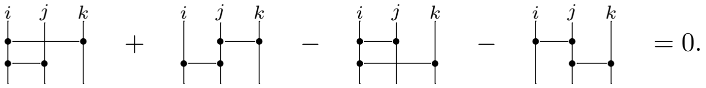
$t_{12}t_{12}t_{36}t_{24} =$



In this chord diagram representation, the commutation relation reads



and the Yang–Baxter relation is equivalent to the identity:



The latter equation is also called the *four term relation* (the *4T relation* for short) in the literature on Vassiliev's invariants.

To complete our account, we check that the rational Malcev completion satisfies the same idempotence property for the pure braid groups as in the case of free groups. We check the following preliminary observation before proving this result:

PROPOSITION 10.0.9. *For the pure braid groups P_r, we have an identity $\mathtt{F}_s\,\hat{P}_r = \Gamma_s\,\hat{P}_r$, for every $s > 0$, where we consider the natural filtration of the Malcev complete group $\hat{P}_r$ on the left-hand side, and the lower central series filtration of the plain (abstract) group underlying $\hat{P}_r$ on the right-hand side.*

PROOF. This proposition follows from a straightforward induction. We just use the identity $\mathtt{F}_s\,\hat{F}_{r-1} = \Gamma_s\,\hat{F}_{r-1}$ for the free groups $F = F_{r-1}$ (see Proposition 8.4.3) and we apply the results of Proposition 8.5.3 to the semi-direct product $P_r = P_{r-1} \ltimes F_{r-1}$. □

PROPOSITION 10.0.10. *If we take $\Bbbk = \mathbb{Q}$ as coefficient ring for our Malcev completion process, then the Malcev completion functor carries the universal morphism*

$\eta : P_r \to \hat{P}_r$ associated to the pure braid group P_r to an isomorphism of Malcev complete groups:

$$\hat{\eta} : \hat{P}_r \xrightarrow{\simeq} \hat{\hat{P}}_r,$$

for every $r \in \mathbb{N}$.

Proof. This proposition follows from a straightforward induction again. We just use the result established in Proposition 8.4.5 for the Malcev completion of free groups, and we apply the results of Proposition 8.5.3 to the semi-direct products $P_r = P_{r-1} \ltimes F_{r-1}$. □

This proposition has also the following corollary which parallels the result of Proposition 8.4.6 about the Malcev completion of free groups with a finite number of generators:

Proposition 10.0.11. *If we take $\Bbbk = \mathbb{Q}$ as coefficient ring for our Malcev completion process, then every group morphism $\psi : \hat{P}_r \to H$, where H is Malcev complete, automatically defines a morphism in the category of Malcev complete groups (see* §8.2*), for any $r \in \mathbb{N}$.*

Proof. This proposition follows from the result of Proposition 10.0.10. We use the same general arguments as in the case of free groups with a finite number of generators which we address in Proposition 8.4.6. □

10.1. The Malcev completion of the braid operads and the Drinfeld–Kohno Lie algebra operad

Recall that the pure braid group on r strands P_r defines the group of automorphisms of any object $p \in \mathtt{Ob}\,\mathsf{B}(r)$ in the component of arity r of the operad of parenthesized braids $\mathsf{B} = \mathsf{PaB}$ and of the operad of colored braids $\mathsf{B} = \mathsf{CoB}$. We accordingly have the identity $\mathtt{Aut}_{\mathsf{B}(r)^{\wedge}}(p) = \hat{P}_r$ when we pass to the Malcev completion of these operads $\mathsf{B}^{\wedge} = \mathsf{PaB}^{\wedge}, \mathsf{CoB}^{\wedge}$, and any morphism set of these operads in groupoids $\mathtt{Mor}_{\mathsf{B}(r)^{\wedge}}(p,q)$ is given by the translation of the Malcev complete group $\hat{P}_r$ by a morphism $\alpha \in \mathtt{Mor}_{\mathsf{B}(r)}(p,q)$.

We study the structure of these Malcev complete operads $\mathsf{B}^{\wedge} = \mathsf{PaB}^{\wedge}, \mathsf{CoB}^{\wedge}$ in this section. Recall that we have a categorical equivalence of operads in groupoids $\omega : \mathsf{PaB} \xrightarrow{\sim} \mathsf{CoB}$ by definition of the parenthesized braid operad PaB as a pullback of the colored braid operad CoB. We have a categorical equivalence of operads in Malcev complete groupoids $\omega : \mathsf{PaB}^{\wedge} \xrightarrow{\sim} \mathsf{CoB}^{\wedge}$ when pass to the completion. We study the operads $\mathsf{B}^{\wedge} = \mathsf{PaB}^{\wedge}, \mathsf{CoB}^{\wedge}$ in parallel and the results which obtain in this section hold for both operads. We are mainly going to check that the Drinfeld–Kohno Lie algebras $\mathfrak{p}(r)$, of which we recall the definition in the previous section, can be equipped with an operad structure, and that the collections $\mathfrak{p}(-)_m$, $m \geq 1$, which we define by taking the components of homogeneous weight of these Lie algebras, determine the fibers of the natural tower decomposition $\mathsf{B}^{\wedge} = \lim_m q_m \mathsf{B}^{\wedge}$ of our operads in Malcev complete groupoids $\mathsf{B}^{\wedge} = \mathsf{PaB}^{\wedge}, \mathsf{CoB}^{\wedge}$. We explain the definition of this operad structure on the collection of the Drinfeld–Kohno Lie algebras first.

10.1.1. *The definition of the operad structure on the Drinfeld–Kohno Lie algebras.* In §7.2.20, we explained that the direct sum of Lie algebras defines the tensor product operation of a symmetric monoidal structure on the category of Lie algebras. We use this symmetric monoidal structure to define the notion of an operad in the category of Lie algebras. We accordingly define a (non-unitary) operad in the category of Lie algebras as a collection of Lie algebras $\mathfrak{g} = \{\mathfrak{g}(r), r > 0\}$ together with an action of the symmetric group Σ_r on $\mathfrak{g}(r)$, for each $r > 0$, a unit morphism $\eta : 0 \to \mathfrak{g}(1)$, and composition operations $\circ_k : \mathfrak{g}(m) \oplus \mathfrak{g}(n) \to \mathfrak{g}(m+n-1)$ such that all these structure operations are formed in the category of Lie algebras and satisfy the axioms of operads in this category. We can also apply this definition to the category of weight graded Lie algebras (the category of Lie algebras in weight graded modules). The collection of the Drinfeld–Kohno Lie algebras $\mathfrak{p} = \{\mathfrak{p}(r), r > 0\}$ which we consider in this paragraph actually forms an operad in this category of weight graded Lie algebras. We define the structure of this operad as follows:

(1) The action of a permutation $s \in \Sigma_r$ on the Lie algebra $\mathfrak{p}(r)$ is determined on generating elements by the formula
$$s_*(t_{ij}) = t_{s(i)s(j)},$$
for each pair $\{i, j\} \subset \{1, \dots, r\}$.

(2) The unit morphism $\eta : 0 \to \mathfrak{p}(1)$ is trivially given by the zero morphism (which is also an isomorphism in this case since we have $\mathfrak{p}(1) = 0$),

(3) The composition operations are the Lie algebra morphisms $\circ_k : \mathfrak{p}(m) \oplus \mathfrak{p}(n) \to \mathfrak{p}(m+n-1)$, $k = 1, \dots, m$, such that
$$t_{ij} \circ_k 0 = \begin{cases} t_{i+n-1 j+n-1}, & \text{if } k < i < j, \\ t_{i j+n-1} + \cdots + t_{i+n-1 j+n-1}, & \text{if } k = i < j, \\ t_{i j+n-1}, & \text{if } i < k < j, \\ t_{ij} + \cdots + t_{i j+n-1}, & \text{if } i < k = j, \\ t_{ij}, & \text{if } i < j < k, \end{cases}$$
for any generating element $t_{ij} \in \mathfrak{p}(m)$, and
$$0 \circ_k t_{ij} = t_{i+k-1 j+k-1} \quad \text{for all } k,$$
for any generating element $t_{ij} \in \mathfrak{p}(n)$, where we use the notation $a \circ_k b$ for the image of an element (a, b) under any of these Lie algebra morphisms $\circ_k : \mathfrak{p}(m) \oplus \mathfrak{p}(n) \to \mathfrak{p}(m+n-1)$.

We give a graphical interpretation of these composition operations in the next section. We then use the chord diagram representation of the elements of the enveloping algebra of the Lie algebras $\mathfrak{p}(r)$ (see §10.0.8).

Recall that the Lie algebra morphisms $\phi : \mathfrak{g} \oplus \mathfrak{h} \to \mathfrak{m}$ are equivalent to pairs of Lie algebra morphisms $(f : \mathfrak{g} \to \mathfrak{m}, g : \mathfrak{h} \to \mathfrak{m})$ such that $[f(\mathfrak{g}), g(\mathfrak{h})] = 0$. In the above definition (3), we implicitly assume that our assignments determine a well-defined Lie algebra morphism on the direct sum $\mathfrak{p}(m) \oplus \mathfrak{p}(n)$. This assertion follows from straightforward verifications by using the commutation relation and the Yang–Baxter relation of the Drinfeld–Kohno Lie algebras. We also readily check that our operations satisfy the operad axioms.

We refer to this operad $\mathfrak{p} = \{\mathfrak{p}(r), r > 0\}$ as the *(weight graded) Drinfeld–Kohno Lie algebra operad.* We will explain later on (in the next section) that the above operations (1-3) extend to the collection of complete Drinfeld–Kohno Lie algebras

$\hat{\mathfrak{p}} = \{\hat{\mathfrak{p}}(r), r > 0\}$ which forms an operad in complete Lie algebras therefore. We use the phrase '*complete Drinfeld–Kohno Lie algebra operad*' to distinguish this operad in complete Lie algebras from the operad in weight graded Lie algebras defined in this paragraph.

10.1.2. *The unitary extension of the Drinfeld–Kohno Lie algebra operad.* We have not been explicit about the component of arity zero of our operad in the definition of the previous paragraph. By convention, we assume that we deal with a non-unitary operad when we use the notation $\mathfrak{p}$. But our definition has an obvious extension in the unitary setting. Hence, we also have a unitary operad in the category of Lie algebras, which defines a unitary extension of our operad $\mathfrak{p}$, and which we denote by $\mathfrak{p}_+$ (with the usual $+$ mark of unitary operads).

Recall that we have to adapt the concepts of §§2.2-2.3 when we work within a symmetric monoidal category, like the category of Lie algebras, of which tensor product does not distribute over colimits (see §§1.1.19-1.1.20). To be explicit, in §§2.2-2.3, we use the convention that the arity 0 term of a non-unitary operad is given by the initial object of the ambient category in order to identify the category of non-unitary operad with a full subcategory of the category of all operads. This correspondence does not work in our situation, and we therefore simply forget about the terms of arity zero when we define non-unitary operads in Lie algebras. In fact, since the initial object (the zero object) 0 is the tensor product unit in the category of Lie algebras, we have $\mathfrak{p}_+(0) = 0$ for the unitary operad $\mathfrak{p}_+$ (and not the converse).

The definition of the partial composition operations with this arity zero term in $\mathfrak{p}_+$ are given by a formal extension of the definition of §10.1.1. The restriction operators $\partial_k : \mathfrak{p}(r) \to \mathfrak{p}(r-1)$, which we use to model these partial composition operations $\circ_k : \mathfrak{p}_+(r) \oplus \mathfrak{p}_+(0) \to \mathfrak{p}_+(r-1)$, $k = 1, \dots, r$, are given on generating elements by:

$$\partial_k(t_{ij}) = \begin{cases} t_{i-1j-1}, & \text{if } k < i < j, \\ 0, & \text{if } k = i < j, \\ t_{ij-1}, & \text{if } i < k < j, \\ 0, & \text{if } i < k = j, \\ t_{ij}, & \text{if } i < j < k. \end{cases}$$

For any $m \geq 1$, we consider the collection $\mathfrak{p}(-)_m = \{\mathfrak{p}(r)_m, r > 0\}$ formed by the homogeneous components of weight m of the Drinfeld–Kohno Lie algebras $\mathfrak{p}(r)$. We immediately see that this collection inherits an additive operad structure from the Drinfeld–Kohno Lie algebra operad. We more precisely get that $\mathfrak{p}(-)_m$ forms an operad in the category of $\Bbbk$-modules where we take the direct sum as symmetric monoidal structure operation instead of the tensor product. In what follows, we use the phrase '*additive operad*' to distinguish this notion of operad in $\Bbbk$-modules from the usual category of operads in $\Bbbk$-modules. We have an obvious extension of these concepts in the context of unitary operads. We consider the operad $\mathfrak{p}_+(-)_m$ such that $\mathfrak{p}_+(0)_m = 0$ to form a unitary extension of the operad $\mathfrak{p}(-)_m$.

We have the following statement:

THEOREM 10.1.3. *The structure group operads of the fibers of the natural tower decomposition* $\mathsf{B}^{\widehat{}} = \lim_m q_m \mathsf{B}^{\widehat{}}$ *of the operads in Malcev complete groupoids* $\mathsf{B}^{\widehat{}} = \mathsf{PaB}^{\widehat{}}, \mathsf{CoB}^{\widehat{}}$ *are isomorphic to the additive operads* $\mathfrak{p}(-)_m$, $m \geq 1$, *which we identify* *with constant local coefficient system operads on the operads in groupoids* $B^{\wedge} = PaB^{\wedge}, CoB^{\wedge}$*, and we have the same result when we pass to the category of unitary operads.*

Explanations and proof. We focus on the non-unitary version of this result. The extension of our constructions to unitary operads is straightforward. Recall that the structure group operad of the mth fiber of the natural tower decomposition $B^{\wedge} = \lim_m q_m B^{\wedge}$ of the Malcev completion $B^{\wedge}$ of an operad B consists of the collection of $\Bbbk$-modules

$$N_{\langle m\rangle}(p) = \mathtt{E}^0_m \, \mathtt{Aut}_{B(r)^{\wedge}}(p),$$

which we associate to any object of our original operad $p \in \mathtt{Ob}\, B(r)$ by taking the mth subquotient $\mathtt{E}^0_m \, \mathtt{Aut}_{B(r)}(p)^{\wedge} = \mathtt{F}_m \, \mathtt{Aut}_{B(r)}(p)^{\wedge} / \mathtt{F}_{m+1} \, \mathtt{Aut}_{B(r)}(p)^{\wedge}$ of the natural filtration of the Malcev complete automorphism group of this object $\mathtt{Aut}_{B(r)^{\wedge}}(p) = \mathtt{Aut}_{B(r)}(p)^{\wedge}$ in the Malcev completion of our operad $B^{\wedge}$. This collection of $\Bbbk$-modules inherits:

- conjugation operations

  $$c_{\alpha} : N_{\langle m\rangle}(p) \to N_{\langle m\rangle}(q),$$

  which we associate to the morphisms $\alpha \in \mathtt{Mor}_{B(r)^{\wedge}}(p, q)$ of the operad $B^{\wedge}$;
- a symmetric structure, which we define by morphisms

  $$s_* : N_{\langle m\rangle}(p) \to N_{\langle m\rangle}(sp)$$

  associated to the permutations $s \in \Sigma_r$, for $p \in \mathtt{Ob}\, B(r)$;
- an operadic unit, which is given by the identity $N_{\langle m\rangle}(1) = 0$ in arity $r = 1$, and additive operadic composition operations

  $$\circ_i : N_{\langle m\rangle}(p) \oplus N_{\langle m\rangle}(q) \to N_{\langle m\rangle}(p \circ_i q),$$

  which are defined for all $p \in \mathtt{Ob}\, B(k)$, $q \in \mathtt{Ob}\, B(l)$, and where $k, l > 0$ and $i = 1, \dots, k$.

These structure operations satisfy natural coherence relations and an obvious generalization of the equivariance, unit and associativity axioms of operads. In §9.2.6, we use the phrase '*local coefficient system operad*' to refer to a structure of this form.

For the operads $B^{\wedge} = PaB^{\wedge}, CoB^{\wedge}$, we have

$$\mathtt{Aut}_{B(r)}(p) = P_r \Rightarrow N_{\langle m\rangle}(p) = \mathtt{E}^0_m \, \mathtt{Aut}_{B(r)^{\wedge}}(p) = \mathtt{E}^0_m \, \hat{P}_r,$$

for all $p \in \mathtt{Ob}\, B(r)$, where we consider the mth subquotient of the natural filtration of the Malcev completion of the pure braid group on r strands $\hat{P}_r$. We now set $\mathfrak{p}(p)_m := \mathfrak{p}(r)_m$, for all $p \in \mathtt{Ob}\, B(r)$ and $r > 0$, in order to identify the operad $\mathfrak{p}(-)_m$ with a constant local coefficient system operad on $B^{\wedge}$.

Recall that we have $\mathtt{Ob}\, CoB(r) = \Pi(r) = \Sigma_r$ so that any object $p \in \mathtt{Ob}\, CoB(r)$ in the component of arity r of the operad $B = CoB$ is associated to a permutation on r letters $w \in \Sigma_r$. In the case $B = PaB$, where we have $\mathtt{Ob}\, PaB(r) = \Omega(r)$, we still have a natural map $\omega : \Omega(r) \to \Pi(r)$ (the forgetting of parenthesization) which we can again use to associate a permutation $w \in \Sigma_r$ to any object $p \in \mathtt{Ob}\, PaB(r)$.

For any such object $p \in \mathtt{Ob}\, B(r)$, we consider the isomorphisms of $\Bbbk$-modules

$$(*)\qquad \begin{tikzcd} \mathfrak{p}(r)_m \arrow[r, "w_*^{-1}"] \arrow[rrr, dotted, bend right, "\upsilon_p"'] & \mathfrak{p}(r)_m \arrow[r, "\upsilon"] & \mathtt{E}^0_m\, \hat{P}_r & N_{\langle m\rangle}(p) \arrow[l, "="'] \end{tikzcd}$$

which we obtain by composing the isomorphism of Theorem 10.0.4 with the action on the Drinfeld–Kohno Lie algebra operad of the inverse of the permutation $w \in \Sigma_r$ associated to our object $p \in \mathtt{Ob}\, B(r)$.

Let $\alpha \in \mathtt{Mor}_{B(r)}(p,q)$ be a morphism in the operad $B = \mathit{PaB}, \mathit{CoB}$. Let $u \in \Sigma_r$ (respectively, $v \in \Sigma_r$) be the permutation associated to the source object p (respectively, to the target object q) of this morphism. By definition of our operads $B = \mathit{PaB}, \mathit{CoB}$, the morphism α is represented by a braid with $w = v^{-1}u$ as underlying permutation. The morphism $c_\alpha : \mathtt{E}^0_m\, \mathtt{Aut}_{B(r)^\wedge}(p) \to \mathtt{E}^0_m\, \mathtt{Aut}_{B(r)^\wedge}(q)$ is identified with the operation induced by the conjugation by this braid when we use the identity $\mathtt{Aut}_{B(r)^\wedge}(p) = \mathtt{Aut}_{B(r)^\wedge}(q) = \hat{P}_r \Rightarrow \mathtt{E}^0_m\, \mathtt{Aut}_{B(r)^\wedge}(p) = \mathtt{E}^0_m\, \mathtt{Aut}_{B(r)^\wedge}(q) = \mathtt{E}^0_m\, \hat{P}_r$ and we forget about the objects associated to our automorphism groups. The result of Proposition 10.0.5 implies that this operation fits in a commutative diagram:

$$\begin{tikzcd} \mathfrak{p}(r)_m \arrow[r, "u_*^{-1}"] \arrow[d, dotted, "id"'] & \mathfrak{p}(r)_m \arrow[r, "\upsilon"] \arrow[d, dotted, "w_*"'] & \mathtt{E}^0_m\, \hat{P}_r \arrow[d, "c_\alpha"'] & N_{\langle m\rangle}(p) \arrow[l, "="'] \arrow[d, "c_\alpha"'] \\ \mathfrak{p}(r)_m \arrow[r, "v_*^{-1}"] & \mathfrak{p}(r)_m \arrow[r, "\upsilon"] & \mathtt{E}^0_m\, \hat{P}_r & N_{\langle m\rangle}(q) \arrow[l, "="'] \end{tikzcd} ,$$

from which we deduce that our maps $(*)$ make the conjugation operation associated to the morphism $\alpha \in \mathtt{Mor}_{B(r)}(p,q)$ on the local coefficient system operad $N_{\langle m\rangle}$ correspond to the identity map on the module $\mathfrak{p}(r)_m$. We extend this correspondence to the morphisms of the Malcev completion of our operads $B^\wedge = \mathit{PaB}^\wedge, \mathit{CoB}^\wedge$ by using that any such morphism $f \in \mathtt{Mor}_{B(r)^\wedge}(p,q)$ admits a decomposition $f = \alpha g$, where $\alpha \in \mathtt{Mor}_{B(r)}(p,q)$ and $g \in \mathtt{Aut}_{B(r)^\wedge}(p) = \hat{P}_r$. We simply use that the conjugation by $g \in \hat{P}_r$ reduces to the identity on $\mathtt{E}^0_m\, \mathtt{Aut}_{B(r)^\wedge}(p) = \mathtt{E}^0_m\, \hat{P}_r$ (see §8.2.2).

We immediately get that our maps intertwine the action of permutations too (recall that this action is trivial on the braids which represent the morphisms of the operads $B = \mathit{PaB}, \mathit{CoB}$). We easily check that the formulas of §10.1.1(3) reflect the operadic composition of the morphisms represented by the pure braids x_{ij} with the morphisms represented by identity braids in the operads $B = \mathit{PaB}, \mathit{CoB}$ when we use the mapping $\upsilon : t_{ij} \mapsto \bar{x}_{ij}$ of Theorem 10.0.4. We use that the composition operation of the operads $B = \mathit{PaB}, \mathit{CoB}$ preserves commutators in automorphism groups and that our maps $(*)$ preserve the action of symmetric groups to conclude that our maps preserve all operadic composition operations. This verification completes the proof that our maps $(*)$ define an isomorphism of local coefficient system operads. □

To complete the study of this section, we prove that the Malcev completion of the operads in groupoids $B = \mathit{PaB}, \mathit{CoB}$ satisfies the following idempotence property:

THEOREM 10.1.4. *If we take $\Bbbk = \mathbb{Q}$ as coefficient ring for our Malcev completion process, then the Malcev completion functor carries the universal morphism $\eta : B \to$*

$B^{\wedge}$ associated to the operad $B = PaB, CoB$ to an isomorphism of operads in Malcev complete groupoids:

$$\hat{\eta} : B^{\wedge} \xrightarrow{\simeq} B^{\wedge\wedge},$$

and we have the same result when we pass to unitary operads.

Proof. We form the following diagram:

$$\begin{tikzcd}
B^{\wedge} \arrow[rr, "id"] \arrow[dr, "\eta"'] & & B^{\wedge}, \\
& B^{\wedge\wedge} \arrow[ur, dotted, "\exists\hat{id}"'] &
\end{tikzcd}$$

where we consider the morphism of operads in Malcev complete groupoids which extends the identity of $B^{\wedge}$, in order to get a morphism $\hat{id}$ which goes in the converse direction as the morphism of the theorem $\hat{\eta}$. We have $\hat{id}\,\hat{\eta} = id$ by adjunction.

Recall that we have $\mathtt{Ob}\, B^{\wedge\wedge} = \mathtt{Ob}\, B^{\wedge} = \mathtt{Ob}\, B$ by definition of our Malcev completion functor on operads in groupoids. The result of Proposition 10.0.10 implies that our operad morphisms define converse bijections on the automorphism groups of objects. We use that the morphism sets of our operads are defined by a translation of these automorphism groups to deduce that our operad morphisms induce converse bijections on any of these morphism sets. We therefore have both $\hat{\eta}\,\hat{id} = id$ and $\hat{id}\,\hat{\eta} = id$ and our conclusion follows. We have a straightforward extension of these arguments when we work in the context of unitary operads. □

This theorem has the following corollary:

Proposition 10.1.5. *Let $B^{\wedge} = PaB^{\wedge}, CoB^{\wedge}$. If we take $\Bbbk = \mathbb{Q}$ as coefficient ring for our Malcev completion process, then every morphism of operads in groupoids $\psi : B^{\wedge} \to Q$, where Q is an operad in Malcev complete groupoids, automatically defines a morphism of operads in Malcev complete groupoids, and we have the same result for the unitary extension of our operads $B^{\wedge}_{+} = PaB^{\wedge}_{+}, CoB^{\wedge}_{+}$.*

Proof. We focus on the non-unitary version of this result again since the extension of our argument lines to unitary operads is straightforward. We consider the universal morphism $\eta : B^{\wedge} \to B^{\wedge\wedge}$ associated to the operad $P = B^{\wedge}$ and the morphism of operads in Malcev complete groupoids $\hat{id} : B^{\wedge\wedge} \to B^{\wedge}$ which extends the identity morphism of $B^{\wedge}$ as in the proof of Theorem 10.1.4. We also have $\hat{id}\,\eta = id$ by adjunction. From the relation $\hat{id}\,\hat{\eta} = id$ together with the observation that $\hat{\eta}$ is an isomorphism of operads in Malcev complete groupoids with $\hat{id}$ as converse isomorphism (see Theorem 10.1.4), we conclude that η is identified with the morphism of operads in Malcev complete groupoids $\hat{id}^{-1} = \hat{\eta}$, and hence that η defines a morphism of operads in Malcev complete groupoids itself. We then form the following diagram:

$$\begin{tikzcd}
B^{\wedge} \arrow[rr, "\psi"] \arrow[dr, "\eta"'] & & Q, \\
& B^{\wedge\wedge} \arrow[ur, dotted, "\exists\hat{\psi}"'] &
\end{tikzcd}$$

where we consider the morphism of operads in Malcev complete groupoids that extends ψ. We eventually obtain that our morphism $\psi = \hat{\psi}\eta$ defines morphism of operads in Malcev complete groupoids by composition. □

10.2. The operad of chord diagrams and Drinfeld's associators

We aim to prove that the isomorphisms of Malcev complete groups $\hat{P}_r \simeq \mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r))$ which we define in Theorem 10.0.7, can be realized by a morphism of operads in Malcev complete groupoids $\phi : \mathsf{PaB}^{\wedge} \to \mathsf{CD}^{\wedge}$ from the Malcev completion of the operad of parenthesized braids $\mathsf{PaB}^{\wedge}$ towards an operad such that $\mathsf{CD}(r)^{\wedge} = \mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r))$, where we regard the group $\mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r))$ as a groupoid with a single object. We call this operad $\mathsf{CD}^{\wedge}$ the *operad of chord diagrams*. This name refers to our representation of the elements of the enveloping algebra $\hat{\mathbb{U}}\hat{\mathfrak{p}}(r)$ in terms of (linear combinations of) chord diagrams on r strands (see §10.0.8).

We divide the proof of the existence of a morphism $\phi : \mathsf{PaB}^{\wedge} \to \mathsf{CD}^{\wedge}$ in several steps. We actually form our morphism in the category of unitary operads $\phi : \mathsf{PaB}^{\wedge}_{+} \to \mathsf{CD}^{\wedge}_{+}$. We therefore consider the (Malcev completion of the) unitary version of the operad of parenthesized braid $\mathsf{PaB}^{\wedge}_{+}$ and a unitary extension of the operad of chord diagrams $\mathsf{CD}^{\wedge}_{+}$. We use that such a morphism ϕ is given by the extension to the completion $\mathsf{PaB}^{\wedge}_{+}$ of a morphism defined on the (ordinary) unitary operad of parenthesized braids $\phi : \mathsf{PaB}_{+} \to \mathsf{CD}^{\wedge}_{+}$. We rely on the observation that PaB_{+} represents the operad governing braided monoidal categories with a strict unit in order to establish that giving a morphism $\phi : \mathsf{PaB}_{+} \to \mathsf{CD}^{\wedge}_{+}$ amounts to fixing a braiding isomorphism and an associativity isomorphism in the morphism sets of the operad of chord diagrams $\mathsf{CD}^{\wedge}_{+}$. We actually retrieve the notion of *associator* introduced by Drinfeld in quantum group theory [57] when we apply this correspondence.

We can therefore reduce the definition of our morphism $\phi : \mathsf{PaB}_{+} \to \mathsf{CD}^{\wedge}_{+}$ to the choice of an element in the set of Drinfeld's associators. We simply survey Drinfeld's definition of an associator from solutions of the Knizhnik–Zamolodchikov differential systems in order to complete our proof of the existence of such a morphism $\phi : \mathsf{PaB}_{+} \to \mathsf{CD}^{\wedge}_{+}$. We call this associator the *Knizhnik–Zamolodchikov associator* in what follows. We will see that the Knizhnik–Zamolodchikov associator is only defined over the field of complex numbers $\Bbbk = \mathbb{C}$. Thus, we still have to refine our arguments in order to prove the existence of morphisms $\phi : \mathsf{PaB}_{+} \to \mathsf{CD}^{\wedge}_{+}$ defined over the field of rational numbers $\Bbbk = \mathbb{Q}$, and over any ground field of characteristic zero $\Bbbk$. We work out this rationality problem in a subsequent section (in §10.4), after studying a natural tower decomposition of the set of Drinfeld's associators.

We explain the definition of the chord diagram operad $\mathsf{CD}^{\wedge}$ in the first part of this section. We first explain the definition of an operad structure on the collection of ordinary enveloping algebras $\mathbb{U}\,\mathfrak{p}(r)$ which we associate to the weight graded Drinfeld–Kohno Lie algebras $\mathfrak{p}(r)$, $r > 0$. We check that this operad structure extends to the complete enveloping algebras $\hat{\mathbb{U}}\hat{\mathfrak{p}}(r)$ afterwards and we define the chord diagram operad $\mathsf{CD}^{\wedge}$ by taking the groups of group-like elements associated to these complete enveloping algebras $\mathsf{CD}^{\wedge}(r) = \mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r))$, for $r > 0$.

10.2.1. *The operad structure on the algebras of chord diagrams.* By Proposition 7.2.23, we have an isomorphism $\mathbb{U}(\mathfrak{g}) \otimes \mathbb{U}(\mathfrak{h}) \simeq \mathbb{U}(\mathfrak{g} \oplus \mathfrak{h})$, for all Lie algebras $\mathfrak{g}, \mathfrak{h} \in \mathcal{L}ie$, so that the enveloping algebra functor defines a symmetric monoidal functor from Lie algebras to Hopf algebras. This result implies, according to the general statement of Proposition 3.1.1, that the collection of enveloping algebras $\mathbb{U}\,\mathfrak{p} =$

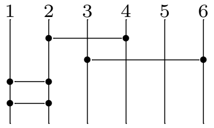
$\{\mathbb{U}\,\mathfrak{p}(r), r > 0\}$ which we associate to the Drinfeld–Kohno Lie algebras $\mathfrak{p}(r)$, $r > 0$, inherits the structure of an operad in the category of Hopf algebras. To be explicit:

- each Hopf algebra $\mathbb{U}\,\mathfrak{p}(r)$ inherits an action of the symmetric group from the Lie algebra $\mathfrak{p}(r)$ by functoriality of the enveloping algebra construction;
- we moreover have the relation $\mathfrak{p}(1) = 0 \Rightarrow \mathbb{U}\,\mathfrak{p}(1) = \Bbbk$ so that our collection has an obvious operadic unit;
- and we consider the Hopf algebra morphisms
$$\mathbb{U}(\mathfrak{p}(m)) \otimes \mathbb{U}(\mathfrak{p}(n)) \xrightarrow{\simeq} \mathbb{U}(\mathfrak{p}(m) \oplus \mathfrak{p}(n)) \xrightarrow{\circ_k} \mathbb{U}(\mathfrak{p}(m+n-1)),$$
induced by the Lie algebra morphism of §10.1.1(3) to define the composition operations of our operad.

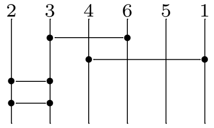
We can obviously extend this construction to the unitary operad setting in order to form a unitary version of our operad $\mathbb{U}\,\mathfrak{p}_+$ with $\mathbb{U}\,\mathfrak{p}_+(0) = \mathbb{U}(0) = \Bbbk$ as arity zero term. We can also use the functoriality of the enveloping algebra construction in order to determine the restriction operators $\partial_k : \mathbb{U}\,\mathfrak{p}(r) \to \mathbb{U}\,\mathfrak{p}(r-1)$, $k = 1, \dots, r$, which reflect the operadic compositions with the unit element $1 \in \mathbb{U}\,\mathfrak{p}_+(0)$ of the extra arity zero term of this operad $\mathbb{U}\,\mathfrak{p}_+$, from the restriction operators of the Drinfeld–Kohno Lie algebras $\partial_k : \mathfrak{p}(r) \to \mathfrak{p}(r-1)$ such as defined in §10.1.2.

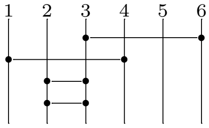
Recall that we can also define the algebras $\mathbb{U}\,\mathfrak{p}(r)$, $r > 0$, by the presentation:
$$\mathbb{U}\,\mathfrak{p}(r) = \mathbb{T}(t_{ij}, 1 \leq i \neq j \leq r)/\langle [t_{ij}, t_{kl}], [t_{ij}, t_{ik} + t_{jk}] \rangle,$$
where we now use the notation $[-,-]$ for the commutator $[a,b] = ab - ba$. Recall also that we can associate any monomial $t_{i_1 j_1} \cdot \ldots \cdot t_{i_m j_m} \in \mathbb{U}\,\mathfrak{p}(r)$ to a chord diagram with r strands in order to get a graphical representation of this associative algebra (see §10.0.8).

In the chord diagram picture, the action of a permutation $s \in \Sigma_r$ on $\mathbb{U}\,\mathfrak{p}(r)$ corresponds to a strand renumbering operation. For instance, the permutation
$$s = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 6 & 5 & 1 \end{pmatrix}$$
acts on the chord diagram given in §10.0.8

$t_{12}t_{12}t_{36}t_{24} =$ 

by:

$s_*(t_{12}t_{12}t_{36}t_{24}) =$ 

$=$ 

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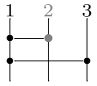
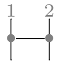
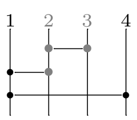
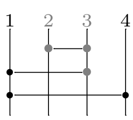
The composition product operation $\circ_k$ is identified with a natural cabling operation where a chord diagram on n strands v is plugged in the kth strand of an input chord diagram u. The composite of these chord diagrams $u \circ_k v$ is the sum of all diagrams obtained by attaching the strings which join the kth strand of u to a strand of v. To give a simple example, we have the formula



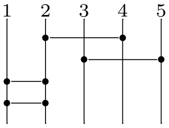
$\circ_2$ 

$=$ 

$+$ 

in $\mathbb{U}\,\mathfrak{p}(4)$. The restriction operator $\partial_k : \mathbb{U}\,\mathfrak{p}(r) \to \mathbb{U}\,\mathfrak{p}(r-1)$ is given by the removal of the kth strand in our diagram. If a chord is attached to this strand, then our restriction operator vanishes. For instance, we have:

$$\partial_5(t_{12}t_{12}t_{36}t_{24}) = $$



,

while we get $\partial_k(t_{12}t_{12}t_{36}t_{24}) = 0$ when $k \neq 5$.

10.2.2. *The operad structure on the completion of the algebras of chord diagrams.* Recall that the complete Drinfeld–Kohno Lie algebras $\hat{\mathfrak{p}}(r)$ are defined by the completion of the weight graded Drinfeld–Kohno Lie algebras $\mathfrak{p}(r)$ with respect to the filtration such that $\mathtt{F}_s\,\mathfrak{p}(r) = \bigoplus_{m\geq s}\mathfrak{p}(r)_m$, for $s \geq 1$. We immediately see that the structure operations of the weight graded Drinfeld–Kohno Lie algebra operad in §§10.1.1-10.1.2 preserve this filtration. We therefore get that the collection of complete Drinfeld–Kohno Lie algebras $\hat{\mathfrak{p}} = \{\hat{\mathfrak{p}}(r), r > 0\}$ inherits the structure of an operad in the category of complete Lie algebras which we can define by the same formulas as in §§10.1.1-10.1.2 on the generating elements of these complete Lie algebras.

We can use the same process as in the previous paragraph to define an operad structure on the collection of complete enveloping algebras $\hat{\mathbb{U}}\hat{\mathfrak{p}} = \{\hat{\mathbb{U}}\hat{\mathfrak{p}}(r), r > 0\}$ from this operad structure on the complete Drinfeld–Kohno Lie algebras. We get that this collection forms an operad in complete Hopf algebras. Recall that these complete enveloping algebras are also given by a presentation:

$$\hat{\mathbb{U}}\hat{\mathfrak{p}}(r) = \hat{\mathbb{T}}(t_{ij}, 1 \leq i \neq j \leq r)/\langle[t_{ij}, t_{kl}], [t_{ij}, t_{ik} + t_{jk}]\rangle,$$

for each $r > 0$, where we replace the ordinary tensor algebra considered in the previous paragraph by the completed one (see §10.0.8). Hence, the elements of the algebra $\hat{\mathbb{U}}\hat{\mathfrak{p}}(r)$ can be represented by formal sums of monomials $t_{i_1 j_1} \cdot \ldots \cdot t_{i_m j_m}$ and we may actually use the same picture as in the previous paragraph to represent the image of such monomials under the operations of our operad structure on this collection of complete enveloping algebras $\hat{\mathbb{U}}\hat{\mathfrak{p}}(r)$, $r > 0$. In the above presentation, the coproduct of the complete enveloping algebra $\hat{\mathbb{U}}\hat{\mathfrak{p}}(r)$ is determined by the formula $\Delta(t_{ij}) = t_{ij} \otimes 1 + 1 \otimes t_{ij}$ for the generating elements $t_{ij} \in \hat{\mathbb{U}}\hat{\mathfrak{p}}(r)$.

We can obviously extend these constructions to get a unitary extension $\hat{\mathfrak{p}}_+$ of the operad in complete Lie algebras $\hat{\mathfrak{p}}$, and a corresponding unitary extension $\hat{\mathbb{U}}\hat{\mathfrak{p}}_+$ of our operad in complete Hopf algebras $\hat{\mathbb{U}}\hat{\mathfrak{p}}$.

10.2.3. *The operad in Hopf groupoids defined by the algebras of chord diagrams.* We can also regard the Hopf algebras $\mathbb{U}\,\mathfrak{p}(r)$ of §10.2.1 as the hom-objects of a collection of Hopf groupoids $\mathsf{A}(r)$ such that

$$\mathtt{Ob}\,\mathsf{A}(r) = pt \quad \text{and} \quad \mathtt{Hom}_{\mathsf{A}(r)}(pt, pt) = \mathbb{U}\,\mathfrak{p}(r),$$

for each $r > 0$. We can provide this collection of Hopf groupoids with the structure of an operad. We define the structure operations of this operad by trivial one-point set maps at the object set level and we use the definitions of §10.2.1 for the hom-objects.

We can also apply the completion process of §9.1 to get an operad in complete Hopf groupoids $\mathsf{A}^{\wedge}$ such that $\mathsf{A}^{\wedge}(r) = \mathsf{A}(r)^{\wedge}$ from this operad in Hopf groupoids A. We then take the completion of the hom-object $\mathtt{Hom}_{\mathsf{A}(r)}(pt, pt) = \mathbb{U}\,\mathfrak{p}(r)$ with respect to the filtration defined by the powers of the augmentation ideal of the enveloping algebra $\mathbb{U}\,\mathfrak{p}(r)$. We readily see that the sth layer of this filtration $\mathbb{I}^s\,\mathbb{U}(\mathfrak{p}(r))$ is identified with the module spanned by monomials $t_{i_1 j_1}\cdot\dots\cdot t_{i_m j_m}$ of length $m\geq s$. We therefore have an identity

$$\mathtt{Hom}_{\mathsf{A}(r)^{\wedge}}(pt,pt)=\hat{\mathbb{U}}\hat{\mathfrak{p}}(r)$$

when we perform this construction. We also obviously retrieve the operad structure of the collection of complete enveloping algebras $\hat{\mathbb{U}}\hat{\mathfrak{p}}(r)$, $r>0$, when we apply the construction of §9.1 to determine the action of permutations and the composition operations $\circ_k:\mathtt{Hom}_{\mathsf{A}(m)^{\wedge}}(pt,pt)\hat{\otimes}\,\mathtt{Hom}_{\mathsf{A}(n)^{\wedge}}(pt,pt)\rightarrow\mathtt{Hom}_{\mathsf{A}(m+n-1)^{\wedge}}(pt,pt)$ associated to the hom-objects of the operad in complete Hopf groupoids $\mathsf{A}^{\wedge}$.

To sum up this discussion, we can identify an operad in (complete) Hopf algebras with an operad in (complete) Hopf groupoids with a single object, and we just checked that we retrieve the operad $\hat{\mathbb{U}}\hat{\mathfrak{p}}=\{\hat{\mathbb{U}}\hat{\mathfrak{p}}(r),r>0\}$ when we apply the completion process of operads in Hopf groupoids §9.1 to the object $\mathbb{U}\,\mathfrak{p}=\{\mathbb{U}\,\mathfrak{p}(r),r>0\}$. In what follows, we just use the notation A (respectively, $\mathsf{A}^{\wedge}$) when we want to regard these operads $\mathbb{U}\,\mathfrak{p}$ (respectively, $\hat{\mathbb{U}}\hat{\mathfrak{p}}$) as operads in (complete) Hopf groupoids with a single object rather than as operads in (complete) Hopf algebras.

We can obviously extend the correspondence studied in this paragraph to unitary operads. We then consider a unitary extension $\mathsf{A}^{\wedge}_{+}$ of the operad $\mathsf{A}^{\wedge}$ with $\mathsf{A}^{\wedge}_{+}(0)=\mathbb{k}$ as term of arity zero.

10.2.4. *The operad of chord diagrams.* The chord diagram operad is the operad in Malcev complete groupoids $\mathsf{CD}^{\wedge}$ which we obtain by applying the group-like element functor to the operad in complete Hopf groupoids defined in the previous paragraph:

$$\mathsf{CD}^{\wedge}=\mathbb{G}(\mathsf{A}^{\wedge})$$

We also use the notation $\mathsf{CD}(r)^{\wedge}=\mathsf{CD}^{\wedge}(r)$ for the components of this operad $\mathsf{CD}^{\wedge}$. We accordingly have:

$$\mathtt{Ob}\,\mathsf{CD}(r)^{\wedge}=pt,\quad\mathtt{Aut}_{\mathsf{CD}(r)^{\wedge}}(pt)=\mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r)),$$

for every $r>0$, by definition of the operad of group-like elements associated to an operad in complete Hopf groupoids (see §3.3), and we provide this collection of groupoids $\mathsf{CD}^{\wedge}(r)=\mathsf{CD}(r)^{\wedge}$, $r>0$, with the structure operations yielded by the operad structure of the enveloping algebras $\hat{\mathbb{U}}\hat{\mathfrak{p}}(r)$ in §§10.2.1-10.2.2. We also abuse notation and write:

$$\mathsf{CD}(r)^{\wedge}=\mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r))$$

when we identify the group $G=\mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r))$ with a groupoid with a single object $pt\in\mathtt{Ob}\,\mathsf{CD}(r)^{\wedge}$. We also deal with a unitary version of the chord diagram operad $\mathsf{CD}^{\wedge}_{+}$, which we associate to the unitary extension $\mathsf{A}^{\wedge}_{+}$ of the operad in complete Hopf groupoids $\mathsf{A}^{\wedge}$, and which has $\mathsf{CD}^{\wedge}_{+}(0)=pt$ as term of arity zero.

Proposition 8.1.5 (see also Lemma 9.1.20) gives a one-to-one correspondence between the elements of the group $u\in\mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r))$ and the exponentials $e^p\in\hat{\mathbb{U}}\hat{\mathfrak{p}}(r)$ of the elements of the complete Drinfeld–Kohno Lie algebras $p\in\hat{\mathfrak{p}}(r)$, $r>0$. We moreover have the formula:

$$e^p\circ_k e^q=e^{p\circ_k q}$$

for the operadic composition operation $\circ_k:\mathsf{CD}(m)^{\wedge}\times\mathsf{CD}(n)^{\wedge}\rightarrow\mathsf{CD}(m+n-1)^{\wedge}$, for every $p\in\hat{\mathfrak{p}}(m)$ and $q\in\hat{\mathfrak{p}}(n)$. We just use that $p\in\hat{\mathfrak{p}}(m)$ and $q\in\hat{\mathfrak{p}}(n)$ define commuting elements in $\hat{\mathbb{U}}\hat{\mathfrak{p}}(m) \otimes \hat{\mathbb{U}}\hat{\mathfrak{p}}(n) \simeq \hat{\mathbb{U}}(\hat{\mathfrak{p}}(m) \oplus \hat{\mathbb{U}}\hat{\mathfrak{p}}(n))$ to get the identity $e^p \otimes e^q = e^{(p,0)+(0,q)} = e^{(p,q)}$ in the set of group-like elements of this enveloping algebra $\hat{\mathbb{U}}(\hat{\mathfrak{p}}(m) \oplus \hat{\mathfrak{p}}(n))$, and we use, in turn, that the operadic composition operations of the chord diagram operad are induced by the corresponding operadic composition operations of the Drinfeld–Kohno Lie algebra operad. We have similar obvious identities for the action of permutations and for the restriction operators. In what follows, we use both the expression of the chord diagram operad in terms of the group of group-like elements $\mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r))$ and this expression in terms of the group of exponentials e^p, $p \in \mathfrak{p}(r)$.

Let us observe that we have a group morphism

$$\rho : \Bbbk^{\times} \to \mathtt{Aut}_{\hat{f}\,\mathcal{G}rd\,\mathcal{O}p}(\mathit{CD}_{+}^{\widehat{\ }})$$

which maps any scalar $\lambda \in \Bbbk^{\times}$ to the automorphism of the unitary operad of chord diagrams $\rho_{\lambda} : \mathit{CD}_{+}^{\widehat{\ }} \to \mathit{CD}_{+}^{\widehat{\ }}$ induced by the map $\rho_{\lambda} : t_{ij} \mapsto \lambda t_{ij}$ on the Drinfeld–Kohno Lie algebras $\mathfrak{p}(r)$. Equivalently, for a group-like element $g \in \mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r))$ represented by a power series $g = g(t_{ij}, 1 \leq i \neq j \leq r)$ on the generators t_{ij}, $1 \leq i \neq j \leq r$, of the chord diagram algebra $\hat{\mathbb{U}}\hat{\mathfrak{p}}(r) = \hat{\mathbb{T}}(t_{ij}, 1 \leq i \neq j \leq r)/\langle [t_{ij}, t_{kl}], [t_{ij}, t_{ik} + t_{jk}] \rangle$, we set

$$\rho_{\lambda}(g) = g(\lambda t_{ij}, 1 \leq i \neq j \leq r),$$

where the expression on the right-hand side represents the result of the substitution operation $\rho_{\lambda} : t_{ij} \mapsto \lambda t_{ij}$ in this power series.

10.2.5. *The definition of operad morphisms from parenthesized braids to chord diagrams.* We aim to establish an operadic counterpart of the result of Theorem 10.0.7 and to define a categorical equivalence of unitary operads in Malcev complete groupoids

$$\phi : \mathit{PaB}_{+}^{\widehat{\ }} \to \mathit{CD}_{+}^{\widehat{\ }}$$

which links the Malcev completion of the operad of parenthesized braids $\mathit{PaB}_{+}^{\widehat{\ }}$ to the operad of chord diagrams $\mathit{CD}_{+}^{\widehat{\ }}$.

By Proposition 9.1.19, any such morphism of unitary operads in the category of Malcev complete groupoids $\phi : \mathit{PaB}_{+}^{\widehat{\ }} \to \mathit{CD}_{+}^{\widehat{\ }}$ occurs as the extension, to the completed operad $\mathit{PaB}_{+}^{\widehat{\ }}$, of a morphism of unitary operads in groupoids $\phi : \mathit{PaB}_{+} \to \mathit{CD}_{+}^{\widehat{\ }}$, where we consider the ordinary operad of parenthesized braids PaB_{+} and we forget about the Malcev complete groupoid structure of the operad of chord diagrams. By Theorem 6.2.4, the construction of such a morphism reduces, in turn, to the definition of a product operation in the chord diagram operad $m = m(x_1, x_2) \in \mathtt{Ob}\,\mathit{CD}(2)^{\wedge}$, of an associativity isomorphism $a = a(x_1, x_2, x_3) \in \mathtt{Mor}_{\mathit{CD}(3)^{\wedge}}(m(m(x_1, x_2), x_3), m(x_1, m(x_2, x_3)))$, and of a braiding $c = c(x_1, x_2) \in \mathtt{Mor}_{\mathit{CD}(2)^{\wedge}}(m(x_1, x_2), m(x_2, x_1))$, which respectively represent the image of the multiplication operation of the parenthesized braid operad $\mu \in \mathtt{Ob}\,\mathit{PaB}(2)$, of the associativity isomorphism $\alpha \in \mathtt{Mor}_{\mathit{PaB}(3)^{\wedge}}((x_1 x_2) x_3, x_1(x_2 x_3))$, and of the braiding $\tau \in \mathtt{Mor}_{\mathit{CD}(2)^{\wedge}}(x_1 x_2, x_2 x_1)$ under our morphism $\phi : \mathit{PaB}_{+} \to \mathit{CD}_{+}^{\widehat{\ }}$.

In the general statement of Theorem 6.2.4, we also consider a unit object $e \in \mathit{Q}(0)$, which occurs in an extension of our construction to operads Q with an arbitrary arity zero term. In our setting, and more generally when we deal with a unitary operad $\mathit{Q} = \mathit{P}_{+}$, this unit object is fixed by the assumption that the arity zero term of our operad reduces to the one-point set $\mathit{P}_{+}(0) = pt$. By definition of the chord diagram operad, we moreover have $\mathtt{Ob}\,\mathit{CD}(r)^{\wedge} = pt$ for all $r > 0$, so that the choice of the product operation $m = \phi(\mu) \in \mathtt{Ob}\,\mathit{CD}(2)^{\wedge}$ is trivial too.

Thus, we only have to specify the associativity isomorphism $a = \phi(\alpha)$ and the braiding $c = \phi(\tau)$ in order to determine our morphism $\phi : \mathsf{PaB}_+ \to \mathsf{CD}_+^{\widehat{}}$. If we make these isomorphisms explicit, then we get the following statement:

Proposition 10.2.6. *A morphism of unitary operads $\phi : \mathsf{PaB}_+ \to \mathsf{CD}_+^{\widehat{}}$ is uniquely determined by a scalar parameter $\kappa \in \Bbbk$ and a group-like element of the complete tensor algebra on two generators $f(\xi_1, \xi_2) \in \mathbb{G}\,\hat{\mathbb{T}}(\xi_1, \xi_2)$ such that we have the following assignments in the morphism sets of the chord diagram operad $\mathsf{CD}_+^{\widehat{}}$:*

$$\phi(\tau) = e^{\kappa t_{12}/2}, \qquad \phi(\alpha) = f(t_{12}, t_{23}),$$

where τ and α respectively denote the braiding and the associativity isomorphism of the parenthesized braid operad PaB_+.

Explanations. We examine the structure of the morphism sets of the operad $\mathsf{CD}_+^{\widehat{}}$ in arity $r = 2, 3$ to determine the form of the braiding $c = c(x_1, x_2)$ and of the associativity isomorphism $a = a(x_1, x_2, x_3)$ which determine our map $\phi : \mathsf{PaB}_+ \to \mathsf{CD}_+^{\widehat{}}$.

In arity 2, we have $\hat{\mathfrak{p}}(2) = \Bbbk t_{12} \Rightarrow \mathtt{Mor}_{\mathsf{CD}(2)^{\widehat{}}}(pt, pt) = \exp(\Bbbk t_{12})$ and our braiding $c = c(x_1, x_2) \in \mathtt{Mor}_{\mathsf{CD}(2)^{\widehat{}}}(pt, pt)$ is therefore given by an expression of the form:

$$c(x_1, x_2) = \exp(\kappa t_{12}/2) \in \mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(2)),$$

for some parameter $\kappa \in \Bbbk$.

In arity 3, we can see that the Lie algebra $\hat{\mathfrak{p}}(3)$ splits as a direct sum $\hat{\mathfrak{p}}(3) = \Bbbk z \oplus \hat{\mathbb{L}}(t_{12}, t_{23})$, where $\Bbbk z$ is a central Lie subalgebra, spanned by the element $z = t_{12} + t_{23} + t_{13}$, and we consider the free complete Lie algebra generated by the elements $t_{12}, t_{23} \in \hat{\mathfrak{p}}(3)$. We therefore have $a(x_1, x_2, x_3) = e^{cz} \cdot e^{p(t_{12}, t_{23})}$, where $c \in \Bbbk$ and $p(t_{12}, t_{23}) \in \hat{\mathbb{L}}(t_{12}, t_{23})$. We also have $\hat{\mathbb{U}}\hat{\mathbb{L}}(t_{12}, t_{23}) = \hat{\mathbb{T}}(t_{12}, t_{23})$, and therefore, we may also write $a(x_1, x_2, x_3) = e^{cz} \cdot f(t_{12}, t_{23})$, where $f(t_{12}, t_{23}) = e^{p(t_{12}, t_{23})}$ represents a group-like element of the complete Hopf algebra $\hat{\mathbb{T}}(t_{12}, t_{23})$.

The unit constraint $a(x_1, e, x_3) = id$ of Theorem 6.2.4 is equivalent to the equation $e^{c \partial_2(z)} \cdot \partial_2 f(t_{12}, t_{23}) = 1$, where we consider the restriction operator $\partial_2 : \hat{\mathfrak{p}}(3) \to \hat{\mathfrak{p}}(2)$. By definition of this restriction operator in the Drinfeld–Kohno Lie algebra operad, we have $\partial_2(t_{12}) = \partial_2(t_{23}) = 0$ while $\partial_2(t_{13}) = t_{12}$ (see §10.1.2). The unit constraint $a(x_1, e, x_3) = id$ therefore reduces to the equation $e^{c t_{12}} = 1$, from which we deduce that the central factor e^{cz} in our expression of the associativity isomorphism $a = a(x_1, x_2, x_3)$ is trivial. Hence, we eventually obtain that this isomorphism is given by an expression of the form:

$$a(x_1, x_2, x_3) = f(t_{12}, t_{23}) \in \mathbb{G}\,\hat{\mathbb{T}}(t_{12}, t_{23}),$$

for some group-like element of the complete tensor algebra on two generators $f(\xi_1, \xi_2) \in \mathbb{G}\,\hat{\mathbb{T}}(\xi_1, \xi_2)$. □

To complete this result, we write down the coherence constraints of Theorem 6.2.4 in terms of this pair $(\kappa, f(\xi_1, \xi_2))$ which we associate to our operad morphism $\phi : \mathsf{PaB}_+ \to \mathsf{CD}_+^{\widehat{}}$. We obtain the following proposition:

Proposition 10.2.7. *The assignments of Proposition* 11.1.3*:*

$$\phi(\tau) = e^{\kappa t_{12}/2}, \qquad \phi(\alpha) = f(t_{12}, t_{23}),$$

where $\kappa \in \Bbbk$ and $f(\xi_1, \xi_2) \in \mathbb{G}\,\hat{\mathbb{T}}(\xi_1, \xi_2)$, determine a well-defined morphism of unitary operads $\phi : \mathsf{PaB}_+ \to \mathsf{CD}_+^{\widehat{}}$ if and only if the power series $f(\xi_1, \xi_2)$ satisfies:

(1) *the unit relations* $f(\xi_1, 0) = 1 = f(0, \xi_2)$,
(2) *the involution relation* $f(\xi_1, \xi_2) \cdot f(\xi_2, \xi_1) = 1$,
(3) *the hexagon equation* $e^{\kappa\xi_1/2} \cdot f(\xi_3, \xi_1) \cdot e^{\kappa\xi_3/2} \cdot f(\xi_2, \xi_3) \cdot e^{\kappa\xi_2/2} \cdot f(\xi_1, \xi_2) = 1$, *where* (ξ_1, ξ_2, ξ_3) *denotes a triple of variables such that* $\xi_1 + \xi_2 + \xi_3 = 0$,
(4) *and the pentagon equation* $f(t_{12}, t_{23}+t_{24}) \cdot f(t_{13}+t_{23}, t_{34}) = f(t_{23}, t_{34}) \cdot f(t_{12}+t_{13}, t_{24}+t_{34}) \cdot f(t_{12}, t_{23})$ *in the group* $\hat{P}_4 = \mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(4))$.

Proof. We go back to the definition of the composition structure of the chord diagram operad (see §10.2.4) in order to make explicit the unit, the pentagon and the hexagon constraint of Theorem 6.2.4(b-c) for the braiding isomorphism $c = e^{\kappa t_{12}/2}$ and for the associativity isomorphism $a = f(t_{12}, t_{23})$ given in the proposition.

We use the expression of the restriction operators on $\mathfrak{p}(3)$ to get the unit relations $f(t_{12}, 0) = 1 = f(0, t_{12})$ in $\mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(2)) = \mathbb{G}\,\hat{\mathbb{T}}(t_{12})$. The equivalence between the pentagon constraint, such as expressed by the commutation of the diagram of Figure 6.1, and the equation of the proposition is also immediate (we just expand the expression of the pentagon equation).

Recall that we have $f(\xi_1, \xi_2) = e^{p(\xi_1,\xi_2)}$ for an element of the free complete Lie algebra on two generators $p(\xi_1, \xi_2) \in \hat{\mathbb{L}}(\xi_1, \xi_2)$. We also have $f(\xi_1, 0) = 1 = f(0, \xi_2) \Leftrightarrow p(\xi_1, 0) = 0 = p(0, \xi_2)$, and this relation is equivalent to the requirement that this Lie power series $p(\xi_1, \xi_2) \in \hat{\mathbb{L}}(\xi_1, \xi_2) = \prod_{m\geq 1} \mathbb{L}(\xi_1, \xi_2)$ has no component in weight $m = 1$. We use this observation in our next argument lines.

The hexagon relations have the following expression:

$$e^{\kappa t_{13}/2} \cdot f(t_{12}, t_{13}) \cdot e^{\kappa t_{12}/2} = f(t_{23}, t_{13}) \cdot e^{\kappa(t_{12}+t_{13})/2} \cdot f(t_{12}, t_{23}),$$
$$e^{\kappa t_{13}/2} \cdot f(t_{13}, t_{23})^{-1} \cdot e^{\kappa t_{23}/2} = f(t_{13}, t_{12})^{-1} \cdot e^{\kappa(t_{13}+t_{23})/2} \cdot f(t_{12}, t_{23})^{-1}.$$

We set $\xi_1 = t_{13}$, $\xi_2 = t_{23}$ in the first equation. We then have $t_{12} = \xi_3 + z$, where we set $\xi_3 = -\xi_1 - \xi_2$ and $z = t_{12}+t_{23}+t_{13}$ denotes the central element of $\mathfrak{p}(3)$ considered in §10.2.5. We use the vanishing relation $[z, -] = 0$ for this central element and that the power series $p(-,-)$ such that $f(-,-) = e^{p(-,-)}$ has no component in weight 1 to obtain that we have the identities $p(\xi_3 + z, -) = p(\xi_3, -) \Rightarrow f(\xi_3 + z, -) = f(\xi_3, -)$ and $p(-, \xi_3 + z) = p(-, \xi_3) \Rightarrow f(-, \xi_3 + z) = f(-, \xi_3)$ in the complete Hopf algebra $\hat{\mathbb{U}}\hat{\mathfrak{p}}(3)$. We can also collect the factors $e^{\kappa z/2}$ in our equation. We eventually get that our first hexagon relation is equivalent to the following equations:

$$e^{\kappa\xi_1/2} \cdot f(\xi_3, \xi_1) \cdot e^{\kappa\xi_3/2} = f(\xi_2, \xi_1) \cdot e^{-\kappa\xi_2/2} \cdot f(\xi_3, \xi_2)$$
$$\Leftrightarrow f(\xi_2, \xi_1) = e^{\kappa\xi_1/2} \cdot f(\xi_3, \xi_1) \cdot e^{\kappa\xi_3/2} \cdot f(\xi_3, \xi_2)^{-1} \cdot e^{\kappa\xi_2/2}.$$

We similarly get that our second hexagon relation, where we set $\xi_1 = t_{12}$ and $\xi_2 = t_{23}$, so that $t_{13} = \xi_3 + z$ for $\xi_3 = -\xi_1 - \xi_2$, is equivalent to the equations:

$$e^{\kappa\xi_3/2} \cdot f(\xi_3, \xi_2)^{-1} \cdot e^{\kappa\xi_2/2} = f(\xi_3, \xi_1)^{-1} \cdot e^{-\kappa\xi_1/2} \cdot f(\xi_1, \xi_2)^{-1}$$
$$\Leftrightarrow f(\xi_1, \xi_2)^{-1} = e^{\kappa\xi_1/2} \cdot f(\xi_3, \xi_1) \cdot e^{\kappa\xi_3/2} \cdot f(\xi_3, \xi_2)^{-1} \cdot e^{\kappa\xi_2/2}.$$

These equations are clearly equivalent to the combination of the identities given in our statement:

$$f(\xi_2, \xi_1) = f(\xi_1, \xi_2)^{-1},$$
$$e^{\kappa\xi_1/2} \cdot f(\xi_3, \xi_1) \cdot e^{\kappa\xi_3/2} \cdot f(\xi_2, \xi_3) \cdot e^{\kappa\xi_2/2} \cdot f(\xi_1, \xi_2) = 1,$$

and this result completes the verification of our assertions. □

We easily see that the scalar parameter $\kappa \in \Bbbk$ which we associate to our morphism of unitary operads $\phi : \mathsf{PaB}_+ \to \mathsf{CD}_+^{\widehat{}}$ in Proposition 10.2.6 is necessarily invertible when we assume that this morphism extends to a categorical equivalence on the Malcev completion of the parenthesized braid operad $\mathsf{PaB}_+^{\widehat{}}$. To check this claim, we use that the morphism of Malcev complete groups $\phi : \mathtt{Aut}_{\mathsf{PaB}(2)^{\wedge}}(\mu) \to \mathtt{Aut}_{\mathsf{CD}(2)^{\wedge}}(pt)$ induced by this morphism $\phi : \mathsf{PaB}_+^{\widehat{}} \to \mathsf{CD}_+^{\widehat{}}$ is identified with the map $\phi : \tau^{2\nu} \mapsto e^{\kappa\nu t_{12}}$ when we use the identities $\mathtt{Aut}_{\mathsf{PaB}(2)^{\wedge}}(\mu) = \hat{P}_2 = \{\tau^{2\nu}, \nu \in \Bbbk\}$ and $\mathtt{Aut}_{\mathsf{CD}(2)^{\wedge}}(pt) = \exp \hat{\mathfrak{p}}(2) = \exp(\Bbbk t_{12})$. We prove in the next proposition that this condition $\kappa \in \Bbbk^{\times}$ actually suffices to ensure that our morphism $\phi : \mathsf{PaB}_+^{\widehat{}} \to \mathsf{CD}_+^{\widehat{}}$ defines a categorical equivalence:

Proposition 10.2.8. *The morphism of unitary operads in Malcev complete groupoids $\phi : \mathsf{PaB}_+^{\widehat{}} \to \mathsf{CD}_+^{\widehat{}}$ which we determine by the assignments of Proposition* 10.2.6 *defines a categorical equivalence of unitary operads in Malcev complete groupoids if and only if the scalar parameter which we associate to this morphism in our correspondence is invertible $\kappa \in \Bbbk^{\times}$.*

Proof. We only examine the "if" part of the proposition since we already checked the "only if" part. We therefore assume $\kappa \in \Bbbk^{\times}$. We fix an object $p \in \mathtt{Ob}\,\mathsf{PaB}(r)$ in the operad of parenthesized braids PaB, for some $r > 0$. We aim to check that the morphism $\phi : \mathsf{PaB}_+^{\widehat{}} \to \mathsf{CD}_+^{\widehat{}}$, which we determine by our assignments $\phi(\tau) = e^{\kappa t_{12}}$, $\phi(\alpha) = f(t_{12}, t_{23})$, induces an isomorphism of Malcev complete groups from the automorphism group of this object in the Malcev completion of the parenthesized braid operad $\mathtt{Aut}_{\mathsf{PaB}(r)^{\wedge}}(p) = \mathtt{Aut}_{\mathsf{PaB}(r)}(p)^{\wedge}$ to the group $\mathsf{CD}(r)^{\wedge} = \mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r))$ which represents the automorphism group of the object $pt \in \mathtt{Ob}\,\mathsf{CD}(r)^{\wedge}$ in the chord diagram operad.

We use the identity $\mathtt{Aut}_{\mathsf{PaB}(r)}(p) = P_r \Rightarrow \mathtt{Aut}_{\mathsf{PaB}(r)^{\wedge}}(p) = \hat{P}_r$, which follows from the definition of the operad $\mathsf{PaB}^{\widehat{}}$, and the isomorphism

$$\upsilon : \mathfrak{p}(r) \xrightarrow{\simeq} \mathrm{E}^0\,\hat{P}_r \tag{1}$$

between the Drinfeld–Kohno Lie algebra $\mathfrak{p}(r)$ and the weight graded Lie algebra associated to the Malcev completion of the pure braid group P_r (see Theorem 10.0.7). Recall that this isomorphism associates the element $\bar{x}_{ij} \in \mathrm{E}^0\,\hat{P}_r$ to the generator t_{ij} of the Drinfeld–Kohno Lie algebra. By Proposition 8.1.5, we also have an isomorphism

$$\mathfrak{p}(r) = \mathrm{E}^0\,\hat{\mathfrak{p}}(r) \xrightarrow{\simeq} \mathrm{E}^0\,\mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r)), \tag{2}$$

which is induced by the exponential correspondence between the complete Lie algebra $\hat{\mathfrak{p}}(r) = \mathbb{P}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r))$ and the group of group-like elements of the complete Hopf algebra $\hat{\mathbb{U}}\hat{\mathfrak{p}}(r)$. We check that the morphism of weight graded Lie algebras $\mathrm{E}^0\,\phi : \mathrm{E}^0\,\mathtt{Aut}_{\mathsf{PaB}(r)^{\wedge}}(p) \to \mathrm{E}^0\,\mathtt{Aut}_{\mathsf{CD}(r)^{\wedge}}(pt)$ induced by our operad morphism $\phi : \mathsf{PaB}_+^{\widehat{}} \to \mathsf{CD}_+^{\widehat{}}$ is an isomorphism to get our result.

For this aim, we use that the element of the automorphism group $\mathtt{Aut}_{\mathsf{PaB}(r)^{\wedge}}(p)$ which corresponds to the generator x_{ij} of the pure braid group P_r can be expressed as a composite morphism:

$$u_{ij} = \beta \cdot \pi(x_1, \dots, \tau^2(x_i, x_j), \dots, \widehat{x_j}, \dots, x_r) \cdot \beta^{-1}, \tag{3}$$

where β is given by a composite of braidings and associativity isomorphisms in PaB, while $\pi(x_1, \dots, \tau^2(x_i, x_j), \dots, \widehat{x_j}, \dots, x_r)$ represents the morphism of $\mathsf{PaB}(r)$

obtained by plugging the morphism $\tau^2 \in \mathtt{Mor}_{\mathit{PaB}(2)}(\mu,\mu)$ in a parenthesized word on $r-1$ variables $\pi \in \Omega(r-1) = \mathtt{Ob}\,\mathit{PaB}(r-1)$. We have

$$\phi(u_{ij}) = \phi(\beta)\cdot e^{\kappa t_{ij}}\cdot\phi(\beta)^{-1}, \tag{4}$$

and we use the general relation $g\cdot u\cdot g^{-1} \equiv u (\mathrm{mod}\,\mathtt{F}_2\,G)$, valid in any Malcev complete group (see §8.2.2 and Proposition 8.2.3), to deduce from this formula that we have the identity $\phi(u_{ij}) \equiv e^{\kappa t_{ij}}$ in $\mathtt{E}^0\,\mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r))$. The exponential element $e^{\kappa t_{ij}}$ corresponds to κt_{ij} in $\hat{\mathfrak{p}}(r)$. Hence, we finally obtain that the map of weight graded Lie algebras $\mathtt{E}^0\,\phi : \mathtt{E}^0\,\mathtt{Aut}_{\mathit{PaB}(r)^{\wedge}}(p) \to \mathtt{E}^0\,\mathtt{Aut}_{\mathit{CD}(r)^{\wedge}}(pt)$ induced by our operad morphism $\phi : \mathit{PaB}^{\wedge}_{+} \to \mathit{CD}^{\wedge}_{+}$ carries the element $\bar{u}_{ij} \in \mathtt{E}^0\,\mathtt{Aut}_{\mathit{PaB}(r)^{\wedge}}(p)$, which corresponds to the class of the pure braid group generator x_{ij} in $\mathtt{E}^0\,\hat{P}_r$, to the multiple κt_{ij} of the generator t_{ij} in the Drinfeld–Kohno Lie algebra $\mathfrak{p}(r)$. Thus, we just retrieve the inverse map of the above isomorphism (1), up to the scalar factor $\kappa \in \Bbbk^{\times}$. We conclude from this relation that our map defines an isomorphism:

$$\mathtt{E}^0\,\phi : \mathtt{E}^0\,\mathtt{Aut}_{\mathit{PaB}(r)^{\wedge}}(p) \xrightarrow{\simeq} \mathtt{E}^0\,\mathtt{Aut}_{\mathit{CD}(r)^{\wedge}}(pt)$$

as requested. □

We eventually get the following theorem:

Theorem 10.2.9 (Equivalence between the operadic definition and Drinfeld's original definition of associators [57, §5]). *The correspondence of Proposition* 10.2.6 *induces a bijection between the categorical equivalences of unitary operads in Malcev complete groupoids $\phi : \mathit{PaB}^{\wedge}_{+} \to \mathit{CD}^{\wedge}_{+}$ and the set of pairs $(\kappa, f(\xi_1,\xi_2))$, where κ is an invertible scalar parameter $\kappa \in \Bbbk^{\times}$, as we require in Proposition* 10.2.8*, and $f(\xi_1,\xi_2)$ is a group-like element of the complete tensor algebra on two generators $f(\xi_1,\xi_2) \in \mathbb{G}\,\hat{\mathbb{T}}(\xi_1,\xi_2)$ which satisfies the unit, involution, hexagon and pentagon relations (*1-4*) of Proposition* 10.2.7*.* □

In [57, §5], the set of Drinfeld's associators associated to the parameter $\lambda \in \Bbbk^{\times}$ is precisely defined as the set of group-like power series $f(\xi_1,\xi_2) \in \mathbb{G}\,\hat{\mathbb{T}}(\xi_1,\xi_2)$ which satisfy these constraints (1-4) of Proposition 10.2.7. The above theorem therefore gives an equivalence between this definition and our approach in terms of categorical equivalences of operads in Malcev complete groupoids $\phi : \mathit{PaB}^{\wedge} \to \mathit{CD}^{\wedge}$. In the rest of this section, we go back to Drinfeld's definition in order to check the existence of associators in the complex coefficient case $\Bbbk = \mathbb{C}$.

10.2.10. *Remark.* Let again $p(\xi_1,\xi_2) = a\xi_1 + b\xi_2 + c[\xi_1,\xi_2] + \cdots \in \hat{\mathbb{L}}(\xi_1,\xi_2)$ be the Lie power series associated to our group-like element $f(\xi_1,\xi_2) = e^{p(\xi_1,\xi_2)}$ in Proposition 10.2.7. In the proof of this statement, we observed that the unit relations $f(\xi_1,0) = 1 = f(0,\xi_2) \Leftrightarrow e^{p(\xi_1,0)} = 1 = e^{p(0,\xi_2)}$ imply that this Lie power series $p(\xi_1,\xi_2)$ has no term in weight one, and hence that we have $a = b = 0$ in the above expansion. By a theorem of Furusho (see [70]), any element $f(\xi_1,\xi_2) \in \mathbb{G}\,\hat{\mathbb{T}}(\xi_1,\xi_2)$ that satisfies the pentagon equation (4) of Proposition 10.2.7 also satisfies the hexagon equation (3) for a parameter $\kappa \in \Bbbk$ determined by the coefficient $c \in \Bbbk$ of our Lie power series.

10.2.11. *The set of Drinfeld's associators.* We adopt the notation $\mathit{Ass}^{\kappa}(\Bbbk)$ for the set of group-like power series $f(\xi_1,\xi_2) \in \mathbb{G}\,\hat{\mathbb{T}}(\xi_1,\xi_2)$ which satisfy the unit, involution, hexagon and pentagon constraints (1-4) of Proposition 10.2.7, for a fixed value of the scalar parameter $\kappa \in \Bbbk$. We refer to this set $\mathit{Ass}^{\kappa}(\Bbbk)$ as the set of *Drinfeld's associators* associated to the parameter $\kappa \in \Bbbk$. We use the study of

Proposition 10.2.6 to identify such a power series with an associativity isomorphism in the chord diagram operad $a(x_1, x_2, x_3) = f(t_{12}, t_{23})$, where we also consider the braiding isomorphism $c(x_1, x_2) = \exp(\kappa t_{12}/2)$ determined by our parameter $\kappa \in \Bbbk$. Recall that we use this correspondence in Proposition 10.2.6-10.2.7 in order to define a bijection between the morphisms of unitary operads in Malcev complete groupoids $\phi : \mathit{PaB}_+^{\wedge} \to \mathit{CD}_+^{\wedge}$ and the pairs $(\kappa, f(\xi_1, \xi_2))$ such that $f(\xi_1, \xi_2) \in \mathit{Ass}^{\kappa}(\Bbbk)$.

We also use the notation

$$\mathit{Ass}(\Bbbk) = \{(\kappa, f(\xi_1, \xi_2)) \in \Bbbk^{\times} \times \mathbb{G}\,\hat{\mathbb{T}}(\xi_1, \xi_2) | f(\xi_1, \xi_2) \in \mathit{Ass}^{\kappa}(\Bbbk)\},$$

for the union of the sets of associators associated an invertible value of this scalar parameter $\kappa \in \Bbbk^{\times}$. We are mainly interested in this case, as we explain in Theorem 10.2.9, since the result of Proposition 10.2.8 implies that the correspondence of Proposition 10.2.6-10.2.7 induces a bijection between the categorical equivalences of unitary operads in Malcev complete groupoids $\phi : \mathit{PaB}_+^{\wedge} \xrightarrow{\sim} \mathit{CD}_+^{\wedge}$ and the elements of this set $\mathit{Ass}(\Bbbk)$.

Let $\lambda \in \Bbbk^{\times}$. We immediately see that the re-scaling operation $\rho_{\lambda} : f(\xi_1, \xi_2) \mapsto f(\lambda\xi_1, \lambda\xi_2)$ carries a group-like power series $f(\xi_1, \xi_2) \in \mathbb{G}\,\hat{\mathbb{T}}(\xi_1, \xi_2)$ which satisfies the equations of an associator in Proposition 10.2.7, for some value of the scalar parameter $\kappa \in \Bbbk^{\times}$, to a group-like power series $\rho_{\lambda} f(\xi_1, \xi_2) = f(\lambda\xi_1, \lambda\xi_2) \in \mathbb{G}\,\hat{\mathbb{T}}(\xi_1, \xi_2)$ which satisfies the equations of an associator for the scalar parameter such that $\kappa' = \lambda\kappa \in \Bbbk^{\times}$. We therefore have bijections:

$$\rho_{\lambda} : \mathit{Ass}^{\kappa}(\Bbbk) \xrightarrow{\simeq} \mathit{Ass}^{\lambda\kappa}(\Bbbk),$$

that relate the sets of associators associated to different invertible scalar values $\kappa \in \Bbbk^{\times}$.

We observed in §10.2.4 that the (unitary) operad of chord diagrams $\mathit{CD}_+^{\wedge}$ inherits an action of the multiplicative group $\Bbbk^{\times}$. We may actually see that the operation $(\rho_{\lambda})_* : \phi \mapsto \rho_{\lambda} \circ \phi$, which we obtain by composing a categorical equivalence of unitary operads in Malcev complete groupoids $\phi : \mathit{PaB}_+^{\wedge} \to \mathit{CD}_+^{\wedge}$ with the automorphism of the chord diagram operad $\rho_{\lambda} : \mathit{CD}_+^{\wedge} \to \mathit{CD}_+^{\wedge}$ associated to a scalar $\lambda \in \Bbbk^{\times}$, corresponds to the map $\rho_{\lambda} : \mathit{Ass}^{\kappa}(\Bbbk) \to \mathit{Ass}^{\lambda\kappa}(\Bbbk)$ given by our re-scaling operation $\rho_{\lambda} : f(\xi_1, \xi_2) \mapsto f(\lambda\xi_1, \lambda\xi_2)$ on the sets of associators $\mathit{Ass}^{\kappa}(\Bbbk)$, $\kappa \in \Bbbk^{\times}$.

We aim to check that these sets $\mathit{Ass}^{\kappa}(\Bbbk)$ are not empty in order to prove the existence of categorical equivalences of unitary operads in Malcev complete groupoids which connect the Malcev completion of the parenthesized braid operad to the operad of chord diagrams. We have the following first result:

THEOREM 10.2.12 (V. Drinfeld [57, §2]). *We have a group-like power series with complex coefficients $f_{KZ}(\xi_1, \xi_2) \in \mathbb{G}\,\hat{\mathbb{T}}(\xi_1, \xi_2)$ which satisfies the constraints (*1-4*) of Proposition* 10.2.7*, for any value $\kappa \neq 0$ of the scalar parameter $\kappa \in \mathbb{C}^{\times}$, and which therefore defines an element of the set of associators $\mathit{Ass}^{\kappa}(\mathbb{C})$ when we take $\Bbbk = \mathbb{C}$ as ground field.*

PROOF (OUTLINE). We adapt the construction of the isomorphism of Theorem 10.0.7, where we use the monodromy of the Knizhnik–Zamolodchikov connection in order to define an isomorphism from the Malcev completion of the pure braid group $\hat{P}_r$ towards the group of group-like elements of the enveloping algebra of the complete Drinfeld–Kohno Lie algebra $\mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r))$, for each $r > 0$. In short, the group-like element $f_{KZ}(\xi_1, \xi_2) \in \mathbb{G}\,\hat{\mathbb{T}}(\xi_1, \xi_2)$ can be defined by using the holonomy of the Knizhnik–Zamolodchikov connection along paths that connect certain asymptotic zones of the space of configurations of 3 points in the plane $\mathsf{F}(\mathbb{C}, 3)$. (We just have to discard logarithmic divergences, which appear in our asymptotic expansions, in order to get a well-defined element in our algebra of chord diagrams.) We can also define this group-like element $f_{KZ}(\xi_1, \xi_2) \in \mathbb{G}\,\hat{\mathbb{T}}(\xi_1, \xi_2)$ directly by using the monodromy of solutions of the differential system associated to the Knizhnik–Zamolodchikov connection. We adopt this approach, which we borrow from [57, §2], and we closely follow the explanations of this reference.

The purpose of our account is just to give an overview of the main steps of the definition of this associator $f_{KZ}(\xi_1, \xi_2)$. Besides Drinfeld's original article [57, §2], we also refer to the books [40, 42, 100, 160] for a detailed account of this construction.

We fix some $r \geq 2$. We consider the system of differential equations

$$\frac{\partial w}{\partial z_i} = \sum_{i \neq j} \hbar \cdot t_{ij} \cdot \frac{w}{z_i - z_j}, \tag{1}$$

for a parameter $\hbar \in \mathbb{C}$, and where w is a function defined on a subdomain of the configuration space $\mathsf{F}(\mathbb{C}, r)$ and with values in the algebra $\hat{\mathbb{U}}\hat{\mathfrak{p}}(r)$.

We see that the equations (1) are invariant under the action of the group of affine transformations $z \mapsto az + b$, where $a \in \mathbb{C}^{\times}$, $b \in \mathbb{C}$, and as a consequence, any solution of this system (1) is determined by a solution of a system depending on $r - 2$ variables (we refer to [160, §12.2] for a nice and detailed analysis of this dependence). When we take $r = 3$, we obtain

$$w(z_1, z_2, z_3) = (z_3 - z_1)^{\hbar(t_{12} + t_{23} + t_{13})} \cdot G\big(\frac{z_2 - z_1}{z_3 - z_1}\big), \tag{2}$$

where $G(z)$ is a solution of the differential equation

$$G'(z) = \hbar \cdot \big(\frac{\xi_1}{z} + \frac{\xi_2}{z - 1}\big) \cdot G(z) \tag{3}$$

in the completed tensor algebra $\hat{\mathbb{T}}(\xi_1, \xi_2) = \hat{\mathbb{T}}(t_{12}, t_{23})$.

Let $C = \{z = x + iy | y \neq 0 \text{ or } 0 < x < 1\}$. The classical theory of Fuchsian equations (see for instance [178, §4.3]) implies that this differential equation (3) has a unique analytic solution $G_0(z)$, defined for $z \in C$, and such that $G_0(z) \sim_{z \to 0} z^{\hbar t_{12}}$. We also have an analytic solution $G_1(z)$ such that $G_1(z) \sim_{z \to 1} (1 - z)^{\hbar t_{12}}$. The solutions w_0 and w_1 of the Knizhnik–Zamolodchikov system (1) associated to these functions are determined by asymptotic behaviors of the form:

$$w_0(z_1, z_2, z_3) \sim (z_2 - z_1)^{\hbar t_{12}} (z_3 - z_1)^{\hbar(t_{13} + t_{23})}, \quad \text{for } |z_2 - z_1| \ll |z_3 - z_1|, \tag{4}$$

$$w_1(z_1, z_2, z_3) \sim (z_3 - z_2)^{\hbar t_{23}} (z_3 - z_1)^{\hbar(t_{12} + t_{13})}, \quad \text{for } |z_3 - z_2| \ll |z_3 - z_1|. \tag{5}$$

The solutions $G_0(z)$ and $G_1(z)$ of equation (3) differ by a constant factor of the variable z. We precisely take this factor, such that $G_1(z) = G_0(z) \cdot f_{KZ}(\xi_1, \xi_2)$, to define our group-like element $f_{KZ}(\xi_1, \xi_2) \in \mathbb{G}\,\hat{\mathbb{T}}(\xi_1, \xi_2)$. This element $f_{KZ}(\xi_1, \xi_2) \in \mathbb{G}\,\hat{\mathbb{T}}(\xi_1, \xi_2)$ can equivalently be characterized by the relation

$$w_1(z_1, z_2, z_3) = w_0(z_1, z_2, z_3) \cdot f_{KZ}(t_{12}, t_{23}), \tag{6}$$

where we consider the solutions of the Knizhnik–Zamolodchikov system (1) with the asymptotic behavior prescribed in (4-5).

We aim to check that our element $f_{KZ}(\xi_1, \xi_2)$ satisfies the constraints of Proposition 10.2.7, for a given scalar parameter $\kappa \in \mathbb{C}$ such that $\hbar = \kappa/2i\pi$. We give a brief survey of the argument lines of [57, §2] to establish this claim. We immediately see that the reduction $t_{12} = t_{13} = 0$ makes our functions w_0 and w_1 equal. We have a similar result when we perform the reduction $t_{13} = t_{23} = 0$. We deduce from these observations that our power series $f_{KZ}(\xi_1, \xi_2)$ satisfies the unit relation $f_{KZ}(\xi_1, 0) = f_{KZ}(0, \xi_2) = 1$. The involution relation $f_{KZ}(\xi_1, \xi_2) \cdot f_{KZ}(\xi_2, \xi_1) = 1$ follows from an easy inspection too. To establish the pentagon equation, we consider asymptotic zones

$$\begin{aligned}
&x_2 - x_1 \ll x_3 - x_1 \ll x_4 - x_1,\\
&x_3 - x_2 \ll x_3 - x_1 \ll x_4 - x_1,\\
&x_2 - x_1 \ll x_4 - x_1 \quad \text{and} \quad x_4 - x_3 \ll x_4 - x_1,\\
&x_3 - x_2 \ll x_4 - x_2 \ll x_4 - x_1,\\
&x_4 - x_3 \ll x_4 - x_2 \ll x_4 - x_1,
\end{aligned} \tag{7}$$

in the range of variation formed by real variables such that $\{x_1 < x_2 < x_3 < x_4\}$. Each of these zone is associated to a vertex of the Mac Lane pentagon (see Figure 6.1), with the rule that we have $x_j - x_i \ll x_l - x_k$ whenever we can retrieve the pattern $x_k \cdots (x_i \cdots x_j) \cdots x_l$ after removing some groupings in the parenthesized word corresponding to our vertex. For instance, we associate the zone such that $x_3 - x_2 \ll x_3 - x_1 \ll x_4 - x_1$ to the word $(x_1(x_2x_3))x_4$. For $r = 4$, the Knizhnik–Zamolodchikov system (1) admits solutions $w_i = w_i(x_1, x_2, x_3, x_4)$, $i = 1, \ldots, 5$, with an asymptotic behavior of the form:

$$\begin{aligned}
&w_1 \sim (x_2 - x_1)^{\hbar t_{12}} \cdot (x_3 - x_1)^{\hbar(t_{13}+t_{23})} \cdot (x_4 - x_1)^{\hbar(t_{14}+t_{24}+t_{34})},\\
&w_2 \sim (x_3 - x_2)^{\hbar t_{23}} \cdot (x_3 - x_1)^{\hbar(t_{12}+t_{13})} \cdot (x_4 - x_1)^{\hbar(t_{14}+t_{24}+t_{34})},\\
&w_3 \sim (x_2 - x_1)^{\hbar t_{12}} \cdot (x_4 - x_3)^{\hbar t_{34}} \cdot (x_4 - x_1)^{\hbar(t_{13}+t_{23}+t_{14}+t_{24})},\\
&w_4 \sim (x_3 - x_2)^{\hbar t_{23}} \cdot (x_4 - x_2)^{\hbar(t_{24}+t_{34})} \cdot (x_4 - x_1)^{\hbar(t_{12}+t_{13}+t_{14})},\\
&w_5 \sim (x_4 - x_3)^{\hbar t_{34}} \cdot (x_4 - x_2)^{\hbar(t_{23}+t_{24})} \cdot (x_4 - x_1)^{\hbar(t_{12}+t_{13}+t_{14})}
\end{aligned} \tag{8}$$

in the zones (7).

The factors $(x_l - x_k)$ which we consider in these asymptotic expansions correspond to the groupings of variables $(x_k \cdots x_l)$ which occur in the parenthesized words associated to the asymptotic zones. The exponent of the factor $(x_l - x_k)$ is the sum $\sum_{ij} t_{ij}$ which ranges over the pairs $i < j$ such that the variables (x_i, x_j) belong to separate groupings $(x_k \cdots x_i \cdots)(\cdots x_j \cdots x_l)$ in the parenthesization of the word $(x_k \ldots x_l)$. The notation $\sim$ asserts that the function w_i, $i = 1, \ldots, 5$, differs from the given expansion by a function $\phi(u, v)$ which depends analytically on the factors $(x_l - x_k)/(x_4 - x_1)$ in our asymptotic zone. The existence of such functions follows from the theory of differential equations (we refer to [60] for the detailed argument). One can prove identities

$$\begin{aligned}
w_2 = w_1 \cdot f_{KZ}(t_{12}, t_{23}), \quad & w_4 = w_2 \cdot f_{KZ}(t_{12} + t_{13}, t_{24} + t_{34}),\\
w_5 = w_4 \cdot f_{KZ}(t_{23}, t_{34}), \quad & w_3 = w_1 \cdot f_{KZ}(t_{13} + t_{23}, t_{34}),\\
& \text{and} \quad w_5 = w_3 \cdot f_{KZ}(t_{13} + t_{23}, t_{34})
\end{aligned} \tag{9}$$

by checking that regularized forms of these functions (where we multiply by some asymptotic factors to eliminate divergences) satisfy the same differential equations, and agree at some initial value of the cyclically ordered quadruple $x_1 < \cdots < x_4$ on the projective line $\mathbb{RP}^1$ (see [57, §2], or [160, §12.4] for a detailed account of this proof). By combining these identities, we get the pentagon equation $f_{KZ}(t_{12}, t_{23} + t_{24}) \cdot f_{KZ}(t_{13} + t_{23}, t_{34}) = f_{KZ}(t_{23}, t_{34}) \cdot f_{KZ}(t_{12} + t_{13}, t_{24} + t_{34}) \cdot f_{KZ}(t_{12}, t_{23})$ of Proposition 10.2.7(4).

The hexagon equation of Proposition 10.2.7(3) is established by the same line of arguments (see for instance [160, §12.4] for full details on this verification), and this verification completes the definition of the Knizhnik-Zamolodchikov associator. ☐

In [175], Tamarkin uses the existence of Drinfeld's associators to prove that the chain operad of little 2-discs is formal as an operad in chain complexes. In the second volume, we will explain that we can upgrade this construction in the context of rational homotopy theory and use the existence of this associator to prove that the operad of little 2-discs is formal in the sense of our rational homotopy theory of operads (see §II.14.2). In [108], Kontsevich gives another proof of the formality of the little discs operads, which works for all dimensions $n \geq 2$. His approach relies on a commutative dg-algebra, formed by a complex of graphs, which defines a model in the sense of rational homotopy theory of the cohomology of the operad of little n-discs. There is another known explicit construction of associator, given in articles of Alekseev-Torossian [6] and Ševera-Willwacher [159], which is related to this proof of the formality of the little discs operads. This associator, usually called the Alekseev-Torossian associator in the literature, is defined by the monodromy of a connection with values in a homotopy Lie algebra (an L_∞-algebra) which is defined by a subcomplex of connected graphs inside the graph complex considered by Kontsevich. We refer to [159] for details on this construction.

The construction outlined in this proof returns a complex solution of the associator existence problem. To be specific, in the case $\kappa = 2i\pi$ (which corresponds to the case $\hbar = 1$ of the construction), one can see that the coefficients of the monomials $\xi_1^{k_1-1}\xi_2\xi_1^{k_2-1}\xi_2 \cdots \xi_1^{k_r-1}\xi_2$ such that $k_1 \geq 2$ in the expansion of $f_{KZ}(\xi_1, \xi_2)$ are given by the value of the multizeta series $\zeta(k_1, \dots, k_r) = \sum_{n_1 > \cdots > n_r > 0} 1/(n_1^{k_1} \cdot \dots \cdot n_r^{k_r})$ (up to a sign). The coefficients of the other monomials can be determined from these multizeta values by a regularization procedure (see [115]). If we fix $\kappa = 1$, then we get a multiple of these numbers by the factor $1/(2i\pi)^{k_1+\cdots+k_r}$. In particular, for $\kappa = 1$, the coefficient of the term $\xi_1^2\xi_2$ in the Knizhnik–Zamolodchikov associator is given by the number $c = \zeta(3)/(2i\pi)^3$ (see [57, §2]). The Alekseev-Torossian associator is defined over the reals, but has non-rational coefficients too.

Nevertheless, we are going to see that the set of associators $Ass(\Bbbk)$ is not empty for every characteristic zero ground field $\Bbbk$. In [57, Proposition 5.1-5.2], a first proof of this rationality statement is given by using that the existence of associators $f \in Ass(\Bbbk)$ is equivalent to the surjectivity of a character map $\lambda : GT(\Bbbk) \to \Bbbk^\times$ on the pro-unipotent version of the Grothendieck–Teichmüller group $GT(\Bbbk)$ which we study in the next chapter. The paper [57] gives another proof of the existence of associators by using the structure of an auxiliary group, the graded Grothendieck–Teichmüller group $GRT(\Bbbk)$, which operates simply and transitively on $Ass(\Bbbk)$. The idea is to consider a tower decomposition of the set of associators $Ass(\Bbbk) = \lim_m Ass_{\langle m \rangle}(\Bbbk)$ and a parallel tower decomposition of the graded Grothendieck–Teichmüller group $\mathit{GRT}(\Bbbk) = \lim_m \mathit{GRT}_{\langle m\rangle}(\Bbbk)$ such that each $\mathit{GRT}_{\langle m\rangle}(\Bbbk)$ forms an (affine) algebraic group acting on $\mathit{Ass}_{\langle m\rangle}(\Bbbk)$.

We are more precisely going to see that each set $\mathit{Ass}_{\langle m\rangle}(\Bbbk)$ forms a torsor under the action of this algebraic group $\mathit{GRT}_{\langle m\rangle}(\Bbbk)$. We will check that each morphism $p_m : \mathit{GRT}_{\langle m\rangle}(\Bbbk) \to \mathit{GRT}_{\langle m-1\rangle}(\Bbbk)$ in the tower decomposition of the graded Grothendieck–Teichmüller group $\mathit{GRT}(\Bbbk)$ is surjective. We can then use the existence of the (complex) Knizhnik–Zamolodchikov associator $f_{KZ}(\xi_1,\xi_2) \in \mathit{Ass}(\mathbb{C})$ in the limit of the tower $\mathit{Ass}(\mathbb{C}) = \lim_m \mathit{Ass}_{\langle m\rangle}(\mathbb{C})$ and the algebraic structure of our objects $\mathit{Ass}_{\langle m\rangle}(\Bbbk)$ to establish that each morphism $p_m : \mathit{Ass}_{\langle m\rangle}(\Bbbk) \to \mathit{Ass}_{\langle m-1\rangle}(\Bbbk)$ in the tower decomposition of the set of associators $\mathit{Ass}(\Bbbk)$ is surjective too, for any choice of characteristic zero ground field $\Bbbk$.

We actually use the natural tower decomposition $\mathsf{CD}^{\wedge}_{+} = \lim_m q_m\,\mathsf{CD}^{\wedge}_{+}$ (which comes from the Malcev complete groupoid structure of the components of the chord diagram operad $\mathsf{CD}^{\wedge}_{+}$) to define the tower decomposition of the set of associators $\mathit{Ass}(\Bbbk)$. The object $\mathit{Ass}_{\langle m\rangle}(\Bbbk)$ explicitly consists of morphisms $\phi : \mathsf{PaB}_{+} \to q_m\,\mathsf{CD}^{\wedge}_{+}$ which induce a categorical equivalence from the quotient $q_m\,\mathsf{PaB}^{\wedge}_{+}$ of the Malcev completion of the parenthesized braid operad $\mathsf{PaB}^{\wedge}_{+}$ to $q_m\,\mathsf{CD}^{\wedge}_{+}$. The tower decomposition of the graded Grothendieck–Teichmüller group $\mathit{GRT}(\Bbbk) = \lim_m \mathit{GRT}_{\langle m\rangle}(\Bbbk)$ is defined analogously. The surjectivity of the morphisms $p_m : \mathit{Ass}_{\langle m\rangle}(\Bbbk) \to \mathit{Ass}_{\langle m-1\rangle}(\Bbbk)$ implies that we can construct our desired operad morphism $\phi : \mathsf{PaB}_{+} \to \mathsf{CD}^{\wedge}_{+}$, which defines a categorical equivalence when we pass to the Malcev completion of the parenthesized braid operad, by solving a sequence of lifting problems in the category of unitary operads in groupoids:

$$\begin{tikzcd}
\vdots & \vdots \arrow[d] \\
 & q_m\,\mathsf{CD}^{\wedge}_{+} \arrow[d] \\
 & q_{m-1}\,\mathsf{CD}^{\wedge}_{+} \arrow[d] \\
\vdots & \vdots \arrow[d] \\
\mathsf{PaB}_{+} \arrow[r] \arrow[uuur, dotted, "\exists?"] \arrow[uur, dotted, "\exists?"] & q_1\,\mathsf{CD}^{\wedge}_{+}
\end{tikzcd}.$$

We explain the definition of the graded Grothendieck–Teichmüller group in the next section. We go back to the construction of associators and we study these tower decompositions of the set of associators and of the graded Grothendieck–Teichmüller group afterwards.

10.3. The graded Grothendieck–Teichmüller group

The graded Grothendieck–Teichmüller group $\mathit{GRT}(\Bbbk)$ was introduced by Drinfeld in [57] and the purpose of this section is to revisit Drinfeld's original definition from an operadic viewpoint, as we did for the definition of associators in the previous section. In short, we define the graded Grothendieck–Teichmüller group $\mathit{GRT}(\Bbbk)$ as a group of operad automorphisms associated to a parenthesized version $\mathsf{PaCD}^{\wedge}_{+}$ of the operad of chord diagrams $\mathsf{CD}^{\wedge}_{+}$. We first explain the definition of this auxiliary operad $\mathsf{PaCD}^{\wedge}_{+}$ which represents the image of the chord diagram operad $\mathsf{CD}^{\wedge}_{+}$ under a pullback operation in the category of operads in Malcev complete groupoids. We deal with unitary operads when we address the definition of the graded Grothendieck–Teichmüller group $GRT(\Bbbk)$. But, as usual, we explain the definition of a non-unitary operad $\mathsf{PaCD}^{\wedge}$ underlying this unitary operad $\mathsf{PaCD}^{\wedge}_{+}$ first, and we check that this non-unitary operad $\mathsf{PaCD}^{\wedge}$ admits a unitary extension afterwards.

In the concluding paragraph of the previous section, we briefly mentioned that $GRT(\Bbbk)$ occurs as the limit of a tower of affine algebraic groups $GRT_{\langle m\rangle}(\Bbbk)$, $m \geq 0$. We say that $GRT(\Bbbk)$ forms a *pro-algebraic group* to express this feature. We can use this pro-algebraic group structure to associate a Lie algebra $\mathfrak{grt}$ to $GRT(\Bbbk)$. We make the definition of this Lie algebra explicit in the next chapter. We will see that this Lie algebra $\mathfrak{grt}$, which we associate to the pro-algebraic group $GRT(\Bbbk)$, admits a decomposition $\mathfrak{grt} = \prod_m \mathfrak{grt}_m$ such that $[\mathfrak{grt}_m, \mathfrak{grt}_n] \subset \mathfrak{grt}_{m+n}$ and is identified with the completion of a Lie algebra in the category of weight graded modules (a weight graded Lie algebra). We go back to this subject in the next chapter. We will see at this moment that the homogeneous components of our Lie algebra $\mathfrak{grt}_m$ are isomorphic to the subquotients $\mathtt{E}^0_m\, GT(\Bbbk) = \mathtt{F}_m\, GT(\Bbbk)/\, \mathtt{F}_{m+1}\, GT(\Bbbk)$ of a pro-unipotent version of the Grothendieck–Teichmüller group $GT(\Bbbk)$. (We have an analogous result for the group $GRT(\Bbbk)$.) This relationship motivates the name 'graded Grothendieck–Teichmüller group' which we give to the group $GRT(\Bbbk)$.

In §6, we mentioned that Bar-Natan uses the name 'parenthesized braid' and the notation PaB_r for objects that differ from the groupoids $\mathsf{PaB}(r)$ in our definition of the operad of parenthesized braids PaB. He actually deals with homogeneous components of the free algebra on one variable associated to this operad (in the linear context). Bar-Natan also considers categories of parenthesized chord diagrams in his work [16]. These categories, which Bar-Natan denotes by PaCD_r, also differ from our categories of parenthesized chord diagrams $\mathsf{PaCD}(r)^{\wedge}$ and consist, as in the case of parenthesized braids, of the homogeneous components of the free algebra on one variable which we associate to the operad $\mathsf{PaCD}^{\wedge}$.

10.3.1. *The operad in complete Hopf groupoids underlying the parenthesized chord diagram operad.* We follow the same overall plan as in our study of the operad of chord diagrams. Recall that we use the notation $\mathsf{A}^{\wedge}$ for the operad in complete Hopf groupoids which underlies the operad of chord diagrams $\mathsf{CD}^{\wedge}$. We define an operad in complete Hopf groupoids $\mathsf{PaA}^{\wedge}$ which underlies our operad of parenthesized chord diagrams $\mathsf{PaCD}^{\wedge}$ first. In §10.2.3, we explain that the operad $\mathsf{A}^{\wedge}$ can also be defined as the completion of an operad in ordinary Hopf groupoids A such that $\mathtt{Ob}\,\mathsf{A}(r) = pt$ and $\mathtt{Hom}_{\mathsf{A}(r)}(pt,pt) = \mathbb{U}\,\mathfrak{p}(r)$, for each $r > 0$. The operad $\mathsf{PaA}^{\wedge}$ is similarly identified with the completion of an operad in ordinary Hopf groupoids which we denote by PaA. But we prefer to define both operads PaA and $\mathsf{PaA}^{\wedge}$ in parallel as pullbacks of the operads in (complete) Hopf groupoids A and $\mathsf{A}^{\wedge}$ which we consider in the definition of the chord diagram operad $\mathsf{CD}^{\wedge}$. For this aim, we use a straightforward extension, to operads in (complete) Hopf groupoids, of the pullback operation of §6.1.5.

To be explicit, we consider the obvious morphism of operads in sets $\omega : \Omega \to \mathtt{Ob}\,\mathsf{A}$, which is defined by the constant map $\Omega(r) \to \mathtt{Ob}\,\mathsf{A}(r) = pt$ in each arity $r > 0$, and which sends the generating element of the magma operad $\mu \in \Omega(2)$ to the unique arity two object $m = \phi(\mu)$ of the operad A. Recall that we have $\mathtt{Ob}\,\mathsf{A}^{\wedge} = \mathtt{Ob}\,\mathsf{A}$ by definition of the completion. We then set

$$\mathsf{PaA} = \omega^*\,\mathsf{A} \quad \text{and} \quad \mathsf{PaA}^{\wedge} = \omega^*\,\mathsf{A}^{\wedge},$$

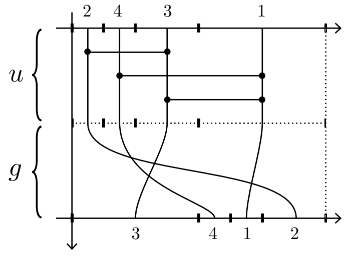


FIGURE 10.1. The picture of a homomorphism in the operad in Hopf groupoids underlying the parenthesized chord diagram operad.

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where $\omega^* \mathit{A}$ (respectively, $\omega^* \mathit{A}\widehat{\ }$) denotes the pullback of the operad A (respectively, $\omega^* \mathit{A}\widehat{\ }$) along the morphism $\omega : \Omega \to \mathtt{Ob}\, \mathit{A}$ (see §6.1.5).

Thus, each Hopf groupoid $\mathit{PaA}(r)$ satisfies $\mathtt{Ob}\, \mathit{PaA}(r) = \Omega(r)$ by construction, and we have the identity of hom-objects

$$\mathtt{Hom}_{\mathit{PaA}(r)}(p, q) = \mathtt{Hom}_{\mathit{A}(r)}(pt, pt) = \mathbb{U}\, \mathfrak{p}(r)$$

for each pair $p, q \in \Omega(r)$. The symmetric group actions, the unit, and the composition operations that define the operad structure of this collection of Hopf groupoids are, as usual, inherited from the magma operad at the object set level and from the corresponding structure operations of the operad A at the hom-object level (see §6.1.5). We have similar identities in the complete setting. We also readily check that the pullback operation commutes with the completion. We can therefore identify the operad $\mathit{PaA}\widehat{\ }$ with the aritywise completion of the operad in plain groupoids PaA, as alluded to at the beginning of this paragraph. We therefore also use the notation $\mathit{PaA}(r)\widehat{\ } = \mathit{PaA}\widehat{\ }(r)$ for the components of this operad $\mathit{PaA}\widehat{\ }$.

We can combine the chord diagram picture of §10.0.8 and the conventions of §6.2.1 to get a graphical representation of the homomorphisms of the operad PaA. We basically use that each homomorphism $f \in \mathtt{Hom}_{\mathit{PaA}(r)}(p, q)$ has a canonical decomposition $f = g \cdot u$ such that $g \in \mathtt{Hom}_{\mathit{PaA}(r)}(p, q)$ is represented by the unit element 1 in the Hopf algebra $\mathbb{U}\, \mathfrak{p}(r)$, and u is an endomorphism of the object $p \in \Omega(r)$ which captures the element of the enveloping algebra corresponding to f when we use the identity $\mathtt{Hom}_{\mathit{PaA}(r)}(p, q) = \mathbb{U}\, \mathfrak{p}(r)$. We represent this factor $u \in \mathtt{Hom}_{\mathit{PaA}(r)}(p, p) = \mathbb{U}\, \mathfrak{p}(r)$ by a chord diagram on r strands arranged on the centers of the diadic decomposition of the interval corresponding to our element $p \in \Omega(r)$. We identify the factor $g \in \mathtt{Hom}_{\mathit{PaA}(r)}(p, q)$ of our morphism with a correspondence, marked by lines in our figure, between the centers of the diadic decompositions associated to p and q. We use a similar picture in the case of the completed operad $\mathit{PaA}\widehat{\ }$. Figure 10.1 gives an instance of this representation for a homomorphism $f = g \cdot u \in \mathtt{Hom}_{\mathit{PaA}(r)}(((x_2 x_4) x_3) x_1, x_3((x_4 x_1) x_2))$.

The morphism $\omega : \Omega \to \mathtt{Ob}\, \mathit{A}$ trivially admits a unitary extension $\omega : \Omega_+ \to \mathtt{Ob}\, \mathit{A}_+$ and we can perform a unitary extension of our pullback construction (see §6.1.5) to get a unitary extension PaA_+ of the operad PaA such that $\mathit{PaA}_+ = \omega^* \mathit{A}_+$. The restriction operators $\partial_k : \mathit{PaA}(r) \to \mathit{PaA}(r-1)$, which reflect the composition operations with the extra arity zero term $\mathit{PaA}_+(0) = pt$ of this unitary operad PaA_+, are given by the restriction operators of the magma operad $\partial_k : \Omega(r) \to \Omega(r-1)$ at the object set level (see §6.1.4 for the explicit definition of these restriction operators) and by a straightforward generalization of the strand removal operation of the algebras of chord diagrams at the hom-object level. We have a similar result in the complete setting.

10.3.2. *The operad of parenthesized chord diagrams.* We define the operad of parenthesized chord diagrams $\mathsf{PaCD}^{\wedge}$ as the operad in Malcev complete groupoids which we obtain by applying the group-like element functor to the operad in complete Hopf groupoids of the previous paragraph:

$$\mathsf{PaCD}^{\wedge} = \mathbb{G}(\mathsf{PaA}^{\wedge}).$$

We accordingly have $\mathtt{Ob}\,\mathsf{PaCD}^{\wedge} = \mathtt{Ob}\,\mathsf{PaA}^{\wedge} = \Omega$. In what follows, we also use the notation $\mathsf{PaCD}(r)^{\wedge} = \mathsf{PaCD}^{\wedge}(r)$ for the components of this operad $\mathsf{PaCD}^{\wedge}$.

Recall that we have $\mathtt{Ob}\,\mathsf{CD}(r)^{\wedge} = \mathtt{Ob}\,\mathsf{A}(r)^{\wedge} = pt$ by definition of the chord diagram operad. We can also trivially identify the operad $\mathsf{PaCD}^{\wedge}$ with the pullback of the chord diagram operad along the same obvious morphism of operads in sets $\omega : \Omega \to \mathtt{Ob}\,\mathsf{CD}^{\wedge}$ as in the construction of the operad $\mathsf{PaA}^{\wedge}$:

$$\mathsf{PaCD}^{\wedge} = \omega^*\,\mathsf{CD}^{\wedge}.$$

We accordingly have $\mathtt{Mor}_{\mathsf{PaCD}(r)^{\wedge}}(p,q) = \mathtt{Mor}_{\mathsf{CD}(r)^{\wedge}}(pt,pt) = \mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r))$, for all $p,q \in \Omega(r)$. Furthermore, we can explicitly determine the symmetric group actions, the unit, and the composition operations which define the operad structure of this collection of groupoids $\mathsf{PaCD}(r)^{\wedge}$ by the corresponding structure operations of the magma operad at the object set level and by the corresponding structure operations of the enveloping algebras $\hat{\mathbb{U}}\hat{\mathfrak{p}}(r)$ at the morphism set level.

We also have an obvious unitary version of the parenthesized chord diagram operad such that $\mathsf{PaCD}_+^{\wedge} = \mathbb{G}(\mathsf{PaA}_+^{\wedge})$ and $\mathsf{PaCD}_+^{\wedge} = \omega^*\,\mathsf{CD}_+^{\wedge}$. The restriction operators $\partial_k : \mathsf{PaCD}(r)^{\wedge} \to \mathsf{PaCD}(r-1)^{\wedge}$ which reflect the composition operations with the extra arity zero term $\mathsf{PaCD}_+^{\wedge}(0) = pt$ in this unitary operad $\mathsf{PaCD}_+^{\wedge}$ are, as in the case of the operad PaA, given by the restriction operators of the magma operad $\partial_k : \Omega(r) \to \Omega(r-1)$ at the object set level and by a straightforward generalization of the strand removal operation of the algebras of chord diagrams at the morphism set level.

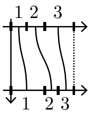
Recall that we use the notation PaS for the operad of parenthesized symmetries which is formed by pulling back the operad of commutative monoids $\Gamma(r) = pt$ (regarded as an operad in discrete groupoids) along the morphism $\omega : \Omega \to \Gamma$. Thus, the operad of parenthesized symmetries PaS is an operad in groupoids with $\mathtt{Ob}\,\mathsf{PaS} = \Omega$ as object set operad, and with $\mathtt{Mor}_{\mathsf{PaS}(r)}(p,q) = pt$ as morphism sets, for all $p,q \in \mathtt{Ob}\,\mathsf{PaS}(r)$, $r > 0$. We have an obvious embedding of operads in groupoids $\sigma : \mathsf{PaS} \hookrightarrow \mathsf{PaCD}^{\wedge}$ which identifies the operad PaS with the suboperad of the operad of parenthesized chord diagrams $\mathsf{PaCD}^{\wedge}$ which has the same objects as this operad in each arity $r > 0$, but whose morphisms reduce to the factors $g \in \mathtt{Mor}_{\mathsf{PaCD}(r)^{\wedge}}(p,q)$ that correspond to the unit element of the enveloping algebras $\hat{\mathbb{U}}\hat{\mathfrak{p}}(r)$ in the decomposition of the morphisms of the parenthesized chord diagram operad $f = g \cdot u$ (see Figure 10.1).

We have a morphism in the converse direction $\rho : \mathsf{PaCD}^{\wedge} \to \mathsf{PaS}$, which carries the factor $u \in \mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r))$ of our morphisms $f = g \cdot u$ to an identity morphism. We trivially have $\rho\sigma = id$ so that the operad of parenthesized symmetries actually forms a retract of the operad of parenthesized chord diagrams. We have an obvious extension of these morphisms and of this retract relation in the unitary setting.

10.3.3. *The associativity isomorphism, the symmetry isomorphism, and the infinitesimal braiding in the Hopf groupoids of parenthesized chord diagrams.* In the previous section, we use that the completed operad of parenthesized braids $\mathsf{PaB}^{\wedge}$ arises as the completion of an operad in plain groupoids PaB to define morphisms on this operad $\phi : \mathsf{PaB}^{\wedge} \to \mathsf{Q}$. In the case of the operad of parenthesized chord diagrams $\mathsf{PaCD}^{\wedge}$, we rather define our morphisms at the level of the operad in complete Hopf groupoids $\mathsf{PaA}^{\wedge}$ which determines $\mathsf{PaCD}^{\wedge}$.

The hom-objects of this operad $\mathsf{PaA}^{\wedge}$ inherit an associativity homomorphism (whose representation is the same as in the parenthesized braid operad case):

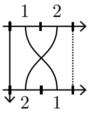
(1) $\alpha =$



$\in \mathtt{Hom}_{\mathsf{PaA}(3)^{\wedge}}((x_1x_2)x_3, x_1(x_2x_3)),$

a symmetry homomorphism:

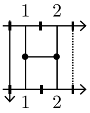
(2) $\tau =$



$\in \mathtt{Hom}_{\mathsf{PaA}(2)^{\wedge}}(x_1x_2, x_2x_1),$

and the following homomorphism:

(3) $\theta =$



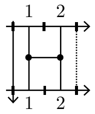
$\in \mathtt{Hom}_{\mathsf{PaA}(2)^{\wedge}}(x_1x_2, x_1x_2),$

which we call the *infinitesimal braiding.* Note that the associativity and the symmetry homomorphisms (1-2) define group-like elements in the hom-objects of the operad $\mathsf{PaA}^{\wedge}$, while the infinitesimal braiding (3) is primitive. Hence, we can identify the associativity and symmetry homomorphisms with morphisms of the operad in groupoids $\mathsf{PaCD}^{\wedge} = \mathbb{G}(\mathsf{PaA}^{\wedge})$, while we have to take the exponential of the infinitesimal braiding in order to get an element of the morphism set $\mathsf{PaCD}(2)^{\wedge}$.

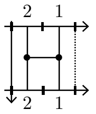
The above associativity and symmetry homomorphisms (1-2) obviously represent the image of the associativity and symmetry isomorphisms of the operad of parenthesized symmetries PaS (see §6.3) in the operad of parenthesized chord diagrams $\mathsf{PaCD}^{\wedge} = \mathbb{G}(\mathsf{PaA}^{\wedge})$. Hence, these homomorphisms satisfy the same relations as the associativity and symmetry isomorphisms of the operad of parenthesized symmetries. In particular, we have the involution relation $\tau(x_1, x_2)\tau(x_2, x_1) = id$ in the operad $\mathsf{PaCD}^{\wedge}$. Recall that the other relations of the associativity and symmetry isomorphisms of the parenthesized symmetry operad reduce to the pentagon relation (see Figure 6.1 for the corresponding diagram) and to the hexagon relations (see Figure 6.6 for the corresponding diagrams). Recall also that the hexagon relations are equivalent to each other when our symmetry isomorphism satisfies the involution relation $\tau(x_1, x_2)\tau(x_2, x_1) = id$.

We now examine the relations satisfied by the infinitesimal braiding (3). We readily see that we have the invariance relation:

(4)



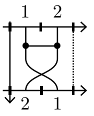
$$= \tau(x_1, x_2)^{-1} \cdot$$



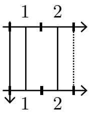
$$\cdot \tau(x_1, x_2),$$

where we consider the image of the infinitesimal braiding under the action of the transposition $t \in \Sigma_2$ on $PaA(2)\hat{}$, and we have the following composition formula:

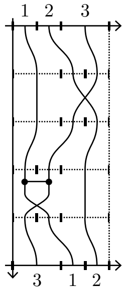
(5)



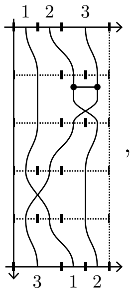
$$\circ_1$$



$$=$$



$$+$$

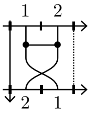


$$,$$

when we consider the composite of the symmetry isomorphism (2) with the infinitesimal braiding (3) in $PaA(2)\hat{}$:

(6)

$$\tau\theta =$$



$$\in \mathtt{Hom}_{PaA(2)\hat{}}(x_1 x_2, x_2 x_1).$$

We refer to the latter equation (5) as the *semi-classical hexagon relation.* We immediately see that the invariance equation reads $\theta(x_1, x_2) = \tau(x_1, x_2)^{-1} \cdot \theta(x_2, x_1) \cdot \tau(x_1, x_2)$, where we use the notation $\theta(x_2, x_1)$ for the image of the infinitesimal braiding $\theta = \theta(x_1, x_2)$ under the action of the transposition $t\theta = \theta(x_2, x_1)$, while the semi-classical hexagon relation reads (we omit the input variables of associativity isomorphisms to simplify the expression of our formula):

$$\begin{aligned} \tau\theta(\mu(x_1, x_2), x_3)) &= \alpha \cdot \mu(\tau\theta(x_1, x_3), x_2) \cdot \alpha^{-1} \cdot \mu(x_1, \tau(x_2, x_3)) \cdot \alpha \\ &\quad + \alpha \cdot \mu(\tau(x_1, x_3), x_2) \cdot \alpha^{-1} \cdot \mu(x_1, \tau\theta(x_2, x_3)) \cdot \alpha. \end{aligned} \tag{7}$$

We still have the relations $\alpha(*, x_1, x_2) = \alpha(x_1, *, x_2) = \alpha(x_1, x_2, *) = id_{\mu(x_1,x_2)}$ and $\tau(*, x_1) = \tau(x_1, *) = id_{x_1}$ (which we can also express by the restriction formulas $\partial_1\alpha = \partial_2\alpha = \partial_3\alpha = id_\mu$ and $\partial_1\tau = \partial_2\tau = id_1$) when we deal with the unitary extension of the parenthesized chord diagram operad $PaCD_+\hat{}$. We also have the identities

$$\theta(*, x_1) = \theta(x_1, *) = 0 \Leftrightarrow \partial_1\theta = \partial_2\theta = 0$$

for the infinitesimal braiding (3).

We have the following analogue of the result of Theorem 6.2.4:

THEOREM 10.3.4 (Bar-Natan [16, Proposition 4.5, Proposition 4.8]). *Let R be an operad in complete Hopf groupoids. Let $Q = \mathbb{G}(R)$ be the operad in complete Malcev groupoids associated to this operad.*

(a) *Let*

$$m = m(x_1, x_2) \in \mathsf{Ob}\, R(2),$$

be an object in the component of arity 2 *of this operad. In what follows, we also set*

$$m(x_1, x_2) = x_1 \square x_2$$

and we use classical algebraic notation (rather than operadic notation) to represent the composites of this object in our operad R*. Let*

$$a = a(x_1, x_2, x_3) \in \mathbb{G}\big(\mathtt{Hom}_{\mathsf{R}(3)}(((x_1 \square x_2) \square x_3), (x_1 \square (x_2 \square x_3)))\big)$$

be a group-like associativity homomorphism which connects the operadic composites

$$\begin{aligned} &(m \circ_1 m)(x_1, x_2, x_3) = (x_1 \square x_2) \square x_3 \in \mathtt{Ob}\, \mathsf{R}(3) \\ \textit{and} \quad &(m \circ_2 m)(x_1, x_2, x_3) = x_1 \square (x_2 \square x_3) \in \mathtt{Ob}\, \mathsf{R}(3) \end{aligned}$$

in the enriched category $\mathsf{R}(3)$*. Let*

$$c = c(x_1, x_2) \in \mathbb{G}\big(\mathtt{Hom}_{\mathsf{R}(2)}(x_1 \square x_2, x_2 \square x_1)\big)$$

be a group-like symmetry homomorphism which connects the object $m(x_1, x_2) = x_1 \square x_2 \in \mathtt{Ob}\, \mathsf{R}(2)$ *to its transposite* $(1\ 2)m(x_1, x_2) = x_2 \square x_1 \in \mathtt{Ob}\, \mathsf{R}(2)$*, and which satisfies the involution relation* $c(x_1, x_2)c(x_2, x_1) = id$*, where* $(1\ 2)c = c(x_2, x_1)$ *represents the image of this homomorphism* $c = c(x_1, x_2)$ *under the action of the transposition* $(1\ 2) \in \Sigma_2$ *on* $\mathsf{R}(2)$*.*

If these homomorphisms $a = a(x_1, x_2, x_3)$ *and* $c = c(x_1, x_2)$ *make the pentagon diagram of Figure* 6.1 *commute in the enriched category* $\mathsf{R}(4)$ *as well as (any one of) the hexagon diagrams of Figure* 6.6 *in* $\mathsf{R}(3)$*, and if we moreover have a primitive homomorphism*

$$h(x_1, x_2) \in \mathbb{P}\big(\mathtt{Hom}_{\mathsf{R}(2)}(x_1 \square x_2, x_2 \square x_1)\big)$$

which satisfies the invariance relation $h(x_1, x_2) = c(x_1, x_2)^{-1} \cdot h(x_2, x_1) \cdot c(x_1, x_2)$ *in* $\mathsf{R}(2)$ *and the semi-classical hexagon relation*

$$\begin{aligned} &c(x_1 \square x_2, x_3) \cdot h(x_1 \square x_2, x_3) \\ &\qquad = a(x_3, x_1, x_2) \cdot (c(x_1, x_3) \square x_2) \cdot (h(x_1, x_3) \square x_2) \cdot a(x_1, x_3, x_2)^{-1} \\ &\qquad\quad \cdot (x_1 \square c(x_2, x_3)) \cdot a(x_1, x_2, x_3) \\ &\qquad + a(x_3, x_1, x_2) \cdot (c(x_1, x_3) \square x_2) \cdot a(x_1, x_3, x_2)^{-1} \\ &\qquad\quad \cdot (x_1 \square c(x_2, x_3)) \cdot (x_1 \square h(x_2, x_3)) \cdot a(x_1, x_2, x_3) \end{aligned}$$

in $\mathsf{R}(3)$ *(where we again use algebraic notation, as in Figure* 6.1*-*6.6*, for the operadic composition of homomorphisms in* R*), then we have a morphism of operads in complete Hopf groupoids* $\phi : \mathsf{PaA}^{\wedge} \to \mathsf{R}$*, uniquely determined by the assignments*

$$\phi(\mu) = x_1 \square x_2, \quad \phi(\alpha) = a(x_1, x_2, x_3), \quad \phi(\tau) = c(x_1, x_2), \quad \phi(\theta) = h(x_1, x_2)$$

in the operad R*, and which, in turn, induces a morphism*

$$\phi : \mathsf{PaCD}^{\wedge} \to \mathsf{Q}$$

in the category of operads in Malcev complete groupoids.

(b) *In the construction of (*a*), if we moreover assume the existence of an object*

$$e \in \mathtt{Ob}\, \mathsf{R}(0)$$

which satisfies the relation $e \mathbin{\square} x_1 = x_1 = x_1 \mathbin{\square} e$ *at the object set level (with the same notation conventions as above), together with the identities*

$$\begin{aligned} &a(e, x_1, x_2) = a(x_1, e, x_2) = a(x_1, x_2, e) = id_{m(x_1,x_2)},\\ &c(e, x_1) = c(x_1, e) = id_{x_1},\\ &h(e, x_1) = h(x_1, e) = 0 \end{aligned}$$

at the hom-object level, then the morphism $\phi : \mathsf{PaA}^{\wedge} \to \mathsf{R}$ *has a unitary extension* $\phi_+ : \mathsf{PaA}_+^{\wedge} \to \mathsf{R}_+$*, which maps the distinguished arity* 0 *element of the unitary operad* PaA_+ *to this object* $e \in \mathtt{Ob}\,\mathsf{R}(0)$*, and this morphism determines a morphism of unitary operads in Malcev complete groupoids yet:*

$$\phi : \mathsf{PaCD}_+^{\wedge} \to \mathsf{Q}_+ .$$

PROOF. We outline a proof of this theorem which parallels the proof of the result of Theorem 6.2.4 about the definition of morphisms on the parenthesized braid operad. The cited reference [16, Proposition 4.5, Proposition 4.8] gives an equivalent result by using the free categories associated to the operad PaA.

We use operadic notation in this proof rather than the algebraic notation which we adopt in the statement of our theorem. We rely on the coherence theorem for the parenthesized symmetry operad PaS, which implies that the associativity isomorphism a and the symmetry isomorphism c of the operad $\mathsf{Q} = \mathbb{G}(\mathsf{R})$ determine a well-defined morphism of operads in groupoids $\phi : \mathsf{PaS} \to \mathsf{Q}$ when the constraints of our theorem hold. We also consider the obvious morphism of operads in sets $\phi : \mathit{\Omega} \to \mathtt{Ob}\,\mathsf{R}$, which underlies this morphism of operads in groupoids at the object set level, and which is determined by the assignment $\phi(\mu) = m$ by using that the magma operad $\mathit{\Omega}$ represents the free operad generated by the operation $\mu \in \mathit{\Omega}(2)$.

We aim to extend the morphism $\phi : \mathsf{PaS} \to \mathsf{Q}$ to the whole operad $\mathsf{PaA}^{\wedge}$ by using that the operad of parenthesized symmetries PaS is identified with a suboperad of group-like homomorphisms in the operad in complete Hopf groupoids $\mathsf{PaA}^{\wedge}$. To be more precise, recall that each homomorphism f of the operad $\mathsf{PaA}^{\wedge}$ admits a canonical decomposition $f = g \cdot u$, where we have $u \in \hat{\mathbb{U}}\hat{\mathfrak{p}}(r)$ and the factor g is a group-like homomorphism which is identified with a morphism of the parenthesized symmetry operad (see §10.3.1).

Step 1: The decomposition of homomorphisms in the operad $\mathsf{PaA}^{\wedge}$. In a first step, we use the presentation of the complete enveloping algebra $\hat{\mathbb{U}}\hat{\mathfrak{p}}(r)$ to express any such homomorphism $f \in \mathtt{Hom}_{\mathsf{PaA}(r)^{\wedge}}(p, q)$ as a composite of (operadic compositions of) the infinitesimal braiding θ and of morphisms which come from the parenthesized symmetry operad PaS. We give an example of such a decomposition in Figure 10.2. We suggest the reader to follow our process on this example. We explicitly pick a factorization $f = f_1 \cdots f_n$ of the homomorphism $f \in \mathtt{Hom}_{\mathsf{PaA}(r)^{\wedge}}(p, q)$ where each factor $f_i \in \mathtt{Hom}_{\mathsf{PaA}(r)^{\wedge}}(p_{i-1}, p_i)$ reduces to a single generating factor $t_{k_i l_i}$ in the complete enveloping algebra $\hat{\mathbb{U}}\hat{\mathfrak{p}}(r)$.

To relate these factors to the infinitesimal braiding, we pick a parenthesized word κ_i which gathers the strands (k_i, l_i) in the magma operad:

$$\kappa_i = \pi_i(x_1, \dots, \mu(x_{k_i}, x_{l_i}), \dots, \widehat{x_{l_i}}, \dots, x_r) \in \mathit{\Omega}(r).$$

We consider the morphisms $\rho_i \in \mathtt{Mor}_{\mathsf{PaS}(r)}(p_{i-1}, \kappa_i)$ and $\sigma_i \in \mathtt{Mor}_{\mathsf{PaS}(r)}(p_i, \kappa_i)$ which connect this parenthesized word κ_i to the objects p_{i-1} and p_i in the parenthesized

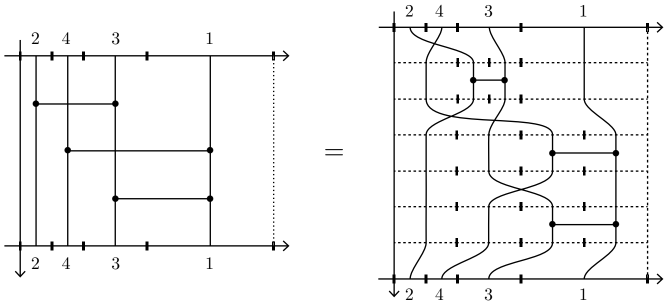


FIGURE 10.2. The decomposition of a homomorphism in the operad PaA.

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symmetry operad. We then get an obvious decomposition of the morphism f_i

$$f_i = \sigma_i^{-1} \cdot \pi_i(x_1, \dots, \theta(x_{k_i}, x_{l_i}), \dots, \widehat{x_{l_i}}, \dots, x_r) \cdot \rho_i$$

where the medium factor reduces to the application of an infinitesimal braiding θ within the parenthesized word κ_i, while ρ_i and σ_i are morphisms coming from the parenthesized symmetry operad PaS.

Step 2: The construction of the morphisms of complete Hopf groupoids $\phi : \mathsf{PaA}(r)\hat{}\to \mathsf{R}(r)$. We use the existence of the decompositions established in Step 1 to determine the value of our morphism $\phi : \mathsf{PaA}(r)\hat{}\to \mathsf{R}(r)$ on any homomorphism $f \in \mathtt{Hom}_{\mathsf{PaA}(r)\hat{}}(p, q)$ of the operad $\mathsf{PaA}\hat{}$ from the value of this morphism $\phi : \mathsf{PaA}(r)\hat{}\to \mathsf{R}(r)$ on the parenthesized symmetry operad $\mathsf{PaS} \subset \mathsf{PaCD}$ and from the assignment $\phi(\theta) = h$ for the infinitesimal braiding θ. For instance, in the case of the morphism of Figure 10.2, we obtain the expression:

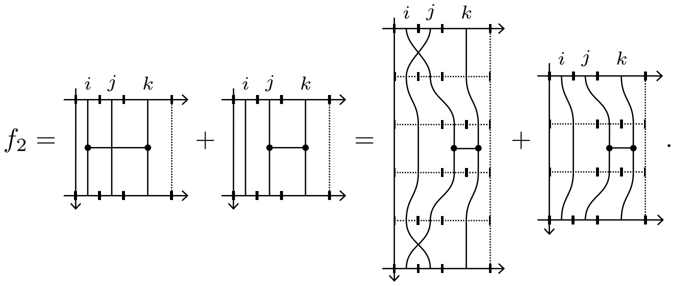
$$\phi(f) = g_1 \cdot m(m(x_2, x_4), h(x_3, x_1)) \cdot g_2 \cdot m(m(x_2, x_3), h(x_4, x_1)) \\ \cdot g_3 \cdot m(m(x_4, h)(x_2, x_3), x_1) \cdot g_4,$$

where g_1, g_2, g_3 and g_4 can be determined by taking appropriate composites of the associativity isomorphism and of the symmetry isomorphism in R.

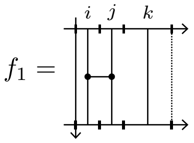
The main purpose of our verifications is to establish that the mapping $\phi : \mathsf{PaA}(r)\hat{}\to \mathsf{R}(r)$ which we determine from these decompositions does not depend on choices. First, we can readily adapt the argument of the proof of Theorem 6.2.4 (Step 2) to check that our homomorphism $\phi(f)$ does not depend on the choice of the intermediate objects κ_i, $i = 1, \dots, n$, of our decomposition. In short, different choices of such parenthesized words are linked by a composite of isomorphisms in the parenthesized symmetry operad which fixes the internal grouping $\mu(x_{k_i}, x_{l_i})$ of our object. In order to check that different choices of decompositions connected by such isomorphisms give equal results, we use the coherence of the definition of our morphism on the parenthesized symmetry operad PaS and the bifunctoriality of the composition products of the operad R. By the way, we readily deduce from the invariance relation $h(x_1, x_2) = c(x_1, x_2)^{-1} \cdot h(x_2, x_1) \cdot c(x_1, x_2)$ that the outcome of our construction does not depend on the choice of the ordering on the pair (k_i, l_i) in our gathering process either.

We then check that our map does not depend on the choice of the monomial decomposition which we use to factorize our homomorphisms in the complete enveloping algebra $\hat{\mathbb{U}}\hat{\mathfrak{p}}(r)$. We can still reduce the proof of this claim to the verification that our map returns equal results when we consider the generating relations of the chord diagram algebra. We can use the same argument as in the proof of Theorem 6.2.4 to check the case of the commutation relation $t_{ij}t_{kl} = t_{kl}t_{ij}$, while the coherence of our map with respect to the $4T$ relation $t_{ij}t_{ik} + t_{ij}t_{jk} = t_{ik}t_{ij} + t_{jk}t_{ij}$ follows from the semi-classical hexagon relation and from the bifunctoriality of the operadic composition product in R. To be explicit, we can focus on the case where we apply this relation within the endomorphism coalgebra of the object $\mu(\mu(x_i, x_j), x_k)$ in a parenthesized word, and we can rephrase the $4T$ relation as the commutation relation $f_1 f_2 = f_2 f_1$ of the homomorphisms $f_1 = t_{ij}$ and $f_2 = t_{ik} + t_{jk}$ in the endomorphism coalgebra of this object $\mu(\mu(x_i, x_j), x_k)$.

We use the following graphical identity to compute the image of the factor $f_2 = t_{ik} + t_{jk}$ under our map $\phi : \mathsf{PaA}(3) \to \mathsf{R}(3)$:



We easily see that the terms of the right-hand side are equivalent to the terms of the semi-classical hexagon relation up to a composition with isomorphisms of the parenthesized symmetry operad (but we already checked that our mapping is coherent with respect to such operations). We can therefore identify the image of f_2 under our map $\phi : \mathsf{PaA}(3) \to \mathsf{R}(3)$ with the operadic composite $\phi(f_2) = h \circ_1 id_m$ returned by the semi-classical hexagon relation. We immediately get, on the other hand, that the image of the homomorphism



under our map is given by the operadic composite $\phi(f_1) = id_m \circ_1 h$ in R. We have $\phi(f_1 f_2) = \phi(f_1)\phi(f_2)$ and $\phi(f_2 f_1) = \phi(f_2)\phi(f_1)$ by construction of our map. We just use the identity

$$(id_m \circ_1 h) \cdot (h \circ_1 id_m) = h \circ_1 h = (h \circ_1 id_m) \cdot (id_m \circ_1 h),$$

which follows from the bifunctoriality of the operadic composition, to conclude that our map $\phi : \mathsf{PaA}(3) \to \mathsf{R}(3)$ carries both sides of our commutation relation $f_1 f_2 = f_2 f_1$ to the same homomorphism of $\mathsf{R}(3)$.

This verification finishes the proof that our construction gives a well-defined map $\phi : \mathsf{PaA}(r) \to \mathsf{R}(r)$ on the complete Hopf groupoid $\mathsf{PaA}(r)$. The independence of the result of our construction with respect to choices also implies that this map preserves categorical composites, and hence does define a morphism in the category of complete Hopf groupoids $\phi : \mathsf{PaA}(r) \to \mathsf{R}(r)$ in each arity $r > 0$.

Step 3: The preservation of operadic composition structures. The morphisms of complete Hopf groupoids $\phi : \mathsf{PaA}(r) \to \mathsf{R}(r)$ constructed in Step 2 trivially preserve the action of symmetric groups on our operads and the operadic unit. We check that these morphisms preserve the composition products too. We can argue as in the proof of Theorem 6.2.4 to reduce our verifications to the case of a composition product of homomorphisms $f \circ_k g$, where f is given by a generator of the chord diagram algebra $f = t_{ij}$ while g is the identity homomorphism $g = id$, or where f is the identity $f = id$ while g is a generator $g = t_{ij}$. The verification of the relation $\phi(f \circ_k g) = \phi(f) \circ_k \phi(g)$ is again immediate in the case $f = id$. Thus we focus on the case where $f = t_{ij}$ and $g = id$ is the identity homomorphism of a parenthesized word $\lambda \in \Omega(n)$. We can still assume that this word reduces to the generating object of the magma operad $\lambda = \mu \in \Omega(2)$ (by the same induction as in the proof of Theorem 6.2.4), and we only face a non-trivial verification when the operadic composition $f \circ_k g$ inserts this homomorphism $g = id_{\mu}$ on the strands $k = i, j$ of the chord diagram $f = t_{ij}$. If we perform the composition $f \circ_k id_{\mu}$, then we retrieve a sum of chord diagrams which is similar to the expression considered in our study of the $4T$ relation in Step 2, and we can rely on the same arguments as in this previous verification to conclude that the image of the homomorphism $f \circ_k id_{\mu}$ under our map is equal to the composite $\phi(f \circ_k id_{\mu}) = \phi(f) \circ_k \phi(id_{\mu})$ in the operad R.

This verification completes the proof of the first assertion of the theorem.

Step 4: The definition of the unitary extension of our morphism. To address the proof of the second assertion of the theorem, we observe again that the assumptions of this assertion are equivalent to the requirement that the assignment $\phi : * \mapsto e$ is coherent with respect to the action of the restriction operators $\partial_k = - \circ_k *$ on the generator of the magma operad $\mu \in \Omega(2)$ and on the homomorphisms α, τ, θ of PaA. We use our decomposition process to deduce from this verification that ϕ carries any restriction operator in PaA to the corresponding composite with the object e in the operad R. ☐

We explain the operadic definition of the graded Grothendieck–Teichmüller group in the next paragraph. We use the previous theorem to relate this operadic approach to Drinfeld's original definition.

10.3.5. *The graded Grothendieck–Teichmüller group as a group of operad automorphisms.* We define the graded Grothendieck-Teichmüller group $GRT(\Bbbk)$ as the group formed by the automorphisms of the unitary operad of parenthesized chord diagrams

$$\phi : \mathsf{PaCD}_{+}^{\wedge} \xrightarrow{\simeq} \mathsf{PaCD}_{+}^{\wedge}$$

which reduce to the identity map on the object sets $\mathtt{Ob}\,\mathsf{PaCD}(r)^{\wedge} = \Omega(r)$ and fix the symmetry isomorphism $\tau \in \mathtt{Mor}_{\mathsf{PaCD}(2)^{\wedge}}(x_1 x_2, x_2 x_1)$. Recall that $\mathsf{PaCD}_{+}^{\wedge}$ is a unitary operad in the category of Malcev complete groupoids by construction, and we naturally assume that our automorphisms ϕ belong to this category of operads in this definition. We can accordingly determine ϕ by giving an automorphism of the operad in complete Hopf groupoids $\mathsf{PaA}_{+}^{\wedge}$ which underlies $\mathsf{PaCD}_{+}^{\wedge}$.

By Theorem 10.3.4, the construction of a morphism $\phi : \mathsf{PaA}_{+}^{\wedge} \xrightarrow{\simeq} \mathsf{PaA}_{+}^{\wedge}$ reduces to the definition of a product operation $m = m(x_1, x_2)$, of an associativity isomorphism $a = a(x_1, x_2, x_3)$, of a symmetry isomorphism $c = c(x_1, x_2)$, and of an infinitesimal braiding $h = h(x_1, x_2)$ which respectively represent the image of the multiplication operation μ, of the associativity isomorphism α, of the braiding τ and of the infinitesimal braiding θ under our morphism $\phi : \mathsf{PaA}_+^{\widehat{}} \to \mathsf{PaA}_+^{\widehat{}}$. In our statement, we also consider a unit object $e \in \mathsf{Q}(0)$. In our setting, this unit object is fixed by the assumption that the arity zero term of our operad reduces to the ground field $\mathsf{PaA}_+(0) = \Bbbk$.

We just fix $m = \mu \Rightarrow \phi(\mu) = \mu$ and $c = \tau \Rightarrow \phi(\tau) = \tau$, since we restrict ourselves to morphisms that reduce to the identity on object sets and fix the braiding in the definition of $GRT(\Bbbk)$. We only leave choice on the definition of the associativity homomorphism $a = a(x_1, x_2, x_3)$ and of the infinitesimal braiding $h = h(x_1, x_2)$ in the hom-objects of the operad $\mathsf{PaA}^{\widehat{}}$. We then have the following statement:

Proposition 10.3.6. *The morphisms of unitary operads in complete Hopf groupoids $\phi : \mathsf{PaA}_+^{\widehat{}} \to \mathsf{PaA}_+^{\widehat{}}$ which reduce to the identity map on the object sets $\mathtt{Ob}\,\mathsf{PaA}(r)^{\widehat{}} = \Omega(r)$, $r > 0$, and fix the symmetry isomorphism*

$$\phi(\tau) = \tau$$

as in the definition of the graded Grothendieck-Teichmüller group (§10.3.5) are uniquely determined by a scalar parameter $\kappa \in \Bbbk$ and a group-like element of the complete tensor algebra on two generators $f(\xi_1, \xi_2) \in \mathbb{G}\,\hat{\mathbb{T}}(\xi_1, \xi_2)$ such that we have the following assignments

$$\phi(\theta) = \lambda\theta, \quad \phi(\alpha) = \alpha \cdot f(t_{12}, t_{23}),$$

in the hom-objects of the operad $\mathsf{PaA}_+^{\widehat{}}$.

Explanations. We examine the structure of the hom-objects of the operad $\mathsf{PaA}_+^{\widehat{}}$ in arity $r = 2, 3$ in order to determine the form of the homomorphisms $h = \phi(\theta)$ and $a = \phi(\alpha)$ which determine our morphism $\phi : \mathsf{PaA}_+^{\widehat{}} \to \mathsf{PaA}_+^{\widehat{}}$.

In arity 2, we have $\hat{\mathfrak{p}}(2) = \Bbbk\, t_{12} \Rightarrow \mathbb{P}(\mathtt{Hom}_{\mathsf{PaA}(2)^{\widehat{}}}(\mu, \mu)) = \mathbb{P}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(2)) = \hat{\mathfrak{p}}(2) = \Bbbk\, t_{12}$. We deduce from this relation that our infinitesimal braiding $h = h(x_1, x_2)$ has an expression of the form

$$h(x_1, x_2) = \lambda t_{12}$$

in $\hat{\mathfrak{p}}(2)$, for some parameter $\lambda \in \Bbbk$.

In arity 3, we use the Lie algebra decomposition $\hat{\mathfrak{p}}(3) = \Bbbk\, z \oplus \hat{\mathbb{L}}(t_{12}, t_{23})$ to obtain, as in the proof of Proposition 10.2.6, that our associativity homomorphism is given by an expression of the form $a(x_1, x_2, x_3) = \alpha \cdot e^{cz} \cdot e^{p(t_{12}, t_{23})}$, for some parameter $c \in \Bbbk$ and a Lie power series $p(t_{12}, t_{23}) \in \hat{\mathbb{L}}(t_{12}, t_{23})$. We can yet identify this exponential Lie power series $f(t_{12}, t_{23}) = e^{p(t_{12}, t_{23})}$ with a group-like element of the complete Hopf algebra $\hat{\mathbb{T}}(t_{12}, t_{23}) = \hat{\mathbb{U}}\hat{\mathbb{L}}(t_{12}, t_{23})$. We still use the unit relation $a(x_1, *, x_3) = 0$ to obtain $c = 0$ and we eventually conclude that our associativity homomorphism has an expression of the form:

$$a(x_1, x_2, x_3) = f(t_{12}, t_{23}) \in \mathbb{G}\,\hat{\mathbb{T}}(t_{12}, t_{23}),$$

for some group-like element of the complete tensor algebra on two generators $f(\xi_1, \xi_2) \in \mathbb{G}\,\hat{\mathbb{T}}(\xi_1, \xi_2)$. □

To complete this result, we still write down the coherence constraints of Theorem 10.3.4 in terms of this pair $(\lambda, f(\xi_1, \xi_2))$ which we associate to our operad morphism $\phi : \mathsf{PaA}_+^{\widehat{}} \to \mathsf{PaA}_+^{\widehat{}}$. We focus on the case where λ is invertible, because we are going to see that we need this assumption $\lambda \in \Bbbk^{\times}$ in order to ensure that our morphism $\phi : \mathsf{PaA}_{+}^{\wedge} \to \mathsf{PaA}_{+}^{\wedge}$ is invertible, and hence, does define an element of the graded Grothendieck–Teichmüller group $GRT(\Bbbk)$. We obtain the following proposition:

Proposition 10.3.7. *The assignments of Proposition* 10.3.6

$$\phi(\tau) = \tau, \quad \phi(\theta) = \lambda\theta, \quad \phi(\alpha) = \alpha \cdot f(t_{12}, t_{23}),$$

where we assume $\lambda \in \Bbbk^{\times}$ and $f(\xi_1, \xi_2) \in \mathbb{G}\hat{\mathbb{T}}(\xi_1, \xi_2)$, determine a well-defined morphism of unitary operads in complete Hopf groupoids $\phi : \mathsf{PaA}_{+}^{\wedge} \to \mathsf{PaA}_{+}^{\wedge}$ if and only if the power series $f(\xi_1, \xi_2)$ satisfies:

1. *the unit relations $f(\xi_1, 0) = 1 = f(0, \xi_2)$,*
2. *the involution relation $f(\xi_1, \xi_2) \cdot f(\xi_2, \xi_1) = 1$,*
3. *the hexagon equation $f(\xi_3, \xi_1) \cdot f(\xi_2, \xi_3) \cdot f(\xi_1, \xi_2) = 1$, where (ξ_1, ξ_2, ξ_3) denotes a triple of variables such that $\xi_1 + \xi_2 + \xi_3 = 0$,*
4. *the semi-classical hexagon equation $\xi_1 + f(\xi_1, \xi_2)^{-1} \cdot \xi_2 \cdot f(\xi_1, \xi_2) + f(\xi_1, \xi_3)^{-1} \cdot \xi_3 \cdot f(\xi_1, \xi_3) = 0$, where (ξ_1, ξ_2, ξ_3) is again a triple of variables such that $\xi_1 + \xi_2 + \xi_3 = 0$,*
5. *and the pentagon equation $f(t_{12}, t_{23} + t_{24}) \cdot f(t_{13} + t_{23}, t_{34}) = f(t_{23}, t_{34}) \cdot f(t_{12} + t_{13}, t_{24} + t_{34}) \cdot f(t_{12}, t_{23})$ in the complete Hopf algebra $\hat{\mathbb{U}}\hat{\mathfrak{p}}(4)$.*

Proof. We go back to the definition of the composition structure of the operad $\mathsf{PaA}^{\wedge}$ in §10.3.1 in order to make explicit the coherence constraints of Theorem 10.3.4 for the braiding $c = \tau$, for the infinitesimal braiding $h = \theta$, and for the associativity isomorphism $a = \alpha \cdot f(t_{12}, t_{23})$ given in this proposition. We use the expression of the restriction operators on $\mathfrak{p}(3)$ to get the unit relations $f(t_{12}, 0) = 1 = f(0, t_{12})$ in $\mathtt{Hom}_{\mathsf{PaA}(2)^{\wedge}}(\mu, \mu) = \mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(2))$ as in the proof of Proposition 10.2.7. We expand the expression of the factors in the pentagon diagram of Figure 6.1 to get the pentagon equation of the proposition similarly.

In our argument lines, we still use that our group-like power series $f(\xi_1, \xi_2) \in \mathbb{G}\hat{\mathbb{T}}(\xi_1, \xi_2)$ can be expressed as the exponential $f(\xi_1, \xi_2) = e^{p(\xi_1, \xi_2)}$ of an element of the free complete Lie algebra on two generators $p(\xi_1, \xi_2) \in \hat{\mathbb{L}}(\xi_1, \xi_2)$, and that the relation $f(\xi_1, 0) = 1 = f(0, \xi_2) \Leftrightarrow p(\xi_1, 0) = 0 = p(0, \xi_2)$ is equivalent to the requirement that this Lie power series $p(\xi_1, \xi_2) \in \hat{\mathbb{L}}(\xi_1, \xi_2) = \prod_{m \geq 1} \mathbb{L}(\xi_1, \xi_2)$ has no component in weight $m = 1$.

Recall that the hexagon constraints are equivalent when the symmetry isomorphism, such as $c = \tau$, satisfies the involution relation $\tau(x_1, x_2)\tau(x_2, x_1) = 1$. We therefore focus on the first hexagon relation, which reads $\mu(x_2, \tau(x_1, x_3)) \cdot \alpha \cdot f(t_{21}, t_{13}) \cdot m(\tau(x_1, x_2), x_3) = \alpha \cdot f(t_{23}, t_{31}) \cdot \tau(x_1, \mu(x_2, x_3)) \cdot \alpha \cdot f(t_{12}, t_{23})$ for the symmetry isomorphism $c = \tau$ and the associativity isomorphism $a = \alpha \cdot f(t_{12}, t_{23})$ of the proposition. We factorize the associativity isomorphisms α and we simplify the permutation operators which occur in this relation to get the following reduced equation

$$f(t_{12}, t_{13}) = f(t_{23}, t_{13}) \cdot f(t_{12}, t_{23}) \tag{1}$$

in $\hat{\mathbb{U}}\hat{\mathfrak{p}}(3)$.

We set $\xi_1 = t_{12}$, $\xi_2 = t_{23}$, and $\xi_3 = -\xi_1 - \xi_2 \Leftrightarrow t_{13} = \xi_3 + z$, where z denotes, as usual, the central element $z = t_{12} + t_{13} + t_{23}$ of the Lie algebra $\mathfrak{p}(3)$. We use the relation $[z, -] = 0$ as in the proof of Proposition 10.2.7 and the observation that the Lie power series $p(-, -)$ underlying $f(-, -)$ has no component in weight

$m = 1$ to get the identities $p(\xi_3 + z, -) = p(\xi_3, -) \Rightarrow f(\xi_3 + z, -) = f(\xi_3, -)$, $p(-, \xi_3 + z) = p(-, \xi_3) \Rightarrow f(-, \xi_3 + z) = f(-, \xi_3)$, and to obtain the formula

$$f(\xi_1, \xi_3) = f(\xi_2, \xi_3) \cdot f(\xi_1, \xi_2) \tag{2}$$

from Equation (1). If we perform the transposition of the variables (ξ_1, ξ_2) in this equation, then we obtain $f(\xi_2, \xi_3) = f(\xi_1, \xi_3) \cdot f(\xi_2, \xi_1)$, If we substitute $f(\xi_2, \xi_3)$ by this expression in the above equation, then we obtain the relation $f(\xi_1, \xi_3) = f(\xi_1, \xi_3) \cdot f(\xi_2, \xi_1) \cdot f(\xi_1, \xi_2)$ which gives the involution relation $f(\xi_2, \xi_1) \cdot f(\xi_1, \xi_2) = 1$ when we mod out the factor $f(\xi_1, \xi_3)$. In turn, we get the hexagon relation of the proposition $f(\xi_3, \xi_1) \cdot f(\xi_2, \xi_3) \cdot f(\xi_1, \xi_2) = 1$ when we carry this factor $f(\xi_1, \xi_3) = f(\xi_3, \xi_1)^{-1}$ to the right hand side in the above equation (2).

We factorize and simplify permutation operators, as in the case of the hexagon relation, to obtain the reduced equation

$$\begin{aligned} \lambda\,(t_{13} + t_{23}) &= f(t_{31}, t_{12}) \cdot \lambda t_{13} \cdot f(t_{13}, t_{32})^{-1} \cdot f(t_{12}, t_{23}) \\ &\quad + f(t_{31}, t_{12}) \cdot f(t_{13}, t_{32})^{-1} \cdot \lambda t_{23} \cdot f(t_{12}, t_{23}) \end{aligned} \tag{3}$$

from the semi-classical hexagon relation of Theorem 10.3.4. We again set $\xi_1 = t_{12}$, $\xi_2 = t_{23}$, $\xi_3 = -\xi_1 - \xi_2 = t_{13} - z \Leftrightarrow t_{13} = z + \xi_3$, and we use the same argument as in the case of the hexagon equation to obtain the formula

$$\begin{aligned} \lambda\,(z - \xi_1) &= f(\xi_3, \xi_1) \cdot \lambda\,(z + \xi_3) \cdot f(\xi_3, \xi_2)^{-1} \cdot f(\xi_1, \xi_2) \\ &\quad + f(\xi_3, \xi_1) \cdot f(\xi_3, \xi_2)^{-1} \cdot \lambda \xi_2 \cdot f(\xi_1, \xi_2) \end{aligned} \tag{4}$$

from the above equation. We can drop the scalar factor from this equation since we assume $\lambda \in \mathbb{k}^{\times}$. We use the involution and the hexagon equation to get the identities $f(\xi_3, \xi_1) = f(\xi_1, \xi_3)^{-1}$, $f(\xi_3, \xi_2)^{-1} \cdot f(\xi_1, \xi_2) = f(\xi_1, \xi_3)$, $f(\xi_3, \xi_1) \cdot f(\xi_3, \xi_2)^{-1} = f(\xi_1, \xi_2)^{-1}$, and we use that z is central to check that these terms vanish in our formula. We eventually get the relation of the proposition $\xi_1 + f(\xi_1, \xi_2)^{-1} \cdot \xi_2 \cdot f(\xi_1, \xi_2) + f(\xi_1, \xi_3)^{-1} \cdot \xi_3 \cdot f(\xi_1, \xi_3) = 0$. ☐

We already mentioned that the scalar parameter $\lambda \in \mathbb{k}$ which we associate to a morphism of unitary operads in Proposition 10.3.6 is necessarily invertible when we assume that this morphism $\phi : \mathit{PaA}^{\wedge}_{+} \to \mathit{PaA}^{\wedge}_{+}$ is an isomorphism. To check this claim, we use that the multiplication by this scalar parameter represents the map induced by our morphism on the $\mathbb{k}$-module $\mathbb{P}(\mathtt{Hom}_{\mathit{PaA}(2)^{\wedge}}(\mu, \mu)) = \mathbb{P}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(2)) = \mathbb{k}\, t_{12}$ in the component of arity two of our operad. We prove that this condition $\lambda \in \mathbb{k}^{\times}$ actually suffices to ensure that our morphism is an automorphism:

PROPOSITION 10.3.8. *The morphism of unitary operads in complete Hopf groupoids* $\phi : \mathit{PaA}^{\wedge}_{+} \to \mathit{PaA}^{\wedge}_{+}$ *which we determine by the assignments of Proposition* 10.3.6 *is an isomorphism if and only if the scalar parameter which we associate to this morphism in our correspondence is invertible* $\lambda \in \mathbb{k}^{\times}$.

PROOF. We adapt the argument lines of the proof of Proposition 10.2.8, where we check that the categorical equivalences of operads in Malcev complete groupoids from the Malcev completion of the parenthesized braid operad to the operad of chord diagrams are characterized by the same condition. We again only examine the "if" part of the proposition since we already checked the "only if" part. We therefore assume $\lambda \in \mathbb{k}^{\times}$.

We fix a parenthesized word $\pi \in \Omega(r)$, $r > 0$. We have $\mathtt{Hom}_{\mathit{PaA}(r)^{\wedge}}(\pi, \pi) = \hat{\mathbb{U}}\hat{\mathfrak{p}}(r)$ and $\mathtt{E}^0\, \mathtt{Hom}_{\mathit{PaA}(r)^{\wedge}}(\pi, \pi) = \mathbb{U}\,\mathfrak{p}(r)$, where we consider the enveloping algebra of the weight graded Lie algebra $\mathfrak{p}(r)$ instead of the complete Drinfeld–Kohno Lie algebra. We just check that our morphism $\phi : \mathsf{PaA}_+^{\widehat{}} \to \mathsf{PaA}_+^{\widehat{}}$ induces an isomorphism on this graded object. We determine the image of the endomorphism of the object π which we associate to the generator t_{ij} of the complete algebra $\hat{\mathbb{U}}\hat{\mathfrak{p}}(r)$. To explain our argument, we adopt the convention to use the notation u_{ij} for this endomorphism of the object π, while we keep the notation t_{ij} for the endomorphism of a parenthesized word where we have gathered the variables (x_i, x_j). We have $u_{ij} = g \cdot t_{ij} \cdot g^{-1}$, where g is the composite of associators and symmetry operators which we use in this gathering process, and $\phi(u_{ij}) = \phi(g) \cdot \lambda t_{ij} \cdot \phi(g)^{-1}$ by construction of our morphism.

We already observed that the unit condition $f(\xi_1, 0) = 1 = f(0, \xi_2)$ in Proposition 10.3.7 implies that the Lie power series $p(\xi_1, \xi_2)$ which we associate to our associativity isomorphism $\phi(\alpha) = \alpha \cdot f(t_{12}, t_{23}) = \alpha \cdot e^{p(t_{12}, t_{23})}$ has no component in weight one. We equivalently have $f(t_{12}, t_{23}) \equiv 1 (\operatorname{mod} \mathtt{F}_2\, \hat{\mathbb{U}}\hat{\mathfrak{p}}(3))$. To perform our verifications, we only need the weaker relation $f(t_{12}, t_{23}) \equiv 1 (\operatorname{mod} \mathtt{F}_1\, \hat{\mathbb{U}}\hat{\mathfrak{p}}(3))$, which implies $\phi(\alpha) = \alpha \cdot f(t_{12}, t_{23}) \equiv \alpha (\operatorname{mod} \mathtt{F}_1\, \hat{\mathbb{U}}\hat{\mathfrak{p}}(3))$. Since we also require $\phi(\tau) = \tau$ in the definition of our morphism $\phi : \mathsf{PaA}_+^{\widehat{}} \to \mathsf{PaA}_+^{\widehat{}}$, we get the relation $\phi(g) \equiv g (\operatorname{mod} \mathtt{F}_1\, \hat{\mathbb{U}}\hat{\mathfrak{p}}(r))$, for our operator g. We consequently have $\phi(u_{ij}) \equiv g \cdot \lambda t_{ij} \cdot g^{-1} \equiv \lambda u_{ij} (\operatorname{mod} \mathtt{F}_2\, \mathbb{U}\, \mathfrak{p}(r))$. We conclude from this computation that ϕ is given by the multiplication by the scalar $\lambda \in \Bbbk^{\times}$ on the weight graded associative algebra $\mathtt{E}^0\, \mathtt{Hom}_{\mathsf{PaA}(r)^{\widehat{}}}(\pi, \pi)$, and hence, defines an isomorphism at this level. The result follows. ☐

We now study the composite $\phi \circ \psi : \mathsf{PaA}_+^{\widehat{}} \to \mathsf{PaA}_+^{\widehat{}}$ of automorphisms $\phi, \psi : \mathsf{PaA}_+^{\widehat{}} \xrightarrow{\simeq} \mathsf{PaA}_+^{\widehat{}}$ in the graded Grothendieck–Teichmüller group $GRT(\Bbbk)$. We assume that these automorphisms are, under the correspondence of Proposition 10.3.6, associated to the pairs $(\lambda, f(\xi_1, \xi_2)), (\mu, g(\xi_1, \xi_2)) \in \Bbbk^{\times} \times \mathbb{G}\hat{\mathbb{T}}(\xi_1, \xi_2)$. We have the following statement:

PROPOSITION 10.3.9. *The composite $\phi \circ \psi : \mathsf{PaA}_+^{\widehat{}} \to \mathsf{PaA}_+^{\widehat{}}$ of the above morphisms $\phi, \psi : \mathsf{PaA}_+^{\widehat{}} \to \mathsf{PaA}_+^{\widehat{}}$ satisfies*

$$(\phi \circ \psi)(\theta) = \lambda\, \mu\, \theta,$$
$$(\phi \circ \psi)(\alpha) = \alpha \cdot f(t_{12}, t_{23}) \cdot g(\lambda\, t_{12}, f(t_{12}, t_{23})^{-1} \cdot \lambda\, t_{23} \cdot f(t_{12}, t_{23})).$$

PROOF. We trivially have $(\phi \circ \psi)(\theta) = \phi(\mu\, \theta) = \lambda\, \mu\, \theta$ and $(\phi \circ \psi)(\alpha) = \phi(\alpha \cdot g(t_{12}, t_{23})) = \phi(\alpha) \cdot \phi(g(t_{12}, t_{23}))$. We use that the factor $g(t_{12}, t_{23})$ in this expression represents an endomorphism of the object $\pi = \mu(\mu(x_1, x_2), x_3)$ in the complete Hopf groupoid $\mathsf{PaA}(3)^{\widehat{}}$. We have $\phi(g(t_{12}, t_{23})) = g(\phi(t_{12}), \phi(t_{23}))$, and from the construction of Theorem 10.3.4, we easily check that we have the formulas $\phi(t_{12}) = \lambda\, t_{12}$ and $\phi(t_{13}) = f(t_{12}, t_{23})^{-1} \cdot \lambda\, t_{23} \cdot f(t_{12}, t_{23})$ in this hom-object $\mathtt{Hom}_{\mathsf{PaA}(3)^{\widehat{}}}(\pi, \pi)$. We eventually get the expression of the proposition for $(\phi \circ \psi)(\alpha)$. ☐

We summarize our results in the following theorem:

THEOREM 10.3.10 (Equivalence between the operadic approach and Drinfeld's definition of the graded Grothendieck–Teichmüller group [57, §5]). *The correspondence of Proposition* 10.3.6 *gives a one-to-one correspondence between the automorphisms of the graded Grothendieck–Teichmüller group $GRT(\Bbbk)$ and the set of pairs $(\lambda, f(\xi_1, \xi_2))$, where λ is an invertible scalar parameter $\lambda \in \Bbbk^{\times}$, as we require in*

Proposition 10.3.8*, and* $f(\xi_1,\xi_2)$ *is a group-like power series* $f(\xi_1,\xi_2)\in\mathbb{G}\,\hat{\mathbb{T}}(\xi_1,\xi_2)$ *which satisfies the unit, involution, hexagon, semi-classical hexagon and pentagon relations (*1-5*) of Proposition* 10.3.7*.*

Furthermore, the composition operation of the group $GRT(\Bbbk)$ *corresponds on this set of pairs to the operation:*

$$(\lambda,f(\xi_1,\xi_2))\star(\mu,g(\xi_1,\xi_2)):=(\lambda\,\mu,f(\xi_1,\xi_2)\cdot g(\lambda\,\xi_1,f(\xi_1,\xi_2)^{-1}\cdot\lambda\,\xi_2\cdot f(\xi_1,\xi_2)))$$

determined in Proposition 10.3.9*.* ☐

We note that the parameter $\lambda\in\Bbbk^{\times}$ does not occur in the equations of Proposition 10.3.7. We give an interpretation of this observation in terms of the group $GRT(\Bbbk)$ in the next paragraph.

10.3.11. *The semi-direct product decomposition of the graded Grothendieck–Teichmüller group.* We have an obvious morphism $\lambda: GRT(\Bbbk)\to\Bbbk^{\times}$ which maps any element of the graded Grothendieck–Teichmüller group $\phi\in GRT(\Bbbk)$ represented by an automorphism $\phi:\mathsf{PaA}^{\widehat{\ }}\to\mathsf{PaA}^{\widehat{\ }}$ to the scalar $\lambda\in\Bbbk^{\times}$ such that $\phi(\theta)=\lambda\theta$. We then set:

$$GRT^1(\Bbbk):=\ker(\lambda: GRT(\Bbbk)\to\Bbbk^{\times}).$$

We can equivalently define this group $GRT^1(\Bbbk)$ as the subgroup of the graded Grothendieck–Teichmüller group formed by the automorphisms $\phi:\mathsf{PaA}_+^{\widehat{\ }}\to\mathsf{PaA}_+^{\widehat{\ }}$ which fix the infinitesimal braiding $\phi(\theta)=\theta$ in addition to the symmetry homomorphism $\phi(\tau)=\tau$.

We explained in §10.2.4 that the multiplicative group operates on the chord diagram operad through automorphisms of unitary operads in Malcev complete groupoids $\rho_\lambda:\mathsf{CD}_+^{\widehat{\ }}\to\mathsf{CD}_+^{\widehat{\ }}$, for each $\lambda\in\Bbbk^{\times}$. We can trivially lift this action to the operad of parenthesized chord diagrams to associate an element of the graded Grothendieck–Teichmüller group $\rho_\lambda\in GRT(\Bbbk)$ to any scalar $\lambda\in\Bbbk^{\times}$. This morphism $\rho:\Bbbk^{\times}\to GRT(\Bbbk)$ clearly defines a section of our map $\lambda: GRT(\Bbbk)\to\Bbbk^{\times}$. The full graded Grothendieck–Teichmüller group accordingly admits a semi-direct product decomposition:

$$GRT(\Bbbk)=\Bbbk^{\times}\ltimes GRT^1(\Bbbk),$$

with the group $GRT^1(\Bbbk)$ as normal factor.

In the description of Proposition 10.3.6-10.3.7 the elements of this subgroup $GRT^1(\Bbbk)$ correspond to the pairs $(\lambda,f(\xi_1,\xi_2))$ such that $\lambda=1$. Furthermore, the section $\rho:\Bbbk^{\times}\to GRT(\Bbbk)$ of our morphism $\lambda: GRT(\Bbbk)\to\Bbbk^{\times}$ maps the scalar $\lambda\in\Bbbk^{\times}$ to the element of the graded Grothendieck–Teichmüller group represented by the pair $(\lambda,1)$ where we consider the trivial power series $f(\xi_1,\xi_2)\equiv 1$. We easily check that the action of the multiplicative group $\Bbbk^{\times}$ on $GRT^1(\Bbbk)$ which we associate to this section corresponds to the operation $c_\lambda: f(\xi_1,\xi_2)\mapsto f(\lambda\,\xi_1,\lambda\,\xi_2)$, $\lambda\in\Bbbk^{\times}$, on the set of group-like power series $f(\xi_1,\xi_2)\in\mathbb{G}\,\hat{\mathbb{T}}(\xi_1,\xi_2)$.

We will explain in the next section that $GRT(\Bbbk)$ admits a decomposition $GRT(\Bbbk)=\lim_m GRT_{\langle m\rangle}(\Bbbk)$, where each $GRT_{\langle m\rangle}(\Bbbk)$ forms an algebraic group. We similarly have $GRT^1(\Bbbk)=\lim_m GRT^1_{\langle m\rangle}(\Bbbk)$ for our subgroup $GRT^1(\Bbbk)$, One may see that the terms of this decomposition $GRT^1_{\langle m\rangle}(\Bbbk)$ are unipotent algebraic groups (see §10.4.5 and §10.4.12). We say that $GRT^1(\Bbbk)$ has a *pro-unipotent* structure to express this feature.

To complete our account, we explain the definition of an action of the graded Grothendieck–Teichmüller group on the set of associators. We rely on the following proposition which is a straightforward variation on the observation of Proposition 6.1.10:

PROPOSITION 10.3.12. *Each categorical equivalence of operads in Malcev complete groupoids $\phi : \mathsf{PaB}^{\widehat{}}_+ \to \mathsf{CD}^{\widehat{}}_+$, which we associate to an element of the set of Drinfeld's associators $Ass(\Bbbk)$, admits a unique lifting*

$$\begin{tikzcd} & \mathsf{PaCD}^{\widehat{}}_+ \arrow[d] \\ \mathsf{PaB}^{\widehat{}}_+ \arrow[ur, dotted, "\exists!"] \arrow[r, "\phi"'] & \mathsf{CD}^{\widehat{}}_+ \end{tikzcd}$$

given by the identity map at the object set level and which defines an isomorphism $\phi : \mathsf{PaB}^{\widehat{}}_+ \xrightarrow{\simeq} \mathsf{PaCD}^{\widehat{}}_+$ from the Malcev completion of the parenthesized braid operad $\mathsf{PaB}^{\widehat{}}_+$ to the operad of parenthesized chord diagrams $\mathsf{PaCD}^{\widehat{}}_+$. □

We use the obvious composition operation

$$\begin{tikzcd} & \mathsf{PaCD}^{\widehat{}}_+ \arrow[d] \arrow[r, "\phi"] & \mathsf{PaCD}^{\widehat{}}_+ \arrow[d] \\ \mathsf{PaB}^{\widehat{}}_+ \arrow[ur, dotted, "\exists!"] \arrow[r, "\psi"] \arrow[rr, dotted, bend right, "\phi\circ\psi"'] & \mathsf{CD}^{\widehat{}}_+ & \mathsf{CD}^{\widehat{}}_+ \end{tikzcd}$$

to determine the action of an automorphism $\phi : \mathsf{PaCD}^{\widehat{}}_+ \xrightarrow{\simeq} \mathsf{PaCD}^{\widehat{}}_+$, which represents an element of the graded Grothendieck–Teichmüller group $GRT(\Bbbk)$, on the categorical equivalence of operads in Malcev complete groupoids $\psi : \mathsf{PaB}^{\widehat{}}_+ \to \mathsf{CD}^{\widehat{}}_+$, which we use to represent an element of the set of Drinfeld's associator $Ass(\Bbbk)$. We easily check that:

PROPOSITION 10.3.13. *The above construction gives a simply transitive action of the graded Grothendieck–Teichmüller group $GRT(\Bbbk)$ on the set of Drinfeld's associator $Ass(\Bbbk)$.*

Furthermore, if we assume that the automorphism $\phi : \mathsf{PaCD}^{\widehat{}}_+ \xrightarrow{\simeq} \mathsf{PaCD}^{\widehat{}}_+$ which defines our element of the graded Grothendieck–Teichmüller group $GRT(\Bbbk)$ is represented by the pair $(\lambda, f(\xi_1, \xi_2))$ in the correspondence of Proposition 10.3.6*, while the categorical equivalence of operads $\psi : \mathsf{PaB}^{\widehat{}}_+ \to \mathsf{CD}^{\widehat{}}_+$, which defines our element of the set of associators $Ass(\Bbbk)$, is represented by the pair $(\mu, g(\xi_1, \xi_2))$ in the correspondence of Proposition* 10.2.6*, then our composition operation on morphisms $\phi \circ \psi : \mathsf{PaB}^{\widehat{}}_+ \to \mathsf{CD}^{\widehat{}}_+$ corresponds to the operation on pairs such that:*

$$(\lambda, f(\xi_1, \xi_2)) \star (\mu, g(\xi_1, \xi_2)) := (\lambda\,\mu, f(\xi_1, \xi_2) \cdot g(\lambda\,\xi_1, f(\xi_1, \xi_2)^{-1} \cdot \lambda\,\xi_2 \cdot f(\xi_1, \xi_2))).$$

PROOF. The first assertion of this proposition is immediate from our construction of this action in terms of the composition of morphisms. To check that the action is simply transitive, we just use that two isomorphisms of operads in Malcev complete groupoids from $\mathsf{PaB}^{\widehat{}}_+$ to $\mathsf{PaCD}^{\widehat{}}_+$ differ by an automorphism of the operad $\mathsf{PaCD}^{\widehat{}}_+$ (which is also defined by the identity map at the object set level when this is the case of our isomorphisms).

The second claim of the proposition follows from the same straightforward inspection as in Proposition 10.3.9, where we determine the product operation of the graded Grothendieck–Teichmüller group. We leave the details of this verification as an exercise. □

We easily see that the subset $Ass^{\kappa}(\Bbbk)$ of the set of associators $Ass(\Bbbk)$ associated to a fixed value of the parameter $\kappa \in \Bbbk^{\times}$ inherits a simply transitive action of the subgroup $GRT^1(\Bbbk)$ of §10.3.11, while we retrieve the bijections $\rho_{\lambda} : Ass^{\kappa}(\Bbbk) \xrightarrow{\simeq} Ass^{\lambda\kappa}(\Bbbk)$ of §10.2.11 when we consider the action of the automorphisms $\rho_{\lambda} \in GRT(\Bbbk)$, $\lambda \in \Bbbk^{\times}$, which we associate to our section of the multiplicative group in $GRT(\Bbbk)$.

10.4. Tower decompositions, the graded Grothendieck–Teichmüller Lie algebra and the existence of rational Drinfeld's associators

In the concluding paragraph of §10.2, we briefly explained that we can use the natural tower decomposition of the chord diagram operad $\mathsf{CD}^{\widehat{}}_{+} = \lim_m q_m\, \mathsf{CD}^{\widehat{}}_{+}$ to get a tower decomposition of the set of associators $Ass(\Bbbk) = \lim_m Ass_{\langle m\rangle}(\Bbbk)$. We are going to use this tower in order to prove the existence of elements defined over any ground field of characteristic zero $\Bbbk$ in the set of associators $Ass(\Bbbk)$.

In the overview of §10.2, we mentioned that the graded Grothendieck–Teichmüller group is also endowed with a decomposition $GRT(\Bbbk) = \lim_m GRT_{\langle m\rangle}(\Bbbk)$, where each $GRT_{\langle m\rangle}(\Bbbk)$ is an algebraic group acting on $Ass_{\langle m\rangle}(\Bbbk)$. We actually prove that the morphisms $p_m : GRT_{\langle m\rangle}(\Bbbk) \to GRT_{\langle m-1\rangle}(\Bbbk)$ which occur in this tower decomposition of the graded Grothendieck–Teichmüller group are surjective in order to establish that the same statement holds for the morphisms $p_m : Ass_{\langle m\rangle}(\Bbbk) \to Ass_{\langle m-1\rangle}(\Bbbk)$ which define the tower decomposition of the set of associators. We explain the definition of these tower decompositions and we check these surjectivity statements in this section. By the way, we determine the Lie algebra $\mathfrak{grt}_{\langle m\rangle}$ of the algebraic groups $GRT_{\langle m\rangle}(\Bbbk)$, as well as the Lie algebra of the pro-algebraic group $GRT(\Bbbk)$, which we define by $\mathfrak{grt} = \lim_m \mathfrak{grt}_{\langle m\rangle}$. We revisit the definition of this Lie algebra $\mathfrak{grt}$ in the next chapter. We then check that this object naturally occurs as the weight graded Lie algebra associated to a pro-unipotent version of the Grothendieck–Teichmüller group (see §11.4).

We examine the definition of the tower decomposition of the chord diagram operad $\mathsf{CD}^{\widehat{}}_{+} = \lim_m q_m\, \mathsf{CD}^{\widehat{}}_{+}$ first. We also deal with an analogous decomposition of the operad of parenthesized braids $\mathsf{PaCD}^{\widehat{}}_{+} = \lim_m q_m\, \mathsf{PaCD}^{\widehat{}}_{+}$ and with a counterpart of these tower decompositions for the operads in complete Hopf groupoids $\mathsf{A}^{\widehat{}}$ and $\mathsf{PaA}^{\widehat{}}$ underlying $\mathsf{CD}^{\widehat{}}_{+}$ and $\mathsf{PaCD}^{\widehat{}}_{+}$. We explain the definition of these decompositions in the context of non-unitary operads first, and we check the extension of our constructions to unitary operads afterwards, as usual.

10.4.1. *The tower decomposition of the chord diagram operad.* In §9.2, we explain a general definition of tower decomposition for operads in the category of Malcev complete groupoids. We can simplify this construction in the case of the chord diagram operad $\mathsf{CD}^{\widehat{}}$ since the components of this operad $\mathsf{CD}(r)^{\widehat{}}$, $r > 0$, are defined by the Malcev complete groups $G = \mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r))$ which we identify with groupoids with a single object. Recall that we also write $\mathsf{CD}(r)^{\widehat{}} = \mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r))$ when we use this identity. We merely set:

$$q_m\, \mathsf{CD}(r)^{\widehat{}} = \mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r)) / \mathtt{F}_{m+1}\, \mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r)),$$

for each arity $r > 0$, where we consider the natural filtration $G = \mathtt{F}_1 G \supset \cdots \supset \mathtt{F}_m G \supset \cdots$ of our Malcev complete group $G = \mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r))$, and we again identify the quotient groups associated to this filtration $q_m G = G/\mathtt{F}_{m+1} G$ with groupoids with a single object. We readily check that the collection $q_m\, \mathsf{CD}^{\wedge} = \{q_m\, \mathsf{CD}(r)^{\wedge}, r > 0\}$ inherits an operad structure so that we have an identity $\mathsf{CD}^{\wedge} = \lim_m q_m\, \mathsf{CD}^{\wedge}$ in the category of operads, because the operations which define the structure of the chord diagram operad $\mathsf{CD}^{\wedge}$ are determine by filtration preserving morphisms (since we form this operad in the category of Malcev complete groups). In what follows, we also write:

$$q_m\, \mathsf{CD}^{\wedge} = \mathsf{CD}^{\wedge}/\mathtt{F}_{m+1}\, \mathsf{CD}^{\wedge}$$

when we use this definition of the operad $q_m\, \mathsf{CD}^{\wedge}$. We moreover set:

$$\mathtt{F}_{m+1}\, \mathsf{CD}(r)^{\wedge} = \mathtt{F}_{m+1}\, \mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r)),$$

for each $r > 0$. We use that this collection of groups $\mathtt{F}_{m+1}\, \mathsf{CD}^{\wedge} = \{\mathtt{F}_{m+1}\, \mathsf{CD}(r)^{\wedge}, r > 0\}$ inherits an operad structure and we regard the object $q_m\, \mathsf{CD}^{\wedge}$ as an aritywise quotient of the operad of chord diagrams $\mathsf{CD}^{\wedge}$ by this suboperad in the category of groups $\mathtt{F}_{m+1}\, \mathsf{CD}^{\wedge}$.

We can also retrieve the natural tower decomposition of the operad of parenthesized chord diagrams $\mathsf{PaCD}^{\wedge} = \lim_m q_m\, \mathsf{PaCD}^{\wedge}$ by applying our pullback process $\mathsf{P} \mapsto \omega^*\, \mathsf{P}$ levelwise to the tower decomposition of the chord diagram operad $\mathsf{CD}^{\wedge} = \lim_m q_m\, \mathsf{CD}^{\wedge}$. We explicitly have $q_m\, \mathsf{PaCD}^{\wedge} = \omega^* q_m\, \mathsf{CD}^{\wedge}$, for each level $m \geq 1$, so that this operad $q_m\, \mathsf{PaCD}^{\wedge} = \{q_m\, \mathsf{PaCD}(r)^{\wedge}, r > 0\}$ is identified with the operad in groupoids which has the magma operad as operad of objects $\mathtt{Ob}\, q_m\, \mathsf{PaCD}^{\wedge} = \Omega$, and whose morphism sets are defined by $\mathtt{Mor}_{q_m\, \mathsf{PaCD}(r)^{\wedge}}(p, q) = \mathtt{Mor}_{q_m\, \mathsf{CD}(r)^{\wedge}}(pt, pt)$, for each pair of parenthesized words $p, q \in \Omega(r)$.

We can moreover use the result of Proposition 8.2.5 to identify the operads $q_m\, \mathsf{CD}^{\wedge}$ with the image of an operad in complete Hopf algebras under the group-like element construction of §9.2. We explicitly have

$$q_m\, \mathsf{CD}^{\wedge} = \mathbb{G}(q_m\, \mathsf{A}^{\wedge}),$$

for the operad in complete Hopf algebras $q_m\, \mathsf{A}^{\wedge} = \{q_m\, \mathsf{A}(r)^{\wedge}, r > 0\}$ such that:

$$q_m\, \mathsf{A}(r)^{\wedge} = \hat{\mathbb{U}}(\hat{\mathfrak{p}}(r)/\mathtt{F}_{m+1}\, \hat{\mathfrak{p}}(r)),$$

for each arity $r > 0$, where we consider the natural filtration $\hat{\mathfrak{g}} = \mathtt{F}_1\, \hat{\mathfrak{g}} \supset \cdots \supset \mathtt{F}_m\, \hat{\mathfrak{g}} \supset \cdots$ of the complete Drinfeld–Kohno Lie algebras $\hat{\mathfrak{g}} = \hat{\mathfrak{p}}(r)$. We again identify these complete Hopf algebras with complete Hopf groupoids with a single object. We still use that the structure morphisms of the Drinfeld–Kohno Lie algebra operad preserve filtrations in order to check that each of these collections of complete Hopf algebras $q_m\, \mathsf{A}^{\wedge} = \{q_m\, \mathsf{A}(r)^{\wedge}, r > 0\}$ inherits an operad structure. We accordingly have a tower decomposition $\mathsf{A}^{\wedge} = \lim_m q_m\, \mathsf{A}^{\wedge}$ in the category of operads in complete Hopf groupoids.

We can apply our pullback process to these operads levelwise in order to retrieve a tower of operads in complete Hopf groupoids $q_m\, \mathsf{PaA}^{\wedge} = \omega^* q_m\, \mathsf{A}$, which have the magma operad as object set operad $\mathtt{Ob}\, q_m\, \mathsf{PaA}^{\wedge} = \Omega$ for all $m \geq 0$, and which define a counterpart, in the category of operads in complete Hopf groupoids, of the tower decomposition of the parenthesized chord operad $\mathsf{PaCD}^{\wedge}$. We explicitly have $q_m\, \mathsf{PaCD}^{\wedge} = \mathbb{G}(q_m\, \mathsf{PaA}^{\wedge})$ for each level $m \geq 1$, and we have the identity $\mathsf{PaA}^{\wedge} = \lim_m q_m\, \mathsf{PaA}^{\wedge}$ which gives $\mathsf{PaCD}^{\wedge} = \lim_m q_m\, \mathsf{PaCD}^{\wedge}$ when we apply the group-like element functor.

We have an obvious extension of these tower decompositions for the unitary operads $\mathsf{CD}^{\wedge}_{+}$, $\mathsf{A}^{\wedge}_{+}$, $\mathsf{PaCD}^{\wedge}_{+}$ and $\mathsf{PaA}^{\wedge}_{+}$.

We now have the following assertions:

Proposition 10.4.2.

(a) *Let* $\phi : \mathsf{PaB}^{\wedge}_{+} \to q_m\,\mathsf{CD}^{\wedge}_{+}$ *be any morphism of operads in Malcev complete groupoids. This morphism admits a factorization:*

$$\begin{tikzcd}
\mathsf{PaB}^{\wedge}_{+} \arrow[rr, "\phi"] \arrow[dr] & & q_m\,\mathsf{CD}^{\wedge}_{+}\ , \\
& q_m\,\mathsf{PaB}^{\wedge}_{+} \arrow[ur, dotted, "\exists!\bar{\phi}"'] &
\end{tikzcd}$$

where we consider the quotient $q_m\,\mathsf{PaB}^{\wedge}_{+}$ *of the Malcev completion of the operad of parenthesized braids* $\mathsf{PaB}^{\wedge}_{+}$.

(b) *Every morphism* $\phi : \mathsf{PaCD}^{\wedge}_{+} \to q_m\,\mathsf{PaCD}^{\wedge}_{+}$ *similarly admits a factorization*

$$\begin{tikzcd}
\mathsf{PaCD}^{\wedge}_{+} \arrow[rr, "\phi"] \arrow[dr] & & q_m\,\mathsf{PaCD}^{\wedge}_{+}\ , \\
& q_m\,\mathsf{PaCD}^{\wedge}_{+} \arrow[ur, dotted, "\exists!\bar{\phi}"'] &
\end{tikzcd}$$

where we consider the quotient $q_m\,\mathsf{PaCD}^{\wedge}_{+}$ *of the operad of parenthesized chord diagrams* $\mathsf{PaCD}^{\wedge}_{+}$, *and we have an analogous statement for the morphism of unitary operads in complete Hopf groupoids* $\phi : \mathsf{PaA}^{\wedge}_{+} \to q_m\,\mathsf{PaA}^{\wedge}_{+}$ *which underlies such a morphism* $\phi : \mathsf{PaCD}^{\wedge}_{+} \to q_m\,\mathsf{PaCD}^{\wedge}_{+}$.

Proof. We apply the general result of Proposition 9.2.5 to the operads $\mathcal{G} = \mathsf{PaB}^{\wedge}_{+}, \mathsf{PaCD}^{\wedge}_{+}$ and $\mathcal{H} = \mathsf{CD}^{\wedge}_{+}, \mathsf{PaCD}^{\wedge}_{+}$ in order to get the assertions of this proposition. ☐

10.4.3. *The tower decomposition of the set of associators.* We now consider the set of operad morphisms $\phi : \mathsf{PaB}^{\wedge}_{+} \to q_{m+1}\,\mathsf{CD}^{\wedge}_{+}$ such that the factorization of Proposition 10.4.2(a) induces a categorical equivalence of operads in Malcev complete groupoids from $q_{m+1}\,\mathsf{PaB}^{\wedge}_{+}$ to $q_{m+1}\,\mathsf{CD}^{\wedge}_{+}$. We still use that any such morphism $\phi : \mathsf{PaB}^{\wedge}_{+} \to q_{m+1}\,\mathsf{CD}^{\wedge}_{+}$ occurs as the unique extension, to the Malcev complete operad $\mathsf{PaB}^{\wedge}_{+}$, of a morphism of operads in groupoids $\phi : \mathsf{PaB}_{+} \to q_{m+1}\,\mathsf{CD}^{\wedge}_{+}$ which is defined on the ordinary operad of parenthesized braids PaB_{+}. Then we can apply the result of Theorem 6.2.4 in order to determine such a morphism by giving the image of the braiding isomorphism and of the associativity isomorphism of PaB_{+} in the operad $q_{m+1}\,\mathsf{CD}^{\wedge}_{+}$. Furthermore, we can obviously adapt the analysis of §10.2 to the case of the operads $\mathsf{Q} = q_{m+1}\,\mathsf{CD}^{\wedge}_{+}$ in order to make this correspondence explicit.

We first get, as in Proposition 10.2.6, that a morphism of unitary operads in groupoids $\phi : \mathsf{PaB}_{+} \to q_{m+1}\,\mathsf{CD}^{\wedge}_{+}$ is determined by a pair $(\kappa, f(\xi_1,\xi_2))$, where κ is a scalar parameter such that $\phi(\tau) = e^{\kappa t_{12}/2}$, and where $f(\xi_1,\xi_2)$ now represents the class of a group-like power series in the quotient group $\mathbb{G}\,\hat{\mathbb{T}}(\xi_1,\xi_2)/\mathtt{F}_{m+2}\,\mathbb{G}\,\hat{\mathbb{T}}(\xi_1,\xi_2)$ so that we have the identity:

$$\phi(\alpha) \equiv f(t_{12},t_{23}) \pmod{\mathtt{F}_{m+2}\,\mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(3))}$$

in $\mathtt{Mor}_{q_{m+1}\,\mathsf{CD}(3)^{\wedge}}(pt, pt) = \mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(3))/\,\mathtt{F}_{m+2}\,\mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(3))$. We also get that this group-like power series $f(\xi_1,\xi_2)$ has to satisfy the relations of Proposition 10.2.7 modulo factors of filtration $\geq m+2$ in the Malcev complete groups where we express these relations. Moreover, we easily check that the morphism of operads in Malcev complete groupoids $\phi : \mathsf{PaB}_+^{\widehat{}} \to q_{m+1}\,\mathsf{CD}_+^{\widehat{}}$ which extends such a morphism $\phi : \mathsf{PaB}_+ \to q_{m+1}\,\mathsf{CD}_+^{\widehat{}}$ induces a categorical equivalence of operads from $q_{m+1}\,\mathsf{PaB}_+^{\widehat{}}$ to $q_{m+1}\,\mathsf{CD}_+^{\widehat{}}$ if and only if the scalar parameter which we associate to this morphism is invertible $\kappa \in \Bbbk^{\times}$.

We define $Ass^{\kappa}_{\langle m\rangle}(\Bbbk)$ as the set of (classes of) group-like power series $f(\xi_1,\xi_2) \in \mathbb{G}\,\hat{\mathbb{T}}(\xi_1,\xi_2)/\,\mathtt{F}_{m+2}\,\mathbb{G}\,\hat{\mathbb{T}}(\xi_1,\xi_2)$ which fulfill the equations of Proposition 10.2.7 modulo factors of filtration $\geq m+2$. We also define $Ass_{\langle m\rangle}(\Bbbk)$ as the union of the sets $Ass^{\kappa}_{\langle m\rangle}(\Bbbk)$ associated to an invertible parameter $\kappa \in \Bbbk^{\times}$. We still have a bijection $\rho_{\lambda} : Ass^{\kappa}_{\langle m\rangle}(\Bbbk) \xrightarrow{\simeq} Ass^{\lambda\kappa}_{\langle m\rangle}(\Bbbk)$, for every invertible scalar $\lambda \in \Bbbk^{\times}$, which is yielded by the re-scaling operation $\rho_{\lambda} : f(\xi_1,\xi_2) \mapsto f(\lambda\,\xi_1,\lambda\,\xi_2)$ at the power series level, and which reflects the action of the group $\Bbbk^{\times}$ on the chord diagram operad $\mathsf{CD}_+^{\widehat{}}$ (see §10.2.4).

We observed in the proof of Proposition 10.2.7 that the unit relations of the definition of an associator forces the vanishing of our power series $f(\xi_1,\xi_2)$ in weight 1. We accordingly get that any morphism $\phi : \mathsf{PaB}_+ \to q_1\,\mathsf{CD}_+^{\widehat{}}$, where we consider the first quotient $q_1\,\mathsf{CD}_+^{\widehat{}}$ of the chord diagram operad $\mathsf{CD}_+^{\widehat{}}$, is uniquely determined by the value of the scalar parameter $\kappa \in \Bbbk$ such that $\phi(\tau) = \exp(\kappa t_{12}/2)$. We equivalently have an identity:

$$\mathtt{Mor}_{\mathcal{G}rd\,\mathcal{O}p}(\mathsf{PaB}_+, q_1\,\mathsf{CD}_+^{\widehat{}}) = \exp(\Bbbk\,t_{12}),$$

when we consider the morphism set associated to these objects PaB_+ and $q_1\,\mathsf{CD}_+^{\widehat{}}$ in the category of unitary operads in groupoids, while the construction of the set $Ass_{\langle m\rangle}(\Bbbk)$ implies that we have a fibered product decomposition:

$$Ass_{\langle m\rangle}(\Bbbk) = \mathtt{Mor}_{\mathcal{G}rd\,\mathcal{O}p}(\mathsf{PaB}_+, q_{m+1}\,\mathsf{CD}_+^{\widehat{}}) \times_{\exp(\Bbbk\,t_{12})} \exp(\Bbbk^{\times}\,t_{12})$$

for each level $m \geq 0$.

We consider the obvious composition with the morphisms of the tower decomposition of the chord diagram operad $\mathsf{CD}_+^{\widehat{}} \to \cdots \to q_m\,\mathsf{CD}_+^{\widehat{}} \to \cdots \to q_1\,\mathsf{CD}_+^{\widehat{}}$ to define the morphisms of our tower decomposition of the set of associators. We easily see that these morphisms correspond to the obvious reduction operation in our power series description of the sets $Ass_{\langle m\rangle}(\Bbbk)$, $m \geq 0$. We have the relation

$$\mathsf{CD}_+^{\widehat{}} = \lim_m q_m\,\mathsf{CD}_+^{\widehat{}} \Rightarrow \mathtt{Mor}_{\mathcal{G}rd\,\mathcal{O}p}(\mathsf{PaB}_+, \mathsf{CD}_+^{\widehat{}}) = \lim_m \mathtt{Mor}_{\mathcal{G}rd\,\mathcal{O}p}(\mathsf{PaB}_+, q_{m+1}\,\mathsf{CD}_+^{\widehat{}})$$

at the morphism set level. From this relation, we readily get the identity:

$$Ass(\Bbbk) = \lim_m Ass_{\langle m\rangle}(\Bbbk)$$

when we pass to the set of associators. We can also deduce this relation from our power series description. We have a similar relation $Ass^{\kappa}(\Bbbk) = \lim_m Ass^{\kappa}_{\langle m\rangle}(\Bbbk)$ for the set of associators $Ass^{\kappa}(\Bbbk)$ associated to a fixed parameter $\kappa \in \Bbbk^{\times}$.

10.4.4. *The tower decomposition of the graded Grothendieck–Teichmüller group.* We proceed as in the previous paragraph to define the tower decomposition of the graded Grothendieck–Teichmüller group. To be explicit, we define the term $GRT_{\langle m\rangle}(\Bbbk)$ of this tower decomposition $GRT(\Bbbk) = \lim_m GRT_{\langle m\rangle}(\Bbbk)$ as the set of morphisms of operads in Malcev complete groupoids $\phi : \mathsf{PaCD}_+^{\widehat{}} \to q_{m+1}\,\mathsf{PaCD}_+^{\widehat{}}$

which are the identity at the object set level, fix the symmetry isomorphism $\phi(\tau) = \tau$, and are such that the factorization of Proposition 10.4.2(b) induces an isomorphism from the operad $q_{m+1}\,\mathsf{PaCD}^{\widehat{}}_{+}$ to itself. We provide this group $GRT_{\langle m\rangle}(\Bbbk)$ with the obvious composition operation, which we form at the level of the automorphism group of the operad $q_{m+1}\,\mathsf{PaCD}^{\widehat{}}_{+}$.

We can also use that a morphism of operads in Malcev complete groupoids $\phi : \mathsf{PaCD}^{\widehat{}}_{+} \to q_{m+1}\,\mathsf{PaCD}^{\widehat{}}_{+}$ is determined by an underlying morphism of operads in complete Hopf groupoids $\phi : \mathsf{PaA}^{\widehat{}}_{+} \to q_{m+1}\,\mathsf{PaA}^{\widehat{}}_{+}$. We then apply the result of Theorem 10.3.4 to determine these morphisms by fixing the image of the infinitesimal braiding and of the associativity homomorphism of $\mathsf{PaA}^{\widehat{}}_{+}$ in the operad $q_{m+1}\,\mathsf{PaA}^{\widehat{}}_{+}$. We adapt the analysis of §10.3 to the case of these operads $\mathsf{Q} = q_{m+1}\,\mathsf{PaA}^{\widehat{}}_{+}$ in order to make this correspondence explicit. We first get, as in Proposition 10.3.6, that our morphisms of unitary operads in complete Hopf groupoids $\phi : \mathsf{PaA}^{\widehat{}}_{+} \to q_{m+1}\,\mathsf{PaA}^{\widehat{}}_{+}$ are determined by pairs $(\lambda, f(\xi_1,\xi_2))$, where λ is a scalar parameter such that $\phi(\theta) = \lambda\theta$, and where $f(\xi_1,\xi_2)$ now represents the class of a group-like power series such that we have the identity:

$$\phi(\alpha) = \alpha\cdot f(t_{12},t_{23})\,(\operatorname{mod}\mathtt{F}_{m+2}\,\mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(3)))$$

in the quotient set:

$$\begin{aligned}\mathbb{G}\big(&\mathtt{Hom}_{q_{m+1}\,\mathsf{PaA}(3)^{\widehat{}}}((x_1x_2)x_3, x_1(x_2x_3))\big)\\ &= \alpha\cdot\mathbb{G}\big(\hat{\mathbb{U}}(\hat{\mathfrak{p}}(3)/\mathtt{F}_{m+2}\,\hat{\mathfrak{p}}(3))\big) = \alpha\cdot\mathbb{G}\big(\hat{\mathbb{U}}\hat{\mathfrak{p}}(3)\big)/\mathtt{F}_{m+2}\,\mathbb{G}\big(\hat{\mathbb{U}}\hat{\mathfrak{p}}(3)\big).\end{aligned}$$

We similarly see that this group-like power series $f(\xi_1,\xi_2)$ has to satisfy the equations of Proposition 10.3.7 modulo factors of filtration $\geq m+2$ in the Malcev complete groups where we express these relations. Moreover, we easily check that this morphism of operads in complete Hopf groupoids $\phi : \mathsf{PaA}^{\widehat{}}_{+} \to q_{m+1}\,\mathsf{PaA}^{\widehat{}}_{+}$ induces an automorphism on the operad $q_{m+1}\,\mathsf{PaA}^{\widehat{}}_{+}$ if and only if the scalar parameter which we associate to this morphism is invertible $\lambda \in \Bbbk^{\times}$.

We consider the obvious composition operation with the morphisms of the tower decomposition of the parenthesized chord diagram operad $\mathsf{PaCD}^{\widehat{}}_{+} \to \cdots \to q_m\,\mathsf{PaCD}^{\widehat{}}_{+} \to \cdots \to q_1\,\mathsf{PaCD}^{\widehat{}}_{+}$ to define the morphisms of our tower decomposition of the graded Grothendieck–Teichmüller group $GRT(\Bbbk) \to \cdots \to GRT_{\langle m\rangle}(\Bbbk) \to \cdots \to GRT_{\langle 0\rangle}(\Bbbk)$. We can identify these morphisms with the obvious reduction operation in our power series description of the groups $GRT_{\langle m\rangle}(\Bbbk)$ and we easily deduce the relation $GRT(\Bbbk) = \lim_m GRT_{\langle m\rangle}(\Bbbk)$ from this description. We also have the identity:

$$GRT_{\langle 0\rangle}(\Bbbk) = \Bbbk^{\times},$$

which still follows from the observation that the unit relations of Proposition 10.3.7 forces the vanishing relation $f(\xi_1,\xi_2) \equiv 1$ in the group $\mathbb{G}\,\hat{\mathbb{T}}(\xi_1,\xi_2)/\mathtt{F}_2\,\mathbb{G}\,\hat{\mathbb{T}}(\xi_1,\xi_2)$ (see the proof of Proposition 10.3.7 for details).

We readily check, from our operadic constructions, that the action of the graded Grothendieck–Teichmüller group $GRT(\Bbbk)$ on the set of associators $Ass(\Bbbk)$ decomposes as a levelwise action of the groups $GRT_{\langle m\rangle}(\Bbbk)$ on the sets $Ass_{\langle m\rangle}(\Bbbk)$ which define the tower decomposition of our object $Ass(\Bbbk) = \lim_m Ass(\Bbbk)$. We easily see that this action is simply transitive at each level too.

We can readily adapt the definition of the tower decomposition $GRT(\Bbbk) = \lim_m GRT_{\langle m\rangle}(\Bbbk)$ to the group $GRT^1(\Bbbk)$ of §10.3.11 which can accordingly be identified with the limit $GRT^1(\Bbbk) = \lim_m GRT^1_{\langle m\rangle}(\Bbbk)$ of a tower of groups such that

$GRT^1_{\langle m\rangle}(\Bbbk) = \ker(\lambda : GRT_{\langle m\rangle}(\Bbbk) \to \Bbbk^\times)$, where we consider an obvious analogue on the group $GRT_{\langle m\rangle}(\Bbbk)$ of the morphism $\lambda : GRT(\Bbbk) \to \Bbbk^\times$ of §10.3.11. We also have an identity $GRT_{\langle m\rangle}(\Bbbk) = \Bbbk^\times \ltimes GRT^1_{\langle m\rangle}(\Bbbk)$ at each level of our tower, and we can check that the action of the group $GRT_{\langle m\rangle}(\Bbbk)$ on $Ass_{\langle m\rangle}(\Bbbk)$ restricts to a simply transitive action of the group $GRT^1_{\langle m\rangle}(\Bbbk)$ on the set $Ass^\kappa_{\langle m\rangle}(\Bbbk)$, for any $\kappa \in \Bbbk^\times$.

10.4.5. *Pro-algebraic group structures.* In §10.3, we use the relation $\hat{\mathbb{T}}(\xi_1,\xi_2) = \hat{\mathbb{U}}\hat{\mathbb{L}}(\xi_1,\xi_2)$ and the exponential mapping to get that the group-like power series $f(\xi_1,\xi_2) \in \mathbb{G}\,\hat{\mathbb{T}}(\xi_1,\xi_2)$ in our description of the graded Grothendieck–Teichmüller group $GRT(\Bbbk)$ is determined by the exponential $f(\xi_1,\xi_2) = e^{p(\xi_1,\xi_2)}$ of a Lie power series $p(\xi_1,\xi_2) \in \hat{\mathbb{L}}(\xi_1,\xi_2)$.

For the quotient group $G = \mathbb{G}\,\hat{\mathbb{T}}(\xi_1,\xi_2)/\,\mathtt{F}_{m+2}\,\mathbb{G}\,\hat{\mathbb{T}}(\xi_1,\xi_2)$, which we use in the description of the groups $GRT_{\langle m\rangle}(\Bbbk)$, we have an identity $G = \mathbb{G}\,\hat{\mathbb{U}}(L)$, where we set $L = \hat{\mathbb{L}}(\xi_1,\xi_2)/\,\mathtt{F}_{m+2}\,\hat{\mathbb{L}}(\xi_1,\xi_2)$ (see Proposition 8.2.5). We deduce from this expression that the class of the group-like power series $f(\xi_1,\xi_2)$ which we consider in our description of $GRT_{\langle m\rangle}(\Bbbk)$ can be represented by the exponential $f(\xi_1,\xi_2) = e^{p(\xi_1,\xi_2)}$ of an element of this truncated free Lie algebra on two generators $p(\xi_1,\xi_2) \in L$. Let us observe that this truncated Lie algebra forms a module of finite rank over the ground field, since we have $L = \prod_{s=1}^{m+1} \mathbb{L}_s(\xi_1,\xi_2)$, where we use the notation $\mathbb{L}_s(\xi_1,\xi_2)$ for the homogeneous component of weight s of the free Lie algebra $\mathbb{L}(\xi_1,\xi_2)$ (as usual).

The elements of the group $GRT_{\langle m\rangle}(\Bbbk)$ are therefore parameterized by a finite number of variables. We easily see that the relations of Proposition 10.3.7 and the composition operation of Proposition 10.3.9 have an algebraic expression in terms of these variables. Each group $GRT_{\langle m\rangle}(\Bbbk)$ therefore inherits the structure of an algebraic group. We can use similar observations to check that $Ass_{\langle m\rangle}(\Bbbk)$ forms an algebraic torsor under the action of this group $GRT_{\langle m\rangle}(\Bbbk)$ in the sense of algebraic group theory (see the textbook [144, §III.4], for instance, for the explicit definition of the notion of a torsor).

In the case of the subgroup $GRT^1_{\langle m\rangle}(\Bbbk) = \ker(\lambda : GRT_{\langle m\rangle}(\Bbbk) \to \Bbbk^\times)$, one can observe further that this algebraic structure provides $GRT^1_{\langle m\rangle}(\Bbbk)$ with the structure of a unipotent algebraic group in the sense of §8.2.9 (see the overview of §10.4.12 for a hint on the proof of this observation). In what follows, we say that the group $GRT(\Bbbk)$ is *pro-algebraic* to assert that this group decomposes as the limit $GRT(\Bbbk) = \lim_m GRT_{\langle m\rangle}(\Bbbk)$ of a tower of algebraic groups $GRT_{\langle m\rangle}(\Bbbk)$, $m \geq 0$, while we say that the group $GRT^1(\Bbbk)$ is *pro-unipotent*.

10.4.6. *The graded Grothendieck–Teichmüller Lie algebra.* We use the notation $\mathfrak{grt}_{\langle m\rangle}$ for the Lie algebra of the algebraic group $GRT_{\langle m\rangle}(\Bbbk)$, for each $m \geq 0$. We also set $\mathfrak{grt} = \lim_m \mathfrak{grt}_{\langle m\rangle}$ to define the Lie algebra of the pro-algebraic group $GRT(\Bbbk)$. We adopt similar conventions for the tower of Lie algebras associated to the groups $GRT^1_{\langle m\rangle}(\Bbbk)$, $m \geq 0$. We have an identity $\mathfrak{grt} = \Bbbk \ltimes \mathfrak{grt}^1$ which reflects the semi-direct product decomposition of the graded Grothendieck–Teichmüller group $GRT(\Bbbk) = \Bbbk^\times \ltimes GRT^1(\Bbbk)$, and we similarly have $\mathfrak{grt}_{\langle m\rangle} = \Bbbk \ltimes \mathfrak{grt}^1_{\langle m\rangle}$, at each level $m \geq 0$.

We can easily get a description of these Lie algebras by using the explicit definition of the elements of the graded Grothendieck–Teichmüller group as pairs

$(\lambda, f(\xi_1,\xi_2))$, where $\lambda\in\Bbbk^{\times}$ and we assume $f(\xi_1,\xi_2) = e^{p(\xi_1,\xi_2)}$ for a Lie power series $p(\xi_1,\xi_2)\in\hat{\mathbb{L}}(\xi_1,\xi_2)$. We use standard methods of algebraic group theory to determine our Lie algebras from this representation (see for instance [180, §12.2]). In short, we consider a group-like element $f_{\epsilon}(\xi_1,\xi_2) = e^{\epsilon p(\xi_1,\xi_2)}$, where ϵ is a formal parameter such that $\epsilon^2 = 0$, and we expand the relations of Proposition 10.3.7 in terms of this parameter ϵ to get the defining equations of our Lie algebra.

We easily obtain that the Lie algebra $\mathfrak{grt}^1$ consists of the Lie power series $p(\xi_1,\xi_2)\in\hat{\mathbb{L}}(\xi_1,\xi_2)$ which satisfy:
(1) the unit relations $p(\xi_1,0) = 0 = p(0,\xi_2)$,
(2) the involution relation $p(\xi_1,\xi_2) + p(\xi_2,\xi_1) = 0$,
(3) the hexagon equation $p(\xi_3,\xi_1) + p(\xi_2,\xi_3) + p(\xi_1,\xi_2) = 0$, where (ξ_1,ξ_2,ξ_3) denotes a triple of variables such that $\xi_1+\xi_2+\xi_3 = 0$,
(4) the semi-classical hexagon equation $[\xi_2, p(\xi_1,\xi_2)] + [\xi_3, p(\xi_1,\xi_3)] = 0$, where (ξ_1,ξ_2,ξ_3) is again a triple of variables such that $\xi_1+\xi_2+\xi_3 = 0$,
(5) and the pentagon equation $p(t_{12}, t_{23}+t_{24}) + p(t_{13}+t_{23}, t_{34}) = p(t_{23}, t_{34}) + p(t_{12}+t_{13}, t_{24}+t_{34}) + p(t_{12}, t_{23})$ in the complete Lie algebra $\hat{\mathfrak{p}}(4)$.

We just consider truncated Lie power series $p(\xi_1,\xi_2)\in\hat{\mathbb{L}}(\xi_1,\xi_2)/\operatorname{F}_{m+2}\hat{\mathbb{L}}(\xi_1,\xi_2)$ in the case of the Lie algebras $\mathfrak{grt}^1_{\langle m\rangle}$, and we assume that the above equations are satisfied modulo terms of filtration $\geq m+2$ in the complete Lie algebras where we express these relations.

We can also get the expression of the Lie bracket on $\mathfrak{grt}^1$ in terms of this description by computing the differential of the adjoint action of the pro-algebraic group $GRT^1(\Bbbk)$ on $\mathfrak{grt}^1$ (see for instance [33, §§3.13-3.14]). We use the notation $\langle -,-\rangle$ for this Lie bracket which differs from the natural Lie bracket $[-,-]$ of Lie power series. We explicitly have the relation:

$$\langle p(\xi_1,\xi_2), q(\xi_1,\xi_2)\rangle = [p(\xi_1,\xi_2), q(\xi_1,\xi_2)] + D_p q(\xi_1,\xi_2) - D_q p(\xi_1,\xi_2),$$

for any $p(\xi_1,\xi_2), q(\xi_1,\xi_2)\in\mathfrak{grt}^1$, where $D_p : \hat{\mathbb{L}}(\xi_1,\xi_2)\to\hat{\mathbb{L}}(\xi_1,\xi_2)$ denotes the derivation of the free complete Lie algebra such that $D_p(\xi_1) = 0$ and $D_p(\xi_2) = [p(\xi_1,\xi_2),\xi_2]$, for any $p = p(\xi_1,\xi_2)\in\hat{\mathbb{L}}(\xi_1,\xi_2)$. For the semi-direct product $\mathfrak{grt} = \Bbbk\ltimes\mathfrak{grt}^1$, we get:

$$\big\langle(\lambda, p(\xi_1,\xi_2)),(\mu, q(\xi_1,\xi_2))\big\rangle = \big(0, \langle p(\xi_1,\xi_2), q(\xi_1,\xi_2)\rangle + \lambda\, Dq(\xi_1,\xi_2) - \mu\, Dp(\xi_1,\xi_2)\big)$$

for any $(\lambda, p(\xi_1,\xi_2)),(\mu, q(\xi_1,\xi_2))\in\mathfrak{grt}$, where $D : \hat{\mathbb{L}}(\xi_1,\xi_2)\to\hat{\mathbb{L}}(\xi_1,\xi_2)$ now denotes the derivation of the free complete Lie algebra $\hat{\mathbb{L}}(\xi_1,\xi_2)$ such that $Dq_n(\xi_1,\xi_2) = nq_n(\xi_1,\xi_2)$, for any Lie polynomial $q_n(\xi_1,\xi_2)\in\mathbb{L}_n(\xi_1,\xi_2)$ of homogeneous weight $n\geq 1$. This Lie bracket is often called the Ihara Lie bracket in the literature, because it corresponds to a Lie bracket introduced by Ihara for the study of actions of the absolute Galois group on a pro-l-version of the fundamental group of $\mathbb{P}^1\setminus\{0,1,\infty\}$ (see [93]).

We do not really use the expression of the Ihara Lie bracket in this book. We only use the defining equations (1-5) of the Lie algebra. We therefore skip the verification of the above formulas and we leave this computation as an exercise.

We are going to see that this Lie algebra $\mathfrak{grt}$ actually decomposes as a product of homogeneous components $\mathfrak{grt} = \prod_m \mathfrak{grt}_m$ and can therefore be identified with the obvious complete Lie algebra associated to a weight graded Lie algebra. We rely on the following observation:

Theorem 10.4.7 (V. Drinfeld [57, Proposition 5.7]). *The semi-classical hexagon equation (4) in the description of the graded Grothendieck–Teichmüller Lie algebra* $\mathfrak{grt}^1$ *in* §10.4.6 *is a consequence of the other relations (1), (2), (3), and (5).*

Proof (ideas and references). We give a summary of the ideas of the cited reference [57, Proposition 5.7]. We consider the Lie power series

$$s(\xi_2, \xi_3) = [\xi_2, p(\xi_1, \xi_2)] + [\xi_3, p(\xi_1, \xi_3)] \in \hat{\mathbb{L}}(\xi_2, \xi_3),$$

where we use the identity $\xi_1 = -\xi_2 - \xi_3$. We trivially have $s(\xi_2, \xi_3) = s(\xi_3, \xi_2)$. This Lie power series $s(\xi_2, \xi_3)$ vanishes in weight 2, because the unit relation (1) implies that the Lie power series $p(\xi_1, \xi_2)$ vanishes in weight 1 (see the proof of Proposition 10.2.7 for the detailed argument).

The first step of the proof of Theorem 10.4.7 consists in proving that this Lie power series satisfies the cocycle relation

$$\partial s(t_{14}, t_{24}, t_{34}) = s(t_{14}, t_{24}) - s(t_{14}, t_{24} + t_{34}) + s(t_{14} + t_{24}, t_{34}) - s(t_{24}, t_{34}) = 0$$

in the complete free Lie algebra $\hat{\mathbb{L}}(t_{14}, t_{24}, t_{34})$. We use a suitable combination of involution, pentagon and hexagon relations to get this result (see [57, Proposition 5.7] and [16, §4.5] for details on this verification). This equation asserts that the Lie power series $s(\xi_2, \xi_3)$ represents a cocycle of degree 2 in Lazard's complex of the Lie analyzer (see [114, §8]). This cocycle is automatically a coboundary, because the degree n cohomology of an analyzer vanishes in weight $m \neq n$ when we work over a field of characteristic zero (see [114, Théorème 10.1bis]), and we already observed that our Lie power series $s(\xi_2, \xi_3)$ has only components of weight $m \neq 2$. Hence, we have the coboundary relation $s(\xi_2, \xi_3) = q(\xi_2 + \xi_3) - q(\xi_2) - q(\xi_3) = \partial q(\xi_2, \xi_3)$, for some Lie power series on one variable $q(\xi) \in \hat{\mathbb{L}}(\xi)$, which necessarily reduces to a term of weight one $q(\xi) = k\xi$ since the complete free Lie algebra on one generator $\hat{\mathbb{L}}(\xi)$ vanishes outside this range. The conclusion $s(\xi_2, \xi_3) = 0$ follows. ☐

We study the weight decomposition of the Lie algebra $\mathfrak{grt}$ more thoroughly in a subsequent paragraph (see §10.4.12). We just use the following consequence of the result of Theorem 10.4.7 for the moment:

Proposition 10.4.8 (V. Drinfeld [57, Proof of Proposition 5.8], see also [16, Proof of Theorem 4]). *The morphisms* $p_m : \mathfrak{grt}_{\langle m\rangle} \to \mathfrak{grt}_{\langle m-1\rangle}$ *in the tower decomposition of the graded Grothendieck–Teichmüller Lie algebra* $\mathfrak{grt}$ *are surjective. The same is true for the morphisms* $p_m : GRT_{\langle m\rangle}(\Bbbk) \to GRT_{\langle m-1\rangle}(\Bbbk)$ *in the tower decomposition of the graded Grothendieck–Teichmüller group* $GRT(\Bbbk)$*, for any characteristic zero ground field* $\Bbbk$*, and we have obvious analogous results for the tower decomposition of the Lie algebra* $\mathfrak{grt}^1$ *of the group* $GRT^1(\Bbbk) = \ker(\lambda : GRT(\Bbbk) \to \Bbbk^{\times})$.

Proof. We can focus on the case of the Lie algebra $\mathfrak{grt}^1$ and of the group $GRT^1(\Bbbk)$ since we have the semi-direct product decompositions $\mathfrak{grt}_{\langle m\rangle} = \Bbbk \ltimes \mathfrak{grt}^1_{\langle m\rangle}$ and $GRT_{\langle m\rangle}(\Bbbk) = \Bbbk^{\times} \ltimes GRT^1_{\langle m\rangle}(\Bbbk)$ for each level $m \geq 0$.

We use that the relations of §10.4.6 decompose as a system of equations of homogeneous weight. We get that the verification of these equations modulo terms of filtration $\geq m + 2$ in the definition of the Lie algebra $\mathfrak{grt}^1_{\langle m\rangle}$ is equivalent to the verification of the homogeneous components of our equations in each weight $n \leq m+1$.

We see that the homogeneous components of weight n of the unit relations (1), of the involution relation (2), of the hexagon relation (3), and of the pentagon relation (5) of §10.4.6 only depend on the homogeneous component of weight n of our Lie power series $p(\xi_1, \xi_2) = \sum_n p_n(\xi_1, \xi_2)$, while the homogeneous component of weight n of the semi-classical hexagon relation (4) depends on the component $p_{n-1}(\xi_1, \xi_2)$ of weight $n-1$. We can trivially extend a truncated Lie power series $p(\xi_1, \xi_2) = \sum_{n=2}^m p_n(\xi_1, \xi_2) \in \hat{\mathbb{L}}(\xi_1, \xi_2)/\mathrm{F}_{m+1}\hat{\mathbb{L}}(\xi_1, \xi_2)$ which satisfies these equations in weight $n \leq m$ by zero $p_{m+1}(\xi_1, \xi_2) = 0$ to get a truncated power series which satisfies these equations in all weights $n \leq m+1$, including $n = m+1$, since the Lie polynomial $p_m(\xi_1, \xi_2)$ satisfies the semi-classical hexagon relation (4) as soon as it satisfies the other relations by Theorem 10.4.7. We conclude that the element of $\mathfrak{grt}^1_{\langle m-1\rangle}$ represented by this truncated power series $p(\xi_1, \xi_2) = \sum_{n=2}^m p_n(\xi_1, \xi_2) \in \hat{\mathbb{L}}(\xi_1, \xi_2)/\mathrm{F}_{m+1}\hat{\mathbb{L}}(\xi_1, \xi_2)$ has a pre-image in $\mathfrak{grt}^1_{\langle m\rangle}$, and hence, that the map $p_m : \mathfrak{grt}^1_{\langle m\rangle} \to \mathfrak{grt}^1_{\langle m-1\rangle}$ is surjective.

By general results of algebraic group theory (see for instance [180, Chapter 11-12]), this statement implies that the group morphism $p_m : GRT^1_{\langle m\rangle}(\Bbbk) \to GRT^1_{\langle m-1\rangle}(\Bbbk)$ is surjective too. $\square$

We use this proposition in the proof of the following statement:

PROPOSITION 10.4.9 (V. Drinfeld [57, Proposition 5.8], see also [16, Theorem 4, Corollary 4.1]). *The morphisms $p_m : Ass^{\kappa}_{\langle m\rangle}(\Bbbk) \to Ass^{\kappa}_{\langle m-1\rangle}(\Bbbk)$ in the tower decomposition of the set of associators $Ass^{\kappa}(\Bbbk) = \lim_m Ass^{\kappa}_{\langle m\rangle}(\Bbbk)$ are surjective, for any choice of characteristic zero ground field $\Bbbk$, and for every value of the scalar parameter $\kappa \in \Bbbk^{\times}$.*

PROOF. We check the case $\Bbbk = \mathbb{C}$ first. We fix $f_{m-1}(\xi_1, \xi_2) \in Ass^{\kappa}_{\langle m-1\rangle}(\mathbb{C})$. We have $f_{m-1}(\xi_1, \xi_2) = \phi_{m-1} \circ f_{KZ}(\xi_1, \xi_2)$ for some $\phi_{m-1} \in GRT^1_{\langle m-1\rangle}(\mathbb{C})$, where we consider the image of the Knizhnik–Zamolodchikov associator $f_{KZ}(\xi_1, \xi_2) \in Ass^{\kappa}(\mathbb{C})$ in $Ass^{\kappa}_{\langle m-1\rangle}(\mathbb{C})$. By Proposition 10.4.8, we have $\phi_{m-1} = p_m(\phi_m)$ for some $\phi_m \in GRT^1_{\langle m\rangle}(\mathbb{C})$, and we can just set $f_m(\xi_1, \xi_2) = \phi_m \circ f_{KZ}(\xi_1, \xi_2)$ to a get a pre-image of $f_{m-1}(\xi_1, \xi_2) \in Ass^{\kappa}_{\langle m-1\rangle}(\mathbb{C})$ in $Ass^{\kappa}_{\langle m\rangle}(\mathbb{C})$.

We now consider the case where $\Bbbk$ is an arbitrary ground field of characteristic zero. We fix $f_{m-1}(\xi_1, \xi_2) \in Ass^{\kappa}_{\langle m-1\rangle}(\Bbbk)$. We aim to prove the existence of a group-like element $f_m(\xi_1, \xi_2) \in \mathbb{G}\hat{\mathbb{T}}(\xi_1, \xi_2)/\mathrm{F}_{m+2}\mathbb{G}\hat{\mathbb{T}}(\xi_1, \xi_2)$ such that $f_m(\xi_1, \xi_2) \equiv f_{m-1}(\xi_1, \xi_2) \pmod{\mathrm{F}_{m+1}\mathbb{G}\hat{\mathbb{T}}(\xi_1, \xi_2)}$ which satisfies the relations of associators up to factors of filtration $\geq m+2$ in the Malcev complete groups where we express these relations (see §10.4.3).

We also have $f_{m-1}(\xi_1, \xi_2) = \exp(\sum_{n=2}^m p_n(\xi_1, \xi_2))$ for a truncated Lie power series $p(\xi_1, \xi_2) = \sum_{n=2}^m p_n(\xi_1, \xi_2) \in \hat{\mathbb{L}}(\xi_1, \xi_2)/\mathrm{F}_{m+1}\hat{\mathbb{L}}(\xi_1, \xi_2)$. We just have to add an extra term $p_{m+1}(\xi_1, \xi_2))$ to this Lie power series in order to get the exponential expression $f_m(\xi_1, \xi_2) = \exp(\sum_{n=2}^{m+1} p_n(\xi_1, \xi_2))$ of our lifting $f_m(\xi_1, \xi_2) \in \mathbb{G}\hat{\mathbb{T}}(\xi_1, \xi_2)/\mathrm{F}_{m+2}\mathbb{G}\hat{\mathbb{T}}(\xi_1, \xi_2)$. We take this extra term $p_{m+1}(\xi_1, \xi_2)$ as unknown variable.

The relation $f_{m-1}(\xi_1, \xi_2) \in Ass^{\kappa}_{\langle m-1\rangle}(\Bbbk)$ implies that our equations are satisfied modulo error terms in the subquotients $\mathrm{E}^0_{m+1} G = \mathrm{F}_{m+1} G/\mathrm{F}_{m+2} G$ of the

Malcev complete groups where we express these relations. We use the exponential correspondence (see §8.1.4 and Proposition 8.2.3) to express these error terms in terms of the graded Lie algebras of our Malcev complete groups. We readily check that the unknown variable $p_{m+1}(\xi_1, \xi_2)$ fits in a system of linear equations with rational coefficients whose second member is algebraically determined by the previous terms $p_n(\xi_1, \xi_2)$, $n \leq m$, of our Lie power series $p(\xi_1, \xi_2)$. We use that this system has a complex solution (according to the first verification of this proof) and we apply arguments of basic linear algebra to conclude from this result that our equations have a solution defined over the ground field $\Bbbk$ as well. $\square$

This proposition has the following immediate corollary:

Theorem 10.4.10 (V. Drinfeld [57, §5]). *The set of associators $Ass^{\kappa}(\Bbbk)$ is not empty, for any choice of characteristic zero ground field $\Bbbk$, and for every value of the scalar parameter $\kappa \in \Bbbk^{\times}$.* $\square$

From which we conclude:

Theorem 10.4.11. *There exists a categorical equivalence of unitary operads in Malcev complete groupoids $\phi : \mathsf{PaB}^{\widehat{}}_{+} \xrightarrow{\sim} \mathsf{CD}^{\widehat{}}_{+}$ between the Malcev completion of the parenthesized braid operad $\mathsf{PaB}^{\widehat{}}_{+}$ and the operad of chord diagrams $\mathsf{CD}^{\widehat{}}_{+}$, for any choice of ground field of characteristic zero $\Bbbk$.* $\square$

10.4.12. *The weight decomposition of the graded Grothendieck–Teichmüller Lie algebra and the filtration of the graded Grothendieck–Teichmüller group.* In the proof of Proposition 10.4.8, we observe that the unit relations (1), the involution relation (2), the hexagon relation (3), and the pentagon relation (5) of the definition of the Lie algebra $\mathfrak{grt}$ §10.4.6 decompose as a system of equations of homogeneous weight n which only depend on the homogeneous components of the same weight n of our Lie power series $p = p(\xi_1, \xi_2)$. The semi-classical hexagon equation §10.4.6(4), does not have this homogeneity property, but we can drop this equation since we checked in Theorem 10.4.7 that the semi-classical hexagon relation follows from the others equations of our system.

We can accordingly decompose the module $\mathfrak{grt}$ as a product of homogeneous components

$$\mathfrak{grt} = \textstyle\prod_{m=0}^{\infty} \mathfrak{grt}_m,$$

where $\mathfrak{grt}_0 = \Bbbk$ captures the scalar factor $\lambda \in \Bbbk$ which we consider in our description of §10.4.6, while $\mathfrak{grt}_m$ consists of the homogeneous Lie polynomials $p_m = p_m(\xi_1, \xi_2) \in \mathbb{L}_m(\xi_1, \xi_2)$ which satisfy our equations when $m \geq 1$. We also have $\mathfrak{grt}^1 = \prod_{m=1}^{\infty} \mathfrak{grt}_m$ when we consider the Lie algebra $\mathfrak{grt}^1 \subset \mathfrak{grt}$. We easily check that the Lie bracket of §10.4.6 satisfies $[\mathfrak{grt}_m, \mathfrak{grt}_n] \subset \mathfrak{grt}_{m+n}$, for all $m, n \geq 0$. We can therefore identify the Lie algebra $\mathfrak{grt}$ with the completion of the weight graded Lie algebra such that $\mathtt{E}^0 \mathfrak{grt} = \bigoplus_{m=0}^{\infty} \mathfrak{grt}_m$ with respect to the obvious filtration. We have a similar observation for the Lie algebra $\mathfrak{grt}^1 \subset \mathfrak{grt}$. We moreover have:

$$\mathfrak{grt}_{\langle m \rangle} = \mathfrak{grt} / \mathtt{F}_{m+1} \mathfrak{grt},$$

for each $m \geq 0$, where we consider the Lie algebra filtration $\mathfrak{grt} = \mathtt{F}_0 \mathfrak{grt} \supset \mathtt{F}_1 \mathfrak{grt} \supset \cdots \supset \mathtt{F}_m \mathfrak{grt} \supset \cdots$ such that $\mathtt{F}_m \mathfrak{grt} = \prod_{n \geq m} \mathfrak{grt}_n$, and the argument line of Proposition 10.4.8 can be rephrased as the application of the splitting formula $\mathfrak{grt}_{\langle m \rangle} = \mathfrak{grt}_{\langle m-1 \rangle} \oplus \mathfrak{grt}_m$ in the category of $\Bbbk$-modules.

We can also relate the weight graded Lie algebra $\mathtt{E}^0\,\mathfrak{grt}^1 = \bigoplus_{m=1}^{\infty}\mathfrak{grt}_m$ to the subquotients of a natural filtration of the group $GRT^1(\Bbbk)$. We just give a brief outline of this relationship in order to complete the account of this paragraph. We give more details in the case of an analogous result for the pro-unipotent Grothendieck–Teichmüller group in the next section. We consider the full graded Grothendieck–Teichmüller group first. The surjectivity of the map $p : GRT(\Bbbk) \to GRT_{\langle m\rangle}(\Bbbk)$ implies that we have the levelwise identity:

$$GRT_{\langle m\rangle}(\Bbbk) = GRT(\Bbbk)/\,\mathtt{F}_{m+1}\,GRT(\Bbbk),$$

for a nested sequence of subgroups of the Grothendieck–Teichmüller group

$$GRT(\Bbbk) = \mathtt{F}_0\,GRT(\Bbbk) \supset \cdots \supset \mathtt{F}_m\,GRT(\Bbbk) \supset \cdots$$

such that $\mathtt{F}_{m+1}\,GRT(\Bbbk) = \ker(p : GRT(\Bbbk) \to GRT_{\langle m\rangle}(\Bbbk))$, for each $m \geq 0$. We have $GRT(\Bbbk) = \lim_m GRT_{\langle m\rangle}(\Bbbk) \Leftrightarrow GRT(\Bbbk) = \lim_m GRT(\Bbbk)/\,\mathtt{F}_m\,GRT(\Bbbk)$, and we may check that this filtration satisfies the commutator relation

$$(\mathtt{F}_m\,GRT(\Bbbk), \mathtt{F}_n\,GRT(\Bbbk)) \subset \mathtt{F}_{m+n}\,GRT(\Bbbk),$$

for all $m, n \geq 0$ (see Proposition 11.4.2 for the detailed verification of an analogous statement in the case of the pro-unipotent Grothendieck–Teichmüller group).

We have $GRT_{\langle 0\rangle}(\Bbbk) = \Bbbk^{\times} \Leftrightarrow GRT^1(\Bbbk) = \mathtt{F}_1\,GRT(\Bbbk)$, and we now focus on the truncated filtration of this subgroup $GRT^1(\Bbbk)$. We can then use the commutator relation $(\mathtt{F}_m\,GRT(\Bbbk), \mathtt{F}_n\,GRT(\Bbbk)) \subset \mathtt{F}_{m+n}\,GRT(\Bbbk)$ to check that the pro-algebraic group $GRT^1(\Bbbk)$ is actually pro-unipotent, as we assert in §10.3.11. This commutator condition moreover implies that the weight graded module $\mathtt{E}^0\,GRT^1(\Bbbk) = \bigoplus_{m\geq 1}\mathtt{E}^0_m\,GRT(\Bbbk)$, where we consider the filtration subquotients $\mathtt{E}^0_m\,GRT(\Bbbk) = \mathtt{F}_m\,GRT(\Bbbk)/\,\mathtt{F}_{m+1}\,GRT(\Bbbk)$, inherits a Lie algebra structure (see §8.2.2). We also have an identity:

$$\mathtt{E}^0_m\,GRT(\Bbbk) = \mathtt{F}_m\,GRT(\Bbbk)/\,\mathtt{F}_{m+1}\,GRT(\Bbbk) = \mathfrak{grt}_m$$

for each weight $m \geq 1$. We moreover have the identity $\mathtt{E}^0\,\mathfrak{grt}^1 = \mathtt{E}^0\,GRT^1(\Bbbk)$ in the category of weight graded Lie algebras when we put these subquotients $\mathtt{E}^0_m\,GRT(\Bbbk)$ together and we again consider the weight graded Lie algebra $\mathtt{E}^0\,\mathfrak{grt}^1$ underlying $\mathfrak{grt}^1$. We give a detailed proof of an analogous relationship $\mathtt{E}^0_m\,GT(\Bbbk) = \mathfrak{grt}_m$ for the subquotients of the pro-unipotent Grothendieck–Teichmüller group in §11.1. We can deduce the above identity from similar argument lines, or from general results about the Lie algebra of unipotent (and of pro-unipotent) algebraic groups.

We explained in §9.2.6 the definition of a general notion of principal fiber for the natural tower decomposition $\mathcal{G} = \lim_m q_m\,\mathcal{G}$ of an operad in Malcev complete groupoids $\mathcal{G}$. We can actually relate the modules $\mathfrak{grt}_m = \mathtt{F}_m\,GRT(\Bbbk)/\,\mathtt{F}_{m+1}\,GRT(\Bbbk)$ to sets of morphisms with values in these principal fibers for the tower decomposition of the operad $\mathsf{PaCD}^{\widehat{}}$ which we consider in our definition of the tower decomposition $GRT(\Bbbk) = \lim_m GRT_{\langle m\rangle}(\Bbbk)$ of the group $GRT(\Bbbk)$. We explicitly have an identity between $\mathfrak{grt}_m$ and the image of the group $GRT_{\langle m\rangle}(\Bbbk)$ in the set of morphisms with values in the principal fiber of the morphism $q_m\,\mathsf{PaCD}^{\widehat{}} \to q_{m-1}\,\mathsf{PaCD}^{\widehat{}}$ at the mth level of our tower decomposition. We have an analogue of this identity for the pro-unipotent Grothendieck–Teichmüller group $GT(\Bbbk)$. We explain this case with full details in the next chapter. We can also relate the module $\mathfrak{grt}_m$ to the fibers of the tower decomposition of the operad $\mathsf{CD}^{\widehat{}}$, which we consider in our definition of the tower decomposition $Ass(\Bbbk) = \lim_m Ass_{\langle m\rangle}(\Bbbk)$ of the set of associators $\mathit{Ass}(\Bbbk)$. We just explain the latter interpretation of the module $\mathfrak{grt}_m$ to complete the account of this section. We determine the principal fibers of the tower decomposition of the chord diagram operad $\widehat{\mathit{CD}} = \lim q_m \widehat{\mathit{CD}}$ first.

10.4.13. *The fibers of the tower decomposition of the chord diagram operad.* In §10.4.1, we explain that the terms $q_m \widehat{\mathit{CD}}$ of the tower decomposition of the chord diagram operad $\widehat{\mathit{CD}} = \lim_m q_m \widehat{\mathit{CD}}$ can be defined by the formula

$$q_m \widehat{\mathit{CD}(r)} = \mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r))/\mathtt{F}_{m+1}\,\mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r)),$$

where, for any arity $r > 0$, we consider the quotient of the Malcev complete group $G = \mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r))$ by the $m+1$st layer of its canonical filtration $\mathtt{F}_{m+1}\,G = \mathtt{F}_{m+1}\,\mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r))$. We then use that the groupoids which form this operad $\widehat{\mathit{CD}}$ are groups, identified with groupoids with a single object, to simplify the general construction of §9.1.13. Recall that we also use the notation $\mathtt{F}_{m+1}\,\widehat{\mathit{CD}(r)} = \mathtt{F}_{m+1}\,\mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r))$ for this collection of subgroups.

We can similarly simplify the definition of the principal fibers of this tower of operads in groupoids. We obtain that these principal fibers are defined by the subquotients of the filtration of our group

$$\mathtt{E}^0_m\,\widehat{\mathit{CD}(r)} = \mathtt{F}_m\,\mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r))/\mathtt{F}_{m+1}\,\mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r)),$$

which we regard as a constant local coefficient system operad over the operad in groupoids with a single object $\widehat{\mathit{CD}}$.

We can moreover check that these objects are isomorphic to the homogeneous components of the weight graded Drinfeld–Kohno Lie algebra operad $\mathfrak{p}(-)_m$. We use that each of these collections $\mathfrak{p}(-)_m = \{\mathfrak{p}(r)_m, r > 0\}$ inherits the structure of an additive operad in $\Bbbk$-modules, and can also be identified with a constant local coefficient system operad over $\widehat{\mathit{CD}}$. Recall that we have $\mathfrak{p}(r)_m = \mathtt{E}^0\,\mathbb{P}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r))$, for each $r > 0$, by the version of the Milnor Moore Theorem for complete Hopf algebras (see Theorem 7.3.26). The exponential map induces an isomorphism from this $\Bbbk$-module $\mathfrak{p}(r)_m$ to the filtration subquotient $\mathtt{E}^0_m\,G = \mathtt{F}_m\,G/\mathtt{F}_{m+1}\,G$ of our group $G = \mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r))$ (see Proposition 8.2.3):

$$\mathfrak{p}(r)_m \xrightarrow[\simeq]{\exp} \mathtt{E}^0_m\,\mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r)) = \mathtt{E}^0_m\,\widehat{\mathit{CD}(r)},$$

and these maps clearly intertwine the structure operations of our operads.

We have an obvious generalizations of these identities in the unitary operad setting. We can moreover use these observations and the result of Theorem 10.4.11 to retrieve the claim of Theorem 10.1.3 because the morphism of unitary operads $\phi : \widehat{\mathit{PaB}}_+ \to \widehat{\mathit{CD}}_+$ which we define in this theorem is a categorical equivalence of operads in Malcev complete groupoids if and only if this morphism induces an isomorphism on the principal fibers of our tower decomposition (see Proposition 9.1.16).

10.4.14. *The fibers of the tower decomposition of the set of associators.* We now consider the morphism sets of operads $\mathtt{Mor}_{\mathcal{G}rd\,\mathcal{O}p}(\mathit{PaB}_+, \mathfrak{p}_+(-)_m)$, where we identify the additive operad $\mathfrak{p}_+(-)_m$ with an operad in groupoids with a single object. We see that this morphism set $\mathtt{Mor}_{\mathcal{G}rd\,\mathcal{O}p}(\mathit{PaB}_+, \mathfrak{p}_+(-)_m)$ inherits a $\Bbbk$-module structure, which is given by the underlying $\Bbbk$-module structure of our target object $\mathfrak{p}_+(-)_m$. We moreover get that this $\Bbbk$-module $\mathtt{Mor}_{\mathcal{G}rd\,\mathcal{O}p}(\mathit{PaB}_+, \mathfrak{p}_+(-)_m)$ operates on the morphism set $\mathtt{Mor}_{\mathcal{G}rd\,\mathcal{O}p}(\mathit{PaB}_+, q_m\,\widehat{\mathit{CD}}_+)$, for each $m \geq 1$, through the aritywise translation action of the group $\mathfrak{p}(r)_m \xrightarrow{\simeq} \mathtt{E}^0_m\,\widehat{\mathit{CD}(r)}$ on $q_m\,\widehat{\mathit{CD}(r)} =$

$\mathit{CD}(r)^{\widehat{}}/\mathtt{F}_{m+1}\,\mathit{CD}(r)^{\widehat{}}$. Besides, a pair of elements in this morphism set $\phi, \psi \in \mathtt{Mor}_{\mathcal{G}rd\,\mathcal{O}p}(\mathit{PaB}_+, q_m\,\mathit{CD}^{\widehat{}}_+)$ have the same image in $\mathtt{Mor}_{\mathcal{G}rd\,\mathcal{O}p}(\mathit{PaB}_+, q_{m-1}\,\mathit{CD}^{\widehat{}}_+)$ if and only if these morphisms differ by the action $\psi = \phi \cdot \exp(\theta)$ of such a morphism $\theta : \mathit{PaB}_+ \to \mathfrak{p}_+(-)_{m+1}$ with value in the additive operad $\mathfrak{p}_+(-)_{m+1}$.

We can use the identity $\mathit{Ass}_{\langle m\rangle}(\Bbbk) = \mathtt{Mor}_{\mathcal{G}rd\,\mathcal{O}p}(\mathit{PaB}_+, q_{m+1}\,\mathit{CD}^{\widehat{}}_+) \times_{\exp(\Bbbk\, t_{12})} \exp(\Bbbk^{\times}\, t_{12})$ (see §10.4.3) to check that similar observations hold for the sets of associators $\mathit{Ass}_{\langle m\rangle}(\Bbbk)$ provided that we restrict ourselves to the case $m \geq 1$. To sum up, we can regard each surjective map of sets $p_m : \mathit{Ass}_{\langle m\rangle}(\Bbbk) \to \mathit{Ass}_{\langle m-1\rangle}(\Bbbk)$ as a principal fibration with this $\Bbbk$-module

$$\mathtt{E}^0_m\,\mathit{Ass}(\Bbbk) = \mathtt{Mor}_{\mathcal{G}rd\,\mathcal{O}p}(\mathit{PaB}_+, \mathfrak{p}_+(-)_{m+1})$$

as structure group. In the formalism of algebraic geometry, we actually get that the variety $\mathit{Ass}_{\langle m\rangle}(\Bbbk)$ over $\mathit{Ass}_{\langle m-1\rangle}(\Bbbk)$ forms a torsor (in the relative sense) under the action of the group $\mathtt{E}^0_m\,\mathit{Ass}(\Bbbk)$.

We have the following additional observation:

PROPOSITION 10.4.15. *We have an identity* $\mathtt{Mor}_{\mathcal{G}rd\,\mathcal{O}p}(\mathit{PaB}_+, \mathfrak{p}_+(-)_{m+1}) = \mathfrak{grt}_m$*, for each* $m \geq 0$*, and hence we have* $\mathtt{E}^0_m\,\mathit{Ass}(\Bbbk) = \mathfrak{grt}_m$*, when* $m \geq 1$.

PROOF. We again use the result of Theorem 6.2.4 to get a description of the morphisms $\phi : \mathit{PaB}_+ \to \mathfrak{p}_+(-)_{m+1}$.

In the case $m = 0$, we easily check that we have $\phi(\tau) = \lambda t_{12}$, for a scalar $\lambda \in \Bbbk$, while the unit relation for the associativity isomorphism α implies $\phi(\alpha) = 0$. We therefore have $\mathtt{Mor}_{\mathcal{G}rd\,\mathcal{O}p}(\mathit{PaB}_+, \mathfrak{p}_+(-)_1) = \Bbbk = \mathfrak{grt}_0$.

In the case $m \geq 1$, we have $\mathfrak{p}(2)_{m+1} = \mathbb{L}(t_{12})_{m+1} = 0$ and $\mathfrak{p}(3) = \Bbbk(t_{12} + t_{23} + t_{13}) \oplus \mathbb{L}(t_{12}, t_{23}) \Rightarrow \mathfrak{p}(3)_{m+1} = \mathbb{L}_{m+1}(t_{12}, t_{23})$. We accordingly get that our morphism is determined by a homogeneous Lie polynomial $p_{m+1}(\xi_1, \xi_2) \in \mathbb{L}(\xi_1, \xi_2)$ such that $\phi(\alpha) = p_{m+1}(t_{12}, t_{23})$, while we trivially have $\phi(\tau) = 0$. We easily check that the coherence constraints of Theorem 6.2.4 are equivalent, for this Lie polynomial, to the unit, involution, hexagon, and pentagon equations (1, 2, 3, 5) of §10.4.6. We then use that the semi-classical equation (4) is implied by these equations (the result of Theorem 10.4.7) to get the identity $\mathtt{Mor}_{\mathcal{G}rd\,\mathcal{O}p}(\mathit{PaB}_+, \mathfrak{p}_+(-)_{m+1}) = \mathfrak{grt}_m$ asserted in the proposition in this case $m \geq 1$. ☐

We can use the levelwise action of the tower decomposition of the graded Grothendieck–Teichmüller groups $GRT(\Bbbk) = \lim_m GRT_{\langle m\rangle}(\Bbbk)$ on the tower decomposition of the set of associators $\mathit{Ass}(\Bbbk) = \lim_m \mathit{Ass}_{\langle m\rangle}(\Bbbk)$ to prolong the identities of this proposition to the subquotients $\mathtt{E}^0_m\,GRT(\Bbbk)$ of our filtration on $GRT(\Bbbk)$. We use a similar idea in the next chapter to get the relation

$$\mathtt{E}^0_m\,GT(\Bbbk) = \mathfrak{grt}_m$$

for the pro-unipotent Grothendieck–Teichmüller group $GT(\Bbbk)$, for each weight $m \geq 1$ (see Theorem 11.4.6).

CHAPTER 11

The Grothendieck–Teichmüller Group

We have several variants of the Grothendieck–Teichmüller group. We mostly deal with a pro-unipotent version of this group $GT(\Bbbk)$, defined over any characteristic zero field $\Bbbk$, and which, for us, occurs as a group of operad automorphisms of the Malcev completion of the unitary operad of parenthesized braids $\mathsf{PaB}^{\widehat{}}_{+}$. We also use the phrase 'Grothendieck–Teichmüller group' (with no further precision) to refer to this pro-unipotent version of the Grothendieck–Teichmüller group in what follows. This pro-unipotent Grothendieck–Teichmüller group $GT(\Bbbk)$ was introduced by Drinfeld in quantum group theory (see [57]). The idea of this definition comes from Grothendieck's program in Galois theory, where the goal is to encode the relations satisfied by the action of the absolute Galois group on curves. The group which encodes this information is a profinite analogue of our pro-unipotent Grothendieck–Teichmüller group. We only give a brief overview of the definition of this profinite Grothendieck–Teichmüller group and of the Grothendieck program in the next chapter.

Our first purpose, in this chapter, is to revisit Drinfeld's definition from an operadic viewpoint and to check the equivalence between our operadic definition of the group $GT(\Bbbk)$ and Drinfeld's original definition. We devote the first section of the chapter §11.1 to this subject.

In §10.3, we already explained the definition of an analogous group, the graded Grothendieck-Teichmüller group $GRT(\Bbbk)$, which represents a (subgroup of the) group of automorphisms of a chord diagram counterpart of the operad of parenthesized braids $\mathsf{PaCD}^{\widehat{}}_{+}$. We mainly use the graded Grothendieck–Teichmüller group as an auxiliary device, in order to prove the existence of rational categorical equivalences of operads in Malcev complete groupoids between the operad of parenthesized braids $\mathsf{PaB}^{\widehat{}}_{+}$ and the operad of chord diagrams $\mathsf{CD}^{\widehat{}}_{+}$, and in order to determine the subquotients of a natural filtration of the Grothendieck–Teichmüller group. We tackle the latter question in this chapter too. To be more specific, we are going to see that the Lie algebra of the graded Grothendieck–Teichmüller group, such as defined in §10.4.6, is identified with the weight graded Lie algebra associated to a natural filtration of the group $GT(\Bbbk)$. (We use this relationship in the next volume when we determine the homotopy automorphisms of E_2-operads.) We are also going to see that the existence of a categorical equivalence of operads in Malcev complete groupoids between the operad of parenthesized braids $\mathsf{PaB}^{\widehat{}}_{+}$ and the operad of chord diagrams $\mathsf{CD}^{\widehat{}}_{+}$ implies the existence of an isomorphism between the pro-unipotent Grothendieck–Teichmüller group $GT(\Bbbk)$ and the graded Grothendieck–Teichmüller group $GRT(\Bbbk)$. We explain this construction in the second section of the chapter §11.2. We then define an analogue, for the group $GT(\Bbbk)$, of the tower decomposition of the graded Grothendieck–Teichmüller group studied in §10.4. We determine the weight graded Lie algebra associated to the group $GT(\Bbbk)$ afterwards, by observing that the components of this weight graded Lie algebra $\mathtt{E}^0_m\, GT(\Bbbk)$ are isomorphic to the kernels of the morphisms $p_m : GT_{\langle m\rangle}(\Bbbk) \to GT_{\langle m-1\rangle}(\Bbbk)$ which occur in our tower decomposition $GT(\Bbbk) = \lim_m GT_{\langle m\rangle}(\Bbbk)$. We devote the third and fourth section of the chapter §§11.3-11.4 to this subject. We work with a fixed ground field of characteristic zero $\Bbbk$ all along this chapter and we carry out all our constructions in this setting.

11.1. The operadic definition of the Grothendieck–Teichmüller group

The pro-unipotent Grothendieck–Teichmüller group $GT(\Bbbk)$ is defined by Drinfeld in [57] as a group of power series that satisfy certain equations in the Malcev completion of the pure braid groups. The goal of this section, as we explain in the introduction of this chapter, is to revisit Drinfeld's approach and to explain that this group $GT(\Bbbk)$ can be interpreted as the group of automorphisms associated to the Malcev completion of the operad of parenthesized braids $\mathsf{PaB}^{\wedge}$. We give this operadic definition first. We prove the equivalence with Drinfeld's original definition afterwards.

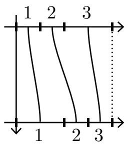
We are going to see that the group $GT(\Bbbk)$ is endowed with a split surjective morphism $\lambda : GT(\Bbbk) \to \Bbbk^{\times}$, like the graded Grothendieck–Teichmüller group $GRT(\Bbbk)$. We accordingly have $GT(\Bbbk) = \Bbbk^{\times} \ltimes GT^1(\Bbbk)$, where we set $GT^1(\Bbbk) = \ker(\lambda : GT(\Bbbk) \to \Bbbk^{\times})$ as in the case of $GRT(\Bbbk)$. This is this group $GT^1(\Bbbk)$ which is actually pro-unipotent. We just abusively refer to $GT(\Bbbk)$ as the *pro-unipotent Grothendieck–Teichmüller group* in order to distinguish this group from the other versions of the Grothendieck–Teichmüller group. This name is also motivated by the consideration of the Malcev completion of the pure braid groups (which form unipotent groups) in the definition of the group $GT(\Bbbk)$.

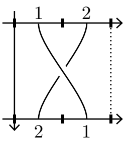
11.1.1. *Recollections on the operad of parenthesized braids.* We consider the Malcev completion of the operad of parenthesized braids $\mathsf{PaB}^{\wedge}$ and the associated unitary operad $\mathsf{PaB}^{\wedge}_{+}$. Recall that we have $\mathsf{PaB}^{\wedge}_{+}(r) = \mathsf{PaB}(r)^{\wedge}$ for all $r > 0$ and $\mathsf{PaB}^{\wedge}_{+}(0) = pt$ by definition of the unitary extension of an operad. For our constructions, we rely on the study of these operads in Malcev complete of groupoids (see §10), and on our study of the ordinary operad of parenthesized braids (see §6).

Briefly recall that we have the identity $\mathtt{Ob}\,\mathsf{PaB}^{\wedge} = \mathtt{Ob}\,\mathsf{PaB} = \Omega$ by construction, where we use the notation Ω for the magma operad (see §6.1.1), and that the automorphism group of any object in $\mathsf{PaB}(r)$ (respectively, in $\mathsf{PaB}(r)^{\wedge}$) is identified with the pure braid group on r strands P_r (respectively, with the Malcev completion of the pure braid group on r strands $\hat{P}_r$). In what follows, we also use the generating elements of the pure braid group P_r given in §10.0.1(1). Recall that we modify the picture of the elements of the braid groups by attaching the strands of our braids to the centers of a diadic decomposition of the interval (instead of the usual equidistant contact points) in order to materialize the source and target of morphisms in the parenthesized braid operad (see §6.2.1). We review specific examples of applications of this representation when we study the definition of elements in the Grothendieck–Teichmüller group (see the proof of Proposition 11.1.3).

Recall also that we use the notation $\mu(x_1, x_2) = x_1 x_2$ for the generator of the magma operad $\Omega(r)$. The morphism sets of the parenthesized braid operad contain an associativity isomorphism, which we denote by α and which makes this operation associative, together with a braiding morphism which we denote by τ and which relates the operation $\mu(x_1, x_2) = x_1 x_2$ to its transposite $(1\ 2)\mu(x_1, x_2) = x_2 x_1$.

Recall that these morphisms are given by the following pictures

$\alpha =$ 

and $\tau =$ 

in our representation of the parenthesized braid operad (see §6.2).

11.1.2. *The Grothendieck–Teichmüller group as a group of operad automorphisms.* We precisely define the Grothendieck–Teichmüller group $GT(\Bbbk)$ as the group formed by the automorphisms of the Malcev completion of the operad of parenthesized braid

$$\psi : \mathit{PaB}^{\wedge}_{+} \xrightarrow{\simeq} \mathit{PaB}^{\wedge}_{+}$$

which reduce to the identity map on the object sets of our operad $\mathtt{Ob}\,\mathit{PaB}^{\wedge}(r) = \Omega(r)$, for $r > 0$.

We use the same ideas as in our study of Drinfeld's associators in §10.2 in order to get an explicit description of these automorphisms which define the elements of the Grothendieck–Teichmüller group $GT(\Bbbk)$. We first use that any morphism of unitary operads in Malcev complete groupoids $\psi : \mathit{PaB}^{\wedge}_{+} \to \mathit{Q}$ occurs as the extension of a morphism

$$\psi : \mathit{PaB}_{+} \to \mathit{Q},$$

where we consider the ordinary operad of parenthesized braids PaB_{+} (Proposition 9.1.19). We then apply the result of Theorem 6.2.4 to reduce the construction of such a morphism $\psi : \mathit{PaB}_{+} \to \mathit{Q}$ to the definition of a product operation

$$m = m(x_1, x_2) \in \mathtt{Ob}\,\mathit{Q}(2),$$

of an associativity isomorphism

$$a = a(x_1, x_2, x_3) \in \mathtt{Mor}_{\mathit{Q}(3)}(m(m(x_1, x_2), x_3), m(x_1, m(x_2, x_3))),$$

and of a braiding

$$c = c(x_1, x_2) \in \mathtt{Mor}_{\mathit{Q}(2)^{\wedge}}(m(x_1, x_2), m(x_2, x_1)),$$

which respectively represent the image of the multiplication operation μ, of the associativity isomorphism α, and of the braiding τ of the parenthesized braid operad under our morphism $\psi : \mathit{PaB}_{+} \to \mathit{Q}$. We just set

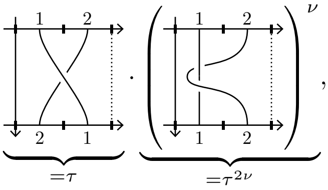
$$m = \mu \Rightarrow \psi(\mu) = \mu$$

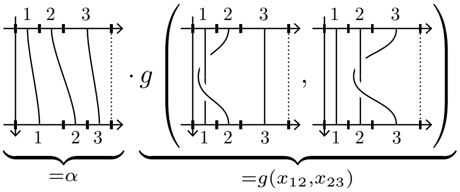
in our case since we assume that our morphism reduces to the identity map at the object set level.

In the general statement of Theorem 6.2.4, we also consider a unit object e in arity 0 in order to define the unitary extension of a morphism $\psi : \mathit{PaB} \to \mathit{Q}$ with values in an arbitrary operad Q. In our case where $\mathit{Q} = \mathit{PaB}^{\wedge}_{+}$ (and, more generally, when we assume that Q is a unitary operads) the image of this arity zero object is just fixed by the assumption that the groupoid $\mathit{PaB}^{\wedge}_{+}(0)$ reduces to the one-point set pt.

Thus, we only have to specify the image of the associativity isomorphism $a = \psi(\alpha)$ and of the braiding $c = \psi(\tau)$ in order to determine our morphisms $\psi : \mathsf{PaB}_+ \to \mathsf{PaB}_+^{\widehat{}}$. We have the following more explicit statement:

Proposition 11.1.3. *A morphism of unitary operads $\psi : \mathsf{PaB}_+ \to \mathsf{PaB}_+^{\widehat{}}$ which is the identity map on object sets is uniquely determined by a scalar parameter $\lambda = 1 + 2\nu \in \Bbbk$ and an element of the Malcev completion of the free group on two generators $g(x_1, x_2) \in \hat{\mathbb{F}}(x_1, x_2)$ such that we have the assignments:*

$\psi(\tau) =$ 

$\psi(\alpha) =$ 

for the symmetry isomorphism $\tau \in \mathtt{Mor}_{\mathsf{PaB}(2)}(x_1 x_2, x_2 x_1)$ and for the associativity isomorphism $\alpha \in \mathtt{Mor}_{\mathsf{PaB}(3)}((x_1 x_2) x_3, x_1(x_2 x_3))$ of the parenthesized braid operad PaB_+.

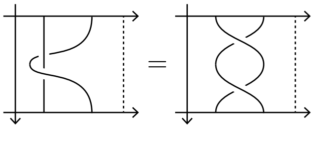
Explanations. We examine the structure of the morphism sets of the operad $Q = \mathsf{PaB}_+^{\widehat{}}$ in arity $r = 2, 3$ in order to determine the form of the isomorphisms

$$a(x_1, x_2, x_3) = \psi(\alpha) \in \mathtt{Mor}_{\mathsf{PaB}(3)^{\widehat{}}}((x_1 x_2) x_3, x_1(x_2 x_3)) \tag{1}$$

$$c(x_1, x_2) = \psi(\tau) \in \mathtt{Mor}_{\mathsf{PaB}(2)^{\widehat{}}}(x_1 x_2, x_2 x_1) \tag{2}$$

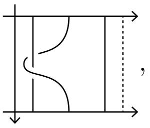
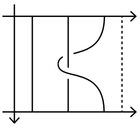
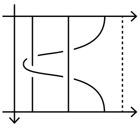
which determine our morphism $\psi : \mathsf{PaB}_+ \to \mathsf{PaB}_+^{\widehat{}}$. We use the same ideas as in Proposition 10.2.6 and Proposition 10.3.6, where we examine the operadic definition of the set of Drinfeld's associators and of the graded Grothendieck–Teichmüller group. We mainly have to deal with the Malcev completion of the pure braid groups instead of the algebras of the chord diagrams. We then use the relation $\mathtt{Aut}_{\mathsf{PaB}(r)^{\widehat{}}}(p) = \hat{P}_r$, which holds for any parenthesized word $p \in \Omega(r)$ (as we recall in §11.1.1), and we combine this relation with translation operations to represent the morphism set $\mathtt{Mor}_{\mathsf{PaB}(r)^{\widehat{}}}(p, q)$ associated to an arbitrary pair of objects $p, q \in \Omega(r)$.

In arity $r = 2$, we have $\mathtt{Aut}_{\mathsf{PaB}(2)^{\widehat{}}}(x_1 x_2, x_1 x_2) = \hat{P}_2$ and we accordingly get $\mathtt{Mor}_{\mathsf{PaB}(2)}(x_1 x_2, x_2 x_1) = \tau \cdot \hat{P}_2$, where we consider the Malcev completion of the pure braid group on 2 strands $\hat{P}_2$ and the translation of this group by the morphism $\tau \in \mathtt{Mor}_{\mathsf{PaB}(2)^{\widehat{}}}(x_1 x_2, x_2 x_1)$ in the groupoid $\mathsf{PaB}(2)^{\widehat{}}$. We moreover have $P_2 = \langle x_{12} \rangle$ for a generating element x_{12} such that $x_{12} = \tau^2$:

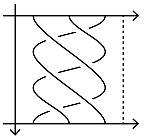
(3) $x_{12} =$ 

(see §10.0.1). We accordingly have $\hat{P}_2 = \{x_{12}^{\nu}, \nu \in \Bbbk\} = \{\tau^{2\nu}, \nu \in \Bbbk\}$, and we get $c(x_1, x_2) = \tau \cdot \tau^{2\nu} = \tau^{\lambda}$, for a formal exponent $\lambda \in \Bbbk$ such that $\lambda = 1 + 2\nu$ for some $\nu \in \Bbbk$.

In arity $r = 3$, we similarly have $\mathtt{Mor}_{\mathsf{PaB}(3)^{\wedge}}((x_1x_2)x_3, (x_1x_2)x_3) = \hat{P}_3$ and we therefore get $\mathtt{Mor}_{\mathsf{PaB}(3)^{\wedge}}((x_1x_2)x_3, (x_1x_2)x_3) = \alpha \cdot \hat{P}_3$, where we consider the translation of the Malcev completion of the pure braid group on 3 strands $\hat{P}_3$ by the morphism $\alpha \in \mathtt{Mor}_{\mathsf{PaB}(3)^{\wedge}}((x_1x_2)x_3, (x_1x_2)x_3)$ in the groupoid $\mathsf{PaB}(3)\hat{}$. Recall that the pure braid group on 3 strands P_3 is generated by the elements:

(4) $x_{12} =$  , $x_{23} =$  , $x_{13} =$  .

In §10.0.1 we also explain that the group P_3 is identified with the semi-direct product of the group P_2, which we identify with the subgroup generated by x_{12} inside P_3, and of a free group, which is generated by the elements x_{13} and x_{23} inside P_3. We can consider the pair of generators (x_{12}, x_{23}) instead of (x_{13}, x_{23}) in this semi-direct product decomposition. We can easily rely on this result to establish that P_3 is also isomorphic to the cartesian product of the free group generated by the elements x_{12} and x_{23} with a central cyclic subgroup $\langle z\rangle$ generated by the element

(5) $z =$ 

such that $z = x_{12}x_{23}x_{13}$ (see for instance [101, §1.3]).

We therefore have $a(x_1, x_2, x_3) = z^{\rho} g(x_{12}, x_{23})$, for some formal exponent $\rho \in \Bbbk$ of the central element z, and where $g(x_{12}, x_{23})$ is an element in the Malcev completion of the free group generated by the pure braids x_{12} and x_{23}. The unit relation $a(x_1, e, x_2) = id_{\mu(x_1,x_2)}$ in the correspondence of Theorem 6.2.4 implies that we have the relation $\partial_2(z)^{\rho} g(1, 1) = \partial_2(z)^{\rho} = 1$ in $\hat{P}_2$, where $\partial_2(z)$ denotes the result of the omission of the second strand in z. We easily get $\partial_2(z) = \tau^2$ and we deduce from this observation that the identity $\partial_2(z)^{\rho} = 1$ implies $\rho = 0$. We conclude from this result that the expression of our element $a = a(x_1, x_2, x_3)$ reduces to the factor $g = g(x_{12}, x_{23})$ in the Malcev completion of the free group $\mathbb{F}(x_{12}, x_{23})$. □

To complete this proposition, we write down the coherence constraints of Theorem 6.2.4 in terms of this pair $(\lambda, F(x_{12}, x_{23}))$ which we associate to a morphism of unitary operads $\psi : \mathsf{PaB}_+ \to \widehat{\mathsf{PaB}_+}$. We obtain the following proposition:

PROPOSITION 11.1.4. *The assignments of Proposition* 11.1.3*:*

$$\psi(\tau) = \tau^{1+2\nu} = \tau^{\lambda}, \quad \psi(\alpha) = \alpha \cdot g(x_{12}, x_{23}),$$

where we assume $\lambda = 1+2\nu \in \Bbbk$ *and* $g(x_1, x_2) \in \hat{\mathbb{F}}(x_1, x_2)$*, determine a well-defined morphism of unitary operads* $\psi : \mathsf{PaB}_+ \to \widehat{\mathsf{PaB}_+}$ *if and only if the element of the Malcev completion of the free group* $g = g(x_1, x_2) \in \hat{\mathbb{F}}(x_1, x_2)$ *satisfies:*

(1) *the unit relations* $g(x_1, 1) = 1 = g(1, x_2)$*,*
(2) *the involution relation* $g(x_1, x_2)g(x_2, x_1) = 1$*,*

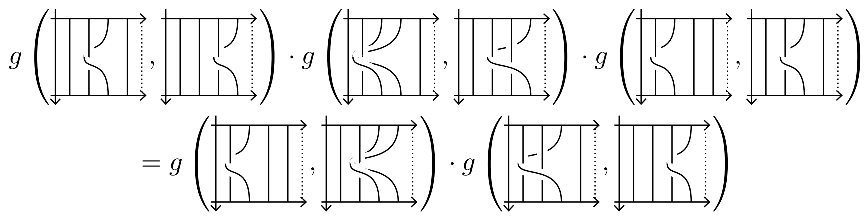


FIGURE 11.1. The pentagon constraints for the element of the Malcev completion of the free group $g = g(x_1, x_2)$ associated to an element of the Grothendieck–Teichmüller group $GT(\Bbbk)$. The relation holds in the Malcev completion of the pure braid group $\hat{P}_4$. The factors of this relation are obtained by applying $g = g(x_1, x_2)$ to the braids $\beta \in P_4$ represented in the picture. The relation also reads $g(x_{23}, x_{34})g(x_{13}x_{12}, x_{34}x_{24})g(x_{12}, x_{23}) = g(x_{12}, x_{24}x_{23})g(x_{23}x_{13}, x_{34})$ when we make explicit the expression of these braids in terms of the generators of the pure braid group.

---

(3) *the hexagon equation* $g(x_1, x_2)x_1^{\nu}g(x_3, x_1)x_3^{\nu}g(x_2, x_3)x_2^{\nu} = 1$, *where* (x_1, x_2, x_3) *denotes a triple of variables such that* $x_3x_2x_1 = 1$,

(4) *and the pentagon relation of Figure* 11.1.

PROOF. The claims of this proposition parallel the results of Proposition 10.2.7 and Proposition 10.3.7, where we examine the coherence constraints which arise from the operadic definition of Drinfeld's associators and from the operadic definition of elements in the graded Grothendieck–Teichmüller group. We make explicit the unit, pentagon and hexagon relations of Theorem 6.2.4(b-c) for the symmetry isomorphism $c = \tau^{1+2\nu}$ and the associativity isomorphism $a = \alpha \cdot g(x_{12}, x_{23})$ given in Proposition 11.1.3.

The reduction of the unit relation of Theorem 6.2.4 to the relation $g(x_1, 1) = 1 = g(1, x_2)$ is immediate. The equivalence between the pentagon relation, expressed by the commutativity of the diagram of Figure 6.1, and the equation of Figure 11.1 is immediate too (we just expand the expression of the factors in this equation).

The hexagon relations, expressed by the commutativity of the diagrams of Figure 6.6, are equivalent to the identities:

$$a = m(1, c)^{-1}a \cdot c(1, m) \cdot a \cdot m(c, 1)^{-1}, \tag{1}$$

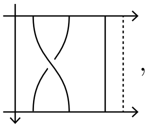
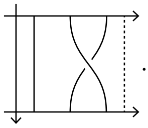
$$a^{-1} = m(c, 1)^{-1} \cdot a^{-1} \cdot c(m, 1) \cdot a^{-1} \cdot m(1, c)^{-1}. \tag{2}$$

The morphism sets where these equations are defined can be identified with $\hat{P}_3$ and we easily get that the above identities reduce to the relations:

$$g(x_{12}, x_{23}) = x_{23}^{-\nu}\tau_2^{-1} \cdot g(x_{12}, x_{23}) \cdot \tau_2(x_{12}x_{23})^{\nu}\tau_1 \cdot g(x_{12}, x_{23})\tau_1^{-1}x_{12}^{-\nu} \tag{3}$$

$$\begin{aligned} g(x_{12}, x_{23})^{-1} = x_{12}^{-\nu}\tau_1^{-1} \cdot g(x_{12}, x_{23})^{-1} \cdot \tau_1\tau_2(x_{23}x_{13})^{\nu} \\ \cdot g(x_{12}, x_{23})^{-1} \cdot x_{23}^{-\nu}\tau_2^{-1} \end{aligned} \tag{4}$$

where we consider the standard generators of the braid group B_3:

(5) $\tau_1 =$  , $\tau_2 =$  .

We can also rewrite Equation (4) as:

$$\begin{aligned}(6)\qquad \tau_2^{-1}g(x_{12},x_{23})^{-1}\tau_2 = \tau_2^{-1}x_{12}^{-\nu}\tau_2\cdot(\tau_1\tau_2)^{-1}g(x_{12},x_{23})^{-1}(\tau_1\tau_2)&\\ \cdot(x_{23}x_{13})^{\nu}\cdot g(x_{12},x_{23})^{-1}\cdot x_{23}^{-\nu}&.\end{aligned}$$

We have the relation $\beta g(x_{12},x_{23})\beta^{-1} = g(\beta x_{12}\beta^{-1},\beta x_{23}\beta^{-1})$ in $\hat{P}_3$ for every $\beta\in B_3$, and by Proposition 10.0.5:

$$(7)\qquad \tau_1\cdot x_{12}\cdot\tau_1^{-1} = x_{12},\qquad \tau_1\cdot x_{23}\cdot\tau_1^{-1} = x_{13},$$

$$(8)\qquad \tau_2^{-1}\cdot x_{12}\cdot\tau_2 = x_{13},\qquad \tau_2^{-1}\cdot x_{23}\cdot\tau_2 = x_{23},$$

$$(9)\qquad (\tau_1\tau_2)^{-1}\cdot x_{12}\cdot(\tau_1\tau_2) = x_{13},\qquad (\tau_1\tau_2)^{-1}\cdot x_{23}\cdot(\tau_1\tau_2) = x_{12}.$$

We therefore get that Equation (3) and Equation (6) are equivalent to the identities:

$$(10)\qquad g(x_{12},x_{23}) = x_{23}^{-\nu}g(x_{13},x_{23})(x_{12}x_{23})^{\nu}g(x_{12},x_{13})x_{12}^{-\nu},$$

$$(11)\qquad g(x_{13},x_{23})^{-1} = x_{13}^{-\nu}g(x_{13},x_{12})^{-1}(x_{23}x_{13})^{\nu}g(x_{12},x_{23})^{-1}x_{23}^{-\nu},$$

which also give:

$$(12)\qquad g(x_{12},x_{13}) = x_{13}^{\nu}g(x_{13},x_{23})^{-1}z^{-\nu}x_{23}^{\nu}g(x_{12},x_{23})x_{12}^{\nu},$$

$$(13)\qquad g(x_{13},x_{12})^{-1} = x_{13}^{\nu}g(x_{13},x_{23})^{-1}z^{-\nu}x_{23}^{\nu}g(x_{12},x_{23})x_{12}^{\nu},$$

where we consider the central element $z = x_{12}x_{23}x_{13}$ of the pure braid group (which we can permute with any formal expression in $\hat{P}_3$) and we use the relations $x_{12}x_{23} = zx_{13}^{-1}$ and $x_{23}x_{13} = x_{12}^{-1}z$ in P_3. We accordingly have:

$$(14)\qquad g(x_{12},x_{13}) = g(x_{13},x_{12})^{-1},$$

and we eventually get that the hexagon relations are equivalent to the combination of this identity with the equation:

$$(15)\qquad g(x_{13},x_{12})x_{13}^{\nu}g(x_{13},x_{23})^{-1}z^{-\nu}x_{23}^{\nu}g(x_{12},x_{23})x_{12}^{\nu} = 1.$$

The elements $x_1 = x_{12}$ and $x_2 = x_{13}$ generate a free group in P_3 (like x_{12} and x_{23}). We also have $z = x_{12}x_{23}x_{13} \Rightarrow z^{-1}x_{23} = x_{12}^{-1}x_{13}^{-1}$. We already used that we can permute this central element z with any formal expression of $\hat{P}_3$, and that we have the identity $(z\gamma)^{\nu} = z^{\nu}\gamma^{\nu}$, for any $\gamma\in\hat{P}_3$. We also have $g(x_{13},z^{-1}x_{23}) = g(x_{13},x_{23})$, because the relation $g(x_1,1) = g(1,x_2) = 1$ implies that the expansion of the element $g(x_1,x_2)\in\hat{\mathbb{F}}(x_1,x_2)$ starts with commutators in the representation of §8.4.2, and the central element z^{-1} obviously vanishes in such commutators. We similarly get $g(x_{12},z^{-1}x_{23}) = g(x_{12},x_{23})$. We can therefore rewrite Equation (15) under the form stated in the proposition by setting $x_1 = x_{12}$, $x_2 = x_{13}$, and $x_3 = z^{-1}x_{23}$. □

We easily see that the scalar parameter $\lambda\in\Bbbk$ which we associate to our morphism of unitary operads $\psi : \mathsf{PaB}_{+} \to \mathsf{PaB}_{+}^{\wedge}$ in Proposition 11.1.3 is necessarily invertible when we assume that this morphism extends to an isomorphism on the Malcev completion of the parenthesized braid operad $\mathsf{PaB}_{+}^{\wedge}$. We use the same argument as in our study of Drinfeld's associators to check this claim: we consider the morphism of Malcev complete groups $\psi : \mathtt{Aut}_{\mathit{PaB}(2)^\wedge}(\mu) \to \mathtt{Aut}_{\mathit{PaB}(2)^\wedge}(\mu)$ induced by our operad morphism $\psi : \mathit{PaB}_+^\wedge \to \mathit{PaB}_+^\wedge$ and we just observe that this morphism is identified with the map $\psi : \tau^{2\kappa} \mapsto \tau^{2\lambda\kappa}$ when we use the identity $\mathtt{Aut}_{\mathit{PaB}(2)^\wedge}(\mu) = \hat{P}_2 = \{\tau^{2\kappa}, \kappa \in \Bbbk\}$. We prove in the next proposition that this condition $\lambda \in \Bbbk^\times$ actually suffices to ensure that this morphism $\psi : \mathit{PaB}_+^\wedge \to \mathit{PaB}_+^\wedge$ defines a categorical equivalence (as in the associator case):

Proposition 11.1.5. *The morphism of unitary operads in Malcev complete groupoids $\psi : \mathit{PaB}_+^\wedge \to \mathit{PaB}_+^\wedge$ which we determine by the assignments of Proposition* 10.2.6 *is an isomorphism if and only if the scalar parameter which we associate to this morphism in our correspondence is invertible $\lambda \in \Bbbk^\times$.*

Proof. We only examine the "if" part of the proposition since we already checked the "only if" part. We therefore assume $\lambda \in \Bbbk^\times$. We fix an object $p \in \mathtt{Ob}\,\mathit{PaB}(r)$ in the operad of parenthesized braids, for some $r > 0$. We aim check that the morphism $\psi : \mathit{PaB}_+^\wedge \to \mathit{PaB}_+^\wedge$, which we determine by our assignments $\psi(\tau) = \tau^\lambda$, $\psi(\alpha) = \alpha \cdot g(x_{12}, x_{23})$, induces an isomorphism of Malcev complete groups on the automorphism group of this object in the Malcev completion of the parenthesized braid operad $\mathtt{Aut}_{\mathit{PaB}(r)^\wedge}(p) = \mathtt{Aut}_{\mathit{PaB}(r)}(p)^\wedge$.

We adapt the arguments of Proposition 10.2.8 and of Proposition 10.3.8 where we establish analogous results for the study of Drinfeld's associators and for the definition of elements of the graded Grothendieck–Teichmüller group. We use the identity $\mathtt{Aut}_{\mathit{PaB}(r)}(p) = P_r \Rightarrow \mathtt{Aut}_{\mathit{PaB}(r)^\wedge}(p) = \hat{P}_r$, which follows from the definition of the operad $\mathit{PaB}^\wedge$, and we use the isomorphism $\upsilon : \mathfrak{p}(r) \xrightarrow{\simeq} \mathtt{E}^0\,\hat{P}_r$ between the Drinfeld–Kohno Lie algebra $\mathfrak{p}(r)$ and the weight graded Lie algebra associated to the Malcev completion of the pure braid group P_r, as in the proof of Proposition 10.2.8. We just apply this relationship on both the source and the target of our morphism (while the target of the morphism studied in Proposition 10.2.8 is identified with the exponential group of the Lie algebra $\mathfrak{p}(r)$). Recall that this isomorphism $\upsilon : \mathfrak{p}(r) \xrightarrow{\simeq} \mathtt{E}^0\,\hat{P}_r$ associates the element $\bar{x}_{ij} \in \mathtt{E}^0\,\hat{P}_r$ to the generator t_{ij} of the Drinfeld–Kohno Lie algebra (see Theorem 10.0.7). We check that the morphism of weight graded Lie algebras $\mathtt{E}^0\,\psi : \mathtt{E}^0\,\mathtt{Aut}_{\mathit{PaB}(r)^\wedge}(p) \to \mathtt{E}^0\,\mathtt{Aut}_{\mathit{PaB}(r)^\wedge}(p)$ induced by our operad morphism $\psi : \mathit{PaB}_+^\wedge \to \mathit{PaB}_+^\wedge$ is an isomorphism to get our result.

We again use that the element of the automorphism group $\mathtt{Aut}_{\mathit{PaB}(r)^\wedge}(p)$ which corresponds to the generator x_{ij} of the pure braid group P_r can be expressed as a composite morphism:

$$u_{ij} = \beta \cdot \pi(x_1, \dots, \tau^2(x_i, x_j), \dots, \widehat{x_j}, \dots, x_r) \cdot \beta^{-1},$$

where β is a composite of braidings and associativity isomorphisms which we use to gather the variables (x_i, x_j) in the word $p = p(x_1, \dots, x_r)$, while the expression $\pi(x_1, \dots, \tau^2(x_i, x_j), \dots, \widehat{x_j}, \dots, x_r)$ represents the morphism of $\mathit{PaB}(r)$ obtained by plugging the morphism $\tau^2 \in \mathtt{Mor}_{\mathit{PaB}(2)}(\mu, \mu)$ in the parenthesized word on $r-1$ variables $\pi \in \Omega(r-1) = \mathtt{Ob}\,\mathit{PaB}(r-1)$ which we form in this gathering process. We have

$$\psi(u_{ij}) = \psi(\beta) \cdot \pi(x_1, \dots, \tau^{2\lambda}(x_i, x_j), \dots, \widehat{x_j}, \dots, x_r) \cdot \psi(\beta)^{-1}.$$

We can moreover write $\psi(\beta) = \beta\gamma$, where γ is an automorphism of the object $\pi(x_1, \dots, \mu(x_i, x_j), \dots, \widehat{x_j}, \dots, x_r)$ in the groupoid $\mathit{PaB}(r)^\wedge$. We again use the general relation $\gamma \cdot u \cdot \gamma^{-1} \equiv u \,(\mathrm{mod}\,\mathtt{F}_2\,G)$, valid in any Malcev complete group (see

Proposition 8.2.3 and §8.2.2), to obtain the relation:

$$\gamma\cdot\pi(x_1,\dots,\tau^{2\lambda}(x_i,x_j),\dots,\widehat{x_j},\dots,x_r)\cdot\gamma^{-1}\equiv\pi(x_1,\dots,\tau^{2\lambda}(x_i,x_j),\dots,\widehat{x_j},\dots,x_r)$$

in the quotient group $\hat{P}_r/\mathtt{F}_2\,\hat{P}_r$. We deduce from this identity that we also have the relation $\psi(u_{ij})\equiv x_{ij}^{\lambda}$ in $\hat{P}_r/\mathtt{F}_2\,\hat{P}_r$ when we consider the element of the group $\hat{P}_r$ which underlies our morphism. We eventually obtain that the map of weight graded Lie algebras $\mathtt{E}^0\,\psi:\mathtt{E}^0\,\mathtt{Aut}_{\mathsf{PaB}(r)^\wedge}(p)\to\mathtt{E}^0\,\mathtt{Aut}_{\mathsf{PaB}(r)^\wedge}(p)$ induced by our operad morphism $\psi:\mathsf{PaB}_+^\wedge\to\mathsf{PaB}_+^\wedge$ is given by the mapping $\mathtt{E}^0\,\psi:\bar{u}_{ij}\mapsto\lambda\bar{u}_{ij}$, for the element $\bar{u}_{ij}\in\mathtt{E}^0\,\mathtt{Aut}_{\mathsf{PaB}(r)^\wedge}(p)$ which corresponds to the class of the pure braid group generator x_{ij} in $\mathtt{E}^0\,\hat{P}_r$ and to the generating element of the Drinfeld–Kohno Lie algebra t_{ij} in $\mathfrak{p}(r)$. We accordingly have $\mathtt{E}^0\,\psi=\lambda\,Id$.

We conclude that this morphism $\mathtt{E}^0\,\psi:\mathtt{E}^0\,\mathtt{Aut}_{\mathsf{PaB}(r)^\wedge}(p)\to\mathtt{E}^0\,\mathtt{Aut}_{\mathsf{PaB}(r)^\wedge}(p)$ is an isomorphism as requested. ☐

We now consider the composite $\phi\circ\psi:\mathsf{PaB}_+^\wedge\to\mathsf{PaB}_+^\wedge$ of automorphisms $\phi,\psi:\mathsf{PaB}_+^\wedge\xrightarrow{\simeq}\mathsf{PaB}_+^\wedge$ in the Grothendieck–Teichmüller group $GT(\Bbbk)$. We assume that these automorphisms are, under the correspondence of Proposition 11.1.3, associated to the pairs $(\lambda,f(x_1,x_2)),(\mu,g(x_1,x_2))\in\Bbbk^\times\times\hat{\mathbb{F}}(x_1,x_2)$. We have the following statement:

Proposition 11.1.6. *The composite $\phi\circ\psi:\mathsf{PaB}_+^\wedge\to\mathsf{PaB}_+^\wedge$ of our morphisms $\phi,\psi:\mathsf{PaB}_+^\wedge\to\mathsf{PaB}_+^\wedge$ satisfies:*

$$(\phi\circ\psi)(\tau)=\tau^{\lambda\,\mu},$$
$$(\phi\circ\psi)(\alpha)=\alpha\cdot f(x_1,x_2)\cdot g(x_1^\lambda,f(x_1,x_2)^{-1}\cdot x_2^\lambda\cdot f(x_1,x_2)).$$

Proof. This proposition follows from a straightforward inspection which is similar to the one carried out in the proof of Proposition 10.3.9 (where we determine the product operation of the graded Grothendieck–Teichmüller group). In short, we just go back to the construction of Theorem 6.2.4 in order to determine the image of the morphisms of the Malcev completion of the parenthesized braid operad $\psi(\tau)=\tau^\lambda$ and $\psi(\alpha)=\alpha\cdot g(x_{12},x_{23})$ under the morphisms $\phi:\mathsf{PaB}_+^\wedge\to\mathsf{PaB}_+^\wedge$, and we easily check that we obtain the result stated in the proposition. ☐

We summarize our results in the following theorem:

Theorem 11.1.7 (Equivalence between the operadic approach and Drinfeld's definition of the Grothendieck–Teichmüller group [57, §4]). *The correspondence of Proposition* 10.3.6 *gives a one-to-one correspondence between the automorphisms of the Grothendieck–Teichmüller group $GT(\Bbbk)$ and the set of pairs $(\lambda,g(x_1,x_2))$, where λ is an invertible scalar parameter $\lambda\in\Bbbk^\times$, as we require in Proposition* 11.1.5, *and $g(x_1,x_2)$ is an element of the Malcev completion of the free group $g(x_1,x_2)\in\hat{\mathbb{F}}(x_1,x_2)$ which satisfies the unit, involution, hexagon, and pentagon relations (*1-4*) of Proposition* 11.1.4.

Furthermore, the composition operation of the group $GT(\Bbbk)$ corresponds on this set of pairs to the operation:

$$(\lambda,f(x_1,x_2))\star(\mu,g(x_1,x_2)):=(\lambda\,\mu,f(x_1,x_2)\cdot g(x_1^\lambda,f(x_1,x_2)^{-1}\cdot x_2^\lambda\cdot f(x_1,x_2)))$$

determined in Proposition 11.1.6. ☐

We mentioned in the introduction of this section that the group $GT(\Bbbk)$ admits a semi-direct product decomposition $GT(\Bbbk) = \Bbbk^{\times} \ltimes GT^1(\Bbbk)$, for a subgroup $GT^1(\Bbbk)$ equipped with a pro-unipotent structure. We explicitly define this group $GT^1(\Bbbk)$ as the kernel of the obvious morphism $\lambda : GT(\Bbbk) \rightarrow \Bbbk^{\times}$ which maps any element of the Grothendieck–Teichmüller group $\psi \in GT(\Bbbk)$ to the scalar factor of the above description $\lambda \in \Bbbk^{\times}$, and the existence of the semi-direct product decomposition $GT(\Bbbk) = \Bbbk^{\times} \ltimes GT^1(\Bbbk)$ is equivalent to the observation that this morphism $\lambda : GT(\Bbbk) \rightarrow \Bbbk^{\times}$ is split surjective. We establish this result in the next section, after checking that the Grothendieck–Teichmüller group $GT(\Bbbk)$ acts simply and transitively on the set of Drinfeld's associators.

11.2. The action on the set of Drinfeld's associators

We explained in the previous chapter that the set of Drinfeld's associators $Ass(\Bbbk)$ inherits a simply transitive action of the graded Grothendieck–Teichmüller group $GRT(\Bbbk)$. The first purpose of this section is to check that an analogous statement holds for the pro-unipotent Grothendieck–Teichmüller group $GT(\Bbbk)$. We observe that this action of $GT(\Bbbk)$ on the set of Drinfeld's associators $Ass(\Bbbk)$ commutes with the action of $GRT(\Bbbk)$ and we use this result to establish that the pro-unipotent Grothendieck–Teichmüller group $GT(\Bbbk)$ is actually isomorphic to the graded Grothendieck–Teichmüller group $GRT(\Bbbk)$. We also use the action of $GT(\Bbbk)$ on associators to construct the semi-direct product decomposition $GT(\Bbbk) = \Bbbk^{\times} \ltimes GT^1(\Bbbk)$ alluded to in the concluding paragraph of the previous section.

We rely on our operadic interpretation of the Grothendieck–Teichmüller group and of the set of associators. We more precisely use that an element of the Grothendieck–Teichmüller group $GT(\Bbbk)$ represents an automorphism $\psi : \mathsf{PaB}^{\wedge}_{+} \xrightarrow{\simeq} \mathsf{PaB}^{\wedge}_{+}$ of the Malcev completion of the operad of parenthesized braids $\mathsf{PaB}^{\wedge}_{+}$, while an element of the set of Drinfeld's associators $Ass(\Bbbk)$ represents a categorical equivalence $\phi : \mathsf{PaB}^{\wedge}_{+} \xrightarrow{\sim} \mathsf{CD}^{\wedge}_{+}$ from this operad $\mathsf{PaB}^{\wedge}_{+}$ to the operad of chord diagrams $\mathsf{CD}^{\wedge}_{+}$. We just consider the obvious composition operation

$$\begin{tikzcd} \mathsf{PaB}^{\wedge}_{+} \arrow[r, "\psi"] \arrow[rr, dotted, bend right, "\phi\circ\psi"'] & \mathsf{PaB}^{\wedge}_{+} \arrow[r, "\phi"] & \mathsf{CD}^{\wedge}_{+} \end{tikzcd}$$

in our morphisms sets and we immediately get our result:

Proposition 11.2.1. *The above construction gives a simply transitive action of the pro-unipotent Grothendieck–Teichmüller group $GT(\Bbbk)$ on the set of Drinfeld's associator $Ass(\Bbbk)$.*

Furthermore, if we assume that the automorphism $\psi : \mathsf{PaB}^{\wedge}_{+} \xrightarrow{\simeq} \mathsf{PaB}^{\wedge}_{+}$ which represents our element of the Grothendieck–Teichmüller group $GT(\Bbbk)$ is associated to the pair $(\lambda, g(x_1, x_2))$ in the correspondence of Proposition 11.1.3*, while the categorical equivalence of operads $\phi : \mathsf{PaB}^{\wedge}_{+} \xrightarrow{\sim} \mathsf{CD}^{\wedge}_{+}$, which represents our element of the set of associators $Ass(\Bbbk)$, is associated by the pair $(\kappa, f(\xi_1, \xi_2))$ in the correspondence of Proposition* 10.2.6*, then our composition operation on morphisms $\phi \circ \psi : \mathsf{PaB}^{\wedge}_{+} \rightarrow \mathsf{CD}^{\wedge}_{+}$ corresponds to the operation on these pairs such that:*

$$(\kappa, f(\xi_1, \xi_2)) \star (\lambda, g(x_1, x_2)) := (\kappa\,\lambda, f(\xi_1, \xi_2) \cdot g(e^{\kappa\xi_1}, f(\xi_1, \xi_2)^{-1} \cdot e^{\lambda\xi_2} \cdot f(\xi_1, \xi_2))).$$

PROOF. The fact that our action is simple and transitive is a consequence of the result of Proposition 10.3.12, where we check that any categorical equivalence of operads in Malcev complete groupoids $\phi : \mathsf{PaB}^{\widehat{}}_{+} \xrightarrow{\sim} \mathsf{CD}^{\widehat{}}_{+}$ uniquely lifts to an isomorphism $\tilde{\phi} : \mathsf{PaB}^{\widehat{}}_{+} \xrightarrow{\cong} \mathsf{PaCD}^{\widehat{}}_{+}$ which is defined by the identity map on the object sets of our operads. Indeed, the action of the Grothendieck–Teichmüller group $GT(\Bbbk)$ on the set of associators $Ass(\Bbbk)$ corresponds to the obvious translation action of the automorphisms of the Grothendieck–Teichmüller group $\psi : \mathsf{PaB}^{\widehat{}}_{+} \xrightarrow{\cong} \mathsf{PaB}^{\widehat{}}_{+}$ on this set of isomorphisms $\tilde{\phi} : \mathsf{PaB}^{\widehat{}}_{+} \xrightarrow{\cong} \mathsf{PaCD}^{\widehat{}}_{+}$, and this translation action is clearly simple and transitive.

To get the second claim of the proposition, we compute the image of the braiding τ and of the associativity isomorphism α under our composite morphisms $\phi \circ \psi : \mathsf{PaB}^{\widehat{}}_{+} \to \mathsf{CD}^{\widehat{}}_{+}$. We get the formula given in the proposition after a straightforward inspection of our correspondence as usual. ☐

We can now check the claims asserted in the introduction of this section:

PROPOSITION 11.2.2.

(a) *Each element of the set of Drinfeld's associators* $\phi \in Ass(\Bbbk)$ *determines a section* $s_{\phi} : \Bbbk^{\times} \to GT(\Bbbk)$ *of the natural morphism* $\lambda : GT(\Bbbk) \to \Bbbk^{\times}$ *(the morphism which maps any element of the Grothendieck–Teichmüller group to the corresponding scalar factor* $\lambda \in \Bbbk^{\times}$ *in the description of Theorem* 11.1.7*).*

(b) *Each element of the set of Drinfeld's associators* $\phi \in Ass(\Bbbk)$ *determines an isomorphism*

$$v_{\phi} : GRT(\Bbbk) \xrightarrow{\cong} GT(\Bbbk)$$

between the graded Grothendieck–Teichmüller group $GRT(\Bbbk)$ *and the pro-unipotent Grothendieck–Teichmüller group* $GT(\Bbbk)$*.*

EXPLANATIONS AND REFERENCES. We already observed in §10.3.11 that the morphism $\lambda : GRT(\Bbbk) \to \Bbbk^{\times}$ analogous to $\lambda : GT(\Bbbk) \to \Bbbk^{\times}$ admits a section when we work with the graded Grothendieck–Teichmüller group $GRT(\Bbbk)$. We can use this result to deduce the first claim of this proposition from the second one, but we prefer to give a proof of our claim which is independent of this statement. We use the action of the multiplicative group on the set of associators. We explain in §10.2.11 that this action corresponds to the composition of the categorical equivalences $\phi : \mathsf{PaB}^{\widehat{}}_{+} \xrightarrow{\sim} \mathsf{CD}^{\widehat{}}_{+}$, which represent the elements of the set of associators, with automorphisms of the chord diagram operad $\rho_{\lambda} : \mathsf{CD}^{\widehat{}}_{+} \xrightarrow{\cong} \mathsf{CD}^{\widehat{}}_{+}$, which we associate to any scalar parameter $\lambda \in \Bbbk^{\times}$. We immediately get that this action of the multiplicative group $\Bbbk^{\times}$ on $Ass(\Bbbk)$ commutes with the action of the Grothendieck–Teichmüller group $GT(\Bbbk)$, because the former is defined by composition of morphisms on the left $\phi \mapsto \rho_{\lambda} \circ \phi$ while the latter is given by a composition of morphisms on the right $\phi \mapsto \phi \circ \psi$, for any categorical equivalence $\phi : \mathsf{PaB}^{\widehat{}}_{+} \xrightarrow{\sim} \mathsf{CD}^{\widehat{}}_{+}$ which we use to represent an element of the set of associators $Ass(\Bbbk)$.

We fix such a categorical equivalence $\phi : \mathsf{PaB}^{\widehat{}}_{+} \xrightarrow{\sim} \mathsf{CD}^{\widehat{}}_{+}$, which represents an element of the set of associators $Ass(\Bbbk)$. To a scalar parameter $\lambda \in \Bbbk^{\times}$, we associate the element of the Grothendieck–Teichmüller group $\psi_{\lambda} \in GT(\Bbbk)$ determined by the equation:

$$\rho_{\lambda} \circ \phi = \phi \circ \psi_{\lambda},$$

where we use that $GT(\Bbbk)$ acts simply and transitively on the set of associators $Ass(\Bbbk)$. We easily check that this mapping $s_{\phi} : \lambda \mapsto \psi_{\lambda}$ defines a group morphism. (We mainly use that our composition actions commute to each other to establish this assertion.) We immediately see that this mapping $s_{\phi} : \lambda \mapsto \psi_{\lambda}$ defines a section of the canonical morphism $\lambda : GT(\Bbbk) \to \Bbbk^{\times}$ too.

We adapt this argument line to associate a group isomorphism $\upsilon_{\phi} : GT(\Bbbk) \xrightarrow{\simeq} GRT(\Bbbk)$ to any associator $\phi \in Ass(\Bbbk)$. We then use the action of the group $GRT(\Bbbk)$ on $Ass(\Bbbk)$. Recall that this action is given by the composition of the automorphisms $\rho : \mathsf{PaCD}^{\wedge}_{+} \to \mathsf{PaCD}^{\wedge}_{+}$ which represent the elements of the graded Grothendieck–Teichmüller group $GRT(\Bbbk)$ with a lifting, to the operad of parenthesized chord diagrams $\mathsf{PaCD}^{\wedge}_{+}$, of the categorical equivalences $\phi : \mathsf{PaB}^{\wedge}_{+} \to \mathsf{CD}^{\wedge}_{+}$ which represent the elements of the set of associators. We easily check that the composition with the automorphisms of the pro-unipotent Grothendieck–Teichmüller group commutes with this lifting operation (we already used this observation in the proof of Proposition 11.2.1). We again get that the action of the groups $GRT(\Bbbk)$ and $GT(\Bbbk)$ on $Ass(\Bbbk)$ commute to each other since we take a composition of morphisms on the left in one case, and a composition on the right in the other case.

We then determine the element of the pro-unipotent Grothendieck–Teichmüller group $\psi_{\rho} \in GT(\Bbbk)$ which we associate to any $\rho \in GRT(\Bbbk)$ by the equation:

$$\rho \circ \phi = \phi \circ \psi_{\rho},$$

where we again use that $GT(\Bbbk)$ acts simply and transitively on the set of associators $Ass(\Bbbk)$. We still check that this mapping $\upsilon_{\phi} : \rho \mapsto \psi_{\rho}$ defines a group morphism, and this morphism is clearly bijective too, since both actions considered in our equation are simple and transitive.

We immediately see that this isomorphism fits in a commutative diagram

$$\begin{tikzcd}
GRT(\Bbbk) \arrow[rr, dotted, "\upsilon_{\phi}", "\simeq"'] \arrow[dr, shift right, "\lambda"'] & & GT(\Bbbk) \arrow[dl, "\lambda"] \\
& \Bbbk^{\times} \arrow[ul, shift right] \arrow[ur, dotted, "s_{\phi}"] &
\end{tikzcd} ,$$

when we consider the morphisms defined on our groups which take values in the multiplicative group $\Bbbk^{\times}$ and their sections. We can therefore retrieve the construction of the first assertion of the proposition from the construction of our second assertion. ☐

The mapping $s : \phi \mapsto s_{\phi}$, given by the construction of this proposition, actually gives a one-to-one correspondence between the set of associators $Ass(\Bbbk)$ and the set of sections of the morphism $\lambda : GT(\Bbbk) \to \Bbbk^{\times}$. In [57, Proposition 5.2-5.3], Drinfeld uses this observation (and arguments of algebraic group theory) in order to give a second proof of the existence of rational associators. In short, one can prove that the existence of a section defined over the field of complex numbers $\Bbbk = \mathbb{C}$ (which we may associate to the Knizhnik–Zamolodchikov associator) implies the existence of a section over any characteristic zero ground field $\Bbbk$.

11.2.3. *The semi-direct product decomposition of the Grothendieck–Teichmüller group.* We already briefly explained that we take the kernel of the obvious morphism $\lambda : GT(\Bbbk) \to \Bbbk^{\times}$ to define an analogue of the subgroup $GRT^{1}(\Bbbk)$ of the graded

Grothendieck–Teichmüller group $GRT(\Bbbk)$ in $GT(\Bbbk)$:

$$GT^1(\Bbbk) := \ker(\lambda : GT(\Bbbk) \to \Bbbk^\times).$$

We can equivalently define this group $GT^1(\Bbbk)$ as the subgroup of the Grothendieck–Teichmüller group $GT(\Bbbk)$ formed by the automorphisms $\psi : \mathsf{PaB}^{\widehat{}}_+ \to \mathsf{PaB}^{\widehat{}}_+$ which fix the braiding isomorphism $\phi(\tau) = \tau$. We clearly see that this subgroup $GT^1(\Bbbk)$ correspond to the pairs $(\lambda, f(x_1, x_2))$ such that $\lambda = 1$ in the description of Proposition 11.1.3.

We use the result of Proposition 11.2.2(a) to get the semi-direct product decomposition of the group $GT(\Bbbk)$ alluded to in the introduction of this chapter:

$$GT(\Bbbk) = \Bbbk^\times \ltimes GT^1(\Bbbk).$$

We also mentioned that the group $GT^1(\Bbbk)$ has a pro-unipotent structure in the introduction of this chapter. In a first step, one can see that the group $GT(\Bbbk)$ admits a tower decomposition $GT(\Bbbk) = \lim_m GT_{\langle m\rangle}(\Bbbk)$, whose terms $GT_{\langle m\rangle}(\Bbbk)$ form (affine) algebraic groups. (We define this decomposition of the group $GT(\Bbbk)$ in the next section.) We have an analogous decomposition $GT^1(\Bbbk) = \lim_m GT^1_{\langle m\rangle}(\Bbbk)$ for our subgroup $GT^1(\Bbbk) \subset GT(\Bbbk)$. Then one can observe that the terms of this tower decomposition of the Grothendieck–Teichmüller group $GT(\Bbbk)$ are identified with the quotient groups $q_m\, GT(\Bbbk) = GT(\Bbbk)/\,\mathtt{F}_{m+1}\, GT(\Bbbk)$ associated to a filtration $GT(\Bbbk) = \mathtt{F}_0\, GT(\Bbbk) \supset \mathtt{F}_1\, GT(\Bbbk) \supset \cdots \supset \mathtt{F}_n\, GT(\Bbbk) \supset \cdots$ which satisfies the commutator condition $(\mathtt{F}_m\, GT(\Bbbk), \mathtt{F}_n\, GT(\Bbbk)) \subset \mathtt{F}_{m+n}\, GT(\Bbbk)$. (We establish this result in §11.4.) We can actually identify $GT^1(\Bbbk)$ with the first layer of this filtration $\mathtt{F}_1\, GT(\Bbbk) \subset GT(\Bbbk)$. We moreover have $GT^1_{\langle m\rangle}(\Bbbk) = GT^1(\Bbbk)/\,\mathtt{F}_{m+1}\, GT(\Bbbk) = \mathtt{F}_1\, GT(\Bbbk)/\,\mathtt{F}_{m+1}\, GT(\Bbbk)$. We just put these results together to conclude that each group $GT^1_{\langle m\rangle}(\Bbbk)$ forms a unipotent algebraic group in the sense of algebraic group theory (see §8.2.9).

We just consider the basic description of our tower decomposition and of the filtration of the group $GT(\Bbbk)$ in what follows. We therefore focus on this side of the subject and we leave the study of the algebraic group structure of our objects to interested readers.

11.3. Tower decompositions

We study a counterpart, for the pro-unipotent Grothendieck–Teichmüller group $GT(\Bbbk)$, of the tower decompositions which we associate to the graded Grothendieck–Teichmüller group $GRT(\Bbbk)$ and to the set of associators $\mathit{Ass}(\Bbbk)$ in §10.4. We check that the action of the pro-unipotent Grothendieck–Teichmüller group on the set of associators, which we define in the previous section, restricts to a levelwise action on this tower, and we use this observation to establish that the results, which we prove for the tower decomposition of the graded Grothendieck–Teichmüller group in §10.4, also hold for the tower decomposition of the pro-unipotent Grothendieck–Teichmüller group. To be specific, we prove that the morphisms $p_m : GT_{\langle m\rangle}(\Bbbk) \to GT_{\langle m-1\rangle}(\Bbbk)$ in this tower are surjective.

We follow the same overall plan as in §10.4. We use the natural tower decomposition of the Malcev completion of the operad of parenthesized braids $\mathsf{PaB}^{\widehat{}}_+ = \lim_m q_m\, \mathsf{PaB}^{\widehat{}}_+$ to give an operadic definition of our tower decomposition of the Grothendieck–Teichmüller group. We give a brief reminder on the definition of this operadic tower decomposition $\mathsf{PaB}^{\widehat{}}_+ = \lim_m q_m\, \mathsf{PaB}^{\widehat{}}_+$ first. Recall that we already studied this particular example of the tower decomposition of an operad in §10.1 and that the weight graded Drinfeld-Khono Lie algebra operad $\mathfrak{p}$ naturally occurs as the object which determines the fibers of this tower of operad morphisms $\cdots \to q_m\,\mathsf{PaB}^{\wedge}_{+} \to q_{m-1}\,\mathsf{PaB}^{\wedge}_{+} \to \cdots \to q_0\,\mathsf{PaB}^{\wedge}_{+}$. We refer to this first study §10.1 for more details on the background of our constructions.

11.3.1. *Reminders on the tower decomposition of the operad of parenthesized braids.* We focus on the case of the non-unitary operad $\mathsf{PaB}^{\wedge}$ underlying $\mathsf{PaB}^{\wedge}_{+}$ since the tower decomposition of such a unitary operad occurs as an obvious unitary extension of the tower decomposition of its underlying non-unitary operad.

Recall that we have the identity $\mathtt{Ob}\,q_m\,\mathsf{PaB}(r) = \mathtt{Ob}\,\mathsf{PaB}(r)$ at the object set level, for each $r > 0$, by definition of the tower decomposition of an operad in Malcev complete groupoids. The automorphism group of an object $p \in \mathtt{Ob}\,\mathsf{PaB}(r)$ in the groupoid $q_m\,\mathsf{PaB}(r)^{\wedge}$is defined by $\mathtt{Aut}_{q_m\,\mathsf{PaB}(r)^{\wedge}}(p) = \hat{P}_r/\,\mathtt{F}_{m+1}\,\hat{P}_r$, for any $m \geq 0$, where we use the identity $\mathtt{Aut}_{\mathsf{PaB}(r)^{\wedge}}(p) = \hat{P}_r$ and we consider the quotients of the Malcev completion of the pure braid group $q_m\hat{P}_r = \hat{P}_r/\,\mathtt{F}_{m+1}\,\hat{P}_r$ by the subgroups of its natural filtration $\hat{P}_r = \mathtt{F}_1\,\hat{P}_r \supset \cdots \supset \mathtt{F}_m\,\hat{P}_r \supset \cdots$. In general, the morphism set $\mathtt{Mor}_{q_m\,\mathsf{PaB}(r)^{\wedge}}(p,q)$, which we associate to a pair of objects $p, q \in \mathtt{Ob}\,\mathsf{PaB}(r)$, is identified with the quotient of the completed morphism set $\mathtt{Mor}_{\mathsf{PaB}(r)^{\wedge}}(p,q) = \mathtt{Mor}_{\mathsf{PaB}(r)}(p,q)^{\wedge}$by the equivalence relation such that $f \equiv g$ if we have $f = g\gamma$ for some $\gamma \in \mathtt{F}_{m+1}\,\hat{P}_r$. (We then use the identity $\hat{P}_r = \mathtt{Aut}_{\mathsf{PaB}(r)^{\wedge}}(p)$.) We use the notation $f \equiv g (\operatorname{mod} \mathtt{F}_{m+1}\,\hat{P}_r)$ for this equivalence relation on $\mathtt{Mor}_{\mathsf{PaB}(r)^{\wedge}}(p,q)$.

In our constructions, we also use the following assertion which, like the analogous statement of Proposition 10.4.2, occurs as a particular case of the general functoriality claims of §9.2.5:

PROPOSITION 11.3.2. *Every morphism of operads in Malcev complete groupoids* $\phi : \mathsf{PaB}^{\wedge}_{+} \to q_m\,\mathsf{PaB}^{\wedge}_{+}$ *with values in the quotient* $q_m\,\mathsf{PaB}^{\wedge}_{+}$ *of the Malcev completion of the operad of parenthesized braids* $\mathsf{PaB}^{\wedge}_{+}$ *admits a factorization:*

$$\begin{tikzcd}
\mathsf{PaB}^{\wedge}_{+} \arrow[rr, "\phi"] \arrow[dr] & & q_m\,\mathsf{PaB}^{\wedge}_{+}\ , \\
& q_m\,\mathsf{PaB}^{\wedge}_{+} \arrow[ur, dotted, "\exists!\bar{\phi}"'] &
\end{tikzcd}$$

where we consider the natural quotient map $\pi_m : \mathsf{PaB}^{\wedge}_{+} \to q_m\,\mathsf{PaB}^{\wedge}_{+}$ *on the left-hand side.* □

11.3.3. *The tower decomposition of the Grothendieck–Teichmüller group.* We define the term $GT_{\langle m\rangle}(\Bbbk)$ of the tower decomposition $GT(\Bbbk) = \lim_m GT_{\langle m\rangle}(\Bbbk)$ of the pro-unipotent Grothendieck–Teichmüller group $GT(\Bbbk)$ as the set of morphisms of operads in Malcev complete groupoids $\phi : \mathsf{PaB}^{\wedge}_{+} \to q_{m+1}\,\mathsf{PaB}^{\wedge}_{+}$ which are given by the identity map at the object set level and whose factorization through the operad $q_{m+1}\,\mathsf{PaB}^{\wedge}_{+}$ defines an automorphism of this object. (We use the result of Proposition 11.3.2 to ensure the existence of this factorization.) We provide this group $GT_{\langle m\rangle}(\Bbbk)$ with the obvious composition operation, which we deduce from the composition of automorphisms for the operad $q_{m+1}\,\mathsf{PaB}^{\wedge}_{+}$.

We again use that each morphism $\phi : \mathsf{PaB}^{\wedge}_{+} \to q_{m+1}\,\mathsf{PaB}^{\wedge}_{+}$ occurs as the extension of a morphism of operads in groupoids $\phi : \mathsf{PaB}_{+} \to q_{m+1}\,\mathsf{PaB}^{\wedge}_{+}$, where we consider the ordinary operad of parenthesized braids PaB_+. The result of Theorem 6.2.4 implies that such a morphism is determined by giving the image of the braiding isomorphism and of the associativity isomorphism of PaB_+ in the operad $q_{m+1}\,\mathsf{PaB}_+^{\widehat{}}$. We can adapt the analysis of §11.1 to the case of the operads $\mathsf{Q} = q_{m+1}\,\mathsf{PaB}_+^{\widehat{}}$ in order to make this correspondence explicit.

We first get that our morphism of unitary operads in Malcev complete groupoids $\phi : \mathsf{PaB}_+^{\widehat{}} \to q_{m+1}\,\mathsf{PaB}_+^{\widehat{}}$ is determined by a pair $(\lambda, f(x_1, x_2))$, where λ is a scalar parameter such that $\phi(\tau) = \tau^{\lambda}$, and where $f(x_1, x_2)$ now represents the class of an element of the Malcev completion of the free group on two generators $\hat{\mathbb{F}}(x_1, x_2)$ such that we have the identity

$$\phi(\alpha) \equiv \alpha \cdot f(x_{12}, x_{23}) \,(\operatorname{mod} \mathtt{F}_{m+2}\,\hat{P}_3)$$

in the morphism set

$$\mathtt{Mor}_{q_{m+1}\,\mathsf{PaB}(3)^{\widehat{}}}((x_1x_2)x_3, x_1(x_2x_3)) = \alpha \cdot \hat{P}_3 / \mathtt{F}_{m+2}\,\hat{P}_3.$$

We then check that this group-like power series $f(x_1, x_2)$ has to satisfy the relations of Proposition 11.1.4 modulo factors of filtration $\geq m + 2$ in the Malcev complete groups where we express these relations. We get, besides, that our morphism of operads in Malcev complete groupoids $\phi : \mathsf{PaB}_+^{\widehat{}} \to q_{m+1}\,\mathsf{PaB}_+^{\widehat{}}$ induces an isomorphism when we pass to the operad $q_{m+1}\,\mathsf{PaB}_+^{\widehat{}}$ if and only if the scalar parameter associated to this morphism is invertible $\lambda \in \mathbb{k}^{\times}$.

We use the obvious composition with the morphisms of the tower decomposition of the Malcev completion of the parenthesized braid operads $\mathsf{PaB}_+^{\widehat{}} \to \cdots \to q_m\,\mathsf{PaB}_+^{\widehat{}} \to \cdots \to q_1\,\mathsf{PaB}_+^{\widehat{}}$ to define the morphisms of our tower decomposition of the Grothendieck–Teichmüller group $GT(\mathbb{k}) \to \cdots \to GT_{\langle m\rangle}(\mathbb{k}) \to \cdots \to GT_{\langle 0\rangle}(\mathbb{k})$. We can identify these morphisms with the obvious reduction operation in the above power series description of the groups $GT_{\langle m\rangle}(\mathbb{k})$ and we can also use this approach to check the relation $GT(\mathbb{k}) = \lim_m GT_{\langle m\rangle}(\mathbb{k})$. We moreover have the identity:

$$GT_{\langle 0\rangle}(\mathbb{k}) = \mathbb{k}^{\times},$$

which still follows from the observation that the unit relations of Proposition 11.1.4 force the vanishing relation $f(x_1, x_2) \equiv 1$ in the group $\hat{\mathbb{F}}(x_1, x_2) / \mathtt{F}_2\,\hat{\mathbb{F}}(x_1, x_2)$ (see the proof of Proposition 11.1.4 for details).

We readily check, from our operadic definitions, that the action of the Grothendieck-Teichmüller group $GT(\mathbb{k})$ on the set of associators $Ass(\mathbb{k})$ decomposes as a levelwise action of the groups $GT_{\langle m\rangle}(\mathbb{k})$ on the sets $Ass_{\langle m\rangle}(\mathbb{k})$ which define the tower decomposition of this object $Ass(\mathbb{k}) = \lim_m Ass(\mathbb{k})$. We easily see that this action is simple and transitive at each level too.

We can readily adapt the definition of the tower decomposition $GT(\mathbb{k}) = \lim_m GT_{\langle m\rangle}(\mathbb{k})$ to the group $GT^1(\mathbb{k})$ of §11.2.3, which can accordingly be identified with the limit $GT^1(\mathbb{k}) = \lim_m GT^1_{\langle m\rangle}(\mathbb{k})$ of a tower of groups such that $GT^1_{\langle m\rangle}(\mathbb{k}) = \ker(\lambda : GT_{\langle m\rangle}(\mathbb{k}) \to \mathbb{k}^{\times})$. We then consider an obvious analogue, for the group $GT_{\langle m\rangle}(\mathbb{k})$, of the morphism $\lambda : GT(\mathbb{k}) \to \mathbb{k}^{\times}$ of §11.2.3. We also have an identity $GT_{\langle m\rangle}(\mathbb{k}) = \mathbb{k}^{\times} \ltimes GT^1_{\langle m\rangle}(\mathbb{k})$ at each level of our tower, and we can check that the action of the group $GT_{\langle m\rangle}(\mathbb{k})$ on $Ass_{\langle m\rangle}(\mathbb{k})$ restricts to a simply transitive action of the group $GT^1_{\langle m\rangle}(\mathbb{k})$ on the set $Ass^{\kappa}_{\langle m\rangle}(\mathbb{k})$, for any $\kappa \in \mathbb{k}^{\times}$.

We have the following result:

Proposition 11.3.4. *The morphisms* $p_m : GT_{\langle m\rangle}(\Bbbk) \to GT_{\langle m-1\rangle}(\Bbbk)$ *which we consider in the tower decomposition of the Grothendieck–Teichmüller group* $GT(\Bbbk) = \lim_m GT_{\langle m\rangle}(\Bbbk)$ *are surjective, for any characteristic zero ground field* $\Bbbk$*, and we have an obvious counterpart of this result for the tower decomposition of the group* $GT^1(\Bbbk) = \ker(\lambda : GT(\Bbbk) \to \Bbbk^{\times})$*.*

Proof. We use the counterpart of these surjectivity claims for the tower decomposition of the set of associators $\mathit{Ass}(\Bbbk) = \lim_m \mathit{Ass}_{\langle m\rangle}(\Bbbk)$. We also fix some element in this set $\phi \in \mathit{Ass}(\Bbbk)$. We assume $\psi_{m-1} \in GT_{\langle m-1\rangle}(\Bbbk)$. We pick an element $\phi_m \in \mathit{Ass}_{\langle m\rangle}(\Bbbk)$ such that $p_m(\phi_m) = \phi \circ \psi_{m-1}$, where we consider the action of ψ_{m-1} on the image of our associator ϕ in $\mathit{Ass}_{\langle m-1\rangle}(\Bbbk)$. We then have $\phi_m = \phi \circ \psi_m$, for some $\psi_m \in GT_{\langle m\rangle}(\Bbbk)$, and $p_m(\phi \circ \psi_m) = \phi \circ p(\psi_m) = \phi \circ \psi_{m-1} \Rightarrow p(\psi_m) = \psi_{m-1}$. We argue similarly in the case of the group $GT^1(\Bbbk)$ by using that our actions fix the value of the parameter $\kappa \in \Bbbk^{\times}$ which we associate to our element of the set of associators $\phi \in \mathit{Ass}(\Bbbk)$. ☐

We moreover have the following statement:

Proposition 11.3.5. *The isomorphism* $\upsilon_{\phi} : \mathit{GRT}(\Bbbk) \xrightarrow{\simeq} GT(\Bbbk)$*, which we associate to an element of the set of associators* $\phi \in \mathit{Ass}(\Bbbk)$ *in Proposition* 11.2.2*, induces a levelwise isomorphism*

$$\upsilon_{\phi} : \mathit{GRT}_{\langle m\rangle}(\Bbbk) \xrightarrow{\simeq} GT_{\langle m\rangle}(\Bbbk)$$

between our tower decomposition of the graded Grothendieck–Teichmüller group $\mathit{GRT}(\Bbbk)$ *and the parallel tower decomposition of the pro-unipotent Grothendieck–Teichmüller group* $GT(\Bbbk)$*. We have a similar result when we pass to the subgroups* $\mathit{GRT}^1(\Bbbk) \subset \mathit{GRT}(\Bbbk)$ *and* $GT^1(\Bbbk) \subset GT(\Bbbk)$*.*

Proof. We can readily adapt the construction of Proposition 11.2.2(b) to associate an isomorphism $\upsilon_{\phi_m} : \mathit{GRT}_{\langle m\rangle}(\Bbbk) \xrightarrow{\simeq} GT_{\langle m\rangle}(\Bbbk)$ to each element $\phi_m \in \mathit{Ass}_{\langle m\rangle}(\Bbbk)$ in the tower decomposition of the set of associators $\mathit{Ass}(\Bbbk) = \mathit{Ass}_{\langle m\rangle}(\Bbbk)$. We easily check that these isomorphisms define a tower decomposition of the isomorphism of Proposition 11.2.2(b) when we take the image of our element of the set of associators $\phi \in \mathit{Ass}(\Bbbk)$ in $\mathit{Ass}_{\langle m\rangle}(\Bbbk)$ for each $\phi_m \in \mathit{Ass}_{\langle m\rangle}(\Bbbk)$. We argue similarly in the case of the pro-unipotent subgroups $\mathit{GRT}^1(\Bbbk) \subset \mathit{GRT}(\Bbbk)$ and $GT^1(\Bbbk) \subset GT(\Bbbk)$. ☐

We can use this result to retrieve that the morphisms $p_m : GT_{\langle m\rangle}(\Bbbk) \to GT_{\langle m-1\rangle}(\Bbbk)$ in the tower decomposition of the Grothendieck–Teichmüller group $GT(\Bbbk) = \lim_m GT_{\langle m\rangle}(\Bbbk)$ are surjective (see Proposition 11.3.4) since we already checked that an analogous statement holds for the morphisms $p_m : \mathit{GRT}_{\langle m\rangle}(\Bbbk) \to \mathit{GRT}_{\langle m-1\rangle}(\Bbbk)$ in the tower decomposition of the graded Grothendieck–Teichmüller group $\mathit{GRT}(\Bbbk) = \lim_m \mathit{GRT}_{\langle m\rangle}(\Bbbk)$ (in Proposition 10.4.8).

11.4. The graded Lie algebra of the Grothendieck–Teichmüller group

The results of the previous section implies that the morphism $p : GT(\Bbbk) \to GT_{\langle m\rangle}(\Bbbk)$ from the Grothendieck–Teichmüller group $GT(\Bbbk)$ towards a term of our tower decomposition $GT(\Bbbk) = \lim_m GT_{\langle m\rangle}(\Bbbk)$ forms a surjective group morphism, for each level $m \geq 0$. We equivalently have a levelwise identity:

$$GT_{\langle m\rangle}(\Bbbk) = GT(\Bbbk)/\,\mathtt{F}_{m+1}\,GT(\Bbbk),$$

for a nested sequence of subgroups of the Grothendieck–Teichmüller group

$$GT(\Bbbk) = \mathtt{F}_0\,GT(\Bbbk) \supset \cdots \supset \mathtt{F}_m\,GT(\Bbbk) \supset \cdots$$

such that $\mathtt{F}_{m+1}\,GT(\Bbbk) = \ker(p : GT(\Bbbk) \to GT_{\langle m\rangle}(\Bbbk))$, for each $m \geq 0$.

We devote this section to the study of this filtration $GT(\Bbbk) = \mathtt{F}_0\,GT(\Bbbk) \supset \cdots \supset \mathtt{F}_m\,GT(\Bbbk) \supset \cdots$ of the Grothendieck–Teichmüller group $GT(\Bbbk)$. We mainly check that the subquotient groups $\mathtt{E}^0_m\,GT(\Bbbk) = \mathtt{F}_m\,GT(\Bbbk)/\,\mathtt{F}_{m+1}\,GT(\Bbbk)$ are isomorphic to the components of the graded Grothendieck–Teichmüller Lie algebra $\mathfrak{grt} = \prod_m \mathfrak{grt}_m$ when $m \geq 1$, while we have an identity $GT(\Bbbk)/\,\mathtt{F}_1\,GT(\Bbbk) = \Bbbk^{\times} \Leftrightarrow \mathtt{F}_1\,GT(\Bbbk) = GT^1(\Bbbk)$ in weight $m = 0$.

We also check that our filtration satisfies the commutator condition of §8.2.1. We can therefore apply the constructions of this previous chapter to get a natural Lie algebra structure on the weight graded object which we form by taking the sum of the subquotients $\mathtt{E}^0_m\,GT(\Bbbk) = \mathtt{F}_m\,GT(\Bbbk)/\,\mathtt{F}_{m+1}\,GT(\Bbbk)$ of the group $GT(\Bbbk)$. We just need to start with the first layer of our filtration $\mathtt{F}_1\,GT(\Bbbk) = GT^1(\Bbbk)$ and to forget about the factor $GT(\Bbbk)/\,\mathtt{F}_1\,GT(\Bbbk) = \Bbbk^{\times}$ in weight $m = 0$ in order to fulfill our general connectedness requirements of §7.3 for Lie algebras in weight graded modules. Thus, we actually consider a weight graded Lie algebra $\mathtt{E}^0\,GT^1(\Bbbk)$ associated to the subgroup $GT^1(\Bbbk) \subset GT(\Bbbk)$ rather than to the full Grothendieck–Teichmüller group $GT(\Bbbk)$. We similarly consider the weight graded Lie algebra $\mathtt{E}^0\,\mathfrak{grt}^1 = \bigoplus_{m\geq 1}\mathfrak{grt}_m$ whose completion gives the Lie algebra $\mathfrak{grt}^1$ associated to the group $GRT^1(\Bbbk)$. In fact, the weight graded module of subquotients $\mathtt{E}^0\,GT^1(\Bbbk)$ is isomorphic to $\mathtt{E}^0\,\mathfrak{grt}^1$ as a weight graded Lie algebra, and not only as a weight graded module. We just give a few hints on the verification of this statement.

We give a self-contained definition of our filtration first. We only use the relationship with the tower decompositions of the previous section in a second step, when we tackle the computation of our filtration subquotients.

11.4.1. *The definition of the filtration of the Grothendieck–Teichmüller group.* We go back to the tower decomposition of the Malcev completion of the operad of parenthesized braids $\mathsf{PaB}^{\wedge}_{+} = \lim_m q_m\,\mathsf{PaB}^{\wedge}_{+}$. We use the conventions recalled in §11.3.1 and we consider the canonical morphism of operads $\pi_m : \mathsf{PaB}^{\wedge}_{+} \to q_m\,\mathsf{PaB}^{\wedge}_{+}$ which we get in this tower decomposition $\mathsf{PaB}^{\wedge}_{+} = \lim_m q_m\,\mathsf{PaB}^{\wedge}_{+}$, for each $m \geq 0$. We set:

$$\mathtt{F}_m\,GT(\Bbbk) := \{\psi \in GT(\Bbbk) | \pi_m \circ \psi = \pi_m\}, \tag{1}$$

where we use the definition of the elements $\psi \in GT(\Bbbk)$ in terms of morphisms of unitary operads in Malcev complete groupoids $\psi : \mathsf{PaB}^{\wedge}_{+} \to \mathsf{PaB}^{\wedge}_{+}$. We can equivalently define $\mathtt{F}_m\,GT(\Bbbk)$ as the subgroup of the Grothendieck–Teichmüller group $GT(\Bbbk)$ formed by the morphisms of unitary operads $\psi : \mathsf{PaB}^{\wedge}_{+} \to \mathsf{PaB}^{\wedge}_{+}$ such that we have

$$\psi(\beta) \equiv \beta (\operatorname{mod} \mathtt{F}_{m+1}\,\hat{P}_r), \tag{2}$$

for every morphism $\beta \in \mathtt{Mor}_{\mathsf{PaB}(r)^{\wedge}}(p,q)$, where $p, q \in \mathtt{Ob}\,\mathsf{PaB}(r)$.

From this second definition, we immediately see that $\mathtt{F}_m\, GT(\Bbbk)$ is preserved by conjugation and hence forms a normal subgroup of $GT(\Bbbk)$. Furthermore, we have $q_0\, \mathit{PaB}^{\wedge} = pt \Rightarrow GT(\Bbbk) = \mathtt{F}_0\, GT(\Bbbk)$, and the following proposition holds:

Proposition 11.4.2. *The collection* $\mathtt{F}_m\, GT(\Bbbk) \subset GT(\Bbbk)$*,* $m \geq 0$*, defines a filtration of the Grothendieck–Teichmüller group* $GT(\Bbbk) = \mathtt{F}_0\, GT(\Bbbk) \supset \mathtt{F}_1\, GT(\Bbbk) \supset \cdots \supset \mathtt{F}_m\, GT(\Bbbk) \supset \cdots$ *such that we have the commutator relation*

$$(\mathtt{F}_m\, GT(\Bbbk), \mathtt{F}_n\, GT(\Bbbk)) \subset \mathtt{F}_{m+n}\, GT(\Bbbk),$$

for all $m, n \geq 0$*. We moreover have* $GT(\Bbbk) = \lim_m GT(\Bbbk)/\, \mathtt{F}_m\, GT(\Bbbk)$*.*

Proof. We use that the morphism sets of the operad $q_m\, \mathit{PaB}^{\wedge}_{+}$ are identified with quotients of the morphisms sets of the operad $\mathit{PaB}^{\wedge}_{+}$ under a natural translation action of the group $\mathtt{F}_{m+1}\, \hat{P}_r \subset \hat{P}_r$ on these morphism sets, where we consider the natural filtration of the Malcev completion of the pure braid groups $\hat{P}_r = \mathtt{F}_1\, \hat{P}_r \supset \cdots \supset \mathtt{F}_m\, \hat{P}_r \supset \cdots$ (see our reminder on the definition of this operadic tower decomposition $\mathit{PaB}^{\wedge}_{+} = \lim_m q_m\, \mathit{PaB}^{\wedge}_{+}$ in §11.3.1).

Recall that $\phi \in \mathtt{F}_m\, GT(\Bbbk)$ if we have the relation $\phi(\beta) \equiv \beta (\operatorname{mod} \mathtt{F}_{m+1}\, \hat{P}_r)$, for every $\beta \in \mathtt{Mor}_{\mathit{PaB}(r)^{\wedge}}(p, q)$. From this property, we get the relation $\gamma \in \mathtt{F}_{n+1}\, \hat{P}_r \Rightarrow \phi(\gamma) \equiv \gamma (\operatorname{mod} \mathtt{F}_{m+n+1}\, \hat{P}_r)$. This claim can be deduced from the observation that γ consists of n-fold commutators of elements of $\hat{P}_r$ (by Proposition 10.0.9), or from the weaker statement that γ consists of n-fold commutators up to factors of higher order in the filtration of the group $\hat{P}_r$ (see Proposition 8.3.3). Recall simply that an element $\phi \in GT(\Bbbk)$ is defined by a morphism of operads in Malcev complete groupoids $\phi : \mathit{PaB}^{\wedge} \to \mathit{PaB}^{\wedge}$, and that this morphism preserves the filtration of the automorphism groups of objects by definition.

We now assume $\phi \in \mathtt{F}_m\, GT(\Bbbk)$ and $\psi \in \mathtt{F}_n\, GT(\Bbbk)$. We fix any morphism $\beta \in \mathtt{Mor}_{\mathit{PaB}(r)^{\wedge}}(p, q)$ in the groupoid $\mathit{PaB}(r)^{\wedge}$. We then have $\psi(\beta) = \beta\delta$ for some $\delta \in \mathtt{F}_{n+1}\, \hat{P}_r$ and $\phi(\delta) = \delta (\operatorname{mod} \mathtt{F}_{m+n+1}\, \hat{P}_r)$. We similarly have $\phi(\beta) = \beta\gamma$ for some $\gamma \in \mathtt{F}_{m+1}\, \hat{P}_r$ and $\psi(\gamma) = \gamma (\operatorname{mod} \mathtt{F}_{m+n+1}\, \hat{P}_r)$, from which we also get $\psi^{-1}(\gamma) = \gamma (\operatorname{mod} \mathtt{F}_{m+n+1}\, \hat{P}_r)$, while we have $\psi(\beta) = \beta\delta \Rightarrow \psi^{-1}(\beta\delta) = \beta$. We use these identities and the relation $\gamma\delta \equiv \delta\gamma (\operatorname{mod} \mathtt{F}_{m+n+1}\, \hat{P}_r)$, which follows from the properties of our filtration on $\hat{P}_r$, to compute the image of the element $\beta \in \mathtt{Mor}_{\mathit{PaB}(r)^{\wedge}}(p, q)$ under the commutator $(\phi, \psi) = \phi^{-1}\psi^{-1}\phi\psi$ of our automorphisms $\phi, \psi : \mathit{PaB}^{\wedge} \to \mathit{PaB}^{\wedge}$. We explicitly get:

$$\begin{aligned} \phi^{-1}\psi^{-1}\phi\psi(\beta) &\equiv \phi^{-1}\psi^{-1}(\beta\gamma\delta) (\operatorname{mod} \mathtt{F}_{m+n+1}\, \hat{P}_r) \\ &\equiv \phi^{-1}\psi^{-1}(\beta\delta\gamma) (\operatorname{mod} \mathtt{F}_{m+n+1}\, \hat{P}_r) \\ &\equiv \phi^{-1}(\beta\gamma) (\operatorname{mod} \mathtt{F}_{m+n+1}\, \hat{P}_r) \\ &\equiv \beta (\operatorname{mod} \mathtt{F}_{m+n+1}\, \hat{P}_r). \end{aligned}$$

We conclude from this computation that we have $(\phi, \psi) \in \mathtt{F}_{m+n}\, GT(\Bbbk)$. We therefore have the relation $(\mathtt{F}_m\, GT(\Bbbk), \mathtt{F}_n\, GT(\Bbbk)) \subset \mathtt{F}_{m+n}\, GT(\Bbbk)$ asserted in our proposition.

We deduce the identity $GT(\Bbbk) = \lim_m GT(\Bbbk)/\, \mathtt{F}_m\, GT(\Bbbk)$ from the relation

$$\mathtt{Mor}_{\hat{f}\,\mathcal{G}rd\,\mathcal{O}p}(\mathit{PaB}^{\wedge}_{+}, \mathit{PaB}^{\wedge}_{+}) = \lim_{m} \mathtt{Mor}_{\hat{f}\,\mathcal{G}rd\,\mathcal{O}p}(\mathit{PaB}^{\wedge}_{+}, q_m\, \mathit{PaB}^{\wedge}_{+}),$$

which is a formal consequence of the identity $\mathsf{PaB}^{\frown}_{+} = \lim_m q_m\, \mathsf{PaB}^{\frown}_{+}$ in the category of unitary operads in Malcev complete groupoids. To be explicit, we fix a collection $\phi_m \in GT(\Bbbk)$, $m \geq 0$, such that we have $\phi_{m+1} = \phi_m \circ \theta_m$, for some $\theta_m \in \mathtt{F}_m\, GT(\Bbbk)$, for every $m \geq 0$. We then have the relation $\pi_m \circ \phi_{m+1} = \pi_m \circ \phi_m \circ \theta_m \circ \phi_m^{-1} \circ \phi_m = \pi_m \circ \phi_m$ in the morphism set $\mathtt{Mor}_{\hat{f}\,\mathcal{G}rd\,\mathcal{O}p}(\mathsf{PaB}^{\frown}_{+}, q_m\, \mathsf{PaB}^{\frown}_{+})$, for every $m \geq 0$, since $\theta_m \in \mathtt{F}_m\, GT(\Bbbk) \Rightarrow \phi_m \circ \theta_m \circ \phi_m^{-1} \in \mathtt{F}_m\, GT(\Bbbk) \Rightarrow \pi_m \circ \phi_m \circ \theta_m \circ \phi_m^{-1} = \pi_m$. We can accordingly form a morphism $\phi \in \mathtt{Mor}_{\hat{f}\,\mathcal{G}rd\,\mathcal{O}p}(\mathsf{PaB}^{\frown}_{+}, \mathsf{PaB}^{\frown}_{+})$ such that $\pi_m \circ \phi_m = \pi_m \circ \phi$ for each $m \geq 0$. We easily check that this morphism is invertible, because we can use the same argument to produce a morphism $\psi \in \mathtt{Mor}_{\hat{f}\,\mathcal{G}rd\,\Lambda\,\mathcal{O}p}(\mathsf{PaB}^{\frown}, \mathsf{PaB}^{\frown})$ such that $\pi_m \circ \phi_m^{-1} = \pi_m \circ \psi$ for each $m \geq 0$, and we readily check that we have the relations $\pi_m \circ \phi \circ \psi = \pi_m = \pi_m \circ \psi \circ \phi$, for all $m \geq 0$, which imply $\phi \circ \psi = id = \psi \circ \phi$. We consequently have $\phi \in GT(\Bbbk)$ and this argument line proves that we do have a one-to-one correspondence between the elements of the group $GT(\Bbbk)$ and the elements of the limit $\lim_m GT(\Bbbk)/\mathtt{F}_m\, GT(\Bbbk)$, which we represent by the collections $\bar{\phi}_m \in GT(\Bbbk)/\mathtt{F}_m\, GT(\Bbbk)$, where we consider the classes of our elements ϕ_m in the quotient groups $GT(\Bbbk)/\mathtt{F}_m\, GT(\Bbbk)$. ☐

In fact, the formula §11.4.1(1) is just a rephrasing of the definition of the kernel $\ker(p : GT(\Bbbk) \to GT_{\langle m-1\rangle}(\Bbbk))$, because the composite $\pi_m \circ \psi : \mathsf{PaB}^{\frown}_{+} \to q_m\, \mathsf{PaB}^{\frown}_{+}$ represents the image of the element $\psi \in GT(\Bbbk)$ in $GT_{\langle m-1\rangle}(\Bbbk)$ (see §11.3.3). We therefore have the relation $\mathtt{F}_{m+1}\, GT(\Bbbk) = \ker(p : GT(\Bbbk) \to GT_{\langle m\rangle}(\Bbbk))$, for each $m \geq 0$, while the surjectivity of the morphisms in the tower decomposition of $GT(\Bbbk)$ implies, as we mention in the introduction of this section, that we have an identity $GT_{\langle m\rangle}(\Bbbk) = GT(\Bbbk)/\mathtt{F}_{m+1}\, GT(\Bbbk)$, for each $m \geq 0$. (We can also use this relation to retrieve the identity $GT(\Bbbk) = \lim_m GT(\Bbbk)/\mathtt{F}_m\, GT(\Bbbk)$ which we proved by a direct inspection in the previous proposition.)

We have the following equivalent statements when we pass to the set of associators:

PROPOSITION 11.4.3.

(a) *We can identify the group* $\mathtt{F}_{m+1}\, GT(\Bbbk) \subset GT(\Bbbk)$ *with the subgroup of the Grothendieck–Teichmüller group formed by the elements which act trivially on the term* $Ass_{\langle m\rangle}(\Bbbk)$ *of the tower decomposition of the set of Drinfeld's associators* $Ass(\Bbbk) = \lim_m Ass_{\langle m\rangle}(\Bbbk)$, *for each* $m \geq 0$.

(b) *We can identify the set* $Ass_{\langle m\rangle}(\Bbbk)$ *with the quotient of the set of associators* $Ass(\Bbbk)$ *by the relation such that* $\rho \equiv \phi$ *when we have* $\rho = \phi \circ \psi$ *for some* $\psi \in \mathtt{F}_{m+1}\, GT(\Bbbk)$.

PROOF. The assertions of this proposition are immediate consequences of the relation $\mathtt{F}_{m+1}\, GT(\Bbbk) = \ker(p : GT(\Bbbk) \to GT_{\langle m\rangle}(\Bbbk))$ and of the observation that the action of the Grothendieck–Teichmüller group $GT(\Bbbk)$ on the set of associators $Ass(\Bbbk)$ decomposes as a levelwise action of the groups $GT_{\langle m\rangle}(\Bbbk)$ on the sets $Ass_{\langle m\rangle}(\Bbbk)$ together with the fact that this action is simple and transitive levelwise. We also use the surjectivity of the map $p : Ass(\Bbbk) \to Ass_{\langle m\rangle}(\Bbbk)$ in the second claim of the proposition. ☐

11.4.4. *From the filtration subquotients of the Grothendieck–Teichmüller group to the fibers of the tower decomposition of the operad of parenthesized braids.* Recall that the collection $\mathfrak{p}(-)_m = \{\mathfrak{p}(r)_m, r > 0\}$, where we consider the homogeneous components of a fixed weight $m \geq 1$ of the Drinfeld–Kohno Lie algebra operad $\mathfrak{p}$, inherits the structure of an additive operad in the category of $\Bbbk$-modules. In what follows, we also consider the obvious unitary extension of this operad $\mathfrak{p}_+(-)_m$, which we define by considering an extra null term in arity zero $\mathfrak{p}_+(0)_m = 0$.

In Theorem 10.1.3, we explain that the additive operads $\mathtt{E}^0_m\,\hat{\mathfrak{p}}$ represent the fibers of the tower decomposition of the Malcev completion of the operad of parenthesized braids $\mathsf{PaB}^{\wedge} = \lim_m q_m\,\mathsf{PaB}^{\wedge}$. We then regard this additive operad $\mathfrak{p}(-)_m$ as a constant local coefficient system operad over the Malcev completion of the operad of parenthesized braids $\mathsf{PaB}^{\wedge}_+$. In short, our claim is that we have a collection of isomorphisms of $\Bbbk$-modules

$$\upsilon_p : \mathfrak{p}(r)_m \xrightarrow{\simeq} \mathtt{E}^0_m\,\mathtt{Aut}_{\mathsf{PaB}(r)^{\wedge}}(p)$$

associated to the objects $p \in \mathtt{Ob}\,\mathsf{PaB}(r)$, such that the additive operad structure of the collection $\mathfrak{p}(-) = \{\mathfrak{p}(r)_m, r > 0\}$ reflects structure operations attached to the subquotients of the automorphism groups $\mathtt{Aut}_{\mathsf{PaB}(r)^{\wedge}}(p)$ in the Malcev completion of the operad of parenthesized braids, while the conjugation operations $c_\beta : \mathtt{E}^0_m\,\mathtt{Aut}_{\mathsf{PaB}(r)^{\wedge}}(p) \rightarrow \mathtt{E}^0_m\,\mathtt{Aut}_{\mathsf{PaB}(r)^{\wedge}}(q)$ which we associate to the morphisms $\beta \in \mathtt{Mor}_{\mathsf{PaB}(r)^{\wedge}}(p, q)$ of the groupoid $\mathsf{PaB}(r)^{\wedge}$, for each $r > 0$, correspond to identity maps on the object $\mathfrak{p}(r)_m$. We have an analogous statement in the unitary setting.

Let $\psi \in \mathtt{F}_m\,GT(\Bbbk)$. Let $\beta \in \mathtt{Mor}_{\mathsf{PaB}(r)^{\wedge}}(p, q)$ with $p, q \in \mathtt{Ob}\,\mathsf{PaB}(r)$ be a morphism in the Malcev completion of the parenthesized braid operad. The relation $\pi_m \circ \psi = \pi_m$ implies that we have an identity:

$$\psi(\beta) \equiv \beta \cdot \upsilon_p \theta_\psi(\beta)$$

in $Mor_{q_{m+1}\,\mathsf{PaB}(r)^{\wedge}}(p, q)$, where we consider the action of an element $\theta_\psi(\beta) \in \mathfrak{p}(r)_{m+1}$ on the class of the morphism β in this groupoid $q_{m+1}\,\mathsf{PaB}(r)^{\wedge}$.

We may see that the map $\theta_\psi : \beta \mapsto \theta_\psi(\beta)$ defines a morphism of Malcev complete groupoids $\theta_\psi : \mathsf{PaB}(r)^{\wedge} \rightarrow \mathfrak{p}(r)_{m+1}$, in each arity $r > 0$, where we use that the $\Bbbk$-module $\mathfrak{p}(r)_{m+1}$ (which we identify with a groupoid with a single object) is equipped with a canonical Malcev complete group structure (see §8.2). These maps $\theta_\psi : \beta \mapsto \theta_\psi(\beta)$ moreover preserve the operad structure which we attach to our objects, and hence define a morphism of operads in Malcev complete groupoids $\theta_\psi : \mathsf{PaB}^{\wedge} \rightarrow \mathfrak{p}(-)_{m+1}$, where we regard the additive operad $\mathfrak{p}(-)_{m+1}$ as an operad in Malcev complete groupoids with a single object. This operad morphism has clearly a unitary extension $\theta_\psi : \mathsf{PaB}^{\wedge}_+ \rightarrow \mathfrak{p}_+(-)_{m+1}$ too.

We then consider the morphism

$$\theta_\psi : \mathsf{PaB}_+ \rightarrow \mathfrak{p}_+(-)_{m+1},$$

defined on the ordinary operad of parenthesized braids PaB_+, which underlies this morphism of operads in Malcev complete groupoids. We only consider this morphism of operads in ordinary groupoids in what follows. We can therefore skip the verification that our maps $\theta_\psi : \beta \mapsto \theta_\psi(\beta)$ define morphisms of Malcev complete groupoids $\theta_\psi : \mathsf{PaB}(r)^{\wedge} \rightarrow \mathfrak{p}(r)_{m+1}$, for each arity $r > 0$. We only need to check that our maps preserve the operad structure of our objects, which is straightforward.

We now assume $m \geq 1$. We check that the map $\theta : \psi \mapsto \theta_\psi$, which carries any element $\psi \in \mathtt{F}_m\,GT(\Bbbk)$ to this operad morphism $\theta_\psi : \mathsf{PaB}_+ \rightarrow \mathfrak{p}_+(-)_{m+1}$, preserves group structures in the sense that we the identity $\theta_{\phi\psi} = \theta_\phi + \theta_\psi$, for each pair $\phi, \psi \in \mathtt{F}_m\,GT(\Bbbk)$, where we consider the obvious additive group operation inherited from the target object $\mathfrak{p}_+(-)_{m+1}$ in the morphism set $\mathtt{Mor}_{\mathcal{G}rd\,\mathcal{O}p}(\mathsf{PaB}_+, \mathfrak{p}_+(-)_{m+1})$. We argue as follows. We observed in the proof of Proposition 11.4.2 that, for an element

$\phi \in \mathtt{F}_m\, GT(\Bbbk)$, we have the relation $\gamma \in \mathtt{F}_{n+1}\, \hat{P}_r \Rightarrow \phi(\gamma) \equiv \gamma (\operatorname{mod} \mathtt{F}_{m+n+1}\, \hat{P}_r)$. For a representative of the element $\gamma = \upsilon_p \theta_\psi(\beta)$, this relation implies $\phi(\upsilon_p \theta_\psi(\beta)) \equiv \upsilon_p \theta_\psi(\beta) (\operatorname{mod} \mathtt{F}_{m+m+1}\, \hat{P}_r)$, and hence, we get $\phi(\upsilon_p \theta_\psi(\beta)) \equiv \upsilon_p \theta_\psi(\beta) (\operatorname{mod} \mathtt{F}_{m+2}\, \hat{P}_r)$ when $m \geq 1$, from which we deduce the relation $\phi \circ \psi(\beta) \equiv \beta \upsilon_p \theta_\phi(\beta) \upsilon_p \theta_\psi(\beta)$. We therefore have the identity $\theta_{\phi\psi}(\beta) = \theta_\phi(\beta) + \theta_\psi(\beta)$ when we pass to the module $\mathfrak{p}(r)_m$, for any $\beta \in \mathtt{Mor}_{\mathsf{PaB}(r)^\wedge}(p,q)$ and $p, q \in \mathtt{Ob}\, \mathsf{PaB}(r)$.

We clearly have $\psi \in \mathtt{F}_{m+1}\, GT(\Bbbk) \Leftrightarrow \theta_\psi = 0$. Hence, the construction of this paragraph gives an injective group morphism:

$$\mathtt{E}^0_m\, GT(\Bbbk) = \mathtt{F}_m\, GT(\Bbbk) / \mathtt{F}_{m+1}\, GT(\Bbbk) \xrightarrow{\theta} \mathtt{Mor}_{\mathcal{G}rd\, \mathcal{O}p}(\mathsf{PaB}_+, \mathfrak{p}_+(-)_{m+1}),$$

where we consider the subquotient associated to our filtration of the group $GT(\Bbbk)$, for each weight $m \geq 1$.

In §11.4.2, we explained that the modules $\mathtt{Mor}_{\mathcal{G}rd\, \mathcal{O}p}(\mathsf{PaB}_+, \mathfrak{p}_+(-)_{m+1})$ determine the fibers of the tower decomposition of the set of associators too. To be more explicit, we observed that the term $Ass_{\langle m \rangle}(\Bbbk)$ of this tower decomposition inherits an action of this module $\mathtt{E}^0_m\, Ass(\Bbbk) = \mathtt{Mor}_{\mathcal{G}rd\, \mathcal{O}p}(\mathsf{PaB}_+, \mathfrak{p}_+(-)_{m+1})$ which is yielded by the aritywise exponential action of the Lie algebras $\mathfrak{p}(r)_{m+1}$ on the Malcev complete group $q_{m+1}\, \mathsf{CD}(r)^\wedge = \mathbb{G}\, \hat{\mathbb{U}}(\mathfrak{p}(r) / \mathtt{F}_{m+2}\, \mathfrak{p}(r))$, for any $m \geq 1$. Furthermore, a pair of elements $\phi, \psi \in Ass_{\langle m \rangle}(\Bbbk)$ have the same image in the next level of our tower $Ass_{\langle m-1 \rangle}(\Bbbk)$ if and only if they differ by the action $\psi = \phi \cdot \exp(\theta)$ of such a morphism $\theta : \mathsf{PaB}_+ \to \mathfrak{p}_+(-)_{m+1}$.

We use the action of the Grothendieck–Teichmüller group on the set of associators to relate the construction of the previous paragraph §11.4.4 to this correspondence $\mathtt{E}^0_m\, Ass(\Bbbk) = \mathtt{Mor}_{\mathcal{G}rd\, \mathcal{O}p}(\mathsf{PaB}_+, \mathfrak{p}_+(-)_{m+1})$, and we get the following statement:

Proposition 11.4.5. *The map constructed in* §11.4.4 *is an isomorphism*

$$\mathtt{E}^0_m\, GT(\Bbbk) = \mathtt{F}_m\, GT(\Bbbk) / \mathtt{F}_{m+1}\, GT(\Bbbk) \xrightarrow{\simeq} \mathtt{Mor}_{\mathcal{G}rd\, \mathcal{O}p}(\mathsf{PaB}_+, \mathfrak{p}_+(-)_{m+1}),$$

for each weight $m \geq 1$.

Proof. We fix an element in the set of associators $\phi \in Ass(\Bbbk)$ and we use the action of the group $GT(\Bbbk)$ on this element. For $\psi \in \mathtt{F}_m\, GT(\Bbbk)$, we have the relation $\phi \circ \psi \equiv \phi \cdot \exp(\rho_\psi)$ in the set $Ass_{\langle m \rangle}(\Bbbk)$, for a morphism $\rho_\psi : \mathsf{PaB}_+ \to \mathfrak{p}_+(-)_{m+1}$. We use that $\mathtt{F}_m\, GT(\Bbbk)$ represents the subgroup of $GT(\Bbbk)$ formed by the elements which act trivially on $Ass_{\langle m-1 \rangle}(\Bbbk)$ while $GT_{\langle m \rangle}(\Bbbk) = GT(\Bbbk) / \mathtt{F}_{m+1}\, GT(\Bbbk)$ acts simply and transitively on $Ass_{\langle m \rangle}(\Bbbk)$ (see §10.4.4) to check that this mapping $\rho : \psi \to \rho_\psi$ induces a bijection from $\mathtt{E}^0_m\, GT(\Bbbk) = \mathtt{F}_m\, GT(\Bbbk) / \mathtt{F}_{m+1}\, GT(\Bbbk)$ to $\mathtt{Mor}_{\mathcal{G}rd\, \mathcal{O}p}(\mathsf{PaB}_+, \mathfrak{p}_+(-)_{m+1})$.

We observed in the proof of Proposition 10.2.8 that the morphism $\phi : \mathsf{PaB}^\wedge_+ \to \mathsf{CD}^\wedge_+$, which our element of the set of associators represents, reduces to the inverse of the standard isomorphism $\upsilon : \mathfrak{p}(r) \xrightarrow{\simeq} \mathtt{E}^0\, \hat{P}_r$ when we pass to the subquotients of the automorphism groups of objects in the operad $\mathsf{PaB}^\wedge_+$ (up to the scalar factor $\kappa \in \Bbbk^\times$ associated to our associators). We can use this observation to check that the map $\theta : \psi \mapsto \theta_\psi$ of §11.4.4 agrees with the above mapping $\rho : \psi \mapsto \rho_\psi$ (up to this scalar factor $\kappa \in \Bbbk^\times$), and hence, induces a bijection from $\mathtt{E}^0_m\, GT(\Bbbk) = \mathtt{F}_m\, GT(\Bbbk) / \mathtt{F}_{m+1}\, GT(\Bbbk)$ to $\mathtt{Mor}_{\mathcal{G}rd\, \mathcal{O}p}(\mathsf{PaB}_+, \mathfrak{p}_+(-)_{m+1})$ as asserted in our proposition. □

In §10.4, we also observe that the module $\mathtt{Mor}_{\mathcal{G}rd\,\mathcal{O}p}(\mathit{PaB}_+, \mathfrak{p}_+(-)_{m+1})$ is isomorphic to the component of weight m of the graded Grothendieck–Teichmüller Lie algebra $\mathfrak{grt}_m$ (see Proposition 10.4.15). By combining this observation with the result of Proposition 11.4.5, we obtain the following statement:

Theorem 11.4.6. *We have $\mathtt{E}^0_m\, GT(\Bbbk) = \mathfrak{grt}_m$, for each weight $m \geq 1$.* □

We already observed that we have an identity $GT^1(\Bbbk) = \ker(p : GT(\Bbbk) \to GT_{\langle 0\rangle}(\Bbbk))$. We therefore have $GT^1(\Bbbk) = \mathtt{F}_1\, GT(\Bbbk)$ and $\mathtt{E}^0_m\, GT^1(\Bbbk) = \mathtt{E}^0_m\, GT(\Bbbk)$ for $m \geq 1$. The result of Proposition 11.4.2 implies that the commutator $(-,-)$ induces a Lie bracket on the weight graded module $\mathtt{E}^0\, GT^1(\Bbbk) = \bigoplus_{m\geq 1} \mathtt{E}^0\, GT(\Bbbk)$. We can check that this Lie bracket agrees with the Lie bracket of the Lie algebra $\mathfrak{grt}$. We can perform this verification by a direct inspection of our constructions. We accordingly have an identity of weight graded Lie algebras $\mathtt{E}^0\, GT^1(\Bbbk) = \mathfrak{grt}^1$, where we consider the Lie subalgebra $\mathfrak{grt}^1$ of the graded Grothendieck–Teichmüller Lie algebra $\mathfrak{grt}$.

We have an analogue of the results of this section for the graded Grothendieck–Teichmüller group. We explicitly have the identity $\mathtt{E}^0\, GRT^1(\Bbbk) = \mathfrak{grt}^1$ in the category of weight graded Lie algebras. We can use the definition of $\mathfrak{grt}^1$ as the Lie algebra associated to the pro-algebraic group $GRT^1(\Bbbk)$ and the observation that this Lie algebra $\mathfrak{grt}^1$, which we define by a construction of algebraic group theory, naturally splits as a weight graded module equipped with a homogeneous Lie structure in order to check this result. We may also retrieve our claim $\mathtt{E}^0\, GT^1(\Bbbk) = \mathfrak{grt}^1$ from this identity, by using that the group isomorphism $v_\phi : GRT(\Bbbk) \xrightarrow{\simeq} GT(\Bbbk)$, which we associate to any associator $\phi \in \mathit{Ass}(\Bbbk)$, induces an isomorphism between the weight graded Lie algebras which we associate to these groups.

CHAPTER 12

A Glimpse at the Grothendieck Program

Beyond the applications to quantum group theory, the definition of the Grothendieck–Teichmüller group by Drinfeld in [57] was motivated by ideas of the Grothendieck program which aims to give a geometrical picture of the absolute Galois group $G_{\mathbb{Q}} = Gal(\bar{\mathbb{Q}}|\,\mathbb{Q})$. The purpose of this concluding chapter is to give an overview of this arithmetic side of the Grothendieck–Teichmüller theory.

In Grothendieck's proposal [83], the fundamental objects are the moduli spaces of marked curves $\mathcal{M}_{gn}$. In this book, we already considered these objects, in the genus zero case, in our study of variations of the little discs operads (see §4.3.5). We mostly deal with this case $g = 0$ yet. We then have

$$\mathcal{M}_{0r+1} = \mathsf{F}(\mathbb{P}^1(\mathbb{C}), r+1)/\,PGL_2(\mathbb{C}),$$

where we consider the diagonal action of the group $PGL_2(\mathbb{C})$ on the configuration space of points in the projective line $\mathbb{P}^1(\mathbb{C})$. In previous chapters, we used the notation $\mathbb{CP}^1$, borrowed from topology, for the projective line. In what follows, we prefer to adopt the notation of algebraic geometry $\mathbb{P}^1(\mathbb{C})$ which stresses the existence of a scheme $\mathbb{P}^1$ underlying this topological space $\mathbb{P}^1(\mathbb{C}) = \mathbb{CP}^1$.

For $r \geq 2$, we have an identity $\mathcal{M}_{0r+1} = \mathsf{F}(\mathbb{P}^1(\mathbb{C})\backslash\{\infty, 0, 1\}, r-2)$ since, for each element of the configuration space $(z_0, \ldots, z_r) \in \mathsf{F}(\mathbb{P}^1(\mathbb{C}), r+1)$, we have one and only one transformation $g \in PGL_2(\mathbb{C})$ which maps the points of our configuration $\underline{z} = (z_0, \ldots, z_r)$ to a configuration of the form $g \cdot \underline{z} = (\infty, 0, 1, z'_3, \ldots, z'_r)$ in the space $\mathsf{F}(\mathbb{P}^1(\mathbb{C}), r+1)$. We can use this identity to regard each space $\mathcal{M}_{0r+1}$, $r \geq 2$, as a scheme defined over $\mathbb{Q}$ in the sense of algebraic geometry. In the particular case $r = 4$, we obtain $\mathcal{M}_{04} = \mathbb{P}^1(\mathbb{C}) \setminus \{\infty, 0, 1\}$. We refer to [87] for a modern textbook on moduli spaces of curves, addressed from the viewpoint of algebraic geometry. We also refer to the book [133] for an account of the connections between moduli spaces, the theory of Gromov-Witten invariants, and the theory of operads.

To start our survey, we recall the construction of an action of the absolute Galois group $G_{\mathbb{Q}} = Gal(\bar{\mathbb{Q}}|\,\mathbb{Q})$ on the profinite fundamental group of the moduli spaces $\hat{\pi}_1(\mathcal{M}_{0r+1})$ and we review the relationship between the definition of this action and the definition of the Grothendieck–Teichmüller group.

The action of the absolute Galois group on the fundamental group of algebraic varieties. We first consider the homotopy exact sequence

$$1 \to \pi_1^{et}(X \times_k k_s) \to \pi_1^{et}(X) \to Gal(k_s|k) \to 1, \tag{1}$$

which relates:

- the étale fundamental group $\pi_1^{et}(X)$ associated to any integral scheme X over a field k,
- the étale fundamental group $\pi_1^{et}(X \times_k k_s)$ associated to the scheme $\bar{X} = X \times_k k_s$, where k_s denotes the separable closure of the field k,

– and the absolute Galois group $Gal(k_s|k)$.

We refer to [82] for the definitions of the étale fundamental group of a scheme and of the above short exact sequence. We also refer to the books [144, §I.5] and [172, §5] for good introductions to these topics.

Let $\mathtt{Out}(\pi_1^{et}(X \times_k k_s))$ be the outer automorphism group of the group $\pi_1^{et}(X \times_k k_s)$. For any $g \in \pi_1^{et}(X)$, we consider the automorphism $c_g : \pi_1^{et}(X \times_k k_s) \to \pi_1^{et}(X \times_k k_s)$ induced by the conjugation operation $c_g(x) = gxg^{-1}$ in the group $\pi_1^{et}(X)$. We have a group morphism $\rho_X : Gal(k_s|k) \to \mathtt{Out}(\pi_1^{et}(X \times_k k_s))$ which we define by applying the map $\rho : g \mapsto c_g$ to the pre-image of the elements of the Galois group $Gal(k_s|k)$ in the étale fundamental group $\pi_1^{et}(X)$.

We assume $k = \mathbb{Q}$, so that $k_s = \bar{\mathbb{Q}}$, and we set $G_{\mathbb{Q}} = Gal(\bar{\mathbb{Q}}|\,\mathbb{Q})$. We consider the analytic space $X(\mathbb{C})$ associated to the scheme X, and we assume that X is locally of finite type over $\mathbb{Q}$. We then have an identity (see [82, §§XII.5.1-5.2]):

$$\pi_1^{et}(X \times_k k_s) = \hat{\pi}_1(X(\mathbb{C})), \tag{2}$$

where $\hat{\pi}_1(X(\mathbb{C}))$ denotes the profinite completion of the fundamental group of the space $X(\mathbb{C})$. We therefore have a group morphism

$$\rho_X : G_{\mathbb{Q}} \to \mathtt{Out}(\hat{\pi}_1(X(\mathbb{C}))), \tag{3}$$

naturally associated to the scheme X, and which we deduce from the homotopy exact sequence (1).

The Teichmüller tower. The main idea of the Grothendieck program [83] is to get information on the absolute Galois group $G_{\mathbb{Q}}$ from the morphisms (3) associated to the moduli spaces $X = \mathcal{M}_{gn}$ by using the geometry of the topological curves $\Sigma = \Sigma_{gn}$ which we associate to the geometrical points of these spaces $C \in \mathcal{M}_{gn}$. The morphism ρ_X is injective for $X = \mathcal{M}_{04} = \mathbb{P}^1 \setminus \{\infty, 0, 1\}$ (by a theorem of Belyi [22], see also [172, §§4.7.6-4.7.7] for an account of the arguments). The issue is therefore to characterize the image of the absolute Galois group $G_{\mathbb{Q}}$ within the fundamental group $\hat{\pi}_1(\mathbb{P}^1(\mathbb{C}) \setminus \{\infty, 0, 1\})$.

The (topological) fundamental group $\pi_1(\mathcal{M}_{gn})$ is identified with the mapping class group Γ_{gn} of the surface Σ_{gn} (up to elements of finite order). We refer to [26, §4] for a classical introduction to this subject. Recall that this group Γ_{gn}, explicitly defined as the group of isotopy classes of orientation preserving diffeomorphisms on Σ_{gn}, is generated by Dehn twists along curves drawn on Σ_{gn}. We can also use decompositions of the surface Σ_{gn} along curves in order to determine this group Γ_{gn} from smaller pieces involving the mapping class group of surfaces with boundary components. The proposal of [83] is to use an algebraic counterpart of the combinatoric of these surface decompositions in order to understand the relations satisfied by the image of the absolute Galois group $G_{\mathbb{Q}}$ in the outer automorphism groups $\mathtt{Out}(\hat{\pi}_1(\mathcal{M}_{gn}))$. These ideas are put in applications in [88] and in [147] with as main outcome a lifting of the morphisms $\rho : G_{\mathbb{Q}} \to \mathtt{Out}(\hat{\pi}_1(\mathcal{M}_{gn}))$ to the automorphism groups $\mathtt{Aut}(\hat{\pi}_1(\mathcal{M}_{gn}))$ and the determination of relations satisfied by the image of the absolute Galois group in $\mathtt{Aut}(\hat{\pi}_1(\mathcal{M}_{gn}))$.

In the previous paragraph, we have not been precise about the base points which are taken in the definition of the étale fundamental groups. In basic references on the subject, the base point is just a fixed geometric point of the scheme. But this choice does not enable us to get a counterpart, at the level of étale fundamental groups, of the operations on mapping class groups associated to surface decompositions. To handle the problems, one idea is to consider the Deligne–Mumford compactification $\overline{\mathcal{M}}_{gn}$ of the moduli space $\mathcal{M}_{gn}$ and to take tangent vectors at the infinity of the compactification as base points for the étale fundamental groups of the schemes $\mathcal{M}_{gn}$. This notion of tangential base point was introduced by P. Deligne in [52, §15]. We also consider fundamental groupoids rather than fundamental groups. In [94, 95], Ihara gives a definition of an action of the absolute Galois group on the tower of fundamental groupoids with tangential base points. He uses this approach to give a proof of the relations satisfied by the image of the absolute Galois group in the étale fundamental groups in the genus zero case $g = 0$.

The definition of the profinite Grothendieck–Teichmüller group. We go back to the case of the space $\mathcal{M}_{04} = \mathbb{P}^1 \setminus \{\infty, 0, 1\}$ and we consider the morphism $\rho = \rho_{\mathbb{P}^1 \setminus \{\infty,0,1\}}$ from the absolute Galois group $G_{\mathbb{Q}}$ to the outer automorphism group of the étale fundamental group of this scheme $\mathbb{P}^1 \setminus \{\infty, 0, 1\}$.

The topological fundamental group associated to this space is identified with the free group $\mathbb{F}(x, y)$, where x (respectively, y) is a loop turning around 0 (respectively, 1). We accordingly have $\pi_1^{et}(\mathbb{P}^1 \setminus \{\infty, 0, 1\}) = \hat{\mathbb{F}}(x, y)$, where the notation $\hat{\mathbb{F}}(x, y)$ now refers to the profinite completion of the free group on two generators (x, y).

In the tangential base point approach of [94, 95], one considers a loop x based at the tangent vector $\overrightarrow{01}$, the image of this loop under the map $\theta(z) = 1 - z$ (which forms a loop $\theta(x)$ based at $\overrightarrow{10}$ in the fundamental groupoid), and the straight path p, which goes from $\overrightarrow{01}$ to $\overrightarrow{10}$. The loops x and $y = p^{-1}\theta(x)p$ correspond to the previously considered generators of the fundamental group $\hat{\pi}_1(\mathbb{P}^1(\mathbb{C}) \setminus \{\infty, 0, 1\})$, based at a point near 0.

We already mentioned that the morphisms $\rho : G_{\mathbb{Q}} \to \mathtt{Out}(\hat{\pi}_1(\mathcal{M}_{gn}))$ can be lifted to the étale fundamental groupoids of the moduli spaces equipped with tangential base points. In the case $(g, n) = (0, 4)$, which we now examine with more details, this approach can also used to prove that the morphism $\rho : G_{\mathbb{Q}} \to \mathtt{Out}(\hat{\mathbb{F}}(x, y))$ admits a lifting to the group of automorphisms of the free group $\hat{\mathbb{F}}(x, y)$. To be more precise, one can prove that we have an automorphism $\rho(\sigma) : \hat{\mathbb{F}}(x, y) \to \hat{\mathbb{F}}(x, y)$, canonically associated to any element $\sigma \in G_{\mathbb{Q}}$, such that we have $\rho(\sigma)(x) = x^{\lambda}$ for our first generator $x \in \pi_1^{et}(\mathbb{P}^1 \setminus \{\infty, 0, 1\})$, where $\lambda = \chi(\sigma)$ denotes the image of σ under the cyclotomic character $\chi : G_{\mathbb{Q}} \to \hat{\mathbb{Z}}^{\times}$, and we have $\rho(\sigma)(y) = f_{\sigma}(x, y)^{-1} y^{\lambda} f_{\sigma}(x, y)$ for our second generator $y \in \pi_1^{et}(\mathbb{P}^1 \setminus \{\infty, 0, 1\})$, for some $f_{\sigma} = f_{\sigma}(x, y) \in \hat{\mathbb{F}}(x, y)$. Furthermore, by using paths in the moduli spaces $\mathcal{M}_{04} = \mathbb{P}^1 \setminus \{\infty, 0, 1\}$ and $\mathcal{M}_{05} = (\mathbb{P}^1 \setminus \{\infty, 0, 1\} \times \mathbb{P}^1 \setminus \{\infty, 0, 1\}) \setminus \Delta$, one can prove that this group element $f_{\sigma}(x, y)$ satisfies profinite analogues of the unit, involution, pentagon and hexagon relations of §11.1. The profinite Grothendieck–Teichmüller group $\widehat{GT}$, such as defined by Drinfeld in [57], precisely consists of the group automorphisms $\phi : \hat{\mathbb{F}}(x, y) \to \hat{\mathbb{F}}(x, y)$ of the form:

$$\begin{aligned} &\phi(x) = x^{\lambda}, \\ &\phi(y) = f(x, y)^{-1} \cdot y^{\lambda} \cdot f(x, y), \end{aligned} \tag{4}$$

where we consider any pair $(\lambda, f(x, y))$ such that $f(x, y)$ satisfies these profinite analogues of the unit, involution, pentagon and hexagon relations of §11.1. The construction of the action of the absolute Galois group $G_{\mathbb{Q}}$ on the étale fundamental group of the moduli spaces $\mathcal{M}_{0r+1}$ therefore yields an injective group morphism $\rho : G_{\mathbb{Q}} \hookrightarrow \widehat{GT}$.

The proof of the Drinfeld involution, pentagon and hexagon relations for the pair $(\lambda, f_\sigma(x,y))$ associated to a Galois group element $\sigma \in G_{\mathbb{Q}}$ is given by Y. Ihara in [94, 95] by using the fundamental groupoid with tangential base points approach. We refer to the work of P. Lochak and L. Schneps [123] for another approach, which relies on a cohomological interpretation of Drinfeld's relations, of this question. Let us mention that the above formulas (4) can also be used to describe the action of the Galois group $G_{\mathbb{Q}}$ on inertia subgroups of a stacky version of the étale fundamental groups of the moduli spaces of curves (see [48, 47] for this subject).

We note that the group $GT\widehat{\ }$ encodes the geometric information captured by the action of the absolute Galois group in genus zero only. We have a generalization of this group, defined by considering the whole collection of moduli spaces $\mathcal{M}_{gn}$, which has been introduced by P. Lochak, H. Nakamura, and L. Schneps in [122]. We do not go further into applications of Grothendieck–Teichmüller groups in the profinite setting. We refer to the cited articles for the reader willing to learn more about this subject.

The category of mixed Tate motives. We now give a brief survey on the definition of an analogue, for the pro-unipotent groups of §11, of this relationship between the absolute Galois group $G_{\mathbb{Q}}$ and the profinite Grothendieck–Teichmüller group $GT\widehat{\ }$. We then replace the absolute Galois group by Galois groups of motives.

Briefly recall that the idea of a motive was introduced by Grothendieck as an attempt to unify the cohomology theories that occur in algebraic geometric: the singular cohomology of the topological space underlying any algebraic variety (the Betti cohomology in the language of algebraic geometry), the de Rham cohomology, the l-adic cohomologies, the crystalline cohomology, and more generally, any suitable cohomology theory that satisfies the Weil axioms. Motives are supposed to form an abelian category under the category of algebraic varieties such that the mapping $M : X \mapsto M(X)$, which assigns a motive $M(X)$ to any algebraic variety X, defines a universal Weil cohomology theory. We refer to [7] for a comprehensive introduction to this subject.

The definition of a conjectural category of pure motives (well suited when we restrict ourselves to smooth projective varieties) was initially proposed by Grothendieck (see [7] for a survey of this approach). For more general varieties, we have Deligne's language of realization systems, which formalizes the structures carried by the images of a motive under a cohomology theory, as well as Hanamura's [86], Levine's [119] and Voevodsky's [179] triangulated categories of mixed motives, which define candidates for the derived category of the abelian category of mixed motives. The definition an abelian category of mixed motives has also been proposed by Nori, by relying on the Tannakian formalism. We refer to [120] for a survey of this construction and to the book [92] for a more detailed account. The Tannakian constructions have a counterpart in Voevodsky's approach which has been studied by Ayoub in [13, 14]. The main outcome of Ayoub's work is a new definition of a Tannakian category of mixed motives which turns to be equivalent to the category of mixed motives defined by Nori (see [43]).

The connection between Grothendieck–Teichmüller groups and motives is made precise in the work of Deligne–Goncharov [54] and in the work of Terasoma [176]. We then mostly consider a category of (rational) mixed Tate motives, which is defined as a subcategory of the category of mixed motives. We follow [54] for our account. We consider varieties and motives defined over a field k of characteristic zero.

Recall that the Tate motive T is classically defined as the tensor inverse $T = L^{-1}$ of a motive L such that $M(\mathbb{P}^1) = \mathbb{1} \oplus L$, where we consider a splitting of the motive associated to the projective line $\mathbb{P}^1$. The triangulated category of rational Tate motives, denoted by $DMT(k)_{\mathbb{Q}}$, is generated by iterated extensions of shifted objects $\mathbb{Q}(n)$ in any of our rational triangulated categories of mixed motives, where $\mathbb{Q}(1) = T \otimes \mathbb{Q}$ denotes the object that represent the Tate motive in this triangulated category and we set $\mathbb{Q}(n) = \mathbb{Q}(1)^{\otimes n}$. This category $DMT(k)_{\mathbb{Q}}$ is actually identified with the derived category of an abelian category $MT(k)$ as soon as the Beilinson-Soulé vanishing conjecture holds, which is at least the case when k is a number field (see [118]). We refer to this category $MT(k)$ as the *abelian category of mixed Tate motives* over k.

Let $\mathcal{O}_S$ be a ring of S-integers in the number field k. In [54], a subcategory of mixed Tate motives over $\mathcal{O}_S$, denoted by $MT(\mathcal{O}_S)$, is also defined within $MT(k)$. One can apply the Tannakian formalism to identify this category $MT(\mathcal{O}_S)$ with the category of representations of an affine group scheme G_ω associated to a realization functor $\omega : MT(\mathcal{O}_S) \to \mathcal{M}od_{\mathbb{Q}}$. Deligne and Goncharov also define an affine group scheme $G_{MT(\mathcal{O}_S)}$ in the category $MT(\mathcal{O}_S)$ such that $\omega(G_{MT(\mathcal{O}_S)}) = \mathtt{Aut}(\omega)$, where $\mathtt{Aut}(\omega)$ denotes the group of natural automorphisms of this realization functor ω on the category $MT(\mathcal{O}_S)$. They refer to this object $G_{MT(\mathcal{O}_S)}$ as the fundamental group of the category of mixed Tate motives $MT(\mathcal{O}_S)$. This is this group $G_{MT(\mathcal{O}_S)}$ and its realizations $\omega(G_{MT(\mathcal{O}_S)}) = \mathtt{Aut}(\omega)$ that replace the absolute Galois in the pro-unipotent setting.

The motivic fundamental group of Tate motives. We now assume $k = \mathbb{Q}$ and $\mathcal{O}_S = \mathbb{Z}$. Deligne–Goncharov [54] and Terasoma [176] have defined a motivic counterpart $\pi_1^{mot}(\mathbb{P}^1 \setminus \{\infty, 0, 1\})$, in the category $MT(\mathbb{Z})$, of the fundamental group of the variety $\mathbb{P}^1 \setminus \{\infty, 0, 1\}$. This motivic fundamental group $\pi_1^{mot}(\mathbb{P}^1 \setminus \{\infty, 0, 1\})$ has a Betti realization $\pi_1^B(\mathbb{P}^1 \setminus \{\infty, 0, 1\})$, which is identified with the prounipotent completion of the fundamental group of the topological space $\mathbb{P}^1(\mathbb{C}) \setminus \{\infty, 0, 1\})$, as well as a de Rham realization $\pi_1^{DR}(\mathbb{P}^1 \setminus \{\infty, 0, 1\})$ (we refer to [52, §10] for the original definition of this de Rham realization of fundamental groups).

We use the notation $G^B_{MT(\mathbb{Z})}$ for the Betti realization of the motivic fundamental group $G_{MT(\mathbb{Z})}$ of the integral category of mixed Tate motives $MT(\mathbb{Z})$. We have a group morphism $\rho : G^B_{MT(\mathbb{Z})} \to \mathtt{Aut}(\pi_1^B(\mathbb{P}^1 \setminus \{\infty, 0, 1\}))$ which defines a motivic analogue, in the Betti realization, of the previously considered morphism $\rho : G_{\mathbb{Q}} \to \mathtt{Aut}(\hat{\pi}_1(\mathbb{P}^1 \setminus \{\infty, 0, 1\}))$, where we consider the usual Galois group $G_{\mathbb{Q}} = Gal(\bar{\mathbb{Q}} | \mathbb{Q})$. We also have $\pi_1^B(\mathbb{P}^1 \setminus \{\infty, 0, 1\}) = \hat{\mathbb{F}}(x, y)$, the prounipotent completion of the free group with two generators, and one can prove, just as in the profinite setting, that the morphism ρ factors through the Grothendieck–Teichmüller group $GT(\mathbb{Q})$ regarded as a subset of the automorphism group of this free group $\hat{\mathbb{F}}(x, y)$ (see [177] for an outline of the arguments).

The obtained morphism

$$\rho : G^B_{MT(\mathbb{Z})} \to GT(\mathbb{Q}) \tag{5}$$

is conjecturally an isomorphism (Deligne–Ihara). A result of F. Brown [37] gives the injectivity of this morphism.

Let us mention that the group $G^B_{MT(\mathbb{Z})}$ is, according to a statement of [54], the semi-direct product of the multiplicative group with the prounipotent completion of a free group on a sequence of generators $s_3, s_5, \dots, s_{2n+1}, \dots$. The Deligne–Ihara conjecture is therefore equivalent to the conjecture that we have an identity between:

- the Lie algebra $\mathfrak{gt}^1$ of the pro-unipotent Grothendieck–Teichmüller group $GT^1(\mathbb{Q})$ (see §11.2.3), or equivalently, the Lie algebra $\mathfrak{grt}^1$ of the graded Grothendieck–Teichmüller group $GRT^1(\mathbb{Q})$ (see §§10.4.5-10.4.6),
- and a free complete Lie algebra $\hat{\mathbb{L}}(s_3, s_5, \dots, s_{2n+1}, \dots)$.

The Knizhnik–Zamolodchikov associator and multizetas. The Knizhnik-Zamolodchikov associator of Theorem 10.2.12 has also an interpretation in terms of a period isomorphism connecting the Betti realization and the de Rham realization of the motivic fundamental group $\pi_1^{mot}(\mathbb{P}^1 \setminus \{\infty, 0, 1\})$ (see [177] for an introduction to this subject).

The Knizhnik–Zamolodchikov associator actually represents a generating power series of the multizeta values

$$\zeta(k_1, \dots, k_r) = \sum_{n_1 > \dots > n_r > 0} \frac{1}{n_1^{k_1} \cdot \dots \cdot n_r^{k_r}} \tag{6}$$

(as we already briefly explained in §10). These numbers $\zeta(k_1, \dots, k_r)$ precisely appear (with a correcting sign) as the coefficients of the terms $x^{k_1-1} y x^{k_2-1} y \cdots y x^{k_r-1}$ in the expansion of the power series of the Knizhnik–Zamolodchikov associator $\Phi(x, y) \in \mathbb{G}\,\hat{\mathbb{T}}(x, y)$ in the completed tensor algebra $\hat{\mathbb{T}}(x, y)$. The other terms of this expansion can be obtained from multizetas by an explicit procedure (see [115]).

Multizeta values are instances of periods in the sense of Kontsevich–Zagier [108, 110]. The multizeta values form an algebra. The result established by F. Brown in [37] actually asserts that a motivic counterpart of this algebra, where we only retain relations underlying an algebraic definition of multizetas in terms of motivic periods (see [80]), is isomorphic to the completed tensor algebra underlying the free Malcev complete group $G^B_{MT(\mathbb{Z})}$. The injectivity of the map (5) in the Deligne–Ihara conjecture follows from this result. This relationship between the motivic Galois group $G^B_{MT(\mathbb{Z})}$ and the algebra of multizetas has also a conceptual interpretation in terms of an action of motivic Galois groups on periods. We just refer to [108] for the definition of such actions in terms of Nori's category of mixed motives (see also the book [92] for a detailed account of this correspondence and for a comprehensive reference on this subject).

Appendices

APPENDIX A

Trees and the Construction of Free Operads

The main purpose of this appendix is to explain an explicit construction of free operads. This construction works in any base symmetric monoidal category $\mathcal{M}$ whose tensor product $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ distributes over colimits (see §0.9).

We basically check that the free operad $\mathbb{O}(M)$ associated to a symmetric sequence M is spanned by tensor products of components of the object M arranged on trees. Intuitively, these trees give the pattern of all composites that can be formed within an operad. The general construction of free objects which we give in this appendix can be handled in the category of all operads without any restriction on the arity zero term. Nevertheless, we prefer to focus on the study of free objects in the category of non-unitary operads, for which we assume $P(0) = \varnothing$, since we mainly deal with this subcategory of operads in this book. By the way, we also examine the definition of free objects in the category of augmented non-unitary Λ-operads which we use to model unitary operad structures (see §I.2.2). We formally check that our plain free operad functor admits a lifting to the category of augmented non-unitary Λ-operads, and we check that this lifting fulfills the natural adjunction relation which we attach to a free object in this category of operads (see §I.2.3).

In §I.1.2, we define the free operad as the left adjoint of the obvious forgetful functor from operads to symmetric sequences. In the verifications which we carry out in this appendix, we rather give an explicit definition of the free operad functor in terms of trees first, and we check that this functor solves the adjunction problem of §I.1.2 in a second step.

We review the formal definition of a tree in a preliminary section (§A.1). We explain the definition of the treewise tensor product of a symmetric sequence and we address the construction of the free operad functor itself afterwards (§§A.2-A.3).

We devote an extra section to the construction of free objects in the category of connected (non-unitary) operads (§A.4). Recall that a non-unitary operad P, for which we assume $P(0) = \varnothing$, is connected in our sense when we also have the relation $P(1) = \mathbb{1}$ in arity one (see §I.1.1.21 and §I.1.2.13). In this setting, we consider the symmetric sequence $\bar{P}$ which obtain by dropping the unit term $P(1) = \mathbb{1}$ from the operad P, and where we take $\bar{P}(1) = \varnothing$ instead. We refer to this object $\bar{P}$ as the *augmentation ideal* of the operad P, because it is identified with the kernel of a natural augmentation $\epsilon : P \to I$ associated to P (at least, when the base category is pointed). Recall that I denotes the initial object of the category of operads. We have $I(1) = \mathbb{1}$ and $I(r) = \varnothing$ for $r > 1$. In §I.1, we call this object I the unit operad (see §I.1.2.3). In §I.1.2, we explain that the free operad functor also induces a left adjoint of the map $\bar{\omega} : P \mapsto \bar{P}$ when we work in the category of connected operads. In §A.4, we go back to the proof of this result, and we check that the free objects of the category of connected operads admit a reduced expansion over a subcategory of reduced trees, where each vertex has at least two ingoing edges. By the way, we also examine the definition of free objects in the context of augmented connected Λ-operads. In short, we just check that this free object functor can be obtained by lifting the plain free object functor, which goes from the category of connected sequences to the category of (ordinary) operads, as in the case of augmented non-unitary Λ-operads, but we have to amend our construction when we work in the connected setting.

To complete our account, we give an explicit definition, by using an extension of our treewise tensor product constructions, of coproducts of the form $\mathsf{P} \vee \Theta(\mathsf{M})$ in the category of operads. We address this subject in a last section (§A.5).

We actually use the formalism of symmetric collections (see §I.2.5) in this appendix rather than the equivalent formalism of symmetric sequences (which we most usually consider in this book). We also use the definition of a (non-unitary) operad P in terms of partial composition products $\circ_{i_k} : \mathsf{P}(\underline{\mathsf{m}}) \otimes \mathsf{P}(\underline{\mathsf{n}}) \to \mathsf{P}(\underline{\mathsf{m}} \circ_{i_k} \underline{\mathsf{n}})$ where $\underline{\mathsf{m}}$ and $\underline{\mathsf{n}}$ are (non-empty) finite sets, we assume $i_k \in \underline{\mathsf{m}}$, and $\underline{\mathsf{m}} \circ_{i_k} \underline{\mathsf{n}}$ refers to the operadic composition of finite sets (see §I.2.5.5).

For simplicity, we assume all through this appendix that we work in a fixed base symmetric monoidal category $\mathcal{M}$ where colimits exist and we assume that the tensor product of this symmetric monoidal category distributes over colimits. Nonetheless, we do not form any colimit until we really tackle the construction of the free operad in §A.3 and all results established in §A.2, among which the equivalence between full composition products and partial composition products (§A.2.10), hold without this extra condition on $\mathcal{M}$.

We do no prove any original result in this appendix, and most constructions which we explain (except the applications to Λ-operads) are borrowed from the literature. In particular, we refer to Ginzburg–Kapranov's article [78] for the treewise definition of the free operad which we use in this monograph. The book [138, §II.1.9] includes a survey of this construction of free operads. We also refer to the book [186] for a detailed survey on these applications of trees in operad theory, and to the book [187] for a comprehensive study of generalizations of trees in the context of PROPs. Trees actually occur in the early developments of the theory of operads. To be specific, let us mention that trees give the shape of the Stasheff associahedra, which Stasheff introduced to formulate his recognition theorem for single loop spaces (see [167]), and of the Boardman–Vogt W-construction, which generalizes the Stasheff operad of associahedra, and which Boardman–Vogt introduced to define homotopy invariant structures in topology (see [28]).

A.1. Trees

The aim of this first section, as we just explain, is to survey fundamental definitions on trees which we use in our construction of free operads. By the way, we check that our trees form an operad. We introduce more constructions on trees later on in this appendix for the purpose of specific applications. We only explain fundamental definitions in this section and all assertions which we make in the course of our account follow from straightforward verifications.

To begin with, we explain the definition of our trees. We essentially adapt a standard description of the structure of a 1-dimensional cell complex, but since we only use the abstract notion of a tree, we only give the abstract definition of our objects. We will give references to the literature for the topological interpretation of the notions which we introduce in this section.

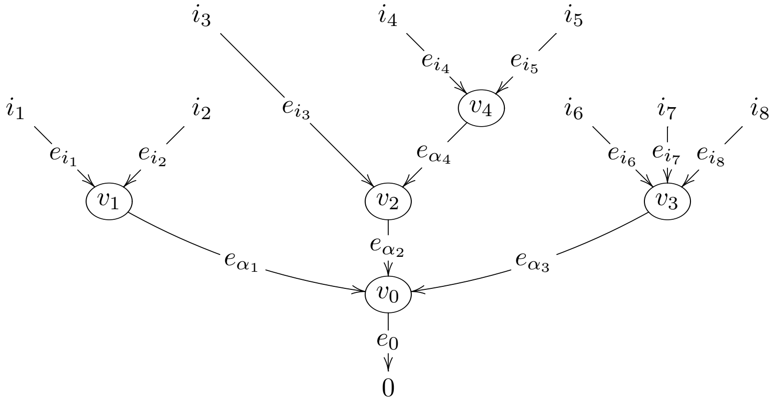


FIGURE A.1. The picture of a tree structure. The input set is $\underline{\mathrm{r}} = \{i_1, \dots, i_8\}$, the vertex set is $V(\underline{\mathrm{T}}) = \{v_0, \dots, v_4\}$, and the edge set is $E(\underline{\mathrm{T}}) = \{e_0, e_{\alpha_1}, \dots, e_{\alpha_4}, e_{i_1}, \dots, e_{i_8}\}$. The notation 0 marks the output of the tree. The edge e_0 is the outgoing edge of the tree, the edges $e_{i_1}, \dots, e_{i_8}$ are the ingoing edges, and the edges $e_{\alpha_1}, \dots, e_{\alpha_4}$ are the inner edges (see §§A.1.1-A.1.2). The edges $e_{\alpha_1}, \dots, e_{\alpha_4}$ form the outgoing edges of the vertices $v_1, \dots, v_4$ (see §A.1.1). The edge e_0 is identified with the outgoing edge of the vertex v_0.

A.1.1. *The formal definition of a tree structure.* To summarize, the trees that we consider in this appendix have a finite number of *inputs*, indexed by a given set $\underline{\mathrm{r}} = \{i_1, \dots, i_r\}$, and an *output*, usually marked by the symbol 0. We use the phrase '$\underline{\mathrm{r}}$-*tree*' when we need to specify the indexing set $\underline{\mathrm{r}}$ of the ingoing edges of our trees.

To begin with an example, we have represented a tree with eight inputs indexed by $\underline{\mathrm{r}} = \{i_1, \dots, i_8\}$ in Figure A.1. The set $V(\underline{\mathrm{T}}) = \{v_0, \dots, v_4\}$ defines the *vertex set* of this tree and the set $E(\underline{\mathrm{T}}) = \{e_0, e_{\alpha_1}, \dots, e_{\alpha_4}, e_{i_1}, \dots, e_{i_8}\}$ defines the *edge set*. In general, we do not specify the "name" of the edges in the representation of a tree, but we prefer to give this indication throughout this preliminary section in order to help the understanding of our general definitions. The "name" of the vertices can be omitted too.

Formally, an $\underline{\mathrm{r}}$-tree $\underline{\mathrm{T}}$ consists of a set of *vertices*, denoted by $V(\underline{\mathrm{T}})$, and a set of *edges* $e \in E(\underline{\mathrm{T}})$, oriented from a *source* $s(e) \in V(\underline{\mathrm{T}}) \amalg \underline{\mathrm{r}}$ towards a *target* $t(e) \in V(\underline{\mathrm{T}}) \amalg \{0\}$, such that the following conditions hold:

(1) There is one and only one edge $e_0 \in E(\underline{\mathrm{T}})$, *the outgoing edge of the tree*, such that $t(e_0) = 0$.
(2) For each $i \in \underline{\mathrm{r}}$, there is one and only one edge $e_i \in E(\underline{\mathrm{T}})$, *the ingoing edge of the tree indexed by* i, such that $s(e_i) = i$.
(3) For each vertex $v \in V(\underline{\mathrm{T}})$, there is one and only one edge $e_v \in E(\underline{\mathrm{T}})$, *the outgoing edge of the vertex* v, such that $s(e_v) = v$.
(4) Each vertex $v \in V(\underline{\mathrm{T}})$ is connected to the output 0 by a chain of edges e_v, $e_{v_{n-1}}$, …, e_{v_1}, e_{v_0} such that $v = s(e_v)$, $t(e_v) = s(e_{v_{n-1}})$, $t(e_{v_{n-1}}) = s(e_{v_{n-2}})$, …, $t(e_{v_2}) = s(e_{v_1})$, $t(e_{v_1}) = s(e_{v_0})$ and $t(e_{v_0}) = 0$.

This definition corresponds to the general shape of free operads with a possible term in arity zero. But as we explain in the introduction of this appendix, we focus on the definition of free objects of the category of non-unitary operads because we most usually deal with non-unitary operads in this monograph. The trees which we consider in this context fulfill the following additional property:

(5) For each vertex $v \in V(\underline{T})$, we have at least one edge $e \in E(\underline{T})$ such that $t(e) = v$.

We take this additional requirement in our definition of a tree. We also say in this appendix that our trees are *open* in order to stress this condition (this terminology is motivated by the topological interpretation of our objects which we explain soon). We just examine examples of trees with a non-empty set of terminal vertices (which do not fulfill the above condition) in a few side remarks.

In the graphical representation, the edges e are materialized by arrows oriented from their source $x = s(e)$ to their target $y = t(e)$. In the example of Figure A.1, the edge e_{i_3} (for instance) satisfies $s(e_{i_3}) = i_3$ and $t(e_{i_3}) = v_2$.

The chain of edges connecting a vertex v to the output 0 in condition (4) has a representation of the form:

$$\begin{tikzcd} \textcircled{v} \arrow[r, "e_v"] & \textcircled{v_l} \arrow[r, "e_{v_l}"] & \cdots \arrow[r, "e_{v_1}"] & \textcircled{v_0} \arrow[r, "e_{v_0}"] & 0 \end{tikzcd} .$$

Note that condition (3) implies that this chain is unique. For an input i, we also have one and only one chain of edges going from i to the output of the tree 0. In general, in an $\underline{r}$-tree $\underline{T}$, we have at most one chain of edges going from a given vertex u to another one v, or from a given input i to a given vertex v. From similar observations, we also obtain that any chain of edges (followed in the upward direction) eventually leads to an input $i \in \underline{r}$ (assuming, as we require in our conventions, that all vertices of a tree have at least one ingoing edge).

In the standard topological language (we refer to [139, §VIII.3, §IX.6] or [166, §3.7]), our trees are open subspaces of oriented contractible regular 1-dimensional finite cell complexes. To be more precise, the trees which we consider in our constructions are identified with the spaces which we form by removing some boundary (0-dimensional) cells in such a cell complex. The 0-dimensional cells which we keep in our subspace define the set of vertices of our tree. The 1-dimensional cells define the edges. In our convention, we remove the maximal vertex of the ambient complex (with respect to the order relation determined by the orientation) when we form our subspace. The unique non-compact 1-dimensional cell of our complex which has this maximal vertex as target represents the outgoing edge of our tree. The 1-dimensional cells whose source is exterior to our tree symmetrically represent the set of ingoing edges of our tree. The extra condition (5), disallowing terminal vertices in our definition of a tree, is equivalent to the requirement that we remove all boundary vertices of the ambient 1-dimensional finite cell complex in our trees.

A.1.2. *Inner edges of trees and conventions on edges.* In what follows, we call *inner edges* of a tree $\underline{T}$ the edges which are neither an ingoing edge nor an outgoing edge, or equivalently, the edges e for which we have $s(e), t(e) \in V(\underline{T})$. We use the notation $\mathring{E}(\underline{T}) \subset E(\underline{T})$ for this subset of inner edges. In the example of Figure A.1, we have $\mathring{E}(\underline{T}) = \{e_{\alpha_1}, e_{\alpha_2}, e_{\alpha_3}, e_{\alpha_4}\}$. The axioms readily imply that the subset of inner edges of a tree is empty if and only if the vertex set of our tree has at most one element. (We analyze the structure of trees with at most one vertex in §A.1.4.)

By convention, we use the letter e (or f, or $\dots$) to denote an edge of any kind. Usually, if we want to specify an ingoing edge of a tree $\underline{\mathsf{T}}$, then we use an expression of the form e_i (or f_i, or $\dots$), with a distinguishing roman index i referring to the input of this edge $i = s(e_i)$. To specify the outgoing edge of a tree, we will use an expression of the form e_0 (or f_0, or $\dots$), with the distinguishing mark 0 which we associate to the output. If we need to specify the outgoing edge of a vertex v in a tree, then we use an expression of the form e_v (or f_v, or $\dots$), with the vertex v as distinguishing index, and we use expressions of the form $e_{\alpha_1}, e_{\alpha_2}, \dots$ (or f_{β_1}, f_{β_2}, or $\dots$), with Greek alphabet indices, when we need to distinguish a set of inner edges in our tree.

A.1.3. *Ingoing edges of vertices and reduced trees.* Recall that the outgoing edge of a vertex v in a tree $\underline{\mathsf{T}}$ is the unique edge e_v such that $s(e_v) = v$. To a vertex, we also associate a set of *ingoing edges* $\underline{\mathsf{r}}_v$ defined by:

$$\underline{\mathsf{r}}_v = \big\{ e \in E(\underline{\mathsf{T}}) \,|\, t(e) = v \big\}.$$

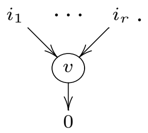
In the case of Figure A.1, we have for instance $\underline{\mathsf{r}}_{v_2} = \{e_{i_3}, e_{\alpha_4}\}$.

The extra condition (5) in our definition of an (open) tree §A.1.1 is equivalent to the requirement that we have $\operatorname{card}(\underline{\mathsf{r}}_v) \geq 1$, for every $v \in V(\underline{\mathsf{T}})$. In the construction of connected free operads and of cofree cooperads, we also consider trees $\underline{\mathsf{T}}$ satisfying $\operatorname{card}(\underline{\mathsf{r}}_v) \geq 2$, for every vertex $v \in V(\underline{\mathsf{T}})$. To depict this situation, we usually say that the tree $\underline{\mathsf{T}}$ is *reduced*. The tree of Figure A.1 for instance is reduced.

A.1.4. *Fundamental examples.* The set of vertices of a tree can be empty. In this case, our tree, which we call '*unit tree*' and for which we adopt the distinguishing notation $\underline{\downarrow}$, has necessarily the form

with a single edge e going straight from the input 1 to the output 0. The input set of such a tree is also necessarily reduced to a single element 1. We may see that the unit tree is essentially unique. We soon introduce a notion of isomorphism of trees to formalize this idea (see §A.1.8). We can precisely check that the trees with an empty set of vertices form an isomorphism class in the category of trees, and when we use the phrase 'unit tree' we actually refer to a representative of this distinguished isomorphism class of trees.

The form of a tree with a single vertex is fully determined by the axioms too. Indeed, for an $\underline{\mathsf{r}}$-tree $\underline{\mathsf{Y}}$ such that $V(\underline{\mathsf{Y}}) = \{v\}$, we necessarily have $E(\underline{\mathsf{Y}}) = \{e_i, i \in \underline{\mathsf{r}}\} \amalg \{e_0\}$, because the outgoing edge of the tree e_0 defines the outgoing edge of our vertex v, the other edges $e \neq e_0$ necessarily arise from an input i, and hence, necessarily form ingoing edges $e = e_i$ of our tree. Moreover, for these edges $e = e_i$, we obviously have $t(e_i) = v$. Hence, we finally obtain that our tree $\underline{\mathsf{Y}}$ has the form:



Throughout this book, we reserve the notation $\underline{\mathsf{Y}}$ and we use the phrase '$\underline{\mathsf{r}}$-*corolla*' to refer to a tree of this form. We also specify the indexing set of the inputs $\underline{\mathsf{r}}$ as a subscript in our notation of a corolla $\underline{\mathsf{Y}} = \underline{\mathsf{Y}}_{\underline{\mathsf{r}}}$ when necessary.

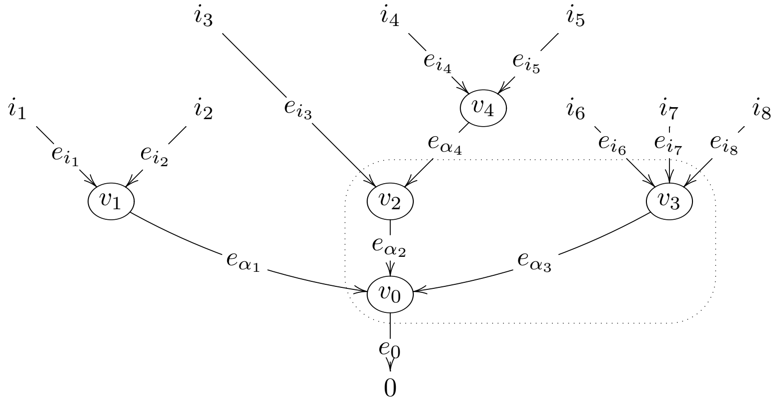


FIGURE A.2. The picture of a subtree drawn in the tree of Figure A.1. The subtree is materialized by the circled array in the picture. The vertices $\{v_0, v_2, v_3\}$ (respectively the edges $\{e_{\alpha_2}, e_{\alpha_3}\}$) included in this array form the set of vertices (respectively, the set of internal edges) of the subtree. The edges $\{e_{\alpha_1}, e_{i_3}, e_{\alpha_4}, e_{i_6}, e_{i_7}, e_{i_8}\}$, getting into the array, form the set of ingoing edges of the subtree, and the edge e_0, getting out, defines the outgoing edge (see §A.1.5).

From the above analysis, we deduce that the set of inputs $\underline{\mathsf{r}}_v$ which we associate to the vertex v of an $\underline{\mathsf{r}}$-corolla $\underline{\mathsf{Y}} = \underline{\mathsf{Y}}_{\underline{\mathsf{r}}}$ is endowed with a canonical bijection $\underline{\mathsf{r}}_v \simeq \underline{\mathsf{r}}$. This observation can be used to check that the $\underline{\mathsf{r}}$-trees with a single vertex $\underline{\mathsf{Y}} = \underline{\mathsf{Y}}_{\underline{\mathsf{r}}}$ form an isomorphism class in the category of $\underline{\mathsf{r}}$-trees (like the trees with an empty set of vertices), for each finite set $\underline{\mathsf{r}}$, though the underlying vertex and edge sets of these trees may themselves vary within the category of sets.

We examine the definition of trees with two vertices in the next section. We use such trees to represent the (partial) composition products of an operad. We already gave an introduction to this treewise representation of the composition products of an operad in §§I.2.1.4-2.1.5 (see also §I.2.5.8). We mainly use our definition of a tree in order to formalize this construction which we informally used in Part I.

A.1.5. *Subtrees.* We have a natural notion of *subtree* associated to our notion of tree. Intuitively, a subtree $\underline{\Sigma}$ of a given $\underline{\mathsf{r}}$-tree $\underline{\mathsf{T}}$ represents an open connected subspace of the cell complex defined by $\underline{\mathsf{T}}$. In a figure, we specify a subtree $\underline{\Sigma}$ by circling an array which circumscribes $\underline{\Sigma}$ in the ambient $\underline{\mathsf{r}}$-tree $\underline{\mathsf{T}}$. An example is given in Figure A.2. Formally, a subtree $\underline{\Sigma}$ of $\underline{\mathsf{T}}$ consists

(1) of a vertex set $V(\underline{\Sigma}) \subset V(\underline{\mathsf{T}})$,
(2) of an edge set $E(\underline{\Sigma}) \subset E(\underline{\mathsf{T}})$,
(3) together with an input set $\underline{\mathsf{r}}_{\underline{\Sigma}} \subset V(\underline{\mathsf{T}}) \amalg \underline{\mathsf{r}}$, disjoint from $V(\underline{\Sigma})$,
(4) and an output element $0_{\underline{\Sigma}} \in V(\underline{\mathsf{T}}) \amalg \{0\}$, also disjoint from $V(\underline{\Sigma})$,

such that we have $0_{\underline{\Sigma}} = t(e)$, for a unique edge $e \in E(\underline{\Sigma})$, and where an edge $e \in E(\underline{\mathsf{T}})$ belongs to $E(\underline{\Sigma}) \subset E(\underline{\mathsf{T}})$ if and only if we have the source and target relations

$$s(e) \in V(\underline{\Sigma}) \amalg \underline{\mathsf{r}}_{\underline{\Sigma}} \quad \text{and} \quad t(e) \in V(\underline{\Sigma}) \amalg \{0_{\underline{\Sigma}}\}$$

in the tree $\underline{\mathrm{T}}$. In the example of Figure A.2, we have:

$$V(\underline{\Sigma}) = \{v_0, v_2, v_3\}, \quad E(\underline{\Sigma}) = \{e_0, e_{\alpha_1}, e_{\alpha_2}, e_{\alpha_3}, e_{\alpha_4}, e_{i_3}, e_{i_6}, e_{i_7}, e_{i_8}\},$$
$$\underline{\mathrm{r}}_{\underline{\Sigma}} = \{v_1, v_4, i_3, i_6, i_7, i_8\}, \quad \text{and} \quad 0_{\underline{\Sigma}} = 0.$$

Our requirements ensure that $\underline{\Sigma}$ inherits a tree structure from the ambient tree $\underline{\mathrm{T}}$, with $\underline{\mathrm{r}}_{\underline{\Sigma}}$ as input set, the element $0_{\underline{\Sigma}}$ as output, and the source and target maps given by the obvious restrictions of the source and target maps of $\underline{\mathrm{T}}$. Indeed, conditions §A.1.1(1-3) in the definition of a tree are clearly fulfilled and so does our extra condition §A.1.1(5) which we use to define our class of open trees. To check the remaining connectedness condition §A.1.1(4), we consider the chain of edges connecting a given vertex $v \in V(\underline{\Sigma})$ to the output 0 of the ambient tree $\underline{\mathrm{T}}$. By following this chain downwards from v, we get edges e satisfying $s(e) \in V(\underline{\Sigma})$ and $e \in E(\underline{\Sigma})$ until we get $t(e) \notin V(\underline{\Sigma})$. Then our requirement $e \in E(\underline{\Sigma}) \Rightarrow t(e) \in V(\underline{\Sigma}) \amalg \{0_{\underline{\Sigma}}\}$ implies that $t(e)$ is necessarily the output $0_{\underline{\Sigma}}$ of the tree $\underline{\Sigma}$. Hence, we do have a chain connecting $v \in V(\underline{\Sigma})$ to $0_{\underline{\Sigma}}$ in $\underline{\Sigma}$.

In our definition, we require that the outgoing edge and all ingoing edges attached to a given vertex v belong to the subtree $\underline{\Sigma}$ when $v \in V(\underline{\Sigma})$. Hence, in the example of Figure A.2, the sets $V = \{v_0\}$ and $E = \{e_0\}$, consisting of the outgoing edge of the tree and its source, do not define an allowable subtree in our sense.

By convention, we will also assume that the vertex set of a subtree $\underline{\Sigma} \subset \underline{\mathrm{T}}$ is not empty $V(\underline{\Sigma}) \neq \varnothing$. This requirement implies that we discard unit trees $\underline{\downarrow}$ from allowable subtrees.

A.1.6. *Subtrees determined by vertices or edges.* We note that a subtree $\underline{\Sigma} \subset \underline{\mathrm{T}}$ is fully determined by the associated edge set $E(\underline{\Sigma}) \subset E(\underline{\mathrm{T}})$ or by the associated vertex set $V(\underline{\Sigma}) \subset V(\underline{\mathrm{T}})$ (provided that this set is non-empty as we require in our conventions). Indeed, if we have $E(\underline{\Sigma}) \subset E(\underline{\mathrm{T}})$, then we can identify $V(\underline{\Sigma})$ with the subset of vertices $v \in V(\underline{\mathrm{T}})$ that satisfy $v = s(e)$ and $v = t(f)$ for some edges $e, f \in E(\underline{\Sigma})$. In the case where $V(\underline{\Sigma}) \subset V(\underline{\mathrm{T}})$ is given, we can identify $E(\underline{\Sigma})$ with the subset of edges $e \in E(\underline{\mathrm{T}})$ satisfying $s(e) \in V(\underline{\Sigma})$ or $t(e) \in V(\underline{\Sigma})$. Then, once we have $V(\underline{\Sigma})$ and $E(\underline{\Sigma})$ together, we can use the relation $\{s(e),\ e \in E(\underline{\Sigma})\} = V(\underline{\Sigma}) \amalg \underline{\mathrm{r}}_{\underline{\Sigma}}$, to determine the input set of our subtree, while the output is determined by the relation $\{t(e),\ e \in E(\underline{\Sigma})\} = V(\underline{\Sigma}) \amalg \{0_{\underline{\Sigma}}\}$.

Naturally, not all subsets $E(\underline{\Sigma}) \subset E(\underline{\mathrm{T}})$ (respectively $V(\underline{\Sigma}) \subset V(\underline{\mathrm{T}})$) are associated to subtrees $\underline{\Sigma} \subset \underline{\mathrm{T}}$. For instance, the edges $\{e_{\alpha_1}, e_{\alpha_4}\}$ in Figure A.1 do not form the edge set of a subtree $\underline{\Sigma} \subset \underline{\mathrm{T}}$. Similarly, the vertex set $\{v_2, v_3, v_4\} \subset V(\underline{\mathrm{T}})$ is not associated to a subtree of the tree of Figure A.1. Nonetheless, in the special case of a one-point set $\{v\} \subset V(\underline{\mathrm{T}})$, we automatically have a subtree $\underline{\mathrm{Y}}_v$, associated to v, such that $V(\underline{\mathrm{Y}}_v) = \{v\}$. The edge set of this subtree $E(\underline{\mathrm{Y}}_v)$ consists of the ingoing edges and of the outgoing edge of the vertex v in the ambient tree $\underline{\mathrm{T}}$. This subtree $\underline{\mathrm{Y}}_v$ obviously forms a corolla, which we call the *star* of the vertex v in $\underline{\mathrm{T}}$.

The subtrees $\underline{\Sigma} \subset \underline{\mathrm{T}}$ such that $\underline{\Sigma} \neq \underline{\mathrm{Y}}$ are also fully determined by their set of inner edges $\mathring{E}(\underline{\Sigma}) \subset \mathring{E}(\underline{\mathrm{T}})$, because under this condition $\underline{\Sigma} \neq \underline{\mathrm{Y}}$, which is equivalent to $\mathring{E}(\underline{\Sigma}) \neq \varnothing$, we have $v \in V(\underline{\Sigma})$ if and only if $v = s(e)$ for some $e \in \mathring{E}(\underline{\Sigma})$ or $v = t(f)$ for some $f \in \mathring{E}(\underline{\Sigma})$. Not all subsets $\mathring{E}(\underline{\Sigma}) \subset \mathring{E}(\underline{\mathrm{T}})$ are associated to subtrees $\underline{\Sigma} \subset \underline{\mathrm{T}}$ again. Nonetheless, in the special case of a one-point set $\{e\}$, $e \in \mathring{E}(\underline{\mathrm{T}})$, we automatically have a subtree $\underline{\Sigma} = \underline{\Gamma}_e$, associated to e, such that $\mathring{E}(\underline{\Gamma}_e) = \{e\}$. In this case, we have $V(\underline{\Gamma}_e) = \{s(e), t(e)\}$ and the edge set $E(\underline{\Gamma}_e)$

consists of the ingoing edges of the vertex $v = s(e)$ together with the ingoing edges and the outgoing edge of the vertex $u = t(e)$ (among which we have the edge e).

A.1.7. *The over-subtree of an edge.* To any edge $e \in E(\underline{\mathsf{T}})$ (which is not an ingoing edge) of an $\underline{\mathsf{r}}$-tree $\underline{\mathsf{T}}$, we also associate an over-tree $\underline{\Upsilon}_e \subset \underline{\mathsf{T}}$ which, in short, consists of the edges and of the vertices that lie above e in the tree $\underline{\mathsf{T}}$. Formally, this tree $\underline{\Upsilon}_e$ consists of the edges and vertices that occur in the open chains

$$(*) \qquad i \xrightarrow{e_i} \bigcirc\!\!\!\!\!v_l \xrightarrow{e_{v_l}} \cdots \xrightarrow{e_{v_1}} \bigcirc\!\!\!\!\!v_0 \xrightarrow{e} t(e)$$

of which last element is our edge $e \in E(\underline{\mathsf{T}})$. The assumption that e is not an ingoing edge implies $v_0 = s(e) \in V(\underline{\mathsf{T}})$ and ensures that this definition returns an allowable subtree of the tree $\underline{\mathsf{T}}$ in our sense $V(\underline{\Upsilon}_e) \neq \varnothing$. The input set of this subtree satisfies $\underline{\mathsf{r}}_{\underline{\Upsilon}_e} \subset \underline{\mathsf{r}}$ and consists of the indices $i \in \underline{\mathsf{r}}$ which label the ingoing edge of our chains $(*)$. We also have $0_{\underline{\Upsilon}_e} = t(e)$.

We have an obvious extension of our definition of the over-tree $\underline{\Upsilon}_e$ in the case where $e = e_i$ is the ingoing edge associated to an input $i \in \underline{\mathsf{n}}$ of the tree $\underline{\mathsf{T}}$. We basically assume that $\underline{\Upsilon}_e$ is the unit tree $i \to 0$ with $i \in \underline{\mathsf{r}}$ as input label in this case $e = e_i$ (and, hence, we just do not get an allowable subtree in our sense). We use the general definition of the over-tree of an edge in our construction of restriction operators on trees.

To give an example, the over-tree of the edge $e = e_{\alpha_2}$ in the tree of Figure A.1 has $E(\underline{\Upsilon}_{e_{\alpha_2}}) = \{e_{\alpha_2}, e_{\alpha_4}, e_{i_3}, e_{i_4}, e_{i_5}\}$ as edge set and $V(\underline{\Upsilon}_{e_{\alpha_2}}) = \{v_2, v_4\}$ as vertex set. We moreover have $\underline{\mathsf{r}}_{\underline{\Upsilon}_{e_{\alpha_2}}} = \{i_3, i_4, i_5\}$.

A.1.8. *Tree isomorphisms.* In the definition of a tree $\underline{\mathsf{T}}$, we give a fixed edge set $E(\underline{\mathsf{T}})$ and a fixed vertex set $V(\underline{\mathsf{T}})$ in order to give a sense to our objects. But the edge and vertex naming is of course not meaningful and we have to consider a natural notion of isomorphism, attached to $\underline{\mathsf{r}}$-trees, in order to handle this non-essential information.

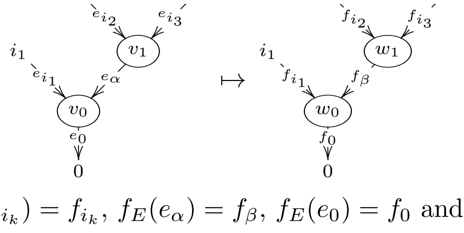
Formally, an *isomorphism of* $\underline{\mathsf{r}}$*-trees* $f : \underline{\mathsf{S}} \xrightarrow{\simeq} \underline{\mathsf{T}}$ consists of a bijection of vertex sets $f_V : V(\underline{\mathsf{S}}) \xrightarrow{\simeq} V(\underline{\mathsf{T}})$ together with a bijection of edge sets $f_E : E(\underline{\mathsf{S}}) \xrightarrow{\simeq} E(\underline{\mathsf{T}})$ that preserves the source and target of edges. If we have $\underline{\mathsf{S}}, \underline{\mathsf{T}} \neq \underline{\downarrow}$, then this preservation requirement reads as follows:

(1) For the outgoing edge $e = e_0$ of the tree $\underline{\mathsf{S}}$, for which we have $t(e) = 0$, we assume $s(f_E(e)) = f_V(s(e))$ and $t(f_E(e)) = 0$.

(2) For an ingoing edge $e = e_i$, which we associate to some input index $i \in \underline{\mathsf{r}}$, so that $s(e) = i$, we assume $s(f_E(e)) = i$ and $t(f_E(e)) = f_V(t(e))$.

(3) For an inner edge $e \in \mathring{E}(\underline{\mathsf{S}})$, for which we have $s(e), t(e) \in V(\underline{\mathsf{S}})$, we assume $s(f_E(e)) = f_E(s(e))$ and $t(f_E(e)) = f_E(t(e))$.

In the graphical representation, applying an isomorphism $f : \underline{\mathsf{S}} \xrightarrow{\simeq} \underline{\mathsf{T}}$ simply amounts to renaming the vertices and edges of our tree, as in the following picture:



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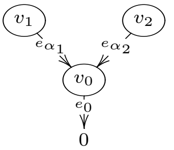
where we apply $f_E(e_{i_k}) = f_{i_k}$, $f_E(e_\alpha) = f_\beta$, $f_E(e_0) = f_0$ and $f_V(v_k) = w_k$.

In the next sections, we use these isomorphisms to handle symmetries in the construction of free operads. In the course of this study, we will check that the group of automorphisms of an $\underline{\mathsf{r}}$-tree $\underline{\mathsf{T}}$ is reduced to the identity as soon as we assume $\operatorname{card}(\underline{\mathsf{r}}_v) \geq 1$, for all $v \in V(\underline{\mathsf{T}})$ (see §A.1.3). This observation will enable us to pick representatives of isomorphism classes of $\underline{\mathsf{r}}$-trees in order to give a reduced expansion of free objects in the category of non-unitary operads. But this reduced expansion is not canonical. Therefore, in general, we prefer not to fix the underlying edge and vertex sets of our trees, and instead, we use isomorphisms in order to relate the tree structures which we need to compare.

Let us mention that non-trivial isomorphism can occur if we drop our extra-condition (5) of the definition of an open tree in §A.1.1. The tree



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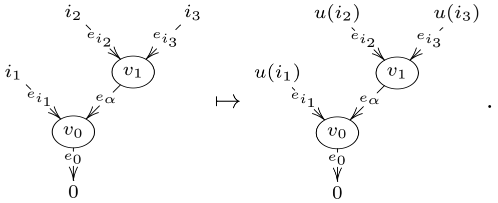
for instance, admits an automorphism $f \neq id$ which we define by the bijection $f_V(v_0) = v_0$, $f_V(v_1) = v_2$, $f_V(v_2) = v_1$ on vertices and by the bijection $f_E(e_0) = e_0$, $f_E(e_{\alpha_1}) = e_{\alpha_2}$, $f_E(e_{\alpha_2}) = e_{\alpha_1}$ on edges. Because of this example, in the general context where we do not assume that the arity zero term of our symmetric collections vanishes, we absolutely need to use tree isomorphisms in order to handle symmetry relations which naturally occur in the construction of free operads.

A.1.9. *The symmetric collection of trees.* The class of $\underline{\mathsf{r}}$-trees, where $\underline{\mathsf{r}}$ is any (non-empty) finite set, is denoted by $\mathcal{T}ree(\underline{\mathsf{r}})$. In what follows, we also use the notation $\mathcal{T}ree(\underline{\mathsf{r}})^{iso}$ when we need to refer to the groupoid formed by $\underline{\mathsf{r}}$-trees and their isomorphisms.

We have an obvious reindexing functor $u_* : \mathcal{T}ree(\underline{\mathsf{m}})^{iso} \to \mathcal{T}ree(\underline{\mathsf{n}})^{iso}$, associated to each bijection of finite sets $u : \underline{\mathsf{m}} \xrightarrow{\simeq} \underline{\mathsf{n}}$, so that the collection of tree groupoids $\mathcal{T}ree(\underline{\mathsf{r}})^{iso}$ forms a symmetric collection $\mathcal{T}ree^{iso}$. Note that the extra condition (5) in our definition of an open tree §A.1.1 implies $\mathcal{T}ree(0) = \varnothing$. We therefore consider a non-unitary symmetric collection of trees $\mathcal{T}ree^{iso} = \{\mathcal{T}ree(\underline{\mathsf{r}}), r > 0\}$ (with no arity-zero term) in our setting.

Formally, the image of a tree $\underline{\mathsf{T}} \in \mathcal{T}ree(\underline{\mathsf{m}})$ under u_* has the same vertex and edge sets as $\underline{\mathsf{T}}$. In $u_*(\underline{\mathsf{T}})$, we just change the source of the ingoing edges of $\underline{\mathsf{T}}$ by an application of the bijection $u : \underline{\mathsf{m}} \xrightarrow{\simeq} \underline{\mathsf{n}}$. In the graphical representation, the image of a tree $\underline{\mathsf{T}} \in \mathcal{T}ree(\underline{\mathsf{m}})$ under the map $u_* : \mathcal{T}ree(\underline{\mathsf{m}}) \to \mathcal{T}ree(\underline{\mathsf{n}})$ is given by an application of our re-indexing bijection u to the input labeling of the tree, as in the following example:



We soon explain the definition of restriction operators $u^* : \mathcal{T}ree(\underline{\mathsf{n}})^{iso} \to \mathcal{T}ree(\underline{\mathsf{m}})^{iso}$, associated to the injective maps $u : \{i_1, \dots, i_m\} \to \{j_1, \dots, j_n\}$, and which extend this symmetric structure of the collection of tree groupoids. We just

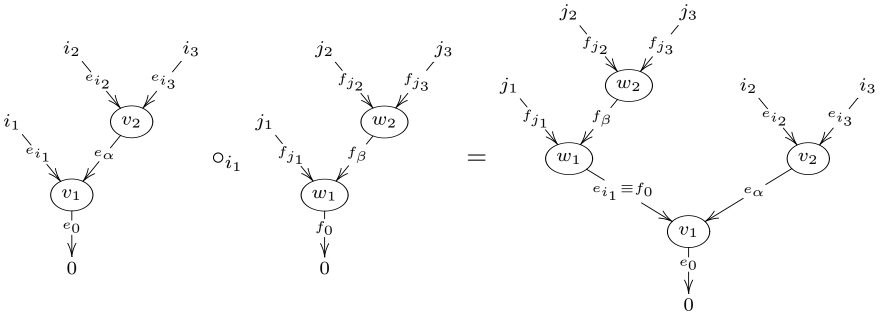


FIGURE A.3. An operadic composition of trees.

check that the groupoids of trees inherit the composition structure of an operad before tackling the construction of these restriction operators.

A.1.10. *The operadic composition of trees.* We explicitly check that we have composition operations

$$\circ_{i_k} : \mathcal{T}ree(\underline{\mathsf{m}})^{iso} \times \mathcal{T}ree(\underline{\mathsf{n}})^{iso} \to \mathcal{T}ree(\underline{\mathsf{m}} \circ_{i_k} \underline{\mathsf{n}})^{iso},$$

defined for all $\underline{\mathsf{m}}, \underline{\mathsf{n}} \neq \underline{\mathsf{0}}$, and for every $i_k \in \underline{\mathsf{m}}$. We will see that these composition operations give the shape of the composition products of free operads.

Intuitively, the operadic composite of an $\underline{\mathsf{m}}$-indexed tree $\underline{\mathsf{S}} \in \mathcal{T}ree(\underline{\mathsf{m}})$ with an $\underline{\mathsf{n}}$-indexed tree $\underline{\mathsf{T}} \in \mathcal{T}ree(\underline{\mathsf{n}})$ at the input $i_k \in \underline{\mathsf{m}}$ is the tree $\underline{\mathsf{S}} \circ_{i_k} \underline{\mathsf{T}} \in \mathcal{T}ree(\underline{\mathsf{m}} \circ_{i_k} \underline{\mathsf{n}})$ which we form by plugging the outgoing edge of $\underline{\mathsf{T}}$ in the ingoing edge of $\underline{\mathsf{S}}$ indexed by i_k. An example is represented Figure A.3.

The vertex set of the composite tree $\underline{\mathsf{S}} \circ_{i_k} \underline{\mathsf{T}}$ is formally defined by the coproduct $V(\underline{\mathsf{S}} \circ_{i_k} \underline{\mathsf{T}}) = V(\underline{\mathsf{S}}) \amalg V(\underline{\mathsf{T}})$. Let f_0 be the outgoing edge of $\underline{\mathsf{T}}$. Let e_{i_k} be the i_kth ingoing edge of $\underline{\mathsf{S}}$. We define the edge set of $\underline{\mathsf{S}} \circ_{i_k} \underline{\mathsf{T}}$ as the quotient $E(\underline{\mathsf{S}} \circ_{i_k} \underline{\mathsf{T}}) = E(\underline{\mathsf{S}}) \amalg E(\underline{\mathsf{T}}) / \equiv$ under the relation $e_{i_k} \equiv f_0$ which identifies f_0 with e_{i_k}. The source (respectively, the target) of an edge e in $\underline{\mathsf{S}} \circ_{i_k} \underline{\mathsf{T}}$ is defined by:

- the source (respectively, the target) of e in $\underline{\mathsf{S}}$ when $e \in E(\underline{\mathsf{S}}) \setminus \{e_{i_k}\}$;
- the source (respectively, the target) of e in $\underline{\mathsf{T}}$ when $e \in E(\underline{\mathsf{T}}) \setminus \{f_0\}$;
- the source of f_0 in $\underline{\mathsf{T}}$ (respectively, the target of e_{i_k} in $\underline{\mathsf{S}}$) when e is the edge produced by the merging operation $e_{i_k} \equiv f_0$.

Note that the set of ingoing edges $\underline{\mathsf{r}}_v$ attached to a vertex $v \in V(\underline{\mathsf{S}})$ (respectively, to a vertex $v \in V(\underline{\mathsf{T}})$) in the composite tree $\underline{\mathsf{S}} \circ_{i_k} \underline{\mathsf{T}}$ is canonically in bijection with the set of ingoing edges of v in $\underline{\mathsf{S}}$ (respectively, in $\underline{\mathsf{T}}$).

The trees $\underline{\mathsf{S}}$ and $\underline{\mathsf{T}}$ are identified with subtrees of the composite $\underline{\mathsf{S}} \circ_{i_k} \underline{\mathsf{T}}$. To be more precise, if we apply the definitions of §§A.1.5-A.1.6, then we readily see that the vertex subset $V(\underline{\mathsf{S}})$ (respectively, $V(\underline{\mathsf{T}})$) in $V(\underline{\mathsf{S}} \circ_{i_k} \underline{\mathsf{T}}) = V(\underline{\mathsf{S}}) \amalg V(\underline{\mathsf{T}})$ determines a subtree of $\underline{\mathsf{S}} \circ_{i_k} \underline{\mathsf{T}}$ together with a natural input set $\underline{\mathsf{r}}_{\mathsf{S}}$ (respectively, $\underline{\mathsf{r}}_{\mathsf{T}}$) canonically in bijection with $\underline{\mathsf{m}}$ (respectively, $\underline{\mathsf{n}}$) and this subtree is canonically isomorphic to $\underline{\mathsf{S}}$ (respectively, $\underline{\mathsf{T}}$) as an $\underline{\mathsf{m}}$-tree (respectively, as an $\underline{\mathsf{n}}$-tree).

We immediately see that our composition operations are equivariant with respect to the action of bijections on the groupoids of trees. The composition of trees also satisfies the associativity relations of operads. We explicitly have the identity:

$$(1)\qquad (\underline{\mathsf{R}} \circ_{i_k} \underline{\mathsf{S}}) \circ_{j_l} \underline{\mathsf{T}} = \underline{\mathsf{R}} \circ_{i_k} (\underline{\mathsf{S}} \circ_{j_l} \underline{\mathsf{T}}),$$

for all trees $\underline{\mathsf{R}} \in \mathcal{T}ree(\underline{\mathsf{r}})$, $\underline{\mathsf{S}} \in \mathcal{T}ree(\underline{\mathsf{s}})$, $\underline{\mathsf{T}} \in \mathcal{T}ree(\underline{\mathsf{t}})$, when we fix $i_k \in \underline{\mathsf{r}}$, $j_l \in \underline{\mathsf{s}}$, and we similarly have:

$$(2)\qquad (\underline{\mathsf{R}} \circ_{i_k} \underline{\mathsf{S}}) \circ_{i_l} \underline{\mathsf{T}} = (\underline{\mathsf{R}} \circ_{i_l} \underline{\mathsf{T}}) \circ_{i_k} \underline{\mathsf{S}},$$

for all trees $\underline{\mathsf{R}} \in \mathcal{T}ree(\underline{\mathsf{r}})$, $\underline{\mathsf{S}} \in \mathcal{T}ree(\underline{\mathsf{s}})$, $\underline{\mathsf{T}} \in \mathcal{T}ree(\underline{\mathsf{t}})$, when we fix $\{i_k \neq i_l\} \subset \underline{\mathsf{r}}$. The composition of trees also satisfies unit relations. In this case, we more precisely get that the composition of trees with the unit tree $\underline{\downarrow} \in \mathcal{T}ree(1)$ of §A.1.4 fulfills the unit relations of operads up to canonical isomorphisms in the category of trees:

$$(3)\qquad \underline{\mathsf{S}} \circ_{i_k} \underline{\downarrow} \simeq \underline{\mathsf{S}} \qquad\text{and}\qquad \underline{\downarrow} \circ_1 \underline{\mathsf{T}} \simeq \underline{\mathsf{T}}\,.$$

These unit isomorphisms are functorial in all possible senses (with respect to the action of tree isomorphisms and with respect to the action of reindexing bijections). The unit isomorphisms of trees moreover satisfy natural coherence relations when we combine them with other composition operations.

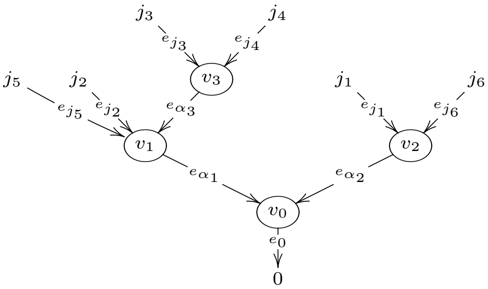
A.1.11. *The restriction operators on trees.* We now explain the definition of restriction operators

$$u^* : \mathcal{T}ree(\underline{\mathsf{n}})^{iso} \to \mathcal{T}ree(\underline{\mathsf{m}})^{iso},$$

which we associate to the injective maps $u \in \mathtt{Mor}_{\mathcal{I}nj}(\underline{\mathsf{m}}, \underline{\mathsf{n}})$ and which we use to shape the restriction structure of free objects in the Λ-operad context. We also have a trivial augmentation $\epsilon : \mathcal{T}ree(\underline{\mathsf{r}})^{iso} \to pt$, for any $r > 0$, where we consider the one-point set groupoid pt. We still identify this augmentation map with an extra restriction operator $\epsilon = o^*$, which we associate to the empty map $o : \underline{\mathsf{0}} \to \underline{\mathsf{r}}$, for any $r > 0$ (see §I.2.2.1).

To define these restriction operators on trees, we elaborate on the ideas of §I.2.3, where we outline the definition of the restriction operators of free objects in the category of augmented non-unitary Λ-operads. In short, the image of a tree $\underline{\mathsf{T}} \in \mathcal{T}ree(\underline{\mathsf{n}})$ under our restriction map $u^* : \mathcal{T}ree(\underline{\mathsf{n}}) \to \mathcal{T}ree(\underline{\mathsf{m}})$ is obtained by re-indexing the inputs of this tree by the indices i_k such that $u(i_k) = j_l$ when $j_l \in \underline{\mathsf{n}}$ lies in the image of our map $u : \{i_1, \dots, i_m\} \to \{j_1, \dots, j_n\}$, by filling in the other inputs with the mark $*$, and by performing a reduction process in order to remove these void inputs in the outcome of our re-indexing operation. We give an instance of application of this procedure in Figure A.4. We then consider the tree:

$$(*)\qquad \underline{\mathsf{T}} =$$



$$\in \mathcal{T}ree(6),$$

which we initially introduced in §I.1.2($*$) to outline the definition of free operads and which we also used in §I.2.3 to outline our construction of free objects of the category of augmented non-unitary Λ-operads. We also consider the map $u \in \mathtt{Mor}_{\mathcal{I}nj}(\underline{\mathsf{3}}, \underline{\mathsf{6}})$ such that $u(i_1) = j_1$, $u(i_2) = j_2$, and $u(i_3) = j_5$.

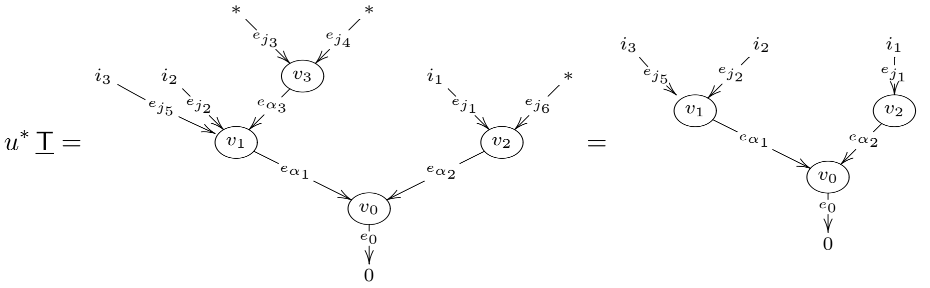
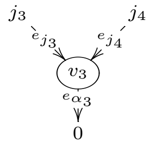


FIGURE A.4. An operadic restriction operator on trees.

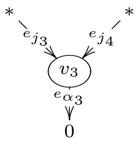
The reduction process, which we use in this construction, basically consists in removing some vertices and edges in the tree $\underline{\mathsf{T}}$. We use the over-trees $\underline{\Upsilon}_e$ which we associate to the edges $e \in E(\underline{\mathsf{T}})$ in order to formalize this construction. We explicitly define the set of edges $E(u^* \underline{\mathsf{T}})$, which we associate to the tree $u^* \underline{\mathsf{T}}$, as the subset of edges $e \in E(\underline{\mathsf{T}})$ satisfying $u^{-1}(\underline{\mathsf{r}}_{\underline{\Upsilon}_e}) \neq \varnothing$, where we consider the preimage under our map $u : \underline{\mathsf{m}} \to \underline{\mathsf{n}}$ of the input set of this over-tree $\underline{\mathsf{r}}_{\underline{\Upsilon}_e}$. We also take the subset of vertices $v \in V(\underline{\mathsf{T}})$ of which outgoing edge $e = e_v$ satisfies $e_v \in E(u^* \underline{\mathsf{T}})$ in order to define the set of vertices $V(u^* \underline{\mathsf{T}})$ of the tree $u^* \underline{\mathsf{T}}$. We define the source and target of edges in $u^* \underline{\mathsf{T}}$ by the obvious restriction of the source and target maps of edges in the tree $\underline{\mathsf{T}}$. We just take $s_{u^* \underline{\mathsf{T}}}(e_{j_l}) = u^{-1}(j_l)$ in the case of an ingoing edge $e = e_{j_l}$, $j_l \in \underline{\mathsf{n}}$. We then consider the pre-image $u^{-1}(j_l) \in \underline{\mathsf{m}}$ of the input label of this edge $j_l \in \underline{\mathsf{n}}$ which is well-defined by definition of the edge set $E(u^* \underline{\mathsf{T}}) \subset E(\underline{\mathsf{T}})$.

In our picture, the sets $u^{-1}(\underline{\mathsf{r}}_{\underline{\Upsilon}_e})$ represent the subsets of inputs, lying over any edge e, which we mark by an element $i_k \in \underline{\mathsf{m}}$ and not by a symbol $*$. In the case of Figure A.4, we have for instance:

$\underline{\Upsilon}_{e_{\alpha_3}} =$



$\mapsto$



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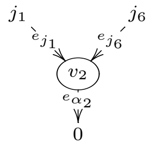
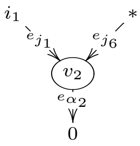
so that $u^{-1}(\underline{\mathsf{r}}_{\underline{\Upsilon}_{e_{\alpha_3}}}) = \varnothing$ and we therefore discard this edge e_{α_3} in $u^* \underline{\mathsf{T}}$, while we have:

$\underline{\Upsilon}_{e_{\alpha_2}} =$



$\mapsto$



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so that $u^{-1}(\underline{\mathsf{r}}_{\underline{\Upsilon}_{e_{\alpha_2}}}) = \{i_1\}$ and we therefore keep the edge e_{α_2} in $u^* \underline{\mathsf{T}}$. Let us observe that the entire over-subtree $\underline{\Upsilon}_e$ is removed by our reduction process as soon as the edge e is discarded.

We readily check that these restriction operators define functors on our isomorphism categories of trees $u^* : \mathcal{T}ree(\underline{\mathsf{n}})^{iso} \to \mathcal{T}ree(\underline{\mathsf{m}})^{iso}$ and fulfill the equivariance relations of §I.2.2 with respect to the operadic composition of trees. We explicitly have the following identity, where we use the conventions and the definition of the operadic composition of injective maps of §I.2.2 (see also §I.2.5.9):

$$(1)\qquad (u^* \underline{\mathsf{S}}) \circ_{i_k} (v^* \underline{\mathsf{T}}) = (u \circ_{u(i_k)} v)^* (\underline{\mathsf{S}} \circ_{u(i_k)} \underline{\mathsf{T}}),$$

for all trees $\underline{\mathsf{S}} \in \mathcal{T}ree(\underline{\mathsf{m}})$, $\underline{\mathsf{T}} \in \mathcal{T}ree(\underline{\mathsf{n}})$, for all maps $u \in \mathtt{Mor}_{\mathcal{I}nj}(\underline{\mathsf{r}}, \underline{\mathsf{m}})$, $v \in \mathtt{Mor}_{\mathcal{I}nj}(\underline{\mathsf{s}}, \underline{\mathsf{n}})$, and for any composition index $i_k \in \underline{\mathsf{r}}$. We similarly have the identity:

$$(2)\qquad \partial_{i_k}(u^* \underline{\mathsf{S}}) = (u \circ_{u(i_k)} o)^* (\underline{\mathsf{S}} \circ_{u(i_k)} \underline{\mathsf{T}}),$$

for all trees $\underline{\mathsf{S}} \in \mathcal{T}ree(\underline{\mathsf{m}})$, $\underline{\mathsf{T}} \in \mathcal{T}ree(\underline{\mathsf{n}})$, for all maps $u \in \mathtt{Mor}_{\mathcal{I}nj}(\underline{\mathsf{r}}, \underline{\mathsf{m}})$, and for each $i_k \in \underline{\mathsf{r}}$. We also retrieve the symmetric structure of our groupoids $\mathcal{T}ree(\underline{\mathsf{r}})$ in the case where we restrict ourselves to bijective maps.

A.1.12. *The operad of reduced trees.* Recall that a tree $\underline{\mathsf{T}} \in \mathcal{T}ree(\underline{\mathsf{r}})$ is said to be *reduced* when we have $\mathrm{card}(\underline{\mathsf{r}}_v) \geq 2$ for all $v \in V(\underline{\mathsf{T}})$. We denote the class of reduced trees by $\widetilde{\mathcal{T}ree}(\underline{\mathsf{r}})$. These classes $\widetilde{\mathcal{T}ree}(\underline{\mathsf{r}})$ obviously form subgroupoids of $\mathcal{T}ree(\underline{\mathsf{r}})$ and are preserved by the action of permutations on trees. Hence, we have a symmetric collection in the category of groupoids $\widetilde{\mathcal{T}ree}$ formed by the classes of reduced trees $\widetilde{\mathcal{T}ree} = \{\widetilde{\mathcal{T}ree}(\underline{\mathsf{r}}), r > 0\}$. In what follows, we also use the notation $\widetilde{\mathcal{T}ree}(\underline{\mathsf{r}})^{iso}$ when we consider these groupoids of reduced trees with their isomorphisms as morphisms.

Since we note in §A.1.10 that the composition of trees preserve the input set of vertices (up to bijection), we immediately see that the classes of reduced trees are preserved by the composition operations of the operad of trees. The unit tree $\downarrow$ belongs to $\widetilde{\mathcal{T}ree}(1)$ as well. In fact, we readily see that the category of reduced 1-trees $\widetilde{\mathcal{T}ree}(1)$ is reduced to (the isomorphism class of) the unit tree $\downarrow$, while we still trivially have $\widetilde{\mathcal{T}ree}(0) = \varnothing$ in arity zero. Thus, the collection $\widetilde{\mathcal{T}ree} = \{\widetilde{\mathcal{T}ree}(\underline{\mathsf{r}}), r > 0\}$ inherits a full operadic composition structure and forms, in a sense, a connected operad in the category of groupoids.

A.1.13. *The restriction operators on reduced trees.* We may observe that the restriction operators of §A.1.11, however, do not preserve the classes of reduced trees $\widetilde{\mathcal{T}ree}(\underline{\mathsf{r}})$. We therefore amend the reduction process of §A.1.11 in order to adapt the definition of these restriction operators to the reduced tree context.

Let $\underline{\mathsf{T}} \in \mathcal{T}ree(\underline{\mathsf{n}})$ and let $u \in \mathtt{Mor}_{\mathcal{I}nj}(\underline{\mathsf{m}}, \underline{\mathsf{n}})$ be any injective map, as in §A.1.11, but where we now assume $m, n > 1$. In the procedure §A.1.11, we just discard the vertices $v \in V(\underline{\mathsf{T}})$ such that $u^{-1}(\underline{\mathsf{T}}_{e_v}) = \varnothing$ from the vertex set of the tree $\underline{\mathsf{T}} \in \mathcal{T}ree(\underline{\mathsf{n}})$. To obtain a reduced tree $u^* \underline{\mathsf{T}} \in \widetilde{\mathcal{T}ree}(\underline{\mathsf{m}})$, we perform an additional reduction operation, which consists in withdrawing the vertices which have a single input in the unreduced tree $u^* \underline{\mathsf{T}} \in \mathcal{T}ree(\underline{\mathsf{m}})$ associated to $\underline{\mathsf{T}} \in \mathcal{T}ree(\underline{\mathsf{n}})$. We also merge the ingoing edge of these vertices with their outgoing edge when we perform this withdrawal operation. We use an obvious quotient of the edge set of the unreduced tree $u^* \underline{\mathsf{T}} \in \mathcal{T}ree(\underline{\mathsf{m}})$ to implement this merging operation in the formal definition of our trees.

We give an instance of application of this modified restriction process in Figure A.5 for the same example of tree $\underline{\mathsf{T}} \in \mathcal{T}ree(6)$ and the same injective map $u \in \mathtt{Mor}_{\mathcal{I}nj}(\underline{3}, \underline{6})$ as in §A.1.11. In comparison with the unreduced restriction operator of §A.1.11, we just discard one additional vertex $v_2 \in V(\underline{\mathsf{T}})$ which had a single input left in the tree depicted in Figure A.4.

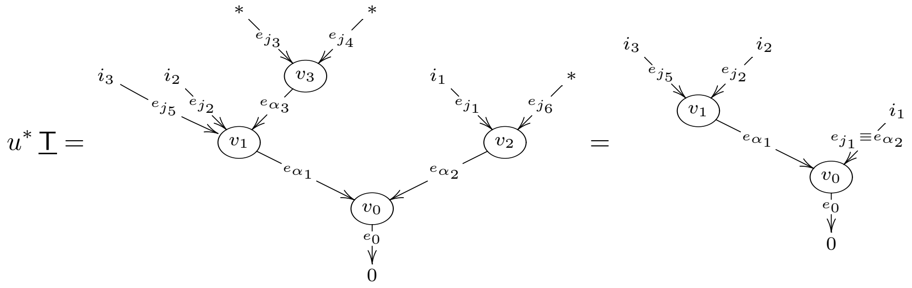


Figure A.5. An operadic restriction operator on reduced trees.

These reduced restriction maps define functors $u^* : \widetilde{\mathcal{T}ree}(\underline{\mathsf{n}})^{iso} \to \widetilde{\mathcal{T}ree}(\underline{\mathsf{m}})^{iso}$ (as in the unreduced case) and still fulfill the equivariance relations of §I.2.2(1-2) with respect to the restriction of operadic composition operations to the groupoids of reduced trees. We also retrieve the symmetric structure of our groupoids $\widetilde{\mathcal{T}ree}(\underline{\mathsf{r}})$, $r > 0$, in the case where we restrict ourselves to bijective maps.

We go back to the study of the operad of reduced trees in §A.4 when we examine the construction of connected free operads.

A.1.14. *The grading of the operad of trees.* In the study of bar complexes, we use a splitting $\mathcal{T}ree(\underline{\mathsf{r}}) = \coprod_{m=0}^{\infty} \mathcal{T}ree_m(\underline{\mathsf{r}})$, where $\mathcal{T}ree_m(\underline{\mathsf{r}})$ is the subcategory of $\mathcal{T}ree(\underline{\mathsf{r}})$ formed by trees with m vertices. The groupoids of reduced trees $\widetilde{\mathcal{T}ree}(\underline{\mathsf{r}})$ inherit a similar splitting $\widetilde{\mathcal{T}ree}(\underline{\mathsf{r}}) = \coprod_{m=0}^{\infty} \widetilde{\mathcal{T}ree}_m(\underline{\mathsf{r}})$, where we set $\widetilde{\mathcal{T}ree}_m(\underline{\mathsf{r}}) = \mathcal{T}ree_m(\underline{\mathsf{r}}) \cap \widetilde{\mathcal{T}ree}(\underline{\mathsf{r}})$.

The observations of §A.1.4 give the structure of the groupoids $\mathcal{T}ree_m(\underline{\mathsf{r}})$ for the first values of $m \in \mathbb{N}$: the category $\mathcal{T}ree_0(\underline{\mathsf{r}})$ is reduced to the (isomorphism class of the) unit tree $\underline{\downarrow}$ in arity $r = 1$, while we have $\mathcal{T}ree_0(\underline{\mathsf{r}}) = \varnothing$ (the empty category) in arity $r > 0$, and the category $\mathcal{T}ree_1(\underline{\mathsf{r}})$ is reduced to the (isomorphism class of the) $\underline{\mathsf{r}}$-corolla $\underline{\mathsf{Y}} = \underline{\mathsf{Y}}_{\underline{\mathsf{r}}}$ in any arity $r > 0$. In the case of reduced trees, we rather obtain $\widetilde{\mathcal{T}ree}_1(\underline{\mathsf{r}}) = \varnothing$ when $r = 0, 1$ and $\widetilde{\mathcal{T}ree}_1(\underline{\mathsf{r}}) \simeq \{\underline{\mathsf{Y}}\}$ otherwise.

The operadic composition of trees clearly splits into components of the form

$$\circ_{i_k} : \mathcal{T}ree_p(\underline{\mathsf{m}}) \times \mathcal{T}ree_q(\underline{\mathsf{n}}) \to \mathcal{T}ree_{p+q}(\underline{\mathsf{m}} \circ_{i_k} \underline{\mathsf{n}}),$$

for all $p, q \geq 0$, since we have $V(\underline{\mathsf{S}} \circ_{i_k} \underline{\mathsf{T}}) = V(\underline{\mathsf{S}}) \amalg V(\underline{\mathsf{T}})$ by definition of the composite $\underline{\mathsf{S}} \circ_{i_k} \underline{\mathsf{T}}$, for any $\underline{\mathsf{S}} \in \mathcal{T}ree(\underline{\mathsf{m}})$ and $\underline{\mathsf{T}} \in \mathcal{T}ree(\underline{\mathsf{m}})$. The restriction operators of §A.1.11 and §A.1.13, on the other hand, do not preserve this grading.

A.2. Treewise tensor products and treewise composites

In the introduction of this appendix, we briefly mentioned that trees give the pattern of the general composition operations that can be formed within an operad. The purpose of this section is to formalize this idea.

To begin with, we explain how to form tensor products over trees. Then we explain the definition of operadic composition operations shaped on these treewise tensor products by using sequences of partial composition operations which we associate to edge contractions in trees. The associativity relations of the partial composition products of an operad actually imply that the outcome of such a sequence of operadic composition operations in a treewise tensor product does not depend on the order in which these composition operations are performed. This coherence statement, which we establish in Theorem A.2.6, represents the crux of our constructions. By the way, we also use this statement to complete the proof of the equivalence between the definition of operads in terms of full composition products (§I.1.1) and the definition of operads in terms of partial composition operations (§I.2.1). To conclude this section, we also explain the definition of restriction operators on treewise tensors and we check the coherence of these restriction operators with respect to our treewise composition products in the context of augmented Λ-operads.

For simplicity, we assume that the symmetric sequences and operads which we consider all through this section are non-unitary (in the sense of §I.1.1.21). Besides, we only consider open trees which fulfill the extra condition (5) of our definition §A.1.1. Most of our constructions however work without these requirements. We only need to restrict ourselves to non-unitary objects when we tackle the definition of restriction operators on treewise tensors.

A.2.1. *Tensor products over trees.* The *treewise tensor product* of a symmetric collection M over an $\underline{\mathsf{r}}$-tree $\underline{\mathsf{T}}$ is a tensor product in the base category formed by attaching a component of M to each vertex $v \in V(\underline{\mathsf{T}})$. The outcome of this construction is an object of the base category. In this appendix (and in the next appendix similarly), we mostly use the notation $\mathsf{M}(\underline{\mathsf{T}})$ for this treewise product in order to stress the functoriality of our construction with respect to the tree $\underline{\mathsf{T}}$. In other chapters, we also use the notation $\mathbb{O}_{\underline{\mathsf{T}}}(\mathsf{M})$ for this object $\mathbb{O}_{\underline{\mathsf{T}}}(\mathsf{M}) = \mathsf{M}(\underline{\mathsf{T}})$, because we rather use the functoriality of our construction with respect to the symmetric collection M and we moreover aim to stress that this treewise tensor product $\mathbb{O}_{\underline{\mathsf{T}}}(\mathsf{M})$ defines a summand of the free operad, which we denote by $\mathbb{O}(\mathsf{M})$.

Recall that we associate a set of ingoing edges $\underline{\mathsf{r}}_v = \{e \in E(\underline{\mathsf{T}}) | t(e) = v\}$ to each vertex $v \in V(\underline{\mathsf{T}})$ of a tree $\underline{\mathsf{T}}$. The treewise tensor product of the symmetric collection M over the tree $\underline{\mathsf{T}}$ is formally defined by the tensor product

$$\mathsf{M}(\underline{\mathsf{T}}) = \bigotimes_{v \in V(\underline{\mathsf{T}})} \mathsf{M}(\underline{\mathsf{r}}_v),$$

which ranges over the vertex set of our tree $\underline{\mathsf{T}}$, and where the input set $\underline{\mathsf{r}}_v$ of a factor $\mathsf{M}(\underline{\mathsf{r}}_v)$ is given by the set of ingoing edges of the corresponding vertex v.

Recall that, in this appendix, we tacitely assume that our collections are non-unitary and hence vanish $\mathsf{M}(0) = \varnothing$ in arity $r = 0$. We therefore do not have to consider trees with terminal vertices in our treewise tensor construction and this observation motivates us to restrict ourselves to open trees, for which we have $\underline{\mathsf{r}}_v \neq \varnothing$ for all $v \in V(\underline{\mathsf{T}})$, in our definition §A.1.1.

In the expression of M, we may replace the edge set $\underline{\mathsf{r}}_v$ by any set $\underline{\mathsf{e}}_v$ equipped with a bijection $u : \underline{\mathsf{e}}_v \xrightarrow{\simeq} \underline{\mathsf{r}}_v$ since we have an isomorphism $u_* : \mathsf{M}(\underline{\mathsf{e}}_v) \xrightarrow{\simeq} \mathsf{M}(\underline{\mathsf{r}}_v)$ associated to u. By using this observation, we readily obtain that any isomorphism of $\underline{\mathsf{r}}$-trees $f : \underline{\mathsf{S}} \to \underline{\mathsf{T}}$ induces a morphism $f_* : \mathsf{M}(\underline{\mathsf{S}}) \to \mathsf{M}(\underline{\mathsf{T}})$ in the base category. Thus, the mapping $\underline{\mathsf{T}} \mapsto \mathsf{M}(\underline{\mathsf{T}})$ defines a functor on the groupoid of $\underline{\mathsf{r}}$-trees $\mathcal{T}ree(\underline{\mathsf{r}})^{iso}$. We have on the other hand $\mathsf{M}(u_* \underline{\mathsf{T}}) = \mathsf{M}(\underline{\mathsf{T}})$ when we consider the treewise tensor product $\mathsf{M}(u_* \underline{\mathsf{T}})$ associated to the image of a tree $\underline{\mathsf{T}} \in \mathcal{T}ree(\underline{\mathsf{r}})$ under the action of an input reindexing bijection $u : \underline{\mathsf{r}} \xrightarrow{\simeq} \underline{\mathsf{s}}$.

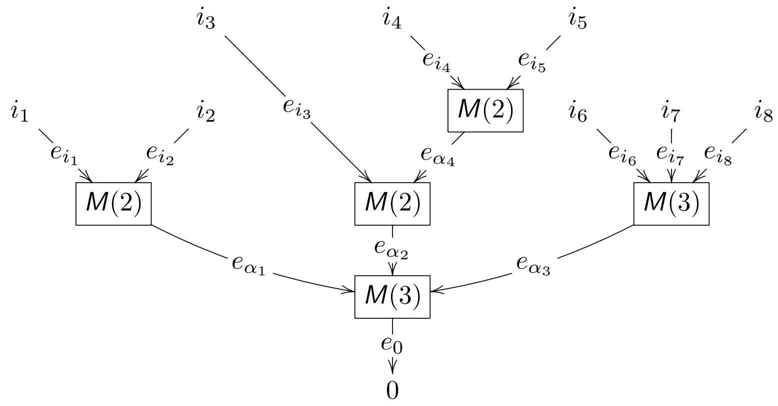


FIGURE A.6. The picture of a treewise tensor product.

Each morphism of collections $f : M \to N$ gives rise to an obvious morphism of treewise tensor products $f_* : M(\underline{T}) \to N(\underline{T})$ too. This morphism commutes with the action of tree isomorphisms and our construction actually gives, for each finite set $\underline{r}$, a functor from the category of collections to the category of functors on the tree groupoid $\mathcal{T}ree(\underline{r})^{iso}$.

Intuitively, we view a treewise tensor product $M(\underline{T})$ as a decoration of the vertices v of the tree $\underline{T}$ by the factors $M(\underline{r}_v)$. In Figure A.6, we give an example of application of this representation for the tree of Figure A.1. In practice, we often restrict ourselves to terms $M(\underline{r})$ of the collection M associated to the standard ordered sets $\underline{r} = \{1 < \cdots < r\}$ in our treewise tensor product construction. In our graphical representation, we just assume that the planar embedding of our figure materializes bijections between these ordered sets and the input sets of the vertices. In the case of Figure A.6, we consider for instance the mapping $u : \underline{3} \to \{e_{\alpha_1}, e_{\alpha_2}, e_{\alpha_3}\}$ such that $u(1) = e_{\alpha_1}$, $u(2) = e_{\alpha_2}$, $u(3) = e_{\alpha_3}$ to get a bijection between the ordered set $\underline{3} = \{1 < 2 < 3\}$ and the input set of the root vertex $\underline{r}_{v_0} = \{e_{\alpha_1}, e_{\alpha_2}, e_{\alpha_3}\}$.

We adopt a similar representation for the elements of the treewise tensor product $\xi = \bigotimes_{v \in V(\underline{T})} \xi_v \in M(\underline{T})$ which are formed by attaching a given element $\xi_v \in M(\underline{r}_v)$ to each vertex $v \in V(\underline{T})$.

We have the following immediate observations:

PROPOSITION A.2.2.

(a) *For the unit tree* $\underline{\downarrow}$*, which has no vertex, we have* $M(\underline{\downarrow}) = \mathbb{1}$*, the unit object of the base category.*

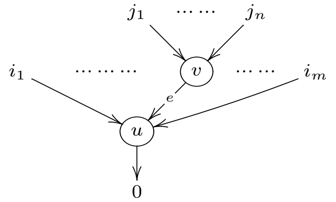
(b) *For the* $\underline{r}$*-corolla* $\underline{Y}$*, which has a single vertex* $v = v_0$ *with* $\underline{r}_v = \underline{r}$ *as input set, we have a canonical isomorphism* $M(\underline{r}) \simeq M(\underline{Y})$*.* ☐

A.2.3. *The partial composition products of an operad.* In §I.2.1.4 and §I.2.5.8, we explain that the (partial) composition operations of an operad

$$(1) \qquad \circ_{i_k} : P(\underline{m}) \otimes P(\underline{n}) \to P(\underline{m} \circ_{i_k} \underline{n})$$

are shaped on composition schemes which we can intuitively depict by using trees with two vertices.

In the formalism of §A.1.1, an $\underline{\mathsf{r}}$-indexed tree $\underline{\Gamma}$ with two vertices u, v, and one inner edge e, oriented from $v = s(e)$ to $u = t(e)$, is determined, up to a canonical isomorphism, by a decomposition of the form $\underline{\mathsf{r}} = \underline{\mathsf{m}} \circ_{i_e} \underline{\mathsf{n}}$ where $\underline{\mathsf{m}} \xrightarrow{\simeq} \underline{\mathsf{r}}_v$ and $\underline{\mathsf{n}} \xrightarrow{\simeq} \underline{\mathsf{r}}_w$. This correspondence is given by the following graphical picture:



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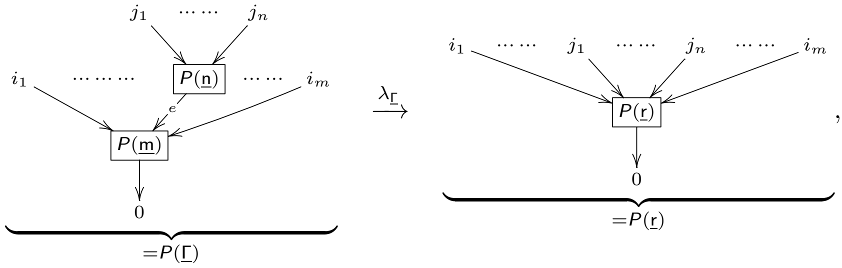
The composition variable of our decomposition $\underline{\mathsf{r}} = \underline{\mathsf{m}} \circ_{i_e} \underline{\mathsf{n}}$ is an extra index $i_e \notin \underline{\mathsf{r}}$ which we associate to the inner edge of the tree. The set $\underline{\mathsf{m}}$ formally consists of this composition variable i_e together with the inputs of the tree $i = s(e_i) \in \underline{\mathsf{r}}$ which we associate to the ingoing edges e_i such that $t(e_i) = u$. The set $\underline{\mathsf{n}}$ consists of the inputs $j = s(e_j) \in \underline{\mathsf{r}}$ which are associated to the ingoing edges of the tree e_j such that $t(e_j) = v$.

This correspondence formally gives an equivalence of categories between the groupoid of $\underline{\mathsf{r}}$-trees with two vertices $\mathcal{T}ree_2(\underline{\mathsf{r}})^{iso}$ and a groupoid whose objects are the decompositions $\underline{\mathsf{r}} = \underline{\mathsf{m}} \circ_{i_e} \underline{\mathsf{n}}$ of the set $\underline{\mathsf{r}}$. Naturally, the action of reindexing bijections on trees corresponds to a natural action of these bijections on the collection of decompositions $\underline{\mathsf{r}} = \underline{\mathsf{m}} \circ_{i_e} \underline{\mathsf{n}}$.

The existence of canonical isomorphisms $\underline{\mathsf{m}} \simeq \underline{\mathsf{r}}_u$ and $\underline{\mathsf{n}} \simeq \underline{\mathsf{r}}_v$ in our correspondence implies that the tensor product of P over an $\underline{\mathsf{r}}$-tree with two vertices $\underline{\Gamma}$ satisfies $P(\underline{\Gamma}) \simeq P(\underline{\mathsf{m}}) \otimes P(\underline{\mathsf{n}})$, where we consider the partition $\underline{\mathsf{r}} = \underline{\mathsf{m}} \circ_{i_e} \underline{\mathsf{n}}$ associated to $\underline{\Gamma}$. Therefore, we formally obtain that the composition operations $\circ_{i_e} : P(\underline{\mathsf{m}}) \otimes P(\underline{\mathsf{n}}) \to P(\underline{\mathsf{m}} \circ_{i_e} \underline{\mathsf{n}})$, which we consider in §I.2.5.8, are equivalent to morphisms

(2)



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shaped on $\underline{\mathsf{r}}$-trees with two vertices $\underline{\Gamma} \in \mathcal{T}ree_2(\underline{\mathsf{r}})$, and which commute with the action of tree isomorphisms and with the action of reindexing bijections.

The aim of the next paragraphs is to extend the definition of these morphisms to arbitrary trees $\underline{T}$ and to rewrite the associativity axiom of operads in terms of these generalized composition operations. The idea is to perform partial composition operations by merging vertices connected by an edge within a tree and to repeat the process. To begin with, we explain the formal definition of this elementary merging process, called an *edge contraction*, and which involves our treewise interpretation of the composition operations of an operad.

A.2.4. *Edge contractions.* In Figure A.7, we have represented an example of application of our edge contraction process. In short, this operation consists in

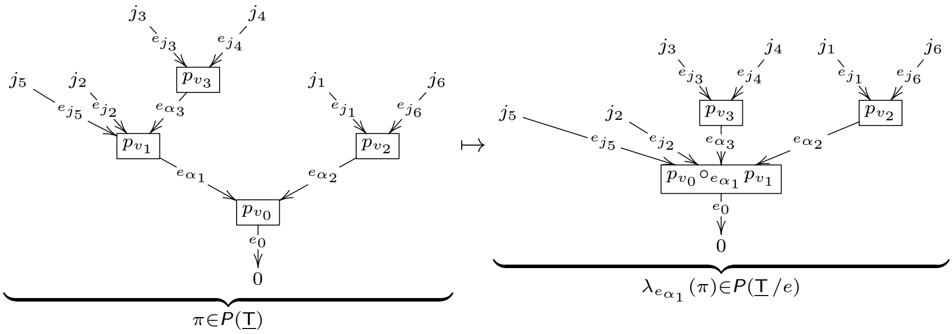


Figure A.7. The picture of a composition along an edge, for a treewise tensor product $\pi \in P(\underline{\mathsf{T}})$ of elements $p_{v_0}, p_{v_1}, p_{v_2}, p_{v_3}$ in an operad P.

merging the source and target of an inner edge (the edge e_{α_1} in our figure), which vanishes in the result of our construction. In this figure, we still consider a treewise tensor product of the same shape as in our first informal account of the definition of free operads (see §I.1.2).

In general, we fix an inner edge $e \in \mathring{E}(\underline{\mathsf{T}})$ in a tree $\underline{\mathsf{T}}$. The tree obtained by contracting the edge e in $\underline{\mathsf{T}}$, which we denote by $\underline{\mathsf{T}}/e$, consists of the edge set $E(\underline{\mathsf{T}}/e) = E(\underline{\mathsf{T}}) \setminus \{e\}$, which we form by removing e from $E(\underline{\mathsf{T}})$, together with the vertex set $V(\underline{\mathsf{T}}/e) = V(\underline{\mathsf{T}})/\{s(e) \equiv t(e)\}$ which we form by identifying the source $v = s(e)$ and the target $u = t(e)$ of our edge e. The source (respectively, the target) of an edge $f \in E(\underline{\mathsf{T}}) \setminus \{e\}$ in the tree $\underline{\mathsf{T}}/e$ is given by the source (respectively, the target) of this edge f in $\underline{\mathsf{T}}$. One can readily check that $\underline{\mathsf{T}}/e$ satisfies the axioms of a tree structure. Hence, our construction gives a well-defined operation on trees.

We still set $v = s(e)$ (respectively, $u = t(e)$) for the source (respectively, the target) of our edge e in the tree $\underline{\mathsf{T}}$. The set of ingoing edges $\underline{\mathsf{r}}_\omega$ of the vertex $x = \omega$ that arises from the merging operation $u \equiv v$ in the tree $\underline{\mathsf{T}}/e$ is clearly identified with the composite $\underline{\mathsf{r}}_\omega = \underline{\mathsf{r}}_u \circ_e \underline{\mathsf{r}}_v$, where we consider the set of ingoing edges of u and v in $\underline{\mathsf{T}}$. The vertices $x \in V(\underline{\mathsf{T}}) \setminus \{u, v\} = V(\underline{\mathsf{T}}/e) \setminus \{\omega\}$, which are untouched by the contraction process, have the same set of ingoing edges in $\underline{\mathsf{T}}/e$ as in $\underline{\mathsf{T}}$. Consequently, for any collection M, we have an identity

$$M(\underline{\mathsf{T}}/e) = M(\underline{\mathsf{r}}_u \circ_e \underline{\mathsf{r}}_v) \otimes \Big(\bigotimes_{x \in V(\underline{\mathsf{T}}/e) \setminus \{\omega\}} M(\underline{\mathsf{r}}_x) \Big),$$

where all expressions $\underline{\mathsf{r}}_x$ refer to sets of ingoing edges in $\underline{\mathsf{T}}$. On the other hand, we have $M(\underline{\mathsf{T}}) = M(\underline{\mathsf{r}}_u) \otimes M(\underline{\mathsf{r}}_v) \otimes \big(\bigotimes_{x \in V(\underline{\mathsf{T}}) \setminus \{u,v\}} M(\underline{\mathsf{r}}_x) \big)$. Hence, in the case of an operad $M = P$, we have a morphism

$$\underbrace{P(\underline{\mathsf{r}}_u) \otimes P(\underline{\mathsf{r}}_v) \otimes \Big(\bigotimes_{x \in V(\underline{\mathsf{T}}) \setminus \{u,v\}} P(\underline{\mathsf{r}}_x) \Big)}_{= P(\underline{\mathsf{T}})} \xrightarrow{\lambda_e} \underbrace{P(\underline{\mathsf{r}}_u \circ_e \underline{\mathsf{r}}_v) \otimes \Big(\bigotimes_{\substack{x \in V(\underline{\mathsf{T}}/e) \setminus \{\omega\} \\ = V(\underline{\mathsf{T}}) \setminus \{u,v\}}} P(\underline{\mathsf{r}}_x) \Big)}_{= P(\underline{\mathsf{T}}/e)},$$

naturally associated to our edge contraction, which we define by applying the composition operation $\circ_e : \mathsf{P}(\underline{\mathsf{r}}_u)\otimes\mathsf{P}(\underline{\mathsf{r}}_v)\to\mathsf{P}(\underline{\mathsf{r}}_u\circ_e\underline{\mathsf{r}}_v)$ to the subfactors attached to the vertices (u,v) in the expression of $\mathsf{P}(\underline{\mathsf{T}})$. In what follows, we refer to this morphism λ_e as the *composite along the edge* e in the treewise tensor product $\mathsf{P}(\underline{\mathsf{T}})$.

In the case of an $\underline{\mathsf{r}}$-tree with two vertices, the contraction of the single inner edge obviously gives an $\underline{\mathsf{r}}$-corolla $\underline{\mathsf{Y}}$ and the operation $\lambda_e : \mathsf{P}(\underline{\mathsf{T}})\to\mathsf{P}(\underline{\mathsf{Y}})$ associated to this contraction is clearly identified with the composition operation studied in §A.2.3, where we use the isomorphism $\mathsf{P}(\underline{\mathsf{r}})\simeq\mathsf{P}(\underline{\mathsf{Y}})$.

In §I.2.1, we observe that the associativity axioms of the partial composition products of an operad can be expressed in terms of commutative diagrams involving composite partial composition products and where we consider tensors arranged on trees with three vertices. For convenience, we recall the form of these diagrams in Figure A.8. We just replace the partial composites of Figure 2.2-2.3 in §I.2.1 by the equivalent edge contractions. We also use our expressions $\underline{\mathsf{r}}_x$, referring to the set of ingoing edges of vertices, instead of abstract indexing sets. The words uv, vw, … denote vertices which arise from merging operations and the expressions $\underline{\mathsf{r}}_{uv}$, $\underline{\mathsf{r}}_{vw}$, … refer to the corresponding input sets.

A.2.5. *The application of multiple edge contractions.* We now consider morphisms defined by multiple applications of the edge contraction process. We start with an $\underline{\mathsf{r}}$-tree $\underline{\mathsf{T}}$ (with an arbitrary number of vertices). By definition of the edge contraction process, the set of inner edges of a tree $\underline{\mathsf{T}}/e_\alpha$, where we have contracted an edge e_α, is simply obtained by removing this edge e_α from the inner edges of $\underline{\mathsf{T}}$. Thus, it makes sense to perform sequences of edge contractions

$$\underline{\mathsf{T}}\mapsto\underline{\mathsf{T}}/e_{\alpha_1}\mapsto\underline{\mathsf{T}}/e_{\alpha_1}/e_{\alpha_2}\mapsto\cdots\mapsto\underline{\mathsf{T}}/e_{\alpha_1}/e_{\alpha_2}/\cdots/e_{\alpha_n},$$

over any subset of inner edges $\{e_{\alpha_1},\dots,e_{\alpha_n}\}\subset\mathring{E}(\underline{\mathsf{T}})$. The lth tree of such a sequence $\underline{\mathsf{T}}/e_{\alpha_1}/\cdots/e_{\alpha_l}$ has

$$\mathring{E}(\underline{\mathsf{T}}/e_{\alpha_1}/\cdots/e_{\alpha_l})=\mathring{E}(\underline{\mathsf{T}})\setminus\{e_{\alpha_1},\cdots,e_{\alpha_l}\}$$

as inner edge set. Moreover, the set of vertices of this tree $V(\underline{\mathsf{T}}/e_{\alpha_1}/\cdots/e_{\alpha_l})$ is identified with the quotient set of $V(\underline{\mathsf{T}})$ under the equivalence relation $\equiv$ generated by $s(e_{\alpha_k})\equiv t(e_{\alpha_k})$, where $k=1,\dots,l$. This inspection shows that the tree $\underline{\mathsf{T}}/e_{\alpha_1}/\dots/e_{\alpha_l}$ does not depend on the choice of an ordering $\{e_{\alpha_1}<\cdots<e_{\alpha_l}\}$ in the sense that all these choices of contraction order give equal results.

We now consider the sequence of treewise tensor products $\mathsf{P}(\underline{\mathsf{T}}/e_{\alpha_1}/\dots/e_{\alpha_l})$ associated to an operad P and the morphisms

$$\mathsf{P}(\underline{\mathsf{T}}/e_{\alpha_1}/\dots/e_{\alpha_{l-1}})\xrightarrow{\lambda_{e_{\alpha_l}}}\mathsf{P}(\underline{\mathsf{T}}/e_{\alpha_1}/\cdots/e_{\alpha_{l-1}}/e_{\alpha_l})$$

determined by our sequence of edge contractions. We have the following statement:

THEOREM A.2.6. *Let P be any (non-unitary) operad. Let $\underline{\mathsf{T}}$ be any tree. Let $\underline{\mathsf{S}}$ be another tree obtained from $\underline{\mathsf{T}}$ by the contraction of a fixed subset of inner edges $\{e_{\alpha_1},\dots,e_{\alpha_l}\}\subset\mathring{E}(\underline{\mathsf{T}})$. The composite morphisms*

$$\mathsf{P}(\underline{\mathsf{T}})\xrightarrow{\lambda_{e_{\alpha_1}}}\mathsf{P}(\underline{\mathsf{T}}/e_{\alpha_1})\xrightarrow{\lambda_{e_{\alpha_2}}}\cdots\xrightarrow{\lambda_{e_{\alpha_l}}}\mathsf{P}(\underline{\mathsf{T}}/e_{\alpha_1}/\cdots/e_{\alpha_l})=\mathsf{P}(\underline{\mathsf{S}}),$$

which we determine by the choice of a contraction order on the set $\{e_{\alpha_1},\dots,e_{\alpha_l}\}$, are all equal.

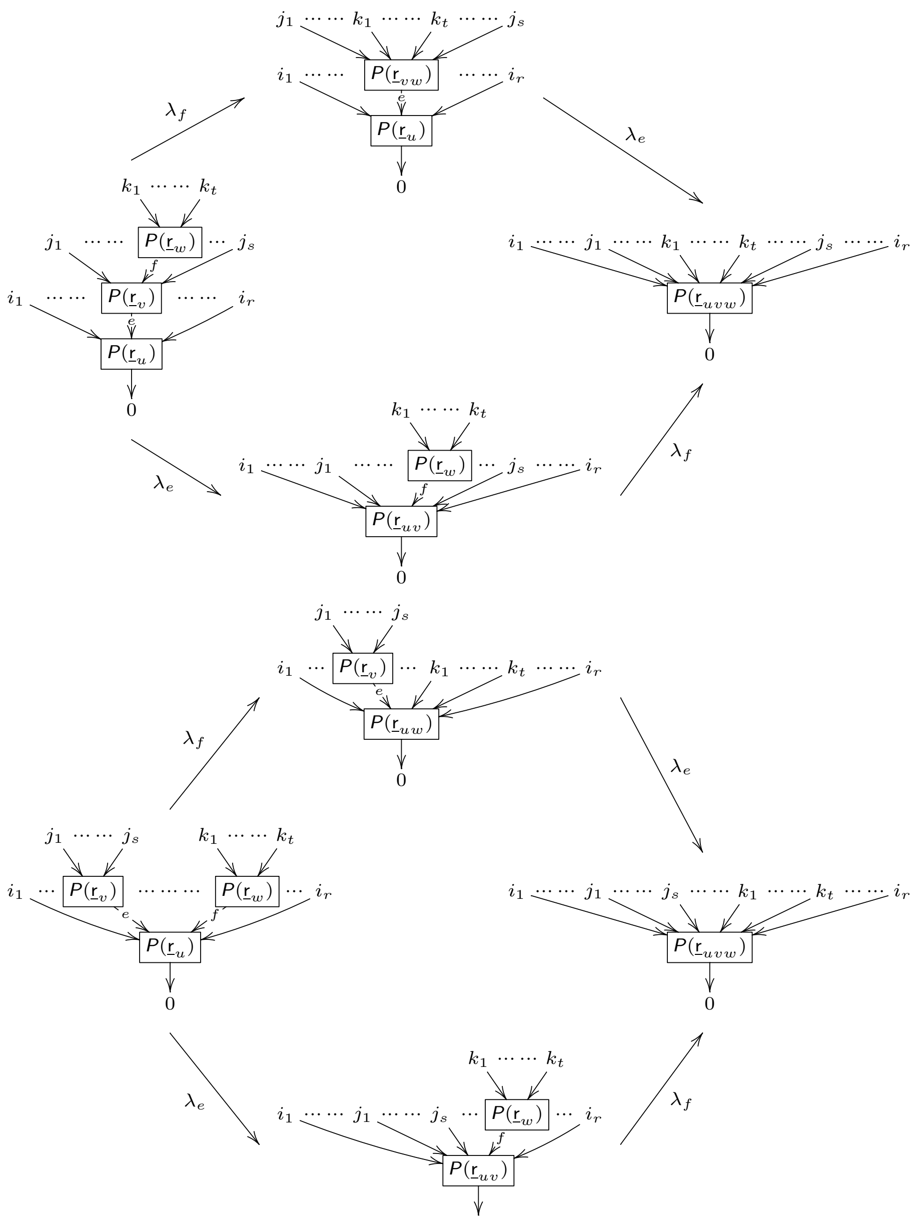


FIGURE A.8. The interpretation of the associativity relation of partial composition products in terms of edge contractions. We use words uv, vw, … to denote the vertices which arise from merging operations and the expressions $\underline{\mathsf{r}}_{uv}$, $\underline{\mathsf{r}}_{vw}$, … refer to the corresponding input sets.

Proof. In the proof of this theorem, we more precisely use the definition of an operad in terms of partial composition operations (see §I.2.1.9). We can also forget about the unit axiom for the moment. We just assume, therefore, that P is a symmetric collection equipped with partial composition operations §A.2.3(1-2) such that the diagrams of Figure A.8 commute, for all 3-fold composition patterns represented in this figure, and our statement holds in this context.

We prove that a permutation of any initial ordering $e_{\alpha_1}/\cdots/e_{\alpha_l}$ does not change the result of our composite morphism. We are left to check this invariance property for the elementary transpositions $e_{\alpha_i}/e_{\alpha_{i+1}} \mapsto e_{\alpha_{i+1}}/e_{\alpha_i}$, which generate all permutations, and hence, to compare composites of edge contractions which fit in diagrams of the form

(1)
$$\begin{tikzcd}
 & & & \cdot \arrow[dr, "\lambda_{e_{\alpha_{i+1}}}"] & & & \\
P(\underline{\mathrm{T}}) \arrow[r, "\lambda_{e_{\alpha_1}}"] & \cdots \arrow[r, "\lambda_{e_{\alpha_{i-1}}}"] & \cdot \arrow[ur, "\lambda_{e_{\alpha_i}}"] \arrow[dr, "\lambda_{e_{\alpha_{i+1}}}"'] & & \cdot \arrow[r, "\lambda_{e_{\alpha_{i+2}}}"] & \cdots \arrow[r, "\lambda_{e_{\alpha_l}}"] & P(\underline{\mathrm{T}}/e_{\alpha_1}/\cdots/e_{\alpha_l}). \\
 & & & \cdot \arrow[ur, "\lambda_{e_{\alpha_i}}"'] & & &
\end{tikzcd}$$

We focus on the diamond diagram that occurs in the middle of this double chain of edge contractions. We set $\underline{\Theta} = \underline{\mathrm{T}}/e_{\alpha_1}/\cdots/e_{\alpha_{i-1}}$ for the tree that occurs at the starting point of this diamond, and, for short, we also set $e = e_{\alpha_i}$, $f = e_{\alpha_{i+1}}$, for the edges which we contract in the diamond. We then deal with a diagram such that:

(2)
$$\begin{tikzcd}
 & P(\underline{\Theta}/e) \arrow[dr, "\lambda_f"] & \\
P(\underline{\Theta}) \arrow[ur, "\lambda_e"] \arrow[dr, "\lambda_f"'] & & P(\underline{\Theta}/e/f) = P(\underline{\Theta}/f/e). \\
 & P(\underline{\Theta}/f) \arrow[ur, "\lambda_e"'] &
\end{tikzcd}$$

The edges e and f can either be disjoint or intersect at a vertex in $\underline{\Theta}$. In the disjoint case, the composition products $\circ_e$ and $\circ_f$ are applied to disjoint factors of the treewise tensor product $P(\underline{\Theta}) = \bigotimes_{v \in V(\underline{\Theta})} P(\underline{\mathrm{r}}_v)$ when we form the morphisms λ_e and λ_f. In this case, the commutativity of (2) follows from the functoriality of tensor products. In the case where our edges intersect at a vertex, we retrieve the configurations of Figure A.8 on the subtree $\underline{\Sigma} \subset \underline{\Theta}$ such that $\mathring{\underline{\Sigma}} = \{e, f\}$, and the commutativity of (2) follows from the assumption that all such diagrams commute.

Thus, our square (2) commutes in all cases, and this verification completes the proof of Theorem A.2.6. ☐

A.2.7. *Treewise composition operations.* We go back to the definition of our contraction process §A.2.5. We assume that $\underline{\mathrm{T}}$ is any tree with at least one vertex and we consider the case where $\{e_{\alpha_1}, \dots, e_{\alpha_l}\}$ is the whole set of inner edges (possibly empty) of this tree $\underline{\mathrm{T}}$. We obtain an $\underline{\mathrm{r}}$-corolla $\underline{\mathrm{Y}} = \underline{\mathrm{T}}/e_{\alpha_1}/\dots/e_{\alpha_l}$ at the end of our construction in this case (this observation follows from the requirement that all vertices of a tree can be connected to the source of the outgoing edge by a chain of inner edges). We still see that this corolla $\underline{\mathrm{Y}} = \underline{\mathrm{T}}/e_{\alpha_1}/\dots/e_{\alpha_l}$, which we obtain by a full application of our edge contraction process, does not depend on the choice of a contraction order (all choices give equal results).

Recall that we have $M(\underline{\mathtt{Y}}) = M(\underline{\mathtt{r}})$, for any collection M. For an operad P, our treewise operations §A.2.5 therefore return a morphism

$$P(\underline{\mathtt{T}}) \xrightarrow{\lambda_{\underline{\mathtt{T}}}} P(\underline{\mathtt{Y}}) = P(\underline{\mathtt{r}}), \tag{1}$$

when we perform a full sequence of edge contractions:

$$P(\underline{\mathtt{T}}) \xrightarrow{\lambda_{e_{\alpha_1}}} P(\underline{\mathtt{T}}/e_{\alpha_1}) \xrightarrow{\lambda_{e_{\alpha_2}}} \cdots \xrightarrow{\lambda_{e_{\alpha_l}}} P(\underline{\mathtt{T}}/e_{\alpha_1}/\cdots/e_{\alpha_l}) = P(\underline{\mathtt{Y}}), \tag{2}$$

for any $\underline{\mathtt{r}}$-tree $\underline{\mathtt{T}} \in \mathcal{T}ree(\underline{\mathtt{r}})$. The result of Theorem A.2.6 implies that this morphism (1) does not depend on the choice of the contraction order in the decomposition (2). We also refer to this morphism, which we denote by $\lambda_{\underline{\mathtt{T}}}$, as the (complete) *treewise composition product* of the operad P over the tree $\underline{\mathtt{T}}$.

We extend the definition of these treewise composition operations to the case where $\underline{\mathtt{T}}$ is the unit tree $\underline{\downarrow}$. We then have $P(\underline{\downarrow}) = \mathbb{1}$. We assume by convention that $\lambda_{\underline{\mathtt{T}}}$ is given by the unit morphism of our operad $\eta : \mathbb{1} \to P(1)$ in this case $\underline{\mathtt{T}} = \underline{\downarrow}$.

A.2.8. *The definition of restriction operators and augmentations on treewise tensors.* Recall that we use the name '(augmented non-unitary) Λ-collection' for the analogue, in the context of diagrams over finite sets, of the notion of an (augmented non-unitary) Λ-sequence (see §2.5.9). We now check that the treewise tensor products of an augmented non-unitary Λ-collection M inherit restriction operators and an augmentation over the unit object of our base symmetric monoidal category. We use the definition of these restriction operators on trees to formalize our construction of free objects in the category of augmented non-unitary Λ-operads (see our outline in §I.2.3). We formally have a treewise restriction operator

$$u^* : M(\underline{\mathtt{T}}) \to M(u^* \underline{\mathtt{T}}), \tag{1}$$

for any injective map $u \in \mathtt{Mor}_{\mathcal{I}nj}(\underline{\mathtt{m}}, \underline{\mathtt{n}})$ and for each $\underline{\mathtt{n}}$-tree $\underline{\mathtt{T}} \in \mathcal{T}ree(\underline{\mathtt{n}})$, where $u^* \underline{\mathtt{T}} \in \mathcal{T}ree(\underline{\mathtt{m}})$ denotes the (unreduced) $\underline{\mathtt{m}}$-tree associated to $\underline{\mathtt{T}}$ under the restriction operator $u^* : \mathcal{T}ree(\underline{\mathtt{n}}) \to \mathcal{T}ree(\underline{\mathtt{m}})$. We have $V(u^* \underline{\mathtt{T}}) \subset V(\underline{\mathtt{T}})$ and $E(u^* \underline{\mathtt{T}}) \subset E(\underline{\mathtt{T}})$ by definition of this tree in §A.1.11. In addition, the set of inputs which we associate to any vertex such that $v \in V(u^* \underline{\mathtt{T}})$ in the tree $u^* \underline{\mathtt{T}}$ is identified with the subset $\underline{\mathtt{r}}_v \cap E(u^* \underline{\mathtt{T}}) \subset E(u^* \underline{\mathtt{T}})$ of the inputs of this vertex $\underline{\mathtt{r}}_v \subset E(\underline{\mathtt{T}})$ in the tree $\underline{\mathtt{T}}$. The vertices $v \in V(\underline{\mathtt{T}})$ which we discard in the tree $u^* \underline{\mathtt{T}}$ are those for which we have $\underline{\mathtt{r}}_v \cap E(u^* \underline{\mathtt{T}}) = \varnothing$.

We form our restriction operator (1) on the treewise tensor product $M(\underline{\mathtt{T}})$ factorwise. We take the internal restriction operator of our collection $u_v^* : M(\underline{\mathtt{r}}_v) \to M(\underline{\mathtt{r}}_v \cap E(u^* \underline{\mathtt{T}}))$, associated to the canonical injection $u_v : \underline{\mathtt{r}}_v \cap E(u^* \underline{\mathtt{T}}) \to \underline{\mathtt{r}}_v$, when $\underline{\mathtt{r}}_v \cap E(u^* \underline{\mathtt{T}}) \neq \varnothing$ and we keep the vertex $v \in V(\underline{\mathtt{T}})$ in the tree $u^* \underline{\mathtt{T}}$. We perform the augmentation $\epsilon : M(\underline{\mathtt{r}}_v) \to \mathbb{1}$ in order to discard the factor $M(\underline{\mathtt{r}}_v)$ from our tensor product otherwise. We give an example of application of this process in Figure A.9. We keep the same tree shape as in §A.1.11, where we explain the definition of restriction operators on trees. Recall that we initially introduced this example of a treewise tensor product in §I.1.2(∗), where we gave a first outline of the construction of free operads. We also used the same treewise tensor in §I.2.3 where we outline the definition of a Λ-operad structure on the free operad associated to an augmented Λ-sequence (see Proposition I.2.3.1). We just retrieve the result of this informal construction in Figure A.9.

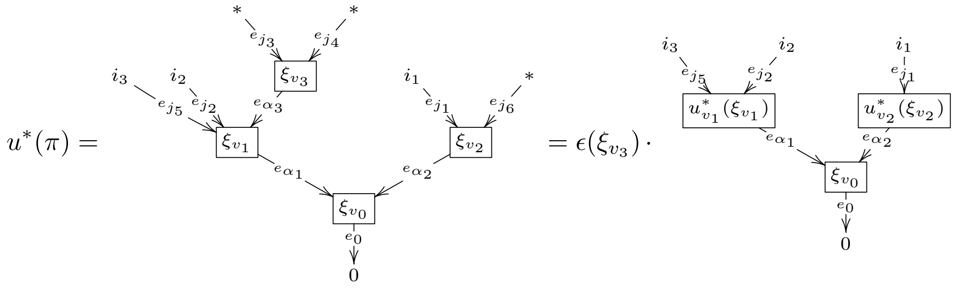


FIGURE A.9. The evaluation of a restriction operator on a treewise tensor $\pi \in M(\underline{\mathtt{T}})$. The outcome of this restriction operator $u^*(\pi) \in M(u^* \underline{\mathtt{T}})$ is shaped on the tree $u^* \underline{\mathtt{T}}$ determined in Figure A.4. The internal restriction operators, which we apply to the factors of this tensor product, are associated to the embeddings $u_{v_2} : \{e_{j_1}\} \to \{e_{j_1}, e_{j_6}\}$ and $u_{v_1} : \{e_{j_5}, e_{j_2}\} \to \{e_{j_5}, e_{j_2}, e_{\alpha_3}\}$ that define the subsets of ingoing of vertices which we keep in the reduced tree $u^* \underline{\mathtt{T}}$. We moreover take the image of the factor ξ_{v_3} under the augmentation of our collection to discard this factor from the outcome of our restriction process.

We also have an augmentation

$$\epsilon : M(\underline{\mathtt{T}}) \to \mathbb{1}, \tag{2}$$

for each $\underline{\mathtt{T}} \in \mathcal{T}ree(\underline{\mathtt{r}})$, which we obtain by applying the augmentation of our collection $\epsilon : M(\underline{\mathtt{r}}_v) \to \mathbb{1}$ to all factors $M(\underline{\mathtt{r}}_v)$ of the treewise tensor product $M(\underline{\mathtt{T}})$. In the example of Figure A.9, we explicitly get $\epsilon(\pi) = \epsilon(\xi_{v_0}) \cdot \epsilon(\xi_{v_1}) \cdot \epsilon(\xi_{v_2}) \cdot \epsilon(\xi_{v_3})$ for the value of this augmentation.

We now examine the equivariance of our edge contraction operations of §A.2.4 with respect these treewise restriction operators. We assume $\underline{\mathtt{T}} \in \mathcal{T}ree(\underline{\mathtt{n}})$ and we again consider an injective map $u \in \mathtt{Mor}_{\mathcal{I}nj}(\underline{\mathtt{m}}, \underline{\mathtt{n}})$ together with the tree $u^* \underline{\mathtt{T}} \in \mathcal{T}ree(\underline{\mathtt{m}})$ which we associate to $\underline{\mathtt{T}}$. Let $e \in \mathring{E}(\underline{\mathtt{T}})$ be any inner edge which we want to contract in the tree $\underline{\mathtt{T}}$. This edge may belong to the subset of edges which we discard in the tree $u^* \underline{\mathtt{T}}$. We have $(u^* \underline{\mathtt{T}})/e = u^*(\underline{\mathtt{T}}/e)$ otherwise. We have such an identity, for instance, when we consider the edge contraction of Figure A.7 together with the restriction operator of Figure A.4. We then have the following statement:

PROPOSITION A.2.9. *Let P be an augmented non-unitary Λ-operad.*

(a) *The edge contraction operations, determined by the composition structure of this operad P, intertwine the treewise restriction operators of* §A.2.8 *in the sense that our edge contraction morphisms make commute the diagrams*

$$\begin{tikzcd}
P(\underline{\mathtt{T}}) \arrow[r, "\lambda_e"] \arrow[d, "u^*"'] & P(\underline{\mathtt{T}}/e) \arrow[d, "u^*"] \\
P(u^* \underline{\mathtt{T}}) \arrow[r, "\lambda_e"] & P(u^*(\underline{\mathtt{T}}/e))
\end{tikzcd}$$

which we can form when we have an injective map $u \in \mathtt{Mor}_{\mathcal{I}nj}(\underline{\mathtt{m}}, \underline{\mathtt{n}})$, *an* $\underline{\mathtt{n}}$*-tree* $\underline{\mathtt{T}}$, *and an edge* $e \in \mathring{E}(\underline{\mathtt{T}})$ *such that* $e \in E(u^* \underline{\mathtt{T}})$, *as well as the diagrams*

$$\begin{tikzcd}
\mathsf{P}(\underline{\mathtt{T}}) \arrow[r, "\lambda_e"] \arrow[d, "u^*"'] & \mathsf{P}(\underline{\mathtt{T}}/e) \arrow[d, "u^*"] \\
\mathsf{P}(u^* \underline{\mathtt{T}}) \arrow[r, "="] & \mathsf{P}(u^*(\underline{\mathtt{T}}/e))
\end{tikzcd}$$

which we get when the edge $e \in \mathring{E}(\underline{\mathtt{T}})$ *is discarded in the tree* $u^* \underline{\mathtt{T}}$.

(b) *The edge contraction operations also preserve the augmentation morphisms of* §A.2.8 *on treewise tensors. We explicitly have a commutative diagram*

$$\begin{tikzcd}
\mathsf{P}(\underline{\mathtt{T}}) \arrow[rr, "\lambda_e"] \arrow[dr, "\epsilon"'] & & \mathsf{P}(\underline{\mathtt{T}}/e) \arrow[dl, "\epsilon"] \\
& \mathbb{1} &
\end{tikzcd} ,$$

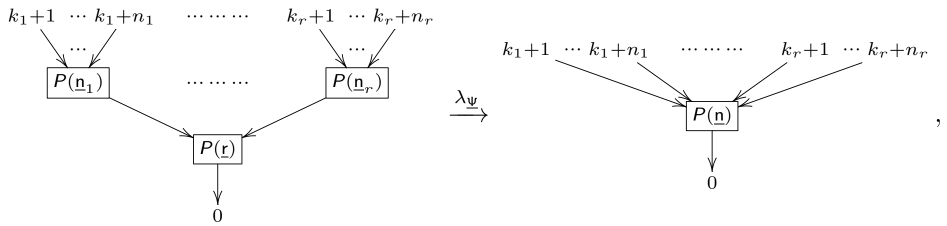
for every $\underline{\mathtt{r}}$*-tree* $\underline{\mathtt{T}}$, *and for any choice of an inner edge* $e \in \mathring{E}(\underline{\mathtt{T}})$ *inside this tree.*

Proof. We readily see that the relations of our first assertion (a) reduce to the equivariance of the composition products with respect to the restriction operators of the operad (see Proposition I.2.2.16) on the factors of the tensor product which we attach to our edge $e \in \mathring{E}(\underline{\mathtt{T}})$. We also use the bi-functoriality of the tensor product to check that our edge contraction operation commutes with the restriction operators which we perform on the other factors of our treewise tensor product. We use the same arguments (in a simplified form) to establish the second assertion of this proposition (b). □

The result of this proposition implies that our treewise composition operations $\lambda_{\underline{\mathtt{T}}} : \mathsf{P}(\underline{\mathtt{T}}) \to \mathsf{P}(\underline{\mathtt{Y}})$ preserve the restriction operators which we attach to our objects when P is an augmented non-unitary Λ-operads. We use this observation in our construction of free objects of the category of augmented non-unitary Λ-operads in the next section.

A.2.10. *The equivalence of operad axioms.* In §I.2.1, we checked that the unit and associativity axioms of the full composition products of an operad imply the unit and associativity relations of the partial composition products $\circ_k$ (see Proposition I.2.1.7). We used this implication to get a correspondence between the definition of an operad in terms of full composition products and the definition of an operad in terms of partial composition operations. We also mentioned that this correspondence defines an isomorphism of categories (see Theorem I.2.1.10) but we put off the proof of this claim.

We can now use that the full composition products are identified with treewise composites of the form



for all $r \in \mathbb{N}$, $n_1, \dots, n_r \in \mathbb{N}$, with $k_i = n_1 + \dots + n_{i-1}$ for $i = 1, \dots, r$, and where we consider the ordered sets $\underline{\mathsf{r}} = \{1 < \dots < r\}$, $\underline{\mathsf{n}}_1 = \{k_1 + 1 < \dots < k_1 + n_1\}$, $\dots$, $\underline{\mathsf{n}}_r = \{k_r + 1 < \dots < k_r + n_r\}$, $\underline{\mathsf{n}} = \{1 < \dots < n_1 + \dots + n_r\}$. We have a similar interpretation, in terms of treewise composition operations, of the composite morphisms occurring in the associativity relation of full composition products of Figure 1.5. We can accordingly use the result of Theorem A.2.6 to get this associativity relation when we start with partial composition operations.

We can also easily establish the validity of the unit relation of full composition products (see Figure 1.6) from the above treewise definition of these operations and from the unit relation of partial composition products. These verifications give the proof of Theorem I.2.1.10. ☐

A.3. The construction of free operads

We now give the construction of the free operad $\mathbb{\Theta}(M)$ associated to a collection $M \in \mathcal{C}oll$. We use the treewise tensor product construction in order to form the underlying symmetric collection of this object $\mathbb{\Theta}(M)$ first. We explain how to provide this collection $\mathbb{\Theta}(M)$ with an operad structure and we check that $\mathbb{\Theta}(M)$ satisfies the universal property of a free object afterwards. Recall that the universal property of a free object implies that the mapping $\mathbb{\Theta} : M \mapsto \mathbb{\Theta}(M)$ defines a left adjoint of the forgetful functor from operads to collections (see §I.1.2). In the course of our construction, we give an explicit description, in terms of the treewise composition products of §A.2, of the morphism $\lambda : \mathbb{\Theta}(P) \to P$, associated to any operad P, which defines the augmentation of this free operad adjunction.

We check, in the second part of this section, that the free operad $\mathbb{\Theta}(M)$ inherits an augmented non-unitary Λ-operad structure and forms a free object in the category of Λ-operads when M is an augmented non-unitary Λ-collection.

The definition of free operads is already well covered by the operad literature. Therefore, we mainly give full details on the definition of the operad morphism $\phi_f : \mathbb{\Theta}(M) \to P$ which we associate to a morphism of symmetric sequences $f : M \to P$ and we just outline the main steps of the proof that this construction of an operad morphism $\phi_f : \mathbb{\Theta}(M) \to P$ from a morphism of symmetric sequences $f : M \to P$ defines a one-to-one correspondence (which is equivalent to the assertion that our free operad $\mathbb{\Theta}(M)$ does satisfy the universal property of a free object).

For simplicity, we still take the convention that our collections and our operads are non-unitary all through this section. Nonetheless, we only really use this condition in the second part of our study when we examine the definition of free objects in the category of augmented non-unitary Λ-operads. We also use the restriction to non-unitary objects to give a reduced construction of free operads at the end of this section.

A.3.1. *The underlying collection of the free operad.* For a fixed collection M, the treewise tensor products $M(\underline{\mathsf{T}})$ define a functor from the category of $\underline{\mathsf{r}}$-trees and isomorphisms $\mathcal{T}ree(\underline{\mathsf{r}})^{iso}$ towards the base category $\mathcal{M}$. We take the colimit of this functor $\underline{\mathsf{T}} \mapsto M(\underline{\mathsf{T}})$ to define the component of the free operad $\mathbb{\Theta}(M)$ associated to a (non-empty) finite set $\underline{\mathsf{r}} \neq \underline{\mathsf{0}}$:

$$\mathbb{\Theta}(M)(\underline{\mathsf{r}}) = \operatorname*{colim}_{\underline{\mathsf{T}} \in \mathcal{T}ree(\underline{\mathsf{r}})^{iso}} M(\underline{\mathsf{T}}).$$

We just use that the category of $\underline{\mathsf{r}}$-trees has a small skeleton, for any fixed set $\underline{\mathsf{r}}$, in order to give a sense to this definition.

Recall that we rather use the notation $\mathsf{M}(\underline{\mathsf{T}})$ for the treewise tensor product of a collection M over a tree $\underline{\mathsf{T}}$ in this appendix. But we also set

$$\mathbb{\Theta}_{\underline{\mathsf{T}}}(\mathsf{M}) := \mathsf{M}(\underline{\mathsf{T}})$$

when we want to emphasize the identity between this object $\mathsf{M}(\underline{\mathsf{T}})$ and a summand of the free operad $\mathbb{\Theta}(\mathsf{M})$ and we mostly use the latter notation in other parts of this monograph.

Each bijection of finite sets $u : \underline{\mathsf{r}} \xrightarrow{\simeq} \underline{\mathsf{s}}$ gives rise to a natural reindexing morphism $u_* : \mathbb{\Theta}(\mathsf{M})(\underline{\mathsf{r}}) \to \mathbb{\Theta}(\mathsf{M})(\underline{\mathsf{s}})$, which is yielded by the reindexing functor on the categories of trees $u_* : \mathcal{T}ree(\underline{\mathsf{r}}) \to \mathcal{T}ree(\underline{\mathsf{s}})$ on which we shape our colimit. Recall simply that we have an identity $\mathsf{M}(u_* \underline{\mathsf{T}}) = \mathsf{M}(\underline{\mathsf{T}})$ when we consider the treewise tensor product $\mathsf{M}(u_* \underline{\mathsf{T}})$ associated to the image of a tree under this reindexing functor $u_* \underline{\mathsf{T}} \in \mathcal{T}ree(\underline{\mathsf{s}})$ (see §A.2.1). Thus, the collection $\{\mathbb{\Theta}(\mathsf{M})(\underline{\mathsf{r}}), r > 0\}$ inherits a natural symmetric structure.

Recall that we have $\mathsf{M}(\underline{\downarrow}) = \mathbb{1}$ when we form a treewise tensor product over the unit tree $\underline{\mathsf{T}} = \underline{\downarrow}$ (Proposition A.2.2). We accordingly have a natural morphism $\eta : \mathbb{1} \to \mathbb{\Theta}(\mathsf{M})(1)$ yielded by the embedding of the term $\mathsf{M}(\underline{\downarrow})$ in the colimit that defines the (component of arity one of our) free operad. Similarly, since we have $\mathsf{M}(\underline{\mathsf{Y}}) = \mathsf{M}(\underline{\mathsf{r}})$ when we form the treewise tensor product over an $\underline{\mathsf{r}}$-corolla $\underline{\mathsf{Y}}$ (see again Proposition A.2.2), we have a natural morphism $\iota : \mathsf{M}(\underline{\mathsf{r}}) \to \mathbb{\Theta}(\mathsf{M})(\underline{\mathsf{r}})$ yielded by the embedding of the term $\mathsf{M}(\underline{\mathsf{Y}})$ in the colimit expression of the object $\mathbb{\Theta}(\mathsf{M})(\underline{\mathsf{r}})$. The collection of these morphisms, where $\underline{\mathsf{r}}$ runs over the category of (non-empty) finite sets and bijections, obviously gives a morphism of symmetric collections $\iota : \mathsf{M} \to \mathbb{\Theta}(\mathsf{M})$ naturally associated to M.

The morphism $\eta : \mathbb{1} \to \mathbb{\Theta}(\mathsf{M})(1)$ will give the unit for the composition structure of the free operad. The morphism $\iota : \mathsf{M} \to \mathbb{\Theta}(\mathsf{M})$ will give the universal morphism which we associate to the definition of a free object in the category of operads, but we still have to provide the collection $\mathbb{\Theta}(\mathsf{M})$ with a composition structure before tackling the verification of the universal property of free objects. For this purpose, we rely on the composition structure of the operad of trees and on the following observation:

OBSERVATION A.3.2. *For a composite tree $\underline{\Theta} = \underline{\mathsf{S}} \circ_{i_k} \underline{\mathsf{T}}$, where $\underline{\mathsf{S}} \in \mathcal{T}ree(\underline{\mathsf{m}})$ and $\underline{\mathsf{T}} \in \mathcal{T}ree(\underline{\mathsf{n}})$, we have a canonical isomorphism*

$$\underbrace{\Big(\bigotimes_{u \in V(\underline{\mathsf{S}})} \mathsf{M}(\underline{\mathsf{r}}_u)\Big)}_{=\mathsf{M}(\underline{\mathsf{S}})} \otimes \underbrace{\Big(\bigotimes_{v \in V(\underline{\mathsf{T}})} \mathsf{M}(\underline{\mathsf{r}}_v)\Big)}_{=\mathsf{M}(\underline{\mathsf{T}})} \xrightarrow{\simeq} \underbrace{\Big(\bigotimes_{w \in V(\underline{\Theta})} \mathsf{M}(\underline{\mathsf{r}}_w)\Big)}_{=\mathsf{M}(\underline{\Theta})}$$

which arises from the construction of the vertex set of $\underline{\Theta}$ as a disjoint union $V(\underline{\Theta}) = V(\underline{\mathsf{S}}) \amalg V(\underline{\mathsf{T}})$.

In what follows, we view this isomorphism as a generalized operadic composition operation, shaped on the composition structure of the operad of trees. Therefore, we also use the notation $\circ_{i_k}$ to refer to it.

A.3.3. *The composition products of the free operad.* We now have natural composition products

$$\underbrace{\operatorname*{colim}_{\underline{\mathsf{S}} \in \mathcal{T}ree(\underline{\mathsf{m}})^{iso}} \mathsf{M}(\underline{\mathsf{S}})}_{=\mathbb{\Theta}(\mathsf{M})(\underline{\mathsf{m}})} \otimes \underbrace{\operatorname*{colim}_{\underline{\mathsf{T}} \in \mathcal{T}ree(\underline{\mathsf{n}})^{iso}} \mathsf{M}(\underline{\mathsf{T}})}_{=\mathbb{\Theta}(\mathsf{M})(\underline{\mathsf{n}})} \xrightarrow{\circ_{i_k}} \underbrace{\operatorname*{colim}_{\underline{\Theta} \in \mathcal{T}ree(\underline{\mathsf{m}} \circ_{i_k} \underline{\mathsf{n}})^{iso}} \mathsf{M}(\underline{\Theta})}_{=\mathbb{\Theta}(\mathsf{M})(\underline{\mathsf{m}} \circ_{i_k} \underline{\mathsf{n}})}, \tag{1}$$

which we form by putting together the isomorphisms

$$\circ_{i_k} : M(\underline{\mathsf{S}}) \otimes M(\underline{\mathsf{T}}) \xrightarrow{\simeq} M(\underline{\mathsf{S}} \circ_{i_k} \underline{\mathsf{T}}) \tag{2}$$

of the previous observation. We readily check that:

Proposition A.3.4. *The above composition products* §A.3.3*(*1*)*

- *preserve the action of bijections of finite sets on our objects,*
- *satisfy the unit axioms of operads, expressed by the diagrams of Figure* 2.1*, with the morphism* $\eta : \mathbb{1} \to \Theta(M)(1)$ *defined in* §A.3.1 *as operadic unit,*
- *and satisfy the associativity axioms, expressed by the diagrams of Figure* 2.2*-*2.3*, similarly.*

Thus, we have a well-defined operad structure on our object $\Theta(M)$*, for any symmetric collection* M*.*

Proof. The isomorphisms of observation A.3.2 are obviously functorial with respect to input reindexing and this assertion immediately implies the functoriality of our composition products. The unit and associativity relations of our composition products can be verified termwise by using that the unit and associativity axioms hold for the operadic composition of trees. (We simply check that the isomorphisms of observation A.3.2 are coherent with respect to these unit and associativity relations which we get at the tree level.) ☐

This proposition completes the definition of the operad $\Theta(M)$. We now have to check that this operad $\Theta(M)$ satisfies the universal property of free objects. In short, we have to establish that any given morphism $f : M \to P$ in the category of collections uniquely extends to a morphism $\phi_f : \Theta(M) \to P$ in the category of operads. We give an explicit definition of this morphism of operads $\phi_f : \Theta(M) \to P$ which we associate to any morphism of symmetric collections $f : M \to P$. We outline the proof that our construction provides the one-to-one correspondence claimed by the universal property of free objects afterwards.

A.3.5. *The explicit construction of morphisms on the free operad.* Thus, we assume that $f : M \to P$ is a morphism in the category of collections with an operad P as target object. For any $\underline{\mathsf{r}}$-tree $\underline{\mathsf{T}}$, where $\underline{\mathsf{r}}$ is any (non-empty) finite set, we form the composite:

$$M(\underline{\mathsf{T}}) \xrightarrow{f_*} P(\underline{\mathsf{T}}) \xrightarrow{\lambda_{\underline{\mathsf{T}}}} P(\underline{\mathsf{r}}), \tag{1}$$

where f_* is the morphism induced by $f : M \to P$ by functoriality of the treewise tensor product construction with respect to morphisms of collections (see §A.2.1) and $\lambda_{\underline{\mathsf{T}}}$ is treewise composition operation associated to the operad P. We put these composite morphisms together to define a morphism associated to $f : M \to P$ on the components of the free operad $\Theta(M)$:

$$\underbrace{\operatorname*{colim}_{\underline{\mathsf{T}} \in \mathcal{T}ree(\underline{\mathsf{r}})^{iso}} M(\underline{\mathsf{T}})}_{=\Theta(M)(\underline{\mathsf{r}})} \xrightarrow{\phi_f} P(\underline{\mathsf{r}}). \tag{2}$$

We easily check that these morphisms define a morphism of symmetric collections $\phi_f : \Theta(M) \to P$. We also have the following proposition:

Proposition A.3.6. *The just defined morphism* $\phi_f : \Theta(M) \to P$ *preserves operadic composition products as well as operadic units. Hence, this morphism* ϕ_f

does define a morphism in the category of operads. Moreover, we have $\phi_f \iota = f$ *when we take the composite of this morphism with the canonical embedding* $\iota : M \to \Theta(M)$ *of* §A.3.1.

PROOF. The coherence result of Theorem A.2.6 and the functoriality of the treewise tensor product construction implies that we have a commutative diagram

$$\begin{tikzcd}
M(\underline{S}) \otimes M(\underline{T}) \arrow[r, "f_* \otimes f_*"] \arrow[d, "\simeq"'] & P(\underline{S}) \otimes P(\underline{T}) \arrow[r, "\lambda_{\underline{S}} \otimes \lambda_{\underline{T}}"] \arrow[d, "\simeq"'] & P(\underline{m}) \otimes P(\underline{n}) \arrow[d, "\circ_{i_k}"] \\
M(\underline{S} \circ_{i_k} \underline{T}) \arrow[r, "f_*"] & P(\underline{S} \circ_{i_k} \underline{T}) \arrow[r, "\lambda_{\underline{S} \circ_{i_k} \underline{T}}"] & P(\underline{m} \circ_{i_k} \underline{n})
\end{tikzcd},$$

for any pair of trees $\underline{S} \in \mathcal{T}ree(\underline{m})$ and $\underline{T} \in \mathcal{T}ree(\underline{n})$. We then use the termwise definition of the morphism ϕ_f and of the composition structure of the free operad to conclude that this morphism ϕ_f preserves the partial composition operations $\circ_{i_k}$.

Recall that the treewise composition operation $\lambda_{\underline{\downarrow}}$ which we associate to the unit tree $\underline{\downarrow}$ is given by the unit morphism of our operad $\eta : \mathbb{1} \to P(1)$. This convention implies that ϕ_f reduces to the unit morphism of the operad P on the term $\mathbb{1} = M(\underline{\downarrow})$ of the free operad $\Theta(M)$, and that our morphism ϕ_f also preserves operadic units. This verification finishes the proof that ϕ_f defines a morphism in the category of operads.

Recall that the morphism $\iota : M \to \Theta(M)$, defined in §A.3.1, is given, in any arity $r > 0$, by the canonical isomorphism between $M(\underline{r})$ and the term attached to the $\underline{r}$-corolla $M(\underline{Y})$ in the expansion of $\Theta(M)(\underline{r})$. Since the treewise composition product of P obviously reduces to the identity on a corolla, we also immediately obtain $\phi_f \iota = f$. □

We also have the following observation:

PROPOSITION A.3.7. *The canonical morphism*

$$M(\underline{T}) \to \operatorname*{colim}_{\underline{T} \in \mathcal{T}ree(\underline{r})^{iso}} M(\underline{T}) = \Theta(M)(\underline{r}),$$

which we associate to each term of our colimit in the definition of the free operad, is identified with the composite:

$$M(\underline{T}) \xrightarrow{\iota_*} \Theta(M)(\underline{T}) \xrightarrow{\lambda_{\underline{T}}} \Theta(M)(\underline{r}),$$

where we consider the natural mapping induced by the morphism of collections $\iota : M \to \Theta(M)$ *at the treewise tensor level, followed by the treewise composition morphism* $\lambda_{\underline{T}}$ *associated to the free operad* $\Theta(M)$.

PROOF. By definition of the free operad, we have the identity $\Theta(M)(\underline{T}) = \bigotimes_{v \in V(\underline{T})} \Theta(M)(\underline{r}_v) = \bigotimes_{v \in V(\underline{T})} \{\operatorname{colim}_{\underline{S}_v \in \mathcal{T}ree(\underline{r}_v)} M(\underline{S}_v)\}$, and we can use the distribution relation of tensor products with respect to colimits in order to obtain that this treewise tensor product $\Theta(M)(\underline{T})$ expands as a colimit of tensor products $\bigotimes_{v \in V(\underline{T})} M(\underline{S}_v)$ such that $\underline{S}_v \in \mathcal{T}ree(\underline{r}_v)$, $v \in V(\underline{T})$. (We revisit the definition of these treewise composition products on free operads in-depth in the next appendix chapter §B.) We see that the morphism $\iota_* : M(\underline{T}) \to \Theta(M)(\underline{T})$ carries $M(\underline{T}) = \bigotimes_v M(\underline{r}_v)$ to the tensor product of the terms $M(\underline{Y}_v)$ associated to the collection of corollas $\underline{S}_v = \underline{Y}_v$, $v \in V(\underline{T})$, in this expansion of the object $\Theta(M)(\underline{T})$.

Then we easily check that the value of the treewise composition operation $\lambda_{\underline{\mathtt{T}}}$ on these terms of the free operad $\Theta(M)$ reduces to the identity morphism of the treewise tensor product $M(\underline{\mathtt{T}})$ inside $\Theta(M)$ ☐

This proposition implies that the treewise tensor products $M(\underline{\mathtt{T}})$ in our construction of the free operad $\Theta(M)$ represent treewise composites of the generating collection M in $\Theta(M)$. Then:

Theorem A.3.8. *The operad $\Theta(M)$, such as defined in* §§A.3.1-A.3.3 *and in Proposition* A.3.4, *does satisfy the universal property of a free object: any morphism of collections $f : M \to P$, where P is an operad, admits a unique factorization*

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "\iota"'] & & P \\ & \Theta(M) \arrow[ur, dotted, "\exists!\phi_f"'] & \end{tikzcd}$$

such that ϕ_f is an operad morphism. This morphism ϕ_f is given by the construction of §A.3.5.

Proof. The existence of this factorization $\phi_f \iota = f$ follows from Proposition A.3.6. The observation of Proposition A.3.7 (together with the functoriality of the treewise composition operation of operads) implies that any such factorization of the morphism of collections $f : M \to P$ is given by the construction of §A.3.5 on the term $M(\underline{\mathtt{T}})$ of the free operad $\Theta(M)$ and this assertion also proves that our factorization is unique. ☐

In §I.1.2, we first define the free operad as a left adjoint of the obvious forgetful functor from operads to collections. Recall that this characterization of free operads is formally equivalent to the statement of the previous theorem. Indeed, the result of Theorem A.3.8 is equivalent to the claim that the mapping $\phi \mapsto \phi\iota$ yields a bijection $\mathtt{Mor}_{\mathcal{O}p}(\Theta(M), P) \xrightarrow{\simeq} \mathtt{Mor}_{\mathcal{C}oll}(M, P)$ and this assertion is nothing but the definition of our adjunction. Therefore:

Theorem A.3.9 (Claim of Theorem I.1.2.1). *Our free operad functor $\Theta : M \mapsto \Theta(M)$, such as defined in* §§A.3.1-A.3.4, *forms a left adjoint of the obvious forgetful functor from the category of (non-unitary) operads to the category of (non-unitary) collections.* ☐

The universal morphism $\iota : M \to \Theta(M)$ corresponds to the identity of the free operad $\Theta(M)$ under the adjunction relation $\mathtt{Mor}_{\mathcal{O}p}(\Theta(M), P) \xrightarrow{\simeq} \mathtt{Mor}_{\mathcal{C}oll}(M, P)$ and represents the unit morphism of this adjunction therefore. For an operad P, we have a morphism in the converse direction $\lambda : \Theta(P) \to P$ which corresponds to the identity of the object P in the category of collections and which defines the augmentation morphism of our adjunction. We have the following explicit description of this natural morphism:

Proposition A.3.10. *In our realization of the free operad $\Theta(P)$ as a colimit $\Theta(P)(\underline{\mathtt{r}}) = \operatorname{colim}_{\underline{\mathtt{T}} \in \mathcal{T}ree(\underline{\mathtt{r}})^{iso}} P(\underline{\mathtt{T}})$, the adjunction augmentation $\lambda : \Theta(P) \to P$ is given termwise by the treewise composition operations $\lambda_{\underline{\mathtt{T}}} : P(\underline{\mathtt{T}}) \to P(\underline{\mathtt{r}})$ of* §A.2.7.

Proof. To check this proposition, we just apply the construction of §A.3.5 to the morphism $\phi_{id} : \Theta(P) \to P$ induced by the identity $id : P \to P$ and which defines the augmentation morphism of our adjunction $\lambda : \Theta(P) \to P$. ☐

A.3.11. *The definition of restriction operators on the free operad.* We now assume that M is equipped with the structure of an augmented (non-unitary) Λ-collection. We then have a restriction operator

$$u^* : \underbrace{\operatorname*{colim}_{\underline{T}\in\mathcal{T}ree(\underline{n})} M(\underline{T})}_{=\mathbb{\Theta}(M)(\underline{n})} \to \underbrace{\operatorname*{colim}_{\underline{S}\in\mathcal{T}ree(\underline{m})} M(\underline{S})}_{=\mathbb{\Theta}(M)(\underline{m})}, \tag{1}$$

associated to any injective map $u \in \mathtt{Mor}_{\mathcal{I}nj}(\underline{m},\underline{n})$, which we form by putting together the treewise restriction operators $u^* : M(\underline{T}) \to M(u^*\,\underline{T})$ of §A.2.8(1), for all $\underline{n}$-trees $\underline{T} \in \mathcal{T}ree(\underline{n})$, and where we consider our restriction operator on trees $u^* : \underline{T} \mapsto u^*\,\underline{T}$ (see §A.1.11).

We also have an augmentation

$$\epsilon : \underbrace{\operatorname*{colim}_{\underline{T}\in\mathcal{T}ree(\underline{n})} M(\underline{T})}_{=\mathbb{\Theta}(M)(\underline{n})} \to \mathbb{1} \tag{2}$$

which we obtain by putting together the treewise restriction augmentations $\epsilon : M(\underline{T}) \to \mathbb{1}$ of §A.2.8(2). We get the following statement:

Proposition A.3.12 (Claims of Proposition I.2.3.1).

(a) *The definitions of the previous paragraph provide the free operad $\mathbb{\Theta}(M)$ with the structure of an augmented non-unitary Λ-operad. The canonical embedding $\iota : M \to \mathbb{\Theta}(M)$, which we associate to the free operad, defines a morphism of augmented non-unitary Λ-collections in this context, and we can also use this property to characterize our augmented non-unitary Λ-operad structure on the object $\mathbb{\Theta}(M)$.*

(b) *Let $f : M \to P$ be a morphism of augmented non-unitary Λ-collections with values in an augmented non-unitary Λ-operad P. The operad morphism $\phi_f : \mathbb{\Theta}(M) \to P$ associated to f (in* §A.3.5*) preserves the extra Λ-operad structure which we attach to our objects and hence defines a morphism in the category of augmented non-unitary Λ-operads.*

Proof. We easily check that the treewise restriction operations of §A.2.8(1) fit in equivariance relations:

$$\begin{tikzcd}
M(\underline{S}) \otimes M(\underline{T}) \arrow[r, "\simeq"] \arrow[d, "u^*\otimes v^*"'] & M(\underline{S}\circ_{u(i_k)}\underline{T}) \arrow[d, "(u\circ_{u(i_k)}v)^*"] \\
M(u^*\,\underline{S}) \otimes M(v^*\,\underline{T}) \arrow[r, "\simeq"] & M((u\circ_{u(i_k)} v)^*(\underline{S}\circ_{u(i_k)}\underline{T}))
\end{tikzcd}, \tag{1}$$

where we consider the composition isomorphism of §A.3.3(2) and we use the equivariance relation $(u^*\,\underline{S}) \circ_{i_k} (v^*\,\underline{T}) = (u \circ_{u(i_k)} v)^*(\underline{S}\circ_{u(i_k)}\underline{T})$ of the composition of trees, for any $\underline{S} \in \mathcal{T}ree(\underline{m})$, $\underline{T} \in \mathcal{T}ree(\underline{n})$, and $u \in \mathtt{Mor}_{\mathcal{I}nj}(\underline{r},\underline{m})$, $v \in \mathtt{Mor}_{\mathcal{I}nj}(\underline{s},\underline{n})$, $i_k \in \underline{r}$ (see §A.1.11). The tree identity $(u^*\,\underline{S}) \circ_{i_k} (v^*\,\underline{T}) = (u \circ_{u(i_k)} v)^*(\underline{S}\circ_{u(i_k)}\underline{T})$ basically implies that we perform the same factorwise augmentation and restriction operators in the treewise restriction maps of our diagram (1), and this observation immediately gives our relation. We similarly have the degenerate equivariance relations:

(2)
$$\begin{tikzcd}
M(\underline{S}) \otimes M(\underline{T}) \arrow[rr, "\simeq"] \arrow[d, "u^* \otimes \epsilon"'] & & M(\underline{S} \circ_{u(i_k)} \underline{T}) \arrow[d, "(u \circ_{u(i_k)} o)^*"] \\
M(u^* \underline{S}) \otimes \mathbb{1} \arrow[r, "\simeq"] & M(u^* \underline{S}) \arrow[r, "\partial_{i_k}"] & M((u \circ_{u(i_k)} o)^*(\underline{S} \circ_{u(i_k)} \underline{T}))
\end{tikzcd} ,$$

where we combine the treewise restriction operator $u^* : M(\underline{S}) \to M(u^* \underline{S})$ associated to a map $u \in \mathtt{Mor}_{\mathcal{I}nj}(\underline{r}, \underline{m})$ with a treewise augmentation $\epsilon : M(\underline{T}) \to \mathbb{1}$. In this relation, we still write ∂_{i_k} for the restriction operator associated to the map ∂^{i_k} that skips the value i_k in the set $\underline{m}$, and we use the identity $\partial_{i_k}(u^* \underline{S}) = (u \circ_{u(i_k)} o)^*(\underline{S} \circ_{u(i_k)} \underline{T})$ at the tree level (see §A.1.11).

The equivariance of the composition operations of free operads with respect to the restriction operators (see Proposition I.2.2.16) immediately follows from these treewise equivariance relations (1-2).

We trivially get a commutative diagram

(3)
$$\begin{tikzcd}
M(\underline{S}) \otimes M(\underline{T}) \arrow[rr, "\simeq"] \arrow[dr, "\epsilon \otimes \epsilon"'] & & M(\underline{S} \circ_{i_k} \underline{T}) \arrow[dl, "\epsilon"] \\
& \mathbb{1} &
\end{tikzcd}$$

when we take the augmentation of a treewise composition operation. We immediately conclude that the composition operations of free operads preserve our augmentation morphisms too, and this verification completes the proof that the free operad $\mathbb{\Theta}(M)$ inherits the extra structure of an augmented non-unitary Λ-operad when M is an augmented non-unitary Λ-collection.

We also immediately see that the treewise restriction operators (respectively, augmentations) of §A.2.8 reduce to the restriction operators (respectively, to the augmentations) of our collection M in the case of a corolla $\underline{Y} = \underline{Y}_{\underline{r}}$ and when we use the identity $M(\underline{r}) = M(\underline{Y})$, for any arity $r > 0$. We deduce from this observation that the morphism $\iota : M \to \mathbb{\Theta}(M)$ defines a morphism of augmented non-unitary Λ-collections, as asserted in our proposition, since we define this morphism by the aritywise identity $M(\underline{r}) = M(\underline{Y})$, for $r > 0$. We refer to the explanations of Proposition I.2.3.1 for the proof that this property uniquely determines our augmented non-unitary Λ-operad structure on $\mathbb{\Theta}(M)$.

We now consider the treewise composition products $\lambda_{\underline{T}} : P(\underline{T}) \to P(\underline{r})$ associated to an augmented non-unitary Λ-operad P. The result of Proposition A.2.9, where we establish the equivariance of the edge-contraction operations with respect to the action of restriction operators, implies that we have a commutative diagram:

(4)
$$\begin{tikzcd}
P(\underline{T}) \arrow[r, "\lambda_{\underline{T}}"] \arrow[d, "u^*"'] & P(\underline{n}) \arrow[d, "u^*"] \\
P(u^* \underline{T}) \arrow[r, "\lambda_{u^* \underline{T}}"] & P(\underline{m})
\end{tikzcd} ,$$

for any $\underline{\mathsf{n}}$-tree $\underline{\mathsf{T}}$ and for any injective map $u \in \mathtt{Mor}_{\mathcal{I}nj}(\underline{\mathsf{m}}, \underline{\mathsf{n}})$. We similarly have a commutative diagram

(5)
$$\begin{tikzcd} P(\underline{\mathsf{T}}) \arrow[rr, "\lambda_{\underline{\mathsf{T}}}"] \arrow[dr, "\epsilon"'] & & P(\underline{\mathsf{r}}) \arrow[dl, "\epsilon"] \\ & \mathbb{1} & \end{tikzcd} ,$$

for any $\underline{\mathsf{r}}$-tree $\underline{\mathsf{T}}$, when we compose our treewise composition product with the augmentation of the operad. We immediately deduce from these statements that the morphism $\lambda : \Theta(P) \to P$, which we obtain by putting the treewise composition products together, defines a morphism of augmented non-unitary Λ-operads.

Recall that this morphism $\lambda : \Theta(P) \to P$ represents the augmentation of the free operad adjunction. We can therefore deduce the second assertion of the proposition from the above verification. Note simply that our augmented non-unitary Λ-operad structure on the free operad is functorial. ☐

This proposition implies (as we explained in §I.2.3) that the plain free operad functor $\Theta : M \mapsto \Theta(M)$ lifts to a free object functor from the category of augmented non-unitary Λ-collections towards the category of augmented non-unitary Λ-operads (see Theorem I.2.3.2).

A.3.13. *The splitting of the free operad.* Recall that the category of trees and isomorphisms $\mathcal{T}ree(\underline{\mathsf{r}})^{iso}$ is equipped with a splitting $\mathcal{T}ree(\underline{\mathsf{r}})^{iso} = \coprod_{m=0}^{\infty} \mathcal{T}ree_m(\underline{\mathsf{r}})^{iso}$, where $\mathcal{T}ree_m(\underline{\mathsf{r}})^{iso}$ is the full subcategory of $\mathcal{T}ree(\underline{\mathsf{r}})^{iso}$ formed by trees with m vertices. The collection $\Theta(M)$ consequently inherits a decomposition $\Theta(M) = \coprod_{m=0}^{\infty} \Theta_m(M)$ such that:

$$\Theta_m(M)(\underline{\mathsf{r}}) = \operatorname*{colim}_{\underline{\mathsf{T}} \in \mathcal{T}ree_m(\underline{\mathsf{r}})^{iso}} M(\underline{\mathsf{T}}),$$

for any $m \geq 0$.

Recall that $\mathcal{T}ree_0(\underline{\mathsf{r}})$ has only one element in arity $r = 1$, namely the unit tree $\underline{\downarrow}$. Therefore, since $M(\underline{\downarrow}) = \mathbb{1}$, we obtain:

$$\Theta_0(M)(\underline{\mathsf{r}}) = \begin{cases} \mathbb{1}, & \text{if } r = 1, \\ \varnothing, & \text{otherwise.} \end{cases}$$

The unit of the free operad $\iota : \mathbb{1} \to \Theta(M)(1)$ can be identified with the embedding of this summand $\Theta_0(M)(1) = \mathbb{1}$ in $\Theta(M)$. We also immediately see that the composition operations of §A.3.3 preserve our weight grading, and accordingly split into components of the form:

$$\Theta_p(M)(\underline{\mathsf{m}}) \otimes \Theta_q(M)(\underline{\mathsf{n}}) \xrightarrow{\circ_{i_k}} \Theta_{p+q}(M)(\underline{\mathsf{m}} \circ_{i_k} \underline{\mathsf{n}}),$$

for $p, q \geq 0$. (We notably use this observation in the definition of the Koszul dual of operads in §C.3.) Let us observe that the restriction operators of §A.3.11 do not preserve the weight grading of this operad, unlike our other structure morphisms.

Recall that $\mathcal{T}ree_1(\underline{\mathsf{r}})$ is reduced to the isomorphism class of the $\underline{\mathsf{r}}$-corolla $\underline{\mathsf{Y}}$. Thus, we have a canonical isomorphism of collections $M \simeq \Theta_1(M)$. We immediately see that the universal morphism $\iota : M \to \Theta(M)$ is given by the identity between M and this summand $\Theta_1(M)$ of the free operad $\Theta(M)$.

We check the following claim to complete the results of this section:

PROPOSITION A.3.14. *The free operad associated to a non-unitary collection $M \in \mathcal{C}oll_{>0}$ has a reduced expansion of the form:*

$$\Theta(M)(\underline{\mathsf{r}}) = \coprod_{[\underline{\mathsf{T}}] \in \pi_0\, \mathcal{T}ree(\underline{\mathsf{r}})^{iso}} M(\underline{\mathsf{T}}),$$

for each finite (non-empty) set $\underline{\mathsf{r}} \neq \underline{\mathsf{0}}$, where the coproduct ranges over (a set of representatives of) isomorphism classes of the category of $\underline{\mathsf{r}}$-trees $[\underline{\mathsf{T}}] \in \pi_0\, \mathcal{T}ree(\underline{\mathsf{r}})^{iso}$.

The non-unitary assumption $M(0) = \varnothing$ becomes crucial in this statement.

PROOF. We deduce this statement from the next proposition, which implies that the isomorphism category of (open) $\underline{\mathsf{r}}$-trees $\mathcal{T}ree(\underline{\mathsf{r}})^{iso}$ in our construction of the free operad §A.3.1 is equivalent to a discrete category with isomorphism classes of reduced $\underline{\mathsf{r}}$-trees as objects. ☐

PROPOSITION A.3.15. *Let $\underline{\mathsf{r}}$ be any (non-empty) finite set. Let $\underline{\mathsf{S}}$ be any (open) $\underline{\mathsf{r}}$-tree, so that we have $\underline{\mathsf{r}}_v \neq \varnothing$, for all $v \in V(\underline{\mathsf{T}})$. The set of isomorphisms $\mathtt{Mor}_{\mathcal{T}ree(\underline{\mathsf{r}})^{iso}}(\underline{\mathsf{S}}, \underline{\mathsf{T}})$, connecting $\underline{\mathsf{S}}$ to another $\underline{\mathsf{r}}$-tree $\underline{\mathsf{T}}$, is either empty or reduced to a point.*

Recall that this proposition fails for general (non-open) trees (see §A.1.8).

PROOF. Let $f, g : \underline{\mathsf{S}} \to \underline{\mathsf{T}}$ be a pair of parallel $\underline{\mathsf{r}}$-tree isomorphisms.

In §A.1.1, we observe that any vertex $v \in V(\underline{\mathsf{S}})$ in an (open) tree $\underline{\mathsf{S}} \in \mathcal{T}ree(\underline{\mathsf{r}})$ is connected to an input of the tree by a chain of edges

$$i \xrightarrow{e_i} \textcircled{v_n} \xrightarrow{e_{\alpha_n}} \cdots \xrightarrow{e_{\alpha_2}} \textcircled{v_1} \xrightarrow{e_{\alpha_1}} \textcircled{v}. \tag{1}$$

If we have $n = 0$ inner edges in this chain, so that v is directly connected to the input i by an ingoing edge of the tree e_i, then we have $s(f_E(e_i)) = i = s(g_E(e_i)) \Rightarrow f_E(e_i) = g_E(e_i)$ and $f_V(v) = t(f_E(e_i)) = t(g_E(e_i)) = g_V(v)$. In the case where v is connected to i by a chain (1) with $n > 0$ inner edges, we assume by induction that f and g agree up to $x = v_1$ on this chain. We then have $s(f_E(e_{\alpha_1})) = f_V(s(e_{\alpha_1})) = f_V(v_1) = g_V(v_1) = g_V(s(e_{\alpha_1})) = s(g_E(e_{\alpha_1}))$, and these identities imply that $f_E(e_{\alpha_1})$ and $g_E(e_{\alpha_1})$ are identified with the outgoing edge of $f_V(v_1) = g_V(v_1)$ in $\underline{\mathsf{T}}$. We accordingly have $f_E(e_{\alpha_1}) = g_E(e_{\alpha_1}) \Rightarrow f_V(v) = t(f_E(e_{\alpha_1})) = t(g_E(e_{\alpha_1})) = g_V(v)$ and we can therefore continue our induction process to conclude that we have $f_V(v) = g_V(v)$, for all $v \in V(\underline{\mathsf{S}})$. By the way we also check that we have $f_E(e) = g_E(e)$, for all $e \in E(\underline{\mathsf{S}})$, since we any edge in an $\underline{\mathsf{r}}$-tree is identified with the outgoing edge of an input label $i \in \underline{\mathsf{r}}$ or with the outgoing edge of a vertex. ☐

A.3.16. *The case of collections equipped with a free symmetric structure.* In the study of the model category of operads in simplicial sets §II.8, we use another reduced expansion of free operads for (non-unitary) symmetric sequences M whose components $M(r)$ are equipped with a free action of the symmetric group Σ_r, for all $r > 0$. In the context of symmetric collections (in an arbitrary symmetric monoidal base category), we assume that the components of our object M have the form:

$$M(\{i_1, \dots, i_r\}) = \mathtt{Mor}_{\mathcal{B}ij}(\{1, \dots, r\}, \{i_1, \dots, i_r\}) \otimes S\,M(r), \tag{0}$$

for some sequence of objects of the base category $S\,M(r)$, $r > 0$, and where the expression $\mathtt{Mor}_{\mathcal{B}ij}(\{1 < \dots < r\}, \{i_1, \dots, i_r\}) \otimes S\,M(r)$ represents the coproduct of copies of the object $\mathsf{S}\,\mathsf{M}(r)$ over the set of bijective maps $f \in \{1 < \dots < r\} \xrightarrow{\simeq} \{i_1, \dots, i_r\}$, for any finite set $\underline{\mathsf{r}} = \{i_1, \dots, i_r\}$ with $\mathrm{card}(\underline{\mathsf{r}}) = r > 0$.

In the context of a concrete symmetric monoidal category, the factors which we use to label the vertices of a treewise tensor $\pi \in \mathsf{M}(\underline{\mathsf{T}})$ are equivalent to pairs (f, ξ_v), where we have $\xi_v \in \mathsf{S}\,\mathsf{M}(r_v)$, $r_v = \mathrm{card}(\underline{\mathsf{r}}_v)$, and f is a bijection (equivalent to an ordering) from the ordered set $\{1 < \dots < r_v\}$ towards the set of ingoing edges of our vertex $\underline{\mathsf{r}}_v = \{e \in E(\underline{\mathsf{T}}) | t(e) = v\}$. If we go back to the topological interpretation, then fixing such an ordering, for any vertex $v \in V(\underline{\mathsf{T}})$, amounts to fixing a planar embedding of the tree $\underline{\mathsf{T}}$. We therefore consider the category $\mathcal{T}ree^o(\underline{\mathsf{r}})^{iso}$ which consists of the $\underline{\mathsf{r}}$-trees $\underline{\mathsf{T}} = \underline{\mathsf{T}}^o$ equipped with a such a planar embedding (referred to by the superscript o in the expression of our objects) together with the order preserving isomorphisms as morphisms. We also use the phrase '*planar* $\underline{\mathsf{r}}$-*tree*' to refer to the objects $\underline{\mathsf{T}} = \underline{\mathsf{T}}^o$ of this category $\mathcal{T}ree^o(\underline{\mathsf{r}})^{iso}$. We now have $\mathcal{T}ree^o(\underline{\mathsf{r}})^{iso} = \mathtt{Mor}_{\mathcal{B}ij}(\{1, \dots, r\}, \underline{\mathsf{r}}) \otimes \mathcal{S}\mathcal{T}ree^o(r)^{iso}$ where $\mathcal{S}\mathcal{T}ree^o(r)^{iso} \subset \mathcal{T}ree^o(r)^{iso}$ is the subcategory formed by these planar r-trees $\underline{\mathsf{T}}^o \in \mathcal{T}ree^o(r)^{iso}$ of which ingoing edges are linearly ordered according to the global orientation of the plan of our figure (and indexed accordingly).

We therefore have

$$\Theta(\mathsf{M})(\{i_1, \dots, i_r\}) = \mathtt{Mor}_{\mathcal{B}ij}(\{1, \dots, r\}, \{i_1, \dots, i_r\}) \otimes \Theta^o(\mathsf{S}\,\mathsf{M})(r), \tag{1}$$

for any finite set $\underline{\mathsf{r}} = \{i_1, \dots, i_r\}$ with $r = \mathrm{card}(\underline{\mathsf{r}}) > 0$, where $\Theta^o(\mathsf{S}\,\mathsf{M})$ is a (non-symmetric) version of the free operad, formally defined by a colimit

$$\Theta^o(\mathsf{S}\,\mathsf{M})(r) = \operatorname*{colim}_{\underline{\mathsf{T}} \in \mathcal{S}\mathcal{T}ree^o(r)^{iso}} \mathsf{S}\,\mathsf{M}(\underline{\mathsf{T}}^o) \tag{2}$$

of the obvious planar analogue of the treewise tensor products of §A.3.1:

$$\mathsf{S}\,\mathsf{M}(\underline{\mathsf{T}}^o) := \bigotimes_{v \in V(\underline{\mathsf{T}})} \mathsf{S}\,\mathsf{M}(\mathrm{card}(\underline{\mathsf{r}}_v)), \tag{3}$$

for any $r > 0$. We moreover have an analogue of the reduced expansion of Proposition A.3.14:

$$\Theta^o(\mathsf{S}\,\mathsf{M})(r) = \coprod_{[\underline{\mathsf{T}}] \in \pi_0\, \mathcal{S}\mathcal{T}ree^o(r)^{iso}} \mathsf{S}\,\mathsf{M}(\underline{\mathsf{T}}^o). \tag{4}$$

We do not actually need the assumption $\mathsf{M}(0) = 0$ in this case: the results of the formulas (1) and (4) hold without assuming $\mathsf{M}(0) = 0$ when the generating collection M is equipped with a free symmetric structure.

A.4. The construction of connected free operads

In the previous sections, we reviewed the general definition of free operads. But, in many applications, we restrict ourselves to operads P such that $\mathsf{P}(0) = \varnothing$ and $\mathsf{P}(1) = \mathbb{1}$. We then say that P is *connected* as an operad (see §I.1.1.21).

We use the notation $\mathcal{O}p_{\varnothing 1}$ for the full subcategory of the category of operads generated by the connected operads. We also consider the category $\mathcal{C}oll_{>1}$ formed by the collections M such that $\mathsf{M}(0) = \mathsf{M}(1) = \varnothing$. We then say that M is connected as a collection.

The main purpose of this section is to check that the free operad functor admits a restriction to the category of connected operads and that a variant of the adjunction relation considered in the previous section holds in this setting.

To be explicit, to a connected operad P, we now associate the connected collection $\bar{P}$ such that:

$$\bar{P}(\underline{r}) = \begin{cases} \varnothing, & \text{if } r = 0, 1, \\ P(\underline{r}), & \text{otherwise,} \end{cases}$$

and which we form by withdrawing the term of arity one of our object $P(1) = \mathbb{1}$. This connected collection $\bar{P}$ obviously forms a subobject of the operad P in the category of collections. In §I.1.2.13, we also explain that any connected operad P is endowed with a canonical augmentation $\epsilon : P \to I$ with values in the unit operad I (at least when our base category is pointed) and which is simply given by the identity of the unit object in arity one $P(1) = I(1) = \mathbb{1}$. The above collection $\bar{P}$ actually represents the kernel of this augmentation (whenever we can define this morphism). Therefore, we usually refer to this object $\bar{P}$ as the *augmentation ideal* of the operad P.

The mapping $P \to \bar{P}$ gives a natural functor $\bar{\omega} : \mathcal{O}p_{\varnothing 1} \to \mathcal{C}oll_{>1}$ from the category of connected operads to the category of connected collections. We get the following result:

Theorem A.4.1. *The free operad $\Theta(M)$ associated to a connected collection $M \in \mathcal{C}oll_{>1}$ is connected as an operad and the mapping $\Theta : M \mapsto \Theta(M)$ defines a left adjoint of the functor $\bar{\omega} : \mathcal{O}p_{\varnothing 1} \to \mathcal{C}oll_{>1}$ which maps a connected operads $P \in \mathcal{O}p_{\varnothing 1}$ to its augmentation ideal $\bar{P} \in \mathcal{C}oll_{>1}$.*

We establish this theorem after two intermediate statements. Firstly, for a connected collection M, we obviously have $M(\underline{T}) = \varnothing$ when the tree $\underline{T}$ is not reduced in the sense of §A.1.12. Indeed, if $\underline{T}$ is not reduced, then we have at least one vertex $v \in V(\underline{T})$ satisfying $\mathrm{card}(\underline{r}_v) = 0$ or 1, and this implies $M(\underline{r}_v) = \varnothing$. Thus:

Proposition A.4.2. *For a connected collection $M \in \mathcal{C}oll_{>1}$, we have an identity*

$$\Theta(M)(\underline{r}) = \operatorname*{colim}_{\underline{T} \in \widetilde{\mathcal{T}ree}(\underline{r})^{iso}} M(\underline{T}),$$

for each arity $r > 0$, where the colimit ranges over $\widetilde{\mathcal{T}ree}(\underline{r})^{iso}$ (the category of reduced $\underline{r}$-trees and isomorphisms between them). □

In §A.1.12, we observe that the category of reduced trees $\widetilde{\mathcal{T}ree}(\underline{r})$ satisfies $\widetilde{\mathcal{T}ree}(0) = \varnothing$ and $\widetilde{\mathcal{T}ree}(1) = \underline{\downarrow}$. The previous proposition therefore implies that the free operad generated by a connected collection $\Theta(M)$ satisfies $\Theta(M)(0) = \varnothing$, $\Theta(M)(1) = M(\underline{\downarrow}) = \mathbb{1}$, and hence, is connected as an operad. Thus, the free operad gives by restriction a functor $\Theta : \mathcal{C}oll_{>1} \to \mathcal{O}p_{\varnothing 1}$ from the category of connected collections $\mathcal{C}oll_{>1}$ towards the category of connected operads $\mathcal{O}p_{\varnothing 1}$, as claimed in Theorem A.4.1. By the way, we can readily adapt the result of Proposition A.3.14 to get a reduced expansion of the free connected operad $\Theta(M)$, where we replace the colimit of the above formula by a coproduct over (a set of representatives of) isomorphism classes of reduced $\underline{r}$-trees:

$$\Theta(M)(\underline{r}) = \coprod_{[\underline{T}] \in \pi_0 \widetilde{\mathcal{T}ree}(\underline{r})^{iso}} M(\underline{T}).$$

We mostly use a dual version of this expression when we examine the definition of cofree cooperads (see §C.1). Let us observe, besides, that the category of reduced

$\underline{\mathrm{r}}$-trees has a finite skeleton (as opposed to the category of $\underline{\mathrm{r}}$-trees), so that the coproduct occurring in this reduced expansion of free connected operads is finite.

We now have the following claim:

LEMMA A.4.3. *For a connected collection M, which satisfies $M(0) = M(1) = \varnothing$, a morphism $f : M \to N$ is necessarily trivial in arity $r = 0, 1$, and any morphism $f : M \to P$, where P is a connected operad, is equivalent to a morphism $f : M \to \bar{P}$ with values in the augmentation ideal of the operad P.* ☐

From this statement, we immediately get that the adjunction relation of the free operad admits an extension

$$(*) \qquad \mathtt{Mor}_{\mathcal{O}p}(\mathbb{F}(M), P) \simeq \mathtt{Mor}_{\mathcal{C}oll}(M, P) \simeq \mathtt{Mor}_{\mathcal{C}oll}(M, \bar{P})$$

when M is a connected collection and P is a connected operad. Recall that $\mathcal{O}p_{\varnothing 1}$ and $\mathcal{C}oll_{>1}$ are defined as full subcategories of $\mathcal{O}p$ and $\mathcal{C}oll$ respectively. We therefore obtain that $\mathbb{F} : \mathcal{C}oll_{>1} \to \mathcal{O}p_{\varnothing 1}$ is left adjoint of the augmentation ideal functor $\bar{\omega} : \mathcal{O}p_{\varnothing 1} \to \mathcal{C}oll_{>1}$, and this result completes of the proof of Theorem A.4.1. ☐

We adopt the short notation $\bar{\mathbb{F}}(M)$ for the augmentation ideal of the free operad $\mathbb{F}(M)$ associated to a connected collection M, and equivalently, for the composite of the free operad functor $\mathbb{F} : \mathcal{C}oll_{>1} \to \mathcal{O}p_{\varnothing 1}$ with the augmentation ideal functor $\bar{\omega} : \mathcal{O}p_{\varnothing 1} \to \mathcal{C}oll_{>1}$.

The constructions of the previous section can easily be adapted to give a representation of the adjunction bijection $(*)$ in our proof of this theorem. In one direction, to an operad morphism $\phi : \mathbb{F}(M) \to P$, we associate the composite $M \xrightarrow{\iota} \mathbb{F}(M) \xrightarrow{\phi} P$ which reduces to a morphism $\phi\iota : M \to \bar{P}$ when M is connected (Lemma A.4.3). In the other direction, when we have a morphism of collections $f : M \to \bar{P}$, we apply the process of §A.3.5 to the obvious morphism $f : M \to \bar{P} \subset P$, in order to obtain the operad morphism $\phi_f : \mathbb{F}(M) \to P$ naturally associated to f.

The unit and augmentation morphisms of our adjunction $\mathbb{F} : \mathcal{C}oll_{>1} \rightleftarrows \mathcal{O}p_{\varnothing 1} : \bar{\omega}$ can readily be made explicit from the constructions of the previous section too. The adjunction unit is now given by a morphism $\iota : M \to \bar{\mathbb{F}}(M)$, associated to any connected collection $M \in \mathcal{C}oll_{>1}$, and with values in the augmentation ideal of the free operad $\bar{\mathbb{F}}(M)$. This morphism is the obvious restriction of the adjunction unit of §A.3, and hence, is still given by the identity of the object $M(\underline{\mathrm{r}})$ with the term $M(\underline{\mathrm{Y}})$ attached to an $\underline{\mathrm{r}}$-corolla $\underline{\mathrm{Y}}$ in the expansion of $\bar{\mathbb{F}}(M)(\underline{\mathrm{r}})$. We just restrict ourselves to the case $r > 1$ of this correspondence when we work in the connected setting.

The adjunction augmentation is now a morphism $\lambda : \mathbb{F}(\bar{P}) \to P$ which we associate to any connected operad $P \in \mathcal{O}p_{\varnothing 1}$. To make this morphism explicit, we first record the following immediate observation:

OBSERVATION A.4.4. *For a connected operad $P \in \mathcal{O}p_{\varnothing 1}$, we have*

$$\bar{P}(\underline{\mathrm{T}}) = \begin{cases} P(\underline{\mathrm{T}}), & \text{if the tree } \underline{\mathrm{T}} \text{ is reduced,} \\ \varnothing, & \text{otherwise.} \end{cases}$$

From which we deduce:

PROPOSITION A.4.5. *The free operad associated to the augmentation ideal $\bar{P} \in \mathcal{C}oll_{>1}$ of a connected operad $P \in \mathcal{O}p_{\varnothing 1}$ satisfies*

$$\mathbb{F}(\bar{P})(\underline{\mathrm{r}}) = \operatorname*{colim}_{\underline{\mathrm{T}} \in \widetilde{\mathcal{T}ree}(\underline{\mathrm{r}})^{iso}} P(\underline{\mathrm{T}}),$$

for any arity $r > 0$*, and the augmentation* $\lambda : \Theta(\bar{P}) \to P$ *of the adjunction* $\Theta : \mathcal{C}oll_{>1} \rightleftarrows \mathcal{O}p_{\varnothing 1} : \bar{\omega}$ *is given, on each term of this colimit* $P(\underline{\mathsf{T}})$*, by the treewise composition product* $\lambda : P(\underline{\mathsf{T}}) \to P(\underline{\mathsf{r}})$ *of* §A.2.7.

Proof. The first assertion of this proposition is an immediate consequence of observation A.4.4. The augmentation of our new adjunction $\lambda : \Theta(\bar{P}) \to P$ is the morphism $\phi_i : \Theta(\bar{P}) \to P$ associated to the obvious embedding $i : \bar{P} \to P$ under the adjunction relation of the previous section. The expression of this morphism $\lambda : \Theta(\bar{P}) \to P$ in terms of treewise composition products immediately follows from the construction of §A.3.5, where we make explicit the mapping $f \mapsto \phi_f$ for a general morphism of symmetric sequences $f : M \to P$. ☐

A.4.6. *The definition of reduced treewise restriction operators and of restriction operators on connected free operads.* We assume from now on that M is an augmented connected Λ-collection. Let $u \in \mathtt{Mor}_{\mathcal{I}nj}(\underline{\mathsf{m}}, \underline{\mathsf{n}})$. We can easily adapt the construction of §A.2.8 to define a reduced restriction operator

$$u^* : M(\underline{\mathsf{T}}) \to M(u^* \underline{\mathsf{T}}) \tag{1}$$

when we consider the restriction functor on the categories of reduced trees $u^* : \widetilde{\mathcal{T}ree}(\underline{\mathsf{n}}) \to \widetilde{\mathcal{T}ree}(\underline{\mathsf{m}})$ (see §A.1.13) rather than the restriction functor of §A.1.11.

Recall that we essentially determine the reduced tree $u^* \underline{\mathsf{T}} \in \widetilde{\mathcal{T}ree}(\underline{\mathsf{m}})$ associated to any $\underline{\mathsf{T}} \in \widetilde{\mathcal{T}ree}(\underline{\mathsf{n}})$ by removing an extra subset of vertices (namely, the vertices with a single ingoing edge) in the outcome of our plain reduction process on trees (see §A.1.13). We basically apply the augmentation $\epsilon : M(\underline{\mathsf{r}}) \to \mathbb{1}$ to delete these extra factors (associated to the vertices with a single ingoing edge) in the result of the restriction operator of §A.2.8 on treewise tensors. We do not touch the other factors, which we obtain by performing an appropriate internal restriction operator of our Λ-collection M.

We give an example of application of this reduced restriction process in Figure A.10. We keep the same tree shape as in §§A.1.11-A.1.13, where we explain the definition of restriction operators on trees, and as in §A.2.8, where we explain the definition of the plain (unreduced) restriction operator on treewise tensors. Recall that we also used this example of a treewise tensor product in §I.2.3, when we outline the definition of a Λ-operad structure on the free operad associated to an augmented Λ-sequence (see Proposition I.2.3.1), and in §I.2.4, when we outline the definition of a Λ-operad structure on the free operad associated to an augmented connected Λ-sequence (see Proposition I.2.4.3). We just retrieve the result of the latter informal construction in Figure A.10.

We put these reduced treewise restriction operators (1) together (as in §A.3.11) in order to get a (reduced) restriction operator associated to any injective map $u \in \mathtt{Mor}_{\mathcal{I}nj}(\underline{\mathsf{m}}, \underline{\mathsf{n}})$ on the connected free operad $\Theta(M)$:

$$\underbrace{\operatorname*{colim}_{\underline{\mathsf{T}} \in \widetilde{\mathcal{T}ree}(\underline{\mathsf{n}})} M(\underline{\mathsf{T}})}_{=\Theta(M)(\underline{\mathsf{n}})} \xrightarrow{u^*} \underbrace{\operatorname*{colim}_{\underline{\mathsf{S}} \in \widetilde{\mathcal{T}ree}(\underline{\mathsf{m}})} M(\underline{\mathsf{S}})}_{=\Theta(M)(\underline{\mathsf{m}})} . \tag{2}$$

We also have an augmentation

$$\epsilon : \underbrace{\operatorname*{colim}_{\underline{\mathsf{T}} \in \widetilde{\mathcal{T}ree}(\underline{\mathsf{r}})} M(\underline{\mathsf{T}})}_{=\Theta(M)(\underline{\mathsf{r}})} \to \mathbb{1}, \tag{3}$$

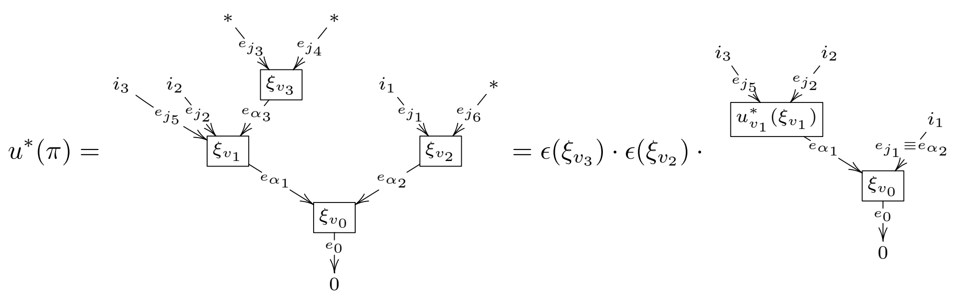


FIGURE A.10. The evaluation of a reduced restriction operator on a treewise tensor $\pi \in M(\underline{\mathrm{T}})$. The outcome of this restriction operator $u^*(\pi) \in M(u^* \underline{\mathrm{T}})$ is shaped on the reduced tree $u^* \underline{\mathrm{T}}$ determined in Figure A.5. The internal restriction operator, which we apply to one factor of this tensor product, is associated to the embedding $u_{v_1} : \{e_{j_5}, e_{j_2}\} \to \{e_{j_5}, e_{j_2}, e_{\alpha_3}\}$ that define the subset of ingoing edges of the vertex v_1 which we keep in the reduced tree $u^* \underline{\mathrm{T}}$.

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which we obtain by putting together the treewise augmentations $\epsilon : M(\underline{\mathrm{T}}) \to \mathbb{1}$ of §A.2.8. (Recall that this treewise augmentation is given by a factorwise application of the augmentation of the object M, and this process can be handled without change in the connected setting.)

We then get the following statement:

PROPOSITION A.4.7 (Claims of Proposition I.2.4.3).

(a) *The definitions of the previous paragraph provide the free connected operad* $\Theta(M)$ *with the structure of an augmented connected* Λ*-operad. The canonical morphism* $\iota : M \to \bar{\Theta}(M)$*, which we associate to the free connected operad, defines a morphism of augmented connected* Λ*-collections in this context, and we can also use this property to characterize the* Λ*-operad structure which we define on the object* $\Theta(M)$*.*

(b) *Let* $f : M \to \bar{P}$ *be a morphism of augmented connected* Λ*-collections with values in (the augmentation ideal of) an augmented connected* Λ*-operad* P*. The operad morphism* $\phi_f : \Theta(M) \to P$ *associated to* f *(in* §A.3.5*) preserves the extra* Λ*-operad structure which we attach to our objects and hence defines a morphism in the category of augmented connected* Λ*-operads.*

PROOF. We use the same verifications as in the proof of Proposition A.3.12. We just replace the plain restriction operators of §A.2.8 by reduced restriction operators. We easily check that our arguments remain valid in this setting. ☐

This proposition also implies (as we explain in §I.2.4) that the plain free operad functor on the category of connected symmetric collections $\Theta : M \mapsto \Theta(M)$ lifts to a free object functor from the category of augmented connected Λ-collections towards the category of augmented connected Λ-operads (see Theorem I.2.4.4).

A.5. The construction of coproducts with free operads

To complete the survey of this chapter, we give an explicit construction of coproducts of the form $\mathsf{P} \vee \Theta(\mathsf{M})$ in the category of operads. We are more precisely going to explain that these objects $\mathsf{P} \vee \Theta(\mathsf{M})$ have an expansion involving a colored version of the treewise tensor products of the previous sections.

In a first step, we address a general version of this construction, which works for any (non-unitary) operad P and for any (non-unitary) collection M. We focus on the case of connected operads in a second step. We check that we can use a reduced version of our coproduct construction when we work in the connected setting.

We will explain that the reduced construction of coproducts which we give in this section can be dualized to give an explicit definition of products in the category of cooperads. We actually crucially use this dual product construction when we define our model structure on the category cooperads. We may therefore regard this section as a preparation for this subsequent definition of a model structure on the category of cooperads.

We first make explicit the colored tree structures which define the shape of our operadic coproducts.

A.5.1. *Semi-alternate two-colored trees.* We consider trees $\underline{\mathsf{T}}$ whose set of vertices is equipped with a partition $V(\underline{\mathsf{T}}) = V_{\bullet}(\underline{\mathsf{T}}) \amalg V_{\circ}(\underline{\mathsf{T}})$ such that $V_{\bullet}(\underline{\mathsf{T}})$ defines a subset of vertices marked with a grey color, and $V_{\circ}(\underline{\mathsf{T}})$ defines a subset of vertices marked with a white color. We can equivalently assume that the set of vertices of our tree $\underline{\mathsf{T}}$ is equipped with a mapping $\underline{\mathsf{c}} : V(\underline{\mathsf{T}}) \to \{\bullet, \circ\}$ to define this coloring:

$$V_{\bullet}(\underline{\mathsf{T}}) = \underline{\mathsf{c}}^{-1}(\bullet), \quad V_{\circ}(\underline{\mathsf{T}}) = \underline{\mathsf{c}}^{-1}(\circ).$$

We generally specify such a vertex coloring $\underline{\mathsf{c}} : V(\underline{\mathsf{T}}) \to \{\bullet, \circ\}$ by adding a subscript $\underline{\mathsf{c}}$ in the notation of our tree $\underline{\mathsf{T}}$. We now say that $\underline{\mathsf{T}}_{\underline{\mathsf{c}}}$ forms a *semi-alternate two-colored tree* when:

(1) For any inner edge $e \in \mathring{E}(\underline{\mathsf{T}})$ with $v = s(e) \in V(\underline{\mathsf{T}})$ and $u = t(e) \in V(\underline{\mathsf{T}})$, we have either $(\underline{\mathsf{c}}(u), \underline{\mathsf{c}}(v)) = (\bullet, \circ)$, or $(\underline{\mathsf{c}}(u), \underline{\mathsf{c}}(v)) = (\circ, \bullet)$, or $(\underline{\mathsf{c}}(u), \underline{\mathsf{c}}(v)) = (\circ, \circ)$, but in all cases $(\underline{\mathsf{c}}(u), \underline{\mathsf{c}}(v)) \neq (\bullet, \bullet)$.

Thus, the white vertices can form non-trivial subtrees in $\underline{\mathsf{T}}_{\underline{\mathsf{c}}}$ but the grey vertices are all isolated.

We give an example of a semi-alternate two-colored structure, shaped on the tree of Figure A.1, in Figure A.11.

We adopt the notation $\mathcal{T}ree_{\bullet\circ}(\underline{\mathsf{r}})$ for the class of semi-alternate two-colored $\underline{\mathsf{r}}$-trees. We also consider isomorphisms of semi-alternate two-colored $\underline{\mathsf{r}}$-trees $f : \underline{\mathsf{S}}_{\underline{\mathsf{c}}} \xrightarrow{\simeq} \underline{\mathsf{T}}_{\underline{\mathsf{d}}}$ which we obviously define as isomorphisms of $\underline{\mathsf{r}}$-trees $f : \underline{\mathsf{S}} \xrightarrow{\simeq} \underline{\mathsf{T}}$ (in the sense of §A.1.8) that preserve the color of vertices:

$$v \in V_{\bullet}(\underline{\mathsf{S}}) \Rightarrow f_V(v) \in V_{\bullet}(\underline{\mathsf{T}}), \quad v \in V_{\circ}(\underline{\mathsf{S}}) \Rightarrow f_V(v) \in V_{\circ}(\underline{\mathsf{T}}).$$

We still use the superscript mark *iso* to denote the category $\mathcal{T}ree_{\bullet\circ}(\underline{\mathsf{r}})^{iso}$ formed by the semi-alternate two-colored $\underline{\mathsf{r}}$-trees and their isomorphisms.

A.5.2. *Semi-alternate treewise tensor products.* Let P be an operad. Let M be a symmetric collection. For any semi-alternate two-colored $\underline{\mathsf{r}}$-tree $\underline{\mathsf{T}}_{\underline{\mathsf{c}}}$ we form the treewise tensor product:

$$\mathsf{M}(\underline{\mathsf{T}}_{\underline{\mathsf{c}}} \mid \mathsf{P}) = \Big(\bigotimes_{v \in V_{\bullet}(\underline{\mathsf{T}})} \mathsf{P}(\underline{\mathsf{r}}_v) \Big) \otimes \Big(\bigotimes_{v \in V_{\circ}(\underline{\mathsf{T}})} \mathsf{M}(\underline{\mathsf{r}}_v) \Big), \tag{1}$$

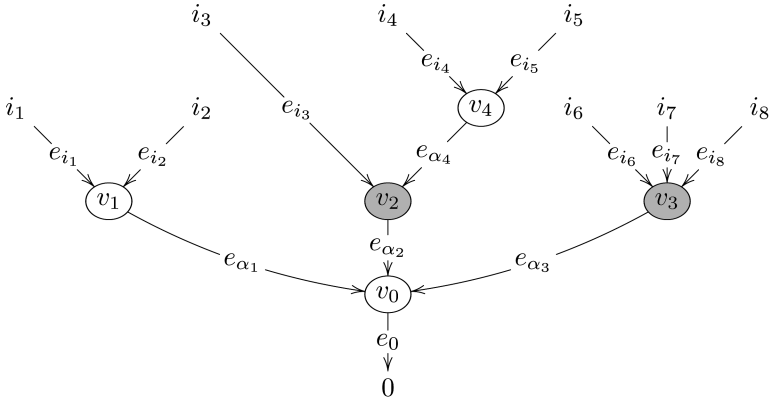

FIGURE A.11. The picture of a semi-alternate two-colored structure on a tree.

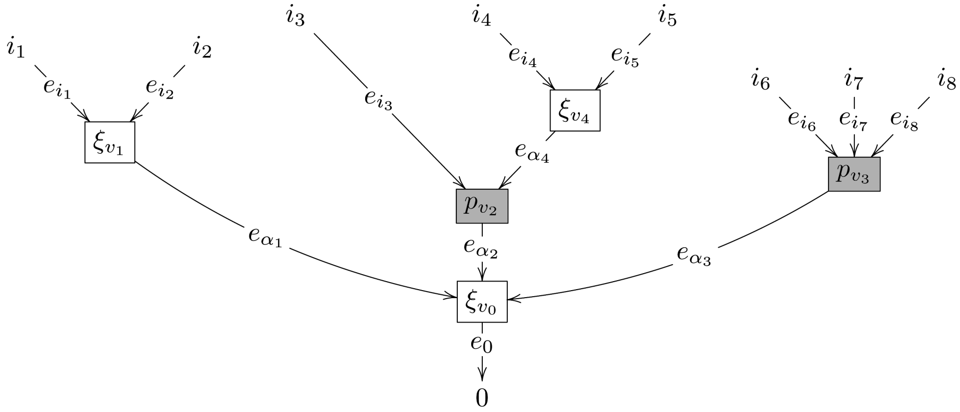

FIGURE A.12. The picture of a semi-alternate treewise tensor. The grey vertices v_2 and v_3 are labeled by operations $p_{v_2} \in \mathsf{P}(2)$ and $p_{v_3} \in \mathsf{P}(3)$ of the operad P, while the white vertices v_0, v_1 and v_4 are labeled by elements $\xi_{v_0} \in \mathsf{M}(3)$, $\xi_{v_1} \in \mathsf{M}(2)$, $\xi_{v_4} \in \mathsf{M}(2)$ of the given symmetric collection M.

where we label the grey vertices $v \in V_{\bullet}(\underline{\mathrm{T}})$ with terms of the operad P and the white vertices $v \in V_{\circ}(\underline{\mathrm{T}})$ with terms of the collection M.

The picture of a semi-alternate treewise tensor $\pi \in \mathsf{M}(\underline{\mathrm{T}}_{\underline{\mathrm{c}}} \mid \mathsf{P})$ is given in Figure A.12. The idea is that such a tensor $\pi \in \mathsf{M}(\underline{\mathrm{T}}_{\underline{\mathrm{c}}} \mid \mathsf{P})$ represents the reduced form of a formal composite of operations $p_v \in \mathsf{P}(\underline{\mathrm{r}}_v)$ with elements of our collection $\xi_v \in \mathsf{M}(\underline{\mathrm{r}}_v)$. We just perform composition operations $p_u \otimes p_v \mapsto p_u \circ_e p_v$ to remove possible adjacent operad factors $p_u, p_v \in \mathsf{P}$ (see Figure A.14 for an instance of application of this reduction process).

In order to complete this reduction process, we still have to keep track of degenerate terms given by the insertion of operadic units in semi-alternate treewise

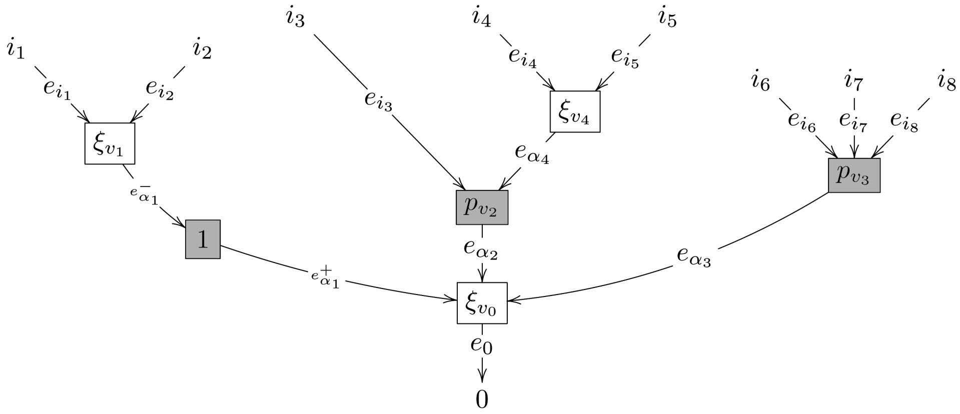

FIGURE A.13. The degenerate semi-alternate treewise tensor obtained by the insertion of an operadic unit $1 \in P(1)$ on the edge e_{α_1} of the treewise tensor of Figure A.12.

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tensor products (see Figure A.13 for an instance of such a degeneration operation). We address this topic in the next paragraph.

A.5.3. *The degeneration of semi-alternate treewise tensors.* We first define degeneration operations on trees which we use to shape our unit insertions. We assume that $e \in E(\underline{\mathsf{T}})$ is an edge satisfying $s(e), t(e) \notin V_{\bullet}(\underline{\mathsf{T}})$ (with possibly $s(e) \in \underline{\mathsf{r}}$ or $t(e) = 0$) in a semi-alternate two-colored $\underline{\mathsf{r}}$-tree $\underline{\mathsf{T}}_{\underline{\mathsf{c}}} \in \mathcal{T}ree_{\bullet\circ}(\underline{\mathsf{r}})$. We then consider a tree $s_e(\underline{\mathsf{T}})_{\underline{\mathsf{c}}}$ formed by inserting a grey vertex v_e on the edge $e \in E(\underline{\mathsf{T}})$. We formally set $V(s_e(\underline{\mathsf{T}})) = V(\underline{\mathsf{T}}) \amalg \{v_e\}$, $E(s_e(\underline{\mathsf{T}})) = E(\underline{\mathsf{T}}) \setminus \{e\} \amalg \{e^-, e^+\}$, with new edges $e^-, e^+ \in E(s_e(\underline{\mathsf{T}}))$, defined by splitting the edge $e \in E(\underline{\mathsf{T}})$ and such that we have $s(e^-) = s(e)$, $t(e^-) = s(e^+) = v_e$, $t(e^+) = t(e)$ in the tree $s_e(\underline{\mathsf{T}})$. We also assign the color $\underline{\mathsf{c}}(v_e) = \bullet$, as required, to the vertex v_e which we insert on the edge e. We now have

$$M(s_e(\underline{\mathsf{T}})_{\underline{\mathsf{c}}} \mid P) = P(\underline{\mathsf{r}}_{v_e}) \otimes \underbrace{\Big(\bigotimes_{v \in V_{\bullet}(\underline{\mathsf{T}})} P(\underline{\mathsf{r}}_v)\Big) \otimes \Big(\bigotimes_{v \in V_{\circ}(\underline{\mathsf{T}})} M(\underline{\mathsf{r}}_v)\Big)}_{= M(\underline{\mathsf{T}}_{\underline{\mathsf{c}}} \mid P)} \tag{1}$$

and we define the degeneration morphism (at the edge $e \in E(\underline{\mathsf{T}})$)

$$s_e : M(\underline{\mathsf{T}}_{\underline{\mathsf{c}}} \mid P) \to M(s_e(\underline{\mathsf{T}})_{\underline{\mathsf{c}}} \mid P) \tag{2}$$

by the insertion of a unit morphism $\eta : \mathbb{1} \to P(1)$ on the extra factor $P(\underline{\mathsf{r}}_{v_e}) = P(1)$ of this tensor product (1)

A.5.4. *The underlying collection of the coproduct with a free operads.* We see that the semi-alternate treewise tensor product construction of §A.5.2, as well as the degeneration operations of §A.5.3, are functorial with respect to the action of the isomorphisms of the groupoids $\mathcal{T}ree_{\bullet\circ}(\underline{\mathsf{r}})^{iso}$. We moreover have an identity $M(u_* \underline{\mathsf{T}}_{\underline{\mathsf{c}}} \mid P) = M(\underline{\mathsf{T}}_{\underline{\mathsf{c}}} \mid P)$ when we apply an input reindexing operation $u_* : \mathcal{T}ree(\underline{\mathsf{r}}) \to \mathcal{T}ree(\underline{\mathsf{s}})$ to any semi-alternate two-colored $\underline{\mathsf{r}}$-tree $\underline{\mathsf{T}}_{\underline{\mathsf{c}}} \in \mathcal{T}ree_{\bullet\circ}(\underline{\mathsf{r}})$.

Let $\mathcal{T}ree_{\bullet\circ}(\underline{\mathsf{r}})^{iso}_{\mathsf{T}}$ be the category obtained by adding formal degeneracy operators $s_e : \underline{\mathsf{T}}_{\underline{\mathsf{c}}} \to s_e(\underline{\mathsf{T}})_{\underline{\mathsf{c}}}$ to the isomorphisms of semi-alternate two-colored $\underline{\mathsf{r}}$-trees.

We set:

$$P \vee \Theta(M)(\underline{\mathsf{r}}) = \operatorname*{colim}_{\underline{\mathsf{T}}_{\underline{\mathsf{c}}} \in \mathcal{T}ree_{\bullet\circ}(\underline{\mathsf{r}})^{iso}_{\mathsf{T}}} M(\underline{\mathsf{T}}_{\underline{\mathsf{c}}} \shortmid P), \tag{1}$$

for any $r > 0$. We mod out by the action of the degeneracy morphisms §A.5.3 and by the action of the isomorphisms of semi-alternate two-colored $\underline{\mathsf{r}}$-trees on semi-alternate treewise tensors when we perform this colimit construction. We aim to check that the objects, which we define by this formula (1) represent the components of the universal operad $P \vee \Theta(M)$ associated to P and $\Theta(M)$.

We have an obvious symmetric structure on our objects given by the termwise action of the input reindexing bijections on trees. We moreover have a natural morphism $i : P(\underline{\mathsf{r}}) \to P \vee \Theta(M)(\underline{\mathsf{r}})$, for any $r > 0$, which identifies the object $P(\underline{\mathsf{r}})$ with the term of the colimit (1) associated to the $\underline{\mathsf{r}}$-corolla $\underline{\mathsf{Y}} = \underline{\mathsf{Y}}_{\underline{\mathsf{r}}}$, where we take a single vertex $v \in V(\underline{\mathsf{Y}})$ colored in grey $\underline{\mathsf{c}}(v) = \bullet$. We similarly have a morphism $j : \Theta(M)(\underline{\mathsf{r}}) \to P \vee \Theta(M)(\underline{\mathsf{r}})$ which we form by identifying the treewise tensor products of the free operad $\Theta(M)$ with the terms of the expansion (1) of which all vertices are colored in white.

We also have an obvious unit morphism $\eta : \mathbb{1} \to P \vee \Theta(M)(1)$ which, as in the case of free operads, identifies the unit object $\mathbb{1}$ with the term of the colimit (1) associated to the unit tree $\underline{\downarrow}$. We define an operadic composition on alternate treewise tensor products in the next paragraph. We then get that the object $P \vee \Theta(M)$ which we define by the above formula (1), inherits a natural operad structure. We check afterwards that this object satisfies the universal property of coproducts in the category of operads.

A.5.5. *The composition of semi-alternate treewise tensors.* We adapt the definition of the operadic composition of treewise tensors in A.3.2.

We consider the composite of semi-alternate two-colored trees $\underline{\mathsf{S}}_{\underline{\mathsf{c}}} \in \mathcal{T}ree_{\bullet\circ}(\underline{\mathsf{m}})$ and $\underline{\mathsf{T}}_{\underline{\mathsf{d}}} \in \mathcal{T}ree_{\bullet\circ}(\underline{\mathsf{n}})$ at some composition index $i_k \in \underline{\mathsf{m}}$. We assume $\underline{\mathsf{S}}, \underline{\mathsf{T}} \neq \underline{\downarrow}$. Let e_{i_k} be the ingoing edge associated to the index $i_k \in \underline{\mathsf{m}}$ in the tree $\underline{\mathsf{S}}$. Let $u = t(e_{i_k})$. Let f_0 be the outgoing edge of the tree $\underline{\mathsf{T}}$. Let $v = s(f_0)$. The vertex set $V(\underline{\mathsf{S}} \circ_{i_k} \underline{\mathsf{T}}) = V(\underline{\mathsf{S}}) \amalg V(\underline{\mathsf{T}})$ of the composite tree $\underline{\mathsf{S}} \circ_{i_k} \underline{\mathsf{T}} \in \mathcal{T}ree(\underline{\mathsf{m}} \circ_{i_k} \underline{\mathsf{n}})$ inherits an obvious coloring such that $V_{\bullet}(\underline{\mathsf{S}} \circ_{i_k} \underline{\mathsf{T}}) = V_{\bullet}(\underline{\mathsf{S}}) \amalg V_{\bullet}(\underline{\mathsf{T}})$ and $V_{\circ}(\underline{\mathsf{S}} \circ_{i_k} \underline{\mathsf{T}}) = V_{\circ}(\underline{\mathsf{S}}) \amalg V_{\circ}(\underline{\mathsf{T}})$, but if we have $\underline{\mathsf{c}}(u) = \bullet$ in $\underline{\mathsf{S}}_{\underline{\mathsf{c}}}$ and $\underline{\mathsf{d}}(v) = \bullet$ in $\underline{\mathsf{T}}_{\underline{\mathsf{d}}}$, then the merging operation $e_{i_k} \equiv f_0$ produces an edge with two adjacent grey vertices (u, v) in the outcome of our treewise composition process. If so, then we just contract this edge in order to restore the alternation condition §A.5.1(1). More formally, we consider a reduced composition product on semi-alternate two-colored trees which we define by:

$$\underline{\mathsf{S}}_{\underline{\mathsf{c}}} \mathbin{\tilde{\circ}_{i_k}} \underline{\mathsf{T}}_{\underline{\mathsf{d}}} = \begin{cases} \underline{\mathsf{S}} \circ_{i_k} \underline{\mathsf{T}} / e_{i_k}, & \text{if the (unreduced) composition operation } \underline{\mathsf{S}} \circ_{i_k} \underline{\mathsf{T}} \\ & \text{produces an edge } e_{i_k} \equiv f_0 \\ & \text{that goes from a grey vertex } s(f_0) \in V_{\bullet}(\underline{\mathsf{T}}) \\ & \text{towards another grey vertex } t(e_{i_k}) \in V_{\bullet}(\underline{\mathsf{S}}), \\ \underline{\mathsf{S}} \circ_{i_k} \underline{\mathsf{T}}, & \text{otherwise.} \end{cases} \tag{1}$$

We now have a natural morphism

$$M(\underline{\mathsf{S}}_{\underline{\mathsf{c}}} \shortmid P) \otimes M(\underline{\mathsf{T}}_{\underline{\mathsf{d}}} \shortmid P) \xrightarrow{\circ_{i_k}} M(\underline{\mathsf{S}}_{\underline{\mathsf{c}}} \mathbin{\tilde{\circ}_{i_k}} \underline{\mathsf{T}}_{\underline{\mathsf{d}}} \shortmid P) \tag{2}$$

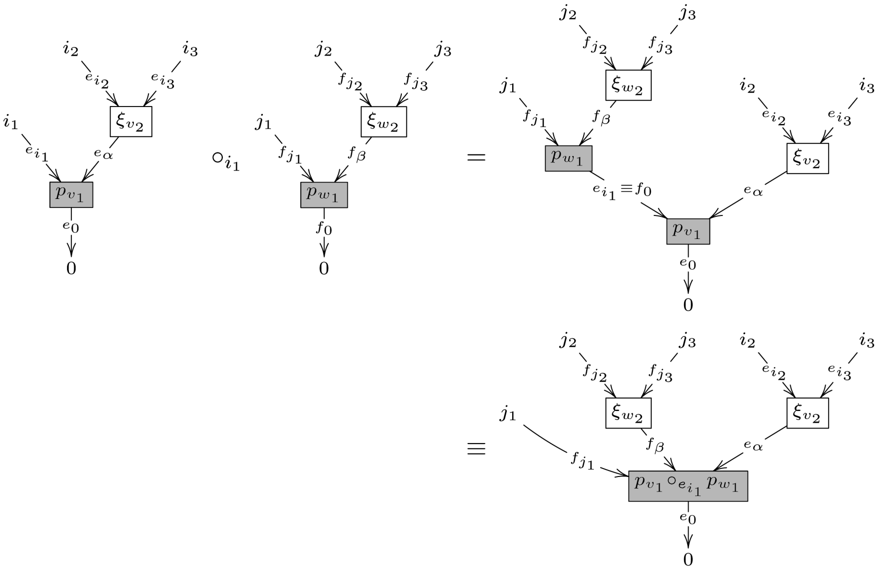

FIGURE A.14. The operadic composition of semi-alternate treewise tensors.

given by the obvious two-colored extension of the isomorphism of observation A.3.2

$$M(\underline{\mathsf{S}}_{\underline{\mathsf{c}}} \shortmid P) \otimes M(\underline{\mathsf{T}}_{\underline{\mathsf{d}}} \shortmid P) \xrightarrow{\simeq} M(\underline{\mathsf{S}}_{\underline{\mathsf{c}}} \circ_{i_k} \underline{\mathsf{T}}_{\underline{\mathsf{d}}} \shortmid P) \tag{3}$$

in the case $\underline{\mathsf{S}}_{\underline{\mathsf{c}}} \tilde{\circ}_{i_k} \underline{\mathsf{T}}_{\underline{\mathsf{d}}} = \underline{\mathsf{S}} \circ_{i_k} \underline{\mathsf{T}}$, and by the composite of this isomorphism with the edge-contraction operation

$$M(\underline{\mathsf{S}}_{\underline{\mathsf{c}}} \circ_{i_k} \underline{\mathsf{T}}_{\underline{\mathsf{d}}} \shortmid P) \xrightarrow{\lambda_{e_{i_k}}} M(\underline{\mathsf{S}}_{\underline{\mathsf{c}}} \circ_{i_k} \underline{\mathsf{T}}_{\underline{\mathsf{d}}} / e_{i_k} \shortmid P) \tag{4}$$

(defined by performing the composite of the adjacent operad factors) in the case $\underline{\mathsf{S}}_{\underline{\mathsf{c}}} \tilde{\circ}_{i_k} \underline{\mathsf{T}}_{\underline{\mathsf{d}}} = \underline{\mathsf{S}} \circ_{i_k} \underline{\mathsf{T}} / e_{i_k}$. We give an example to illustrate this reduced composition process in Figure A.14.

These reduced composition operations are clearly functorial with respect to the action of isomorphisms of semi-alternate two-colored trees. The unit relations of operads imply that our reduced composition operations commute with the degeneracy operations of §A.5.3 as well. We therefore have composition operations

$$\underbrace{\operatorname*{colim}_{\underline{\mathsf{S}}_{\underline{\mathsf{c}}} \in \mathcal{T}ree_{\bullet\circ}(\underline{\mathsf{m}})^{iso}_{\mathsf{T}}} M(\underline{\mathsf{S}}_{\underline{\mathsf{c}}} \shortmid P)}_{= P \vee \Theta(M)(\underline{\mathsf{m}})} \otimes \underbrace{\operatorname*{colim}_{\underline{\mathsf{T}}_{\underline{\mathsf{d}}} \in \mathcal{T}ree_{\bullet\circ}(\underline{\mathsf{n}})^{iso}_{\mathsf{T}}} M(\underline{\mathsf{T}}_{\underline{\mathsf{d}}} \shortmid P)}_{= P \vee \Theta(M)(\underline{\mathsf{n}})} \xrightarrow{\circ_{i_k}} \underbrace{\operatorname*{colim}_{\underline{\Theta}_{\underline{\mathsf{u}}} \in \mathcal{T}ree_{\bullet\circ}(\underline{\mathsf{m}} \circ_{i_k} \underline{\mathsf{n}})^{iso}_{\mathsf{T}}} M(\underline{\Theta}_{\underline{\mathsf{u}}} \shortmid P)}_{= P \vee \Theta(M)(\underline{\mathsf{m}} \circ_{i_k} \underline{\mathsf{n}})}, \tag{5}$$

which we define by putting together the reduced treewise composition operations (2) on our colimit.

We readily check that:

PROPOSITION A.5.6. *The above composition products* §A.5.5*(*5*):*

- *preserve the action of bijections of finite sets on our objects,*
- *satisfy the unit axioms of operads, expressed by the diagrams of Figure* I.2.1, *with the morphism* $\eta : \mathbb{1} \to P \vee \Theta(M)(1)$ *defined in* §A.5.4 *as operadic unit,*
- *and satisfy the associativity axioms, expressed by the diagrams of Figure* I.2.2-2.3, *similarly.*

Thus, we have a well-defined operad structure on $P \vee \Theta(M)$.

Proof. This proposition follows from a straightforward extension of the verifications of Proposition A.3.4 where we check the validity of the definition of our composition structure on free operads. We essentially use that the composition products of §A.5.5 are defined as quotients of (a two-colored version of) the treewise composition operations of free operads §A.3.3, and we rely on the equivariance, associativity, and unit relations of the composition products of the operad P to check that the equivariance, associativity, and unit relations of our treewise composition operations remain satisfied when we pass to this quotient. ☐

We easily see that the morphism $i : P \to P \vee \Theta(M)$ in §A.5.4 preserves composition structures (and defines a morphism of operads therefore), because the treewise composition operations of §A.5.5 reduce to the internal composition products of the operad P on the terms $P(\underline{r}) = P(\underline{Y})$ associated to grey colored corollas $\underline{Y} = \underline{Y}_{\underline{r}}$ in the expression of our object $P \vee \Theta(M)$.

We similarly check that our second morphism $j : \Theta(M) \to P \vee \Theta(M)$ in §A.5.4 is a morphism of operads. We use in this case that the treewise composition operations of §A.5.5 reduce to the treewise composition operations of the free operad on the terms $M(\underline{T})$ associated to monochrome white trees $\underline{T}$.

We have the following statement:

Theorem A.5.7. *The operad* $P \vee \Theta(M)$, *such as defined in* §§A.5.4-A.5.5 *and in Proposition* A.5.6, *does satisfy the universal property of a coproduct: for any pair of morphisms* $\phi : P \to Q$ *and* $\psi : \Theta(M) \to Q$, *we have a unique morphism* $(\phi, \psi) : P \vee \Theta(M) \to Q$ *which makes commute the diagram*

$$\begin{tikzcd}
P \arrow[r, "i"] \arrow[dr, "\phi"'] & P \vee \Theta(M) \arrow[d, dotted, "\exists!(\phi,\psi)"] & \Theta(M) \arrow[l, "j"'] \arrow[dl, "\psi"] \\
 & Q &
\end{tikzcd}$$

in the category of operads.

Proof. We use a two-colored generalization of the construction of morphisms on free operads (see §A.3.5). We have a natural morphism of treewise tensors

$$\underbrace{\Big(\bigotimes_{v\in V_\bullet(\underline{T})} P(\underline{r}_v)\Big)\otimes\Big(\bigotimes_{v\in V_\circ(\underline{T})} M(\underline{r}_v)\Big)}_{=M(\underline{T}_{\underline{c}}|P)} \xrightarrow{(\phi_*,\psi_*)} \underbrace{\Big(\bigotimes_{v\in V_\bullet(\underline{T})} Q(\underline{r}_v)\Big)\otimes\Big(\bigotimes_{v\in V_\circ(\underline{T})} Q(\underline{r}_v)\Big)}_{=Q(\underline{T})}, \tag{1}$$

for each semi-alternate two-colored tree $\underline{T} \in \mathcal{T}ree_{\bullet\circ}(\underline{r})$, which we form by taking the tensor product of the maps $\phi : P(\underline{r}_v) \to Q(\underline{r}_v)$ on the factors $P(\underline{r}_v)$ associated to grey vertices $v \in V_\bullet(\underline{T})$ together with the tensor product of the maps $f = \psi|_M : M(\underline{r}_v) \to Q(\underline{r}_v)$ on the factors $M(\underline{r}_v)$ associated to white vertices $v \in V_\circ(\underline{T})$. We compose these morphisms with the treewise composition products of the operad Q. We then get a morphism

$$M(\underline{\mathsf{T}}_{\underline{\mathsf{c}}} \shortmid P) \to Q(\underline{\mathsf{T}}) \to Q(\underline{\mathsf{r}}), \tag{2}$$

defined for each semi-alternate two-colored $\underline{\mathsf{r}}$-trees $\underline{\mathsf{T}} \in \mathcal{T}ree_{\bullet\circ}(\underline{\mathsf{r}})$, and which is clearly natural with respect to the action of isomorphisms. We easily check, by using the unit relations of operads in Q, that these morphisms are functorial with respect to the degeneracy operators of §A.5.3 too. We can therefore put these morphisms together to get a morphism

$$\underbrace{\operatorname{colim}_{\underline{\mathsf{T}}_{\underline{\mathsf{c}}} \in \mathcal{T}ree_{\bullet\circ}(\underline{\mathsf{r}})^{iso}_{\mathsf{T}}} M(\underline{\mathsf{T}}_{\underline{\mathsf{c}}} \shortmid P)}_{= P \vee \Theta(M)(\underline{\mathsf{r}})} \xrightarrow{(\phi,\psi)} Q(\underline{\mathsf{r}}) \tag{3}$$

associated to the pair (ϕ, ψ) and defined on each component of our operad $P \vee \Theta(M)$.

We then use a straightforward extension of the arguments of Proposition A.3.6 to check that this construction returns a well-defined operad morphism $(\phi, \psi) : P \vee \Theta(M) \to Q$. We still rely on the result of Theorem A.2.6 (the coherence of the edge-contraction process) to establish this claim. We just use that $\phi : P \to Q$ preserves the composition products of our operads in order to check that our morphism $(\phi, \psi) : P \vee \Theta(M) \to Q$ carries the reduction operation of a semi-alternate treewise composition product §A.5.5(4) to an appropriate composition operation in the operad Q. We clearly have the relations $(\phi, \psi)i = \phi$ and $(\phi, \psi)j = \psi$ too, and we use the same arguments as in the proof of Theorem A.3.8 to establish that these factorization relations uniquely determine our morphism $(\phi, \psi) : P \vee \Theta(M) \to Q$, as asserted by our theorem. ☐

A.5.8. *Maximal degeneration of semi-alternate two-colored trees.* In §A.3, we observed that the isomorphism category of (open) $\underline{\mathsf{r}}$-trees $\mathcal{T}ree(\underline{\mathsf{r}})^{iso}$ is equivalent to a discrete category in any arity $r > 0$ (see Proposition A.3.15) and we use this result to establish that free non-unitary operads admit a reduced expansion where no quotient occurs (see Proposition A.3.14).

We aim to establish a similar result for the coproducts $P \vee \Theta(M)$. We still get, from the result of Proposition A.3.15, that the automorphism group of an object $\underline{\mathsf{T}}_{\underline{\mathsf{c}}}$ in the category of semi-alternate two-colored trees is either empty or reduced to a point, but we now have to handle the additional degeneracy operators occurring in our colimit. We can actually observe that any semi-alternate two-colored tree $\underline{\mathsf{T}}_{\underline{\mathsf{c}}}$ admits a maximal degeneration $\hat{\underline{\mathsf{T}}}_{\underline{\mathsf{c}}}$, obtained by degenerating all edges e satisfying $s(e), t(e) \notin V_{\bullet}(\underline{\mathsf{T}})$ (and allowable for a degeneration therefore) in the tree $\underline{\mathsf{T}}$ (see Figure A.15 for an example of application of this process). We get, as a consequence, that the category of isomorphisms of semi-alternate two-colored $\underline{\mathsf{r}}$-trees and degeneracies $\mathcal{T}ree_{\bullet\circ}(\underline{\mathsf{r}})^{iso}_{\mathsf{T}}$ splits as a coproduct of categories with terminal objects, which are precisely the maximal degenerations of semi-alternate two-colored $\underline{\mathsf{r}}$-trees $\hat{\underline{\mathsf{T}}}_{\underline{\mathsf{c}}}$, for any arity $r > 0$. Let $\mathcal{T}ree_{\widehat{\bullet\circ}}(\underline{\mathsf{r}})^{iso}_{\mathsf{T}} \subset \mathcal{T}ree_{\bullet\circ}(\underline{\mathsf{r}})^{iso}_{\mathsf{T}}$ denote the subcategory generated by these maximal objects inside $\mathcal{T}ree_{\bullet\circ}(\underline{\mathsf{r}})^{iso}_{\mathsf{T}}$. We obtain (as in Proposition A.3.14) that:

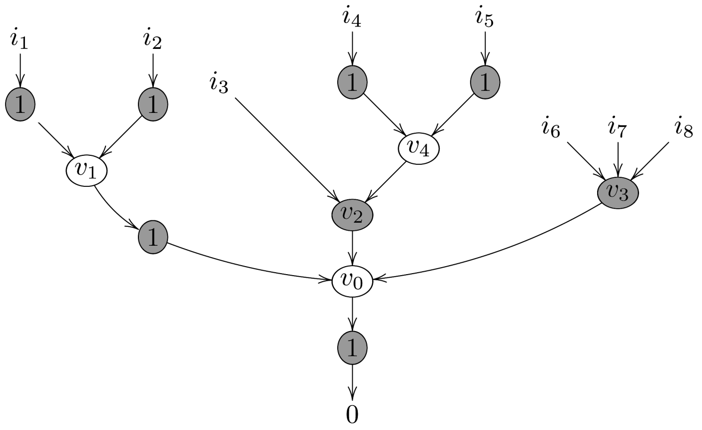

FIGURE A.15. The maximal degeneration of the tree of Figure A.11. To simplify the picture, we have marked all added vertices with a 1 symbol, and we have dropped the name of the edges in this figure.

PROPOSITION A.5.9. *The coproduct $P \vee \Theta(M)$ of a non-unitary operad P with the free operad on a non-unitary collection M has a reduced expansion such that:*

$$P \vee \Theta(M)(\underline{\mathsf{r}}) = \coprod_{[\underline{\hat{\mathsf{T}}}_{\underline{\mathsf{c}}}] \in \pi_0 \mathcal{T}ree_{\widehat{\bullet\circ}}(\underline{\mathsf{r}})^{iso}_{\mathsf{T}}} M(\underline{\hat{\mathsf{T}}}_{\underline{\mathsf{c}}} \wr P),$$

for each finite (non-empty) set $\underline{\mathsf{r}} \neq \underline{0}$, where the coproduct ranges over (a set of representatives of) isomorphism classes of maximal objects $\underline{\hat{\mathsf{T}}}_{\underline{\mathsf{c}}} \in \mathcal{T}ree_{\widehat{\bullet\circ}}(\underline{\mathsf{r}})^{iso}_{\mathsf{T}}$ in the category of semi-alternate two-colored $\underline{\mathsf{r}}$-trees and degeneracies $\mathcal{T}ree_{\bullet\circ}(\underline{\mathsf{r}})^{iso}_{\mathsf{T}}$. ☐

A.5.10. *The case of connected operads.* We actually deal with another representation of the coproduct $P \vee \Theta(M)$ than the one given in §A.5.4 in the case where P is a connected operad and M is a connected collection.

We then consider, for any finite (non-empty) set $\underline{\mathsf{r}}$, the subcategory of the isomorphism category of semi-alternate two-colored $\underline{\mathsf{r}}$-trees $\widetilde{\mathcal{T}ree}_{\bullet\circ}(\underline{\mathsf{r}})^{iso} \subset \mathcal{T}ree_{\bullet\circ}(\underline{\mathsf{r}})^{iso}$ of which objects $\underline{\mathsf{T}}_{\underline{\mathsf{c}}} \in \widetilde{\mathcal{T}ree}_{\bullet\circ}(\underline{\mathsf{r}})^{iso}$ are reduced trees in the sense of §A.1.12. We have $M(0) = M(1) = \varnothing \Rightarrow M(\underline{\mathsf{r}}_v) = \varnothing$ when a white vertex $v \in V_{\circ}(\underline{\mathsf{T}})$ in a general semi-alternate two-colored $\underline{\mathsf{r}}$-tree $\underline{\mathsf{T}}_{\underline{\mathsf{c}}} \in \mathcal{T}ree_{\bullet\circ}(\underline{\mathsf{r}})$ satisfies $\mathrm{card}(\underline{\mathsf{r}}_v) \leq 1$. We moreover see that the assumption $P(1) = \mathbb{1}$ implies that the degeneracy operators of §A.5.3 are isomorphisms when P is connected. These observations readily imply that the colimit of our initial construction §A.5.4(1) reduces to a colimit over the isomorphism category of semi-alternate two-colored reduced $\underline{\mathsf{r}}$-trees

$$P \vee \Theta(M)(\underline{\mathsf{r}}) = \operatorname*{colim}_{\underline{\mathsf{T}}_{\underline{\mathsf{c}}} \in \widetilde{\mathcal{T}ree}_{\bullet\circ}(\underline{\mathsf{r}})^{iso}} M(\underline{\mathsf{T}}_{\underline{\mathsf{c}}} \wr P) \tag{1}$$

when we work in the connected setting. We also have a variant of the reduced expansion of Proposition A.5.9 where we take a coproduct over (representatives of) isomorphism classes of reduced semi-alternate two-colored $\underline{\mathsf{r}}$-trees (instead of the maximal objects which we consider in this previous statement):

$$P \vee \Theta(M)(\underline{r}) = \coprod_{[\underline{T}_{\underline{c}}] \in \pi_0 \widetilde{\mathcal{T}ree}_{\bullet\circ}(\underline{r})^{iso}} M(\underline{T}_{\underline{c}} \wr P), \tag{2}$$

We actually use a dual version of these reduced expressions in our definition of the model category of cooperads in §II.9.2.

A.5.11. *Remark: The definition of restriction operators on alternate treewise tensors and on operad coproducts.* We explained in §I.2.3 that the obvious forgetful functor $\tau : \Lambda\,\mathcal{O}p_{\varnothing} \,/\, Com \to \mathcal{O}p_{\varnothing}$ from the category of augmented non-unitary Λ-operads $\Lambda\,\mathcal{O}p_{\varnothing} \,/\, Com$ towards the category of plain non-unitary operads $\mathcal{O}p_{\varnothing}$ preserves (and actually creates) colimits. We deduced this result from the lifting of the free object functor (which we formally establish in this appendix, in Proposition A.3.12) and from a realization of colimits in terms of a reflexive coequalizer of free objects. This general statement implies that the coproducts $P \vee \Theta(M)$ which we consider in this section inherit an augmented non-unitary Λ-operad structure when P is so and M is an augmented non-unitary Λ-collection. We can actually made explicit the restriction operators $u^* : P \vee \Theta(M)(\underline{n}) \to P \vee \Theta(M)(\underline{m})$ as well as the augmentations $\epsilon : P \vee \Theta(M)(\underline{r}) \to \mathbb{1}$ of this extra structure on the semi-alternate treewise tensor products of our construction. We follow the same procedure as in the case of the treewise tensor product of free operads. We just note that the treewise restriction operators of §A.2.8(1) do not break our alternation condition §A.5.1(1) when we apply the definition to semi-alternate treewise tensors (because the reduction operations of §A.2.8 remove entire over-subtrees) and preserve our degeneracy operators §A.5.3(2).

We can proceed similarly to extend the augmented non-unitary Λ-operad structure of free operads to coproducts $P \vee \Theta(M)$ in the connected setting.

APPENDIX B

The Cotriple Resolution of Operads

The free operad functor $\mathbb{O} : \mathcal{S}eq \to \mathcal{O}p$ gives, after composition with the forgetful functor $\omega : \mathcal{O}p \to \mathcal{S}eq$, a functor from symmetric sequences to symmetric sequences:

$$\omega\,\mathbb{O} : \mathcal{S}eq \to \mathcal{S}eq\,.$$

By performing this composite the other way round, we obtain a functor from operads to operads:

$$\mathbb{O}\,\omega : \mathcal{O}p \to \mathcal{O}p\,.$$

We often omit forgetful functors in the expression of our constructions. We accordingly use the single free operad functor notation $\mathbb{O}$ for any of the above composites whenever our source (respectively, target) category is specified by the context.

The adjunction relation of free operads implies that the endofunctor of the category of symmetric sequences $\mathbb{O} = \omega\,\mathbb{O} : \mathcal{S}eq \to \mathcal{S}eq$ is equipped with morphisms $\iota : Id \to \mathbb{O}$ and $\mu : \mathbb{O} \circ \mathbb{O} \to \mathbb{O}$ which satisfy natural unit and associativity relations (we just review the precise definition of an analogue of these morphisms for connected free operads in §B.1.1). These morphisms provide the functor $\mathbb{O} : \mathcal{S}eq \to \mathcal{S}eq$ with the structure of a *monad* (some authors use the name '*triple*' to refer to this notion). The second endofunctor $\mathbb{O} = \mathbb{O}\,\omega : \mathcal{O}p \to \mathcal{O}p$ which we deduce from the free operad adjunction is dually equipped with morphisms $\lambda : \mathbb{O} \to Id$ and $\nu : \mathbb{O} \to \mathbb{O} \circ \mathbb{O}$ which provide our object with the structure of a *comonad* (or *cotriple*). We have similar results for the free non-unitary operad functor $\mathbb{O} : \mathcal{S}eq_{>0} \to \mathcal{O}p_{\varnothing}$.

In the context of connected operads, we consider the endofunctor

$$\bar{\mathbb{O}} = \bar{\omega}\,\mathbb{O} : \mathcal{S}eq_{>1} \to \mathcal{S}eq_{>1}$$

which we form by composing the free operad functor $\mathbb{O} : M \mapsto \mathbb{O}(M)$ with the augmentation ideal functor $\bar{\omega} : \mathcal{O}p_{\varnothing 1} \to \mathcal{S}eq_{>1}$ (rather than the plain forgetful functor $\omega : \mathcal{O}p \to \mathcal{S}eq$). This functor also inherits a monad structure, while the functor

$$\mathbb{O}\,\bar{\omega} : \mathcal{O}p_{\varnothing 1} \to \mathcal{O}p_{\varnothing 1},$$

which we obtain by performing our composite the other way round, inherits the structure of a comonad. Recall that we also use the short notation $\bar{\mathbb{O}} : \mathcal{S}eq_{>1} \to \mathcal{S}eq_{>1}$ for the first of these composites $\bar{\mathbb{O}} = \bar{\omega}\,\mathbb{O}$.

The main purpose of this appendix is to make explicit the definition of a simplicial resolution of operads, the *cotriple resolution*, which we construct by using these monad and comonad structures deduced from the free operad adjunction. We need more insights into the connected version of this construction. We therefore focus on this case in what follows. We just give a few observations on the extension of our results to general (non-connected) operads in the course of our account.

The cotriple resolution of a connected operad $\mathtt{Res}_\bullet(\mathsf{P})$ is a simplicial object of the category of connected operads defined by the composite:

$$\mathtt{Res}_n(\mathsf{P}) = \mathbb{\Theta} \circ \underbrace{\bar{\omega}\,\mathbb{\Theta} \circ \cdots \circ \bar{\omega}\,\mathbb{\Theta}}_{n} \circ \bar{\omega}(\mathsf{P}),$$

for any $n \geq 0$, where we consider the free connected operad monad $\bar{\mathbb{\Theta}} = \bar{\omega}\,\mathbb{\Theta} : \mathcal{S}eq_{>1} \to \mathcal{S}eq_{>1}$ as middle factors. The free operad functor $\mathbb{\Theta} : \mathcal{S}eq_{>1} \to \mathcal{O}p_{\varnothing 1}$, which we take as a front factor in this composite, serves as left coefficients for our construction (in a sense which we explain later on) and the augmentation ideal of the operad $\bar{\mathsf{P}} = \bar{\omega}\,\mathsf{P} \in \mathcal{S}eq_{>1}$ serves as right coefficients. We use the monad structure of the functor $\bar{\mathbb{\Theta}}$ to formulate the definition of the faces and of the degeneracy operators of the cotriple resolution.

We review all these definitions with full details in this appendix. We are also going to give an explicit description, in terms of treewise tensors equipped with extra structures, of the cotriple resolution. In short, our idea is that the elements of composite free operad functors (whenever the notion of an element makes sense) can be represented by tensors arranged on trees equipped with nested subtree decompositions which reflect the composition of the treewise tensors of free operads. We prove that these decompositions can be determined by introducing a suitable notion of morphism for trees. We then prove that the cotriple resolution of an operad is shaped on the nerve of this category of trees.

We define our notion of a tree morphism in a preliminary section of this appendix (§B.0). We give our treewise interpretation of the cotriple resolution of operads afterwards (§B.1). We review the abstract definition of this resolution in the course of this examination. We also revisit the definition of the free connected operad monad at this moment.

We go back to the general definition of a monad in the concluding section of the appendix (§B.2). We check, to complete our account, that the structure of an operad can be defined in terms of the monad $\mathbb{\Theta} : \mathcal{S}eq \to \mathcal{S}eq$. We formally have a natural notion of an algebra associated a monad and our claim is that the category of operads is isomorphic to the category of algebras over this monad $\mathbb{\Theta} : \mathcal{S}eq \to \mathcal{S}eq$ which we deduce from the free operad functor. We have a similar result for the category of connected operads. In the language of category theory, authors say that the category of (connected) operads is *monadic* (or *triplable*).

This appendix, like the previous one, is mainly a detailed review of ideas of the literature. We notably refer to Livernet's article [121] for the description, in terms of the nerve of the category of trees, of the cotriple resolution of operads, and we refer to Getzler-Jones's article [77] for the statement that the category of operads is monadic.

We still prefer to work with the category of symmetric collections $\mathcal{C}oll$ rather than with the equivalent category of symmetric sequences $\mathcal{S}eq$ in this appendix. We actually heavily rely on the formalism of the previous chapter for our constructions.

B.0. Tree morphisms

The purpose of this first section is to explain the definition of the notion of tree morphism which we use in our description of the cotriple resolution of operads.

We fix an indexing set $\underline{\mathrm{r}}$. In §A, we used the notation $\mathcal{T}ree(\underline{\mathrm{r}})$ for the class of $\underline{\mathrm{r}}$-trees. We now use this notation $\mathcal{T}ree(\underline{\mathrm{r}})$ for the category formed by the class of $\underline{\mathrm{r}}$-trees as objects together with our general notion of morphism of $\underline{\mathrm{r}}$-trees as

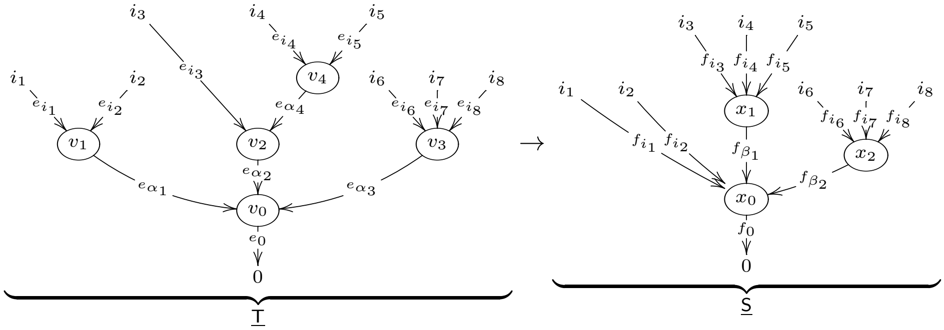

FIGURE B.1. The picture of a morphism of $\underline{\mathsf{r}}$-trees $f : \underline{\mathsf{T}} \to \underline{\mathsf{S}}$ with $\underline{\mathsf{r}} = \{i_1, \dots, i_8\}$. This morphism is defined by the mapping such that $f_E(e_0) = e_0$, $f_E(e_{i_k}) = f_{i_k}$, $k = 1, \dots, 8$, and $f_E(e_{\alpha_1}) = x_0$, $f_E(e_{\alpha_2}) = f_{\beta_1}$, $f_E(e_{\alpha_3}) = f_{\beta_2}$, $f_E(e_{\alpha_4}) = x_1$ on the edge set of the tree $\underline{\mathsf{T}}$, while we consider the correspondence $f_V(v_0) = f_V(v_1) = x_0$, $f_V(v_2) = f_V(v_4) = x_1$ and $f_V(v_3) = x_2$ on the vertex set.

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morphisms. We will see that the groupoid $\mathcal{T}ree(\underline{\mathsf{r}})^{iso}$, which we consider in our definition of free operads in §A.1.8, is actually identified with the isomorphism subcategory of this category $\mathcal{T}ree(\underline{\mathsf{r}})$. Recall that we generally assume that our trees are open for simplicity (see §A.1.1). We keep this convention in this appendix. We may see, however, that our notion of a tree morphism makes sense without this condition.

We just get special results in the case of the full subcategories $\widetilde{\mathcal{T}ree}(\underline{\mathsf{r}}) \subset \mathcal{T}ree(\underline{\mathsf{r}})$ which we associate to the class of reduced trees (see §A.1.12). We examine the structure of these categories in-depth in the second part of this section. We will precisely check that this category $\widetilde{\mathcal{T}ree}(\underline{\mathsf{r}})$ forms a poset unlike the category of all (open) trees $\mathcal{T}ree(\underline{\mathsf{r}})$.

B.0.1. *The notion of a tree morphism.* We give an example of a morphism of $\underline{\mathsf{r}}$-trees, with $\mathrm{card}(\underline{\mathsf{r}}) = 8$, in Figure B.1.

We adapt the definition of a tree isomorphism (see §A.1.8) in order to allow edge contractions on vertices. We more precisely define a morphism of $\underline{\mathsf{r}}$-trees $f : \underline{\mathsf{T}} \to \underline{\mathsf{S}}$, with $\underline{\mathsf{S}}, \underline{\mathsf{T}} \neq \downarrow$, by giving a map of vertex sets $f_V : V(\underline{\mathsf{T}}) \to V(\underline{\mathsf{S}})$ together with a map on edge sets $f_E : E(\underline{\mathsf{T}}) \to E(\underline{\mathsf{S}}) \amalg V(\underline{\mathsf{S}})$ such that:

(0) The vertex map $f_V : V(\underline{\mathsf{T}}) \to V(\underline{\mathsf{S}})$ is surjective and the edge map $f_E : E(\underline{\mathsf{T}}) \to E(\underline{\mathsf{S}}) \amalg V(\underline{\mathsf{S}})$ induces a bijection from the subset $f_E^{-1}(E(\underline{\mathsf{S}})) \subset E(\underline{\mathsf{T}})$ onto $E(\underline{\mathsf{S}})$.

(1) For the outgoing edge $e = e_0$ of the tree $\underline{\mathsf{T}}$, for which we have $t(e) = 0$, we assume $f_E(e) \in E(\underline{\mathsf{S}})$ and $s(f_E(e)) = f_V(s(e))$, $t(f_E(e)) = 0$, $s(f_E(e)) = f_V(s(e))$ and $t(f_E(e)) = 0$.

(2) For an ingoing edge $e = e_i$, which we associate to some input index $i \in \underline{\mathsf{r}}$ of the tree $\underline{\mathsf{T}}$ so that $s(e) = i$, we assume $f_E(e) \in E(\underline{\mathsf{S}})$ and $s(f_E(e)) = i$, $t(f_E(e)) = f_V(t(e))$.

(3) For an inner edge $e \in \mathring{E}(\underline{\mathsf{T}})$, for which we have $s(e), t(e) \in V(\underline{\mathsf{T}})$, we assume that we have either $f_E(e) = f_V(s(e)) = f_V(t(e)) \in V(\underline{\mathsf{S}})$ or $f_E(e) \in E(\underline{\mathsf{S}})$ and

in the latter case we assume that we have the relations $s(f_E(e)) = f_V(s(e))$ and $t(f_E(e)) = f_V(t(e))$.

In the case where $\underline{\mathsf{S}}$ or $\underline{\mathsf{T}}$ is the unit tree $\underline{\downarrow}$, we only consider the isomorphisms $\underline{\mathsf{S}} \xrightarrow{\simeq} \underline{\mathsf{T}}$ as morphisms.

In what follows, we use the notation $\mathcal{T}ree(\underline{\mathsf{r}})$, already introduced for the class of $\underline{\mathsf{r}}$-trees, to refer to the category formed by the $\underline{\mathsf{r}}$-trees as objects and the above notion of morphism.

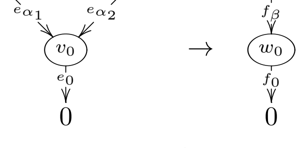
The isomorphisms of $\underline{\mathsf{r}}$-trees, such as defined in §A.1.8, clearly form a subclass of the class of morphisms which we define in this paragraph. Recall that we use the notation $\mathcal{T}ree(\underline{\mathsf{r}})^{iso}$ for the category where we only take these isomorphisms as morphisms. We immediately see that $\mathcal{T}ree(\underline{\mathsf{r}})^{iso}$ represents the isomorphism subcategory of the category $\mathcal{T}ree(\underline{\mathsf{r}})$.

We readily check that the action of input re-indexing bijections on trees is functorial with respect to our class of morphisms. We have the same result for the restriction operators $u^* : \mathcal{T}ree(\underline{\mathsf{n}}) \to \mathcal{T}ree(\underline{\mathsf{m}})$ (see §A.1.11) and for the reduced version of these restriction operators $\widetilde{u^*} : \widetilde{\mathcal{T}ree}(\underline{\mathsf{n}}) \to \widetilde{\mathcal{T}ree}(\underline{\mathsf{m}})$ when we consider the full subcategories of reduced trees $\widetilde{\mathcal{T}ree}(\underline{\mathsf{r}}) \subset \mathcal{T}ree(\underline{\mathsf{r}})$ (see §A.1.13). We also have the following easy observation:

Proposition B.0.2. *The partial composition operations of the operad of trees* $\circ_i : \mathcal{T}ree(\underline{\mathsf{m}}) \times \mathcal{T}ree(\underline{\mathsf{n}}) \to \mathcal{T}ree(\underline{\mathsf{m}} \circ_i \underline{\mathsf{n}})$*, such as defined in* §A.1.10*, are functorial with respect to all tree morphisms and not only with respect to the isomorphisms of* §A.1.8*. Hence, the full categories of trees* $\mathcal{T}ree(\underline{\mathsf{r}})$*,* $r > 0$*, where we take all our morphisms, form an operad in the category of categories.*

Proof. We use the relation $V(\underline{\mathsf{S}} \circ_i \underline{\mathsf{T}}) = V(\underline{\mathsf{S}}) \amalg V(\underline{\mathsf{T}})$ in the definition of the composition of trees in order to determine the vertex map of a morphism on a composite tree. We use the definition of the edge set $E(\underline{\mathsf{S}} \circ_i \underline{\mathsf{T}})$ as the quotient of the coproduct $E(\underline{\mathsf{S}}) \amalg E(\underline{\mathsf{T}})$ under the relation that merges the outgoing edge of the tree $\underline{\mathsf{T}}$ with the ith ingoing edge of the tree $\underline{\mathsf{S}}$ in order to determine the edge map of such a morphism. We have to check that this procedure yields a well-defined morphism on a composite tree $\underline{\mathsf{S}} \circ_i \underline{\mathsf{T}}$ whenever we have a well-defined morphism of trees on each factor of this composition. This verification is straightforward. □

The requirements of §B.0.1 are modeled on the definition of a cellular map between topological cell complexes, except the bijectivity condition (0) which is not a standard requirement. Thus, not all cellular maps in the classical sense are tree morphisms in our sense. To summarize the main point, our definition allows us to contract edges onto vertices in tree morphisms, but not to merge edges onto edges. For instance, the map



which merges the edges e_{α_1} and e_{α_2} onto f_β is not an admissible morphism of trees in our sense. We can retrieve this observation from the result of the following proposition, which gives an alternate description of our morphisms of $\underline{\mathsf{r}}$-trees:

Proposition B.0.3. *Let $f : \underline{\mathsf{T}} \to \underline{\mathsf{S}}$ be a morphism of $\underline{\mathsf{r}}$-trees, determined by a vertex map $f_V : V(\underline{\mathsf{T}}) \to V(\underline{\mathsf{S}})$ together with a map $f_E : E(\underline{\mathsf{T}}) \to E(\underline{\mathsf{S}}) \amalg V(\underline{\mathsf{S}})$ on the edge set $E(\underline{\mathsf{T}})$ as we explain in our definition* §B.0.1.

(a) *Let $x \in V(\underline{\mathsf{S}})$. The vertex and edge sets $f_V^{-1}(x) \subset V(\underline{\mathsf{T}})$, $f_E^{-1}(x) \subset E(\underline{\mathsf{T}})$, define the set of vertices and the set of inner edges of a subtree $\underline{\Sigma}_x \subset \underline{\mathsf{T}}$ associated to our vertex $x \in V(\underline{\mathsf{S}})$.*

The set of ingoing edges (respectively, the outgoing edge) of this subtree is also given by the (bijective) pre-image, in $f_E^{-1}(E(\underline{\mathsf{S}})) \subset E(\underline{\mathsf{T}})$, of the set of ingoing edges (respectively, of the outgoing edge) of our vertex x in the tree $\underline{\mathsf{S}}$.

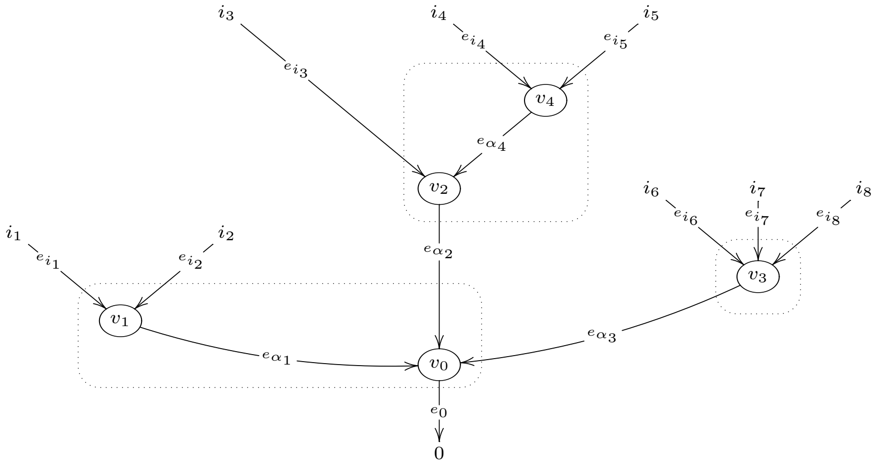
(b) *These subtrees $\underline{\Sigma}_x \subset \underline{\mathsf{T}}$ and $x \in V(\underline{\mathsf{S}})$, taken together, define a decomposition of the tree $\underline{\mathsf{T}}$ in the sense that we have a canonical isomorphism*

$$\tilde{f} : \lambda_{\underline{\mathsf{S}}}(\underline{\Sigma}_x, x \in V(\underline{\mathsf{S}})) \xrightarrow{\simeq} \underline{\mathsf{T}}$$

when we take the treewise composite $\lambda_{\underline{\mathsf{S}}} : \mathcal{T}ree(\underline{\mathsf{S}}) \to \mathcal{T}ree(\underline{\mathsf{r}})$ of the collection $\underline{\Sigma}_x \in \mathcal{T}ree(\underline{\mathsf{r}}_x)$, $x \in V(\underline{\mathsf{S}})$, in the operad of trees.

Explanations. To illustrate our process, in the case of the morphism of Figure B.1, the construction of the proposition returns the subtree decomposition such that:

(1)



,

where we use circled arrays to materialize the subtrees of this decomposition (as in Figure A.2). In this picture, the tree $\underline{\mathsf{S}}$, which represents the target object of our morphism $f : \underline{\mathsf{T}} \to \underline{\mathsf{S}}$, can be retrieved by identifying the circled subtrees to vertices and by keeping the edges which do not lie inside any such subtree.

Let $x \in V(\underline{\mathsf{S}})$. Recall that we take $V(\underline{\Sigma}_x) = f_V^{-1}(x)$ to define the vertex set of the subtree $\underline{\Sigma}_x \subset \underline{\mathsf{T}}$ in our statement. The edge set $E(\underline{\Sigma}_x)$ of this subtree $\underline{\Sigma}_x \subset \underline{\mathsf{T}}$ actually consists of the edges $e \in E(\underline{\mathsf{T}})$ such that $f_E(e) = x$ or $f_E(e) \in E(\underline{\mathsf{S}})$ and we have either $s(f_E(e)) = x$, in which case e represents the outgoing edge of $\underline{\Sigma}_x$, or $t(f_E(e)) = x$, in which case e defines an outgoing edge of our subtree $\underline{\Sigma}_x$. To define the output $0_{\underline{\Sigma}_x} \in V(\underline{\mathsf{T}}) \amalg \{0\}$ of our subtree $\underline{\Sigma}_x$, we take the pre-image, under the map $f_V \amalg \{0\} : V(\underline{\mathsf{T}}) \amalg \{0\} \to V(\underline{\mathsf{S}}) \amalg \{0\}$, of the target of the outgoing edge of the vertex $x \in V(\underline{\mathsf{S}})$. To define the set of inputs $\underline{\mathsf{r}}_{\underline{\Sigma}_x} \subset V(\underline{\mathsf{T}}) \amalg \underline{\mathsf{r}}$ of $\underline{\Sigma}_x$, we take the pre-image, under the map $f_V \amalg \underline{\mathsf{r}} : V(\underline{\mathsf{T}}) \amalg \underline{\mathsf{r}} \to V(\underline{\mathsf{S}}) \amalg \underline{\mathsf{r}}$, of the source of the ingoing edges of the vertex $x \in V(\underline{\mathsf{S}})$.

The verification of the conditions of §A.1.5 for the definition of a subtree follows from a straightforward inspection. Let us mention that we use the bijectivity condition of our definition of a tree morphism §B.0.1(0) to ensure that we have one and only one edge e such that $t(e) = 0_{\underline{\Sigma}_x}$ in our set $E(\underline{\Sigma}_x)$. Moreover, we have $f_E(e) = x \Leftrightarrow f_E(s(e)) = f_E(t(e)) = x \Leftrightarrow s(e), t(e) \in V(\underline{\Sigma}_x)$ and we conclude from these relations that $f_E^{-1}(x) \subset E(\underline{\Sigma}_x)$ represent the set of inner edges of our subtree $\underline{\Sigma}_x$.

The isomorphism in assertion (b) is given by the identity $V(\lambda_{\underline{\mathsf{S}}}(\underline{\Sigma}_x, x \in V(\underline{\mathsf{S}}))) = \coprod_{x \in V(\underline{\mathsf{S}})} V(\underline{\Sigma}_x)$ and is induced by the canonical embeddings $E(\underline{\Sigma}_x) \subset E(\underline{\mathsf{T}})$, $x \in V(\underline{\mathsf{S}})$, at the edge set level. We just note that the edges of the trees $\underline{\Sigma}_x$ which we merge in our abstract treewise composition operation $\underline{\Theta} = \lambda_{\underline{\mathsf{S}}}(\underline{\Sigma}_x, x \in V(\underline{\mathsf{S}}))$ are the edges of the coproduct $\coprod_{x \in E(\underline{\mathsf{S}})} E(\underline{\Sigma}_x)$ which become equal in the set $\bigcup_{x \in E(\underline{\mathsf{S}})} E(\underline{\Sigma}_x) = E(\underline{\mathsf{T}})$.

To complete our result, let us observe that any treewise composite of trees $\underline{\Theta} = \lambda_{\underline{\mathsf{S}}}(\underline{\Sigma}_x, x \in V(\underline{\mathsf{S}}))$ is equipped with a canonical morphism $\omega : \lambda_{\underline{\mathsf{S}}}(\underline{\Sigma}_x, x \in V(\underline{\mathsf{S}})) \to \underline{\mathsf{S}}$. The isomorphism which we define in this proposition actually fits in a commutative diagram

(2)
$$\begin{tikzcd}
\lambda_{\underline{\mathsf{S}}}(\underline{\Sigma}_x, x \in V(\underline{\mathsf{S}})) \arrow[rr] \arrow[dr, dotted, "\simeq"'] & & \underline{\mathsf{S}} \\
& \underline{\mathsf{T}} \arrow[ur, "f"'] &
\end{tikzcd}$$

in the category of $\underline{\mathsf{r}}$-trees.

We explain in the next paragraph that our morphisms are equivalent to composites of isomorphisms and of multiple edge contractions. We can easily make explicit the decompositions that correspond to the morphisms $\underline{\mathsf{T}} \to \underline{\mathsf{T}}/e$ which we associate to a single edge contraction $\underline{\mathsf{S}} = \underline{\mathsf{T}}/e$. We use this particular case of our correspondence in the study of cofree cooperads in §C.

We formally consider the subtree $\underline{\Gamma}_e \subset \underline{\mathsf{T}}$, associated to our edge $e \in \mathring{E}(\underline{\mathsf{T}})$, such that $\mathring{E}(\underline{\Gamma}_e) = \{e\}$ (see §A.1.6), and the corollas $\underline{\mathsf{Y}}_x \subset \underline{\mathsf{T}}$ such that $V(\underline{\mathsf{Y}}_x) = \{x\}$, for the vertices $x \neq s(e), t(e)$ (see also §A.1.6). Recall that we have $V(\underline{\mathsf{T}}/e) = V(\underline{\mathsf{T}})/\{s(e) \equiv t(e)\}$. Let ω be the vertex obtained by the merging operation $s(e) \equiv t(e)$ in the tree $\underline{\mathsf{S}} = \underline{\mathsf{T}}/e$. We now have $\underline{\mathsf{T}} = \lambda_{\underline{\mathsf{T}}/e}(\underline{\Gamma}_e, \underline{\mathsf{Y}}_x, x \neq \omega)$, where we consider the treewise composition of the subtree $\underline{\Gamma}_e \subset \underline{\mathsf{T}}$ at position $x = \omega$ together with the corollas $\underline{\mathsf{Y}}_x \subset \underline{\mathsf{T}}$ at the other places $x \neq \omega$ of the tree $\underline{\mathsf{S}} = \underline{\mathsf{T}}/e$. The morphism $\underline{\mathsf{T}} \to \underline{\mathsf{T}}/e$ corresponds to this decomposition. □

B.0.4. *Tree morphisms and edge contractions.* The previous proposition gives a correspondence, between subtree decompositions and morphisms, which we use in our description of the cotriple resolution of connected operads. The morphism sets of the categories of trees have a further description by generators and relations. For our purpose, we just check that any morphism of $\underline{\mathsf{r}}$-trees is identified with a composite of edge contractions and of an isomorphism.

In one direction, a (possibly multiple) edge contraction $\underline{\mathsf{T}} \mapsto \underline{\mathsf{T}}/e_{\alpha_1}/\dots/e_{\alpha_n}$ naturally determines a morphism

$$\underline{\mathsf{T}} \to \underline{\mathsf{T}}/e_{\alpha_1}/\dots/e_{\alpha_n}$$

since the edges $e \in \{e_{\alpha_1}, \dots, e_{\alpha_n}\}$ are carried to a vertex $s(e_{\alpha_i}) \equiv t(e_{\alpha_i})$ by the contraction process while the other edges $e \in E(\underline{\mathsf{T}}) \setminus \{e_{\alpha_1}, \dots, e_{\alpha_n}\}$ are preserved.

The other way round, to a morphism of $\underline{\mathsf{r}}$-trees $f : \underline{\mathsf{T}} \to \underline{\mathsf{S}}$, we can associate the subset of edges $\{e_{\alpha_1}, \dots, e_{\alpha_n}\} \subset E(\underline{\mathsf{T}})$ which the mapping $f_E : E(\underline{\mathsf{T}}) \to E(\underline{\mathsf{S}}) \amalg V(\underline{\mathsf{S}})$ carries to a vertex of the tree $\underline{\mathsf{S}}$. We then have an obvious quotient morphism fitting in a diagram:

(1)
$$\begin{tikzcd}
\underline{\mathsf{T}} \arrow[r] \arrow[dr, "f"'] & \underline{\mathsf{T}}/e_{\alpha_1}/\dots/e_{\alpha_n} \arrow[d, dotted, "\simeq"', "\exists!"] \\
 & \underline{\mathsf{S}}
\end{tikzcd}$$

in the category of $\underline{\mathsf{r}}$-trees. This morphism is clearly bijective on vertices and edges, and hence, forms an isomorphism in the category of $\underline{\mathsf{r}}$-trees, as required.

This factorization of a morphism of $\underline{\mathsf{r}}$-trees (1), where we take a (multiple) edge contraction followed by an isomorphism, is clearly unique. We readily see that the commutation relation $\underline{\mathsf{T}}/e/f = \underline{\mathsf{T}}/f/e$ together with the commutation relations expressed by the functoriality of the edge contraction operations $\underline{\mathsf{T}} \mapsto \underline{\mathsf{T}}/e$ under the action of tree isomorphisms determine all relations between our morphisms in the category of $\underline{\mathsf{r}}$-trees. We mainly use this description of morphisms in order to establish that:

Proposition B.0.5. *For a (non-unitary) operad P, the mapping which assigns the object $\mathsf{P}(\underline{\mathsf{T}})$ to any $\underline{\mathsf{r}}$-tree $\underline{\mathsf{T}} \in \mathcal{T}ree(\underline{\mathsf{r}})$ in* §A.2.1 *extends to a functor on the category which as the $\underline{\mathsf{r}}$-trees as objects together with the class of morphisms defined in* §B.0.1 *as morphisms, for any finite (non-empty) set $\underline{\mathsf{r}} \neq \underline{\mathsf{0}}$.*

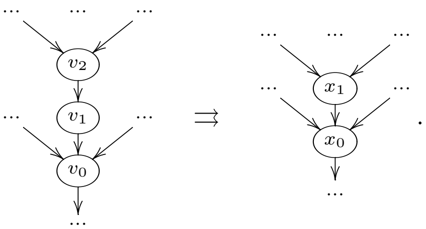
Explanations. To any edge contraction operation $f : \underline{\mathsf{T}} \to \underline{\mathsf{T}}/e$, we associate the morphism $\lambda_e : \mathsf{P}(\underline{\mathsf{T}}) \to \mathsf{P}(\underline{\mathsf{T}}/e)$, defined in §A.2.4, and which we determine from the composition structure of our operad.

We just use the coherence result of Theorem A.2.6 (together with the functoriality of the edge contraction operations with respect to the action of isomorphisms) in order to extend this mapping coherently to the composites §B.0.4(1) which, according to the observations of the previous paragraph, represent the morphisms of the category of $\underline{\mathsf{r}}$-trees, for any finite set $\underline{\mathsf{r}}$. ☐

The next theorem implies that a morphism of reduced $\underline{\mathsf{r}}$-trees $f : \underline{\mathsf{T}} \to \underline{\mathsf{S}}$ is unambiguously specified by giving the source $\underline{\mathsf{T}}$ and the target object $\underline{\mathsf{S}}$ of the morphism. For this reason, in what follows, we generally do not name the morphisms of reduced $\underline{\mathsf{r}}$-trees, and we adopt the generic notation $\lambda_* : \mathsf{P}(\underline{\mathsf{T}}) \to \mathsf{P}(\underline{\mathsf{S}})$ for the treewise composition operation which we associate to any such morphism $\underline{\mathsf{T}} \to \underline{\mathsf{S}}$ when we can assume $\underline{\mathsf{S}}, \underline{\mathsf{T}} \in \widetilde{\mathcal{T}ree}(\underline{\mathsf{r}})$.

Theorem B.0.6. *Let $\underline{\mathsf{r}}$ be any finite set. The morphism set $\mathtt{Mor}_{\mathcal{T}ree(\underline{\mathsf{r}})}(\underline{\mathsf{T}}, \underline{\mathsf{S}})$ associated to reduced $\underline{\mathsf{r}}$-trees $\underline{\mathsf{T}}, \underline{\mathsf{S}} \in \widetilde{\mathcal{T}ree}(\underline{\mathsf{r}})$ is either empty or reduced to a point. The category of reduced $\underline{\mathsf{r}}$-tree $\widetilde{\mathcal{T}ree}(\underline{\mathsf{r}})$ therefore forms a posets, for any arity $r > 0$.*

PROOF. Let us stress that the assumption $\underline{\mathsf{T}} \in \widetilde{\mathcal{T}ree}(\underline{\mathsf{r}})$ is crucial as we see from the following picture:

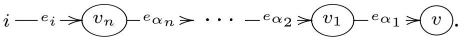


But we do not really need to assume that the tree $\underline{\mathsf{S}}$ is reduced and our arguments remain valid without any condition on the $\underline{\mathsf{r}}$-tree $\underline{\mathsf{S}}$.

Thus, we just assume that $f, g : \underline{\mathsf{T}} \to \underline{\mathsf{S}}$ is a pair of parallel morphisms of $\underline{\mathsf{r}}$-trees, where the tree $\underline{\mathsf{T}}$ is reduced. We explicitly have $\operatorname{card}(\underline{\mathsf{r}}_v) \geq 2$, for all vertices $v \in V(\underline{\mathsf{T}})$, and we aim to prove that in this situation we necessarily have $f = g$. We suggest the reader to follow our argument line on the example of Figure B.1.

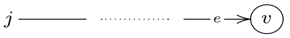
We still use that any vertex $v \in V(\underline{\mathsf{T}})$ in an (open) tree $\underline{\mathsf{T}} \in \mathcal{T}ree(\underline{\mathsf{r}})$ is connected to an input of the tree by a chain of edges

(1)



If we have $n = 0$ inner edges in this chain, so that v is directly connected to the input i by an ingoing edge of the tree e_i, then we get $s(f_E(e_i)) = i = s(g_E(e_i)) \Rightarrow f_E(e_i) = g_E(e_i)$ and $f_V(v) = t(f_E(e_i)) = t(g_E(e_i)) = g_V(v)$ by the same argument line as in Proposition A.3.15 (where we prove that the isomorphism categories of trees are equivalent to discrete categories). Thus, we now consider the case where v is connected to i by a chain (1) with $n > 0$ inner edges, and we assume by induction that f and g agree up to the vertex $x = v_1$ on this chain. By assumption, any vertex of the tree $\underline{\mathsf{T}}$, including v, has not only one, but at least two ingoing edges. We can accordingly pick a second chain of edges

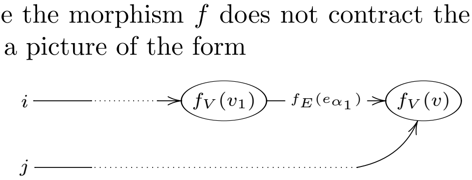
(2)



connecting v to an input $j \neq i$ and starting with an edge such that $e \neq e_{\alpha_1}$ in $\underline{\mathsf{r}}_v$. In §A.1.1, we record that we have at most one chain of edges connecting a given input j to a given vertex v in an $\underline{\mathsf{r}}$-tree $\underline{\mathsf{T}}$. The requirement $e \neq e_{\alpha_1}$ ensures, as a consequence, that we have no chain of edges connecting the input j to the vertex v_1 in the tree $\underline{\mathsf{T}}$.

In the case where the morphism f does not contract the edge e_{α_1} to a vertex in the tree $\underline{\mathsf{S}}$, we get a picture of the form

(3)



in our target object. Recall that in our definition of a tree morphism §B.0.1(0), we require that the pre-image of an edge consists of one and only one element. Therefore no edge of the chain (2) can be mapped to $f_E(e_{\alpha_1})$ inside the tree $\underline{\mathsf{S}}$ and this observation explains our picture (3). We still get, moreover, that no chain of edges connects the input j to the vertex $f_V(v_1) = g_V(v_1)$ in the tree $\underline{\mathsf{S}}$. Now, since the morphism g maps (2) to a chain of edges connecting the input j to the vertex

$g_V(v)$, we necessarily have $g_V(v) \neq g_V(v_1)$, and we deduce from this relation that the edge e_{α_1} is not contracted by the morphism g too.

Thus, in the case where f (or g symmetrically) does not contract the edge e_{α_1}, we obtain that both morphisms f and g map the edge e_{α_1} to an edge of the tree $\underline{\mathsf{S}}$. This edge is necessarily identified with the outgoing edge of $f_V(v_1) = g_V(v_1)$ in both cases. Hence, we necessarily have $f_E(e_{\alpha_1}) = g_E(e_{\alpha_1})$ in $E(\underline{\mathsf{S}})$ and as a consequence, we also get $f_V(v) = t(f_E(e_{\alpha_1})) = t(g_E(e_{\alpha_1})) = g_V(v)$.

In the contrary case where both f and g contract the edge e_{α_1} to a vertex of the tree $\underline{\mathsf{S}}$, we have $f_V(v) = f_V(v_1) = f_E(e_{\alpha_1}) = g_E(e_{\alpha_1}) = g_V(v_1) = g_V(v_0)$. Thus, we trivially get that f and g agree up to the next stage of our chain (1) in this case too. This verification completes our induction.

To complete the proof of our theorem, we just have to record that the morphisms f and g agree on the outgoing edge of our tree too, since we assume (in our definition) that a morphism preserves the outgoing edge of the trees. The morphisms f and g accordingly agree on the whole tree $\underline{\mathsf{T}}$, as asserted. $\square$

B.1. The definition of the cotriple resolution of operads

The purpose of this section is to give an explicit definition of the cotriple resolution of operads, as we explain in the introduction of this appendix. We mainly examine the case of connected operads (and of augmented connected Λ-operads by the way). Recall that we use a monad structure, which we associate to the augmentation ideal of the free operad functor $\bar{\mathbb{\Theta}} = \bar{\omega}\,\mathbb{\Theta}$, in order to define our resolution in this case. We entirely review the abstract definition of the cotriple resolution first. We prove afterwards that the cotriple resolution can be explicitly described by using treewise tensors shaped on the nerve of the category of reduced trees.

We could actually rely on the same approach (without much change) in order to describe the cotriple resolution of any operad P equipped with an augmentation over the unit object I (we just need to consider general trees instead of reduced trees). We can still adapt our construction in order to get an explicit description of the cotriple resolution of general operads (in this case, we do not assume that our objects are equipped with an augmentation). We then have to consider extra degeneration operations which models the insertion of operadic units that occur in the free operad $\mathbb{\Theta}$, but which we could neglect in the augmented case. We just give an outline of this extension of our constructions at the end of this section.

To start with, we quickly explain the definition of a monad structure on the augmentation ideal of the free operad functor $\bar{\mathbb{\Theta}} = \bar{\omega}\,\mathbb{\Theta}$.

B.1.1. *The free connected operad monad.* This monad structure is defined by a unit morphism $\iota : \mathit{Id} \to \bar{\mathbb{\Theta}}$ and a multiplication operation $\mu : \bar{\mathbb{\Theta}} \circ \bar{\mathbb{\Theta}} \to \bar{\mathbb{\Theta}}$ on the composite functor $\bar{\mathbb{\Theta}} = \bar{\omega}\,\mathbb{\Theta}$. In the definition of a monad structure (we go back to the general definition in the next section), we also require that these natural transformations satisfy an obvious analogue, in the category of functors, of the standard unit and associativity relations of monoids.

The unit and multiplication operations of our monad are determined by the adjunction relation $\mathbb{\Theta} : \mathcal{C}oll_{>1} \rightleftarrows \mathcal{O}p_{\varnothing 1} : \bar{\omega}$ between the free connected operad functor $\mathbb{\Theta} : M \mapsto \mathbb{\Theta}(M)$ and the augmentation ideal functor on the category of connected operads $\bar{\omega} : \mathcal{O}p_{\varnothing 1} \to \mathcal{C}oll_{>1}$. The monad unit $\iota : \mathit{Id} \to \bar{\mathbb{\Theta}}$ is simply given by the unit morphism $\iota : M \to \bar{\mathbb{\Theta}}(M)$ of this adjunction. The monad multiplication

$\mu : \bar{\mathbb{\Theta}} \circ \bar{\mathbb{\Theta}} \to \bar{\mathbb{\Theta}}$ is formed by applying the adjunction augmentation $\lambda : \mathbb{\Theta}(\bar{P}) \to P$ to free operads $P = \mathbb{\Theta}(M)$ and by taking the morphism induced by this transformation on augmentation ideals afterwards. The unit and associativity relations are expressed by the commutativity of the diagrams:

$$\begin{tikzcd}
Id \circ \bar{\mathbb{\Theta}} \arrow[r, "\iota \circ \bar{\mathbb{\Theta}}"] \arrow[dr, "="'] & \bar{\mathbb{\Theta}} \circ \bar{\mathbb{\Theta}} \arrow[d, "\mu"] & \bar{\mathbb{\Theta}} \circ Id \arrow[l, "\bar{\mathbb{\Theta}} \circ \iota"'] \arrow[dl, "="] \\
& \bar{\mathbb{\Theta}} &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\bar{\mathbb{\Theta}} \circ \bar{\mathbb{\Theta}} \circ \bar{\mathbb{\Theta}} \arrow[r, "\mu \circ \bar{\mathbb{\Theta}}"] \arrow[d, "\bar{\mathbb{\Theta}} \circ \mu"'] & \bar{\mathbb{\Theta}} \circ \bar{\mathbb{\Theta}} \arrow[d, "\mu"] \\
\bar{\mathbb{\Theta}} \circ \bar{\mathbb{\Theta}} \arrow[r, "\mu"] & \bar{\mathbb{\Theta}}
\end{tikzcd},$$

and follow from general relations between the unit and the augmentation morphism of an adjunction.

We can also use the natural transformation $\lambda \circ \mathbb{\Theta} : \mathbb{\Theta}\,\bar{\omega}\,\mathbb{\Theta}(M) \to \mathbb{\Theta}(M)$ (without passing to augmentation ideals) in order to get a morphism $\rho : \mathbb{\Theta} \circ \bar{\mathbb{\Theta}} \to \bar{\mathbb{\Theta}}$ which defines a right action of the monad $\bar{\mathbb{\Theta}} : \mathcal{C}oll_{>1} \to \mathcal{C}oll_{>1}$ on the free connected operad functor $\mathbb{\Theta} : \mathcal{C}oll_{>1} \to \mathcal{O}p_{\varnothing 1}$. We still have obvious diagrams asserting that this morphism $\rho : \mathbb{\Theta} \circ \bar{\mathbb{\Theta}} \to \bar{\mathbb{\Theta}}$ fulfills natural unit and associativity relations with respect to the structure morphisms of our monad.

The natural transformation $\lambda : \mathbb{\Theta}(\bar{P}) \to P$, which defines the augmentation of our adjunction, induces, on the other hand, a morphism $\lambda : \bar{\mathbb{\Theta}}(\bar{P}) \to \bar{P}$ which, in some natural sense, defines a left action of the monad $\bar{\mathbb{\Theta}} : \mathcal{C}oll_{>1} \to \mathcal{C}oll_{>1}$ on the augmentation ideal $\bar{P} \in \mathcal{C}oll_{>1}$ of any operad $P \in \mathcal{O}p_{\varnothing 1}$. We readily check, yet again, that this morphism $\lambda : \bar{\mathbb{\Theta}}(\bar{P}) \to \bar{P}$ fulfills natural unit and associativity relations with respect to the structure morphisms of our monad.

We use these observations to define the face and degeneracy operators of the cotriple resolution of a connected operad.

B.1.2. *The categorical definition of the cotriple resolution.* The cotriple resolution of a connected operad $P \in \mathcal{O}p_{\varnothing 1}$ is a simplicial object of the category of connected operads $\mathtt{Res}_\bullet(P) \in s\,\mathcal{O}p_{\varnothing 1}$ defined in dimension $n \in \mathbb{N}$ by the expression:

$$\mathtt{Res}_n(P) = \mathbb{\Theta} \circ \underbrace{\bar{\mathbb{\Theta}} \circ \cdots \circ \bar{\mathbb{\Theta}}}_{n}(\bar{P}),$$

where we consider the free operad functor $\mathbb{\Theta} : \mathcal{C}oll_{>1} \to \mathcal{O}p_{\varnothing 1}$, followed by an n-fold composite of the free connected operad monad $\bar{\mathbb{\Theta}} : \mathcal{C}oll_{>1} \to \mathcal{C}oll_{>1}$, and we apply this composite functor to the augmentation ideal of our object $\bar{P} \in \mathcal{C}oll_{>1}$. We number the free connected operad monad factors of this composite from 1 to n, and from the left to the right, as in the following expression:

$$\mathtt{Res}_n(P) = \mathbb{\Theta} \circ \underset{1}{\bar{\mathbb{\Theta}}} \circ \cdots \circ \underset{n}{\bar{\mathbb{\Theta}}}(\bar{P}).$$

The face operators $d_i : \mathtt{Res}_n(P) \to \mathtt{Res}_{n-1}(P)$, $i = 0, \dots, n$, of this simplicial object are yielded by:

- the morphism $\rho : \mathbb{\Theta} \circ \bar{\mathbb{\Theta}} \to \mathbb{\Theta}$ which defines the right action of the monad $\bar{\mathbb{\Theta}}$ on the free operad functor in the case $i = 0$;
- the multiplication $\mu : \bar{\mathbb{\Theta}} \circ \bar{\mathbb{\Theta}} \to \bar{\mathbb{\Theta}}$ on the $(i, i+1)$th monadic factors of our composite in the case $i = 1, \dots, n-1$;
- and the morphism $\lambda : \bar{\mathbb{\Theta}}(\bar{P}) \to \bar{P}$ which defines the left action of the monad $\bar{\mathbb{\Theta}}$ on the object $\bar{P} \in \mathcal{C}oll_{>1}$ in the case $i = n$.

The degeneracy operator $s_j : \mathtt{Res}_n(\mathsf{P}) \to \mathtt{Res}_{n+1}(\mathsf{P})$ is given by the insertion of a monad unit $\iota : Id \to \bar{\Theta}$ on the $j+1$st monad factor of the object $\mathtt{Res}_n(\mathsf{P})$, for any $j = 0, \dots, n$.

The simplicial relations §0.4(4) can be deduced from the unit and associativity relations of the monadic structures which we consider in this definition of our face (respectively, degeneracy) operators.

The simplicial object $\mathtt{Res}_\bullet(\mathsf{P}) \in s\,\mathcal{O}p_{\varnothing 1}$ also inherits a natural transformation $\epsilon : \mathtt{Res}_0(\mathsf{P}) \to \mathsf{P}$ defined by the augmentation morphism of our adjunction $\lambda : \Theta(\bar{\mathsf{P}}) \to \mathsf{P}$ and such that $\epsilon d_0 = \epsilon d_1$. This relation $\epsilon d_0 = \epsilon d_1$ implies that the natural transformation $\epsilon : \mathtt{Res}_0(\mathsf{P}) \to \mathsf{P}$ extends to a morphism $\epsilon : \mathtt{Res}_n(\mathsf{P}) \to \mathsf{P}$ in any dimension $n \in \mathbb{N}$ so that we have $\epsilon u^* = \epsilon$ for any simplicial operator $u^* : \mathtt{Res}_n(\mathsf{P}) \to \mathtt{Res}_m(\mathsf{P})$ associated to a morphism $u \in \mathtt{Mor}_\Delta(\underline{m}, \underline{n})$ in the simplicial category Δ. Equivalently, we get that the simplicial object $\mathtt{Res}_\bullet(\mathsf{P}) \in s\,\mathcal{O}p_{\varnothing 1}$ is equipped with a natural augmentation

$$\epsilon : \mathtt{Res}_\bullet(\mathsf{P}) \to \mathsf{P}$$

over the constant simplicial object defined by our operad $\mathsf{P} \in \mathcal{O}p_{\varnothing 1}$.

B.1.3. *The definition of extra-degeneracies.* If we take the augmentation ideal of the cotriple resolution, then we get a simplicial object of the category of collections $\overline{\mathtt{Res}}_\bullet(\mathsf{P}) \in s\,\mathcal{C}oll_{>1}$ such that

$$\overline{\mathtt{Res}}_n(\mathsf{P}) = \bar{\Theta} \circ \underbrace{\bar{\Theta} \circ \cdots \circ \bar{\Theta}}_{n}(\bar{\mathsf{P}}),$$

for any $n \in \mathbb{N}$. In this setting, we can also consider the insertion of a monadic unit $\iota : Id \to \bar{\Theta}$ in front of our functor expression. We then get an extra-degeneracy $s_{-1} : \overline{\mathtt{Res}}_n(\mathsf{P}) \to \overline{\mathtt{Res}}_{n+1}(\mathsf{P})$, associated to our object, and which satisfies an obvious extension of the relations of degeneracies in a simplicial object.

We moreover have a morphism $\eta : \bar{\mathsf{P}} \to \bar{\Theta}(\bar{\mathsf{P}})$, such that $\epsilon\eta = id$, $\eta\epsilon = d_1 s_{-1}$, which is given by the unit morphism $\iota : \mathsf{M} \to \bar{\Theta}(\mathsf{M})$ of the adjunction of free connected operads for the object $\mathsf{M} = \bar{\mathsf{P}}$.

In our study of simplicial operads, we use the augmentation $\epsilon : \mathtt{Res}_\bullet(\mathsf{P}) \to \mathsf{P}$ to define a morphism $\epsilon : |\mathtt{Res}_\bullet(\mathsf{P})| \to \mathsf{P}$, where we consider the geometric realization of the simplicial object $\mathtt{Res}_\bullet(\mathsf{P})$ in the category of operads in simplicial sets, and we use the existence of the extra-degeneracies to check that this morphism defines a weak-equivalence.

We tackle this subject in §II.8 (in the context of general non-unitary operads). We focus on general structure properties of the cotriple resolution for the moment. We start with the following treewise description of the components of the cotriple resolution:

Proposition B.1.4. *Let $n \in \mathbb{N}$. We have an identity:*

$$\mathtt{Res}_n(\mathsf{P})(\underline{\mathtt{r}}) = \operatorname*{colim}_{\substack{\underline{\mathtt{T}}_0 \leftarrow \cdots \leftarrow \underline{\mathtt{T}}_n \\ \underline{\mathtt{T}}_0, \dots, \underline{\mathtt{T}}_n \in \widetilde{\mathcal{T}ree}(\underline{\mathtt{r}})^{iso}}} \mathsf{P}(\underline{\mathtt{T}}_n),$$

for any finite (non-empty) set $\underline{\mathtt{r}}$, where our colimit ranges over the category formed by:

- *the chains of $\underline{\mathtt{r}}$-tree morphisms $\underline{\mathtt{T}}_0 \longleftarrow \underline{\mathtt{T}}_1 \longleftarrow \cdots \longleftarrow \underline{\mathtt{T}}_n$ as objects,*

– *together with the diagrams of fill-in isomorphisms between such chains*

$$\begin{tikzcd}
\underline{\mathsf{S}}_0 \arrow[d, dotted, "\simeq"'] & \underline{\mathsf{S}}_n \arrow[l] \arrow[d, dotted, "\simeq"'] & \cdots \arrow[l] & \underline{\mathsf{S}}_n \arrow[l] \arrow[d, dotted, "\simeq"'] \\
\underline{\mathsf{T}}_0 & \underline{\mathsf{T}}_1 \arrow[l] & \cdots \arrow[l] & \underline{\mathsf{T}}_n \arrow[l]
\end{tikzcd}$$

as morphisms.

For short, we generally do not specify the range of the objects $\underline{\mathsf{T}}_i \in \widetilde{\mathcal{T}ree}(\underline{\mathsf{r}})^{iso}$ in the subsequent expressions of the colimit of this proposition.

EXPLANATIONS. In the case $n = 0$, we retrieve the treewise definition of the free connected operad:

$$\mathbb{F}(\bar{P}) = \operatorname*{colim}_{\underline{\mathsf{T}}_0} P(\underline{\mathsf{T}}_0).$$

We elaborate on the observation of Proposition B.0.3, where we give an interpretation of our tree morphisms in terms of subtree decompositions, in order to extend the correspondence of this treewise definition of free operads to the composite functors occurring in our statement.

In what follows, we represent the elements of the colimit of the proposition by pairs (λ, π), where λ denotes a chain of tree morphisms $\underline{\mathsf{T}}_0 \leftarrow \underline{\mathsf{T}}_1 \leftarrow \cdots \leftarrow \underline{\mathsf{T}}_n$ and we assume $\pi \in P(\underline{\mathsf{T}}_n)$. This element $\pi \in P(\underline{\mathsf{T}}_n)$ explicitly consists of a tensor with factors indexed by the vertices of the last tree of our chain $\pi = \otimes_{v \in V(\underline{\mathsf{T}}_n)} p_v \in \otimes_{v \in V(\underline{\mathsf{T}}_n)} P(\underline{\mathsf{r}}_v)$. We need to assume that we work in a concrete symmetric monoidal category to give a sense to this tensor product in terms of elements, but we can still use the pair representation in a general categorical setting, where the notion of an element does not necessarily make sense. We then assume that our element $\pi \in P(\underline{\mathsf{T}}_n)$ (by abusing this expression) represents a tensor product of abstract variables in this context.

By induction, the construction of Proposition B.0.3 implies that giving a chain of tree morphisms $\underline{\mathsf{T}}_0 \leftarrow \underline{\mathsf{T}}_1 \leftarrow \cdots \leftarrow \underline{\mathsf{T}}_n$ amounts to defining n nested subtree decompositions, numbered from 0 to $n-1$, on the treewise tensor product which represents our element $\pi \in P(\underline{\mathsf{T}}_n)$ (see Figure B.3). We proceed inductively in order to associate elements of free operad composites to the components of these decompositions.

We give the picture of a pair (λ, π) and of the corresponding nested subtree decomposition in Figure B.2-B.3. We can follow the definition of our correspondence on this example. We assume that we have already defined a mapping when we have $n-1$ levels of tree morphisms:

$$\operatorname*{colim}_{\underline{\mathsf{T}}_0 \leftarrow \cdots \leftarrow \underline{\mathsf{T}}_{n-1}} P(\underline{\mathsf{T}}_{n-1}) \xrightarrow{\simeq} \mathbb{F} \circ \underbrace{\bar{\mathbb{F}} \circ \cdots \circ \bar{\mathbb{F}}}_{n-1}(\bar{P})(\underline{\mathsf{r}})$$

for any connected operad $P \in \mathcal{O}p_{\varnothing 1}$ (which we take as a variable in our construction). We extend the construction of this mapping to the case where we have a chain of trees with n levels $\underline{\mathsf{T}}_0 \leftarrow \cdots \leftarrow \underline{\mathsf{T}}_n$. We accordingly assume $\pi \in P(\underline{\mathsf{T}}_n)$ in what follows.

We use the decomposition $\lambda_{\underline{\mathsf{T}}_{n-1}}(\underline{\Sigma}_x, x \in V(\underline{\mathsf{T}}_{n-1})) \xrightarrow{\simeq} \underline{\mathsf{T}}_n$ equivalent to the morphism $\underline{\mathsf{T}}_{n-1} \leftarrow \underline{\mathsf{T}}_n$ (see Proposition B.0.3) and which corresponds to the top level decomposition of our picture (see Figure B.3). We then have $V(\underline{\mathsf{T}}_n) =$

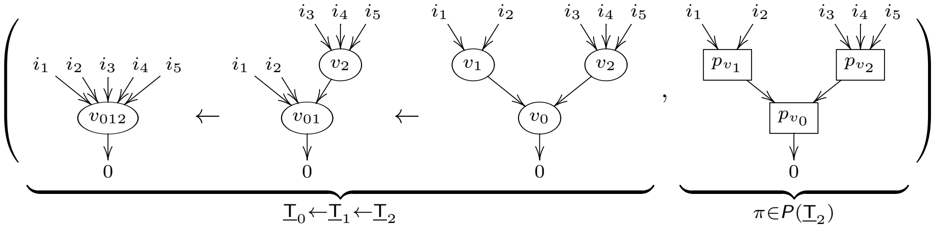


FIGURE B.2. The representation of an element of the cotriple resolution in terms of a pair (λ, π) consisting of a treewise tensor π together with a chain of tree morphisms λ.

$$(\underline{\mathrm{T}}_0 \leftarrow \underline{\mathrm{T}}_1 \leftarrow \underline{\mathrm{T}}_2, \pi) =$$

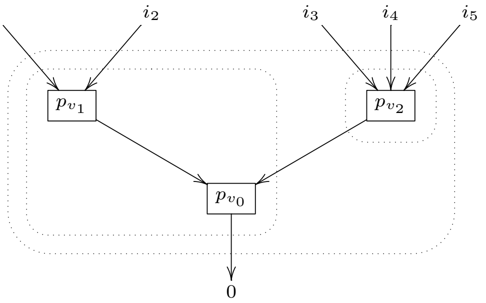
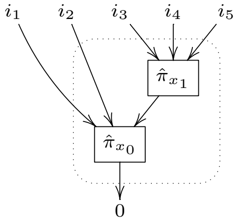


FIGURE B.3. The picture of the nested decompositions which correspond to the chain of tree morphisms of Figure B.2. The tree $\underline{\mathrm{T}}_2$ gives the shape of our treewise tensor. The morphism $\underline{\mathrm{T}}_1 \leftarrow \underline{\mathrm{T}}_2$ determines the top (internal) decomposition of our figure. The tree $\underline{\mathrm{T}}_1$ can be retrieved by collapsing the subtrees of this decomposition to vertices. The morphism $\underline{\mathrm{T}}_0 \leftarrow \underline{\mathrm{T}}_2$ determines the lowest (external) decomposition, which reduces to a single component in this example. The inclusion relations between our decompositions are equivalent to the factorization relations given in our chain.

$\coprod_{x\in V(\underline{\mathrm{T}}_{n-1})} V(\underline{\Sigma}_x)$. We gather the factors $p_v \in P(\underline{\mathrm{r}}_v)$, $v \in V(\underline{\Sigma}_x)$, associated to each component of this decomposition $\underline{\Sigma}_x \subset \underline{\mathrm{T}}$ in the expression of our treewise tensor $\pi = \otimes_{v\in V(\underline{\mathrm{T}}_n)} p_v \in P(\underline{\mathrm{T}}_n)$. We accordingly get a factorization of this tensor $\hat{\pi} = \otimes_{x\in V(\underline{\mathrm{T}}_{n-1})} \hat{\pi}_x$, shaped on the tree $\underline{\mathrm{T}}_{n-1}$, and of which factors are treewise tensors $\hat{\pi}_x = \otimes_{v\in V(\underline{\Sigma}_x)} p_v \in P(\underline{\Sigma}_x)$ which represent elements of the free operad $Q = \mathbb{\Theta}(\bar{P})$. We then use our inductive construction to assign an element of the composite $\mathbb{\Theta} \circ \bar{\mathbb{\Theta}} \circ \cdots \circ \bar{\mathbb{\Theta}}(\bar{Q})$ to the pair $(\underline{\mathrm{T}}_0 \leftarrow \cdots \leftarrow \underline{\mathrm{T}}_{n-1}, \hat{\pi})$, where we use our factorization to regard $\hat{\pi} = \pi \in P(\underline{\mathrm{T}}_n)$ as an element of the treewise tensor product $Q(\underline{\mathrm{T}}_{n-1})$. We can easily define a map the other way round in order to formally check that this correspondence defines an isomorphism.

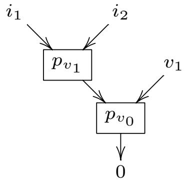
For instance, in the case of Figure B.2, we get the picture:

$(\underline{\mathsf{T}}_0 \leftarrow \underline{\mathsf{T}}_1, \hat{\pi}) =$

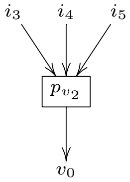


with $\hat{\pi}_{x_0}, \hat{\pi}_{x_1} \in \mathbb{\Theta}(\bar{P})$ such that:

$\hat{\pi}_{x_0} =$



and $\hat{\pi}_{x_1} =$



.

Thus, we basically take the pieces of the top decomposition of the picture of Figure B.3 to get these free operads elements $\hat{\pi}_{x_0}, \hat{\pi}_{x_1} \in \mathbb{\Theta}(\bar{P})$, while the lower level part of our structure gives the composite tree shape, corresponding to the composite functor $\mathbb{\Theta}\circ\bar{\mathbb{\Theta}}(-)$, in which we plug these elements.

We have an obvious action of bijections $u \in \mathtt{Mor}_{\mathcal{B}ij}(\underline{\mathsf{r}}, \underline{\mathsf{s}})$ on the colimit of the proposition, which is given termwise by the expression:

$$u_*(\underline{\mathsf{T}}_0 \leftarrow \cdots \leftarrow \underline{\mathsf{T}}_n, \pi) = (u_* \underline{\mathsf{T}}_0 \leftarrow \cdots \leftarrow u_* \underline{\mathsf{T}}_n, u_*\pi),$$

where we consider the natural input re-indexing operation on trees $u_* : \widetilde{\mathcal{T}ree}(\underline{\mathsf{r}}) \to \widetilde{\mathcal{T}ree}(\underline{\mathsf{s}})$, together with the morphism $u_* : P(\underline{\mathsf{T}}_n) \to P(u_* \underline{\mathsf{T}}_n)$ which we deduce from the functoriality of the treewise tensor product construction. We easily check that this termwise action of bijections on our colimit corresponds to the natural symmetric structure of the collection $\mathtt{Res}_n(P) = \mathbb{\Theta}\circ\bar{\mathbb{\Theta}}\circ\cdots\circ\bar{\mathbb{\Theta}}(\bar{P})$, for any $n \in \mathbb{N}$. We also have operadic composition operations such that:

$$(\underline{\mathsf{S}}_0 \leftarrow \cdots \leftarrow \underline{\mathsf{S}}_n, \pi) \circ_i (\underline{\mathsf{T}}_0 \leftarrow \cdots \leftarrow \underline{\mathsf{T}}_n, \rho) = (\underline{\mathsf{S}}_0 \circ_i \underline{\mathsf{T}}_0 \leftarrow \cdots \leftarrow \underline{\mathsf{S}}_n \circ_i \underline{\mathsf{T}}_n, \pi \circ_i \rho),$$

and which we form on the terms of our colimits by using the (functorial) composition structure of the operad of trees $\circ_i : \mathcal{T}ree(\underline{\mathsf{m}}) \times \mathcal{T}ree(\underline{\mathsf{n}}) \to \mathcal{T}ree(\underline{\mathsf{m}} \circ_i \underline{\mathsf{n}})$ together with the isomorphisms $\bar{P}(\underline{\mathsf{S}}_n) \otimes \bar{P}(\underline{\mathsf{T}}_n) \xrightarrow{\simeq} \bar{P}(\underline{\mathsf{S}}_n \circ_i \underline{\mathsf{T}}_n)$. We easily check that these termwise composition operations correspond to the natural composition operations of the operad $\mathtt{Res}_n(P) = \mathbb{\Theta}\circ\bar{\mathbb{\Theta}}\circ\cdots\circ\bar{\mathbb{\Theta}}(\bar{P})$ too, for any dimension $n \in \mathbb{N}$. In the nested subtree decomposition picture, these operadic composition operations correspond to the obvious extension of the treewise composition operations of free operads (we essentially have to retain the decomposition markings attached to each factor of our composite).

We still have an obvious representative of an operadic unit in our colimit, given by the degenerate sequence of unit trees $(\underline{\downarrow} \leftarrow \cdots \leftarrow \underline{\downarrow}, 1)$ (the single element that we can form in arity one). The formula of the proposition therefore gives a full representation of the operad structure of our object $\mathtt{Res}_n(P) = \mathbb{\Theta}\circ\bar{\mathbb{\Theta}}\circ\cdots\circ\bar{\mathbb{\Theta}}(\bar{P})$ in any dimension $n \in \mathbb{N}$. $\square$

We now have the following correspondence for the faces and degeneracies of the cotriple resolution:

Proposition B.1.5.

(a) *In the colimit expression of Proposition* B.1.4*, the face operators of the cotriple resolution* $d_i : \mathtt{Res}_n(P) \to \mathtt{Res}_{n-1}(P)$ *are given by the formula:*

$$d_i(\underline{\mathrm{T}}_0 \leftarrow \cdots \leftarrow \underline{\mathrm{T}}_n, \pi) = (\underline{\mathrm{T}}_0 \leftarrow \cdots \leftarrow \widehat{\underline{\mathrm{T}}_i} \leftarrow \cdots \leftarrow \underline{\mathrm{T}}_n, \pi)$$

in the case $i = 0, \dots, n-1$*, and by the formula:*

$$d_n(\underline{\mathrm{T}}_0 \leftarrow \cdots \leftarrow \underline{\mathrm{T}}_n, \pi) = (\underline{\mathrm{T}}_0 \leftarrow \cdots \leftarrow \underline{\mathrm{T}}_{n-1}, \lambda_*(\pi))$$

in the case $i = n$*, where we also consider the treewise composition operation* $\lambda_* : P(\underline{\mathrm{T}}_n) \to P(\underline{\mathrm{T}}_{n-1})$ *associated to the top tree morphism* $\underline{\mathrm{T}}_n \to \underline{\mathrm{T}}_{n-1}$*.*

(b) *The degeneracies* $s_j : \mathtt{Res}_n(P) \to \mathtt{Res}_{n+1}(P)$*, on the other hand, are given termwise by the formula:*

$$s_j(\underline{\mathrm{T}}_0 \leftarrow \cdots \leftarrow \underline{\mathrm{T}}_n, \pi) = (\underline{\mathrm{T}}_0 \leftarrow \cdots \leftarrow \underline{\mathrm{T}}_j \leftarrow \underline{\mathrm{T}}_j \leftarrow \cdots \leftarrow \underline{\mathrm{T}}_n, \pi),$$

for all $j = 0, \dots, n$*.*

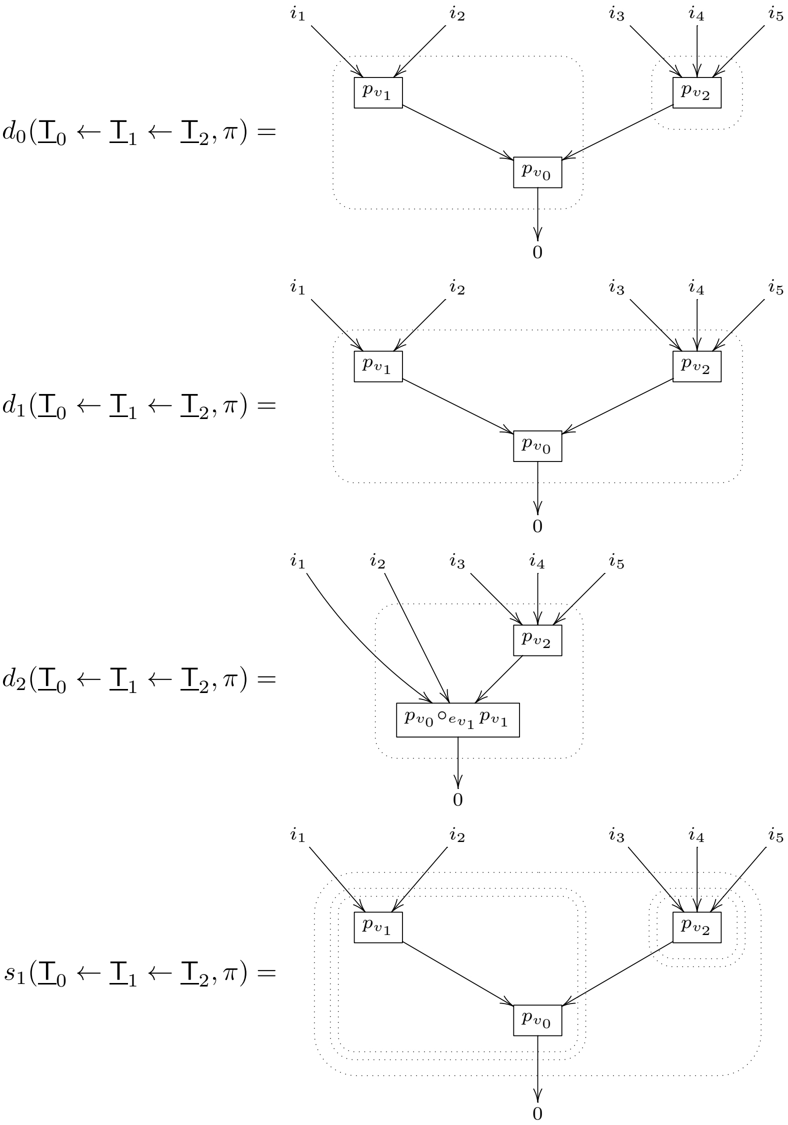
Explanations. The formulas of this proposition have an obvious interpretation when we use the representation of the cotriple resolution in terms of treewise tensors equipped with nested subtree decompositions. We make this picture explicit in Figure B.4. We use this representation to explain our process. Our claims follow from a straightforward inspection of the definition of the correspondence of Proposition B.1.4 and we go back to the notation of this proposition.

To get the top face of our element $(\underline{\mathrm{T}}_0 \leftarrow \cdots \leftarrow \underline{\mathrm{T}}_n, \pi)$ in the cotriple resolution $\mathtt{Res}_\bullet(P)$, we basically have to take the image, under the adjunction augmentation $\lambda : \mathbb{\Theta}(\bar{P}) \to P$, of the elements of the free operad $\hat{\pi}_x \in \mathbb{\Theta}(\bar{P})$, $x \in V(\underline{\mathrm{T}}_{n-1})$, which arise from the factorization of the treewise tensor $\pi \in P(\underline{\mathrm{T}}_n)$. For this purpose, we evaluate the treewise composition operations $\lambda_{\underline{\Sigma}_x} : P(\underline{\Sigma}_x) \to P(\underline{\mathrm{r}}_{\underline{\Sigma}_x})$ on the treewise tensors $\hat{\pi}_x \in P(\underline{\Sigma}_x)$ which define these elements of the free operad $\hat{\pi}_x \in \mathbb{\Theta}(\bar{P})$, $x \in V(\underline{\mathrm{T}}_{n-1})$. We just retrieve the composition operation $\lambda_* : P(\underline{\mathrm{T}}_n) \to P(\underline{\mathrm{T}}_{n-1})$ associated to the tree morphism $\underline{\mathrm{T}}_n \to \underline{\mathrm{T}}_{n-1}$ when we assemble these local composition operations on the tree $\underline{\mathrm{T}}_{n-1}$ that connect our subfactors $\hat{\pi}_x \in P(\underline{\Sigma}_x)$ in the treewise tensor $\hat{\pi} = \pi \in P(\underline{\mathrm{T}}_n)$. In the nested decomposition picture, we just see that the performance of the treewise composition operations $\lambda_{\underline{\Sigma}_x} : P(\underline{\Sigma}_x) \to P(\underline{\mathrm{r}}_{\underline{\Sigma}_x})$, $x \in V(\underline{\mathrm{T}}_{n-1})$, corresponds to the composition of the factors lying within the subtrees of the top level decomposition.

To get the lower face operators d_i, $i < n$, we proceed similarly with the element determined by the chain $(\underline{\mathrm{T}}_0 \leftarrow \cdots \leftarrow \underline{\mathrm{T}}_i, \hat{\pi}^{(n-i)})$, where $\hat{\pi}^{(n-i)}$ now denotes a treewise tensor product $\hat{\pi}^{(n-i)} = \bigotimes_{v \in V(\underline{\mathrm{T}}_i)} \hat{\pi}_v^{(n-i)}$ such that:

$$\hat{\pi}_v^{(n-i)} \in \mathbb{\Theta} \circ \underbrace{\bar{\mathbb{\Theta}} \circ \cdots \circ \bar{\mathbb{\Theta}}}_{n-i-1}(\bar{P})(\underline{\mathrm{r}}_v), \tag{1}$$

for all $v \in V(\underline{\mathrm{T}}_i)$. In the nested decomposition picture, these objects $\hat{\pi}_v^{(n-i)}$, $v \in V(\underline{\mathrm{T}}_i)$, are represented by the components of the decomposition of level i of the treewise tensor $\pi \in P(\underline{\mathrm{T}}_n)$ together with the nested decompositions defined by the subparts of the decompositions of level $i+1, \dots, n$ that sit inside each of these components of our object.



$$d_0(\underline{T}_0 \leftarrow \underline{T}_1 \leftarrow \underline{T}_2, \pi) =$$

$$d_1(\underline{T}_0 \leftarrow \underline{T}_1 \leftarrow \underline{T}_2, \pi) =$$

$$d_2(\underline{T}_0 \leftarrow \underline{T}_1 \leftarrow \underline{T}_2, \pi) =$$

$$s_1(\underline{T}_0 \leftarrow \underline{T}_1 \leftarrow \underline{T}_2, \pi) =$$

FIGURE B.4. The faces and an example of a degeneracy for the element of the cotriple resolution depicted in Figure B.3. The top face operator is obtained by performing the treewise composition operations of the operad within the components of the top subtree decomposition. We then drop this dummy term from our chain of decompositions. The other faces are simply given by the omission of terms in our subtree decomposition while the degeneracies are given by the repetition of terms.

Recall that the monadic multiplication $\mu : \bar{\Theta} \circ \bar{\Theta} \to \bar{\Theta}$, which we use to define our face operators $d_i : \mathtt{Res}_n(\mathsf{P}) \to \mathtt{Res}_{n-1}(\mathsf{P})$ when $i = 1, \dots, n-1$, is essentially given by an application of the augmentation of adjunction $\lambda : \Theta(\bar{\mathsf{R}}) \to \mathsf{R}$ to the free operad $\mathsf{R} = \Theta(-)$ and so does the structure morphism $\rho : \Theta \circ \bar{\Theta} \to \Theta$ which determines the 0-face operation $d_0 : \mathtt{Res}_n(\mathsf{P}) \to \mathtt{Res}_{n-1}(\mathsf{P})$. Thus, we can argue as in the case of the top face d_n to determine the face operators $d_i : \mathtt{Res}_n(\mathsf{P}) \to \mathtt{Res}_{n-1}(\mathsf{P})$ such that $i = 0, \dots, n-1$.

In short, we have to take the image, under the treewise composition operations $\lambda_{\underline{\Sigma}_x} : \mathsf{R}(\underline{\Sigma}_x) \to \mathsf{R}(\underline{\mathtt{r}}_{\underline{\Sigma}_x})$ of the subfactors (1) which lie in the components $\underline{\Sigma}_x \subset \underline{\mathtt{T}}_i$ of the decomposition of the tree $\underline{\mathtt{T}}_i$ defined by the morphism $\underline{\mathtt{T}}_{i-1} \leftarrow \underline{\mathtt{T}}_i$ (or of all these subfactors in the case $i = 0$). We just assume that $\mathsf{R}(-)$ is the (composite) free operad occurring in (1). But in the definition of the free operad, the composition products are given by simple grafting operations on treewise tensors (see §A.3.3). The mapping $\lambda_* : \mathsf{R}(\underline{\mathtt{T}}_i) \to \mathsf{R}(\underline{\mathtt{T}}_{i-1})$, which we obtain by taking the tensor product of these composition operations $\lambda_{\underline{\Sigma}_x} : \mathsf{R}(\underline{\Sigma}_x) \to \mathsf{R}(\underline{\mathtt{r}}_{\underline{\Sigma}_x})$, $x \in V(\underline{\mathtt{T}}_{i-1})$, is therefore given by the omission of the factorization operation which we perform to get the tensor $\hat{\pi}^{n-i}$ from $\hat{\pi}^{n-i-1}$ while we leave the rest of the structure unchanged.

We conclude from this study that the face operators $d_i : \mathtt{Res}_n(\mathsf{P}) \to \mathtt{Res}_{n-1}(\mathsf{P})$ such that $i = 0, \dots, n-1$ are equivalent to the withdrawal of a term $\underline{\mathtt{T}}_i$ from the chain $\underline{\mathtt{T}}_0 \leftarrow \cdots \leftarrow \underline{\mathtt{T}}_n$ in the treewise representation of the cotriple resolution, as asserted in the proposition. In the nested decomposition picture, this operation reduces to the removal of the decomposition of level i in our sequence (see Figure B.4).

We proceed similarly to determine the expression of the degeneracy operators. We just use that the unit morphism of our monad structure $\iota : Id \to \bar{\Theta}$ correspond in this case to the insertion of trivial (identity) extra factorizations in our treewise tensors. In the nested decomposition picture, this operation is equivalent to the duplication of a decomposition (see Figure B.4). □

B.1.6. *Remarks: the treewise expression of the augmentation and of the extra-degeneracies of the cotriple resolution.* The expression of the face operators in assertion (a) of the previous proposition has an obvious extension in dimension zero. In this case, we just retrieve the treewise expression of the adjunction augmentation of the free operad $\lambda : \Theta(\bar{\mathsf{P}}) \to \mathsf{P}$ which, by definition, gives the augmentation morphism of the cotriple resolution $\epsilon : \mathtt{Res}_0(\mathsf{P}) \to \mathsf{P}$ (see §B.1.2).

The formula of assertion (b) for the degeneracies has an extension for the extra-degeneracies too. We explicitly get $s_{-1}(\underline{\mathtt{T}}_0 \leftarrow \cdots \leftarrow \underline{\mathtt{T}}_n, \pi) = (\underline{\mathtt{Y}} \leftarrow \underline{\mathtt{T}}_0 \leftarrow \cdots \leftarrow \underline{\mathtt{T}}_n, \pi)$ in this case, for any $(\underline{\mathtt{T}}_0 \leftarrow \cdots \leftarrow \underline{\mathtt{T}}_n, \pi) \in \mathtt{Res}_n(\mathsf{P})$. The insertion of the corolla $\underline{\mathtt{Y}} \in \widetilde{\mathcal{T}ree}(\underline{\mathtt{r}})$ which we perform in this formula obviously corresponds to the insertion of the adjunction unit $\iota : Id \to \bar{\Theta}$ in front of our composite functor in our categorical definition of the extra-degeneracy (see §B.1.3) and reflects our treewise definition of this natural transformation (see §A.3.1). We similarly get the expression $\eta(\pi) = (\underline{\mathtt{Y}}, \pi)$ for the section of the augmentation $\eta : \mathsf{P} \to \mathtt{Res}_0(\mathsf{P})$.

We can also easily determine the expression of general simplicial operators $u^* : \mathtt{Res}_n(\mathsf{P}) \to \mathtt{Res}_m(\mathsf{P})$ on the cotriple resolution $\mathtt{Res}_{\bullet}(\mathsf{P})$ from the formulas of

Proposition B.1.5. We get the following statement:

PROPOSITION B.1.7. *Let* $u \in \mathtt{Mor}_{\Delta}(\underline{m}, \underline{n})$ *be any morphism in the simplicial category* Δ*. We have the formula:*

$$u^*(\underline{\mathrm{T}}_0 \leftarrow \cdots \leftarrow \underline{\mathrm{T}}_n, \pi) = (\underline{\mathrm{T}}_{u(0)} \leftarrow \cdots \leftarrow \underline{\mathrm{T}}_{u(m)}, \lambda_*(\pi)),$$

for any pair $(\underline{\mathrm{T}}_0 \leftarrow \cdots \leftarrow \underline{\mathrm{T}}_n, \pi)$*, which represents an element of the cotriple resolution* $\mathtt{Res}_{\bullet}(P)$ *with* $\pi \in P(\underline{\mathrm{T}}_n)$*. The treewise tensor* $\lambda_*(\pi) \in P(\underline{\mathrm{T}}_{u(m)})$*, in the outcome of this operation, is obtained by performing the treewise composition operation* $\lambda_* : P(\underline{\mathrm{T}}_n) \to P(\underline{\mathrm{T}}_{u(m)})$ *naturally associated to the tree morphism* $\underline{\mathrm{T}}_n \to \underline{\mathrm{T}}_{u(m)}$*.* □

B.1.8. *The quasi-free structure of the cotriple resolution.* The cotriple resolution $\mathtt{Res}_{\bullet}(P)$ forms a free operad dimensionwise by construction. We explicitly have $\mathtt{Res}_n(P) = \Theta(\bar{\Theta}^n(\bar{P}))$, for any $n \in \mathbb{N}$, where we set:

$$(1) \qquad \bar{\Theta}^n(\bar{P}) = \underbrace{\bar{\Theta} \circ \cdots \circ \bar{\Theta}}_{n}(\bar{P}),$$

for short. In what follows, we also use the notation $\bar{\Theta}^{\bullet}(\bar{P})$ to refer to the collection of these objects $\bar{\Theta}^n(\bar{P})$, $n \in \mathbb{N}$, taken as a whole.

We use the unit morphism of the free operad adjunction $\iota : \bar{\Theta}^n(\bar{P}) \to \Theta(\bar{\Theta}^n(\bar{P}))$ to identify the symmetric collection $\bar{\Theta}^n(\bar{P})$ with a subobject of the free operad $\mathtt{Res}_n(P) = \Theta(\bar{\Theta}^n(\bar{P}))$, for each $n \in \mathbb{N}$. We immediately see that the faces of the cotriple resolution $d_i : \mathtt{Res}_n(P) \to \mathtt{Res}_{n-1}(P)$ satisfy the relation $d_i(\bar{\Theta}^n(\bar{P})) \subset \bar{\Theta}^{n-1}(\bar{P})$ when $i > 0$. We equivalently get that the collection $\bar{\Theta}^{\bullet}(\bar{P})$ inherits face operators $d_i : \bar{\Theta}^n(\bar{P}) \to \bar{\Theta}^{n-1}(\bar{P})$, for $i = 1, \dots, n$, and that the face operators of the cotriple resolution $d_i : \mathtt{Res}_n(P) \to \mathtt{Res}_{n-1}(P)$ such that $i > 0$ are identified with the morphisms of free operads induced by these morphisms on the collection $\bar{\Theta}^{\bullet}(\bar{P})$. We similarly have $s_j(\bar{\Theta}^n(\bar{P})) \subset \bar{\Theta}^{n+1}(\bar{P})$ for the degeneracies $s_j : \mathtt{Res}_n(P) \to \mathtt{Res}_{n+1}(P)$, for all $j = 0, \dots, n$. We equivalently get that the collection $\bar{\Theta}^{\bullet}(\bar{P})$ inherits degeneracy operators $s_j : \bar{\Theta}^n(\bar{P}) \to \bar{\Theta}^{n+1}(\bar{P})$, for $j = 0, \dots, n$, and the degeneracy operators of the cotriple resolution $s_j : \mathtt{Res}_n(P) \to \mathtt{Res}_{n+1}(P)$ are also identified with the morphisms of free operads induced by these morphisms on the collection $\bar{\Theta}^{\bullet}(\bar{P})$. We can define these face and degeneracy operators by the same categorical expression as the face and degeneracy operators of the cotriple resolution (see §B.1.2). We just discard the front free operad functor from our construction.

We readily see, on the other hand, that the object $\bar{\Theta}^{\bullet}(\bar{P})$ is not preserved by the 0th face operator of the cotriple resolution $d_0 : \mathtt{Res}_n(P) \to \mathtt{Res}_{n-1}(P)$, and hence, we can not fully identify the cotriple resolution $\mathtt{Res}_{\bullet}(P)$ with a free operad associated to a simplicial object of the category of collections. We actually get that the cotriple resolution $\mathtt{Res}_{\bullet}(P)$ forms a *quasi-free object* in the category of operads. We make the definition of a quasi-free operad more explicit in §II.8, where we use this notion to give an explicit description of the class of cofibrant objects in the model category of operads in simplicial sets.

We can also express the structure of the object $\bar{\Theta}^{\bullet}(\bar{P})$ in terms of an action of (a subcategory of) the simplicial category Δ. We basically get that the definition of face operations d_i for $i > 0$ and of degeneracy operations s_j on $\bar{\Theta}^{\bullet}(\bar{P})$ is equivalent to the definition of simplicial operators $u^* : \bar{\Theta}^n(\bar{P}) \to \bar{\Theta}^m(\bar{P})$ associated to the morphisms of the simplicial category $u \in \mathtt{Mor}_{\Delta}(\underline{m}, \underline{n})$ such that $u(0) = 0$.

We can readily adapt the construction of Proposition B.1.4 to get an expression, in terms of treewise structures, of the object $\bar{\Theta}^{\bullet}(\bar{P})$. We explicitly have:

$$\bar{\Theta}^{n}(\bar{P})(\underline{\mathsf{r}}) = \operatorname*{colim}_{\underline{\mathsf{Y}}=\underline{\mathsf{T}}_0 \leftarrow \underline{\mathsf{T}}_1 \leftarrow \cdots \leftarrow \underline{\mathsf{T}}_n} P(\underline{\mathsf{T}}_n), \tag{2}$$

for any arity $r \geq 2$ (where our collection is defined) and for any dimension $n \geq 0$. We just assume that the initial term of such a sequence $\underline{\mathsf{T}}_0$ is given by the $\underline{\mathsf{r}}$-corolla $\underline{\mathsf{Y}} = \underline{\mathsf{Y}}_{\underline{\mathsf{r}}}$. We could equivalently discard this object $\underline{\mathsf{T}}_0 = \underline{\mathsf{Y}}$ (without changing the value of our colimit), because the corolla $\underline{\mathsf{Y}}$ is terminal in the category of reduced $\underline{\mathsf{r}}$-trees. We prefer not to do so, because the canonical morphism $\iota : \bar{\Theta}^{n}(\bar{P}) \to \Theta(\bar{\Theta}^{n}(\bar{P}))$ is given by the obvious inclusion of the indexing categories of our colimits when we take the above expression (2) for our object $\bar{\Theta}^{n}(\bar{P})$, for any $n \in \mathbb{N}$. We can moreover use the same expression as in Proposition B.1.7

$$u^*(\underline{\mathsf{Y}} \leftarrow \underline{\mathsf{T}}_1 \leftarrow \cdots \leftarrow \underline{\mathsf{T}}_n, \pi) = (\underline{\mathsf{Y}} \leftarrow \underline{\mathsf{T}}_{u(1)} \leftarrow \cdots \leftarrow \underline{\mathsf{T}}_{u(m)}, \lambda_*(\pi)) \tag{3}$$

with the convention that we have $\underline{\mathsf{T}}_0 = \underline{\mathsf{Y}}$ in order to determine the action of any simplicial map $u \in \mathtt{Mor}_{\Delta}(\underline{m}, \underline{n})$ satisfying $u(0) = 0$ on the collection $\bar{\Theta}^{\bullet}(\bar{P})$.

B.1.9. *The latching object construction.* To any simplicial object in a category $A_{\bullet} \in s\,\mathcal{C}$ we associate a collection of latching objects $\mathtt{L}_n(A) \in \mathcal{C}$, $n \in \mathbb{N}$, whose purpose is to collect the information carried by the degeneracy operators inside each component of our object A_n, $n \in \mathbb{N}$. We formally have:

$$\mathtt{L}_n(A) = \operatorname*{colim}_{\substack{u \in \mathtt{Mor}_{\Delta^-}(\underline{n}, \underline{k}) \\ k<n}} A_k, \tag{1}$$

for any $n \in \mathbb{N}$, where the colimit ranges over the category of surjective non-decreasing maps $u : \underline{n} \to \underline{k}$ satisfying $k < n$, and where a copy of the object A_k is assigned to any such $u \in \mathtt{Mor}_{\Delta^-}(\underline{n}, \underline{k})$. (Recall that Δ^- denotes the subcategory of the simplicial category Δ which has the subclass of surjective non-decreasing maps as morphisms.) We have an equivalent construction where we take all non-decreasing maps $u : \underline{n} \to \underline{k}$ such that $k < n$ (and not only the surjective ones), but we rather use the above expression in this appendix, because our main purpose is to make explicit the latching object of the cotriple resolution. We give a more comprehensive survey on latching objects and simplicial structures in §II.3.

Recall simply that when we form our colimit (1) we consider surjective non-decreasing maps $f : \underline{k} \to \underline{l}$ such that $v = fu$ in order to relate the terms associated to any pair of surjective non-decreasing maps $u : \underline{n} \to \underline{k}$ and $v : \underline{n} \to \underline{l}$. We also have a natural morphism $\lambda : \mathtt{L}_n(A) \to A_n$, defined for any $n \in \mathbb{N}$ and usually called the latching morphism, which is given by the natural transformation $u^* : A_k \to A_n$ associated to any map $u \in \mathtt{Mor}_{\Delta^-}(\underline{k}, \underline{n})$ on the colimit (1). We have the following result:

Proposition B.1.10. *The terms of the cotriple resolution* $\mathtt{Res}_{\bullet}(P)$ *admit a coproduct decomposition in the category of (connected) operads*

$$\mathtt{Res}_n(P) = \mathtt{L}_n\,\mathtt{Res}_{\bullet}(P) \vee \Theta(\mathtt{N}_n\,\bar{\Theta}^{\bullet}(\bar{P}))$$

so that the latching object $\mathtt{L}_n\,\mathtt{Res}_{\bullet}(P)$ *is identified with a subobject of the operad* $\mathtt{Res}_n(P)$*, for each dimension* $n \in \mathbb{N}$.

The collection $\mathtt{N}_n\,\bar{\Theta}^{\bullet}(\bar{P}) \subset \bar{\Theta}^{n}(\bar{P})$*, which occurs in this formula, is defined by the colimit:*

$$\mathtt{N}_n\,\bar{\Theta}^{n}(\bar{P})(\underline{\mathsf{r}}) = \operatorname*{colim}_{\underline{\mathsf{Y}} \xleftarrow{\not\cong} \underline{\mathsf{T}}_1 \xleftarrow{\not\cong} \cdots \xleftarrow{\not\cong} \underline{\mathsf{T}}_n} P(\underline{\mathsf{T}}_n)$$

for any finite set $\underline{\mathsf{r}}$ *of cardinal* $r \geq 2$*, where, in comparison with the expression of* §B.1.8*(2), we restrict ourselves to the chains of morphisms of* $\underline{\mathsf{r}}$*-trees* $\underline{\mathsf{Y}} = \underline{\mathsf{T}}_0 \leftarrow \underline{\mathsf{T}}_1 \leftarrow \cdots \leftarrow \underline{\mathsf{T}}_n$ *satisfying* $\underline{\mathsf{T}}_j \not\simeq \underline{\mathsf{T}}_{j+1}$*, for all* $j = 0, \dots, n-1$*.*

Explanations. We use that the morphisms $u \in \mathtt{Mor}_{\Delta^-}(\underline{k}, \underline{l})$ (the surjective non-decreasing maps) which we consider in the expression of the latching objects $\mathtt{L}_n \mathtt{Res}_\bullet(\mathsf{P})$ necessarily have $u(0) = 0$ and act on the cotriple resolution $\mathtt{Res}_\bullet(\mathsf{P})$ by morphisms of free operads $u^* : \Theta(\bar{\Theta}^k(\bar{\mathsf{P}})) \to \Theta(\bar{\Theta}^l(\bar{\mathsf{P}}))$ which we determine by the expression of §B.1.8(3) on the generating collection of our object $\bar{\Theta}^k(\bar{\mathsf{P}}) \subset \Theta(\bar{\Theta}^k(\bar{\mathsf{P}}))$. We accordingly have the relation:

$$\mathtt{L}_n \mathtt{Res}_\bullet(\mathsf{P}) = \Theta(\mathtt{L}_n \bar{\Theta}^\bullet(\bar{\mathsf{P}})), \tag{1}$$

where we perform the latching object construction on the generating collection of our operad:

$$\mathtt{L}_n \bar{\Theta}^\bullet(\bar{\mathsf{P}})(\underline{\mathsf{r}}) = \operatorname*{colim}_{\substack{u \in \mathtt{Mor}_{\Delta^-}(\underline{n}, \underline{k}) \\ k < n}} \Bigl(\underbrace{\operatorname*{colim}_{\underline{\mathsf{Y}} = \underline{\mathsf{T}}_0 \leftarrow \underline{\mathsf{T}}_1 \leftarrow \cdots \leftarrow \underline{\mathsf{T}}_k} \mathsf{P}(\underline{\mathsf{T}}_k)}_{= \bar{\Theta}^k(\bar{\mathsf{P}})(\underline{\mathsf{r}})} \Bigr). \tag{2}$$

We then have the immediate splitting formula:

$$\underbrace{\operatorname*{colim}_{\underline{\mathsf{Y}} = \underline{\mathsf{T}}_0 \leftarrow \underline{\mathsf{T}}_1 \leftarrow \cdots \leftarrow \underline{\mathsf{T}}_n} \mathsf{P}(\underline{\mathsf{T}}_n)}_{= \bar{\Theta}^n(\bar{\mathsf{P}})(\underline{\mathsf{r}})} = \underbrace{\Bigl(\operatorname*{colim}_{\substack{\underline{\mathsf{Y}} = \underline{\mathsf{T}}_0 \leftarrow \underline{\mathsf{T}}_1 \leftarrow \cdots \leftarrow \underline{\mathsf{T}}_n \\ \exists j \,|\, \underline{\mathsf{T}}_j \simeq \underline{\mathsf{T}}_{j+1}}} \mathsf{P}(\underline{\mathsf{T}}_n) \Bigr)}_{\simeq \mathtt{L}_n \bar{\Theta}^\bullet(\bar{\mathsf{P}})(\underline{\mathsf{r}})} \amalg \underbrace{\Bigl(\operatorname*{colim}_{\substack{\underline{\mathsf{Y}} = \underline{\mathsf{T}}_0 \leftarrow \underline{\mathsf{T}}_1 \leftarrow \cdots \leftarrow \underline{\mathsf{T}}_n \\ \forall j \,|\, \underline{\mathsf{T}}_j \not\simeq \underline{\mathsf{T}}_{j+1}}} \mathsf{P}(\underline{\mathsf{T}}_n) \Bigr)}_{=: \mathtt{N}_n \bar{\Theta}^\bullet(\bar{\mathsf{P}})(\underline{\mathsf{r}})}$$

from which we get the result of our proposition, just because the first subcolimit of this decomposition has the same value (exercise) as the latching object colimit of (2). □

B.1.11. *The cotriple resolution of augmented connected* Λ*-operads.* The constructions of this section have a straightforward extension to augmented connected Λ-operads. We then consider our lifting of the free operad functor $\Theta : \mathsf{M} \mapsto \Theta(\mathsf{M})$ to the category of augmented connected Λ-operads (see §§A.4.6-A.4.7 and Proposition I.2.4.3) and the adjunction relation $\Theta : \Lambda\, \mathcal{C}oll_{>1} / \mathsf{Com} \rightleftarrows \Lambda\, \mathcal{O}p_{\varnothing 1} / \mathsf{Com} : \bar{\omega}$ between the category of augmented connected Λ-collections $\Lambda\, \mathcal{C}oll_{>1} / \mathsf{Com}$ and the category of augmented connected Λ-operads $\Lambda\, \mathcal{O}p_{\varnothing 1} / \mathsf{Com}$ (see Theorem I.2.4.4).

The object $\mathtt{Res}_\bullet(\mathsf{P})$ which we consider in this case $\mathsf{P} \in \Lambda\, \mathcal{O}p_{\varnothing 1} / \mathsf{Com}$ is accordingly obtained by performing the cotriple resolution construction in the category of plain (connected) operads first, and by using that the free operad $\Theta(\mathsf{M})$ associated to any augmented connected Λ-collection $\mathsf{M} \in \Lambda\, \mathcal{O}p_{\varnothing 1} / \mathsf{Com}$ inherits natural restriction operators $u^* : \Theta(\mathsf{M})(\underline{\mathsf{l}}) \to \Theta(\mathsf{M})(\underline{\mathsf{k}})$, $u \in \mathtt{Mor}_{\mathcal{I}nj}(\underline{\mathsf{k}}, \underline{\mathsf{l}})$ and natural augmentations $\epsilon : \Theta(\mathsf{M})(\underline{\mathsf{r}}) \to \mathbb{1}$, $r > 0$ (see §A.2.8).

We can easily make the expression of these operations explicit on the treewise representation of Proposition B.1.4. We basically get the same formula $u^*(\underline{\mathsf{T}}_0 \leftarrow \cdots \leftarrow \underline{\mathsf{T}}_n, \pi) = (u^* \underline{\mathsf{T}}_0 \leftarrow \cdots \leftarrow u^* \underline{\mathsf{T}}_n, u^*(\pi))$ for the restriction operators $u^* : \mathtt{Res}_n(\mathsf{P})(\underline{\mathsf{l}}) \to \mathtt{Res}_n(\mathsf{P})(\underline{\mathsf{k}})$ as in the case of the action of permutations, but we now consider the restriction operators on the category of reduced trees $u^* : \widetilde{\mathcal{T}ree}(\underline{\mathsf{l}}) \to \widetilde{\mathcal{T}ree}(\underline{\mathsf{k}})$, such as defined in §A.1.13, together with the corresponding reduced restriction operators on treewise tensor products $u^* : \mathsf{P}(\underline{\mathsf{T}}_n) \to \mathsf{P}(u^* \underline{\mathsf{T}}_n)$ which we define in §A.4.6. We use the functoriality of the restriction operators $u^* : \widetilde{\mathcal{T}ree}(\underline{\mathsf{l}}) \to$

$\widetilde{\mathcal{T}ree}(\underline{\mathsf{k}})$ with respect to our notion of a tree morphism in order to give a sense to the above formula.

The augmentation $\epsilon : \mathtt{Res}_n(\mathsf{P})(\underline{\mathsf{r}}) \to \mathbb{1}$ is given by the obvious formula $\epsilon(\underline{\mathsf{T}}_0 \leftarrow \cdots \leftarrow \underline{\mathsf{T}}_n, \pi) = \epsilon(\pi)$, for any arity $r > 0$, where we forget about our chain of tree-morphisms and we take the image of the tensor $\pi \in \mathsf{P}(\underline{\mathsf{T}}_n)$ under the treewise augmentation operation $\epsilon : \mathsf{P}(\underline{\mathsf{T}}_n) \to \mathbb{1}$ of §A.2.8. The augmentation $\epsilon : \mathtt{Res}_\bullet(\mathsf{P}) \to \mathsf{Com}$, regarded as a whole operad morphism, is also identified with the composite of the morphism $\mathtt{Res}_\bullet(\epsilon) : \mathtt{Res}_\bullet(\mathsf{P}) \to \mathtt{Res}_\bullet(\mathsf{Com})$, which we deduce from the functoriality of the cotriple resolution construction, and of the natural augmentation $\epsilon : \mathtt{Res}_\bullet(\mathsf{Com}) \to \mathsf{Com}$, which we attach to the cotriple resolution of the commutative operad.

B.1.12. *The treewise representation of the cotriple resolution for general (non-augmented) operads.* We can adapt the constructions of this section to the case of the cotriple resolution of general (non-connected and non-augmented) operads. We just outline the modifications which we have to perform in this setting. We still restrict ourselves to non-unitary operads for simplicity (though the general process does not change when we have operations in arity zero).

We then have to consider the standard free operad adjunction $\mathbb{\Theta} : \mathcal{C}oll_{>0} \to \mathcal{O}p_{\varnothing} : \omega$ and a monad structure which we associate to the full free operad functor $\mathbb{\Theta} : \mathcal{C}oll_{>0} \to \mathcal{C}oll_{>0}$. We explicitly deal with a unit morphism $\iota : Id \to \mathbb{\Theta}$, given by the standard unit morphism of the free operad adjunction, and a multiplication $\mu : \mathbb{\Theta} \circ \mathbb{\Theta} \to \mathbb{\Theta}$ defined by applying the augmentation of this adjunction to a free operad $\mathsf{P} = \mathbb{\Theta}(-)$. For $\mathsf{P} \in \mathcal{O}p_{\varnothing}$, we now form a simplicial object such that:

$$\mathtt{Res}_n(\mathsf{P}) = \mathbb{\Theta} \circ \underbrace{\mathbb{\Theta} \circ \cdots \circ \mathbb{\Theta}}_{n}(\mathsf{P}), \tag{1}$$

for any dimension $n \in \mathbb{N}$, and where the face operators $d_i : \mathtt{Res}_n(\mathsf{P}) \to \mathtt{Res}_{n-1}(\mathsf{P})$ are determined by the monadic multiplication of the $(i, i+1)$th free operad factors $\mu : \mathbb{\Theta} \circ \mathbb{\Theta} \to \mathbb{\Theta}$ for $i = 0, \dots, n-1$, and by the plain adjunction augmentation of free operads $\lambda : \mathbb{\Theta}(\mathsf{P}) \to \mathsf{P}$ for $i = n$. We just regard the front free operad functor of this expression as a functor from the category of collections to the category of operads $\mathbb{\Theta} : \mathcal{C}oll_{>0} \to \mathcal{O}p_{\varnothing}$ and, when we form the 0th face $d_0 : \mathtt{Res}_n(\mathsf{P}) \to \mathtt{Res}_{n-1}(\mathsf{P})$, we identify our monadic multiplication operation with a right action of the free operad monad $\rho : \mathbb{\Theta} \circ \mathbb{\Theta} \to \mathbb{\Theta}$ on this functor $\mathbb{\Theta} : \mathcal{C}oll_{>0} \to \mathcal{O}p_{\varnothing}$. The degeneracy morphisms of our object $s_j : \mathtt{Res}_n(\mathsf{P}) \to \mathtt{Res}_{n+1}(\mathsf{P})$ are given by the insertion of adjunction units $\iota : Id \to \mathbb{\Theta}$ (as in the connected operad case).

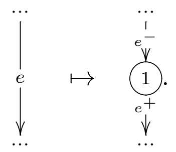
We still have an augmentation $\epsilon : \mathtt{Res}_\bullet(\mathsf{P}) \to \mathsf{P}$ defined by the augmentation of the free operad adjunction $\lambda : \mathbb{\Theta}(\mathsf{P}) \to \mathsf{P}$ in dimension zero $\mathtt{Res}_0(\mathsf{P}) = \mathbb{\Theta}(\mathsf{P})$. We can moreover define a section of this augmentation $\eta : \mathsf{P} \to \mathbb{\Theta}(\mathsf{P})$ in the category of symmetric collections and extra-degeneracies $s_{-1} : \mathtt{Res}_n(\mathsf{P}) \to \mathtt{Res}_{n+1}(\mathsf{P})$ (as in the connected operad case again).

For this cotriple resolution, we have a version of the treewise expansion of Proposition B.1.4:

$$\mathtt{Res}_n(\mathsf{P})(\underline{\mathsf{r}}) = \operatorname*{colim}_{\underline{\mathsf{T}}_0 \leftarrow \cdots \leftarrow \underline{\mathsf{T}}_n} \mathsf{P}(\underline{\mathsf{T}}_n). \tag{2}$$

We essentially need to enlarge our class of morphisms in order to keep track of the operadic units 1 which naturally occur as soon as we consider the full free operad functor $\mathbb{\Theta}(-)$. For this aim, we consider degeneration maps $s_e : \underline{\mathsf{T}} \mapsto s_e(\underline{\mathsf{T}})$ given by the insertion of an extra one-input vertex on any edge $e \in E(\underline{\mathsf{T}})$ of a given tree

$\underline{\mathsf{T}} \in \mathcal{T}ree(\underline{\mathsf{r}})$:



We add these maps as formal morphisms $s_e : \underline{\mathsf{T}} \to s_e(\underline{\mathsf{T}})$ to our category of $\underline{\mathsf{r}}$-trees. We also assume that the degeneration maps and the ordinary tree morphisms satisfy natural relations which reflect the identities which we get when we insert an operadic unit in the treewise composition products of an operad. We write $\mathcal{T}ree(\underline{\mathsf{r}})_{\mathsf{T}}$ for this extended category of $\underline{\mathsf{r}}$-trees.

To each degeneration map $s_e : \underline{\mathsf{T}} \to s_e(\underline{\mathsf{T}})$, we associate the morphism of treewise tensors $s_e : \mathsf{P}(\underline{\mathsf{T}}) \to \mathsf{P}(s_e(\underline{\mathsf{T}}))$ defined by inserting an operadic unit $1 \in \mathsf{P}(1)$ on the extra vertex of the tree $s_e(\underline{\mathsf{T}})$ (compare with the construction of §A.5.3). The mapping $\mathsf{P} : \underline{\mathsf{T}} \mapsto \mathsf{P}(\underline{\mathsf{T}})$ accordingly extends to a functor on the category $\mathcal{T}ree(\underline{\mathsf{r}})_{\mathsf{T}}$, for any arity $r > 0$. We precisely use morphisms $f : \underline{\mathsf{T}}_{i+1} \to \underline{\mathsf{T}}_i$ of the extended category $\mathcal{T}ree(\underline{\mathsf{r}})_{\mathsf{T}}$ together with these extended treewise composition operations $f_* : \mathsf{P}(\underline{\mathsf{T}}_{i+1}) \to \mathsf{P}(\underline{\mathsf{T}}_i)$ which we associate to such morphisms in order to get the treewise definition (2) of the cotriple resolution of plain operads (1).

The observations of §B.1.8 and the result of Proposition B.1.10 work for the resolution of general (non-unitary) operads when we take this extended version of our tree categories. We can also extend the treewise definition of restriction operators and augmentations in §B.1.11 when we assume $\mathsf{P} \in \Lambda\,\mathcal{O}p_{\varnothing} / \mathsf{Com}$. We then consider the un-reduced restriction operators on trees (see §A.1.11) and on treewise tensors (see §A.2.8).

B.2. The monadic definition of operads

To complete the account of this appendix, we quickly review the general definition of a monad and we revisit the monadic interpretation of the structure of an operad.

B.2.1. *Monads.* In general, a *monad* on a category $\mathcal{C}$ is a functor $\mathbb{F} : \mathcal{C} \to \mathcal{C}$ together with natural transformations $\iota : Id \to \mathbb{F}$ and $\mu : \mathbb{F} \circ \mathbb{F} \to \mathbb{F}$ which make the diagrams

$$\begin{tikzcd}
Id \circ \mathbb{F} \arrow[r, "\iota\circ\mathbb{F}"] \arrow[dr, "="'] & \mathbb{F} \circ \mathbb{F} \arrow[d, "\mu"] & \mathbb{F} \circ Id \arrow[l, "\mathbb{F}\circ\iota"'] \arrow[dl, "="] \\
& \mathbb{F} &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathbb{F} \circ \mathbb{F} \circ \mathbb{F} \arrow[r, "\mu\circ\mathbb{F}"] \arrow[d, "\mathbb{F}\circ\mu"'] & \mathbb{F} \circ \mathbb{F} \arrow[d, "\mu"] \\
\mathbb{F} \circ \mathbb{F} \arrow[r, "\mu"] & \mathbb{F}
\end{tikzcd}$$

commute. These diagrams express a version, for functors, of the usual unit and associativity relations of monoids. In fact, a monad can be identified with a monoid in the category of functors $\mathbb{F} : \mathcal{C} \to \mathcal{C}$ where we take the composition of functors as the (non-symmetric) tensor product operation of a monoidal structure.

To a monad $\mathbb{F}$, we associate a category of algebras, consisting of the objects $A \in \mathcal{C}$ equipped with a morphism $\lambda : \mathbb{F}(A) \to A$ such that the diagrams

$$\begin{tikzcd} Id(A) \arrow[r, "\iota(A)"] \arrow[dr, "="'] & \mathbb{F}(A) \arrow[d, "\lambda"] \\ & A \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathbb{F}\circ\mathbb{F}(A) \arrow[r, "\mu(A)"] \arrow[d, "\mathbb{F}(\lambda)"'] & \mathbb{F}(A) \arrow[d, "\lambda"] \\ \mathbb{F}(A) \arrow[r, "\lambda"] & A \end{tikzcd}$$

commute. We say that the morphism $\lambda : \mathbb{F}(A) \to A$ defines a *left action of the monad* $\mathbb{F}$ on the object $A \in \mathcal{C}$. Naturally, a morphism of $\mathbb{F}$-algebras is a morphism in the base category $f : A \to B$ which makes commute the diagram

$$\begin{tikzcd} \mathbb{F}(A) \arrow[r, "\mathbb{F}(f)"] \arrow[d, "\lambda"'] & \mathbb{F}(B) \arrow[d, "\lambda"] \\ A \arrow[r, "f"] & B \end{tikzcd} ,$$

where we use the generic notation λ to refer to all actions of the monad $\mathbb{F}$.

B.2.2. *The monad associated to an adjunction.* The monad associated to the free operad functor $\mathbb{\Theta} : \mathcal{C}oll_{>0} \to \mathcal{O}p_{\varnothing}$ (and the monad associated to the connected free operad similarly $\mathbb{\Theta} : \mathcal{C}oll_{>1} \to \mathcal{O}p_{\varnothing 1}$) is defined by applying a general construction of category theory, which we now review, to the free operad adjunction.

We generally start with an adjunction relation $\mathbb{A} : \mathcal{C} \rightleftarrows \mathcal{A} : \omega$, where $\mathbb{A} : \mathcal{C} \to \mathcal{A}$ represents a free object functor with values in a category $\mathcal{A}$. We take the composite $\mathbb{F} = \omega\,\mathbb{A}$ to define the underlying functor of our monad $\mathbb{F} : \mathcal{C} \to \mathcal{C}$. The monad unit $\iota : Id \to \mathbb{F}$ is given by the unit morphism of our adjunction. The monad multiplication $\mu : \mathbb{F}\circ\mathbb{F} \to \mathbb{F}$ is formed by applying the adjunction augmentation $\lambda : \mathbb{A}(\omega A) \to A$ to the free objects $A = \mathbb{A}(X)$ and by taking the image of this morphism under the functor $\omega : \mathcal{A} \to \mathcal{C}$. The general structure relations between adjunction units and adjunction augmentations formally imply that these morphisms satisfy the unit and associativity relations of §B.2.1 (see [130, §VI.1]).

We clearly retrieve the construction of §B.1.1 when we apply this definition of a monad to the free connected operad adjunction $\mathbb{A} = \mathbb{\Theta} : \mathcal{C}oll_{>1} \rightleftarrows \mathcal{O}p_{\varnothing 1} : \bar{\omega}$. We then have $\mathbb{F} = \bar{\omega}\,\mathbb{\Theta} = \bar{\mathbb{\Theta}}$. We similarly retrieve the monad structure of the free operad, such as defined in §B.1.12, when we apply our construction to the adjunction $\mathbb{A} = \mathbb{\Theta} : \mathcal{C}oll_{>0} \rightleftarrows \mathcal{O}p_{\varnothing} : \omega$. We then have $\mathbb{F} = \mathbb{\Theta}$, where we regard the free operad $\mathbb{\Theta}(-)$ as an endofunctor of the category of (non-unitary) collections $\mathcal{C}oll_{>0}$ (by forgetting about composition structures).

The augmentation of an adjunction $\lambda : \mathbb{A}(\omega A) \to A$ generally induces a morphism $\lambda : \omega\,\mathbb{A}(\omega A) \to \omega A$ which provides the object $X = \omega A$ associated to any $A \in \mathcal{A}$ with the structure of an algebra over the monad $\mathbb{F} = \omega\,\mathbb{A}$ (we still readily deduce the unit and associativity relations of monad actions for this morphism from the general structure relations between adjunction units and adjunction augmentations). Thus, we have a well-defined functor from the category $\mathcal{A}$ to the category of algebras over the monad $\mathbb{F}$.

We retrieve the monad action $\lambda : \bar{\mathbb{\Theta}}(\mathsf{P}) \to \bar{\mathsf{P}}$ of §B.1 when we apply this definition to connected operads $\mathsf{P} \in \mathcal{O}p_{\varnothing 1}$. We similarly retrieve the monad action $\lambda : \mathbb{\Theta}(\mathsf{P}) \to \mathsf{P}$ defined by the plain adjunction augmentation of the free operad when we assume $\mathsf{P} \in \mathcal{O}p_{\varnothing}$.

We have the following additional statement (which is not fulfilled by all adjunctions) in the operad case:

THEOREM B.2.3.

(a) *The adjunction* $\mathbb{\Theta} : \mathcal{C}oll_{>0} \rightleftarrows \mathcal{O}p_{\varnothing} : \omega$ *is monadic in the sense that the functor which maps a (non-unitary) operad* $\mathsf{P} \in \mathcal{O}p_{\varnothing}$ *to the associated algebra over the free operad monad* $\mathbb{\Theta}$ *defines an equivalence of categories.*

(b) *The adjunction* $\mathbb{\Theta} : \mathcal{C}oll_{>1} \rightleftarrows \mathcal{O}p_{\varnothing 1} : \bar{\omega}$ *for connected operads is monadic too: the functor which carries a connected operad* $\mathsf{P} \in \mathcal{O}p_{\varnothing 1}$ *to the associated augmentation ideal* $\bar{\mathsf{P}} \in \mathcal{C}oll_{>1}$ *induces an equivalence of categories from the category of connected operads to the category of algebras over the monad* $\bar{\mathbb{\Theta}}$.

PROOF (OUTLINE). We generally say that an adjunction $\mathbb{A} : \mathcal{C} \rightleftarrows \mathcal{A} : \omega$ is *monadic* when we have such an equivalence of categories between $\mathcal{A}$ and the category of algebras over the monad $\mathbb{A} = \omega\,\mathbb{F}$.

This theorem is stated as a remark. We therefore just outline the proof of these equivalences of categories. We still focus on the connected operad case. We mainly use a version, in dimension one, of the treewise representation of the cotriple resolution of connected operads. We basically identify an element $\pi \in \bar{\mathbb{\Theta}} \circ \bar{\mathbb{\Theta}}(\mathsf{M})$ with a treewise tensor $\pi \in \mathsf{M}(\underline{\mathsf{T}})$ together with a subtree decomposition of the form considered in Proposition B.0.3 (see also Proposition B.1.4) and which we can also determine by a tree morphism $\underline{\mathsf{T}} \to \underline{\mathsf{S}}$. The monad multiplication $\mu : \bar{\mathbb{\Theta}} \circ \bar{\mathbb{\Theta}}(\mathsf{M}) \to \bar{\mathbb{\Theta}}(\mathsf{M})$ is the mapping which forgets about this extra decomposition (compare with the expression of faces on the cotriple resolution in Proposition B.1.5).

The structure of an algebra over the monad $\bar{\mathbb{\Theta}}$ is determined by treewise composition operations $\lambda_{\underline{\mathsf{T}}} : \mathsf{M}(\underline{\mathsf{T}}) \to \mathsf{M}(\underline{\mathsf{r}})$, defined for any $\underline{\mathsf{T}} \in \widetilde{\mathcal{T}ree}(\underline{\mathsf{r}})$ and for any arity $r \geq 2$, since we construct the underlying functor of the free operad $\mathbb{\Theta}(\mathsf{M})$ by taking a colimit of these treewise tensor products $\mathsf{M}(\underline{\mathsf{T}})$ over the isomorphism categories of reduced trees. To express the associativity relation of this monadic algebra structure, we have to plug our morphism $\lambda : \bar{\mathbb{F}}(\mathsf{M}) \to \mathsf{M}$ in the composite functor $\bar{\mathbb{\Theta}} \circ \bar{\mathbb{\Theta}}(\mathsf{M})$. This operation is equivalent to the performance of treewise composition operations $\lambda_{\underline{\mathsf{T}}} : \mathsf{M}(\underline{\mathsf{T}}) \to \mathsf{M}(\underline{\mathsf{r}})$ within the components of the subtree decompositions which we associate to the shape of the treewise tensors $\pi \in \bar{\mathbb{\Theta}} \circ \bar{\mathbb{\Theta}}(\mathsf{M})$. (We use the same process as in the definition of the top face of the cotriple resolution in Proposition B.1.5.)

We now consider the treewise composition operations $\lambda_{\underline{\Gamma}} : \mathsf{M}(\underline{\Gamma}) \to \mathsf{M}(\underline{\mathsf{r}})$ associated to two-fold trees $\underline{\Gamma}$. We can just use the equivalence between these morphisms $\lambda_{\underline{\Gamma}} : \mathsf{M}(\underline{\Gamma}) \to \mathsf{M}(\underline{\mathsf{r}})$ and partial composites $\circ_{i_k} : \mathsf{M}(\underline{\mathsf{m}}) \otimes \mathsf{M}(\underline{\mathsf{n}}) \to \mathsf{M}(\underline{\mathsf{m}} \circ_{i_e} \underline{\mathsf{n}})$, to determine a partial composition structure from a monad action $\lambda : \bar{\mathbb{\Theta}}(\mathsf{M}) \to \mathsf{M}$. The associativity of monad actions implies that these composition operations fulfill the associativity axioms of operads, such as expressed in the diagrams of Figure A.8 (we consider the different subtree decompositions of the trees occurring in this figure and we use our treewise interpretation of the monad multiplication). Then we just formally add a unit term $\mathbb{1}$ in arity one in order to form an operad (with unit) from our algebra M over the monad $\bar{\mathbb{\Theta}}$. Thus, we also have a mapping from the category of algebras over the monad $\bar{\mathbb{\Theta}}$ to the category of connected operads, and this mapping is clearly a right inverse of the natural functor considered in the proposition.

The other way round, when we have an algebra over the monad $\bar{\mathbb{\Theta}}$, we can readily check that the monad action $\lambda : \bar{\mathbb{\Theta}}(\mathsf{M}) \to \mathsf{M}$ is fully determined by the terms $\lambda_{\underline{\Gamma}} : M(\underline{\Gamma}) \to M(\underline{r})$ which we associate to (reduced) trees with two vertices $\underline{\Gamma} \in \widetilde{\mathcal{T}ree}_2(\underline{r})$. We still use that the morphism $\lambda : \bar{\Theta}(P) \to P$ is defined by putting the treewise composition operations $\lambda_{\underline{T}} : P(\underline{T}) \to P(\underline{r})$. We can apply the edge contraction process of §§A.2.4-A.2.5, which we now interpret in terms of iterated monad actions (by using the treewise interpretation of our monadic structures and suitable subtree decompositions), to determine these morphisms $\lambda_{\underline{T}} : M(\underline{T}) \to M(\underline{r})$ from the two-fold composites $\lambda_{\underline{\Gamma}} : M(\underline{\Gamma}) \to M(\underline{r})$. Therefore, we obtain that our mapping from the category of algebras over the monad $\bar{\Theta}$ towards the category of connected operads is both a left and a right inverse of the natural functor from the category of connected operads to the category of algebras over the monad $\bar{\Theta}$. The claim of our theorem follows. ☐

B.2.4. *The comonad associated with the free (connected) operad adjunction.* The structure of a comonad is dual to a monad and consists of a functor $\mathbb{C} : \mathcal{C} \to \mathcal{C}$ on a given category $\mathcal{C}$ equipped with natural transformations $\pi : \mathbb{C} \to Id$ and $\nu : \mathbb{C} \to \mathbb{C} \circ \mathbb{C}$ that satisfy the opposite of the unit and associativity relations of monads (simply reverse arrow directions in the diagram of §B.2.1). Similarly, a coalgebra over a comonad $\mathbb{C}$ is an object $A \in \mathcal{C}$ equipped with a structure morphism $\rho : A \to \mathbb{C}(A)$ which satisfies the opposite of the unit and associativity relations of §B.2.1 for algebras over monads.

We can also form a comonad structure from any adjunction relation $\mathbb{A} : \mathcal{C} \rightleftarrows \mathcal{A} : \omega$. We just compose the functors of §B.2.2 the other way round to get this result. We have for instance a comonad $\mathbb{C} = \Theta\,\bar{\omega} : \mathcal{O}p_{\varnothing 1} \to \mathcal{O}p_{\varnothing 1}$ which we associate to the free connected operad adjunction $\Theta : \mathcal{C}oll_{>1} \to \mathcal{O}p_{\varnothing 1} : \bar{\omega}$. The adjunction augmentation $\lambda : \Theta(\bar{P}) \to P$ yields the counit of this comonad $\pi = \lambda : \Theta\,\bar{\omega} \to Id$, while the adjunction unit $\iota : M \to \bar{\Theta}(M)$ induces an operad morphism $\Theta(\iota(M)) : \Theta(M) \to \Theta(\bar{\Theta}(M))$ and we apply this natural morphism to the object $M = \bar{P}$, for any $P \in \mathcal{O}p_{\varnothing 1}$, in order to get the comultiplication of our comonad $\nu : \Theta\,\bar{\omega} \to \Theta\,\bar{\omega} \circ \Theta\,\bar{\omega}$.

The cotriple resolution $\mathtt{Res}_\bullet(P)$ can be defined in terms of this comonad rather than in terms of the monad structure of §B.1.1. We just change the functor groupings in the expression of §B.1.2:

$$\mathtt{Res}_n(P) = \Theta \circ \underbrace{\bar{\omega}\,\Theta \circ \cdots \circ \bar{\omega}\,\Theta}_{n} \circ \bar{\omega}(P) \quad \Leftrightarrow \quad \mathtt{Res}_n(P) = \Theta\,\bar{\omega} \circ \underbrace{\Theta\,\bar{\omega} \circ \cdots \circ \Theta\,\bar{\omega}}_{n}(P)$$

and we use the obvious applications of the structure morphisms of our comonad to retrieve the face and degeneracy operators of §B.1.2. We can make similar observations for general (non-connected) operads. This definition motivates the name '*cotriple*' (for comonad) given to our resolution.

Glossary of Notation

Background

Fundamental objects

$\Bbbk$: the ground ring
$\mathbb{D}^n$: the unit n-disc, see §I.4.1.1
$\mathbb{\Delta}^n$: the topological n-simplex, see §0.3, §II.1.3.4
pt: the one-point set (also denoted by $*$ when regarded as a terminal object)
Δ: the simplicial category, see §0.3, §II.1.3.2
Δ^n: the n-simplex object of the category of simplicial sets, see §0.3, §II.1.3.4

Generic categorical notation

$\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, ...: general categories
$\mathcal{I}$, $\mathcal{J}$, ...: indexing categories, as well as the set of generating cofibrations and the set of generating acyclic cofibrations in a cofibrantly generated model category, see §II.4.1.3
$\mathcal{F}$, $\mathcal{G}$, ...: some classes of morphisms in a category
$\mathcal{M}$, $\mathcal{N}$, ...: (symmetric) monoidal categories, see §0.8
$\mathbb{1}$: the unit object of a (symmetric) monoidal category, see §0.8
eq: the equalizer of parallel arrows in a category
coeq: the coequalizer of parallel arrows in a category

Fundamental categories

$\mathcal{M}od$: the category of modules over the ground ring
$\mathcal{S}et$: the category of sets
$\mathcal{T}op$: the category of topological spaces, see §II.1.3
$s\mathcal{S}et$: the category of simplicial sets, see §0.3, §II.1.3
$\mathcal{G}rp$: the category of groups
$\mathcal{G}rd$: the category of groupoids, see §I.5.2.1
$\mathcal{C}at$: the category of small categories, see §I.5.2.1
$\mathcal{A}b$: the category of abelian groups

Categories of algebras and of coalgebras

$\mathcal{C}om$: the category of non-unitary commutative algebras
$\mathcal{A}s$: the category of non-unitary associative algebras
$\mathcal{L}ie$: the category of Lie algebras
$\mathcal{A}s_+$: the category of unitary associative algebras
$\mathcal{C}om_+$: the category of unitary commutative algebras, see §I.3.0.1
$\mathcal{C}om^c_+$: the category of counitary cocommutative coalgebras, see §I.3.0.4

$\mathcal{H}opf\mathcal{A}lg$: the category of Hopf algebras (defined as the category of bialgebras equipped with an antipode operation), see §I.7.1.8

$\mathcal{H}opf\mathcal{G}rd$: the category of Hopf groupoids (defined as the category of small categories equipped with an antipode operation), see §I.9.0.2

Functors and constructions for filtered objects

$\mathtt{F}_s$: the sth layer of a decreasing filtration

$\mathtt{E}^0_s$: the sth subquotient of a filtered object, see §I.7.3.6 (also used to denote the sth fiber of a tower of set maps in the context of homotopy spectral sequences, see §III.1.1.7)

$\mathtt{E}^0$: the weight graded object associated to a filtered object in a category (e.g. the weight graded module associated to a filtered module, see §I.7.3.6, the weight graded Lie algebra associated to a Malcev complete group, see §I.8.2.2, ...)

$(-)\hat{}$: the completion functor on a category of objects equipped with a decreasing filtration, as well as the Malcev completion for groups and groupoids, see §I.7.3.4, §I.8.3 (also the rationalization functor on spaces, see the section about the constructions of homotopy theory in this glossary)

Functors and constructions on algebras and coalgebras

$\mathbb{S}$: the symmetric algebra functor (in any symmetric monoidal category), see §I.7.2.4

$\mathbb{T}$: the tensor algebra functor (in any symmetric monoidal category), see §I.7.2.4

$\mathbb{L}$: the free Lie algebra functor (in any $\mathbb{Q}$-additive symmetric monoidal category and in abelian groups), see §I.7.2.3

$\mathbb{U}$: the enveloping algebra functor (on the category of Lie algebras in any $\mathbb{Q}$-additive symmetric monoidal category), see §I.7.2.7

$\hat{\mathbb{S}}$, $\hat{\mathbb{T}}$, ...: the complete variants of the symmetric algebra functor, of the tensor algebra functor, ... in the context of a category of complete filtered modules, see §I.7.3.22

$\mathbb{G}$: the group-like element functor on coalgebras, see §I.7.1.14, and on complete Hopf coalgebras, see §I.8.1.2

$\mathbb{P}$: the primitive element functor on Hopf algebras, see §I.7.2.11

$\mathbb{I}(-)$: the augmentation ideal of Hopf algebras, see §I.8.1.1

Categorical prefixes

dg: prefix for a category of differential graded objects in a category (e.g. the category of dg-modules $dg\,\mathcal{M}od$, see §0.1, §II.5.0.1)

dg_*, dg^*: prefix for the chain graded and cochain graded variants of the categories of differential graded objects (e.g. the category of chain graded dg-modules $dg_*\,\mathcal{M}od$, see §II.5.0.1, the category of cochain graded dg-modules $dg^*\,\mathcal{M}od$, see §II.5.0.1, §II.5.1, and the category of unitary commutative cochain dg-algebras $dg^*\,\mathcal{C}om_+$, see §II.6.1.1, ...)

gr: prefix for a category of graded objects in a category when the grading underlies a differential graded structure (e.g. the category of graded modules $gr\,\mathcal{M}od$, see §0.1, see §II.5.0.2, ...)

s: prefix for a category of simplicial objects in a category (e.g. the category of simplicial modules $s\,\mathcal{M}od$, see §0.6, §II.5.0.4, the category of simplicial sets $s\mathcal{S}et$, see §0.3, §II.1.3, ...)

c: prefix for a category of cosimplicial objects in a category (e.g. the category of cosimplicial modules $c\,\mathcal{M}od$, see §0.6, §II.5.0.4, the category of cosimplicial unitary commutative algebras $c\,\mathcal{C}om_+$, see §II.6.1.3, ...)
f: prefix for a category of filtered objects in a category (e.g. the category of filtered modules $f\,\mathcal{M}od$, see §I.7.3.1)
$\hat{f}$: prefix for a category of complete filtered objects in a category (e.g. the category of complete filtered modules $\hat{f}\,\mathcal{M}od$, see §I.7.3.4, the category of Malcev complete groups $\hat{f}\,\mathcal{G}rp$, see §I.8.2). Note that the categories of complete Hopf algebras $\hat{f}\,\mathcal{H}opf\mathcal{A}lg$ and of complete Lie algebras $\hat{f}\,\mathcal{L}ie$ consist of Hopf algebras and Lie algebras in complete filtered modules that satisfy an extra connectedness requirement and a similar convention is made for the category of complete Hopf groupoids $\hat{f}\,\mathcal{H}opf\mathcal{G}rd$, see §I.7.3.15, §I.7.3.20, §I.9.1.2
w: prefix for a category of weight graded objects in a category (e.g. the category of weight graded modules $w\,\mathcal{M}od$, see §I.7.3.5)

Morphisms, hom-objects, duals, and analogous constructions

`Mor`: the notation for the morphism sets of any category (e.g. $\mathtt{Mor}_{\mathcal{M}od}(-,-)$ for the morphism sets of the category of modules over the ground ring $\mathcal{M}od$)
`Aut`: the notation for the automorphism group of an object in a category
`Hom`: the notation for the hom-objects of an enriched category structure (not to be confused with the morphism sets), see §0.12
`D`: the duality functor for ordinary modules, dg-modules, simplicial modules and cosimplicial modules, see §II.5.0.13
$(-)^\vee$: the dual of individual objects, or of objects equipped with extra structures (algebras, operads, ...), see §II.5.0.13
`Der`: the modules of derivations (for algebras, operads, ...), see §III.2.1
`Map`, $\mathtt{Aut}^h$: see the section of this glossary about the constructions of homotopy theory

Constructions of homotopy theory

Fundamental constructions in model categories

$\mathtt{Ho}(-)$: the homotopy of a model category, see §II.1.2
$\mathtt{Aut}^h$: the notation for the homotopy automorphism space of an object in a model category, see §II.2.2
`Map`: the notation for the mapping spaces of a pair of objects in simplicial model categories and in general model categories, see §II.2.1, §II.3.2.11

Fundamental simplicial and cosimplicial constructions

`B`: the classifying space construction for groups, groupoids, categories, ..., see §I.5.2.3 (also the bar construction of algebras and of operads, see the relevant sections of this glossary)
$\mathtt{sk}_r$: the rth skeleton of a simplicial set, of a simplicial and of a cosimplicial object in a model category, see §II.1.3.8, §II.3.1.7, §II.3.1.17
`Tot`: the totalization of cosimplicial spaces, of cosimplicial objects in a model category, see §II.3.3.13

$|-|$: the geometric realization of simplicial sets, of simplicial objects in a model category, see §0.5, §II.1.3.5, §II.3.3.5
$\mathtt{Diag}$: the diagonal complex of a bisimplicial set, of a bisimplicial object and of a bicosimplicial object in a model category, see §II.3.3.19
$\mathtt{L}_r(X)$: the rth latching object of a simplicial object in a category, see §II.3.1.14
$\mathtt{M}_r(X)$: the rth matching object of a simplicial object in a category, see §II.3.1.15 (also the matching objects of Λ-sequences, see the section about operads and related structures of this glossary)
$\mathtt{L}^r(X)$, $\mathtt{M}^r(X)$: the cosimplicial variants of the matching and matching object constructions, see §II.3.1.3, §II.3.1.5

Differential graded constructions

$\mathbb{b}^m$, $\mathbb{e}^m$: notation for particular homogeneous elements (of upper degree m) notably used to define the generating (acyclic) cofibrations of the category of cochain graded dg-modules, see §II.5.1.2
$\mathbb{b}_m$, $\mathbb{e}_m$: same as $\mathbb{b}^m$ and $\mathbb{e}^m$ but in the chain graded context
$\mathbb{B}^m$: source objects of the generating cofibrations of the category of cochain graded dg-modules, see §II.5.1.2
$\mathbb{E}^m$: target objects of the generating (acyclic) cofibrations of the category of cochain graded dg-modules, see §II.5.1.2
$\mathbb{B}_m$, $\mathbb{E}_m$: dual objects of the dg-modules $\mathbb{B}^m$ and $\mathbb{E}^m$
$\mathbb{\sigma}$: notation for particular homogeneous elements used in the definition of suspension functors on dg-modules, see §C.2.3
$\mathbb{p}_r$, $\mathbb{p}_r^s$: notation for particular homogeneous elements used in the definition of the operadic suspension functor for operads in dg-modules, see §II.4.1.1
$\mathtt{Cyl}$: the standard cylinder object functor on the category of dg-modules, see §II.13.1.10
$\mathtt{B}$: the bar construction for algebras, see §II.6.3 (also the classifying space of groups, categories, and the bar construction of operads, see the relevant sections of this glossary)
τ_*: the right adjoint $\tau_* : dg\,\mathcal{M}od \to dg_*\,\mathcal{M}od$ of the embedding $\iota : dg_*\,\mathcal{M}od \hookrightarrow dg\,\mathcal{M}od$ of the category of chain graded dg-modules $dg_*\,\mathcal{M}od$ into the category of all dg-modules $dg\,\mathcal{M}od$, see §II.5.3.2
τ^*: the left adjoint $\tau^* : dg\,\mathcal{M}od \to dg^*\,\mathcal{M}od$ of the embedding $\iota : dg^*\,\mathcal{M}od \hookrightarrow dg\,\mathcal{M}od$ of the category of cochain graded dg-modules $dg^*\,\mathcal{M}od$ into the category of all dg-modules $dg\,\mathcal{M}od$, see §II.5.0.1
$(-)_\flat$: the forgetful functor from dg-modules to graded modules, see §0.1

The Dold–Kan correspondence

$\mathtt{N}_*$: the normalized chain complex functor on the category of simplicial modules, see §0.6, §II.5.0.5
$\mathtt{N}^*$: the conormalized cochain complex functor on the category of cosimplicial modules, see §II.5.0.9
$\mathtt{\Gamma}_\bullet$: the Dold–Kan functor on the category of chain graded dg-modules, see §II.5.0.6
$\mathtt{\Gamma}^\bullet$: the cosimplicial version of the Dold–Kan functor on the category of cochain graded dg-modules, see §II.5.0.9

Constructions of rational homotopy theory

$(-)^{\wedge}$: the rationalization functor on spaces, see §II.7.2.3, and on operads in simplicial sets, see §II.10.2, §II.12.2 (also the completion of filtered objects, see the section of this glossary about the background of our constructions)
Ω^*: the Sullivan cochain dg-algebra functor on simplicial sets, see §II.7.1
$\Omega^*_\sharp$: the operadic upgrade of the cochain dg-algebra functor on operads in simplicial sets, see §II.10.1, §II.12.1
$\mathtt{G}_\bullet$: the functor from cochain dg-algebras to simplicial sets, see §II.7.2
$\mathtt{MC}_\bullet$: the Maurer–Cartan spaces associated to (complete) Lie algebras, see §II.13.1.8

Operads and related structures

Indexing of operads

Σ_r: the symmetric group on r letters
Σ: the category of finite ordinals and permutations, see §I.2.2.3
Λ: the category of finite ordinals and injections, see §I.2.2.2
Λ^+: the category of finite ordinals and increasing injections, see §I.2.2.2
$\Sigma_{>0}$, $\Sigma_{>1}$, $\Lambda_{>0}$, $\Lambda_{>1}$, $\dots$: the full subcategory of the category Σ, Λ, $\dots$ generated by the ordinals of cardinal $r > 0$, $r > 1$, see §I.2.2.2, §I.2.4.1
$\mathcal{B}ij$: the category of finite sets and bijections, see §I.2.5.1
$\mathcal{I}nj$: the category of finite sets and injections, see §I.2.5.9
$\mathcal{B}ij_{>0}$, $\mathcal{B}ij_{>1}$, $\mathcal{I}nj_{>0}$, $\mathcal{I}nj_{>1}$, $\dots$: the full subcategory of the categories $\mathcal{B}ij$, $\mathcal{I}nj$, $\dots$ generated by the finite sets of cardinal $r > 0$, $r > 1$, see §I.2.5.9
$\underline{\mathsf{m}}$, $\underline{\mathsf{n}}$, $\dots$, $\underline{\mathsf{r}}$, $\dots$: generic notation for finite ordinals $\underline{\mathsf{r}} = \{1 < \cdots < r\}$ or for finite sets $\underline{\mathsf{r}} = \{i_1, \dots, i_r\}$ used to index the terms of operads, symmetric sequences and Λ-sequences
$\underline{0}$, $\underline{1}$, $\underline{2}$, $\dots$: the empty ordinal, the ordinal of cardinal one $\underline{1} = \{1\}$, of cardinal two $\underline{2} = \{1 < 2\}$, $\dots$

Categories of operads and related

$\mathcal{O}p$: the category of (symmetric) operads, see §I.1.1.2
$\mathcal{O}p_\varnothing$: the category of non-unitary (symmetric) operads, see §I.1.1.20
$\mathcal{O}p_{\varnothing 1}$: the category of connected (symmetric) operads, see §I.1.1.21
$\mathcal{O}p^c_{\varnothing 1}$: the category of (symmetric) cooperads, see §II.9.1.8
$\Lambda\,\mathcal{O}p_\varnothing\,/\,\mathsf{Com}$: the category of augmented non-unitary Λ-operads (the postfix expression $-/\,\mathsf{Com}$ can be discarded when the augmentation is trivial), see §I.2.2.17
$\Lambda\,\mathcal{O}p_{\varnothing 1}\,/\,\mathsf{Com}$: the category of augmented connected Λ-operads (the postfix expression $-/\,\mathsf{Com}$ can be discarded when the augmentation is trivial), see §I.2.4
$\mathcal{S}eq$: the category of symmetric sequences, see §I.1.2
$\mathcal{S}eq_{>0}$: the category of non-unitary symmetric sequences, see §I.1.2.13
$\mathcal{S}eq_{>1}$: the category of connected symmetric sequences, see §I.1.2.13
$\mathcal{S}eq^c$, $\mathcal{S}eq^c_{>0}$, $\mathcal{S}eq^c_{>1}$: same as $\mathcal{S}eq$, $\mathcal{S}eq_{>0}$, $\mathcal{S}eq_{>1}$ but used instead of this notation in the context of cooperads
$\Lambda\,\mathcal{S}eq$: the category of Λ-sequences, see §I.2.3
$\Lambda\,\mathcal{S}eq_{>0}$: the category of non-unitary Λ-sequences, see §I.2.3
$\Lambda\,\mathcal{S}eq_{>1}$: the category of connected Λ-sequences, see §I.2.4.1
$\Lambda\,\mathcal{S}eq^c$: the category of covariant Λ-sequences

$\Lambda\,\mathcal{S}eq^c_{>0}$: the category of covariant non-unitary Λ-sequences
$\Lambda\,\mathcal{S}eq^c_{>1}$: the category of covariant connected Λ-sequences, see §II.11.1.7
$\mathcal{C}oll$: the category of (symmetric) collections, see §I.2.5.1
$\mathcal{C}oll_{>0}$: the category of non-unitary (symmetric) collections
$\mathcal{C}oll_{>1}$: the category of connected (symmetric) collections

Categories of Hopf operads and related

$\mathcal{H}opf\,\mathcal{O}p$: the category of Hopf operads (defined as the category of operads in counitary cocommutative coalgebras), see §I.3.2
$\mathcal{H}opf\,\mathcal{O}p_{\varnothing}$, $\mathcal{H}opf\,\mathcal{O}p_{\varnothing 1}$: the non-unitary and connected variants of the category of Hopf operads
$\mathcal{H}opf\,\Lambda\,\mathcal{O}p_{\varnothing}$, $\mathcal{H}opf\,\Lambda\,\mathcal{O}p_{\varnothing 1}$: the Λ-operad variants of the categories of non-unitary and connected Hopf operads, see §I.3.2.15
$\mathcal{H}opf\,\mathcal{S}eq$: the category of Hopf symmetric sequences (defined as the category of symmetric sequences in counitary cocommutative coalgebras), see §I.3.2.6
$\mathcal{H}opf\,\mathcal{S}eq_{>0}$, $\mathcal{H}opf\,\mathcal{S}eq_{>1}$: the non-unitary and connected variants of the category of Hopf symmetric sequences
$\mathcal{H}opf\,\Lambda\,\mathcal{S}eq_{>0}$, $\mathcal{H}opf\,\Lambda\,\mathcal{S}eq_{>1}$: the Λ-sequence variants of the categories of non-unitary and connected Hopf symmetric sequences
$\mathcal{H}opf\,\mathcal{O}p^c_{\varnothing 1}$: the category of Hopf cooperads (defined as the category of cooperads in unitary commutative algebras), see §II.9.3.1
$\mathcal{H}opf\,\Lambda\,\mathcal{O}p^c_{\varnothing 1}$: the category of Hopf Λ-cooperads (defined as the category of cooperads in unitary commutative algebras), see §II.11.4.1
$\mathcal{H}opf\,\mathcal{S}eq^c_{>1}$: the category of connected Hopf symmetric sequences underlying Hopf cooperads (defined as the category of symmetric sequences in unitary commutative algebras), see §II.9.3.1
$\mathcal{H}opf\,\Lambda\,\mathcal{S}eq^c_{>1}$: the category of connected Hopf Λ-sequences underlying Hopf Λ-cooperads (defined as the category of Λ-sequences in unitary commutative algebras), see §II.11.4.1

Notation of operads

P, Q, …: generic notation for operads (of any kind)
M, N, …: generic notation for symmetric sequences, Λ-sequences, covariant Λ-sequences
C, D, …: generic notation for cooperads (of any kind)
C_n: the operad of little n-cubes, see §I.4.1.3
D_n: the operad of little n-discs, see §I.4.1.7
As: the (non-unitary) associative operad, see §I.1.1.16, §I.1.2.6, §I.1.2.10
Com: the (non-unitary) commutative operad, see §I.1.1.16, §I.1.2.6, §§I.1.2.10-2.1.11
Lie: the Lie operad, see §I.1.2.10
Pois: the Poisson operad, see §I.1.2.12
Gerst_n: the n-Gerstenhaber operad (defined as a graded variant of the Poisson operad), see §I.4.2.13
Com^c: the commutative cooperad, see §II.9.1.3
CoS, PaS, CoB, PaB, …: see the section about the applications of operads to the definition of Grothendieck–Teichmüller groups

Constructions on operads and on cooperads

τ: the truncation functors from non-unitary operads to connected operads and from augmented non-unitary Λ-operads to augmented connected Λ-operads, see §I.1.2.15, Proposition I.2.4.5

$\mathbb{\Theta}$: the free operad functor, see §A.3

$\mathbb{\Theta}^c$: the cofree cooperad functor, see §C.1

$\mathbb{\Theta}_{\underline{\mathsf{T}}}(\mathsf{M})$: the treewise tensor product of a symmetric sequence M over a tree $\underline{\mathsf{T}}$ when regarded as a term of the free operad and of the cofree cooperad (same as the object denoted by $\mathsf{M}(\underline{\mathsf{T}})$ in the section about trees), see §A.2

$\Sigma\mathsf{F}^r$: the rth free symmetric sequence, see §II.8.1.2

$\Lambda\mathsf{F}^r$: the rth free Λ-sequence, see §II.8.3.6

$\partial\Lambda\mathsf{F}^r$: the boundary of the rth free Λ-sequence, see §II.8.3.7

$\partial'\Lambda\mathsf{F}^r$: the boundary of the rth free Λ-sequence in the context of connected Λ-sequences, see §II.12.0.1

$\mathtt{Res}_{\bullet}$: the cotriple resolution functor on operads, see §B.1, §II.8.5

$\mathtt{B}$: the bar construction of operads, see §C.2 (also the classifying space of groups, categories, and the bar construction of algebras, see the relevant sections of this glossary)

$\mathtt{B}^c$: the cobar construction of cooperads, see §C.2

$\mathtt{K}$: the Koszul dual of operads, see §C.3

$\mathtt{M}(\mathsf{M})(r)$: the rth matching object of a Λ-sequence, see §II.8.3.1

$\mathtt{ar}_{\leq s}$: the sth layer of the arity filtration of a Λ-sequence, see Proof of Theorem II.8.3.20

$\mathtt{ar}^{\sharp}_{\leq s}$: the operadic upgrade of the arity filtration, see Proof of Theorem II.8.4.12

$\mathtt{cosk}^{\Lambda}_r$: the rth Λ-coskeleton of a Λ-sequence, see §II.8.3.3, of an augmented non-unitary Λ-operad, see Proof of Theorem II.8.4.12

Trees

$\mathcal{T}ree(\underline{\mathsf{r}})$: the category of $\underline{\mathsf{r}}$-trees (where $\underline{\mathsf{r}}$ is the indexing set of the inputs of the trees), see §A.1

$\mathcal{T}ree$: the operad of trees, see §A.1

$\widetilde{\mathcal{T}ree}(\underline{\mathsf{r}})$: the category of reduced $\underline{\mathsf{r}}$-trees (where $\underline{\mathsf{r}}$ is the indexing set of the inputs of the trees), see §A.1.12

$\widetilde{\mathcal{T}ree}$: the operad of reduced trees, see §A.1.12

$\mathcal{T}ree^{\circ}(\underline{\mathsf{r}})$: the category of planar $\underline{\mathsf{r}}$-trees (where $\underline{\mathsf{r}}$ is the indexing set of the inputs of the trees), see §A.3.16

$\mathcal{T}ree^{\circ}$: the operad of planar trees, see §A.3.16

$\underline{\mathsf{S}}, \underline{\mathsf{T}}, \dots$: generic notation for trees

$\underline{\downarrow}$: the unit tree (the tree with no vertex), see §A.1.4

$\underline{\mathsf{Y}}$: the notation of a corolla (a tree with a single vertex), see §A.1.4

$\underline{\Gamma}$: the notation of a tree with two vertices, see §A.2.3

$V(\underline{\mathsf{T}})$: the vertex set of a tree

$E(\underline{\mathsf{T}})$: the edge set of a tree

$\mathring{E}(\underline{\mathsf{T}})$: the set of inner edges of a tree

$\underline{\mathsf{r}}_v$: the set of ingoing edges of a vertex in a tree

$\mathsf{M}(\underline{\mathsf{T}})$: the treewise tensor product of a symmetric sequence M over a tree $\underline{\mathsf{T}}$ (same as the object denoted by $\mathbb{\Theta}_{\underline{\mathsf{T}}}(\mathsf{M})$ in the section about constructions on operads and on cooperads), see §A.2

$\lambda_{\underline{\mathrm{T}}}$: the treewise composition products associated to an operad, see §A.2.7
$\rho_{\underline{\mathrm{T}}}$: the treewise composition coproducts associated to a cooperad, see §C.1.5

From operads to Grothendieck–Teichmüller groups

Permutations, braids, and related objects

Σ_r: the symmetric group on r letters
B_r: the Artin braid group on r strands, see §I.5.0
P_r: the pure braid group on r strands, see §I.5.0
$\mathfrak{p}(r)$: the rth Drinfeld–Kohno Lie algebras (the Lie algebra of infinitesimal braids on r strands), see §I.10.0.2
$\hat{\mathfrak{p}}(r)$: the complete Drinfeld–Kohno Lie algebra, see §I.10.0.6
$\mathfrak{p}_n(r)$: the graded variants of the Drinfeld–Kohno Lie algebras (with $\mathfrak{p}(r) = \mathfrak{p}_2(r)$), see §II.14.1.1
$\mathfrak{p}$: the Drinfeld–Kohno Lie algebra operad, see §I.10.1.1
$\mathfrak{p}_n$: the graded variants of the Drinfeld–Kohno Lie algebra operad (with $\mathfrak{p} = \mathfrak{p}_2$), see §II.14.1.1
$\hat{\mathfrak{p}}$: the complete Drinfeld–Kohno Lie algebra operad, see §I.10.2.2
CoS: the operad of colored symmetries, see §I.6.3
PaS: the operad of parenthesized symmetries, see §I.6.3
CoB: the operad of colored braids, see §§I.5.2.8-5.2.11, §I.6.2.7
PaB: the operad of parenthesized braids, see §I.6.2
$\mathsf{CoB}^{\widehat{}}$, $\mathsf{PaB}^{\widehat{}}$: the Malcev completion of the colored and parenthesized braid operads, see §I.10.1
$\mathsf{CD}^{\widehat{}}$: the operad of chord diagrams, see §I.10.2.4
$\mathsf{PaCD}^{\widehat{}}$: the operad of parenthesized chord diagrams, see §I.10.3.2

Grothendieck–Teichmüller groups and related objects

$Ass(\Bbbk)$: the set of Drinfeld's associators, see §I.10.2.11
$GT(\Bbbk)$: the pro-unipotent Grothendieck–Teichmüller group, see §I.11.1
GRT: the graded Grothendieck–Teichmüller group, see §I.10.3
$GT^{\widehat{}}$: the profinite Grothendieck–Teichmüller group
$\mathfrak{grt}$: the graded Grothendieck–Teichmüller Lie algebra, see §I.10.4.6, §I.11.4

Bibliography

[1] Eiichi Abe. *Hopf algebras*, volume 74 of *Cambridge Tracts in Mathematics.* Cambridge University Press, Cambridge-New York, 1980. Translated from the Japanese by Hisae Kinoshita and Hiroko Tanaka.

[2] J. Adámek, J. Rosický, and E. M. Vitale. What are sifted colimits? *Theory Appl. Categ.*, 23:No. 13, 251–260, 2010.

[3] John Frank Adams. *Infinite loop spaces*, volume 90 of *Annals of Mathematics Studies.* Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1978.

[4] Marcelo Aguiar and Swapneel Mahajan. *Coxeter groups and Hopf algebras*, volume 23 of *Fields Institute Monographs.* American Mathematical Society, Providence, RI, 2006. With a foreword by Nantel Bergeron.

[5] Marcelo Aguiar and Swapneel Mahajan. *Monoidal functors, species and Hopf algebras*, volume 29 of *CRM Monograph Series.* American Mathematical Society, Providence, RI, 2010. With forewords by Kenneth Brown and Stephen Chase and André Joyal.

[6] Anton Alekseev and Charles Torossian. Kontsevich deformation quantization and flat connections. *Comm. Math. Phys.*, 300(1):47–64, 2010.

[7] Yves André. *Une introduction aux motifs (motifs purs, motifs mixtes, périodes)*, volume 17 of *Panoramas et Synthèses [Panoramas and Syntheses].* Société Mathématique de France, Paris, 2004.

[8] V. I. Arnol′d. The cohomology ring of the group of dyed braids. *Mat. Zametki*, 5:227–231, 1969.

[9] Gregory Arone and Victor Turchin. On the rational homology of high-dimensional analogues of spaces of long knots. *Geom. Topol.*, 18(3):1261–1322, 2014.

[10] E. Artin. Theorie der Zöpfe. *Hamb. Abh.*, 4:47–72, 1925.

[11] E. Artin. Theory of braids. *Ann. of Math. (2)*, 48:101–126, 1947.

[12] Scott Axelrod and I. M. Singer. Chern-Simons perturbation theory. II. *J. Differential Geom.*, 39(1):173–213, 1994.

[13] Joseph Ayoub. L'algèbre de Hopf et le groupe de Galois motiviques d'un corps de caractéristique nulle, I. *J. Reine Angew. Math.*, 693:1–149, 2014.

[14] Joseph Ayoub. L'algèbre de Hopf et le groupe de Galois motiviques d'un corps de caractéristique nulle, II. *J. Reine Angew. Math.*, 693:151–226, 2014.

[15] C. Balteanu, Z. Fiedorowicz, R. Schwänzl, and R. Vogt. Iterated monoidal categories. *Adv. Math.*, 176(2):277–349, 2003.

[16] Dror Bar-Natan. On associators and the Grothendieck-Teichmuller group. I. *Selecta Math. (N.S.)*, 4(2):183–212, 1998.

[17] M. G. Barratt and Peter J. Eccles. Γ^+-structures. I. A free group functor for stable homotopy theory. *Topology*, 13:25–45, 1974.

[18] M. A. Batanin. Monoidal globular categories as a natural environment for the theory of weak n-categories. *Adv. Math.*, 136(1):39–103, 1998.

[19] M. A. Batanin. Symmetrisation of n-operads and compactification of real configuration spaces. *Adv. Math.*, 211(2):684–725, 2007.

[20] M. A. Batanin. The Eckmann-Hilton argument and higher operads. *Adv. Math.*, 217(1):334–385, 2008.

[21] Michael A. Batanin. Locally constant n-operads as higher braided operads. *J. Noncommut. Geom.*, 4(2):237–263, 2010.

[22] G. V. Belyĭ. Galois extensions of a maximal cyclotomic field. *Izv. Akad. Nauk SSSR Ser. Mat.*, 43(2):267–276, 479, 1979.

[23] Clemens Berger. Opérades cellulaires et espaces de lacets itérés. *Ann. Inst. Fourier (Grenoble)*, 46(4):1125–1157, 1996.

[24] Clemens Berger. Combinatorial models for real configuration spaces and E_n-operads. In *Operads: Proceedings of Renaissance Conferences (Hartford, CT/Luminy, 1995)*, volume 202 of *Contemp. Math.*, pages 37–52. Amer. Math. Soc., Providence, RI, 1997.

[25] Clemens Berger and Ieke Moerdijk. Axiomatic homotopy theory for operads. *Comment. Math. Helv.*, 78(4):805–831, 2003.

[26] Joan S. Birman. *Braids, links, and mapping class groups.* Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1974. Annals of Mathematics Studies, No. 82.

[27] J. M. Boardman and R. M. Vogt. Homotopy-everything H-spaces. *Bull. Amer. Math. Soc.*, 74:1117–1122, 1968.

[28] J. M. Boardman and R. M. Vogt. *Homotopy invariant algebraic structures on topological spaces.* Lecture Notes in Mathematics, Vol. 347. Springer-Verlag, Berlin-New York, 1973.

[29] Pedro Boavida de Brito and Michael Weiss. Spaces of smooth embeddings and configuration categories. Preprint `arXiv:1502.01640`, 2015.

[30] Francis Borceux. *Handbook of categorical algebra. 1*, volume 50 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, Cambridge, 1994. Basic category theory.

[31] Francis Borceux. *Handbook of categorical algebra. 2*, volume 51 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, Cambridge, 1994. Categories and structures.

[32] Francis Borceux and George Janelidze. *Galois theories*, volume 72 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 2001.

[33] Armand Borel. *Linear algebraic groups*, volume 126 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, second edition, 1991.

[34] N. Bourbaki. *Éléments de mathématique. Groupes et algèbres de Lie. Chapitres 2 et 3.* Springer-Verlag, Berlin, 2006. Reprint of the 1972 original [Hermann, Paris; MR0573068].

[35] N. Bourbaki. *Éléments de mathématique. Algèbre. Chapitres 1 à 3.* Springer-Verlag, Berlin, 2007. Reprint of the 1970 original [Hermann, Paris; MR0274237].

[36] A. K. Bousfield and V. K. A. M. Gugenheim. On PL de Rham theory and rational homotopy type. *Mem. Amer. Math. Soc.*, 8(179):ix+94, 1976.

[37] Francis Brown. Mixed Tate motives over $\mathbb{Z}$. *Ann. of Math. (2)*, 175(2):949–976, 2012.

[38] Damien Calaque, Tony Pantev, Bertrand Toën, Michel Vaquié, and Gabriele Vezzosi. Shifted Poisson structures and deformation quantization. Preprint `arXiv:1506.03699`, 2015.

[39] Pierre Cartier. A primer of Hopf algebras. Prépublication Inst. Hautes Études Sci., 2006.

[40] Vyjayanthi Chari and Andrew Pressley. *A guide to quantum groups.* Cambridge University Press, Cambridge, 1994.

[41] Kuo Tsai Chen. Iterated path integrals. *Bull. Amer. Math. Soc.*, 83(5):831–879, 1977.

[42] S. Chmutov, S. Duzhin, and J. Mostovoy. *Introduction to Vassiliev knot invariants.* Cambridge University Press, Cambridge, 2012.

[43] Utsav Choudhury and Martin Gallauer Alves de Souza. An isomorphism of motivic galois groups. 2014. Preprint `arXiv:1410.6104`.

[44] F. R. Cohen, J. P. May, and L. R. Taylor. Splitting of certain spaces CX. *Math. Proc. Cambridge Philos. Soc.*, 84(3):465–496, 1978.

[45] Frederick R. Cohen. The homology of $\mathcal{C}_{n+1}$-spaces, $n \geq 0$. In *The homology of iterated loop spaces*, Lecture Notes in Mathematics, Vol. 533, pages 207–351. Springer-Verlag, Berlin-New York, 1976.

[46] Frederick R. Cohen, Thomas J. Lada, and J. Peter May. *The homology of iterated loop spaces.* Lecture Notes in Mathematics, Vol. 533. Springer-Verlag, Berlin-New York, 1976.

[47] Benjamin Collas and Sylvain Maugeais. On galois action on stack inertia of moduli spaces of curves. 2014. Preprint `arXiv:1412.4644`.

[48] Benjamin Collas and Sylvain Maugeais. Composantes irréductibles de lieux spéciaux d'espaces de modules de courbes, action galoisienne en genre quelconque. *Ann. Inst. Fourier (Grenoble)*, 65(1):245–276, 2015.

[49] Alain Connes and Dirk Kreimer. Hopf algebras, renormalization and noncommutative geometry. *Comm. Math. Phys.*, 199(1):203–242, 1998.

[50] Kevin Costello and Owen Gwilliam. Factorization algebras in quantum field theory. Volume 31 of *New Mathematical Monographs.* Cambridge Univ. Press, Cambridge, 2016.

[51] Edward B. Curtis. Simplicial homotopy theory. *Advances in Math.*, 6:107–209 (1971), 1971.

[52] P. Deligne. Le groupe fondamental de la droite projective moins trois points. In *Galois groups over* $\mathbf{Q}$ *(Berkeley, CA, 1987)*, volume 16 of *Math. Sci. Res. Inst. Publ.*, pages 79–297. Springer, New York, 1989.

[53] P. Deligne and D. Mumford. The irreducibility of the space of curves of given genus. *Inst. Hautes Études Sci. Publ. Math.*, (36):75–109, 1969.

[54] Pierre Deligne and Alexander B. Goncharov. Groupes fondamentaux motiviques de Tate mixte. *Ann. Sci. École Norm. Sup. (4)*, 38(1):1–56, 2005.

[55] Michel Demazure and Pierre Gabriel. *Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs.* Masson & Cie, Éditeur, Paris; North-Holland Publishing Co., Amsterdam, 1970. Avec un appendice ıt Corps de classes local par Michiel Hazewinkel.

[56] V. G. Drinfel′d. Quantum groups. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986)*, pages 798–820.

Amer. Math. Soc., Providence, RI, 1987.

[57] V. G. Drinfel′d. On quasitriangular quasi-Hopf algebras and on a group that is closely connected with Gal($\overline{\mathbf{Q}}/\mathbf{Q}$). *Algebra i Analiz*, 2(4):149–181, 1990.

[58] Gabriel C. Drummond-Cole. Homotopically trivializing the circle in the framed little disks. *J. Topol.*, 7(3):641–676, 2014.

[59] William Dwyer and Kathryn Hess. Long knots and maps between operads. *Geom. Topol.*, 16(2):919–955, 2012.

[60] Pavel I. Etingof, Igor B. Frenkel, and Alexander A. Kirillov, Jr. *Lectures on representation theory and Knizhnik-Zamolodchikov equations*, volume 58 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1998.

[61] Yves Félix, Stephen Halperin, and Jean-Claude Thomas. *Rational homotopy theory*, volume 205 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2001.

[62] Z. Fiedorowicz. The symmetric bar construction. Preprint, 1996.

[63] R. Fox and L. Neuwirth. The braid groups. *Math. Scand.*, 10:119–126, 1962.

[64] John Francis. The tangent complex and Hochschild cohomology of $\mathcal{E}_n$-rings. *Compos. Math.*, 149(3):430–480, 2013.

[65] Benoit Fresse. Koszul duality of operads and homology of partition posets. In *Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory*, volume 346 of *Contemp. Math.*, pages 115–215. Amer. Math. Soc., Providence, RI, 2004.

[66] Benoit Fresse. *Modules over operads and functors*, volume 1967 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2009.

[67] Benoit Fresse, Victor Turchin, and Thomas Willwacher. Mapping spaces of the E_n operads. In preparation, 2015.

[68] Benoit Fresse and Thomas Willwacher. The intrinsic formality of E_n-operads. Preprint `arXiv:1503.08699`, 2015.

[69] William Fulton and Robert MacPherson. A compactification of configuration spaces. *Ann. of Math. (2)*, 139(1):183–225, 1994.

[70] Hidekazu Furusho. Pentagon and hexagon equations. *Ann. of Math. (2)*, 171(1):545–556, 2010.

[71] Peter Gabriel and Friedrich Ulmer. *Lokal präsentierbare Kategorien.* Lecture Notes in Mathematics, Vol. 221. Springer-Verlag, Berlin-New York, 1971.

[72] Giovanni Gaiffi. Models for real subspace arrangements and stratified manifolds. *Int. Math. Res. Not.*, (12):627–656, 2003.

[73] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky. *Discriminants, resultants and multidimensional determinants.* Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2008. Reprint of the 1994 edition.

[74] Murray Gerstenhaber. The cohomology structure of an associative ring. *Ann. of Math. (2)*, 78:267–288, 1963.

[75] E. Getzler. Batalin-Vilkovisky algebras and two-dimensional topological field theories. *Comm. Math. Phys.*, 159(2):265–285, 1994.

[76] E. Getzler. Operads and moduli spaces of genus 0 Riemann surfaces. In *The moduli space of curves (Texel Island, 1994)*, volume 129 of *Progr. Math.*, pages 199–230. Birkhäuser Boston, Boston, MA, 1995.

[77] Ezra Getzler and John D.S. Jones. Operads, homotopy algebra and iterated integrals for double loop spaces. Preprint `arXiv:hep-th/9403055`, 1994.

[78] Victor Ginzburg and Mikhail Kapranov. Koszul duality for operads. *Duke Math. J.*, 76(1):203–272, 1994.

[79] Paul G. Goerss and John F. Jardine. *Simplicial homotopy theory.* Modern Birkhäuser Classics. Birkhäuser Verlag, Basel, 2009. Reprint of the 1999 edition [MR1711612].

[80] A. B. Goncharov. Galois symmetries of fundamental groupoids and noncommutative geometry. *Duke Math. J.*, 128(2):209–284, 2005.

[81] Pierre-Paul Grivel. Une histoire du théorème de Poincaré-Birkhoff-Witt. *Expo. Math.*, 22(2):145–184, 2004.

[82] Alexander Grothendieck. *Revêtements étales et groupe fondamental. Fasc. I: Exposés 1 à 5*, volume 1960/61 of *Séminaire de Géométrie Algébrique.* Institut des Hautes Études Scientifiques, Paris, 1963.

[83] Alexandre Grothendieck. Esquisse d'un programme. In *Geometric Galois actions, 1*, volume 242 of *London Math. Soc. Lecture Note Ser.*, pages 5–48. Cambridge Univ. Press, Cambridge, 1997. With an English translation on pp. 243–283.

[84] P. Hall. A Contribution to the Theory of Groups of Prime-Power Order. *Proc. London Math. Soc.*, S2-36(1):29, 1933.

[85] Stephen Halperin and James Stasheff. Obstructions to homotopy equivalences. *Adv. in Math.*, 32(3):233–279, 1979.

[86] Masaki Hanamura. Mixed motives and algebraic cycles. I. *Math. Res. Lett.*, 2(6):811–821, 1995.

[87] Joe Harris and Ian Morrison. *Moduli of curves*, volume 187 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1998.

[88] Allen Hatcher, Pierre Lochak, and Leila Schneps. On the Teichmüller tower of mapping class groups. *J. Reine Angew. Math.*, 521:1–24, 2000.

[89] Philip S. Hirschhorn. *Model categories and their localizations*, volume 99 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 2003.

[90] Goeffroy Horel. Profinite completion of operads and the Grothendieck-Teichmüller group. Preprint `arXiv:1504.01605`, 2005.

[91] Mark Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 1999.

[92] Annette Huber and Stefan Müller-Stach. Periods and Nori motives. Volume 65 of *Ergebnisse der Mathematik und ihrer Grenzgebiete.* Springer International Publishing, Switzerland, 2017.

[93] Yasutaka Ihara. The Galois representation arising from $\mathbf{P}^1 - \{0, 1, \infty\}$ and Tate twists of even degree. In *Galois groups over* $\mathbf{Q}$ *(Berkeley, CA, 1987)*, volume 16 of *Math. Sci. Res. Inst. Publ.*, pages 299–313. Springer, New York, 1989.

[94] Yasutaka Ihara. Braids, Galois groups, and some arithmetic functions. In *Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990)*, pages 99–120. Math. Soc. Japan, Tokyo, 1991.

[95] Yasutaka Ihara. On the embedding of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ into $\widehat{\mathrm{GT}}$. In *The Grothendieck theory of dessins d'enfants (Luminy, 1993)*, volume 200 of *London Math. Soc. Lecture Note Ser.*, pages 289–321. Cambridge Univ. Press, Cambridge, 1994. With an appendix: the action of the absolute Galois group on the moduli space of spheres with four marked points by Michel Emsalem

and Pierre Lochak.

[96] I. M. James. Reduced product spaces. *Ann. of Math. (2)*, 62:170–197, 1955.

[97] P. T. Johnstone. Adjoint lifting theorems for categories of algebras. *Bull. London Math. Soc.*, 7(3):294–297, 1975.

[98] André Joyal. Foncteurs analytiques et espèces de structures. In *Combinatoire énumérative (Montreal, Que., 1985/Quebec, Que., 1985)*, volume 1234 of *Lecture Notes in Math.*, pages 126–159. Springer, Berlin, 1986.

[99] André Joyal and Ross Street. Braided tensor categories. *Adv. Math.*, 102(1):20–78, 1993.

[100] Christian Kassel. *Quantum groups*, volume 155 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1995.

[101] Christian Kassel and Vladimir Turaev. *Braid groups*, volume 247 of *Graduate Texts in Mathematics.* Springer, New York, 2008. With the graphical assistance of Olivier Dodane.

[102] Sean Keel. Intersection theory of moduli space of stable n-pointed curves of genus zero. *Trans. Amer. Math. Soc.*, 330(2):545–574, 1992.

[103] G. M. Kelly. On the operads of J. P. May. *Repr. Theory Appl. Categ.*, (13):1–13, 2005.

[104] V. G. Knizhnik and A. B. Zamolodchikov. Current algebra and Wess-Zumino model in two dimensions. *Nuclear Phys. B*, 247(1):83–103, 1984.

[105] Finn F. Knudsen. The projectivity of the moduli space of stable curves. II and III. *Math. Scand.*, 52(2):161–212, 1983.

[106] Toshitake Kohno. Série de Poincaré-Koszul associée aux groupes de tresses pures. *Invent. Math.*, 82(1):57–75, 1985.

[107] Toshitake Kohno. Bar complex, configuration spaces and finite type invariants for braids. *Topology Appl.*, 157(1):2–9, 2010.

[108] Maxim Kontsevich. Operads and motives in deformation quantization. *Lett. Math. Phys.*, 48(1):35–72, 1999. Moshé Flato (1937–1998).

[109] Maxim Kontsevich and Yan Soibelman. Deformations of algebras over operads and the Deligne conjecture. In *Conférence Moshé Flato 1999, Vol. I (Dijon)*, volume 21 of *Math. Phys. Stud.*, pages 255–307. Kluwer Acad. Publ., Dordrecht, 2000.

[110] Maxim Kontsevich and Don Zagier. Periods. In *Mathematics unlimited—2001 and beyond*, pages 771–808. Springer, Berlin, 2001.

[111] C. Lair. Sur le profil d'esquissabilité des catégories modelables (accessibles) possédant les noyaux. *Diagrammes*, 38:19–78, 1997.

[112] Pascal Lambrechts, Victor Turchin, and Ismar Volić. The rational homology of spaces of long knots in codimension > 2. *Geom. Topol.*, 14(4):2151–2187, 2010.

[113] Michel Lazard. Sur les groupes nilpotents et les anneaux de Lie. *Ann. Sci. Ecole Norm. Sup. (3)*, 71:101–190, 1954.

[114] Michel Lazard. Lois de groupes et analyseurs. *Ann. Sci. Ecole Norm. Sup. (3)*, 72:299–400, 1955.

[115] Thang Tu Quoc Le and Jun Murakami. Kontsevich's integral for the Kauffman polynomial. *Nagoya Math. J.*, 142:39–65, 1996.

[116] Carl W. Lee. The associahedron and triangulations of the n-gon. *European J. Combin.*, 10(6):551–560, 1989.

[117] Tom Leinster. *Higher operads, higher categories*, volume 298 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 2004.

[118] Marc Levine. Tate motives and the vanishing conjectures for algebraic K-theory. In *Algebraic K-theory and algebraic topology (Lake Louise, AB, 1991)*, volume 407 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 167–188. Kluwer Acad. Publ., Dordrecht, 1993.

[119] Marc Levine. *Mixed motives*, volume 57 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 1998.

[120] Marc Levine. Mixed motives. In *Handbook of K-theory. Vol. 1, 2*, pages 429–521. Springer, Berlin, 2005.

[121] Muriel Livernet. Koszul duality of the category of trees and bar constructions for operads. In *Operads and universal algebra*, volume 9 of *Nankai Ser. Pure Appl. Math. Theoret. Phys.*, pages 107–138. World Sci. Publ., Hackensack, NJ, 2012.

[122] Pierre Lochak, Hiroaki Nakamura, and Leila Schneps. On a new version of the Grothendieck-Teichmüller group. *C. R. Acad. Sci. Paris Sér. I Math.*, 325(1):11–16, 1997.

[123] Pierre Lochak and Leila Schneps. A cohomological interpretation of the Grothendieck-Teichmüller group. *Invent. Math.*, 127(3):571–600, 1997. With an appendix by C. Scheiderer.

[124] Jean-Louis Loday. Série de Hausdorff, idempotents eulériens et algèbres de Hopf. *Exposition. Math.*, 12(2):165–178, 1994.

[125] Jean-Louis Loday. Realization of the Stasheff polytope. *Arch. Math. (Basel)*, 83(3):267–278, 2004.

[126] Jean-Louis Loday and Bruno Vallette. *Algebraic operads*, volume 346 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer, Heidelberg, 2012.

[127] Jacob Lurie. On the classification of topological field theories. In *Current developments in mathematics, 2008*, pages 129–280. Int. Press, Somerville, MA, 2009.

[128] Jacob Lurie. Higher algebra. Book project, 2014.

[129] Saunders Mac Lane. *Homology.* Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1975 edition.

[130] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, second edition, 1998.

[131] Wilhelm Magnus, Abraham Karrass, and Donald Solitar. *Combinatorial group theory.* Dover Publications, Inc., Mineola, NY, second edition, 2004. Presentations of groups in terms of generators and relations.

[132] A. I. Mal′cev. Nilpotent torsion-free groups. *Izvestiya Akad. Nauk. SSSR. Ser. Mat.*, 13:201–212, 1949.

[133] Yuri I. Manin. *Frobenius manifolds, quantum cohomology, and moduli spaces*, volume 47 of *American Mathematical Society Colloquium Publications.* American Mathematical Society, Providence, RI, 1999.

[134] Martin Markl. Distributive laws and Koszulness. *Ann. Inst. Fourier (Grenoble)*, 46(2):307–323, 1996.

[135] Martin Markl. Models for operads. *Comm. Algebra*, 24(4):1471–1500, 1996.

[136] Martin Markl. Operads and PROPs. In *Handbook of algebra. Vol. 5*, volume 5 of *Handb. Algebr.*, pages 87–140. Elsevier/North-Holland, Amsterdam, 2008.

[137] Martin Markl. *Deformation theory of algebras and their diagrams*, volume 116 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2012.

[138] Martin Markl, Steve Shnider, and Jim Stasheff. *Operads in algebra, topology and physics*, volume 96 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002.

[139] William S. Massey. *A basic course in algebraic topology*, volume 127 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991.

[140] J. P. May. *The geometry of iterated loop spaces*. Springer-Verlag, Berlin-New York, 1972. Lectures Notes in Mathematics, Vol. 271.

[141] J. Peter May. *Simplicial objects in algebraic topology*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1992. Reprint of the 1967 original.

[142] James E. McClure and Jeffrey H. Smith. A solution of Deligne's Hochschild cohomology conjecture. In *Recent progress in homotopy theory (Baltimore, MD, 2000)*, volume 293 of *Contemp. Math.*, pages 153–193. Amer. Math. Soc., Providence, RI, 2002.

[143] James E. McClure and Jeffrey H. Smith. Multivariable cochain operations and little n-cubes. *J. Amer. Math. Soc.*, 16(3):681–704, 2003.

[144] James S. Milne. *Étale cohomology*, volume 33 of *Princeton Mathematical Series*. Princeton University Press, Princeton, N.J., 1980.

[145] John W. Milnor and John C. Moore. On the structure of Hopf algebras. *Ann. of Math. (2)*, 81:211–264, 1965.

[146] Shigeyuki Morita. *Geometry of differential forms*, volume 201 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 2001. Translated from the two-volume Japanese original (1997, 1998) by Teruko Nagase and Katsumi Nomizu, Iwanami Series in Modern Mathematics.

[147] Hiroaki Nakamura and Leila Schneps. On a subgroup of the Grothendieck-Teichmüller group acting on the tower of profinite Teichmüller modular groups. *Invent. Math.*, 141(3):503–560, 2000.

[148] Tony Pantev, Bertrand Toën, Michel Vaquié, and Gabriele Vezzosi. Shifted symplectic structures. *Publ. Math. Inst. Hautes Études Sci.*, 117:271–328, 2013.

[149] Frédéric Patras. *Homothéties simpliciales*. Thèse de doctorat, Université Paris 7, 1992.

[150] Frédéric Patras. La décomposition en poids des algèbres de Hopf. *Ann. Inst. Fourier (Grenoble)*, 43(4):1067–1087, 1993.

[151] Daniel Quillen. Rational homotopy theory. *Ann. of Math. (2)*, 90:205–295, 1969.

[152] Daniel G. Quillen. *Homotopical algebra*. Lecture Notes in Mathematics, No. 43. Springer-Verlag, Berlin-New York, 1967.

[153] Daniel G. Quillen. On the associated graded ring of a group ring. *J. Algebra*, 10:411–418, 1968.

[154] Christophe Reutenauer. Theorem of Poincaré-Birkhoff-Witt, logarithm and symmetric group representations of degrees equal to Stirling numbers. In *Combinatoire énumérative (Montreal, Que., 1985/Quebec, Que., 1985)*, volume 1234 of *Lecture Notes in Math.*, pages 267–284. Springer, Berlin, 1986.

[155] Christophe Reutenauer. *Free Lie algebras*, volume 7 of *London Mathematical Society Monographs. New Series.* The Clarendon Press, Oxford University Press, New York, 1993. Oxford Science Publications.

[156] Charles Rezk. *Spaces of algebra structures and cohomology of operads.* PhD thesis, Massachusetts Institute of Technology, 1996.

[157] Paolo Salvatore. Configuration spaces with summable labels. In *Cohomological methods in homotopy theory (Bellaterra, 1998)*, volume 196 of *Progr. Math.*, pages 375–395. Birkhäuser, Basel, 2001.

[158] Paolo Salvatore and Nathalie Wahl. Framed discs operads and Batalin-Vilkovisky algebras. *Q. J. Math.*, 54(2):213–231, 2003.

[159] Pavol Ševera and Thomas Willwacher. Equivalence of formalities of the little discs operad. *Duke Math. J.*, 160(1):175–206, 2011.

[160] Steven Shnider and Shlomo Sternberg. *Quantum groups.* Graduate Texts in Mathematical Physics, II. International Press, Cambridge, MA, 1993. From coalgebras to Drinfel′d algebras, A guided tour.

[161] Dev P. Sinha. Manifold-theoretic compactifications of configuration spaces. *Selecta Math. (N.S.)*, 10(3):391–428, 2004.

[162] Dev P. Sinha. Operads and knot spaces. *J. Amer. Math. Soc.*, 19(2):461–486, 2006.

[163] Dev P. Sinha. The (non-equivariant) homology of the little disks operad. In *OPERADS 2009*, volume 26 of *Sémin. Congr.*, pages 253–279. Soc. Math. France, Paris, 2013.

[164] V. A. Smirnov. On the chain complex of an iterated loop space. *Izv. Akad. Nauk SSSR Ser. Mat.*, 53(5):1108–1119, 1135–1136, 1989.

[165] Jeffrey Henderson Smith. Simplicial group models for $\Omega^n S^n X$. *Israel J. Math.*, 66(1-3):330–350, 1989.

[166] Edwin H. Spanier. *Algebraic topology.* Springer-Verlag, New York-Berlin, 1981. Corrected reprint.

[167] James Dillon Stasheff. Homotopy associativity of H-spaces. I, II. *Trans. Amer. Math. Soc. 108 (1963), 275-292; ibid.*, 108:293–312, 1963.

[168] N. E. Steenrod. *Cohomology operations.* Lectures by N. E. Steenrod written and revised by D. B. A. Epstein. Annals of Mathematics Studies, No. 50. Princeton University Press, Princeton, N.J., 1962.

[169] Richard Steiner. A canonical operad pair. *Math. Proc. Cambridge Philos. Soc.*, 86(3):443–449, 1979.

[170] Dennis Sullivan. Infinitesimal computations in topology. *Inst. Hautes Études Sci. Publ. Math.*, 47:269–331 (1978), 1977.

[171] Moss E. Sweedler. *Hopf algebras.* Mathematics Lecture Note Series. W. A. Benjamin, Inc., New York, 1969.

[172] Tamás Szamuely. *Galois groups and fundamental groups*, volume 117 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 2009.

[173] Dmitry E. Tamarkin. Another proof of M. Kontsevich formality theorem. Preprint `arXiv:math.QA/9803025`, 1998.

[174] Dmitry E. Tamarkin. Action of the Grothendieck-Teichmüller group on the operad of Gerstenhaber algebras. Preprint `arXiv:math/0202039`, 2002.

[175] Dmitry E. Tamarkin. Formality of chain operad of little discs. *Lett. Math. Phys.*, 66(1-2):65–72, 2003.

[176] Tomohide Terasoma. Mixed Tate motives and multiple zeta values. *Invent. Math.*, 149(2):339–369, 2002.

[177] Tomohide Terasoma. Geometry of multiple zeta values. In *International Congress of Mathematicians. Vol. II*, pages 627–635. Eur. Math. Soc., Zürich, 2006.

[178] Gerald Teschl. *Ordinary differential equations and dynamical systems*, volume 140 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2012.

[179] Vladimir Voevodsky. Triangulated categories of motives over a field. In *Cycles, transfers, and motivic homology theories*, volume 143 of *Ann. of Math. Stud.*, pages 188–238. Princeton Univ. Press, Princeton, NJ, 2000.

[180] William C. Waterhouse. *Introduction to affine group schemes*, volume 66 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1979.

[181] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

[182] George W. Whitehead. *Elements of homotopy theory*, volume 61 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1978.

[183] David Wigner. An identity in the free Lie algebra. *Proc. Amer. Math. Soc.*, 106(3):639–640, 1989.

[184] Thomas Willwacher. M. Kontsevich's graph complex and the Grothendieck-Teichmüller Lie algebra. *Invent. Math.*, 200(3):671–760, 2015.

[185] Miguel A. Xicoténcatl. The Lie algebra of the pure braid group. *Bol. Soc. Mat. Mexicana (3)*, 6(1):55–62, 2000.

[186] Donald Yau. *Colored operads*, volume 170 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2016.

[187] Donald Yau and Mark W. Johnson. *A foundation for PROPs, algebras, and modules*, volume 203 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2015.

[188] Wenbin Zhang. Group operads and homotopy theory. Preprint `arXiv:1111.7090`, 2011.

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