FEUILLE DE TRAVAUX DIRIGÉS 4

OPÉRADES BIS

Exercice 1 (Diassociative algebras).
By definition, a *diassociative algebra* is a $\mathbb{K}$-module A equipped with two linear maps

$$\dashv\ : A \otimes A \to A \quad \text{and} \quad \vdash\ : A \otimes A \to A\ ,$$

called the *left operation* and the *right operation* respectively, satisfying the following five relations

$$\begin{cases} (x \dashv y) \dashv z &= x \dashv (y \dashv z)\ , \\ (x \dashv y) \dashv z &= x \dashv (y \vdash z)\ , \\ (x \vdash y) \dashv z &= x \vdash (y \dashv z)\ , \\ (x \dashv y) \vdash z &= x \vdash (y \vdash z)\ , \\ (x \vdash y) \vdash z &= x \vdash (y \vdash z)\ . \end{cases}$$

for any $x, y, z \in A$.

(1) Make explicit the free diassociative algebra $\mathrm{Di}(V)$ on a $\mathbb{K}$-module V.

(2) From this result, describe the nonsymmetric operad Di which encodes diassociative algebras using the classical (or equivalently monoidal) definition.

(3) Describe the nonsymmetric operad Di, using the partial composition products.

(4) From the definition of a diassociative algebra, define a nonsymmetric operad Di', by means of generators and relations, which encodes diassociative algebras.

(5) Prove by hand that Di' is isomorphic to Di.

(6) Construct, in two different ways, a morphism of nonsymmetric operads $f : \mathrm{Di} \to \mathrm{As}$.

(7) Describe the induced pullback functor

$$f^* \ :\ \textsf{associative algebras} \to \textsf{diassociative algebras}\ .$$

(8) Describe its left adjoint functor

$$f_! \ :\ \textsf{diassociative algebras} \to \textsf{associative algebras}\ .$$

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Exercice 2 (Duplicial algebras).
By definition, a *duplicial algebra* is a $\mathbb{K}$-module A equipped with two linear maps

$$\lhd : A \otimes A \to A \quad \text{and} \quad \rhd : A \otimes A \to A\ ,$$

satisfying the following three relations

$$\begin{cases} (x \lhd y) \lhd z &= x \lhd (y \lhd z)\ , \\ (x \rhd y) \lhd z &= x \rhd (y \lhd z)\ , \\ (x \rhd y) \rhd z &= x \rhd (y \rhd z)\ , \end{cases}$$

for any $x, y, z \in A$. We denote by Dupl the nonsymmetric operad encoding duplicial algebras.

(1) Describe a canonical morphism of ns operads $\mathrm{Dupl} \to \mathrm{Di}$.

(2) We consider the set PBT_n of planar binary trees with n leaves, for $n \geqslant 2$. We endow the free $\mathbb{K}$-module $\bigoplus_{n\geqslant 1} \mathbb{K}[\mathrm{PBT}_n]$ with operations $\triangleleft$ and $\triangleright$ defined by : $t \triangleleft s$ is the planar binary tree obtained by grafting the tree s at the last leaf of the tree t and $t \triangleright s$ is the planar binary tree obtained by grafting the tree t at the first vertex of the tree s.

Show that this defines a duplicial algebra.

(3) Show that this duplicial algebra is free on one generator.

(4) Describe the nonsymmetric operad Dupl.

(5) Describe the morphism of Question (1) on the elements of Dupl.

Exercice 3 (Dendriform algebras).
By definition, a *dendriform algebra* is a $\mathbb{K}$-module A equipped with two linear maps

$$\prec\, : A \otimes A \to A \quad \text{and} \quad \succ\, : A \otimes A \to A\,,$$

satisfying the following three relations

$$\begin{cases} (x \prec y) \prec z &= x \prec (y \prec z) + x \prec (y \succ z)\,, \\ (x \succ y) \prec z &= x \succ (y \prec z)\,, \\ (x \prec y) \succ z + (x \succ y) \succ z &= x \succ (y \succ z)\,, \end{cases}$$

for any $x, y, z \in A$. We denote by Dend the nonsymmetric operad encoding dendriform algebras.

(1) Show that $a * b := a \prec b + a \succ b$ defines a morphism of ns operads $\mathrm{As} \to \mathrm{Dend}$.

(2) We consider the set PBT_n of planar binary trees with n leaves, for $n \geqslant 2$, with the exception that, for $n = 1$, this set admits only one element, the trivial tree $\mathrm{PBT}_1 = \{|\}$. We endow the free $\mathbb{K}$-module $\bigoplus_{n\geqslant 1} \mathbb{K}[\mathrm{PBT}_n]$ with operations $\prec$ and $\succ$ defined recursively by the following formulae

$$t \prec s := t^l \vee (t^r * s)\,,$$
$$t \succ s := (t * s^l) \vee s^r\,,$$

where

$$t = t^r \vee t^l = \overset{t^l \quad t^r}{\curlyvee}\,, \qquad s = s^r \vee s^l = \overset{s^l \quad s^r}{\curlyvee}\,, \qquad \text{and} \qquad | * t = t = t * |\,.$$

Show that this defines a dendriform algebra.

(3) Show that this dendriform algebra is free on one generator.

(4) Computation the dimension of $\mathrm{Dend}(n)$.

Exercice 4 (Alternative presentation for the operad Ass). We consider the (symmetric) operad $\mathrm{Ass}(n) := \mathbb{K}[\mathbb{S}_n]$ encoding associative algebras from Exercise 6 of Sheet 3.
The regular representation $\mathbb{K}[\mathbb{S}_2]$ decomposes as a direct sum of the trivial representation and the signature representation of $\mathbb{S}_2$. In other words, if we represent the canonical basis of $\mathbb{K}[\mathbb{S}_2]$ by

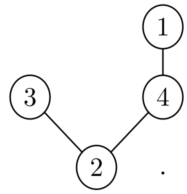
$$\mathrm{id} = \overset{1 \quad 2}{\curlyvee} \qquad \text{and} \qquad (1\,2) = \overset{2 \quad 1}{\curlyvee}\,,$$

another basis of $\mathbb{K}[\mathbb{S}_2]$ is given by

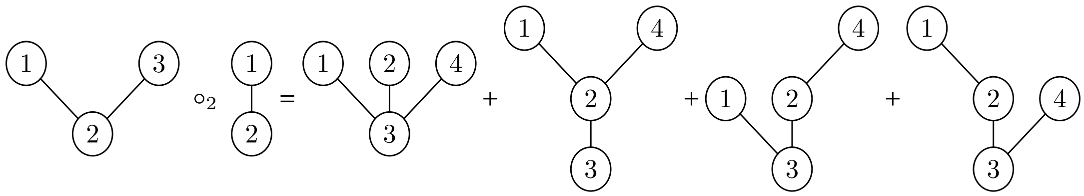
$$\overset{1 \quad 2}{\underset{*}{\curlyvee}} := \overset{1 \quad 2}{\curlyvee} + \overset{2 \quad 1}{\curlyvee} \qquad \text{and} \qquad \overset{1 \quad 2}{\underset{[\,,\,]}{\curlyvee}} := \overset{1 \quad 2}{\curlyvee} - \overset{2 \quad 1}{\curlyvee}\,.$$

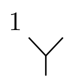
Give another presentation of the (symmetric) operad Ass using $*$ and $[\,,\,]$ for generators.

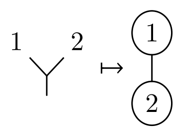
Exercice 5 (The operad preLie).
For any $n \geqslant 1$, we consider the set RT_n of rooted trees (in space) with n vertices labelled bijectively by $\{1, \ldots, n\}$ and with no leaf, like for instance



We denote by $\mathcal{RT}_n$ the free $\mathbb{K}$-module spanned by RT_n, which acquires an action of the symmetric group $\mathbb{S}_n$ by permutation of the indices. One defines partial composition products as follows. Let t and s be two rooted trees. One let $t \circ_i s$ be the sum of all possible ways to insert the tree s at the i^{th} vertex of t : one replaces the i^{th} vertex of t by s and one attaches all the subtrees grafted in t above the vertex i to s in all possible ways. We relabel the vertices accordingly : the vertices labelled by $1, \ldots, i-1$ in t remain unchanged, the labels of s are all shifted by $i-1$, and the labels of t greater than $i+1$ are all shifted by the number of vertices of s minus 1. Here is an example :



(1) Show that $\mathcal{RT} = (\{\mathcal{RT}_n\}_{n\in\mathbb{N}}, \circ_i, \textcircled{1})$ is an operad.
(2) We consider the operad preLie defined by generators and relations which encodes preLie algebras. Let us denote by  its generator. Show that the assignment



defines a morphism of operads $\mathrm{preLie} \to \mathcal{RT}$.
(3) Prove that this is an isomorphism of operads.
(4) Show that the assignment $a \star b := a \prec b - a \succ b$ defines a functor from

$$f^* \ : \ \textsf{dendriform algebras} \to \textsf{preLie algebras}\ .$$

Exercice 6 (Monomial algebras). Let $V := \mathbb{K}\{x_1, \ldots, x_n\}$ be the free $\mathbb{K}$-module on n generators and let $R \subset \{x_1 \otimes x_1, x_1 \otimes x_2, \ldots, x_n \otimes x_{n-1}, x_n \otimes x_n\}$ be a subset of quadratic monomials.

(1) Describe the quadratric algebra $A(V, R)$.
(2) Describe the quadratric coalgebra $C(V, R)$.
(3) Describe the quadratric coalgebra $A^{¡}$.
(4) Compute the homology groups of the Koszul complex $A^{¡} \otimes_{\kappa} A$.
(5) Describe the quasi-free resolution $\Omega A^{¡} \xrightarrow{\sim} A$.

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