The cyclic Deligne conjecture for spaces, chain complexes and Hopf algebras[1]

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[1] joint work with Michael Batanin (Sydney)

Hochschild cochains

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Hochschild cochains

Definition

For a (unital associative) K-algebra A and A-bimodule M, the *Hochschild cochain complex* of A with coefficients in M is given by

$$C^n(A;M) = \mathrm{Hom}_K(A^{\otimes n}, M), \quad n \geq 0,$$

where for $f \in C^n(A;M)$,

$$(\partial_i f)(a_1, \ldots, a_{n+1}) = \begin{cases} a_1 f(a_2, \ldots, a_n) & i = 0; \\ f(a_1, \ldots, a_i a_{i+1}, \ldots, a_n) & i = 1, \ldots, n; \\ f(a_1, \ldots, a_n) a_{n+1} & i = n+1. \end{cases}$$

$$(s_i f)(a_1, \ldots, a_{n-1}) = f(a_1, \ldots, a_i, 1_A, a_{i+1}, \ldots, a_{n-1}).$$

The Hochschild cohomology $HH^\bullet(A;M)$ is the cohomology of the cochain complex of the cosimplicial K-module $C^\bullet(A;M)$.

Cup and brace operations on $C^\bullet(A;A)$

There is a *cup product*

$$- \cup - : C^m(A;A) \otimes_K C^n(A;A) \to C^{m+n}(A;A)$$

$$(f \cup g)(a_1, \ldots, a_{m+n}) = f(a_1, \ldots, a_m) g(a_{m+1}, \ldots, a_{m+n})$$

and a *brace operation*

$$-\{-\} : C^m(A;A) \otimes_K C^n(A;A) \to C^{m+n-1}(A;A)$$

where $f\{g\}(a_1, \ldots, a_{m+n-1})$ is defined by

$$\sum_{1 \le i \le m} (-1)^{(i-1)(n-1)} f(a_1, \ldots, a_{i-1}, g(a_i, \ldots, a_{i+n-1}), a_{i+n}, \ldots, a_{m+n-1}).$$

The bracket $\{f, g\} = f\{g\} - (-1)^{(|f|-1)(|g|-1)} g\{f\}$ induces a Lie bracket of degree -1 on $HH^\bullet(A;A)$.

Definition

A *Gerstenhaber K-algebra* $(H, \cup, \{-,-\})$ is a graded-commutative K-algebra with Lie bracket of degree -1 such that

$$\{f, g \cup h\} = \{f, g\} \cup h + (-1)^{|f|(|g|-1)} g \cup \{f, h\}.$$

Proposition (Gerstenhaber '63)

For any algebra A, the Hochschild cohomology $HH^\bullet(A; A)$ is a Gerstenhaber algebra.

Theorem (F. Cohen '72)

For any field K, the homology $H_\bullet(D_2; K)$ of the little disks operad is the operad for Gerstenhaber K-algebras.

Corollary

For any based space $(X, *)$, the homology $H_\bullet(\Omega^2 X; K)$ is a Gerstenhaber K-algebra.

Connes' coboundary on $C^\bullet(A; A^*)$

For $A^* = \mathrm{Hom}_K(A, K)$, the adjunction

$$\mathrm{Hom}_K(A^{\otimes n}, A^*) \cong \mathrm{Hom}_K(A^{\otimes n+1}, K)$$

induces a cyclic operator τ_n on $C^n(A; A^*)$ of order $n+1$. These cyclic operators are compatible with the simplicial operators:

$$\tau_{n+1}\partial_i = \partial_{i-1}\tau_n \quad i > 0, \quad \tau_{n-1}s_i = s_{i-1}\tau_n \quad i > 0.$$

It results a covariant functor on Connes' *cyclic category*

$$\Delta C \to \mathrm{Mod}_K : [n] \mapsto C^n(A; A^*).$$

In particular, $C^\bullet(A; A^*)$ is a *mixed complex*

$$C^0(A; A^*) \leftrightarrows C^1(A, A^*) \leftrightarrows C^2(A; A^*) \leftrightarrows \cdots$$

and $HH^\bullet(A; A^*)$ has a differential Δ of degree -1:

$$\Delta^n : HH^n(A, A^*) \to HH^{n-1}(A; A^*).$$

Definition

A *Batalin-Vilkovisky algebra* is a Gerstenhaber algebra $(H, \cup, \{-,-\})$ with a differential Δ of degree -1 such that

$$(-1)^{|f|}\{f,g\} = \Delta(f \cup g) - (\Delta f \cup g) - (-1)^{|f|}(f \cup \Delta g).$$

A *symmetric K-algebra* A is a K-algebra equipped with an isomorphism of A-bimodules $A \cong A^*$, i.e. a symmetric exact pairing $<-,->: A \otimes_K A \to K$ such that $<ab, c> = <a, bc>$.

Proposition (Menichi '04)

For any symmetric algebra A, the Hochschild cohomology $HH^\bullet(A, A)$ is a Batalin-Vilkovisky algebra.

Theorem (Getzler '94)

For any field K, the homology $H_\bullet(fD_2, K)$ of the framed little disks operad is the operad for Batalin-Vilkovisky K-algebras.

Theorem (MS '02, KS '02, Vo '02, Ta '04, BF '04)

The Hochschild cochain complex of an algebra A admits a $C_\bullet(D_2)$-action inducing the Gerstenhaber structure on $HH^\bullet(A;A)$.

Theorem (KS '06, TZ '06, Ka '07, BB '09)

The Hochschild cochain complex of a symmetric algebra A admits a $C_\bullet(fD_2)$-action inducing the BV-structure on $HH^\bullet(A;A)$.

Proposition (Gerstenhaber-Voronov '95, Menichi '04)

The Hochschild cochain complex of A is isomorphic to the *deformation complex* of the *endomorphism operad* End_A of A.
If A is symmetric, then End_A is multiplicative cyclic.

Proof.

$C^n(A;A) = \mathrm{Hom}(A^{\otimes n}, A) = \mathrm{End}_A(n) \ni \mu_n$. For $f \in \mathrm{End}_A(n)$,
$\partial_0 f = \mu_2 \circ_1 f$, $\partial_n f = \mu_2 \circ_0 f$, $\partial_i f = f \circ_i \mu_2$ if $0 < i < n$.
If A is symmetric then End_A is cyclic and $\tau_n(\mu_n) = \mu_n$. □

Multiplicative operads

Definition

A multiplicative (cyclic) operad is a non-symmetric (cyclic) operad $\mathcal{O}$ equipped with a map of (cyclic) operads $\mathcal{A}ss \to \mathcal{O}$.
A muliplicative (cyclic) operad $\mathcal{O}$ has an underlying cosimplicial (cocyclic) object $\mathcal{O}^\bullet$. In a closed monoidal category $\mathcal{E}$ equipped with $\delta : \Delta \to \mathcal{E}$, the *deformation complex* of $\mathcal{O}$ is $\underline{\mathrm{Hom}}_\Delta(\delta^\bullet, \mathcal{O}^\bullet)$.

Example

For $\mathcal{E} = \mathrm{Ch}(\mathbb{Z})$ and $\delta_\mathbb{Z} : \Delta \to \mathrm{Ch}(\mathbb{Z}) : [n] \mapsto N_*(\Delta[n]; \mathbb{Z})$ we get

$$C^\bullet(A; A) = \underline{\mathrm{Hom}}_\Delta(\delta_\mathbb{Z}^\bullet, \mathrm{End}_A^\bullet).$$

Theorem (Kaufmann '07, BB '09)

For any multiplicative chain operad $\mathcal{O}$, the deformation complex of $\mathcal{O}$ admits a $C_\bullet(D_2)$-action. If $\mathcal{O}$ is multiplicative cyclic, this action extends to a $C_\bullet(fD_2)$-action.

Let $\mathcal{L}_2(n_1, \ldots, n_k; n)$ be the set of iso-classes of planar rooted trees with n leaves and a bipartite vertex-set such that:

1. one part of the vertex-set is in bijection with $\{1, \ldots, k\}$;
2. the vertex with label i has arity n_i;
3. each edge has at least one labelled extremity;
4. unlabelled vertices have arity $\neq 1$.

Let $C_{[n]} = \mathbb{Z}/(n+1)\mathbb{Z}$ and put

$$\mathcal{L}_2^{cyc}(n_1, \ldots, n_k; n) = \mathcal{L}_2(n_1, \ldots, n_k; n) \times C_{[n_1]} \times \cdots \times C_{[n_k]}.$$

$\mathcal{L}_2$ and $\mathcal{L}_2^{cyc}$ are $\mathbb{N}$-coloured operads for an evident substitution of trees into trees; in $\mathcal{L}_2^{cyc}$, the cyclic permutations distinguish for each labelled vertex one of its incident edges, the neutral element stands for the edge closest to the root of the tree.

Lemma

$\mathcal{L}_2$-algebras are multiplicative operads; $\mathcal{L}_2^{cyc}$-algebras are multiplicative cyclic operads. The category of unary operations of $\mathcal{L}_2$ (resp. $\mathcal{L}_2^{cyc}$) is Δ (resp. ΔC).

Unary operations of a coloured operad act covariantly on inputs and contravariantly on the output; therefore:

$$\mathcal{L}_2(-,\cdots,-;-) : \Delta^{\mathrm{op}} \times \cdots \times \Delta^{\mathrm{op}} \times \Delta \to \mathrm{Sets}.$$

Given $\delta_{\mathbb{Z}} : \Delta \to \mathrm{Ch}(\mathbb{Z})$ we can *realize* multisimplicially, and *totalize* the resulting cosimplicial chain complex. This yields

$$\xi(\mathcal{L}_2,\delta_{\mathbb{Z}})(k) := \underline{\mathrm{Hom}}_{\Delta}(\delta^{\bullet}, |\mathcal{L}_2(\overbrace{-,\cdots,-}^{k};\bullet)|_{\delta_{\mathbb{Z}}^{\otimes k}}), \quad k \geq 0.$$

Proposition (Day-Street '03, McClure-Smith '04, BB '09)

$\xi(\mathcal{L}_2,\delta_{\mathbb{Z}})$ (resp. $\xi(\mathcal{L}_2^{cyc},\delta_{\mathbb{Z}}^{cyc})$) is a chain operad acting on the deformation complex of any multiplicative (cyclic) operad.

Theorem (BB '09)

As chain operads we have $\xi(\mathcal{L}_2,\delta_{\mathbb{Z}}) \sim C_\bullet(D_2)$ and $\xi(\mathcal{L}_2^{cyc},\delta_{\mathbb{Z}}^{cyc}) \sim C_\bullet(fD_2)$.

Theorem (cf. Gerstenhaber-Schack '92, Menichi '04)

The cobar complex ΩA of a bialgebra (resp. involutive Hopf algebra) A has an action by $\xi(\mathcal{L}_2, \delta_{\mathbb{Z}})$ (resp. $\xi(\mathcal{L}_2^{cyc}, \delta_{\mathbb{Z}}^{cyc})$). Its homology $H_\bullet(\Omega A; \mathbb{Z})$ is a Gerstenhaber (resp. BV-) algebra.

Proof.

The bialgebra A is a comonoid in the monoidal category of A-modules. Therefore: $(\Omega A)_n = A^{\otimes n} \cong \mathrm{Hom}_A(A, A^{\otimes n})$. This $\mathbb{Z}$-linear operad is multiplicative via the diagonal of A. If A has an involutive antipode then the operad is multiplicative cyclic. □

Remark

(a) $\Omega C_\bullet(\Omega X; \mathbb{Z}) \sim C_\bullet(\Omega^2 X; \mathbb{Z})$ (Adams). The $\xi(\mathcal{L}_2, \delta_{\mathbb{Z}})$-action on $\Omega C_\bullet(\Omega X; \mathbb{Z})$ corresponds to the $C_\bullet(D_2)$-action on $C_\bullet(\Omega^2 X; \mathbb{Z})$.
(b) If A is involutive, the $\xi(\mathcal{L}_2^{cyc}, \delta_{\mathbb{Z}}^{cyc})$-action induces a cocyclic structure on ΩA yielding $HC^\bullet(A)$ of Connes-Moscovici '99.
(c) $\xi(\mathcal{L}_2, \delta_{\mathbb{Z}})$ contains the second filtration stage of the *surjection operad* of MS '03, BF '04 as a suboperad. Cyclic extension ?

There is a cosimplicial resp. cocyclic space

$\delta_{top} : \Delta \to \mathrm{Top} : [n] \mapsto \Delta^n$ resp. $\delta_{top}^{cyc} : \Delta C \to \mathrm{Top} : [n] \mapsto \Delta^n \times S^1$.

Theorem (McClure-Smith '04, Salvatore '09, BB '09)

The operad $\xi(\mathcal{L}_2, \delta_{top})$ is weakly equivalent to D_2 and acts on the deformation complex of multiplicative operads in spaces.
The operad $\xi(\mathcal{L}_2^{cyc}, \delta_{top}^{cyc})$ is weakly equivalent to fD_2 and acts on the deformation complex of multiplicative cyclic operads in spaces.

Remark (cf. Markl '99, Salvatore-Wahl '03, Salvatore '09)

$fD_2(k) \cong D_2(k) \times (S^1)^k$, $\xi(\mathcal{L}_2^{cyc}, \delta_{top}^{cyc})(k) \cong \xi(\mathcal{L}_2, \delta_{top})(k) \times (S^1)^k$.
For $n = 1$:
$fD(1) \cong D(1) \rtimes S^1$, $\underline{\mathrm{Hom}}_{\Delta C}(\delta_{top}^{cyc}, \delta_{top}^{cyc}) \cong \underline{\mathrm{Hom}}_{\Delta}(\delta_{top}, \delta_{top}) \boxtimes S^1$.

Proposition (Sinha '06)

The simplicial 2-sphere $S^2 = \Delta[2]/\partial\Delta[2]$ is an $\mathcal{L}_2$-coalgebra in finite pointed sets. For a based space $(X, *)$, $\Omega^2 X$ is the deformation complex of the multiplicative operad $(X, *)^{(S^2, *)}$.

Braid and ribbon-braid groups

$\mathfrak{S}_k$ denotes the *permutation group* on k letters. $\mathfrak{S}_k^{\pm}$ denotes the *signed permutation group* on k letters.
$\mathfrak{S}_k^{\pm} = \mathfrak{S}_k \wr \mathfrak{S}_2 = \mathfrak{S}_k \ltimes (\mathfrak{S}_2)^k$ acts on $fD_2(k) = D_2(k) \times (S^1)^k$.

Definition (Braid and ribbon-braid groups on k strands)

$$\begin{array}{rclrcl} B_k & = & \pi_1(D_2(k)/\mathfrak{S}_k) & RB_k & = & \pi_1(fD_2(k)/\mathfrak{S}_k^{\pm}) \\ PB_k & = & \pi_1(D_2(k)) & PRB_k & = & \pi_1(fD_2(k)) \end{array}$$

Proposition (Asphericity of $D_2(k)$ and $fD_2(k)$)

$$\begin{array}{rclrcl} D_2(k)/\mathfrak{S}_k & = & K(B_k, 1) & fD_2(k)/\mathfrak{S}_k^{\pm} & = & K(RB_k, 1) \\ D_2(k) & = & K(PB_k, 1) & fD_2(k) & = & K(PRB_k, 1) \end{array}$$

Corollary

The coverings $D_2(k) \to D_2(k)/\mathfrak{S}_k$ and $fD_2(k) \to fD_2(k)/\mathfrak{S}_k^{\pm}$ are classified by the short exact sequences $1 \to PB_k \to B_k \to \mathfrak{S}_k \to 1$ and $1 \to PRB_k \to RB_k \to \mathfrak{S}_k^{\pm} \to 1$.

Problem

Describe the operad structure of D_2 (resp. fD_2) in terms of the pure braid (resp. ribbon-braid) groups.

$B_k = < s_1, \ldots, s_{k-1} \,|\, (s_i s_j)^2 = 1 \text{ if } |i-j| > 1 \text{ and } (s_i s_{i+1})^3 = 1 >$

The *pure Artin group* $PB_k = Ker(B_k \to \mathfrak{S}_k) \cong \pi_1(\mathbb{C}^k - \mathcal{A}_{\mathfrak{S}_k})$ where $\mathcal{A}_{\mathfrak{S}_k}$ is the complexified *braid arrangement*.

The *Salvetti complex* $Sal_{\mathfrak{S}_k}$ is a partially ordered set of the same equivariant homotopy type as $\mathbb{C}^k - \mathcal{A}_{\mathfrak{S}_k}$.

$$Sal_{-} : (\textit{Coxeter groups}) \to (\textit{posets})$$

is a functor commuting with finite products. Thus, $(PB_k)_{k\geq 0}$ is a categorical operad. Similarly, $(PRB_k)_{k\geq 0}$ is a categorical operad.

Proposition

$D_2 \sim K(PB, 1)$ and $fD_2 \sim K(PRB, 1)$ as operads. Moreover, PB-algebras are braided strict monoidal categories; PRB-algebras are ribbon-braided (i.e. balanced) strict monoidal categories.

Corollary (B '98, Salvatore-Wahl '03)

The nerve of a braided (resp. ribbon-braided) strict monoidal category is E_2 (resp. framed E_2).

Consider the cosimplicial category

$$\delta_{\mathrm{Cat}} : \Delta \to \mathrm{Cat} : [n] \mapsto [n][n]^{-1}$$

Proposition

There are weak equivalences of categorical operads

$$PB \xrightarrow{\sim} \xi(\mathcal{L}_2, \delta_{\mathrm{Cat}}) \quad \text{and} \quad PRB \xrightarrow{\sim} \xi(\mathcal{L}_2^{cyc}, \delta_{\mathrm{Cat}}).$$

Definition

A central element of a monoidal category $\mathcal{E}$ is a pair (A, c_A) where $c_{A,-} : A \otimes - \cong - \otimes A$ and $c_{A,B\otimes C} = (1_B \otimes c_{A,C}) \circ (c_{A,B} \otimes 1_C)$. The center $\mathcal{Z}\mathcal{E}$ is the category of central elements.

Proposition

For $\mathcal{E} = \mathrm{Mod}_H$, $\mathcal{Z}\mathcal{E} \simeq \mathrm{Mod}_{DH}$ where DH is the Drinfeld double of the Hopf algebra H.

Proposition (Street '04)

$\mathcal{Z}\mathcal{E} = \underline{\mathrm{Hom}}_\Delta(\delta_{\mathrm{Cat}}, \mathrm{End}_\mathcal{E})$

Corollary

The center of a monoidal category is braided monoidal; in particular, the Drinfeld double of a Hopf algebra is "braided".

Definition

An involutive category is a closed monoidal category $\mathcal{E}$ such that the duality functor $(-)^* = \underline{\mathrm{Hom}}(-, I)$ is self-adjoint. A Hopf algebra H is called quasi-involutive if Mod_H is involutive.

Proposition

The category $\mathcal{E}_f$ of symmetric duality objects of an involutive category $\mathcal{E}$ has a multiplicative cyclic endomorphism-operad $\mathrm{End}_{\mathcal{E}_f}$.

Corollary

The center of $\mathcal{E}_f$ is ribbon-braided; in particular, the Drinfeld double of a quasi-involutive Hopf algebra is "ribbon-braided".