EXAM

7 JANVIER 2021

INSTRUCTIONS. The presentation and the quality of the redaction, *the clarity and the precision of the exposition* will play an important part in the evaluation of the copy. Any answer given without justification will receive no point. Only handwritten notes and paper documents are allowed.

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Exercice 1 (Fiber and loop space).

(1) Let (X, x_0) be a pointed topological space. Show that the map

$$\mathrm{Path}(X) \coloneqq \left\{\varphi : I \to X \mid \varphi(0) = x_0\right\} \to X \ , \quad \varphi \mapsto \varphi(1)$$

is a fibration with $\mathrm{Path}(X)$ contractible.

(2) Let $F \to E \to B$ be a fibration where (B, b_0) is a pointed path connected topological space, where $F \coloneqq p^{-1}(b_0)$ is the fiber, and where E is contractible. Show that there exists a weak homotopy equivalence $F \xrightarrow{\sim} \Omega B$.

HINT : We will admit the functoriality of the long exact sequence associated to a fibration.

(3) Show that $\Omega\,\mathbb{P}^\infty\mathbb{C}$ is homotopy equivalent to S^1.

HINT : One can use Moore's theorem which asserts that the loop space of any CW-complex is homotopy equivalent to a CW-complex.

(4) Compute the homotopy groups of $\mathbb{P}^\infty\mathbb{C}$.

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Exercice 2 (Fibrant resolution).

To any poset $(P, \leqslant)$, one associates the simplicial complex

$$\Delta(P, \leqslant) \coloneqq (P, \{\lambda_0 < \cdots < \lambda_k\})$$

made up of chains $\lambda_0 < \cdots < \lambda_k$ of elements of P, for any $k \geqslant 0$.

(1) For any $n \geqslant 0$, we consider the totally ordered set $[n] \coloneqq \left(\{0, \ldots, n\}, \leqslant\right)$. What is the simplicial complex $\Delta([n])$?

To any simplicial complex $(V, \mathfrak{X})$, one associates a simplicial set $\mathrm{bary}(V, \mathfrak{X})$ whose k-simplicies

$$\mathrm{bary}(V, \mathfrak{X})_k \coloneqq \{F_0 \subseteq \cdots \subseteq F_k\}$$

are the chains of faces $F_i \in \mathfrak{X}$ and whose face and degeneracy maps are giving by

$$d_i(F_0 \subseteq \cdots \subseteq F_k) \coloneqq (F_0 \subseteq \cdots \subseteq F_{i-1} \subseteq F_{i+1} \subseteq \cdots \subseteq F_k) \quad \text{and}$$
$$s_i(F_0 \subseteq \cdots \subseteq F_k) \coloneqq (F_0 \subseteq \cdots \subseteq F_i \subseteq F_i \subseteq \cdots \subseteq F_k) \ .$$

For any $n \geqslant 0$, we consider the image of $[n]$ by the composite of these two constructions :

$$\mathrm{sd}\,\Delta^n \coloneqq \mathrm{bary}\big(\Delta([n])\big) \ .$$

(2) Describe the simplicial set $\mathrm{sd}\,\Delta^n$, for $n \geqslant 0$, and represent graphically its geometric realisation $\left|\mathrm{sd}\,\Delta^2\right|$ with its cellular structure.

(3) Show that the maps $\delta_i\colon [n-1] \to [n]$ and $\sigma_i\colon [n+1] \to [n]$ defined by

$$\delta_i(j) := \begin{cases} j & \text{for} \quad j < i\,, \\ j+1 & \text{for} \quad j \geqslant i\,, \end{cases} \qquad \text{and} \quad \sigma_i(j) = \begin{cases} j & \text{for} \quad j \leqslant i\,, \\ j-1 & \text{for} \quad j > i\,, \end{cases}$$

induce a cosimplicial simplicial set structure on the simplicial sets $\mathrm{sd}\,\Delta^n$, for $n \geqslant 0$, under the formula

$$\bar{\delta}_i(F_0 \subseteq \cdots \subseteq F_k) = \big(\delta_i(F_0) \subseteq \cdots \subseteq \delta_i(F_k)\big) \quad \text{and} \quad \bar{\sigma}_i(F_0 \subseteq \cdots \subseteq F_k) = \big(\sigma_i(F_0) \subseteq \cdots \subseteq \sigma_i(F_k)\big)\ .$$

(4) We denote this cosimplicial simplicial set by $\mathrm{sd}\,\Delta^\bullet\colon \Delta \to \mathsf{sSet}$. Show that this functor extends to an endofunctor $\mathrm{sd}\colon \mathsf{sSet} \to \mathsf{sSet}$ via Yoneda's embedding and that it admits a right adjoint $\mathrm{Ex}\colon \mathsf{sSet} \to \mathsf{sSet}$ given by $\mathrm{Ex}\,\mathfrak{X} := \mathrm{Hom}_{\mathsf{sSet}}(\mathrm{sd}\,\Delta^\bullet, \mathfrak{X})$.

(5) Describe the simplicial set $\mathrm{sd}\left(\Delta^2/\partial\Delta^2\right)$ and represent graphically its geometric realisation $\left|\mathrm{sd}\left(\Delta^2/\partial\Delta^2\right)\right|$ with its cellular structure. Does $\mathrm{sd}\left(\Delta^2/\partial\Delta^2\right)$ give a triangulation of the sphere ?

(6) Represent graphically the geometric realisation $\left|\mathrm{sd}^2\left(\Delta^2\right)\right|$ and $\left|\mathrm{sd}^2\left(\Delta^2/\partial\Delta^2\right)\right|$ with their cellular structures. Does $\mathrm{sd}^2\left(\Delta^2/\partial\Delta^2\right)$ give a triangulation of the sphere ?

(7) Describe the n-simplicies of $\mathrm{Ex}\,\mathfrak{X}$ in terms of the n-simplicies of $\mathfrak{X}$.
Indication : One could write the simplicial set $\mathrm{sd}\,\Delta^n$ like a coequalizer of the form

$$\coprod_{?} \Delta^{n-1} \underset{?}{\overset{?}{\rightrightarrows}} \coprod_{\omega \in \mathbb{S}_{[n]}} \Delta^n \twoheadrightarrow \mathrm{sd}\,\Delta^n\ ,$$

where $\mathbb{S}_{[n]} \cong \mathbb{S}_{n+1}$ is the set of bijections of $[n]$.

(8) For any $n \geqslant 0$, we consider the morphism $\varepsilon_n\colon \mathrm{sd}\,\Delta^n \to \Delta^n$ of simplicial sets defined by

$$\varepsilon_n\left(F_0 \subseteq \cdots \subseteq F_k\right) := \left(\max F_0 \leqslant \cdots \leqslant \max F_k\right)\ .$$

Show that this is a simplicial homotopy equivalence.

We denote by $\varepsilon_\bullet\ :\ \mathrm{sd}\,\Delta^\bullet \to \Delta^\bullet$ the induced morphism of cosimplicial simplicial sets. Pulling back with this latter one, we get a natural transformation of functors $\eta\colon \mathrm{id}_{\mathsf{sSet}} \to \mathrm{Ex}$:

$$\eta_{\mathfrak{X}} := (\varepsilon_\bullet)^* \quad : \quad \mathfrak{X} \cong \mathrm{Hom}_{\mathsf{sSet}}(\Delta^\bullet, \mathfrak{X}) \quad \to \quad \mathrm{Hom}_{\mathsf{sSet}}(\mathrm{sd}\,\Delta^\bullet, \mathfrak{X}) = \mathrm{Ex}\,\mathfrak{X}\ .$$

(9) Show that, for any simplicial set $\mathfrak{X}$ and for any morphism $\lambda\colon \Lambda^n_k \to \mathrm{Ex}\,\mathfrak{X}$, there exists a morphism $\Delta^n \to \mathrm{Ex}^2\,\mathfrak{X}$ of simplicial sets making commutative the following diagram

$$\begin{array}{ccc} \Lambda^n_k & \xrightarrow{\ \lambda\ } & \mathrm{Ex}\,\mathfrak{X} \\ \big\downarrow & & \big\downarrow{\scriptstyle \eta_{\mathrm{Ex}\,\mathfrak{X}}} \\ \Delta^n & \dashrightarrow & \mathrm{Ex}^2\,\mathfrak{X}\ . \end{array}$$

Hint : One can admit that there exists a morphism of simplicial sets $\mathrm{sd}\,\Delta^n \to \mathrm{Ex}\,\mathrm{sd}\,\Lambda^n_k$ factorizing the morphism $\eta_{\mathrm{sd}\,\Lambda^n_k}$ in the following way

$$\begin{array}{ccc} \mathrm{sd}\,\Lambda^n_k & \xrightarrow{\ \eta_{\mathrm{sd}\,\Lambda^n_k}\ } & \mathrm{Ex}\,\mathrm{sd}\,\Lambda^n_k \\ \big\downarrow & \nearrow & \\ \mathrm{sd}\,\Delta^n & & . \end{array}$$

(10) We consider

$$\mathrm{Ex}^\infty\,\mathfrak{X} := \mathrm{colim}\left(\ \mathfrak{X} \xrightarrow{\ \eta_{\mathfrak{X}}\ } \mathrm{Ex}\,\mathfrak{X} \xrightarrow{\ \eta_{\mathrm{Ex}\,\mathfrak{X}}\ } \mathrm{Ex}^2\,\mathfrak{X} \xrightarrow{\ \eta_{\mathrm{Ex}^2\,\mathfrak{X}}\ } \cdots\ \right)\ .$$

Show that $\mathrm{Ex}^\infty\,\mathfrak{X}$ is a Kan complex, for any simplicial set $\mathfrak{X}$.

(11) [BONUS] Show the hint of Question (9).

(12) [BONUS] Show that the canonical morphism $\theta\colon \mathfrak{X} \xrightarrow{\sim} \mathrm{Ex}^\infty\,\mathfrak{X}$ is a weak homotopy equivalence, for any simplicial set $\mathfrak{X}$, that is $\pi_n(|\theta|)\colon \pi_n(|\mathfrak{X}|, x) \cong \pi_n(|\mathrm{Ex}^\infty\,\mathfrak{X}|, |f|(x))$ are isomorphisms, for $n \geqslant 1$ and $x \in |\mathfrak{X}|$, and a bijection for $n = 0$.