EXAM

INSTRUCTIONS. The presentation and the quality of the redaction, *the clarity and the precision of the exposition* will play an important part in the evaluation of the copy. Any answer given without justification will receive no point. Only handwritten notes and paper documents are allowed. All electronic devices are forbidden.

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Exercise 1 (Configuration spaces). For $n \geqslant 1$, the *configuration space* of n-points of a topological space X is the sub-space

$$\mathrm{Conf}_n(X) := \{(x_1, \ldots, x_n) \in X^n \,;\, x_i \neq x_j \text{ pour } i \neq j\}$$

of X^n.

(1) Describe the homotopy type of the configuration spaces of points $\mathrm{Conf}_n(\mathbb{R})$ of $\mathbb{R}$, for $n \geqslant 1$.

(2) Show that the continuous map

$$\begin{array}{rccc} p_i \colon & \mathrm{Conf}_n(\mathbb{R}^d) & \to & \mathrm{Conf}_{n-1}(\mathbb{R}^d) \\ & (x_1, \ldots, x_n) & \mapsto & (x_1, \ldots, x_{i-1}, x_{i+1}, x_n)\,, \end{array}$$

which forget the i^{th} point, is a fibration, for $n, d \geqslant 1$ and describe its fiber.

(3) Show that, for $n \geqslant 1$, the configuration spaces $\mathrm{Conf}_n(\mathbb{R}^2)$ are path connected and that the higher homotopy groups $\pi_k(\mathrm{Conf}_n(\mathbb{R}^2)) \cong \{0\}$ are trivial, for $k \geqslant 2$.

(4) Show that the configuration space of points $\mathrm{Conf}_2(\mathbb{R}^d) \sim S^{d-1}$ is homotopically equivalent to the $d-1$-dimensional sphere, for all $d \geqslant 1$.

(5) Show that the fibration $p_i \colon \mathrm{Conf}_n(\mathbb{R}^d) \twoheadrightarrow \mathrm{Conf}_{n-1}(\mathbb{R}^d)$ admits a section, that is a continuous map $s \colon \mathrm{Conf}_{n-1}(\mathbb{R}^d) \to \mathrm{Conf}_n(\mathbb{R}^d)$ such that $p_i \circ s = \mathrm{id}_{\mathrm{Conf}_{n-1}(\mathbb{R}^d)}$.

(6) Show that there exists an isomorphism (bijection for $k = 0$)

$$\pi_k\left(\mathrm{Conf}_n(\mathbb{R}^d)\right) \cong \prod_{j=1}^{n-1} \pi_k\left(\bigvee_j S^{d-1}\right),$$

for $d \geqslant 3$, $n \geqslant 1$ and $k \geqslant 0$.

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Exercise 2 (∞-catégorie). In an ∞-category $\mathfrak{X}$, the *objects* are the 0-simplices $x \in X_0$, and the *morphisms* are the 1-simplices $f \in X_1$. A morphism $f \in X_1$ has *source* $x := d_1(f)$ and *target* $y := d_0(f)$; it is then denoted by $f \colon x \to y$. For any object $x \in X_0$, the *identity morphism of* x is the 1-simplex $\mathrm{id}_x := s_0(x) : x \to x$, which is the image of x under the degeneracy map $s_0 : X_0 \to X_1$.

Two morphisms $f, g \colon x \to y$ are *left homotopic*, denoted by $f \sim_G g$, if there exists a 2-simplex $\sigma \in X_2$ such that

$$\partial\sigma = (g, f, \mathrm{id}_x) \qquad \begin{tikzcd} & x \arrow[dr, "g"] & \\ x \arrow[ur, "\mathrm{id}_x"] \arrow[rr, "f"'] & \sigma & y \end{tikzcd}$$

and they are *right homotopic*, denoted $f \sim_D g$, if there exists a 2-simplex $\tau \in X_2$ such that

$$\partial\tau = (\mathrm{id}_y, g, f) \qquad \begin{tikzcd} & y \arrow[dr, "\mathrm{id}_y"] & \\ x \arrow[ur, "f"] \arrow[rr, "g"'] & \tau & y \end{tikzcd} .$$

From now on, we assume that these two relations are equivalent.

(1) Show that the left homotopy relation on the set of morphisms from x to y is an equivalence relation for any pair of objects $x, y \in X_0$.

We denote by $\iota\colon \Delta^0 \sqcup \Delta^0 \to \Delta^1$ the morphism of simplicial sets defined by the coproduct $\iota \coloneqq \iota_0 \sqcup \iota_1$ of the two morphisms $\iota_i\colon \Delta^0 \to \Delta^1$ defined by $0 \in \Delta^0_0 \mapsto i \in \Delta^1_0$, for $i = 0, 1$. We denote by $\mathfrak{Y}^{\mathfrak{X}} = \mathfrak{Hom}(\mathfrak{X}, \mathfrak{Y}) = \mathrm{Hom}_{\Delta\mathsf{Ens}}(\mathfrak{X} \times \Delta^\bullet, \mathfrak{Y})$ the mapping space from $\mathfrak{X}$ to $\mathfrak{Y}$. For any pair of objects $x, y \in X_0$, the *space of morphisms from x to y* is defined as the simplicial set $\mathfrak{X}(x, y)$, obtained as the pullback

$$\begin{tikzcd}
\mathfrak{X}(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathfrak{X}^{\Delta^1} \arrow[d, "\iota^*"] \\
\Delta^0 \arrow[r, "(x,y)"] & \mathfrak{X}^{\Delta^0 \sqcup \Delta^0}
\end{tikzcd} ,$$

where the morphism (x, y) of simplicial sets sends the 0-simplex of Δ^0 to the 0-simplex $\Delta^0 \sqcup \Delta^0 \to \mathfrak{X}$, whose image of the left copy Δ^0_0 is mapped to x, and the right copy of Δ^0_0 is mapped to y.

(2) Provide a description of the 0-simplices of $\mathfrak{X}(x, y)$ in terms of morphisms in the ∞-category $\mathfrak{X}$.

(3) Provide a description of the 1-simplices of $\mathfrak{X}(x, y)$.

We assume that $\mathfrak{X}(x, y)$ is a Kan complex for any pair of objects $x, y \in X_0$.

(4) Describe the homotopy relation $\sim$ between the 0-simplices of the morphism space $\mathfrak{X}(x, y)$ from x to y.

(5) Show that the homotopy relation $\sim$ is equivalent to the left homotopy relation $\sim_G$.

The *homotopy category* $\mathsf{Ho}(\mathfrak{X})$ of an ∞-category $\mathfrak{X}$ is defined as follows: its objects are the objects X_0 of $\mathfrak{X}$, and its morphisms are the equivalence classes $[f]$ of morphisms $f\colon x \to y$ under the homotopy relation defined above. The *composition* $[g] \circ [f]\colon x \to z$ is defined by filling the Λ^2_1-horn $(g, -, f)$ in $\mathfrak{X}$ with a 2-simplex $\omega \in X_2$ and by setting

$$[g] \circ [f] \coloneqq [d_1(\omega)] .$$

(6) Show that this composition is well-defined and associative.

The *identity morphisms* in the homotopy category are the equivalence classes $[\mathrm{id}_x] = [s_0(x)]$ of the identities of the ∞-category $\mathfrak{X}$. From now on, we assume that these data form a category. The *fundamental category of an ∞-category* $\mathfrak{X}$ is the fundamental category $\tau_1(\mathfrak{X})$ of its underlying simplicial set.

(7) Show that there is a natural isomorphism $\tau_1(\mathfrak{X}) \cong \mathsf{Ho}(\mathfrak{X})$ between the fundamental category and the homotopy category of ∞-categories.

A *functor* between two ∞-categories $\mathfrak{X}$ and $\mathfrak{Y}$ is a morphism of simplicial sets $F\colon \mathfrak{X} \to \mathfrak{Y}$. We assume that the *space of functors* $\mathfrak{Y}^{\mathfrak{X}}$ is an ∞-category. A *natural transformation* $\alpha\colon F \to G$ between two functors $F, G\colon \mathfrak{X} \to \mathfrak{Y}$ is a morphism from F to G in the space of functors $\mathfrak{Y}^{\mathfrak{X}}$.

(8) Show that there is a natural isomorphism of simplicial sets

$$\mathfrak{N}\mathsf{Fun}(\mathsf{C}, \mathsf{D}) \cong \mathfrak{N}\mathsf{D}^{\mathfrak{N}\mathsf{C}} ,$$

where C, D are two categories, where $\mathsf{Fun}(\mathsf{C}, \mathsf{D})$ is the category of functors from C to D, and where $\mathfrak{N}$ is the nerve of a category.