Homotopy Batalin-Vilkovisky algebras

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Overview

1. Homotopy algebras and operads
2. Definition of homotopy BV-algebra
3. Homotopy and deformation theory of algebras

Overview

Associative algebras and homotopy [Stasheff 63]

X, Y homotopy equivalent topological spaces: $X \sim Y$

$X \times X \xrightarrow{\mu} X$ associative product $\xrightarrow{\text{transfer}} Y \times Y \xrightarrow{\nu} Y$ but

ν associative up to homotopy

$\Longrightarrow$ "The disadvantage of topological groups and monoids is that they do not live in homotopy theory"

[Mac Lane's seminar at the University of Chicago 1967]

Homotopy associative algebras [Stasheff 63]

Let (A, d_A) be a dg module. We consider $\mathrm{Hom}(A^{\otimes n}, A)$ with derivative $\partial(f) := d_A \circ f - (-1)^{|f|} f \circ d_{A^{\otimes n}}$

Definition (Homotopy associative algebra or A_∞-algebra)

Family of operations $\{\mu_n : A^{\otimes n} \to A\}_{n \geq 2}$, degree $|\mu_n| = n - 2$, such that

$$\sum_{\substack{2 \leq k \leq n-1 \\ 1 \leq i \leq k}} \pm \mu_k \circ \left(\mathrm{id}^{\otimes(i-1)} \otimes \mu_{n-k+1} \otimes \mathrm{id}^{\otimes(k-i)}\right) = \partial(\mu_n)$$

in $\mathrm{Hom}(A^{\otimes n}, A)$ for all $n \geq 2$.

Example: $\mu_2 \circ (\mu_2 \otimes \mathrm{id}) - \mu_2 \circ (\mathrm{id} \otimes \mu_2) = \partial(\mu_3)$ The left hand sight relation holds up to the homotopy μ_n.

▷ Associative algebra = A_∞-algebra with $\mu_n = 0$, $n \geq 3$.

Homotopy theory for A_∞-algebras

Definition (A_∞-morphism between A_∞-algebras)

Family of maps $\{f_n : A^{\otimes n} \to B\}_{n\geq 1}$, of degree $|f_n| = n-1$, satisfying some condition. **Notation** $f_\bullet : A \to B$

Proposition

$(A_\infty - algebras, A_\infty - morphisms)$ forms a category.

▷ Notion of A_∞-*homotopy* relation $\sim$ between A_∞-morphisms.
$\Longrightarrow$ Abstract homotopy theory for A_∞-algebras

Definition (A_∞-homotopy equivalence)

$f_\bullet : A \to B$, A_∞-morphism such that
$\exists\ g_\bullet : B \to A$, A_∞-morphism satisfying
$f \circ g \sim \mathrm{id}_B$ and $g \circ f \sim \mathrm{id}_A$: A_∞-homotopy equivalence

Homotopy invariant property for A_∞-algebras

Let $(V, d_V) \xrightarrow{f} (W, d_W)$ be a chain homotopy equivalence.

Theorem (...)

Any A_∞-algebra structure on W transfers to an A_∞-algebra structure on V such that

- *f extends to an A_∞-morphisms with $f_1 = f$, which is a A_∞-homotopy equivalence.*

▷ Explicit formula based on planar trees [Kontsevich-Soibelman]

Application: $V = H(A) \rightarrowtail A = W$

transferred A_∞-operations on $H(A)$ = Massey products

Example: Associative algebra $A = (C^\bullet(X), \cup)$ of singular cochain complex of a space X. $\Longrightarrow$ the original Massey product.

▷ Allow to reconstruct the homotopy class of A in $\mathrm{Ho}(A_\infty\text{-alg})$.

Other homotopy algebras

Other examples

- "Homotopy everything (commutative) algebras" or E_∞-algebras (C_∞-algebras, $\mathbb{K}$ field of characteristic 0) [p-adic homotopy, Mandell]
- Homotopy Lie algebras or L_∞-algebras [Deformation-Quantization of Poisson manifold, Kontsevich]
- Gerstenhaber algebras up to homotopy or G_∞-algebras [Deligne conjecture, Getzler-Jones, Tamarkin-Tsygan, Ginot]

$$\begin{tikzcd}
\text{an operad } \mathcal{P} \arrow[d, dotted] \\
\text{a category of algebras}
\end{tikzcd}$$

Homotopy theory for operads

Theorem (Getzler-Jones, Hinich, Berger-Moerdijk, Spitzweck)

There is a cofibrantly generated model category structure on dg operads transferred from that of dg modules.

$$\begin{tikzcd}
\text{an operad } \mathcal{P} \arrow[d, dotted] & \mathcal{P}_\infty \ :\ \text{cofibrant replacement} \arrow[l, "\sim"'] \arrow[d, dotted] \\
\text{category of algebras} \arrow[r, hook] & \text{category of } \textit{homotopy} \text{ algebras}
\end{tikzcd}$$

Proposition (Boardmann-Vogt, Berger-Moerdijk)

Let $\mathcal{P}_\infty$ be a cofibrant dg operad. Under some assumptions, $\mathcal{P}_\infty$-algebra structures transfer through weak equivalences.

▷ Homotopy invariant property for algebras over cofibrant operads.

Cofibrant operads

Definition (Quasi-free operad)

Quasi-free operad $(\mathcal{P}, d)$:= dg operad which is free $\mathcal{P} \cong \mathcal{F}(\mathcal{C})$ when forgetting the differential.

Proposition

$\{$*projective modules*$\}$ = $\{$*direct summand of free modules*$\}$
$\{$**cofibrant operads**$\}$ = $\{$**retract of quasi-free operads**$\}$ $\Longrightarrow$ *free modules are projective* $\Longrightarrow$ **quasi-free operads are cofibrant**

$\Longrightarrow$ Look for quasi-free resolutions: $(\mathcal{F}(\mathcal{C}), d) \xrightarrow{\sim} \mathcal{P}$

Quasi-free operads $(\mathcal{F}(\mathcal{C}), d)$

Data: generators $\mathcal{C}$ and differential d

Proposition

Since d is a derivation, it is characterized by its restriction $d_{|\mathcal{C}} : \mathcal{C} \to \mathcal{F}(\mathcal{C})$.

$d^2 = 0 \iff$ algebraic structure on $\mathcal{C}$ *(homotopy cooperad)* **Data:**

- the shape of $\mathcal{C}$ $=$ underlying $\mathbb{S}$-module
- $d \iff$ algebraic structure on $\mathcal{C}$

Quasi-free resolution $(\mathcal{F}(\mathcal{C}), d) \xrightarrow{\sim} \mathcal{P}$: operadic syzygies

General problems

- How to make cofibrant replacements for operads explicit ?
  $\Longleftrightarrow$ Explicit definition of homotopy $\mathcal{P}$-algebras
- How to make the transferred $\mathcal{P}_\infty$-algebra structure explicit ?
- How to define $\mathcal{P}_\infty$-morphism "in general" ?
- Describe the homotopy category of $\mathcal{P}_\infty$-algebras
- Define the deformation theory of $\mathcal{P}_\infty$-algebras

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Gerstenhaber algebras

Let (A, d_A) be a dg module.

Definition (Gerstenhaber algebra)

- symmetric product $\bullet \ : \ A \odot A \to A$, degree 0,
- skew-symmetric bracket $[\ ,\ ] \ : \ A \wedge A \to A$, degree $+1$

such that

- $\bullet$ associative,
- $[\ ,\ ]$ satisfies the Jacobi identity,
- $\bullet$ and $[\ ,\ ]$ satisfy the Leibniz relation

$$[\text{-} \bullet \text{-}, \text{-}] = ([\text{-}, \text{-}] \bullet \text{-}).(23) + (\text{-} \bullet [\text{-}, \text{-}]),$$

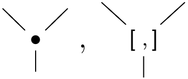
Example: Hochschild cohomology $HH(A, A)$ is a Gerstenhaber algebra.

Questions:

- Define the notion of Gerstenhaber algebra up to homotopy
- Extend the operations defined by Gerstenhaber on CH(A,A), which induce the Gerstenhaber algebra on HH(A,A), to a Gerstenhaber algebra up to homotopy. [=Deligne conjecture] =Question dual to that of Massey products.

Gerstenhaber operad

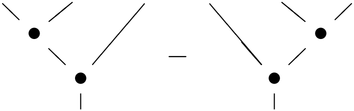
Let $\mathcal{G}$ be the operad of Gerstenhaber algebras. **Presentation:**

$\mathcal{G} = \mathcal{F}\Big($ 

$\Big)/(R) =$ Free operad/ideal generated by R

$$R = \{\mathrm{Assoc}(\bullet), \mathrm{Jacobi}([\ ,\ ]), \mathrm{Leibniz}(\bullet, [\ ,\ ])\}$$

Quadratic presentation:

$\mathrm{Assoc}(\bullet) =$ 

Quadratic := R writes with 2-vertices trees

Koszul duality theory

Dual notion of operad : cooperad [reverse the arrows] **Example:** a coalgebra is a cooperad concentrated in arity 1.

Theorem (Ginzburg-Kapranov, Getzler-Jones)

There exist adjoint functors

$$B \ : \ \{dg\ operads\} \leftrightharpoons \{dg\ cooperads\} \ : \ \Omega$$

called bar and cobar constructions

Quadratic operad $\mathcal{P} \xrightarrow{\text{Koszul dual cooperad}}$ cooperad $\mathcal{P}^{¡}$ with $\mathcal{P}_\infty := \Omega\mathcal{P}^{¡} = \mathcal{F}(s^{-1}\bar{\mathcal{P}}^{¡}) \to \mathcal{P}$ morphism of dg operads

Definition (Koszul operad)

$\mathcal{P}$ Koszul if $\mathcal{P}_\infty = \Omega\mathcal{P}^{¡} \xrightarrow{\sim} \mathcal{P}$ quasi-isomorphism: cofibrant replacement

Applied Koszul duality theory

- Computation of $\mathcal{P}^{\text{¡}}$

Proposition (Ginzburg-Kapranov, Getzler-Jones)

Let $\mathcal{P} = \mathcal{F}(V)/(R)$ be a quadratic operad with $dim(V) < +\infty$. The linear dual $\mathcal{P}^{\text{¡}*}$ is a quadratic operad and the suspension of $\mathcal{P}^{\text{¡}*} \cong \mathcal{P}^! := \mathcal{F}(V^* \otimes sgn)/(R^\perp)$.

Example: $\mathcal{G}^! = \mathcal{G}$: Koszul auto-dual operad **Method:** Compute $\mathcal{P}^!$ and its operadic structure with this formula, then dualize everything to get $\mathcal{P}^{\text{¡}}$ and its cooperadic structure.

- Methods to prove that $\mathcal{P}$ is Koszul

Proposition (Getzler-Jones, Markl)

The Gerstenhaber operad $\mathcal{G}$ is Koszul.

Homotopy Gerstenhaber algebras or G_∞-algebras

$$\mathcal{G} = \mathcal{G}^! = \mathcal{C}om \circ \mathcal{L}ie^1$$

Proposition (Ree 58)

$$\mathcal{L}ie(A) \cong \bigoplus_n A^{\otimes n}/Shuffles =: \bigoplus_n \overline{A^{\otimes n}}$$

Definition-Proposition (Homotopy Gerstenhaber algebra)

$$m_{p_1,\dots,p_n} : \overline{A^{\otimes p_1}} \wedge \cdots \wedge \overline{A^{\otimes p_n}} \longrightarrow A, \qquad n,\ p_1, \dots, p_n \geq 1,$$

such that

$$\sum \pm\, m_{t,p_{j_1},\dots,p_{j_{n-r}}}\Big(\sum m^{p_{i_1},\dots,p_{i_r}}_{q_1,\dots,q_r}(a_{i_1}\wedge\cdots\wedge a_{i_r})\wedge a_{j_1}\wedge\cdots\wedge a_{j_{n-r}}\Big) = 0.$$

Applied G_∞-algebras

Proposition (Getzler-Jones)

Let $\mathcal{P}$ *be a Koszul operad. A* $\mathcal{P}_\infty$*-algebra structure on* $A \Longleftrightarrow$ *a square-zero coderivation on the cofree* $\mathcal{P}^{¡}$*-coalgebra* $\mathcal{P}^{¡}(A)$*.*

Application: $\mathcal{G}_\infty$-algebra structures on $A \Longleftrightarrow$ square-zero derivations on the cofree Gerstenhaber coalgebra $\mathcal{G}^*(sA)$.

Theorem (Tamarkin-*Deligne Conjecture*)

There is a $\mathcal{G}_\infty$*-algebra structure on Hochschild cochain complex* $CH(A, A)$ *which lifts the Gerstenhaber algebra structure on* $HH(A, A)$*.*

Batalin-Vilkovisky algebras or BV-algebras

Definition (Batalin-Vilkovisky algebra)

A Gerstenhaber algebra $(A, \bullet, [\ ,\ ])$ with ▷ unary operator $\Delta\ :\ A \to A$, degree $+1$

such that

▷ $\Delta^2 = 0$, ▷ $[\ ,\ ] =$ obstruction to Δ being a derivation with respect to $\bullet$

$$[\text{-},\text{-}]\ =\ \Delta \circ (\text{-} \bullet \text{-})\ -\ (\Delta(\text{-}) \bullet \text{-})\ -\ (\text{-} \bullet \Delta(\text{-})),$$

$$\Downarrow$$

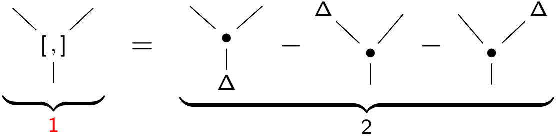
▷ Δ derivation with respect to $[\ ,\ ]$

$$\Delta([\text{-},\text{-}])\ =\ [\Delta(\text{-}),\text{-}]\ +\ [\text{-},\Delta(\text{-})]$$

BV-operad

Problem: The operad BV is not quadratic

$$[\text{-},\text{-}] \;=\; \Delta\circ(\text{-}\bullet\text{-}) - (\Delta(\text{-})\bullet\text{-}) - (\text{-}\bullet\Delta(\text{-}))\ \text{writes}$$



Quadratic Batalin-Vilkovisky algebras or qBV-algebras

Definition (quadratic Batalin-Vilkovisky algebra)

A Gerstenhaber algebra $(A, \bullet, [\ ,\ ])$ with ▷ unary operator $\Delta\ :\ A \to A$, degree $+1$ such that

▷ $\Delta^2 = 0$, ▷ Δ derivation with respect to $\bullet$

$$\Delta \circ (\text{-} \bullet \text{-}) \ - \ (\Delta(\text{-}) \bullet \text{-}) \ - \ (\text{-} \bullet \Delta(\text{-})) = 0,$$

does not imply any more

▷ Δ derivation with respect to $[\ ,\ ]$

$$\Delta([\text{-}, \text{-}]) \ = \ [\Delta(\text{-}), \text{-}] \ + \ [\text{-}, \Delta(\text{-})]$$

The operad qBV is quadratic by definition.

qBV-operad and Homotopy quadratic BV-algebras

Proposition

The operad qBV is Koszul. $\Longrightarrow qBV_\infty := \Omega\, qBV^{i} \xrightarrow{\sim} qBV$ *quasi-free resolution*

Lemma

$$qBV^! \cong \mathbb{K}[\delta] \otimes \mathcal{C}om \circ \mathcal{L}ie^1 = \mathbb{K}[\delta] \otimes \mathcal{G}^!, \quad |\delta| = 2$$

Definition-Proposition (Homotopy qBV-algebra)

$$m^d_{p_1,\ldots,p_n} : \overline{A^{\otimes p_1}} \wedge \cdots \wedge \overline{A^{\otimes p_n}} \longrightarrow A, \qquad n,\ p_1,\ldots,p_n \geq 1, d \geq 0$$

such that

$$\sum \pm\, m^d_{t,p_{j_1},\ldots,p_{j_{n-r}}}\Big(\sum m^{d',p_{i_1},\ldots,p_{i_r}}_{q_1,\ldots,q_r}(a_{i_1}\wedge\cdots\wedge a_{i_r})\wedge a_{j_1}\wedge\cdots\wedge a_{j_{n-r}}\Big) = 0.$$

Applied Homotopy quadratic BV-algebras

$\mathcal{G}$-alg $\rightarrowtail$ qBV-alg $\mathcal{G}_\infty$-alg $\rightarrowtail$ qBV_∞-alg

Proposition ("Extended Deligne conjecture")

A associative algebra with a unit 1

$$\begin{aligned}\Delta \ : \ f \in \mathrm{Hom}(A^{\otimes n}, A) \ \ \mapsto \ \ & \sum_{i=1}^{n} f \circ \left(id^{\otimes(i-1)} \otimes 1 \otimes id^{\otimes(n-i)} \right) \\ & \in \mathrm{Hom}(A^{\otimes n-1}, A)\end{aligned}$$

defines a qBV-algebra structure on $HH(A, A)$*, which extends the Gerstenhaber algebra structure.*
It lifts to a homotopy qBV-algebra structure on $CH(A, A)$*.*

Idea: Add a suitable inner differential $d_1 : qBV^{¡} \to qBV^{¡}$ and consider $\Omega(qBV^{¡}, d_1) \xrightarrow{?\sim?} BV$.

Koszul duality theory revisited

Let $\mathcal{P} = \mathcal{F}(V)/(R)$ be a quadratic and linear presentation. Let $q : \mathcal{F}(V) \twoheadrightarrow \mathcal{F}(V)^{(2)}$ the quadratic projection and $q\mathcal{P} := F(V)/(qR)$, the quadratic analogue of $\mathcal{P}$.

Definition (Quadratic-Linear Koszul operad)

$\mathcal{P} = \mathcal{F}(V)/(R)$ is a Koszul operad if

- $R \cap V = \{0\} \iff V$ is “minimal”
- $(R \otimes V + V \otimes R) \cap \mathcal{F}(V)^{(2)} \subset R \cap \mathcal{F}(V)^{(2)} \iff$ R is “maximal”
- $q\mathcal{P}$ is Koszul

R maximal corresponds to $\Downarrow$ in the example of BV. Quadratic Koszul operad are also Koszul in the above definition.

The inner differential

Lemma

- $R \cap V = \{0\} \Longrightarrow R = Graph(\varphi : qR \to V)$.
- $(R \otimes V + V \otimes R) \cap \mathcal{F}(V)^{(2)} \subset R \cap \mathcal{F}(V)^{(2)} \Longleftrightarrow$ *φ induces a square-zero coderivation d_φ on $q\mathcal{P}^i$.*

Proof.

$q\mathcal{P}^{\mathrm{i}} \rightarrowtail \mathcal{F}^c(sV)$ $\exists!$ coderivation on $\mathcal{F}^c(sV)$ which extends

$$\mathcal{F}^c(sV) \twoheadrightarrow s^2 qR \xrightarrow{s^{-1}\varphi} sV$$

It squares to 0 and descents to $q\mathcal{P}^{\mathrm{i}}$ iff
$(R \otimes V + V \otimes R) \cap \mathcal{F}(V)^{(2)} \subset R \cap \mathcal{F}(V)^{(2)}$ □

Cofibrant resolutions for Koszul operads

Definition (Koszul dual dg cooperad)

$$\mathcal{P}^{\text{i}} := (q\mathcal{P}^{\text{i}}, d_\varphi)$$

Theorem

When $\mathcal{P}$ is a quadratic-linear Koszul operad, then

$$\mathcal{P}_\infty := \Omega\,\mathcal{P}^{i} \xrightarrow{\sim} \mathcal{P}$$

Theorem

The operad BV is Koszul

$$\Longrightarrow \quad BV_\infty := \Omega\, BV^{i} = \Omega(qBV^{i}, d_\varphi) \xrightarrow{\sim} BV$$

Homotopy Batalin-Vilkovisky algebras

$$qBV^! \cong \mathbb{K}[\delta] \otimes \mathcal{C}om \circ \mathcal{L}ie^1 = \mathbb{K}[\delta] \otimes \mathcal{G}^!$$

Proposition

$d_\varphi(\delta^d \otimes L_1 \odot \cdots \odot L_t) = \sum_{i=1}^t \pm \delta^{d-1} \otimes L_1 \odot \cdots \odot L_i' \odot L_i'' \odot \cdots \odot L_t$, where $L_i' \odot L_i''$ = image of L_i under the decomposition map of the cooperad ${\mathcal{L}ie^1}^*$.

Definition-Proposition (BV_∞-algebra)

$$m^d_{p_1,\ldots,p_n} : \overline{A^{\otimes p_1}} \wedge \cdots \wedge \overline{A^{\otimes p_n}} \longrightarrow A, \qquad n,\ p_1, \ldots, p_n \geq 1, d \geq 0$$

s.t.

$$\sum \pm\, m^d_{t, p_{j_1}, \ldots, p_{j_{n-r}}} \big(\sum m^{d', p_{i_1}, \ldots, p_{i_r}}_{q_1, \ldots, q_r} (a_{i_1} \wedge \cdots \wedge a_{i_r}) \wedge a_{j_1} \wedge \cdots \wedge a_{j_{n-r}}\big)$$
$$+ \sum \pm\, m^{d-1}_{p_1, \ldots, p', p_r - p', \ldots, p_n} (a_1 \wedge \cdots \wedge \overline{a^r_{p'}} \wedge \overline{a^r_{p_r - p'}} \wedge \cdots \wedge a_n) = 0$$

Applied Homotopy Batalin-Vilkovisky algebras

$\mathcal{G}$-alg $\rightarrowtail$ BV-alg $\mathcal{G}_\infty$-alg $\rightarrowtail$ BV_∞-alg

Theorem (Ginzburg, Tradler, Menichi)

A: Frobenius algebra algebra Hochschild cohomology $HH(A, A)$ is a BV-algebra

Theorem (Cyclic Deligne conjecture)

This structure lifts to a BV_∞-algebra structure on $CH(A, A)$, such that the first operations are the Gerstenhaber operations and the operator Δ.

Comparison with the other definitions

Proposition

A homotopy commutative Batalin-Vilkovisky algebra [Kravchenko] is a homotopy Batalin-Vilkovisky algebra such that all the operations vanish except m_2^0 *and the* $m^d_{1,\ldots,1}$.

Proposition

A homotopy Batalin-Vilkovisky algebra in this sense is a particular example of a homotopy BV-algebra in the sense of [Tamarkin-Tsygan].

This notion also differs from that of Beilinson-Drinfeld in the context of chiral algebras.

PBW isomorphism

The free operad $\mathcal{F}(V)$ filtered $\Longrightarrow \mathcal{P} = \mathcal{F}(V)/(R)$ filtered. There is a morphism of operads $q\mathcal{P} \to \mathrm{gr}\mathcal{P}$.

Theorem (PBW isomorphism)

For any Koszul operad $\mathcal{P}$, we have an isomorphism of operads

$$q\mathcal{P} {\color{red}\cong} gr\mathcal{P}$$

Applications :

- $qBV \cong \mathrm{gr}\, BV$ $(\cong BV$ as an $\mathbb{S} - module)$.
- Provides a small chain complex to compute $H_\bullet^{BV}(A)$.

Relation with the framed little disk operad

fD:=*framed little disk operad*

Proposition (Getzler 94)

$$H_\bullet(fD) \cong BV,$$

Getzler When S^1 acts on X, $H_\bullet(\Omega^2 X)$ carries a BV-algebra

S.-W. fD detects spaces of type $\Omega^2 X$, with S^1 acting on X.

Theorem

When S^1 acts on X, $C_\bullet(\Omega^2 X)$ carries a BV_∞-algebra structure which induces the BV-algebra structure on $H_\bullet(\Omega^2 X)$ (Pontryagin product and Browder bracket).

Formality of the framed little disk operad

Theorem (Giansiracusa-Salvatore-Severa 09)

The framed little disk operad is formal: $C_\bullet(fD) \xleftarrow{\sim} \cdots \xrightarrow{\sim} H_\bullet(fD)$.

BV_∞ cofibrant $\Rightarrow$

$$\begin{tikzcd}
 & & C_\bullet(fD) \\
 & & \vdots \arrow[u, "\sim"'] \arrow[d, "\sim"] \\
BV_\infty \arrow[rr, "\sim"] \arrow[uurr, dashed, "\sim"] & & BV \cong H_\bullet(fD)
\end{tikzcd}$$

Relation with TCFT

a Topological Conformal Field Theory := algebra over the prop(erad) $C_\bullet(\mathcal{R})$ of Riemann surfaces R := suboperad of $\mathcal{R}$ composed by Riemann spheres with one output (holomorphic disk).

Theorem

$\exists$ *quasi-isomorphisms of dg operads*

$$\begin{tikzcd}
C_\bullet(R) \arrow[r, shift right, two heads] & C_\bullet(fD) \arrow[l, shift right, tail] \\
BV_\infty \arrow[u] \arrow[ur] \arrow[r] & BV \cong H_\bullet(fD). \arrow[u, dotted, leftrightarrow]
\end{tikzcd}$$

Relation with TCFT

Theorem

Any TCFT (Topological Conformal Field Theory) carries a homotopy BV-algebra structure which lifts the BV-algebra structure of Getzler on its homology.

$\Rightarrow$ Purely algebraic description of part of a TCFT structure

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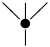
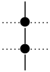
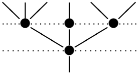
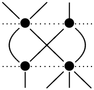
Operads and props

Operations			
Composition			
Monoidal category	$(\text{Vect}, \otimes)$	$(\mathbb{S}\text{-Mod}, \circ)$	$(\mathbb{S}\text{-biMod}, \boxtimes_c)$
Monoid	$R \otimes R \to R$ Associative Algebra	$\mathcal{P} \circ \mathcal{P} \to \mathcal{P}$ *Operad*	$\mathcal{P} \boxtimes_c \mathcal{P} \to \mathcal{P}$ *Properad*
Modules	Modules	Algebras	Bialgebras
Examples		Associative, Lie, Gerstenhaber algebras	bialgebras, Lie bialgebras
Free monoid	Ladders (Tensor module)	Trees	Connected graphs

Properads $\rightsquigarrow$ prop (not a monoid, but a **2-monoid** !)

$\Longrightarrow$ homological constructions much difficult.

General theory

Generalized Koszul duality theory

- includes unary operations [V. Ph.D. Thesis]
- no restriction on the homological degree of $\mathcal{P}$
- works for properads, i.e. operations with several outputs

Homotopy theory for $\mathcal{P}_\infty$-algebras

When $\mathcal{P}$ is a Koszul operad

- Define the notion of $\mathcal{P}_\infty$-morphism and $\mathcal{P}_\infty$-homotopy
- Category of $\mathcal{P}_\infty$-algebras $=$ Category of fibrant objects of $\mathcal{P}^{\text{i}}$-coalgebras $\Longrightarrow$ understand the homotopy category $\mathrm{Ho}(\mathcal{P}_\infty\text{-alg})$

Example: BV_∞-algebras

Deformation theory

$\mathcal{P}$ be a Koszul properad and (A, d) dg module

Proposition (Deligne philosophy on deformation theory)

$\exists$ *dg Lie algebra* $(\mathfrak{g}_{\mathcal{P}_\infty} := \mathrm{Hom}_{\mathbb{S}}(\mathcal{P}^{i}, \mathrm{End}_A), [\ ,\ ], \partial)$ *such that*
Maurer-Cartan elements $\Longleftrightarrow$ $\mathcal{P}_\infty$*-algebra structures on* A

Definition (Cohomology of $\mathcal{P}_\infty$-algebras=tangent homology)

Given a $\mathcal{P}_\infty$-algebra structure $\alpha \in \mathrm{MC}(\mathfrak{g})$ twisted dg Lie algebra
$\mathfrak{g}^{\alpha}_{\mathcal{P}_\infty} := (\mathfrak{g}, [\ ,\ ], \partial + [-, \alpha])$

$H(\mathfrak{g}^{\alpha})=$ Obstructions to deformations of $\mathcal{P}_\infty$-algebra structures

Proposition

$$\mathfrak{g}_{BV_\infty} \cong \mathfrak{g}_{\mathcal{G}_\infty} \otimes \mathbb{K}[[\delta]] \quad \textit{isomorphism of Lie algebras}$$

Obstruction theory

In general $\mathfrak{g}$ is triangulated [Sullivan]

Proposition

When $\mathcal{P}$ is a Koszul operad, $\mathfrak{g}$ and $\mathfrak{g}^\alpha$ are graded by an extra weight. $\Longrightarrow$ the Maurer-Cartan equation splits with respect to this weight.

Theorem (Generalized Lian-Zuckerman conjecture)

For any topological vertex algebra A with $\mathbb{N}$-graded conformal weight there exists an explicit BV_∞-algebra structure on A which extends Lian-Zuckerman operations on A and which lifts the BV-algebra structure on $H(A)$.

Relative obstruction theory

Let α be a $\mathcal{G}_\infty$-algebra structure on A.

Theorem

The obstructions to lift α to a BV_∞-algebra structure live in

$$H_{-2n}(\mathfrak{g}_{\mathcal{G}_\infty}), \ \ n \geq 1$$

negative 2-periodic cohomology of the $\mathcal{G}_\infty$-algebra A

Transfer theorem: the theory

Theorem (Merkulov-V.)

There is a cofibrantly generated model category structure on dg properads transferred from that of dg modules.

Remark: There is NO model category structure on algebras over a properad which is not an operad (no coproduct).

Homotopy equivalence:

$$\begin{tikzcd} (V, d_V) \arrow[loop left, "h'"] \arrow[r, shift left, "i"] & (W, d_W) \arrow[l, shift left, "p"] \arrow[loop right, "h"] \end{tikzcd}$$

$$\mathrm{Id}_W - ip = d_W h + h d_W, \quad \mathrm{Id}_V - pi = d_V h' + h' d_V$$

Theorem

Any $\mathcal{P}_\infty$-algebra structure on W transfers to V.

Transfer theorem: Explicit formulae

Lemma (Van der Laan, V.)

$\exists$ *homotopy morphism of properads between* End_W *and* End_V
$\Longleftrightarrow \quad \Phi : B(\mathrm{End}_W) \to B(\mathrm{End}_V)$*: morphism of dg coproperads*

$\mathcal{P}_\infty$-algebra structures

$$\mathrm{Hom}_{dg\ properad}(\Omega(\mathcal{P}^{¡}), \mathrm{End}_W) \cong \mathrm{Hom}_{dg\ coproperad}(\mathcal{P}^{¡}, B(\mathrm{End}_W))$$

Transfer:

$$\begin{tikzcd}
\mathcal{P}^{¡} \arrow[r] \arrow[dr, dashed] & B(\mathrm{End}_W) \arrow[d, "\Phi"] \\
 & B(\mathrm{End}_V)
\end{tikzcd}$$

Let $\alpha : \mathcal{P}^{\text{i}} \to B(\mathrm{End}_W)$ be a $\mathcal{P}_\infty$-algebra structure on W.

Theorem

Explicit formula for a transferred $\mathcal{P}_\infty$-algebra structure on V:

$$\mathcal{P}^{i} \xrightarrow{\alpha} B(\mathrm{End}_W) \xrightarrow{\Phi} B(\mathrm{End}_V)$$

Applications:

$V = H(A) \rightarrowtail A = W$, explicit formulae for Massey products

- $\mathcal{P} = Ass$: conceptual explanation for Kontsevich-Soibelman formulae
- $\mathcal{P} = InvBiLie$: Cieliebak-Fukaya-Latschev formulae (Symplectic field theory, Floer homology and String topology)
- $\mathcal{P} = BV$: Massey products for BV and BV_∞-algebras.
- Works in general (even in a non-Koszul case)

References

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Available at: `http://math.unice.fr/∼brunov/`

Thank you for your attention !