HOMOTOPY THEORIES

BRUNO VALLETTE

"In the 20th century, any mathematical theory was built upon set theory.
In the 21th century, new mathematical theories will be built upon a homotopy theory."

Yuri I. Manin

Abstract. These are lecture notes of a 5th-year course given at the universities of Paris in 2018-2020 and 2024-2025. The purpose is to develop the first two examples of homotopy theories: topological spaces and simplicial sets. The approach chosen is to present them in a certain uniform way that will lead to their abstractization, as performed by Quillen under the notion of a model category structure. The second goal is to provide the audience with enough material on simplicial sets to open the gates to the study of higher category theory via the notion of an ∞-category.

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Introduction

The study of topological spaces is that of sets equipped with a way to compare how points are close from one another. We do not care here whether this can be done in a smooth way, so we will leave geometry aside. Examples of topological spaces abound in nature: circle, surfaces, cup of coffee, doughnut, etc. One can study their properties like connectivity, number of holes, etc.

What does it mean to say that two topological spaces are "the same" or "equivalent", which would imply that they have the same properties? The general mathematical method to answer such a question is to coin the group of "symmetries", whose action encodes the way to pass from one to another. Given such an action, one identifies the similar objects to be the ones lying in the same orbit.

The first definition that comes to mind is rather categorical in nature: the canonical notion of maps between topological spaces is the one of continuous maps, and one can consider the associated class of "isomorphisms", that is homeomorphisms. Alright ... but this is actually too strong. For instance, a circle and a doughnut are actually the "same thing" for us since, when the latter one is made up of dough before passing in the oven, one can shrink it to a circle. But, removing two distinct points from the circle, one gets two connected components, which will never happen for the doughnut, so they cannot be homeomorphic ...

A group action induces an equivalence relation, so we can try to weaken this tentative first definition by considering a well chosen equivalence relation. Pursing the above analogy, we view topological spaces as made up of clay that we can stretch, compress but not cut, namely as objects that we can continuously deform. We would like to declare equivalent two spaces obtained from one another in this way and this would definitely preserve the connectivity, number of holes, etc.

In order to make this precise, one has to focus on maps, not on objects, and to introduce the notion of a homotopy for continuous maps. This leads to the notion of homotopy equivalent spaces, that are spaces related by a map which admits an inverse *up to homotopy*. We just opened Pandora's box! Indeed, this seemingly simple move from "strong" equivalence relation (homeomorphism) to "weak" equivalence relation (homotopy equivalence) will have drastic and exciting consequences.

When one says equivalence relation, one often tries to classify the associated equivalence classes. A proper mathematical way to do this is by introducing invariants, that is "something" (number, group, vector spaces, etc.) which is invariant under the equivalence relation. Then one hopes to get a *complete set* of invariants. For instance, a complete set of invariant of finite dimensional vector spaces under isomorphisms is given by the dimensions. Since at least Poincaré, algebraic topologists have tried to coin such faithful invariants characterising the homotopy type of topological spaces like Betti numbers, (co)homology groups, homotopy groups, etc. Over the 20th century ... they failed! Then they considered some higher algebraic structures made up of collections of operations of any arity, now called E_∞-algebras. And, at the beginning of the 21st century, they showed [Man06] that they have reached here a rich enough world which detects with accuracy the homotopy type of topological spaces.

Does this "solve" the entire question of classifying topological spaces? No! And, in some sense, it cannot: keep in mind that nobody knows how to compute the homotopy groups of the first topological spaces that are the spheres. So there was no hope to turn this transcendental problem into a simple one. However, since this was a relevant programme of study, algebraic topologists developed many revolutionary mathematical tools along the way. First, category theory, created by Eilenberg–MacLane [EM45] in 1942-45, provides us with the right setting to express the functoriality of constructions, like the invariants of topological spaces. (Grothendieck, the master of categories, introduced and developed extensively this notion in algebraic geometry). In order to uniformise algebraically homotopy theories (topological spaces, simplicial sets, differential graded algebras, rational homotopy theory, etc.), Quillen came up with the conceptual notion of a model category [Qui67]. Operad theory [LV12] (shameless self promotion) arose from the study of iterated loop spaces. At the beginning of the 21th century, it was finally understood how a certain homotopy property imposed on simplicial sets was giving rise to a suitable notion of higher category: ∞-categories [Lur09]. Notice that already the work of Eilenberg–MacLane was motivated by higher categorical questions: they came up with the notion of a category in order to compare topological invariants, which is achieved by natural transformations that are some kind of 2-morphisms.

The present course will not be viewed by its author as a way to teach once again any new material, but rather as a way to introduce his young fellows to the current methods of mathematical research in order to help them upgrade from students to researchers. We will present the two homotopy theories of topological spaces and simplicial sets in order exhibit their commun underlying structures and to ease the way to model category structures, that will be developed extensively in the forthcoming course "Homotopy II" by Grégory Ginot. In Section 1, we will make explicit the main tools and properties of the homotopy theory for topological spaces, for which we refer to classical references like [May99, tD08, Hat02]. These three manuscripts have different bright advantages. J.P. May's book [May99] provides the literature with a very concise treatment and focuses on the important points. The long book [tD08] by T. tom Dieck gives full details and proofs. And the clear book [Hat02] of A. Hatcher offers many pictures. In these notes, we tried to merge these respective three main points. Also the categorical presentation giving here follows the development of the algebraic topology during the second half of the 20th century. Category theory was created by algebraic topologists as a need to have a suitable language to express their ideas. In the other way round, algebraic topology was then developed using the universal properties of category theory. The way Section 2 on the homotopy theory of simplicial sets is developed is rather new as it is guided by the quest of a suitable definition of higher category. Here again, we tried to merge the various advantages of the existing texts on the subject: introduce the notion of a simplicial set from simplicial complexes *with pictures*, provide combinatorial descriptions, and fully use the categorical language.

Why should future algebraic geometers or representation theorists, for instance, follow this course: because the recent developments of the ideas triggered by homotopy theory ultimately lead to suitable notions of higher category theory which will provide them with the required framework for their own domains of research. To name but a few, the notion of stable ∞-category bypasses that of a triangulated category, the main goals of the Langlands programme can finally be formulated using the language of ∞-categories, derived algebraic geometry can give a meaning to the notion of a tangent space at a singular point, and algebraic ∞-groupoids give rise to the salient tools of Lie theory and deformation theory.

Prerequisistes. The following notions have been studied in the previous course on "Homologie, cohomologie et faisceaux" by Bernhard Keller and will thus be used without recollection here.

- CATEGORY THEORY: category, functor, adjunction, limits and colimits, homotopy category (of chain complexes).
- ALGEBRAIC TOPOLOGY: topological space, homeomorphism, homotopy equivalence, fundamental group(oid).

Convention. To simplify the presentation of these notes, "space" will mean "topological space" and "map" will mean " continuous map". In the core of these notes, we will go from the more general to the more particular as we will start by working with any topological spaces, to restrict ourself to locally compact or compactly generated weak Hausdorff spaces from Section 1.4. Finally, we will consider CW-complexes and then simplicial complexes and simplicial sets. The main topic of the present notes is algebraic topology. As a consequence, some point-set topology issues will arise but the treatment we will give them will be far from being exhaustive.

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1.1. **Homotopy equivalence and category theory**.

1.1.1. *Homotopy categories.* The underlying category we will be working in is that of topological spaces (objects) with continuous maps (morphisms). We will denote it by Top. In this category, the isomorphisms are the homeomorphisms, that is continuous maps $f\colon X \to Y$ which admit a continuous inverse $g\colon Y \to X$ satisfying $gf = \mathrm{id}_X$ and $fg = \mathrm{id}_Y$.

Example. Let us recall the following classical topological spaces, for $n \in \mathbb{N}$:

- the n-dimensional real vector space $\mathbb{R}^n$,
- the n-dimensional disk D^n,
- the n-dimensional sphere $\partial D^{n+1} = S^n$,
- the cubes I^n, where $I := [0, 1]$ stands for the interval,
- the geometrical n-simplex $|\Delta^n|$,
- the boundary of the geometrical n-simplex $\partial|\Delta^n|$,
- the n-dimensional real projective space $\mathbb{P}^n\mathbb{R}$,
- the $2n$-dimensional complex projective space $\mathbb{P}^n\mathbb{C}$.

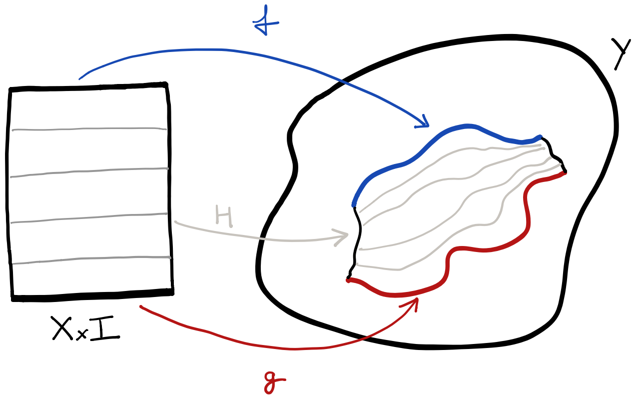
As explained in the introduction, comparing topological spaces under homeomorphisms is too restrictive. Instead, we will use the following more relaxed notion which will meets our needs.

Definition 1.1.1 (Homotopy). Let $f, g\colon X \to Y$ be two continuous maps in $\mathsf{Top}(X, Y)$. A ***homotopy*** from f to g is a continuous map $H\colon X \times I \to Y$ such that $H(-, 0) = f$ and $H(-, 1) = g$, which is equivalent to the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "i_0"] \arrow[dr, bend right, "f"'] & X \times I \arrow[d, "H"] & X \arrow[l, "i_1"'] \arrow[dl, bend left, "g"] \\
& Y &
\end{tikzcd} .$$

Two maps f and g are ***homotopic*** if there exists a homotopy from f to g; in this case, we use the notation $f \sim g$. A continuous map $f : X \to Y$ is ***null homotopic*** if it is homotopic to a constant map.

We denote the homotopy equivalence class of a continuous map f by $[f]$.



Proposition 1.1.2. *For any pair of topological spaces X and Y, the homotopy relation $\sim$ is an equivalence relation on the set* $\mathsf{Top}(X, Y)$.

Proof. A homotopy from a map f to itself is given by $H(x, t) := f(x)$. If $H := X \times I \to Y$ stands for a homotopy from f to g, then a homotopy from g to f is given by $(x, t) \mapsto H(x, -t)$. Let $H : X \times I \to Y$ be a homotopy from f to g and let $K : X \times I \to Y$ be a homotopy from g to h, then a direct homotopy from f to h is given by

$$(x, t) \longmapsto \begin{cases} H(x, 2t) & \text{for } 0 \leqslant t \leqslant \frac{1}{2} , \\ K(x, 2t - 1) & \text{for } \frac{1}{2} \leqslant t \leqslant 1 . \end{cases}$$

□

Here is the naive first definition of a category of topological spaces up to the homotopy equivalence relations on continuous maps.

Definition-Proposition 1.1.3 (Homotopy category hoTop). The *homotopy category* hoTop is defined by

OBJECTS: topological spaces,
MORPHISMS: cosets $[X, Y] := \mathsf{Top}(X, Y)/\sim$.

The composite is defined on representatives of classes:

$$\begin{array}{ccc} [X,Y]\times[Y,Z] & \xrightarrow{\circ} & [X,Z] \\ ([f],[g]) & \longmapsto & [gf]\ . \end{array}$$

Proof. It is enough to show that the composite is well defined since then the associativity and unital axioms are straightforward consequences of the associativity and unital axioms of the category Top. It is well defined, because, if one considers $f, f', g, g' : X \to Y$ such that $[f] = [f']$ and $[g] = [g']$, then $gf \sim g'f'$. Indeed, if we denote by $H : X\times I \to Y$ a homotopy from f to f' and by $K : Y\times I \to Z$ a homotopy from g to g', then $(x,t) \mapsto K(H(x,t),t)$ defines a homotopy from gf to $g'f'$. □

Like in the definition of the group structure on the coset G/N of a group G by a normal subgroup N, the above definition satisfies the following property: the category structure on hoTop is the unique one which makes the assignment $\mathsf{Top} \to \mathsf{hoTop}$ given by

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X \arrow[d, "{[f]}"] \\ Y \arrow[r] & Y \end{tikzcd}$$

into a functor.

One might now ask what are the isomorphisms in the homotopy category hoTop?

Definition 1.1.4 (Homotopy equivalence). A *homotopy equivalence* is a continuous map denoted by $f : X \xrightarrow{\sim} Y$ such that the induced map $[f]$ is an isomorphism in the homotopy category hoTop, i.e. there exists a continuous map $g : Y \to X$ such that $gf \sim \mathrm{id}_X$ and $fg \sim \mathrm{id}_Y$.

Definition 1.1.5 (Homotopy equivalent spaces). Two topological spaces X and Y are *homotopy equivalent* if they are isomorphic in the homotopy category hoTop; this means that there exists a homotopy equivalence $f : X \xrightarrow{\sim} Y$. In this case, we use the notation $X \sim Y$ and we say that X and Y have *the same homotopy type.*

This defines an equivalence relation among topological spaces. A topological space X is *contractible* if it is homotopy equivalent to a point: $X \sim *$.

EXAMPLE (DEFORMATION RETRACT). Here is a practical way to prove that two spaces are homotopy equivalent. Let $X \subset Y$ be a pair of topological spaces. The space X is a *deformation retract* of Y if there exists a *retraction* r

$$X \overset{i}{\hookrightarrow} Y \overset{r}{\to} X,$$

satisfying

$$ri = \mathrm{id}_X \quad \text{and} \quad ir \sim \mathrm{id}_Y\ .$$

The most important example is that of the cylinder construction $i_0 : X \hookrightarrow X\times I$, given by $x \mapsto (x,0)$. The retraction $r : X\times I \to X$ is given by $(x,t) \mapsto x$ and the homotopy $H : (X\times I)\times I \to X\times I$ between $i_o r$ and $\mathrm{id}_{X\times I}$ is given by $H(x,t,s) = (x,ts)$.

A *topological invariant* is "a natural assignment which sends homotopy equivalences to isomorphisms"; what a long and vague sentence to say: it is a functor $F : \mathsf{Top} \to \mathsf{C}$ which factors through the canonical "projection" functor

$$\begin{tikzcd} \mathsf{Top} \arrow[r] \arrow[dr, bend right, "F"'] & \mathsf{hoTop} \arrow[d, dotted, "\exists !"] \\ & \mathsf{C}\ . \end{tikzcd}$$

This is the case for the various topological invariants that you have encountered so far: homology groups $\mathrm{H}_\bullet$, cohomology groups $\mathrm{H}^\bullet$, and homotopy groups $\pi_\bullet$, for instance. One can try to characterise the homotopy category by this universal property.

Definition 1.1.6 (Universal homotopy property). A category HoTop satisfies the ***universal homotopy property*** if it comes equipped with a functor $\mathsf{Top} \to \mathsf{HoTop}$ which sends homotopy equivalences to isomorphisms and such that any functor $f\colon \mathsf{Top} \to \mathsf{C}$ sending homotopy equivalences to isomorphisms factors uniquely through it:

$$\begin{tikzcd} \mathsf{Top} \arrow[r] \arrow[dr, "F"'] & \mathsf{HoTop} \arrow[d, dotted, "\exists! \widetilde{F}"] \\ & \mathsf{C} . \end{tikzcd}$$

As usual for universal property, when such a category HoTop exists, it is unique up to unique isomorphism. So far it is not obvious that the homotopy category hoTop satisfies the universal homotopy property: why would two homotopy equivalent maps $f, g\colon X \to Y$ induce the same map $F(f) = F(g)$ in C?

This raises the question how to build the category HoTop and the present situation is actually quite general: given a category T and a class $\mathscr{W}$ of morphisms stable by composition and containing the isomorphisms, one can study the existence of the category $\mathsf{T}[\mathscr{W}^{-1}]$ satisfying the universal property with respect to $\mathscr{W}$:

$$\begin{tikzcd} \mathsf{T} \arrow[r] \arrow[dr, "F"'] & \mathsf{T}[\mathscr{W}^{-1}] \arrow[d, dotted, "\exists! \widetilde{F}"] \\ & \mathsf{C} . \end{tikzcd}$$

A tentative construction is given by a ***localisation*** process similar to the one which produces the ring of rational numbers from the ring of integers by formally inverting non-zero elements. The objects of $\mathsf{T}[\mathscr{W}^{-1}]$ are the same as the ones of T. The "sets" of morphisms $\mathsf{T}[\mathscr{W}^{-1}](X,Y)$ are given by considering strings of morphisms of T and formal inverse of morphisms from $\mathscr{W}$

$$\begin{tikzcd} X \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r, dotted] & \bullet \arrow[l, bend right, "\sim"'] \arrow[r] & \bullet \cdots & \cdots \bullet \arrow[r] & \bullet \arrow[r, dotted] & \bullet \arrow[l, bend right, "\sim"'] \arrow[r, dotted] & \bullet \arrow[l, bend right, "\sim"'] \arrow[r] & Y \end{tikzcd}$$

modulo the equivalence relation generated by

$$X \xrightarrow{f} Y \xrightarrow{g} Z \ \approx\ X \xrightarrow{gf} Z, \quad \begin{tikzcd} X \arrow[r, "\sim", "f"'] & Y \arrow[r, dotted] & X \arrow[l, bend right, "\sim"', "f"] \end{tikzcd} \ \approx\ \mathrm{id}_X \ \text{ et } \ \begin{tikzcd} Y \arrow[r, dotted] & X \arrow[l, bend right, "\sim"', "f"] \arrow[r, "\sim", "f"'] & Y \end{tikzcd} \ \approx\ \mathrm{id}_Y \ .$$

Be careful that nothing ensures that such a construction renders actually a category as morphisms might not form proper sets. But when it exists, it is called the *localized category* ot T at the class $\mathscr{W}$. In the previous course, you have already encountered such a general construction in the example of the category $\mathsf{dg\text{-}mod}$ of differential graded modules, with $\mathscr{W}$ the class of quasi-isomorphisms. In this case, the localized category $\mathsf{dg\text{-}mod}[\mathscr{W}^{-1}]$ is the derived category.

Proposition 1.1.7. *The homotopy category* hoTop *satisfies the universal homotopy property; as a consequence it is isomorphic to the localized category*

$$\mathsf{hoTop} \cong \mathsf{Top}[\mathrm{h{-}eq}^{-1}] \,,$$

where the notation $\mathrm{h{-}eq}$ *stands for the class of homotopy equivalences.*

Proof. Let $F\colon \mathsf{Top} \to \mathsf{C}$ be a functor which sends homotopy equivalences to isomorphisms. It factors through the homotopy category hoTop if and only if the assignment

$$\left(X \xrightarrow{f} Y\right) \ \mapsto\ \left(F(X) \xrightarrow{F(f)} F(Y)\right)$$

does not depend of f up to homotopy. In this case, the factorisation is unique. Let $f, g\colon X \to Y$ be two maps related by a homotopy $H\colon X \times I \to Y$. The commutative diagram

$$\begin{tikzcd} X \arrow[r, hook, "\sim", "i_0"'] \arrow[dr, "f"'] & X \times I \arrow[d, "H"] & X \arrow[l, hook', "\sim"', "i_1"] \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

induces the commutative diagram

$$\begin{tikzcd}
F(X) \arrow[r, "\cong", "F(i_0)"'] \arrow[dr, "F(f)"'] & F(X\times I) \arrow[d, "F(H)"] & F(X) \arrow[l, "\cong"', "F(i_1)"] \arrow[dl, "F(g)"] \\
& Y &
\end{tikzcd} .$$

We consider the abovementioned retract $r : X \times I \to X$ of i_0 defined by $r(x,t) := x$. Since $F(i_0)$ is an isomorphism and since $F(i_0)F(r) = \mathrm{id}_{F(X\times I)}$, then $F(r)$ is the inverse of $F(i_0)$ in the category C . Since the retractions for i_0 and i_1 are equal to r, then $F(r)$ is also the inverse of $F(i_1)$. This prove that

$$F(f) = F(i_0)F(i_1)^{-1}F(g) = F(i_0)F(r)F(g) = F(g) ,$$

which concludes the proof. □

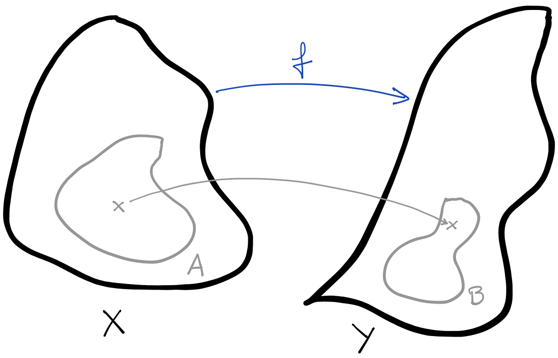
Remark. One can actually go one step further and show that the localized category and the homotopy category are equivalent to the sub-category of the latter one made up of CW-complexes only. This will be the subject of Section 1.7.

1.1.2. *Relative and pointed versions.* In the sequel, we will actually need the following relative version of the category of topological spaces.

Definition 1.1.8 (The relative category $\mathsf{Top}_{(2)}$). The *relative category* $\mathsf{Top}_{(2)}$ is defined by

Objects: pairs of spaces (X, A) such that $A \subset X$,

Morphisms: a morphism $f : (X, A) \to (Y, B)$ is a continuous map $f : X \to Y$ such that $f(A) \subset B$.



This category admits a key sub-category where the sub-spaces A are all made up of one point.

Definition 1.1.9 (The category of pointed topological spaces Top_*). The category Top_* of *pointed topological spaces* is defined by

Objects: pairs (X, x), where $x \in X$,

Morphisms: a morphism $f : (X, x_0) \to (Y, y_0)$ is a continuous map $f : X \to Y$ such that $y_0 = f(x_0)$.

In order to compare maps in the relative category $\mathsf{Top}_{(2)}$, we could consider the following notion of homotopy: a continuous map $H : X \times I \to Y$ such that $H(A, t) \subset B$, for all $t \in I$. in the same way as above, this would define a relative homotopy category $\mathsf{hoTop}_{(2)}$. This notion of a homotopy once applied to the sub-category of pointed topological spaces gives the relevant notion of *pointed homotopy*: a continuous map $H : X \times I \to Y$ such that $H(x_0, t) = y_0$, for all $t \in I$. This defines the *pointed homotopy category* hoTop_*.

Inspired by this latter case, we actually consider the following more strict version of a relative homotopy.

Definition 1.1.10 (Relative homotopy). Let A be a subspace of X and let $f, g : X \to Y$ two continuous maps such that $f_{|A} = g_{|A}$. The map f is *homotopic to* g *relatively to* A if there exists a homotopy $H : X \times I \to Y$ between f and g such that $H(a, t) = f(a) = g(a)$, for all $t \in I$ and for all $a \in A$. In this case, we use the notation $f \sim g$ rel A.

Remark. By definition, this notion can compare only maps which agree on their given subspaces; as such it is too narrow to compare the morphisms of the relative category $\mathsf{hoTop}_{(2)}$.

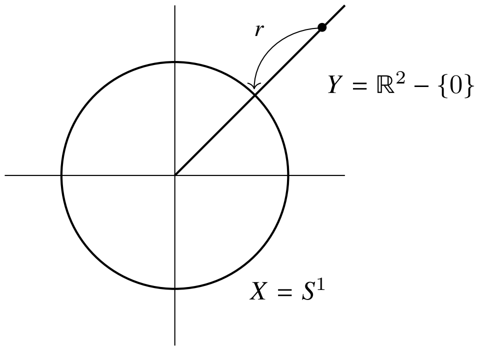
Example (Strong deformation retract). It turns out that many deformation retract actually carry a stronger property. A subspace $X \subset Y$ is a *strong deformation retract* of Y if there exists a retraction r

$$X \overset{i}{\hookrightarrow} Y \overset{r}{\to} X \ ,$$

satisfying

$$ri = \mathrm{id}_X \quad \text{and} \quad ir \sim \mathrm{id}_Y \text{ rel } X \ .$$

The example given above $i_0 \colon X \hookrightarrow X \times I$ is actually a strong deformation retract as $H(x, 0, s) = (x, 0)$. Another classical example is given by $X = S^1$ and $Y = \mathbb{R}^2 - \{0\}$, where the retraction is given by $r \colon y \mapsto y/\|y\|$.



Remark. There exists further generalizations like $\mathsf{Top}_{(3)}$ which is made up of triples $A \subset B \subset X$, etc.

1.1.3. *Toward higher homotopy categories.* One salient drawback of the homotopy category hoTop is that it forgets the data of homotopies between maps. To bypass this, one might want to consider, for any pair (X, Y) of topological spaces, the tentative category $\Pi(X, Y)$ whose objects are maps from X to Y and whose morphisms are homotopies between maps. But the composite of homotopies fails to be associative since it implies two different reparametrisations of the interval.

This phenomenon can already be seen of the level of the the fundamental groupoid, where is solved as follows. Recall that a groupoid is a category where every morphism is an isomorphism and that the fundamental groupoid $\Pi(X)$ of a topological space X is the groupoid whose objects are the points of X and whose morphisms are *homotopy equivalences* of paths from x to x', that is maps $\varphi \colon I \to X$ such that $\varphi(0) = x$ and $\varphi(1) = x'$. Notice the crucial fact that: with the identification under the homotopy equivalence relation, one now gets an associative and unital composition of paths.

One can merge the above two approaches by first considering all the topological spaces as objects, continuous maps as 1-morphisms and homotopies as 2-morphisms. However, in order to get an associative and unital composition for these latter ones, one has to impose a suitable homotopy relation on homotopies: we consider the coset of homotopies, that is continuous maps $X \times I \to Y$, under the homotopy relation relative to $X \times \partial I$, where $\partial I = \{0, 1\}$ is the boundary of the interval I. Namely, if H and K are two homotopies from $f \colon X \to Y$ to $g \colon X \to Y$, then a homotopy from H to K relative to $X \times \partial I$ preserves f and g.

Definition-Proposition 1.1.11 (Homotopy 2-category of topological spaces). The following data

- Objects: topological spaces X,
- 1-Morphisms: continuous maps $f \colon X \to Y$,
- 2-Morphisms: classes of homotopies from $f \colon X \to Y$ to $g \colon X \to Y$ modulo homotopies relative to $X \times \partial I$,

forms a 2-category called the *homotopy 2-category of topological spaces.*

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow"] & Y \end{tikzcd}$$

Proof. This definition-proposition is given as an enlightening example to show how works the quest for higher structures. The details of the proof are neither uninteresting but nor expected here. We refer the reader to any textbook containing the precise definition of a 2-category: a category enriched in small categories. It is then straightforward to check the various axioms in the present case as we have settled the right definitions. On the way, notice that the compatibility axiom between the horizontal and the vertical composite of 2-morphisms

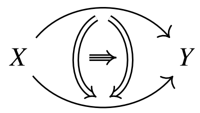
$$\begin{tikzcd}
\mathrm{X} \arrow[r, bend left=60, ""{name=U1}] \arrow[r, bend right=60, ""'{name=D1}] \arrow[r, ""{name=M1}] & \mathrm{Y} \arrow[r, bend left=60, ""{name=U2}] \arrow[r, bend right=60, ""'{name=D2}] \arrow[r, ""{name=M2}] & \mathrm{Z}
\arrow[from=U1, to=M1, phantom, "\Downarrow H"] \arrow[from=M1, to=D1, phantom, "\Downarrow K"] \arrow[from=U2, to=M2, phantom, "\Downarrow H'"] \arrow[from=M2, to=D2, phantom, "\Downarrow K'"]
\end{tikzcd}$$

amounts to the interchange law

$$\left([H] \circ_h [H']\right) \circ_v \left([K] \circ_h [K']\right) = \left([H] \circ_v [K]\right) \circ_v \left([H'] \circ_v [K']\right),$$

which will play a key role later on in Section 1.3. □

How can one pursue this story higher up? On the one hand, it is rather obvious that considering spaces, maps, homotopies, homotopies of homotopies, etc. is a good way to obtain a certain kind of "∞-category" but how can one make this precise? What are exactly the higher coherences to expect between n-morphisms and m-morphisms? And ... should one be doing this? Maybe because the full data of all the higher homotopy groups characterises the homotopy type of well-behaved spaces like the CW-complexes for instance, see Theorem 1.7.11.



1.2. **Categorical topological constructions**.

1.2.1. *Categorical constructions in* Top. Let us first recall basic categorical constructions. Consider the diagram category $\mathsf{D}\colon \bullet \quad \bullet$ made up of two objects and only the identity maps. The category $\mathsf{Func}(\mathsf{D}, \mathsf{Top})$ of functors from D to Top amounts to pairs of topological spaces. To any such functor, we associate the category $\mathsf{Cone}(F)$ whose objects are the data $X \leftarrow C \rightarrow Y$ and whose morphisms are given by continuous maps $C \to D$ such that the following diagram is commutative.

$$\begin{tikzcd}
 & C \arrow[dl] \arrow[dr] \arrow[dd] & \\
X & & Y \\
 & D \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Definition 1.2.1 (Product). The ***product*** $X \times Y$ of two topological spaces is the terminal object in the category $\mathsf{Cone}(F)$:

$$\begin{tikzcd}
 & C \arrow[dl] \arrow[dr] \arrow[dd, dotted, "\exists!"] & \\
X & & Y \\
 & X \times Y \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Set-theoretically, it is given by the cartesian product endowed with the product topology.

Dually, we consider the category $\mathsf{Cocone}(F)$ of cocones $X \to C \leftarrow Y$.

Definition 1.2.2 (Coproduct). The ***coproduct*** $X \amalg Y$ of two topological spaces is the initial object in the category $\mathsf{Cocone}(F)$:

$$\begin{tikzcd}
 & X \amalg Y \arrow[dd, dotted, "\exists!"] & \\
X \arrow[ur] \arrow[dr] & & Y \arrow[ul] \arrow[dl] \\
 & C &
\end{tikzcd}$$

Set-theoretically, it is given by the union with the induced topology.

Let us consider now the diagram category $\mathsf{D} : \bullet \to \bullet \leftarrow \bullet$ and the associated category of functors $\mathsf{Func}(\mathsf{D}, \mathsf{Top})$. An object in this latter category amounts to the data of three topological spaces related by two maps $X \to Z \leftarrow Y$. The category $\mathsf{Cone}(F)$ of cones over such a functor is made up of objects and maps subject to the following commutative diagram.

$$\begin{tikzcd} C \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

Definition 1.2.3 (Pullback). The *pullback* $X \times_Z Y$ of two topological spaces along two maps $f \colon X \to Z$ and $g \colon Y \to Z$ is the terminal object in the category $\mathsf{Cone}(F)$:

$$\begin{tikzcd} C \arrow[dr, dotted, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & X \underset{Z}{\times} Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"] & Z \end{tikzcd}$$

Set-theoretically, it is given by

$$X \underset{Z}{\times} Y = \big\{(x, y) \mid f(x) = g(y)\big\}$$

endowed with the induced topology of a sub-space of a product.

Remark. The notion of a pullback is also called fibre product, fibered product, or Cartesian square, in the literature. When the map g is a (canonical) epimorphism, we use the notation $X \underset{f}{\times} Y$.

Passing to the opposite category, we now consider the diagram category $\mathsf{D}^{\mathsf{op}} : \bullet \leftarrow \bullet \to \bullet$, the associated category of functors $\mathsf{Func}(\mathsf{D}^{\mathsf{op}}, \mathsf{Top})$, and the category $\mathsf{Cocone}(F)$ of cocones over such functors.

Definition 1.2.4 (Pushout). The *pushout* $X \underset{Z}{\amalg} Y$ of two topological spaces along along two maps $f \colon X \to Z$ and $g \colon Y \to Z$ is the initial object in the category $\mathsf{Cocone}(F)$:

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\ulcorner"] & Y \arrow[d] \arrow[ddr, bend left] & \\ X \arrow[r] \arrow[drr, bend right] & X \underset{Z}{\amalg} Y \arrow[dr, dotted, "\exists!"] & \\ & & C \end{tikzcd}$$

Set-theoretically, it is given by

$$X \underset{Z}{\amalg} Y = \frac{X \amalg Y}{f(x) \sim g(y)}$$

endowed with the coset topology.

Remark. The notion of a pushout is also called fibered coproduct, fibered sum, cocartesian square or amalgamated sum, in the literature. When the map f is a (canonical) monomorphism, we rather use the notation $X \underset{g}{\amalg} Y$.

On a set-theoretical level, we have the following classical adjunction,

$$- \times Y : \mathsf{Set} \rightleftarrows^{\perp} \mathsf{Set} \ : \ (-)^Y$$

given by the natural bijection

$$\mathsf{Set}(X \times Y, Z) \underset{\text{nat.}}{\cong} \mathsf{Set}\left(X, \mathsf{Set}(Y, Z)\right) \ . \tag{1}$$

Remark. Such a monoidal category is called *cartesian* since the monoidal product is the categorical product. It is called *closed* since it admits internal homs. In cartesian closed monoidal categories, the internal homs are usually called *exponential objects* and denoted under the exponential notation Z^Y. When this is the case, the category is enriched over itself.

We would like to promote this onto the topological level. In order to do so, we first need to endow the mapping set $\mathsf{Set}(Y, Z)$ with a suitable topology that we will denote by Z^Y. The following one is commonly used.

Definition-Proposition 1.2.5 (Compact-open topology). Let X and Y be two topological spaces. For any compact set K of X and any open set U of Y, we consider the set

$$W(K, U) := \{f : X \to Y \mid f(K) \subset U\}.$$

The sets defined by finite intersections of such $W(K, U)$ give us a basis of a topology for $Y^X = \mathsf{Set}(X, Y)$, called *the compact-open topology.*

Proof. This is an easy exercise of point-set topology. It is enough to see that the sets $W(K, U)$, for $K \subset X$ compact and $U \subset Y$ open, cover the entire mapping space Y^X. □

Exercise. Show that, for any continuous map $f \colon X \to Y$, the pullback map $f^* : Z^Y \to Z^X$ and the push-forward map $f_* : X^Z \to Y^Z$ are continuous.

The compact-open topology does not always behave nicely with respect to all topological spaces. As a consequence, we will often have to restrict ourselves to some sub-categories.

Definition 1.2.6 (Locally compact). A topological space is *locally compact* if any neighbourhood of a point contains a compact neighbourhood.

Example. All the topological spaces considered in this course are locally compact: $\mathbb{R}^n$, D^n, $\partial D^{n+1} = S^n$, I^n, $|\Delta^n|$, $\partial|\Delta^n|$, $\mathbb{P}^n\mathbb{R}$, $\mathbb{P}^n\mathbb{C}$, etc.

Proposition 1.2.7 (Evaluation map). *When X is a locally compact space, the* evaluation map, *defined by*

$$\begin{array}{cccc} \mathrm{ev} & : & Y^X \times X & \longrightarrow & Y \\ & & (f, x) & \longmapsto & f(x) \end{array}$$

is continuous.

Proof. Let U be an open set of Y and let $(f, x) \in Y^X \times X$ be such that $\mathrm{ev}(f, x) = f(x) \in U$. Since f is continuous, $f^{-1}(U)$ is open in X and since X is locally compact, there exists a compact neighborhood K of x in $f^{-1}(U)$. Let V be an open set of K containing x. Then (f, x) lives in the open set $W(K, U) \times V$ of $Y^X \times X$ which is included in $\mathrm{ev}^{-1}(U)$. □

Theorem 1.2.8. *For any locally compact space Y, the following pair of functors are adjoint.*

$$- \times Y : \mathsf{Top} \underset{\perp}{\rightleftarrows} \mathsf{Top} \ : \ (-)^Y$$

This means that the natural bijection (1) *holds on the level of the topological categories*

$$\mathsf{Top}(X \times Y, Z) \underset{\text{nat.}}{\cong} \mathsf{Top}\left(X, Z^Y\right) \ . \tag{2}$$

Proof. It is enough to check that a map $f \colon X \times Y \to Z$ is continuous if and only if the induced map $\breve{f} \colon X \to Z^Y$ is continuous.

(⇒) Let K be a compact set of Y and let U be an open set of Z. Any element $x \in \breve{f}^{-1}(W(K, U)) = \{x \in X \mid f(x, K) \subset U\}$ satisfies $\{x\} \times K \subset f^{-1}(U)$. Since f in continuous, this latter set is open, so there exists an open neighbourhood V of x in X such that $V \times K \subset f^{-1}(U)$, by the properties of the product topology.

(⇐) The map $f \colon X \times Y \to Z$ is the composite of the following two maps

$$f \ : \ X \times Y \xrightarrow{\breve{f} \times \mathrm{id}} Z^Y \times Y \xrightarrow{e} Z \ ,$$

which are continuous by Proposition 1.2.7.

□

In other words, the category of locally compact topological spaces is cartesian closed monoidal. One natural question to study from this situation is: can we internalize this adjunction, that is can we have a natural homeomorphism between $Z^{X \times Y} = \mathsf{Top}(X \times Y, Z)$ and $\left(Z^Y\right)^X = \mathsf{Top}\left(X, Z^Y\right)$? Otherwise stated, is the category of topological spaces *enriched* over itself? This is actually the case for any closed monoidal category. However, this cannot hold for the entire category Top but for a big enough subcategory of it.

Theorem 1.2.9 (Exponential law). *Let X and Y be locally compact topological spaces. Then the natural isomorphism* (2) *is an homeomorphism*

$$Z^{X\times Y} \underset{\substack{\text{nat.}\\ \text{homeo.}}}{\cong} \left(Z^Y\right)^X .$$

Proof. The proof is categorial in nature as its states that in any closed monoidal category the natural bijection is internal.

We first need to prove that the adjunction map

$$\breve{\ } \ :\ Z^{X\times Y} = \mathsf{Top}(X\times Y, Z) \to \left(Z^Y\right)^X = \mathsf{Top}\left(X, Z^Y\right)$$

which sends $f\colon X\times Y\to Z$ to $\breve{f}\colon X\to Z^Y$ is continuous. This follows from the following steps. Notice first that, since X and Y are locally compact, so does their product $X\times Y$. This implies that the evaluation map

$$\mathrm{ev}_{X\times Y,Z}\ :\ Z^{X\times Y}\times(X\times Y)\to Z$$

is continuous, by Proposition 1.2.7. The proof of Theorem 1.2.8 shows that, viewing it as a map from $\left(Z^{X\times Y}\times X\right)\times Y$ to Z, its first adjoint map

$$\breve{\mathrm{ev}}_{X\times Y,Z}\ :\ Z^{X\times Y}\times X\to Z^Y$$

is continuous and so does its second adjoint map

$$\breve{\breve{\mathrm{ev}}}_{X\times Y,Z}\ :\ Z^{X\times Y}\to \left(Z^Y\right)^X ,$$

which is equal to the adjunction map $\breve{\ }$.

Applying Theorem 1.2.8 to all these maps, one gets the following bijections, whose composite is equal to the push-forward by the adjunction map $\breve{\ }$:

$$\mathsf{Top}\left(A, Z^{X\times Y}\right)\cong \mathsf{Top}(A\times X\times Y, Z)\cong \mathsf{Top}\left(A\times X, Z^Y\right)\cong \mathsf{Top}\left(A, \left(Z^Y\right)^X\right) ,$$

for any topological space A. Since the category Top of topological spaces is locally small, the Yoneda functor $\mathsf{Top}\to\mathsf{Fun}\left(\mathsf{Top}^{\mathsf{op}},\mathsf{Set}\right)$ given by $W\mapsto (A\mapsto \mathsf{Top}(A,W))$ is fully faithful, which implies that the continuous map $\breve{\ }: Z^{X\times Y}\to\left(Z^Y\right)^X$ is an homeomorphism. □

Theorem 1.2.10. *For any locally compact space Y, the following pair of functors are adjoint.*

$$-\times Y : \mathsf{hoTop} \underset{\perp}{\rightleftarrows} \mathsf{hoTop}\ :\ (-)^Y .$$

Proof. We first have to check that these functors are well defined. Let $H\colon X\times I\to X'$ be a homotopy between $f\colon X\to X'$ and $g\colon X\to X'$. The map $X\times Y\times I\to X'\times Y$ defined by $(x,y,t)\mapsto (H(x,t),y)$ is a homotopy between $f\times\mathrm{id}_Y$ and $g\times\mathrm{id}_Y$. Similarly, the map $X^Y\times I\to {X'}^Y$ defined by $(\varphi,t)\mapsto \left(y\mapsto H(\varphi(y),t)\right)$ is a homotopy between f_* and g_* . This assignment defines a continuous map since it corresponds to the composite

$$X^Y\times I\times Y\cong X^Y\times Y\times I\xrightarrow{\mathrm{ev}\times\mathrm{id}_I} X\times I\xrightarrow{H} Y,$$

under the adjunction of Theorem 1.2.8. So both functors $-\times Y$ and $(-)^Y$ pass to the homotopy category.

It remains to show that the bijection of the adjunction of Theorem 1.2.8 also passes to the homotopy category. This is giving by the fact that $f,g\colon X\times Y\to Z$ are homotopic if and only if $\breve{f},\breve{g}\colon X\to Z^Y$ are homotopic. Let $H\colon X\times Y\times I\to Z$ be a homotopy between f and g, then the assignment $X\times I\to Z^Y$ given by $(x,t)\mapsto \left(y\mapsto H(x,y,t)\right)$ is a homotopy between $\breve{f}$ and $\breve{g}$. In the other way round, let $K\colon X\times I\to Z^Y$ be a homotopy between $\breve{f}$ and $\breve{g}$, the assignment $X\times Y\times I\to Z$ given by $(x,y,t)\mapsto K(x,t)(y)$ is a homotopy between f and g: it is equal to the composite

$$X\times Y\times I\cong X\times I\times Y\xrightarrow{K\times\mathrm{id}_Y} Z^Y\times Y\xrightarrow{\mathrm{ev}} Z,$$

so it is continuous since Y is locally compact by Proposition 1.2.7. □

There is exist a more relaxed notion, called *compactly generated weak Hausdorff space* under which the above properties hold. We refer the reader to [May99, Chapter 5] for more details. **Such spaces form a subcategory of** Top **which is one of the favorite categories of algebraic topologists and in which we will work implicitly here in order to avoid pathologies like the present one and to get a clean presentation.** The details of definitions are not relevant for this course. In order to understand some assumptions that we will make later on, just have in mind that one needs to consider a different topology on the product $X \times Y$ of compactly generated weak Hausdorff spaces, but this topology agrees with the classical one when X or Y is locally compact.

Theorem 1.2.8 gives in particular a natural bijection

$$\mathsf{Top}(X \times I, Y) \cong \mathsf{Top}\left(X, Y^I\right) \ ,$$

which is a natural homeomorphism when X is locally compact. This opens the door to a tentative equivalent definition of a homotopy, originally defined with a *cylinder object* $X \times I$, now by means of a *path object* Y^I.

Definition 1.2.11 (Cohomotopy). Let $f, g : X \to Y$ be two continuous maps. A *cohomotopy* between f and g is a continuous map $\breve{H} \colon X \to Y^I$ such that the following diagram commutes

$$\begin{tikzcd} & X \arrow[dl, bend right, "f"'] \arrow[d, "\breve{H}"] \arrow[dr, bend left, "g"] & \\ Y & Y^I \arrow[l, "e_0"] \arrow[r, "e_1"'] & Y \ , \end{tikzcd}$$

where $e_0 := e(-0)$ and $e_1 := e(-, 1)$.

Proposition 1.2.12. *Two maps are homotopic if and and only if they are cohomotopic.*

Proof. This is a direct corollary of Theorem 1.2.8 since the interval I is locally compact. □

1.2.2. *Categorical constructions in* Top_*. Let us now try to settle similar constructions and results in the category of pointed topological spaces Top_*. At the beginning, only the construction of the coproduct has to be modified.

Proposition 1.2.13. *In the category* Top_* *of pointed topological spaces, the following objects provide us to the relevant limits and colimits.*

Product: $(X, x_0) \times (Y, y_0) = (X \times Y, (x_0, y_0))$.
Coproduct: *the* wedge *or* bouquet

$$(X, x_0) \vee (Y, y_0) := \left(\frac{X \amalg Y}{x_0 \sim y_0}, x_0 \sim y_0\right) .$$

Pullback: $(X, x_0) \times_{(Z, z_0)} (Y, y_0) = (X \underset{Z}{\times} Y, (x_0, y_0))$.
Pushout: $(X, x_0) \underset{(Z, z_0)}{\amalg} (Y, y_0) = (X \underset{Z}{\amalg} Y, x_0 \sim y_0)$.

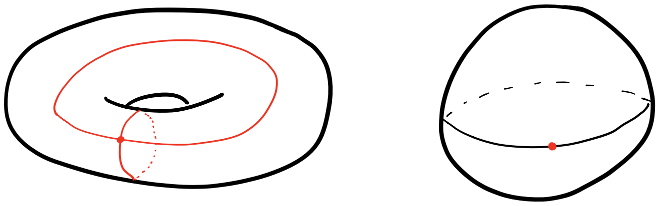
Proof. The various verifications are straightforward and thus left to the reader. □

Remark. When the base point will be understood, we will lighten the notation and not write it all the time; for instance, we will simply write $X \vee Y$ for the wedge.

Next question: what is the base point of $\mathsf{Top}_*(Y, Z)$, when (Y, y_0) and (Z, z_0) are pointed? The answer is easy: the constant map $\mathrm{cst} \colon y \mapsto z_0$. But then, the pointed product of spaces does not induce and natural bijection between $\mathsf{Set}_*(X \times Y, Z)$ and $\mathsf{Set}_*\left(X, \mathsf{Set}_*(Y, Z)\right)$; so we have to refine it.

Definition 1.2.14 (Smash product). Let (X, x_0) and (Y, y_0) be two pointed spaces. The *smash product* of X and Y is defined by

$$X \wedge Y := \frac{X \times Y}{X \times \{y_0\} \cup \{x_0\} \times Y} \ .$$



$$S^1 \wedge S^1 \simeq S^2$$

Remark. One can embed the wedge into the product $X \vee Y \hookrightarrow X \times Y$ under the assignment $x \mapsto (x, y_0)$ and $y \mapsto (x_0, y)$. Under this embedding, the smash product is equivalently given by

$$X \wedge Y = \frac{X \times Y}{X \vee Y} \ .$$

The smash product is associative up to homeomorphism (for compactly generated weak Hausdorff spaces), that is $(X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z)$ and the "point" $S^0 = \{*, \star\}$ is the unit of it $S^0 \wedge X \cong X$.

In the pointed case, since the context is obvious, we denote by $Z^Y = \mathsf{Top}_*(Y, Z)$ the subspace of $\mathsf{Top}(Y, Z)$ made up of pointed maps that we equipp with the compact-open topology.

Theorem 1.2.15 (Exponential law). *For any locally compact pointed space (Y, y_0), the following pair of functors are adjoint*

$$\begin{tikzcd} - \wedge Y : \mathsf{Top}_* \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathsf{Top}_* : (-)^Y \arrow[l, shift left] \end{tikzcd} ,$$

meaning that there exists a natural bijection

$$\mathsf{Top}_*(X \wedge Y, Z) \underset{\text{nat.}}{\cong} \mathsf{Top}_*\left(X, Z^Y\right) \ .$$

Moreover, when (X, x_0) is also locally compact, this induces a natural pointed homeomorphism

$$Z^{X \wedge Y} \underset{\substack{\text{nat.}\\ \text{homeo.}}}{\cong} \left(Z^Y\right)^X \ .$$

Proof. By definition, the smash product provides us with a natural bijection

$$\mathsf{Set}_*(X \wedge Y, Z) \underset{\text{nat.}}{\cong} \mathsf{Set}_*\left(X, Z^Y\right) ,$$

which satisfises all the required properties in the pointed case by the same arguments as given above. □

Remark. If one compares the present situation with the one of vector spaces, then the smash product should be viewed as a tensor product rather than a product, as the product in the category of pointed topological spaces is given by the underlying product of sets.

Corollary 1.2.16. *For any locally compact pointed space Y, the following pair of functors are adjoint.*

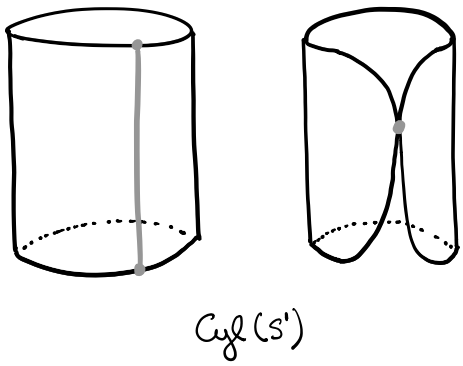
$$\begin{tikzcd} - \wedge Y : \mathsf{hoTop}_* \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathsf{hoTop}_* : (-)^Y \arrow[l, shift left] \end{tikzcd} .$$

Proof. It is a direct corollary of Theorem 1.2.15 with the arguments given in the proof of Theorem 1.2.10. □

We saw that the data of a homotopy amounts to a map in Top from the construction $X \times I$ to Y. What is the analogous "cylinder" construction in the category Top_* of pointed topological spaces?

Definition 1.2.17 (Cylinder of a pointed topological space). The *cylinder* of a pointed topological space (X, x_0) is defined by:

$$\mathrm{Cyl}(X) := \frac{X \times I}{\{x_0\} \times I} \ .$$



Proposition 1.2.18. *The data of a pointed homotopy between pointed maps from X to Y is equivalent to a map in* Top_* *from the cylinder* $\mathrm{Cyl}(X)$ *to* Y*:*

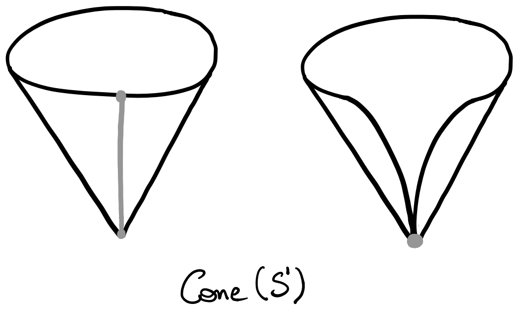
$$\left\{ \begin{array}{c} H : X \times I \to Y \\ H(x_0, t) = y_0 \end{array} \right\} \cong \mathsf{Top}_* \left(\mathrm{Cyl}(X), Y\right) .$$

Proof. The proof is straightforward and thus left to the reader. □

Let us pursue one step further and try to find a universal construction which (co)represents the data of a homotopy from the constant map.

Definition 1.2.19 (Cone of a pointed topological space)**.** The *cone* of a pointed topological space (X, x_0) is defined by:

$$\mathrm{Cone}(X) := \frac{X \times I}{\{x_0\} \times I \cup X \times \{0\}} .$$



Proposition 1.2.20. *The data of a pointed homotopy from the constant map to a pointed map from X to Y is equivalent to a map in* Top_* *from the cone* $\mathrm{Cone}(X)$ *to* Y*:*

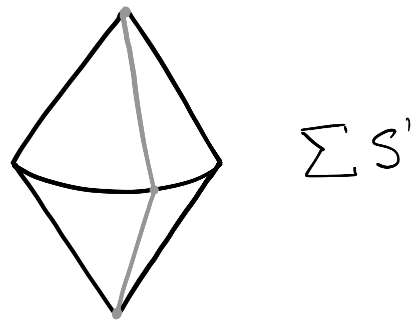
$$\left\{ \begin{array}{c} H : X \times I \to Y \\ H(x_0, t) = y_0 \\ H(x, 0) = y_0 \end{array} \right\} \cong \mathsf{Top}_* \left(\mathrm{Cone}(X), Y\right) .$$

Proof. The proof is straightforward and thus left to the reader. □

Let us now address the more symmetric question: what is the universal construction which (co)represents the data of a homotopy from and to the constant map.

Definition 1.2.21 (Suspension of a pointed topological space)**.** The *suspension* of a pointed topological space (X, x_0) is defined by:

$$\Sigma X := \frac{X \times I}{\{x_0\} \times I \cup X \times \partial I} .$$



Proposition 1.2.22. *The data of a pointed homotopy from and to the constant map from X to Y is equivalent to a map in* Top_* *from the suspension* ΣX *to* Y*:*

$$\left\{ \begin{array}{c} H : X \times I \to Y \\ H(x_0, t) = y_0 \\ H(x, 0) = y_0 \\ H(x, 1) = y_0 \end{array} \right\} \cong \mathsf{Top}_*\left(\Sigma X, Y\right) \ .$$

Proof. The proof is straightforward and thus left to the reader. ☐

These three constructions are functorial and, since they are defined by successive quotients, the natural "projections"

$$\mathrm{Cyl}(X) \twoheadrightarrow \mathrm{Cone}(X) \twoheadrightarrow \Sigma X$$

induce natural transformations

$$\mathrm{Cyl} \Longrightarrow \mathrm{Cone} \Longrightarrow \Sigma \ .$$

Now one can start looking for the analogous constructions on the right-hand side of the adjunction of Theorem 1.2.8 in the category of pointed spaces. Notice that one main drawback of the ubiquitous interval I is that it is not canonically pointed. However the exponential object $X^I = \mathsf{Top}(I, X)$ is always pointed by the constant map as soon as X is.

Proposition 1.2.23. *The cylinder and the exponential object form a pair of adjoint functors*

$$\mathrm{Cyl} : \mathsf{Top}_* \underset{\perp}{\rightleftarrows} \mathsf{Top}_* \ : \ (-)^I \ ,$$

whose natural bijection

$$\mathsf{Top}_*(\mathrm{Cyl}(X), Y) \underset{\substack{\text{nat.}\\ \text{homeo.}}}{\cong} \mathsf{Top}_*\left(X, Y^I\right)$$

is an homeomorphism as soon as the pointed space (X, x_0) *is locally compact.*

Proof. By the definition of the cylinder, we get a natural bijection on the pointed set level

$$\mathsf{Set}_*(\mathrm{Cyl}(X), Y) \underset{\text{nat.}}{\cong} \mathsf{Set}_*\left(X, Y^I\right) \ .$$

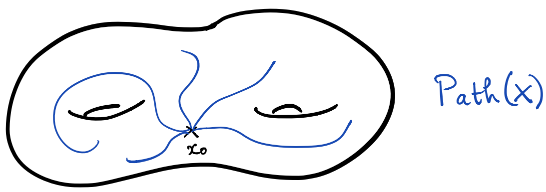
The rest follows directly from the arguments given above since the interval i is locally compact. ☐

This shows that the data of a pointed homotopy between two pointed maps $f, g\colon : (X, x_0) \to (Y, y_0)$ is equivalent to a pointed map $\breve{H}\colon (X, x_0) \to (Y^I, \mathrm{cst})$ satisfying the following commutative diagram in Top_* :

$$\begin{tikzcd} & X \arrow[dl, bend right, "f"'] \arrow[d, "\breve{H}"] \arrow[dr, bend left, "g"] & \\ Y & Y^I \arrow[l, "e_0"] \arrow[r, "e_1"'] & Y \ . \end{tikzcd}$$

Definition 1.2.24 (Path space). The *path space* of a pointed topological space (X, x_0) is defined by

$$\mathrm{Path}(X) \coloneqq \{\varphi\colon I \to X \mid \varphi(0) = x_0\} \subset X^I .$$



Proposition 1.2.25. *The cone and the path space define a pair of adjoint functors*

$$\mathrm{Cone} : \mathsf{Top}_* \underset{\perp}{\rightleftarrows} \mathsf{Top}_* : \mathrm{Path}\ ,$$

whose the natural bijection

$$\mathsf{Top}_*(\mathrm{Cone}(X), Y) \underset{\substack{\text{nat.}\\ \text{homeo.}}}{\cong} \mathsf{Top}_*(X, \mathrm{Path}(Y))$$

is an homeomorphism as soon as the pointed space (X, x_0) is locally compact.

Proof. The same arguments as above apply. □

So the data of a pointed homotopy from the constant map to a pointed map $f \colon : X \to Y$ is equivalent to a pointed map $\breve{H} \colon X \to \mathrm{Path}(Y)$ such that $e_1\left(\breve{H}\right) = f$.

Definition 1.2.26 (Loop space). The *loop space* of a pointed topological space (X, x_0) is defined by

$$\Omega X := \left\{\varphi \colon I \to X \mid \varphi(0) = x_0 = \varphi(1)\right\} \subset X^I .$$

The loop space is equivalently defined by

$$\Omega X \cong \mathsf{Top}_*(S^1, X)\ .$$

Proposition 1.2.27. *The suspension and the loop space define a pair of adjoint functors*

$$\Sigma : \mathsf{Top}_* \underset{\perp}{\rightleftarrows} \mathsf{Top}_* : \Omega\ ,$$

whose the natural bijection

$$\mathsf{Top}_*(\Sigma X, Y) \underset{\substack{\text{nat.}\\ \text{homeo.}}}{\cong} \mathsf{Top}_*(X, \Omega Y)$$

is an homeomorphism as soon as the pointed space (X, x_0) is locally compact.

Proof. The same arguments as above apply. □

This shows that the data of a pointed homotopy between from and to the constant map is equivalent to a pointed map $X \to \Omega Y$.

The canonical embeddings

$$X^I \hookrightarrow \mathrm{Path}(X) \hookrightarrow \Omega X$$

induce natural transformations

$$(-)^I \Longrightarrow \mathrm{Path} \Longrightarrow \Omega\ .$$

Corollary 1.2.28. *The suspension and the loop space define a pair of adjoint functors on the level of the pointed homotopy category*

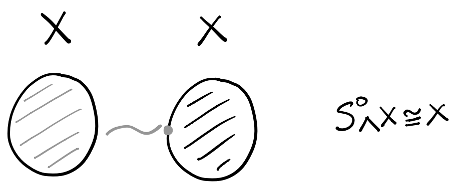
$$\Sigma : \mathsf{hoTop}_* \underset{\perp}{\rightleftarrows} \mathsf{hoTop}_* : \Omega\ .$$

Proof. It is a direct corollary of Proposition 1.2.27 with the arguments given in the proof of Theorem 1.2.10. □

05/11/25

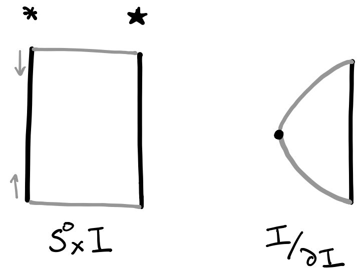
1.3. **Higher homotopy groups.**

1.3.1. *Suspension and smash product.* Before to proceed even further, let us get more acquainted with the notions of smash product and suspension. We already noticed that $S^0 := \{*, \star\}$ is the unit for the smash product: $S^0 \wedge X \cong X$.



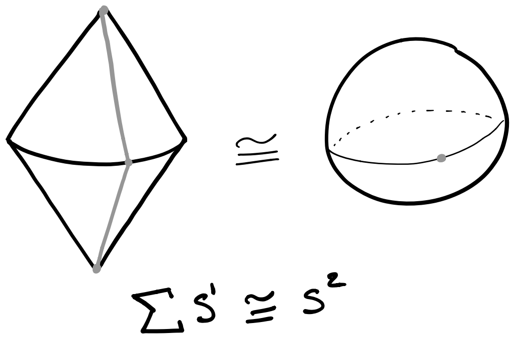
The first computations of the suspension show

$$\Sigma S^0 \cong S^1 \cong I/\partial I$$



and

$$\Sigma^2 S^0 \cong \Sigma S^1 \cong S^2 \cong I^2/\partial I^2 \ .$$



For any pointed topological space X, one notices that

$$\Sigma X \cong X \wedge I/\partial I \ .$$

Proposition 1.3.1. *The iterated suspension of any pointed topological space X is given by*

$$\Sigma^n X \ \cong \ X \wedge I^n/\partial I^n \ ,$$

for $n \geqslant 1$.

Proof. Let us prove this assertion by induction on $n \geqslant 1$. The case $n = 1$ holds by the remark above. Suppose now that the result holds for n, that is $\Sigma^n X \ \cong \ X \wedge I^n/\partial I^n$, and let us show it for $n+1$ as follows. We have

$$\begin{aligned} \Sigma^{n+1} X = \Sigma\,(\Sigma^n X) &\underset{\text{ind.}}{\cong} \Sigma(X \wedge I^n/\partial I^n) \\ &\cong (X \wedge I^n/\partial I^n) \wedge I/\partial I \\ &\cong X \wedge (I^n/\partial I^n \wedge I/\partial I) \\ &\cong X \wedge I^{n+1}/\partial I^{n+1}. \end{aligned}$$

In the third line, we used the fact that I is (locally) compact for the associativity of the smash product. In the last line, we used the identification

$$\frac{I^n}{\partial I^n} \wedge \frac{I}{\partial I} \cong \frac{I^n \times I}{\partial I^n \times I \cup I^n \times \partial I} \cong \frac{I^{n+1}}{\partial I^{n+1}} \ .$$

□

Corollary 1.3.2. *For any $n \geqslant 1$, the following pointed topological spaces are homeomorphic*

$$\Sigma^n S^0 \cong I^n/\partial I^n \cong S^n \ .$$

Proof. The first pointed homeomorphism is a direct consequence of Proposition 1.3.1 and the fact that S^0 is the unit for the smash product. For the second homeomorphism, it remains to show that $I^n/\partial I^n$ is a model for the n-dimensional sphere. Recall that this latter one is given by

$$S^n := \partial D^{n+1} \cong \partial I^{n+1} \cong I^n/\partial I^n.$$

Indeed, the boundary ∂I^{n+1} of the $n+1$-dimensional cube is made up of $2(n+1)$ faces homeomorphic to I^n. Chose one of them and stretch its boundary to cover the $2(n+1)-1$ other faces. This gives the last displayed homeomorphism. □

Remark (Sphere spectrum). Corollary 1.3.2 shows that the sequence of spheres can be obtained by iterating the suspension operation:

$$S^0 \xrightarrow{\Sigma} S^1 \xrightarrow{\Sigma} S^2 \xrightarrow{\Sigma} S^3 \xrightarrow{\Sigma} \cdots .$$

This universal object is called *the sphere spectrum* and it plays a seminal role in stable homotopy theory. The category of spectra is the good category to represent cohomology theories, and the sphere spectrum is the unit in this category.

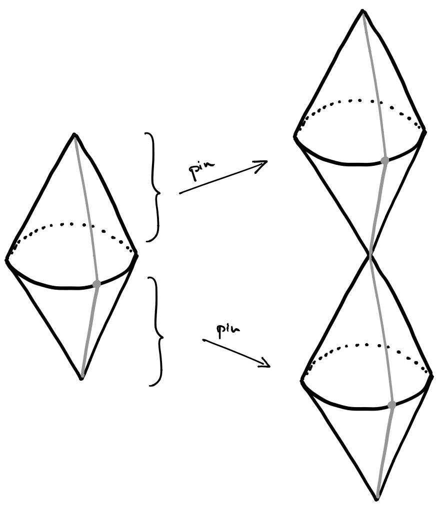
By a slight abuse of notations, we still denote this induced functor by $\Sigma : \mathsf{hoTop}_* \to \mathsf{hoTop}_*$.

1.3.2. *Definition of the higher homotopy groups.* recall the definition of the fundamental group

Definition 1.3.3 (Pinch map). The *pinch map* is the map

$$\mathrm{pin} \ : \ \Sigma X \longrightarrow \Sigma X \vee \Sigma X$$

where the image of (x, t) is $(x, 2t)$ in the first copy of ΣX, when $0 \leqslant t \leqslant \frac{1}{2}$, and $(x, 2t-1)$ in the second copy of ΣX, when $\frac{1}{2} \leqslant t \leqslant 1$.



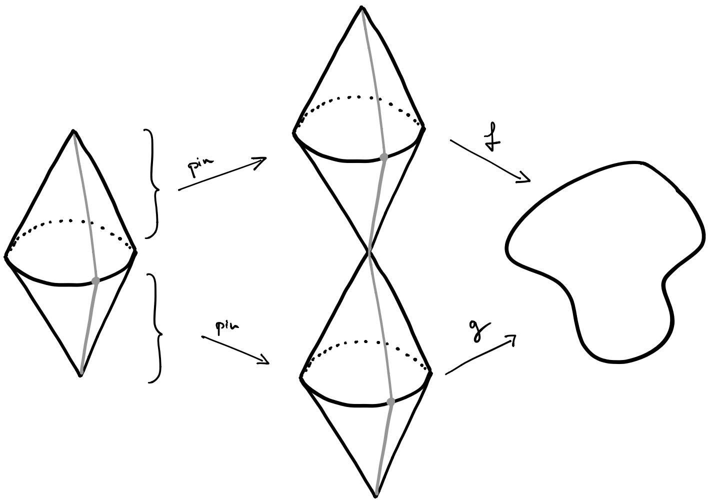
Pulling back along the pinch map, one gets a "convolution type" binary product denoted $\cdot$ on the space $\mathsf{Top}_*(\Sigma X, Y)$:

$$\begin{tikzcd}
\mathsf{Top}_*(\Sigma X, Y) \times \mathsf{Top}_*(\Sigma X, Y) \arrow[r, "\cong"] \arrow[dr, bend right, "\cdot"'] & \mathsf{Top}_*(\Sigma X \vee \Sigma X, Y) \arrow[d, "\mathrm{pin}^*"] \\
& \mathsf{Top}_*(\Sigma X, Y) ,
\end{tikzcd}$$

explicitly given by

$$f \cdot g : (x, t) \longmapsto \begin{cases} f(x, 2t) & \text{for } 0 \leqslant t \leqslant \frac{1}{2} , \\ g(x, 2t-1) & \text{for } \frac{1}{2} \leqslant t \leqslant 1 , \end{cases} \tag{3}$$

for two pointed continuous map $f, g \colon \Sigma X \to Y$.



$$f \cdot g \colon \Sigma X \xrightarrow{\text{pin}} \Sigma X \vee \Sigma X \xrightarrow{f \vee g} Y$$

Due to the parametrization, the product $\cdot$ is *neither* associative *nor* unital on $\mathsf{Top}_*(\Sigma X, Y)$, but it will satisfies these properties in the level of the pointed homotopy category.

Proposition 1.3.4. *The product $\cdot$ induces a group structure on $\big([\Sigma X, Y]_*, \cdot, [\mathrm{cst}]\big)$, where $[\Sigma X, Y]_* = \mathsf{hoTop}_*(\Sigma X, Y)$.*

Proof. Let us first show that the above formula (3) does not depend on the choice of representatives in the homotopy category. Let $H \colon \Sigma X \times I \to Y$ be a pointed homotopy from $f \colon \Sigma X \to Y$ to $f' \colon \Sigma X \to Y$. The assignment

$$\begin{array}{ccl} \Sigma X \times I & \longrightarrow & Y \\ (x, t, s) & \longmapsto & \begin{cases} H(x, 2t, s) & \text{for } 0 \leqslant t \leqslant \frac{1}{2}\,, \\ g(x, 2t-1) & \text{for } \frac{1}{2} \leqslant t \leqslant 1\,, \end{cases} \end{array}$$

defines a pointed homotopy from $f \cdot g$ to $f' \cdot g$. The proof of the same property on the right-hand side is similar.

Let us now prove that this product defines a group structure.

Unit: For any pointed map $f \colon \Sigma X \to Y$, the following assignment

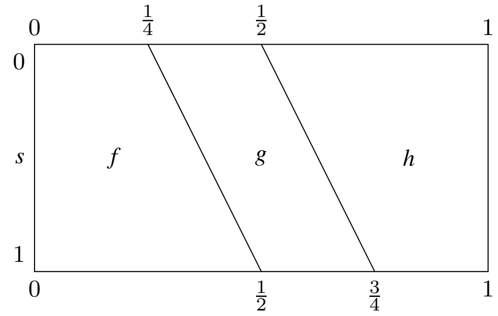
$$\begin{array}{ccl} \Sigma X \times I & \longrightarrow & Y \\ (x, t, s) & \longmapsto & \begin{cases} f\left(x, \frac{2}{1+s} t\right) & \text{for } 0 \leqslant t \leqslant \frac{1+s}{2}\,, \\ y_0 & \text{for } \frac{1+s}{2} \leqslant t \leqslant 1\,, \end{cases} \end{array}$$

is a pointed homotopy from $f \cdot \mathrm{cst}$ to f. The proof for the other side is similar.

Associativity: The assignment

$$\begin{array}{ccl} \Sigma X \times I & \longrightarrow & Y \\ (x, t, s) & \longmapsto & \begin{cases} f\left(x, \frac{4}{1+s} t\right) & \text{for } 0 \leqslant t \leqslant \frac{1+s}{4}\,, \\ g\left(x, 4\left(t - \frac{1+s}{4}\right)\right) & \text{for } \frac{1+s}{4} \leqslant t \leqslant \frac{2+s}{4}\,, \\ h\left(x, \frac{4}{2-s}\left(t - \frac{2+s}{4}\right)\right) & \text{for } \frac{2+s}{4} \leqslant t \leqslant 1\,, \end{cases} \end{array}$$

is a pointed homotopy from $(f \cdot g) \cdot h$ to $f \cdot (g \cdot h)$.



Inverse: For any pointed map $f\colon \Sigma X \to Y$, we consider the pointed map $f^{-1}\colon \Sigma X \to Y$ defined by $(x,t) \mapsto f(x, 1-t)$. The assignment

$$\begin{array}{ccl} \Sigma X \times I & \longrightarrow & Y \\ (x,t,s) & \longmapsto & \begin{cases} f(x, 2(1-s)t) & \text{for } 0 \leqslant t \leqslant \frac{1}{2} , \\ f(x, 2(1-s)(1-t)) & \text{for } \frac{1}{2} \leqslant t \leqslant 1 , \end{cases} \end{array}$$

is a pointed homotopy from $f \cdot f^{-1}$ to cst .

□

Remark. Recall that a map in $\mathsf{Top}_*(\Sigma X, Y)$ is a pointed homotopy from and to the constant map between X and Y and that the product $\cdot$ is a parametrized composite of them. One can see that a pointed homotopy between such maps amounts to a homotopy relative to $X \times \partial I$. Under this interpretation, the above proposition is related to the 2-groupoid mentioned in Section 1.1.3.

The range of ideas used here might make the reader think at the definition of the fundamental group as the loop spaces identified up to homotopy equipped with the concatenation of paths. This is not a surprise as

$$\left(\left[\Sigma S^0, Y\right]_* \cong \left[S^0, \Omega Y\right]_* \cong \left[S^1, Y\right]_*, \cdot, [\mathrm{cst}]\right) \cong \pi_1(Y, y_0) ,$$

by the suspension-loop space adjunction of Corollary 1.2.28. However Proposition 1.3.4 allows us to go further and to define the higher homotopy groups.

Definition 1.3.5 (n-th homotopy group). For $n \geqslant 1$, the *n-th homotopy group* of a pointed topological space X is defined by

$$\left(\pi_n(X, x_0), \cdot, 0\right) := \left(\left[\Sigma^n S^0, X\right]_* \cong [S^n, X]_* \cong [(I^n, \partial I^n), (X, x_0)], \cdot, [\mathrm{cst}]\right) .$$

For $n = 0$, we consider the set $\pi_0(X) := \left[S^0, X\right]_*$ of connected components of X . The suspension-loop space adjunction of Corollary 1.2.28 shows that the underlying set of the nth homotopy group is equivalently given by

$$\pi_n(X, x_0) \cong \pi_{n-1}(\Omega X) \cong \cdots \cong \pi_1(\Omega^{n-1} X) \cong \pi_0(\Omega^n X) .$$

Proposition 1.3.6. *The n-th homotopy group is a homotopy invariant: any pointed homotopy equivalence $f\colon X \xrightarrow{\sim} Y$ induces isomorphisms $\pi_n(f)\colon \pi_n(X, x_0) \xrightarrow{\cong} \pi_n(Y, f(x_0))$, for $n \geqslant 1$, and a bijection, for $n = 0$.*

Proof. This is straightforward from the definition. □

Example. When a space X is contractible, all its homotopy groups are trivial, i.e. $\pi_n(X) \cong 0$, for $n \geqslant 0$. When a space X is discrete, all its homotopy groups are trivial, i.e. $\pi_n(X) \cong 0$, for $n \geqslant 1$, and $\pi_0(X) \cong X$.

Theorem 1.3.7. *For any $n \geqslant 2$, the group $([\Sigma^n X, Y], \cdot, [\mathrm{cst}])$ is abelian. For instance, the n-homotopy group $\left(\pi_n(X, x_0), \cdot, 0\right)$ is abelian, for $n \geqslant 2$.*

The proof relies on the following Eckmann–Hilton principle. Let us recall that a magma is a binary product which is not required to satisfy any relation *a priori*.

Lemma 1.3.8 (Eckman-Hilton principle). *Let $(A, +_1, u_1)$ and $(A, +_2, u_2)$ be two unital magmas on a set A. If they satisfy the interchanging law*

$$(x +_1 y) +_2 (x' +_1 y') = (x +_2 x') +_1 (y +_2 y')$$

then they are equal, that is $+_1 = +_2$ and $u_1 = u_2$, and the binary product is associative and commutative.

Proof. This follows from the following direct computations:

$$\begin{array}{rcccl} u_2 & = & (u_2 +_1 u_1) +_2 (u_1 +_1 u_2) = (u_2 +_2 u_1) +_1 (u_1 +_2 u_2) & = & u_1 , \\ x +_2 y & = & (x +_1 u) +_2 (u +_1 y) = (x +_2 u) +_1 (u +_2 y) & = & x +_1 y , \\ x + y & = & (u + x) + (y + u) = (u + y) + (x + u) & = & y + x , \\ (x + y) + z & = & (x + y) + (u + z) = (x + u) + (y + z) & = & x + (y + z) . \end{array}$$

□

Proof of Theorem 1.3.7. Since the nth iteration of the suspension is a coset of $X \times I^n$, any map from $\Sigma^n X$ in the homotopy category can be represented by a map from $X \times I^n$:

$$\mathsf{Top}(X \times I^n, Y) \longleftarrow \mathsf{Top}_*(\Sigma^n X, Y) \twoheadrightarrow [\Sigma^n X, Y]_* \ .$$

Given two maps $f, g : X \times I^n \to Y$, we consider the i-th sum, for $1 \leqslant i \leqslant n$, defined by

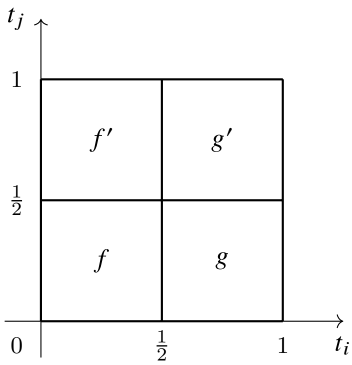
$$f +_i g : (x, t_1, \ldots, t_i, \ldots, t_n) \longmapsto \begin{cases} f(x, t_1, \ldots, 2t_i, \ldots, t_n) & , \text{ for } 0 \leqslant t_i \leqslant \frac{1}{2} \ , \\ g(x, t_1, \ldots, 2t_i - 1, \ldots, t_n) & , \text{ for } \frac{1}{2} \leqslant t_i \leqslant 1 \ . \end{cases}$$

This induces well defined unital binary product on $[\Sigma^n X, Y]_*$ that satisfy the interchanging law:

$$(f +_i g) +_j (f' +_i g') = (f +_j f') +_i (g +_j g') \ ,$$

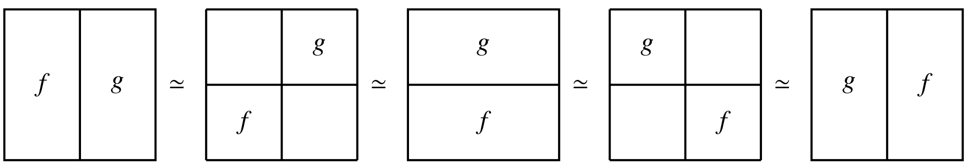
for all $1 \leqslant i \neq j \leqslant n$. Indeed, both side are equal to

$$(x, t_1, \ldots, t_i, \ldots, t_n) \longmapsto \begin{cases} f(x, t_1, \ldots, 2t_i, \ldots, 2t_j, \ldots, t_n) & , \text{ for } 0 \leqslant t_i \leqslant \frac{1}{2} \text{ and } 0 \leqslant t_j \leqslant \frac{1}{2} \ , \\ g(x, t_1, \ldots, 2t_i - 1, \ldots, 2t_j, \ldots, t_n) & , \text{ for } \frac{1}{2} \leqslant t_i \leqslant 1 \text{ and } 0 \leqslant t_j \leqslant \frac{1}{2} \ , \\ f'(x, t_1, \ldots, 2t_i, \ldots, 2t_j - 1, \ldots, t_n) & , \text{ for } 0 \leqslant t_i \leqslant \frac{1}{2} \text{ and } \frac{1}{2} \leqslant t_j \leqslant 1 \ , \\ g'(x, t_1, \ldots, 2t_i - 1, \ldots, 2t_j - 1, \ldots, t_n) & , \text{ for } \frac{1}{2} \leqslant t_i \leqslant 1 \text{ and } \frac{1}{2} \leqslant t_j \leqslant 1 \ . \end{cases}$$



Finally, we conclude by applying the Eckmann–Hilton principle of Lemma 1.3.8. □

The fact that the fondamental group $\pi_1(X)$ is not abelian is a consequence of "low dimensional pathology", like the ones that we often encounter in mathematics. For $n \geqslant 2$, there is certain degree of freedom which allows us to "move" elements around and to make the product $+$ commutative. Indeed, the lines of computation in the proof of the Eckmann–Hilton principle (Lemma 1.3.8) correspond here to the the following picture.



Proposition 1.3.9. *The map* $\Sigma \colon ([\Sigma X, Y], \cdot, [\mathrm{cst}]) \to ([\Sigma^2 X, \Sigma Y], \cdot, [\mathrm{cst}])$ *is a group morphism.*

Proof. Since the suspension functor Σ coincides with the smash product with $I/\partial I$, it preserves pointed homotopies. Therefore it induces a endofunctor of the homotopy category, so this map is well defined. The definition of the product $\cdot$ with the pinch map shows that this is a group morphism. □

1.4. **Fibre and cofibre sequences.** In this section, we work in the category of locally compact or compactly generated weak Hausdorff pointed spaces.

1.4.1. *Cofibre sequence.* Pointed topological maps are in general not "exact", so we will "derive" them in a certain way.

Definition 1.4.1 (Exact sequence). A short sequence $(A, a) \xrightarrow{\alpha} (B, b) \xrightarrow{\beta} (C, c)$ in Set_* is *exact* if $\alpha(A) = \beta^{-1}(c)$. A long sequence is exact when all its short sequences are exact.

Definition 1.4.2 (h-coexact sequence). A short sequence $U \xrightarrow{f} V \xrightarrow{g} W$ of pointed continuous maps is *h-coexact* if the short sequence

$$[U, Z]_* \xleftarrow{f^*} [V, Z]_* \xleftarrow{g^*} [W, Z]_*$$

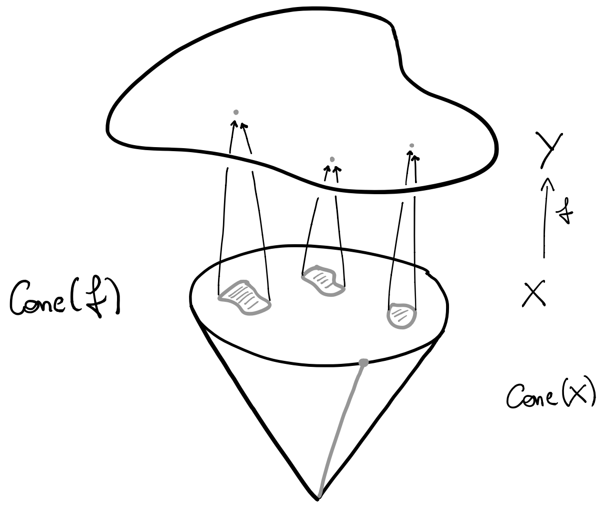
is exact, for all pointed topological space Z. A long sequence is h-coexact when all its short sequences are h-coexact.

Notice that this condition amounts to saying that, for all pointed continuous map $\psi : V \to Z$, the composition $\psi f : U \to V \to Z$ is null homotopic if and only if there exists $\varphi : W \to Z$ such that $\psi \sim \varphi g$ rel $\{*\}$. When it is the case, this implies that gf is null homotopic (consider $Z = W$ and $\varphi = \mathrm{id}_W$) . In the other way round, this condition is suffisant to imply that $g^* ([W, Z]_*) \subset (f^*)^{-1} ([\mathrm{cst}])$.

Nothing ensures that the canonical sequence $X \xrightarrow{f} Y \twoheadrightarrow Y/X$ is h-coexact. So, let us refine the construction on the right-hand side.

Definition 1.4.3 (Mapping cone). The *mapping cone* $\mathrm{Cone}(f)$ of a pointed continuous map $f : X \to Y$ is defined by the following pushout:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_1"'] \arrow[dr, phantom, "\ulcorner"] & Y \arrow[d, "f_1"] \\ \mathrm{Cone}(X) \arrow[r] & \mathrm{Cone}(f) \end{tikzcd} \qquad \text{that is } \mathrm{Cone}(f) = \mathrm{Cone}(X) \underset{f}{\cup} Y \cong \frac{\mathrm{Cone}(X) \vee Y}{(x, 1) \sim f(x)} .$$



Remark. Notice that $f_1 : Y \hookrightarrow \mathrm{Cone}(f)$ is an embedding, i.e. a monomorphism which induces an homeomorphism onto its image.

Notice that the composite $f_1 f$ is null homotopic: consider the homotopy $H : X \times I \to \mathrm{Cone}(f)$ given by $H(x, t) := (x, t)$. The universal property of the pushout

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_1"'] \arrow[dr, phantom, "\ulcorner"] & Y \arrow[d, "f_1"] \arrow[ddr, bend left, "g"] & \\ \mathrm{Cone}(X) \arrow[r, "j"'] \arrow[drr, bend right, "h"'] & \mathrm{Cone}(f) \arrow[dr, dotted, "\exists ! \Phi"] & \\ & & Z . \end{tikzcd}$$

implies that the data of a map $\Phi : \mathrm{Cone}(f) \to Z$ is equivalent to the data of two maps $g : Y \to Z$ and $h : \mathrm{Cone}(f) \to Z$, such that $g = \Phi f_1$ and $h = \Phi j$, which coincides with a homotopy from the constant map to the composite gf . All together, this shows that the sequence $X \xrightarrow{f} Y \xrightarrow{f_1} \mathrm{Cone}(f)$ is h-coexact.

The next theorem tells us how to pursue this h-coexact short sequence into a long one.

Theorem 1.4.4 (Cofibre sequence). *For any pointed continuous map $f\colon X \to Y$, the following sequence is h-coexact :*

$$X \xrightarrow{f} Y \xrightarrow{f_1} \mathrm{Cone}(f) \xrightarrow{p(f)} \Sigma X \xrightarrow{\Sigma f} \Sigma Y \xrightarrow{\Sigma f_1} \Sigma\mathrm{Cone}(f) \xrightarrow{\Sigma p(f)} \Sigma^2 X \xrightarrow{\Sigma^2 f} \Sigma^2 Y \xrightarrow{\Sigma^2 f_1} \Sigma^2\mathrm{Cone}(f) \xrightarrow{\Sigma^2 p(f)} \cdots,$$

where $p(f)\colon \mathrm{Cone}(f) \twoheadrightarrow \mathrm{Cone}(f)/f_1(Y) \cong \Sigma X$ is the canonical map. For any pointed topological space $Z \in \mathsf{Top}_*$, the following sequence

$$\begin{tikzcd}
{[X,Z]_*} & {[Y,Z]_*} \arrow[l] & {[\mathrm{Cone}(f),Z]_*} \arrow[l] & {[\Sigma X,Z]_*} \arrow[l] & {[\Sigma Y,Z]_*} \arrow[l] & {[\Sigma\mathrm{Cone}(f),Z]_*} \arrow[l] & {[\Sigma^2 X,Z]} \arrow[l] & {[\Sigma^2 Y,Z]} \arrow[l] & {[\Sigma^2\mathrm{Cone}(f),Z]} \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

is exact in the category of pointed sets for the first line, in the category of groups for the second line, and then in the category of abelian groups.

Remark. The cofibre sequence of Theorem 1.4.4 lead D. Puppe and J.-L. Verdier to the notion of a *triangulated category* and more recently J. Lurie to the notion of *stable ∞-category*.

Proof. We already proved above that any sequence of the form

$$X \xrightarrow{f} Y \xrightarrow{f_1} \mathrm{Cone}(f)$$

is h-coexact. This implies that the sequence

$$X \xrightarrow{f} Y \xrightarrow{f_1} \mathrm{Cone}(f) \xrightarrow{f_2} \mathrm{Cone}(f_1) \xrightarrow{f_3} \mathrm{Cone}(f_2)$$

is h-coexact.

From the various definitions, it is straightforward to check that the following diagram is commutative i_1^X

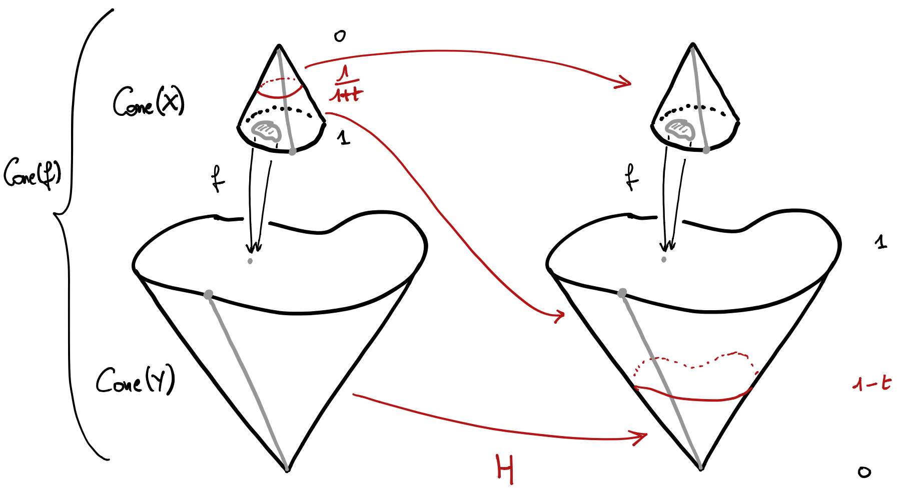
$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "i_1"] & Y \arrow[r, "i_1"] \arrow[d, "f_1"] \arrow[dl, phantom, "\ulcorner"] & \mathrm{Cone}(Y) \arrow[d, "j_1"] \arrow[dl, phantom, "\ulcorner"] \\
\mathrm{Cone}(X) \arrow[r, "j"] \arrow[d, "p"] & \mathrm{Cone}(f) \arrow[r, "f_2"] \arrow[d, "p(f)"] & \mathrm{Cone}(f_1) \arrow[d, "q(f)"] \\
\Sigma X \cong \mathrm{Cone}(X)/i_1(X) \arrow[r, "\cong"] & \Sigma X \cong \mathrm{Cone}(f)/f_1(Y) \arrow[r, "\cong"] & \Sigma X \cong \mathrm{Cone}(f_1)/j_1(\mathrm{Cone}(Y))
\end{tikzcd}$$

and that its bottom horizontal maps are homeomorphisms with the suspension ΣX of the space X.

Let us now show that the map $q(f)$ is a homotopy equivalence. Notice first that

$$\mathrm{Cone}(f_1) \cong \frac{\mathrm{Cone}(X) \amalg \mathrm{Cone}(Y)}{(x,1) \sim (f(x),1)} \cong \frac{X \times I \amalg Y \times I}{\{x_0\} \times I \cup X \times \{0\} \cup \{y_0\} \times I \cup Y \times \{0\} \cup \{(x,1) \sim (f(x),1)\}}\,.$$

We consider the following homotopy $H\colon \mathrm{Cone}(f_1) \times I \to \mathrm{Cone}(f_1)$ whose idea amounts to retracting the cone of Y onto its $s = 0$ base point and in the same time expending the cone of X inside it.



Explicitly, the map H is defined by changer les roles de s et t

$$H((y,s),t) := (y,(1-t)s)\ , \quad \text{for } y \in Y, \text{ and } s,t \in I\ ,$$

$$H((x,s),t) := \begin{cases} (x,(1+t)s)\ , & \text{for } x \in X, t \in I, \text{ and } s \in \left[0, \frac{1}{1+t}\right]\ , \\ (f(x), 2-(1+t)s)\ , & \text{for } x \in X, t \in I, \text{ and } s \in \left[\frac{1}{1+t}, 1\right]\ . \end{cases}$$

The abovementioned description of $\mathrm{Cone}(f_1)$ shows that this assignment is compatible with the various identifications, so it is continuous. At time $t=0$, we get $H(-,0) = \mathrm{id}_{\mathrm{Cone}(f_1)}$. Considering the map $\mathfrak{z}(f)\colon \Sigma X \to \mathrm{Cone}(f_1)$ defined by $\mathfrak{z}(f)(x,s) := H((x,s),1))$, we get $H(-,1) = \mathfrak{z}(f) \circ q(f)$ at time $t=1$. definir cette application avant Now the map $K\colon \Sigma X \times I \to \Sigma X$ defined by

$$K((x,s),t) := \begin{cases} \left(x, \frac{2}{1+t}s\right)\ , & \text{for } s \in \left[0, \frac{1+t}{2}\right]\ , \\ *\ , & \text{for } s \in \left[\frac{1+t}{2}, 1\right]\ , \end{cases}$$

is a homotopy from $q(f) \circ \mathfrak{z}(f)$ to $\mathrm{id}_{\Sigma X}$.

We denote by $\tau : \Sigma X \to \Sigma X$ the orientation reversing homeomorphism defined by $(x,t) \mapsto (x, 1-t)$ and we consider the following diagram

$$\begin{tikzcd} \mathrm{Cone}(f) \arrow[r, "f_2"] \arrow[dr, "p(f)"'] & \mathrm{Cone}(f_1) \arrow[r, "f_3"] \arrow[d, "\sim"', "q(f)"] \arrow[dr, "p(f_1)"'] & \mathrm{Cone}(f_2) \arrow[d, "\sim"', "q(f_1)"] \\ & \Sigma X \arrow[r, "\Sigma f \circ \tau"] & \Sigma Y\ . \end{tikzcd}$$

We have already seen that the triangles on the left-hand side and on the right-hand side are commutative. The middle triangle is homotopy commutative, that is $\Sigma f \circ \tau \circ q(f) \sim p(f_1)$. Since $\mathfrak{z}(f)$ is homotopy inverse to $q(f)$ and since τ is its own inverse, this latter fact is equivalent to Σf homotopy equivalent to $p(f_1) \circ \mathfrak{z}(f) \circ \tau$. One proves this with a homotopy $L\colon \Sigma X \times I \to \Sigma Y$, analogous to K:

$$L((x,s),t) := \begin{cases} \left(f(x), \frac{2}{1+t}s\right)\ , & \text{for } s \in \left[0, \frac{1+t}{2}\right]\ , \\ *\ , & \text{for } s \in \left[\frac{1+t}{2}, 1\right]\ . \end{cases}$$

So we get the following commutative diagram

$$\begin{tikzcd} {[Y,Z]_*} & {[\mathrm{Cone}(f),Z]_*} \arrow[l, "f_1^*"'] & {[\mathrm{Cone}(f_1),Z]_*} \arrow[l, "f_2^*"'] & {[\mathrm{Cone}(f_2),Z]_*} \arrow[l, "f_3^*"'] \\ & & {[\Sigma X,Z]_*} \arrow[ul, "p(f)^*"'] \arrow[u, "\cong", "q(f)^*"'] & {[\Sigma Y,Z]_*}\ , \arrow[l, "(\Sigma f \circ \tau)^*"] \arrow[ul, "p(f_1)^*"'] \arrow[u, "\cong", "q(f_1)^*"'] \end{tikzcd}$$

for any $Z \in \mathsf{Top}_*$. Since the top sequence is exact, so is the bottom sequence. Finally, this proves that the sequence

$$X \xrightarrow{f} Y \xrightarrow{f_1} \mathrm{Cone}(f) \xrightarrow{p(f)} \Sigma X \xrightarrow{\Sigma f} \Sigma Y$$

is h-coexact.

Finally, we apply this result to Σf, $\Sigma^2 f$, etc. In order to be able to conclude the proof with this, we need the existence of an homeomorphism $\chi : \mathrm{Cone}(\Sigma f) \xrightarrow{\cong} \Sigma\mathrm{Cone}(f)$, which satisfies

$$\chi \circ (\Sigma f)_1 = \Sigma f_1 \quad \text{and} \quad (12) \circ p(\Sigma f) = \Sigma p(f) \circ \chi \,,$$

where $(12) := \Sigma^2 X \xrightarrow{\cong} \Sigma^2 X$ defined by $(x, s, t) \mapsto (x, t, s)$. We begin by noticing that the assignment $(x, s, t) \mapsto (x, t, s)$ defines an homeomorphism

$$\overline{\chi} : \ \mathrm{Cone}(\Sigma X) \xrightarrow{\cong} \Sigma\mathrm{Cone}(X)$$

which satisfies $\overline{\chi} \circ i_1^{\Sigma X} = \Sigma i_1^X$. Considering the diagram defining the cone of Σf and the image under Σ of the diagram defining the cone of f

$$\begin{tikzcd}
\Sigma X \arrow[r, "\Sigma f"] \arrow[d, "i_1^{\Sigma X}"'] \arrow[dd, bend right=60, "\Sigma i_1^X"'] & \Sigma Y \arrow[d, "(\Sigma f)_1"] \arrow[dd, bend left=40, "\Sigma f_1"] \\
\mathrm{Cone}(\Sigma X) \arrow[r] \arrow[d, "\overline{\chi}"'] \arrow[ur, phantom, "\ulcorner"] & \mathrm{Cone}(\Sigma f) \arrow[d, dotted, "\exists! \chi"] \\
\Sigma\mathrm{Cone}(X) \arrow[r, "\Sigma j"'] & \Sigma\mathrm{Cone}(f) \,,
\end{tikzcd}$$

we get a map $\chi \colon \mathrm{Cone}(\Sigma f) \to \Sigma\mathrm{Cone}(f)$ satisfying $\chi \circ (\Sigma f)_1 = \Sigma f_1$. Since the suspension functor Σ is a left adjoint by Proposition 1.2.27, it preserves colimits and thus pushouts. This implies that χ is an homeomorphism. It remains to check $(12) \circ p(\Sigma f) = \Sigma p(f) \circ \chi$, which is straightforward since both are given by $(x, s, t) \mapsto (x, t, s)$. □

1.4.2. *Fibre sequence.* A dual fibre sequence is obtained by the same arguments applied to dual constructions under the adjunction $\Sigma \dashv \Omega$.

Definition 1.4.5 (h-exact sequence). A short sequence $U \xrightarrow{f} V \xrightarrow{g} W$ of pointed continuous maps is *h-exact* if the short sequence

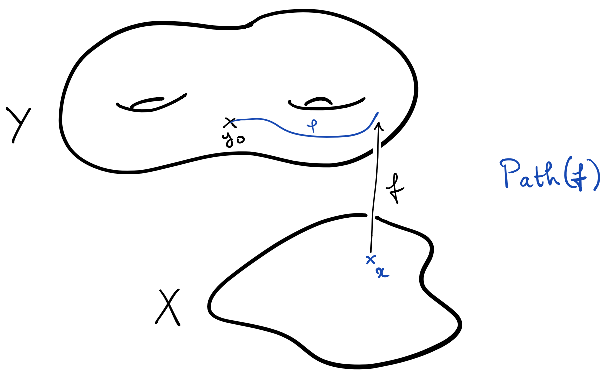
$$[Z, U]_* \xrightarrow{f_*} [Z, V]_* \xrightarrow{g_*} [Z, W]_*$$

is exact, for all topological space Z. A long sequence is h-exact when all its short sequences are h-exact.

Nothing ensures that the canonical sequence $(f^{-1}(y_0), x_0) \hookrightarrow (X, x_0) \xrightarrow{f} (Y, y_0)$ is h-exact. So let us refine the construction on the left-hand side.

Definition 1.4.6 (Path space of a map). The *path space* $\mathrm{Path}(f)$ of a pointed continuous map $f \colon X \to Y$ is defined by the following pullback:

$$\begin{tikzcd}
\mathrm{Path}(f) \arrow[r] \arrow[d, "f^1"'] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Path}(Y) \arrow[d, "e_1"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}
\qquad \text{that is } \mathrm{Path}(f) \cong X \underset{f}{\times} \mathrm{Path}(Y) \ \cong \ \{(x, \varphi \colon I \to Y) \mid f(x) = \varphi(1)\} \ .$$



Remark. The map $f^1 \colon \mathrm{Path}(f) \twoheadrightarrow X$ is a quotient map, i.e. a set U in X is open if and only if its inverse image is an open of $\mathrm{Path}(f)$.

Notice that the composite ff^1 is null homotopic. The universal property of the pullback

$$\begin{tikzcd} Z \arrow[rrd, bend left, "h"] \arrow[rd, dotted, "\exists!\Psi"] \arrow[rdd, bend right, "g"'] & & \\ & \mathrm{Path}(f) \arrow[r, "q"] \arrow[d, "f^1"'] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Path}(Y) \arrow[d, "e_1"] \\ & X \arrow[r, "f"'] & Y \end{tikzcd}$$

implies that the data of a map $\Psi \colon Z \to \mathrm{Path}(f)$ is equivalent to the data of two maps $g \colon Z \to X$ and $h \colon Z \to \mathrm{Path}(f)$, such that $g = f^1\Psi$ and $h = q\Psi$, which coincides with a homotopy from the constant map to the composite fg . All together, this shows that the sequence $\mathrm{Path}(f) \to X \xrightarrow{f} Y$ is h-exact.

The next theorem tells us how to pursue this h-exact short sequence into a long one, in a way dual to Theorem 1.4.4.

Theorem 1.4.7 (Fibre sequence). *For any pointed continuous map $f \colon X \to Y$, the following sequence is h-exact:*

$$\cdots \longrightarrow \Omega^2\mathrm{Path}(f) \xrightarrow{\Omega^2 f_1} \Omega^2 X \xrightarrow{\Omega^2 f} \Omega^2 Y \xrightarrow{\Omega i(f)} \Omega\mathrm{Path}(f) \xrightarrow{\Omega f_1} \Omega X \xrightarrow{\Omega f} \Omega Y \xrightarrow{i(f)} \mathrm{Path}(f) \xrightarrow{f^1} X \xrightarrow{f} Y\ ,$$

where $i(f) \colon \Omega Y \to \mathrm{Path}(f)$ is the map which sends a loop φ onto (x_0, φ) . , For any pointed topological space $Z \in \mathsf{Top}_*$, the following sequence

$$\begin{array}{l} \to [Z, \mathrm{Path}(f)]_* \longrightarrow [Z, X]_* \longrightarrow [Z, Y]_* \\ \to [Z, \Omega\mathrm{Path}(f)]_* \longrightarrow [Z, \Omega X]_* \longrightarrow [Z, \Omega Y]_* \\ \cdots \to [Z, \Omega^2\mathrm{Path}(f)]_* \longrightarrow [Z, \Omega^2 X]_* \longrightarrow [Z, \Omega^2 Y]_* \end{array}$$

is exact in the category of pointed sets for the first line, in the category of groups for the second line, and then in the category of abelian groups.

Proof. The proof is dual that of Theorem 1.4.4. □

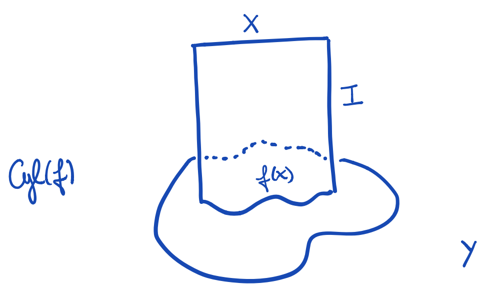
10/11/25

1.5. **Fibrations and cofibrations.** We introduce here two classes of maps, cofibrations and fibrations, which are the fundamental tools of homotopy theory. The (co)fibre sequence applies efficiently to them and they lie at the core of the modern axiomatic treatment of homotopical algebra.

1.5.1. *Cofibrations.* Let us "come back" to the general non-necessarily pointed world. The following construction provides us with a non-pointed notion analogous to the mapping cone.

Definition 1.5.1 (Mapping cylinder). The *mapping cylinder* $\mathrm{Cyl}(f)$ of a continuous map $f \colon X \to Y$ is defined by the following pushout:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_0"'] \arrow[dr, phantom, "\ulcorner"] & Y \arrow[d, "f_0"] \\ X \times I \arrow[r, "j"'] & \mathrm{Cyl}(f) \end{tikzcd} \quad \text{that is } \mathrm{Cyl}(f) \cong \frac{(X \times I) \amalg Y}{(x, 0) \sim f(x)}\ .$$



The space X naturally embeds into the mapping cylinder under $i_1 \colon X \hookrightarrow \mathrm{Cyl}(f)$; so does Y under $f_0 \colon Y \hookrightarrow \mathrm{Cyl}(f)$. This latter map admits the retraction $P \colon \mathrm{Cyl}(f) \to Y$ defined by $(x, t) \mapsto f(x)$ and $y \mapsto y$. Considering the homotopy $H \colon \mathrm{Cyl}(f) \times I \to I$ given by $(x, s, t) \mapsto (x, st)$ and $(y, t) \mapsto y$, one can see that this actually forms a strong deformation retract. Hence the canonical retraction

$$P \ : \ \mathrm{Cyl}(f) \xrightarrow{\sim} \mathrel{\mkern-14mu}\twoheadrightarrow Y$$

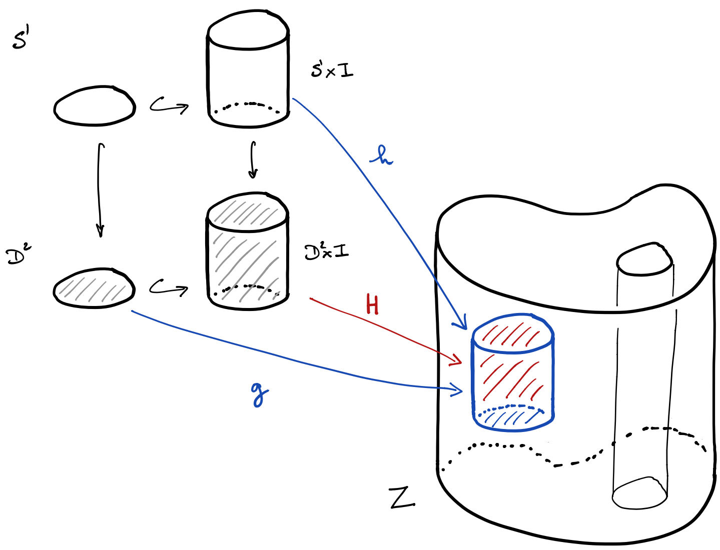
is a quotient map and a homotopy equivalence. In the end, this shows that any map $f \colon X \to X$ factors through an embedding followed by a quotient map which is a homotopy equivalence:

$$\begin{tikzcd}
X \arrow[r, hook, "i_1"'] \arrow[rr, bend left, "f"] & \mathrm{Cyl}(f) \arrow[r, two heads, "\sim", "P"'] & Y
\end{tikzcd} .$$

This is a remarkable property, but one might want to go further and look for a universal property satisfied by the map i_1.

Definition 1.5.2 (Homotopy extension property (HEP))**.** A map $i \colon A \to X$ satisfies the *homotopy extension property (HEP)* with respect to a space Z if, for any pair of maps $g \colon X \to Z$ and $h \colon A \times I \to Z$ such that $h\, i_0^A = g\, i$, there exists a map $H : X \times I \to Z$ such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "i_0^A"] \arrow[d, "i"'] & A \times I \arrow[d, "i \times \mathrm{id}"] \arrow[ddr, bend left, "h"] & \\
X \arrow[r, "i_0^X"'] \arrow[drr, bend right, "g"'] & X \times I \arrow[dr, dotted, "\exists H"] & \\
& & Z
\end{tikzcd} .$$



As it is obvious from the above picture, the data of the map H needs not be unique. We say that the map H extends h with initial condition g . Recall that a map $F \colon X \to Y$ is said to *extend* a map $f \colon A \to Y$ along a map $i \colon A \to X$ when $f = Fi$.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"'] & Y \\
X \arrow[ur, "F"'] &
\end{tikzcd}$$

Proposition 1.5.3. *The homotopy extension property for $i \colon A \to X$ with respect to a space Z and for a pair of maps $g \colon X \to Z$ and $\check{h} \colon A \to Z^I$ is equivalent to the existence of a map $\check{H} \colon X \to Z^I$ factorising the* *following commutative diagram.*

$$
\begin{tikzcd}
A \arrow[r, "\check{h}"] \arrow[d, "i"'] & Z^I \arrow[d, "e_0"] \\
X \arrow[r, "g"'] \arrow[ur, dotted, "\exists \check{H}"] & Z
\end{tikzcd}
$$

Proof. This is a direct corollary of the $(-\times I) \dashv (-)^I$-adjonction of Theorem 1.2.8. □

In this case, the map $\check{H}$ extends $\check{h}$ along i, which explains the chosen terminology.

Definition 1.5.4 (Cofibration). A map $i\colon A \to X$ is *a cofibration* if it satisfies the homotopy extension property with respect to any space. In this case, we denote it by

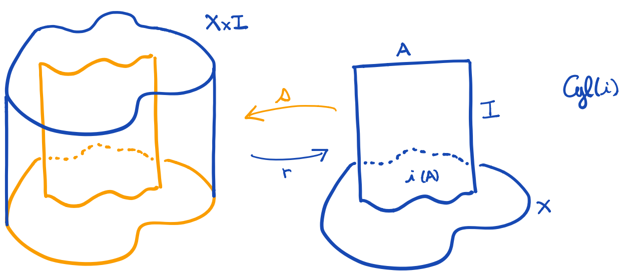
$$
i\colon A \rightarrowtail X\ .
$$

Example. One can directly see from the definition that homeomorphisms are cofibrations and that cofibrations are stable under composition.

In order to have a practical way to settle that more maps are cofibrations, let us analyse the homotopy extension property on the test space provided by the mapping cylinder $Z = \mathrm{Cyl}(i)$:

$$
\begin{tikzcd}
A \arrow[r, "i_0^A"] \arrow[d, "i"'] & A \times I \arrow[d, "i\times\mathrm{id}"] \arrow[ddr, bend left, "j"] & \\
X \arrow[r, "i_0^X"'] \arrow[drr, bend right, "i_0"'] & X \times I \arrow[dr, dotted, "\exists r"'] & \\
& & \mathrm{Cyl}(i) \arrow[ul, "\exists!\, \mathfrak{z}"'] \ .
\end{tikzcd}
$$

By the pushout property, there exists a unique map $\mathfrak{z}\colon \mathrm{Cyl}(i) \to X \times I$. When it is satisfied, the homotopy extension property provides us with a retraction r of it.



This example indicates that it is enough to check the homotopy extension property on the mapping cylinder only.

Proposition 1.5.5. *Let $i\colon A \to X$ be a continuous map. The following assertions are equivalent.*

(1) *The map i is a cofibration.*
(2) *The map i satisfies the homotopy extension property with respect to its mapping cylinder* $\mathrm{Cyl}(i)$.
(3) *The map $\mathfrak{z}\colon \mathrm{Cyl}(i) \hookrightarrow X \times I$ admits a retraction.*

Proof. From the arguments given above, it remains to prove (3) $\Rightarrow$ (1). Suppose that the map $\mathfrak{z}$ admits a retraction r and that we are giving a space Z and maps $g\colon X \to Z$ and $h\colon A \times X \to Z$ such that $h\, i_0^A = g\, i$. The pushout property defining the mapping cylinder provides us with a map

$\Phi\colon \mathrm{Cyl}(i) \to Z$ such that $h = \Phi j$ and $g = \Phi i_0$.

$$
\begin{tikzcd}
A \arrow[r, "i_0^A"] \arrow[d, "i"'] & A \times I \arrow[d, "i\times\mathrm{id}"] \arrow[ddr, bend left, "j"] \arrow[dddrr, bend left, "h"] & & \\
X \arrow[r, "i_0^X"'] \arrow[ddrr, bend right, "i_0"] \arrow[dddrrr, bend right, "g"'] & X \times I \arrow[dr, shift right, "\exists\, r"'] & & \\
 & & \mathrm{Cyl}(i). \arrow[ul, shift right, "\exists!\, \mathfrak{s}"'] \arrow[dr, dotted, "\exists\, \Phi"] & \\
 & & & Z\, .
\end{tikzcd}
$$

We claim that the map $H \coloneqq \Phi r$ satisfies the homotopy extension property. Indeed, we have

$$
h = \Phi j = \Phi\, r\, \mathfrak{s}\, j = H(i \times \mathrm{id}) \quad \text{and} \quad g = \Phi i_0 = \Phi\, r\, \mathfrak{s} i_0 = H\, i_0^X\ .
$$

□

Proposition 1.5.6. *Any cofibration $i\colon A \rightarrowtail X$ is an embedding. Moreover, its image $i(A)$ is closed when X is Hausdorff.*

Proof. Since $i\colon A \rightarrowtail X$ is a cofibration, it satisfies the homotopy extension property with respect to its mapping cylinder and its canonical maps, under the dual form: there exists a map $\breve{H}$ factorising the commutative square

$$
\begin{tikzcd}
A \arrow[r, "\breve{h}"] \arrow[d, tail, "i"'] & \mathrm{Cyl}(i)^I \arrow[d, "e_0"] \\
X \arrow[r, "i_0"'] \arrow[ur, dotted, "\exists \breve{H}"] & \mathrm{Cyl}(i)
\end{tikzcd}
$$

The map $\lambda\colon A \hookrightarrow \mathrm{Cyl}(i)$ defined by $a \mapsto \breve{h}(a)\left(\frac{1}{2}\right)$ is an embedding, that is an injection which induces an homeomorphism onto its image. The map $\Lambda\colon X \to \mathrm{Cyl}(i)$ defined by $x \mapsto \breve{H}(x)\left(\frac{1}{2}\right)$ satisfies $\Lambda i = \lambda$, which implies that i is injective and thus bijective onto its image. Under the following tilde notations for the various restrictions

$$
\tilde{i}\colon A \to i(A)\ , \quad \tilde{\lambda}\colon A \to \lambda(A)\ , \quad \tilde{\Lambda}\colon i(A) \to \lambda(A)\ ,
$$

we get $\tilde{\lambda}^{-1}\tilde{\Lambda}\,\tilde{i} = \mathrm{id}_A$. This shows that the inverse of $\tilde{i}$ is the composite $\tilde{\lambda}^{-1}\tilde{\Lambda}$, which is continuous; so $\tilde{i}$ is an homeomorphism onto its image.

Since X is Hausdorff, then so does $X{\times}I$. Since $i\colon A \rightarrowtail X$ is a cofibration, the section $\mathfrak{s}\colon \mathrm{Cyl}(i) \rightarrowtail X{\times}I$ admits a retraction $r\colon\ \colon X \times I \to \mathrm{Cyl}(i)$ by Proposition 1.5.5. Recall that the set of points on which two maps $f, g\colon Y \to Z$ agree is closed in Y when Z is Hausdorff. Here the identity of $X \times I$ and $\mathfrak{s}r$ agree on the image of $\mathfrak{s}$, which is thus closed in $X \times I$. This shows that $i(A)$ is closed in X. Indeed, given any point $x \notin i(A)$, we consider the point $\left(x, \frac{1}{2}\right) \notin \mathfrak{s}(\mathrm{Cyl}(i))$ and an open neighbourhood of it of the form $U \times V$, where U is an open set of X and V an open set in I, in the complement of $\mathfrak{s}(\mathrm{Cyl}(i))$ in $X \times I$. So U is an open neighbourhood of x in the complement of $i(A)$ in X. □

This property justifies *a posteriori* the notation chosen for a cofibration. It also shows that being a cofibration is not a homotopy equivalent notion: the identity map of a contractible space is homotopy equivalent to a constant map, which is not a cofibration since not injective in general.

Remark. The result of Proposition 1.5.6 holds in more generality for compactly generated weak Hausdorff spaces, see `https://mathoverflow.net/questions/221183`.

This result raises the question: when is the inclusion $i\colon A \hookrightarrow X$ associated to a sub-set $A \subset X$ a cofibration? Here we would like to consider the subset $X \times \{0\} \cup A \times I$ of $X \times I$ but we have to be careful: it is indeed in bijection with the mapping cylinder $\mathrm{Cyl}(i)$ but it fails to be homeomorphic to it. Nevertheless, the criterion similar to Point (3) of Proposition 1.5.5 holds here.

Proposition 1.5.7. *Let $A \subset X$ be a set of X. The inclusion $i\colon A \hookrightarrow X$ is a cofibration if and only if there exists a retraction $r\colon X \times I \to X \times \{0\} \cup A \times I$ to the canonical inclusion $\colon X \times \{0\} \cup A \times I \hookrightarrow X \times I$.*

Proof.

($\Rightarrow$) The canonical map $X \times \{0\} \sqcup A \times I \to X \times \{0\} \cup A \times I$ induces a continuous bijection $\Psi\colon \mathrm{Cyl}(i) \to X \times \{0\} \cup A \times I$, which fails to be an homeomorphism in general. Since i is a cofibration, the inclusion $\jmath\colon \mathrm{Cyl}(i) \hookrightarrow X \times I$ admits a retraction $r\colon X \times I \to \mathrm{Cyl}(i)$ by Point (3) of Proposition 1.5.5. Finally the composite Ψr is a retraction to the canonical inclusion $:X \times \{0\} \cup A \times I \hookrightarrow X \times I$.

($\Leftarrow$) Let us prove it when A is closed. Suppose that there exists a retraction $r\colon X \times I \to X \times \{0\} \cup A \times I$. We consider the map $\Phi\colon X \times \{0\} \cup A \times I \to Z$ defined by $(x, 0) \mapsto g(x)$, for $(x, 0) \in X \times \{0\}$, and by $(a, t) \longmapsto h(a, t)$, for $(a, t) \in A \times I$. When A is closed, this map is continuous since it is given by two continuous maps defined on two closed subsets that agree on their intersection. The composite Φr is a solution to the homotopy extension property.

$$\begin{tikzcd}
A \arrow[r, "i_0^A"] \arrow[d, hook, "i"'] & A \times I \arrow[d, "i \times \mathrm{id}"] \arrow[ddr, bend left, "j"] \arrow[dddrr, bend left, "h"] & & \\
X \arrow[r, "i_0^X"'] \arrow[dr, bend right, "i_0"] \arrow[ddrrr, bend right, "g"'] & X \times I \arrow[dr, "r"] & & \\
& & X \times \{0\} \cup A \times I . \arrow[dr, dotted, "\exists \Phi"] & \\
& & & Z \, .
\end{tikzcd}$$

A clear proof of the general case is given in [Sm68, Theorem 2].

□

Example. Proposition 1.5.7 allows us to prove that the maps $S^{n-1} \hookrightarrow D^n$ and $\partial I^n \hookrightarrow I^n$, for $n \geqslant 1$, are cofibrations. They are important cofibrations since they satisfy a certain generating property; we refer the reader to the Quillen model structures [Qui67] on topological spaces and simplicial sets for more details.

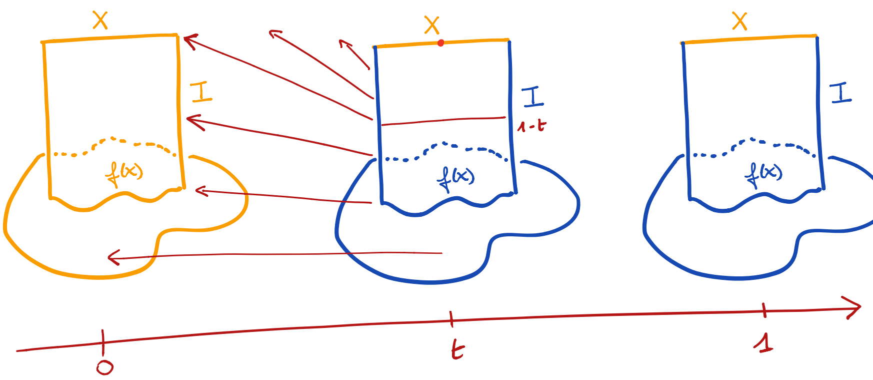
Theorem 1.5.8. *Any continuous map $f : X \to Y$ factors canonically through*

$$\begin{tikzcd}
X \arrow[r, hook, "i_1"'] \arrow[rr, bend left, "f"] & \mathrm{Cyl}(f) \arrow[r, two heads, "\sim", "P"'] & Y \, .
\end{tikzcd}$$

where P is a homotopy equivalence and i_1 is a cofibration.

Proof. It remains to prove that i_1 is a cofibration. Since $X \hookrightarrow \mathrm{Cyl}(f)$ is a (closed) embedding, we can use of Proposition 1.5.7: the map $\mathrm{Cyl}(f) \times \{0\} \cup X \times I \hookrightarrow \mathrm{Cyl}(f) \times I$ admits a retraction given by:

$$\begin{array}{rcll}
r & : & \mathrm{Cyl}(f) \times I & \longrightarrow \quad \mathrm{Cyl}(f) \times \{0\} \cup X \times I \\
& & (x, s, t) & \longmapsto \quad \begin{cases} \left(x, \frac{s}{1-t}, 0\right) \in \mathrm{Cyl}(f) \times \{0\} \, , & \text{for } x \in X \, , \ 0 \leqslant s \leqslant 1 - t \, , \\ (x, s - 1 + t) \in X \times I \, , & \text{for } x \in X \, , \ 1 - t \leqslant s \leqslant 1 \, , \end{cases} \\
& & (y, t) & \longmapsto \quad (y, 0) \in \mathrm{Cyl}(f) \times \{0\} \, , \quad \text{for } y \in Y \, .
\end{array}$$



□

Remark. Theorem 1.5.8 shows that, up to homotopy, one can canonically replace any map by a cofibration.

Proposition 1.5.9 (Cobase change). *For any cofibration $i\colon A \rightarrowtail X$ and any map $f\colon A \to B$, the map $j\colon B \rightarrowtail X \cup_f B$ obtained by* cobase change *under the pushout is a cofibration.*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, tail, "i"'] & B \arrow[d, tail, "j"] \\
X \arrow[r] & X \cup_f B \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Proof. We check directly the definition under the dual form of the homotopy extension property given in Proposition 1.5.3. Let Z be a space and let $g\colon X \to Z$ and $\breve{h}\colon B \times X \to Z$ be two maps such that $e_0\breve{h} = gj$. Since i is a cofibration, the homotopy extension property applied to the outer square of the following diagram provides us with a map $K\colon X \to Z^I$ satisfying $Ki = \breve{h}f$ and $e_0K = gk$.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, tail, "i"'] & B \arrow[r, "\breve{h}"] \arrow[d, "j"'] & Z^I \arrow[d, "e_0"] \\
X \arrow[r, "k"'] \arrow[urr, dotted, "\exists K"] & X \cup_f B \arrow[r, "g"'] \arrow[ur, dotted, "\exists ! \breve{H}"'] \arrow[ul, phantom, "\ulcorner", very near start] & Z
\end{tikzcd}$$

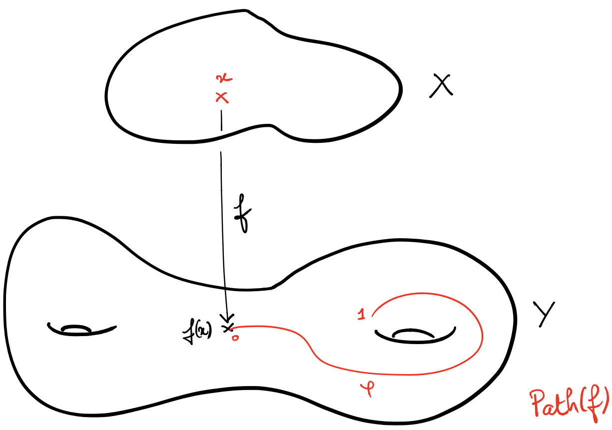
Then the pushout property defining $X \cup_f B$ produces a (unique) map $\breve{H}\colon X \cup_f B \to Z^I$ satisfying $\breve{H}k = K$ and $\breve{H}j = \breve{h}$. Since $e_0\breve{H}j = e_0\breve{h} = gj$ and $e_0\breve{H}k = e_0K = gk$, we get $g = e_0\breve{H}$, by the uniqueness property of the pushout $X \cup_f B$. □

12/11/25 1.5.2. *Fibrations.* Let us now look for a notion dual to cofibrations under the $(-\times I) \dashv (-)^I$-adjunction of Theorem 1.2.8.

The first step amounts to defining a notion of mapping path space in the non-necessarily pointed case and dual to the mapping cylinder.

Definition 1.5.10 (Mapping path space). The *mapping path space* $\mathrm{Path}(f)$ of a continuous map $f\colon X \to Y$ is defined by the following pullback:

$$\begin{tikzcd}
\mathrm{Path}(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y^I \arrow[d, "e_0"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd} \qquad \text{that is } \mathrm{Path}(f) \cong X \underset{f}{\times} Y^I \cong \{(x, \varphi\colon I \to Y) \mid f(x) = \varphi(0)\} \ .$$



Any map $f\colon X \to Y$ factors through

$$\begin{tikzcd}
X \arrow[r, hook, "i"'] \arrow[rr, bend left, "f"] & \mathrm{Path}(f) \arrow[r, two heads, "p"'] & Y
\end{tikzcd} \ ,$$

where i is defined by $x \mapsto (x, \mathrm{cst}_{f(x)})$, the latter element stands for the constant map equal to $f(x)$, and where p is defined by $(x, \varphi) \mapsto \varphi(1)$. Considering the retraction $r\colon \mathrm{Path}(f) \to X$, given by $(x, \varphi) \mapsto x$, and the homotopy $H\colon \mathrm{Path}(f) \times I \to \mathrm{Path}(f)$, given by $(x, \varphi, t) \mapsto (x, s \mapsto \varphi(st))$, one can see that the map i is a strong deformation retract and thus a homotopy equivalence. Now one might want to go further and look for a universal property satisfied by the map p .

Definition 1.5.11 (Homotopy Lifting Property (HLP)). A map $p : E \to B$ satisfies the *homotopy lifting property (HLP)* with respect to a space Z if, for any pair of maps $g\colon Z \to E$ and $\breve{h}\colon Z \to B^I$, such that $p\, g = e_0\, \breve{h}$, there exists a map $\breve{H}\colon Z \to E^I$ such that the following diagram commutes.

$$\begin{tikzcd}
Z \arrow[drr, bend left, "g"] \arrow[dr, dotted, "\exists \breve{H}"] \arrow[ddr, bend right, "\breve{h}"'] & & \\
& E^I \arrow[r, "e_0"] \arrow[d, "p_*"] & E \arrow[d, "p"] \\
& B^I \arrow[r, "e_0"] & B
\end{tikzcd}$$

In this case, the data of the map $\breve{H}$ is in general not unique. We say that "the map $\breve{H}$ is a lifting of $\breve{h}$ with initial condition g" . Recall that a map $F\colon X \to E$ is said to *lift* a map $f\colon X \to B$ along a map $p\colon E \to B$ when $f = pF$.

$$\begin{tikzcd}
& E \arrow[d, "p"] \\
X \arrow[ur, "F"] \arrow[r, "f"] & B
\end{tikzcd}$$

Proposition 1.5.12. *The homotopy lifting property for $p\colon E \to B$ with respect to a space Z and for a pair of maps $g\colon Z \to E$ and $h\colon Z \to B^I$ is equivalent to the existence of a map $H\colon X \to Z^I$ factorising the following commutative diagram.*

$$\begin{tikzcd}
Z \arrow[r, "g"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\
Z \times I \arrow[r, "h"'] \arrow[ur, dotted, "\exists H"] & B
\end{tikzcd}$$

Proof. This is a direct corollary of the $(- \times I) \dashv (-)^I$-adjonction of Theorem 1.2.8. □

In this case, the map H lifts h along p, which explains the chosen terminology.

Definition 1.5.13 (Hurewicz and Serre fibrations). A continuous map $p : E \to B$ is a

- *(Hurewicz) fibration* if it satisfies the homotopy lifting property with respect to any space Z ,
- *Serre fibration* if it satisfies the homotopy lifting property with respect to any cube I^n, $n \geqslant 0$.

The first case is simply called a *fibration* and denoted by $p\colon E \twoheadrightarrow B$.

Obviously a fibration is a Serre fibration; this latter notion is enough to treat the case of homotopy groups, see Theorem 1.6.3.

Example. One can directly see from the definition that homeomorphisms, projections $B \times F \to B$, and constant maps $E \twoheadrightarrow \{*\}$ are fibrations. It is also straightforward to check that fibrations are stable under composition.

Theorem 1.5.14. *Any continuous map $f : X \to Y$ factors canonically through*

$$\begin{tikzcd}
X \arrow[r, hook, "\sim", "i"'] \arrow[rr, bend left, "f"] & \mathrm{Path}(f) \arrow[r, two heads, "p"'] & Y
\end{tikzcd} ,$$

where i is a homotopy equivalence and where p is a fibration.

Proof. It remains to prove that p is a fibration. Let us check directly the homotopy lifting property with respect to any space Z on the equivalent definition given in Proposition 1.5.12. So let $g : Z \to \mathrm{Path}(f)$

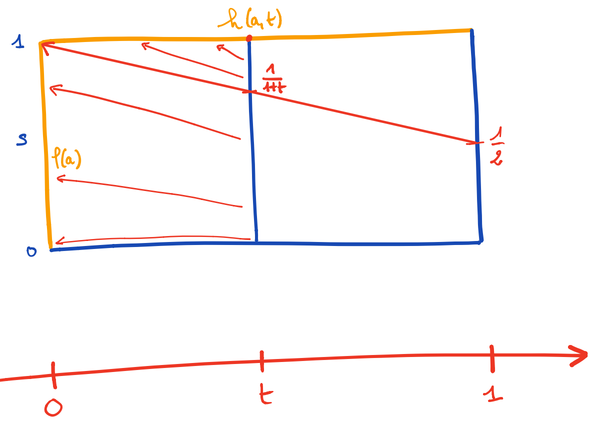
and $h : Z \times I \to Y$ be maps such that the following diagram commutes:

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "i_0"'] & \mathrm{Path}(f) \arrow[d, "p"] \\ Z \times I \arrow[r, "h"'] \arrow[ur, dotted, "H"] & Y . \end{tikzcd}$$

We write the data of the map g by $z \mapsto (x_z, \varphi_z)$, where $\varphi_z(0) = f(x_z)$. Under these notations, the commutativity of the above square amounts to $\varphi_z(1) = h(z, 0)$. This shows that the map $H : Z \times I \to \mathrm{Path}(f)$ given by $(z, t) \mapsto (x_{z,t}, \varphi_{z,t})$, where

$$x_{z,t} \coloneqq x_z ,$$
$$\varphi_{z,t}(s) \coloneqq \begin{cases} \varphi_z((1+t)s) , & \text{for } 0 \leqslant s \leqslant \frac{1}{1+t} , \\ h(z, (1+t)s - 1) , & \text{for } \frac{1}{1+t} \leqslant s \leqslant 1 , \end{cases}$$

is well defined and continuous.



(dessin à reprendre en changeant $a \mapsto z$.) The commutativity of the upper left triangle amounts to $x_{z,0} = x_z$ and $\varphi_{z,0} = \varphi_z$, and the commutativity of the lower right triangle amounts to $\varphi_{z,t}(1) = h(z, t)$. ☐

Remark. Theorem 1.5.14 shows that, up to homotopy, one can canonically replace any map by a fibration.

The above definition of a fibration can be simplified by looking only at the case of the test space provided by the mapping path space $Z = \mathrm{Path}(p)$:

$$\begin{tikzcd} \mathrm{Path}(p) \arrow[drr, bend left, "pr_1"] \arrow[ddr, bend right, "pr_2"'] \arrow[dr, dotted, "\exists \sigma"'] & & \\ & E^I \arrow[ul, "\exists ! \rho"'] \arrow[r, "e_0"] \arrow[d, "p_*"] & E \arrow[d, "p"] \\ & B^I \arrow[r, "e_0"] & B \end{tikzcd}$$

where $pr_1(x, \varphi) \coloneqq x$ and $pr_2(x, \varphi) \coloneqq \varphi$ stand for the respective projections. By the pullback property, there exists a unique map $\rho : E^I \to \mathrm{Path}(p)$ making the diagram commutative. When it is satisfied, the homotopy lifting property provides us with a section of it, that is a map $\sigma : \mathrm{Path}(p) \to E^I$ satisfying $\rho\sigma = \mathrm{id}_{\mathrm{Path}(p)}$.

Proposition 1.5.15. *Let $p : E \to B$ be a continuous map. The following assertions are equivalent.*

(1) *The map p is a fibration.*
(2) *The map p satisfies the homotopy lifting property with respect to its path space* $\mathrm{Path}(p)$.
(3) *The map $\rho : E^I \to \mathrm{Path}(p)$ admits a section.*

Proof. It remains to prove (3) $\Rightarrow$ (1) which can be archived using the dual form of the arguments given in the proof of Proposition 1.5.5. ☐

It is straightforward to see that a fibration $p\colon E \to B$ with B path connected is surjective. It is moreover a quotient map by [Sm68, Theorem 1]. This justifies *a posteriori* the notation $\twoheadrightarrow$ chosen for fibrations. This also shows that being a fibration is not a homotopy equivalent notion: the identity map of a contractible space is homotopy equivalent to a constant map, which is not a fibration since not surjective in general.

We do not have at hand an easy characterisation of fibrations as Proposition 1.5.7 for cofibrations. However, we will give below several ways to produce automatically fibrations and then two classical classes of maps (coverings and fibre bundles) which are known to be fibrations.

Proposition 1.5.16 (Base change). *For any fibration $p\colon E \twoheadrightarrow B$ and any map $f\colon A \to B$, the map $q\colon B \rightarrowtail X \cup_f B$ obtained by* base change *under the pullback is a fibration.*

$$\begin{tikzcd} A \times_f E \arrow[r] \arrow[d, two heads, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, two heads, "p"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Proof. The proof is completely dual to that of Proposition 1.5.9. □

Proposition 1.5.17.

(1) *Let $i : A \rightarrowtail X$ be a cofibration between locally compact spaces. The map $i^*\colon Z^X \twoheadrightarrow Z^A$ is a fibration, for any space Z.*

(2) *Let $p : E \twoheadrightarrow B$ be a fibration. The map $p_*\colon E^Z \twoheadrightarrow B^Z$ is fibration, for any locally compact space Z.*

Proof. The proof is left as an exercise. □

Example.

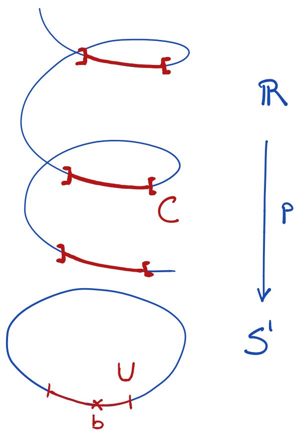
- Since $i\colon \partial I \rightarrowtail I$ is a cofibration between locally compact spaces, the map $X^I \twoheadrightarrow X^{\partial I} \cong X \times X$, given by $\varphi \mapsto (\varphi(0), \varphi(1))$, is a fibration.
- Similarly, the evaluation map $e_t\colon Y^I \twoheadrightarrow Y$, is a fibration, for any $t \in I$ and any Y locally compact, since the embedding $i_t\colon \{*\} \hookrightarrow I$ is a cofibration. For any map $f\colon X \to Y$, the projection map $pr_1\colon \mathrm{Path}(f) \twoheadrightarrow X$ onto the first component is a fibration by base change (Proposition 1.5.16).

$$\begin{tikzcd} \mathrm{Path}(f) \arrow[r] \arrow[d, two heads, "pr_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y^I \arrow[d, two heads, "e_0"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Definition 1.5.18 (Covering). A *covering* (*cover* or *covering space*) of a space B is a surjective map $p : E \twoheadrightarrow B$ satisfying the property: for all b in B, there exists an open neighbourhood U of b such that the restriction $p|_C : C \xrightarrow{\cong} U$ of p to any connected component C of $p^{-1}(U)$ is an homeomorphism.
In this case, E is called the *total space*, B the *base space*, and $F_b := p^{-1}b$ the *fibre* of the covering at b.

Remark. A covering is called a *revêtement* in French.

Example. The paradigm of coverings is the exponential map from the real line onto the circle:



$$\begin{aligned} \mathbb{R} &\longrightarrow S^1\,, \\ \theta &\longmapsto e^{2i\pi\theta}\,. \end{aligned}$$

In this case, the fibre is constant and equal to $\mathbb{Z}$.

Another classical covering is provided by the map $p\colon S^n \twoheadrightarrow P^n\mathbb{R}$ which sends a point $x \in S^n \subset \mathbb{R}^{n+1}$ to the line $[x] \in P^n\mathbb{R}$ supported by x. In this case, the fibre is constant and equal to $\mathbb{Z}/2\mathbb{Z}$ since the real projective space is homeomorphic to $P^n\mathbb{R} \cong S^n/\{\pm 1\}$.

The universal property satisfied by coverings is the following one.

Proposition 1.5.19 (Unique path lifting property). *Let $p\colon E \twoheadrightarrow B$ be a covering. For any path $\varphi\colon I \to B$ and for any $x \in E$ satisfying $\varphi(0) = p(x)$, there exists a unique path $\psi\colon I \to E$ lifting φ and starting at x, that is $p\psi = \varphi$ and $\psi(0) = x$.*

We will need the following point-set topological property.

Lemma 1.5.20 (Lebesgue's number lemma). *Let (X, d) be a metric compact space and let $\{U_i\}_{i \in \mathcal{J}}$ be an open cover of X. There exist a positive real number $\varepsilon > 0$ such that for any $x \in X$, there exists $i \in \mathcal{J}$ satisfying $B(x, \varepsilon) \subset U_i$.*

Proof of Lemma 1.5.20. The proof of this point-set topological property is left as an exercise. ◻

Proof of Proposition 1.5.19. Let $\varphi\colon I \to B$ be a path in B such that $\varphi(0) = p(x)$, with $x \in E$. For any point $t \in I$, we consider an open neighbourhood $U_{\varphi(t)}$ of $\varphi(t)$ in B satisfying the defining property of a covering. Since φ is continuous, $\left\{\varphi^{-1}\left(U_{\varphi(t)}\right)\right\}_{t\in I}$ is an open cover of the metric compact space I. Therefore, it admits a Lebesgue number $\varepsilon > 0$. We divide the interval $I = [0, 1]$ into $t_0 := 0 < t_1 < \cdots < t_{n-1} < t_n := 1$ such that $t_k - t_{k-1} < \varepsilon$, for any $1 \leqslant k \leqslant n$. It is then straightforward to prove by induction on k that there exists a unique map $\psi_k\colon [0, t_k] \to E$ satisfying $\psi_k(0) = x$ and $p\psi_k = \varphi|_{[0,t_k]}$. Indeed for $k = 0$, this is trivial. Suppose that the result holds for k and let us prove it for $k + 1$. We consider $s \in t$ such $B(t_k, \varepsilon) \subset \varphi^{-1}\left(U_{\varphi(s)}\right)$. Let C be the connected component of $p^{-1}\left(U_{\varphi(s)}\right)$ containing $\psi_k(t_k)$. So the restriction $p|_C\colon C \cong U_{\varphi(s)}$ is an homeomorphism. This proves that the following map

$$\psi_{k+1}(t) := \begin{cases} \psi_k(t)\,, & \text{for } 0 \leqslant t \leqslant t_k\,, \\ (p|_C)^{-1}\left(\varphi(t)\right), & \text{for } t_k \leqslant s \leqslant t_{k+1}\,, \end{cases}$$

defines a unique lifting. ◻

Theorem 1.5.21. *Any covering $p\colon E \to B$ is a fibration with a unique section $\sigma\colon \mathrm{Path}(p) \to E^I$.*

Proof. Proposition 1.5.19 shows that Point (3) of Proposition 1.5.15 is satisfied by a unique section σ of the map $\rho\colon E^I \to \mathrm{Path}(p)$. ◻

Notice that each fiber of a covering is discrete. In the next case, we will relax this assumption and require instead the fibre to be constant.

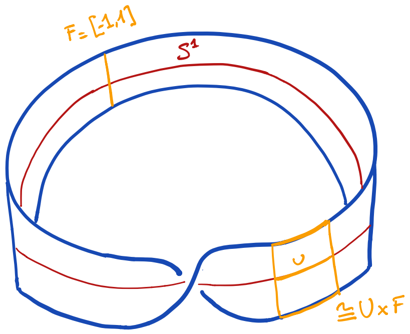
Definition 1.5.22 (Fibre bundle). A *fibre bundle* structure on a space E *with fibre* F is a surjective map $p\colon E \twoheadrightarrow B$ such that for all b in B, there exist an open neighbourhood U of b and an homeomorphism $h\colon p^{-1}(U) \xrightarrow{\cong} U \times F$ satisfying the following commutative diagram:

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\cong", "h"'] \arrow[dr, "p"'] & & U \times F \arrow[dl, "\mathrm{proj}"] \\ & U & \end{tikzcd}.$$

Each homeomorphism h is called a *local trivialisation*. Fiber bundles are often denoted simply by $F \to E \to B$.

Products $B \times F \twoheadrightarrow B$ are trivial fibre bundle; their local trivialisation maps are equal to the identity on $U \times F$.

Example. The first non-trivial example of a fibre bundle is given by the Möbius strip



$$[-1,1] \longrightarrow \frac{I \times [-1,1]}{(0,v) \sim (1,-v)} \longrightarrow S^1 \ .$$

Notice that any fibre bundle with a discrete fibre F is a covering. In the other way round, a covering with fibres of same cardinality is a fibre bundle; this is for instance the case when B is connected.

The next result shows that these two classes of maps provide us with fibrations. Let us recall that a space is *paracompact* when every open cover admits a locally finite refinement: at every point, there exists an open neighbourhood that intersects only of finite numbers of its elements. Compact spaces and CW-complexes are paracompact.

Theorem 1.5.23. *Any fibre bundle $f\colon E \twoheadrightarrow B$ with a paracompact base B is a fibration.*

Proof. The non-trivial proof of this theorem relies on point-set topology; it is thus skipped here. We refer the reader to [May99, Chapter 7, Section 4] for full details. ◻

The relationship between these various notions is summarized into the following table.

$$\boxed{\text{covering} \underset{\substack{\text{discrete}\\ \text{fibre}}}{\subset} \text{fibre bundle} \underset{\substack{\text{constant}\\ \text{fibre}}}{\subset} \text{fibration} \underset{\substack{\text{HLP}\\ \text{wrt } I^n}}{\subset} \text{Serre fibration}}$$

This raises the question of the shape of the fibres in any fibration.

Proposition 1.5.24. *For any fibration $p\colon E \twoheadrightarrow B$, the fibers $p^{-1}(b)$ over each path component are all homotopy equivalent.*

Proof. [Hat02, Proposition 4.61]. TBC ◻

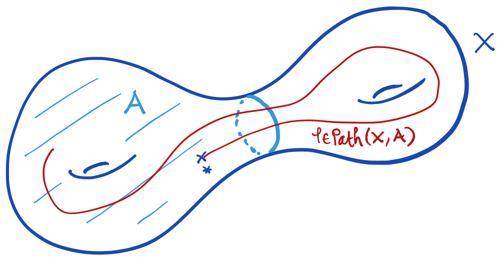
1.6. **Computations of homotopy groups.** The homotopy groups are paradoxal objects as they are easily defined, see Section 1.3.2, but hardly computable. So this makes them exciting objects of study. In this section, we apply the preceeding notions of a fibration and fiber sequence in order to get a powerful tool to compute some of them.

The fiber sequence of Theorem 1.4.7 associated to the embedding $i\colon A \hookrightarrow X$ of a pair $(A,*) \subset (X,*)$ of pointed spaces provides us with the following long h-exact sequence:

$$\cdots \longrightarrow \Omega^2\mathrm{Path}(i) \longrightarrow \Omega^2 A \xrightarrow{\Omega^2 i} \Omega^2 X \longrightarrow \Omega\mathrm{Path}(i) \longrightarrow \Omega A \xrightarrow{\Omega i} \Omega X \longrightarrow \mathrm{Path}(i) \longrightarrow A \xrightarrow{i} X \ .$$

In this case, the path space of i is equal to

$$\mathrm{Path}(X,A) := \mathrm{Path}(i) = \left\{\varphi \in X^I \mid \varphi(0) = *,\ \varphi(1) \in A\right\} \ .$$



Definition 1.6.1 (Relative homotopy groups). For any $n \geqslant 1$, we consider

$$\pi_n(X,A) := \pi_{n-1}(\mathrm{Path}(X,A)) \cong \pi_0\left(\Omega^{n-1}\mathrm{Path}(X,A)\right) \ ,$$

with their group structure, for $n = 2$, and their abelian group structure, for $n \geqslant 3$. They are called the *relative homotopy groups of a pair* (X,A).

Remark. When $A = \{*\}$ is made up of the sole base point, the definition coincides with the classical one $\pi_n(X, \{*\}) \cong \pi_n(X, *)$ as $\mathrm{Path}(X, \{*\}) = \Omega X$.

Theorem 1.6.2 (Long exact sequence associated to a pair). *For any pair $* \in A \subset X$ of pointed topological spaces, there exists a long exact sequence:*

$$\cdots \longrightarrow \pi_2(A) \xrightarrow{\pi_2(i)} \pi_2(X) \longrightarrow \pi_2(X, A) \longrightarrow \pi_1(A) \xrightarrow{\pi_1(i)} \pi_1(X) \longrightarrow \pi_1(X, A) \longrightarrow \pi_0(A) \xrightarrow{\pi_0(i)} \pi_0(X) .$$

Proof. This long exact sequence is obtained from the aforementioned h-exact sequence by applying the functor $[S^0, -]_*$. □

The long exact sequence of Theorem 1.6.2 admits the following more amenable form.

Theorem 1.6.3 (Long exact sequence associated to a fibration). *Let $p : E \twoheadrightarrow B$ be a Serre fibration, with (B, b_0), a path connected pointed space. Denoting by $F := p^{-1}(b_0)$ the fibre of p, there exists a long exact sequence:*

$$\cdots \longrightarrow \pi_2(F) \longrightarrow \pi_2(E) \longrightarrow \pi_2(B) \longrightarrow \pi_1(F) \longrightarrow \pi_1(E) \longrightarrow \pi_1(B) \longrightarrow \pi_0(F) \longrightarrow \pi_0(E) \longrightarrow \{*\} .$$

Remark. Notice the similarity with the long exact sequence of homology groups associated to a short exact sequence of chain complexes: the Serre fibration $F \to E \twoheadrightarrow B$ plays the role of the short exact sequence of topological spaces which induces a long exact sequence of homotopy groups this time.

Proof. TBC □

Corollary 1.6.4. *Any pointed covering $p\colon E \twoheadrightarrow B$ induces isomorphisms $\pi_n(p)\colon \pi_n(E) \cong \pi_n(B)$, for $n \geqslant 2$.*

Proof. This is a direct corollary of Theorem 1.5.21 and Theorem 1.6.3 as the long exact sequence is equal to

$$\cdots \longrightarrow \pi_3(F) \cong 0 \longrightarrow \pi_3(E) \xrightarrow{\cong} \pi_3(B) \longrightarrow \pi_2(F) \cong 0 \longrightarrow \pi_2(E) \xrightarrow{\cong} \pi_2(B) \longrightarrow \pi_1(F) \cong 0 \longrightarrow \ .$$

□

Proposition 1.6.5. *We have $\pi_n(S^1) \cong 0$, for any $n \geqslant 2$.*

Proof. This is proved using the covering $\mathbb{R} \twoheadrightarrow S^1$ given by $\theta \mapsto e^{i\theta}$, where $\mathbb{R}$ is contractible. So by Corollary 1.6.4, we get $\pi_n(S^1) \cong \pi_n(\mathbb{R}) \cong 0$, for $n \geqslant 2$. □

One can endow coverings over a fixed base B with a category structure where morphisms from $p\colon E \twoheadrightarrow B$ to $p'\colon E' \twoheadrightarrow B$ are are continuous map $f\colon E \to E'$ such that $p'f = p$:

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & B & \end{tikzcd} .$$

In this context, we consider the group $\mathrm{Aut}(p)$ of automorphisms of a covering p.

Theorem 1.6.6. *Let $p\colon E \twoheadrightarrow B$ be a pointed covering from a simply connected and locally path connected space E. The fibre $F = p^{-1}(*)$ is in bijection with $\mathrm{Aut}(p)$ and*

$$\pi_1(B) \cong \mathrm{Aut}(p) .$$

Proof. [GH81, Theorem 5.8] TBC □

Corollary 1.6.7. *The fundamental group of S^1 is isomorphic to $\mathbb{Z}$:*

$$\pi_1(S^1) \cong \mathbb{Z} .$$

Proof. This is a direct corollary of Theorem 1.6.6 applied to the covering $p\colon \mathbb{R} \twoheadrightarrow S^1$. The fiber is $\mathbb{Z}$ and any automorphism of p is of the form $\mathbb{R} \to \mathbb{R}$, $x \mapsto x + 2\pi k$, with $k \in \mathbb{Z}$. □

Proposition 1.6.8. *We have $\pi_n(S^d) \cong 0$, for any $n < d$.*

Proof. TBC □

To go further, we need to consider more elaborate fibrations.

Definition 1.6.9 (Complex Hopf fibration). For any $d \geqslant 1$, the *complex Hopf fibration* is the fiber bundle

$$S^1 \to S^{2d+1} \to \mathbb{P}^d\mathbb{C}$$

defined by $(x_0, y_0, \ldots, x_d, y_d) \mapsto [x_0 + iy_0 : \cdots : x_d + iy_d]$, for any $d \geqslant 1$.

The classical Hopf fibration coincides to the case $d = 1$:

$$S^1 \to S^3 \to S^2 \cong \mathbb{P}^1\mathbb{C} \ .$$

Proposition 1.6.10. *For any $d \geqslant 1$, we have*

$$\begin{aligned}
&\pi_1(\mathbb{P}^d\mathbb{C}) \cong 0 \ , \ \pi_2(\mathbb{P}^d\mathbb{C}) \cong \mathbb{Z} \ , \\
&\pi_n(\mathbb{P}^d\mathbb{C}) \cong 0 \ , \ \textit{for } 3 \leqslant n \leqslant 2d \ , \\
&\pi_n(\mathbb{P}^d\mathbb{C}) \cong \pi_n(S^{2d+1}) \ , \ \textit{for } n \geqslant 2d+1 \ .
\end{aligned}$$

Proof. By the preceding results, the long exact sequence of Theorem 1.6.3 associated to the complex Hopf fibration is equal to

$$\cdots \to \pi_{2d+3}(S^{2d+1}) \xrightarrow{\cong} \pi_{2d+3}(\mathbb{P}^d\mathbb{C}) \to \underbrace{\pi_{2d+2}(S^1)}_{0} \to \pi_{2d+2}(S^{2d+1}) \xrightarrow{\cong} \pi_{2d+2}(\mathbb{P}^d\mathbb{C}) \to \underbrace{\pi_{2d+1}(S^1)}_{0} \longrightarrow$$

$$\longrightarrow \pi_{2d+1}(S^{2d+1}) \xrightarrow{\cong} \pi_{2d+1}(\mathbb{P}^d\mathbb{C}) \longrightarrow \underbrace{\pi_{2d}(S^1)}_{0} \longrightarrow \underbrace{\pi_{2d}(S^{2d+1})}_{0} \xrightarrow{\cong} \pi_{2d}(\mathbb{P}^d\mathbb{C}) \longrightarrow \underbrace{\pi_{2d-1}(S^1)}_{0} \longrightarrow \cdots$$

$$\cdots \longrightarrow \underbrace{\pi_2(S^1)}_{0} \longrightarrow \underbrace{\pi_2(S^{2d+1})}_{0} \longrightarrow \pi_2(\mathbb{P}^d\mathbb{C}) \xrightarrow{\cong} \underbrace{\pi_1(S^1)}_{\mathbb{Z}} \longrightarrow \underbrace{\pi_1(S^{2d+1})}_{0} \longrightarrow \pi_1(\mathbb{P}^d\mathbb{C}) \longrightarrow \{*\} \ ,$$

where the isomorphism on the level of the fundamental group is given by Theorem 1.6.6. ☐

Corollary 1.6.11. *We have $\pi_2(S^2) \cong \mathbb{Z}$ and $\pi_n(S^2) \cong \pi_n(S^3)$, for $n \geqslant 3$.*

Proof. This is the direct application of Proposition 1.6.10 to the case $d = 1$. ☐

The complex Hopf fibration admits the following real analogue.

Definition 1.6.12 (Real Hopf fibration). For any $d \geqslant 1$, the *real Hopf fibration* is the covering

$$\mathbb{Z}/2\mathbb{Z} \to S^d \to \mathbb{P}^d\mathbb{R}$$

defined by $(x_1, \ldots, x_d) \mapsto [x_1 : \cdots : x_d]$, for any $d \geqslant 1$.

Proposition 1.6.13. *For any $d \geqslant 1$, we have*

$$\begin{aligned}
&\pi_1(\mathbb{P}^d\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z} \ , \\
&\pi_n(\mathbb{P}^d\mathbb{R}) \cong 0 \ , \ \textit{for } 2 \leqslant n \leqslant d-1 \ , \\
&\pi_n(\mathbb{P}^d\mathbb{R}) \cong \pi_n(S^d) \ , \ \textit{for } n \geqslant d \ .
\end{aligned}$$

Proof. By the preceding results, the long exact sequence of Theorem 1.6.3 associated to the real Hopf fibration is equal to

$$\cdots \to \pi_{d+2}(S^d) \xrightarrow{\cong} \pi_{d+2}(\mathbb{P}^d\mathbb{R}) \to 0 \to \pi_{d+1}(S^d) \xrightarrow{\cong} \pi_{d+1}(\mathbb{P}^d\mathbb{R}) \longrightarrow 0 \longrightarrow$$

$$\longrightarrow \pi_d(S^d) \xrightarrow{\cong} \pi_d(\mathbb{P}^d\mathbb{R}) \longrightarrow 0 \longrightarrow 0 \longrightarrow \pi_{d-1}(\mathbb{P}^d\mathbb{R}) \longrightarrow 0 \longrightarrow \cdots$$

$$\cdots \longrightarrow 0 \longrightarrow \pi_2(\mathbb{P}^d\mathbb{R}) \longrightarrow 0 \longrightarrow 0 \longrightarrow \pi_1(\mathbb{P}^d\mathbb{R}) \xrightarrow{\cong} \mathbb{Z}/2\mathbb{Z} \to \{*\} \ .$$

☐

Since the 1-dimensional real projective space $\mathbb{P}^1\mathbb{R} \cong S^1$ is homeomorphic to the circle, it shares with it the same homotopy groups.

Theorem 1.6.14. *For any $n \geqslant 1$, we have $\pi_n(S^n) \cong \pi_n(\mathbb{P}^n\mathbb{R}) \cong \pi_{2n+1}(\mathbb{P}^n\mathbb{C}) \cong \mathbb{Z}$.*

Proof. TBC.
The last two isomorphisms come from Proposition 1.6.10 and Proposition 1.6.13 respectively. ☐

Here is a first "concrete" application of these computations of homotopy groups.

Corollary 1.6.15. *For $0 < k < n$, the embedding $\mathbb{P}^k\mathbb{R} \hookrightarrow \mathbb{P}^n\mathbb{R}$ does not admit a retraction.*

Proof. Let us denote by i the embedding and let us suppose that it admits a retraction $r\colon \mathbb{P}^n\mathbb{R} \twoheadrightarrow \mathbb{P}^k\mathbb{R}$. This implies that $\pi_k(r)\circ\pi_k(i) = \mathrm{id}_{\pi_k(\mathbb{P}^k\mathbb{R})}$. So $\pi_k(i)\colon \pi_k\left(\mathbb{P}^k\mathbb{R}\right) \cong \mathbb{Z} \hookrightarrow \pi_k\left(\mathbb{P}^n\mathbb{R}\right) \cong 0$ is injective, which is impossible by Proposition 1.6.13 and Theorem 1.6.14. □

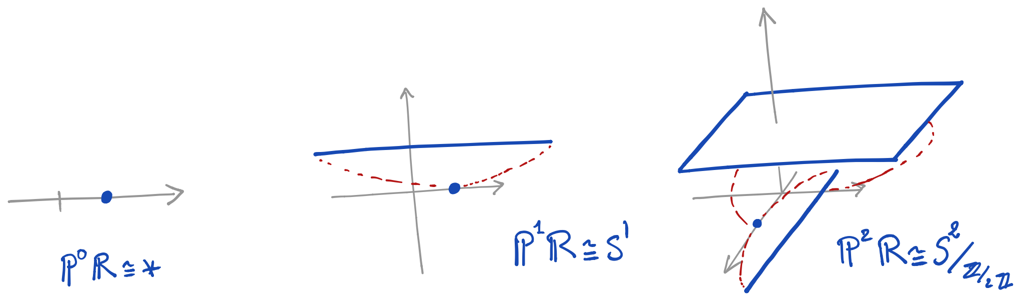
1.7. **Cell complexes**. So far we have been studying topological spaces from top to bottom, that is we considered all of them, established general properties, and ultimately looked at some examples. Let us now take the dual bottom-to-top approach and consider the particular case of topological spaces which are obtained by consecutive cells gluing. Historical, one of the first motivating example is the Euler characteristic (1752): for all polygonal decomposition of the sphere S^2, the alternating sum of the number of vertices, edges, and polygons is invariantly equal to

$$\chi\left(S^2\right) = \eta(0) - \eta(1) + \eta(2) = 2.$$

In this section, we will focus of the type of such cell complexes given by *CW-complexes*. There is actually no loss of generality as any space can be approximated by a CW-complex. However their homotopy theory is much more simple since, in some sense, their homotopy groups provide us with a complete collection of invariants.

1.7.1. *Definitions.* The following key example will pave the way to a precise definition for "consecutive cells gluing".

Example (Toy model). Let us consider the real projective spaces, which are defined as the sets $\mathbb{P}^n\mathbb{R} = \mathbb{P}\mathbb{R}^{n+1}$ of lines of $\mathbb{R}^{n+1}$, for any $n \geqslant 0$. Since there are equivalently defined as the cosets of non-zero points of $\mathbb{R}^{n+1}$ under the action of $\mathbb{R}\backslash\{0\}$, its is equipped with the coset topology. So, one can see that $\mathbb{P}^0\mathbb{R}$ is a point, that $\mathbb{P}^1\mathbb{R}$ is a line glued to this point at each extremities forming a circle, that $\mathbb{P}^2\mathbb{R}$ is a plane glued along this circle, etc.



Definition 1.7.1 (CW complex). A *CW complex* is a topological space X equipped with an homeomorphism to a colimit

$$\varnothing =: X^{(-1)} \subset X^{(0)} \subset \cdots \subset X^{(n)} \subset \cdots \bigcup_{n\in\mathbb{N}} X^{(n)} = \operatorname*{colim}_{n\in\mathbb{N}} X^{(n)} \cong X\ ,$$

where

$$\begin{tikzcd}
\coprod_{J_n} S^{n-1} = J_n \times S^{n-1} \arrow[rr, "\varphi_n"] \arrow[d, hook] & & X^{(n-1)} \arrow[d] \\
\coprod_{J_n} D^n = J_n \times D^n \arrow[r, "\Phi_n"] & X^{(n)} = X^{(n-1)} \coprod_{\varphi_n} (J_n \times D^n) \cong \dfrac{X^{(n-1)} \coprod (J_n \times D^n)}{\varphi_n(j,x) \sim (j,x)} &
\end{tikzcd} ,$$

with discrete spaces J_n, for any $n \in \mathbb{N}$. The maps φ_n are called the *attaching maps* and the maps Φ_n are called the *characteristic maps*. The sub-space $X^{(n)}$ is called the *n-skeleton* of X. A CW-complex is *finite dimensional* if $X = X^{(n)}$ for some $n \geqslant 0$, and it is *finite* when $\coprod_{n\in\mathbb{N}} J_n$ is finite.

Remark. For infinite dimensional CW-complexes, we consider the *colimit topology* (also called *weak topology*) where any set $U \subset X$ is open if $U \cap X^{(n)}$ is open for any n in $\mathbb{N}$. The underling idea of this topology is to test the properties on all components $X^{(n)}$: a map $f\colon X \to Y$ is continuous if and only if all its restrictions $f|_{X^{(n)}}\colon X^{(n)} \to Y$ are continuous, for $n \in \mathbb{N}$. In this topology, one often only needs to "approximate X with $X^{(n)}$ for n large enough": a map $f\colon K \to X$ from a compact space K to X is continuous if and only if $f\colon K \to X^{(n)} \subset X$ is continuous for some n.

Let us unfold this definition a little bit. One starts from a set J_0 indexing the base points

$$\begin{tikzcd} \varnothing \arrow[r, "\varphi_0"] \arrow[d] & X^{(-1)} = \varnothing \arrow[d] \\ J_0 \times \{*\} \arrow[r, "\Phi_0"] \arrow[ur, phantom, "\ulcorner"] & X^{(0)} = J_0 \times \{*\} \ , \end{tikzcd}$$

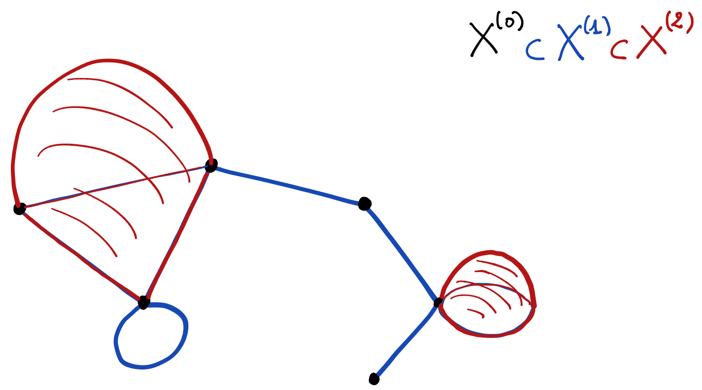
as $S^{-1} = \emptyset$ by convention. Then one glues the endpoints of intervals indexed by J_1 on them:

$$\begin{tikzcd} J_1 \times \{0,1\} \arrow[r, "\varphi_1"] \arrow[d, hook] & X^{(0)} \arrow[d] \\ J_1 \times I \arrow[r, "\Phi_1"] \arrow[ur, phantom, "\ulcorner"] & X^{(1)} \cong \dfrac{X^{(0)} \coprod (J_1 \times I)}{\varphi_1(j,0) \sim (j,0),\ \varphi_1(j,1) \sim (j,1)} \ , \end{tikzcd}$$

as $D^1 = I$ and $S^0 = \partial D^1 = \{0,1\}$. This produces a graph $X^{(1)}$ on which one glues J_2 copies of disks D^2 along their boundary under the attaching map φ_2:

$$\begin{tikzcd} J_2 \times S^1 \arrow[r, "\varphi_2"] \arrow[d, hook] & X^{(1)} \arrow[d] \\ J_2 \times D^2 \arrow[r, "\Phi_2"] \arrow[ur, phantom, "\ulcorner"] & X^{(2)} \cong \dfrac{X^{(1)} \coprod (J_2 \times D^2)}{\varphi_2(j,x) \sim (j,0)} \ , \end{tikzcd}$$

And so on, and so forth ...



Example.

- Spheres: For any $n \geqslant 1$, the n-dimensional sphere is a finite CW-complex $S^n \cong \{*\} \amalg_f D^n$ made up of two cells, a point and an n-dimensional disk D^n, with attaching map $f : S^{n-1} = \partial D^n \to \{*\}$. (Notice that we have already seen this model previously when we considered $S^n \cong D^n / \partial D^n \cong I^n / \partial I^n$.)
- Graphs: The definition of a graph is a 1-dimensional CW-complex. It is a tree when it is simply connected.
- Real projective spaces: For any $d \geqslant 0$, the d-dimensional real projective space is a finite CW-complex made up of one cell in each dimension from 0 to d:

$$\mathbb{P}^d\mathbb{R} \cong D^0 \amalg_{\varphi_1} D^1 \amalg_{\varphi_2} \cdots \amalg_{\varphi_d} D^d \ .$$

  The infinite dimension real projective space is the colimit of the finite dimensional ones: $\mathbb{P}^\infty\mathbb{R} := \operatorname{colim}_{n\in\mathbb{N}} \mathbb{P}^n\mathbb{R}$.
- Complex projective spaces: For any $d \geqslant 0$, the $2d{+}2$-dimensional complex projective space is a finite CW-complex made up of one cell in each dimension from 0 to d:

$$\mathbb{P}^d\mathbb{C} \cong D^0 \amalg_{\varphi_2} D^2 \amalg_{\varphi_4} \cdots \amalg_{\varphi_{2d+2}} D^{2d+2} \ .$$

where the complex Hopf fibration $\varphi_{2n+2}\colon S^{2n+1} \twoheadrightarrow \mathbb{P}^n\mathbb{C}$ is the attaching map. The infinite dimension complex projective space is the colimit of the finite dimensional ones: $\mathbb{P}^\infty\mathbb{C} := \operatorname{colim}_{n\in\mathbb{N}} \mathbb{P}^n\mathbb{C}$.

- ⋄ TORUS: The torus $S^1 \times S^1$ is a finite CW-complex made up of one vertex, two 1-cells, and one 2-cells.

REMARK. The terminology "CW-complex" comes from "C" for "Closed finiteness", as the boundary of any cell intersects a finite number of other cells, and "W" for "Weak topology".

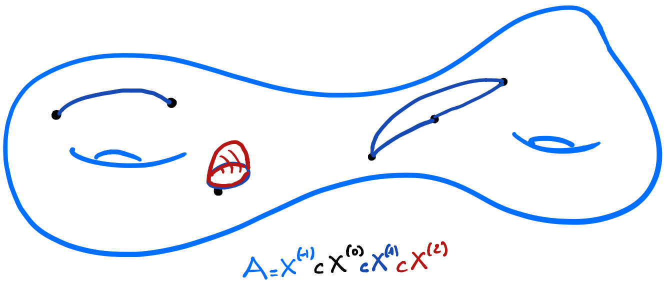
Proposition 1.7.2. *Let X, Y be two CW complexes such that either X or Y are locally compact either X and Y have both a countable number of cells. In this case, the product $X \times Y$ is a CW complex.*

Proof. exercise □

Definition 1.7.3 (Relative CW complex)**.** A pair (X, A) of topological spaces is a *relative CW complex* if X is homeomorphic to a colimit

$$A =: X^{(-1)} \subset X^{(0)} \subset \cdots \subset X^{(n)} \subset \cdots \bigcup_{n\in\mathbb{N}} X^{(n)} = \operatorname*{colim}_{n\in\mathbb{N}} X^{(n)} \cong X\ ,$$

where each embedding $X^{(n)} \hookrightarrow X^{(n+1)}$ is obtained as in the definition of a CW complex.



revoir la figure : à la première étape, on ajoute des points de manière disjointe. This means that instead of starting from the empty set and considering points, intervals, etc., one starts here from a given topological space A on which one puts points, attach interval, etc. A relative CW complex of the form $(X, \varnothing)$ is a CW complex. X.

Definition 1.7.4 (CW subcomplex and CW pair)**.** A subspace $A \subset X$ of a CW complex X is a *CW subcomplex* if it is obtained by the restrictions of the attaching maps of X to subcollections $K_n \times S^{n-1}$, with $K_n \subset J_n$, for any $n \geqslant 0$. A *CW pair* is a pair (X, A) where A is a CW subcomplex of X.

REMARK. A CW pair (X, A) is also a relative CW complex, by the reverse does not hold true.

EXERCISE. Let (X, A) be a CW pair. Show that X/A is a CW complex.

Proposition 1.7.5. *For any relative CW complex (X, A), the embedding $A \rightarrowtail X$ is a cofibration.*

Proof. We saw in Section 1.5.1 that the embedding $S^{n-1} \hookrightarrow D^n$ is a cofibration. Since cofibrations are stable under coproducts (straightforward) and cobase change (Proposition 1.5.9), every map $X^{(n-1)} \rightarrowtail X^{(n)}$ is a cofibration, for $n \geqslant 0$.

$$\begin{tikzcd}
\coprod_{J_n} S^{n-1} \arrow[r, "\varphi_n"] \arrow[d, tail] & X^{(n-1)} \arrow[d, tail] \\
\coprod_{J_n} D^n \arrow[r, "\Phi_n"] \arrow[ur, phantom, "\ulcorner"] & X^{(n)}
\end{tikzcd}$$

Since cofibrations are stable under composition, every map $A \rightarrowtail X^{(n)}$ is a cofibration, for $n \geqslant 0$. The arguments of the proof of the stablity under composition pass to the colimit, so $A \rightarrowtail X$ is a cofibration.

$$\begin{tikzcd}
A \arrow[r, "\breve{h}"] \arrow[d, tail] \arrow[dddd, bend right] & Z^I \arrow[dddd, "e_0"] \\
X^{(0)} \arrow[d, tail] \arrow[ur, dotted] \arrow[ddd, bend right] & \\
X^{(1)} \arrow[d, tail] \arrow[uur, dotted] \arrow[dd, bend right] & \\
\vdots & \\
X \arrow[r, "g"'] \arrow[uuuur, dashed, "\exists \breve{H}"] & Z
\end{tikzcd}$$

□

Definition 1.7.6 (Euler characteristic). The *Euler characteristic* of a finite CW complex X is defined by the alternating sum

$$\chi(X) := \sum_{n \in \mathbb{N}} (-1)^n |J_n| \ .$$

Nothing guarantees *a priori* that two finite CW decomposition of a space X will carry the same Euler characteristic.

Proposition 1.7.7. *The Euler characteristic is well defined and homotopy invariant.*

Proof. The most natural proof of these two facts relies on the cellular homology, see [tD08, Section 12.4] for instance. □

Considering continuous maps between CW complexes, one gets a full subcategory of topological spaces. It is however desirable to consider maps which respect the cellular structures.

Definition 1.7.8 (Cellular map). A continuous map $f : X \to Y$ between CW complexes is *cellular* if it satisfises

$$f\left(X^{(n)}\right) \subset Y^{(n)} \ ,$$

for all $n \in \mathbb{N}$.

Proposition 1.7.9. *For any cellular map $f : X \to Y$ between two CW complexes, the factorisation*

$$\begin{tikzcd}
X \arrow[r, tail, "i_1"'] \arrow[rr, bend left, "f"] & \mathrm{Cyl}(f) \arrow[r, two heads, "\sim", "p"'] & Y
\end{tikzcd}$$

considered in Section 1.5.1 is made up of cellular maps.

Proof. The point first amounts to endowing the cylinder construction $\mathrm{Cyl}(f)$ with a CW complex structure. TBC:exercise □

1.7.2. *Whitehead theorem.* We have seen in Proposition 1.3.6 that the notion of homotopy groups is homotopy invariant, that is, if $f : X \to Y$ is a homotopy equivalence, then $\pi_n(f) : \pi_n(X) \cong \pi_n(Y)$ is an isomorphism, for any $n \geqslant 0$. One can ask the reverse question, that is consider the continuous maps which induces isomorphisms on the level of the homotopy groups. Since homotopy groups detects a huge amount of the homotopy type of spaces, they should be of particular interest.

Definition 1.7.10 (Weak homotopy equivalence). A continuous map $f : X \to Y$ is a *weak homotopy equivalence* when the maps

$$\pi_n(f) : \pi_n(X, x) \xrightarrow{\cong} \pi_n(Y, (f(x))$$

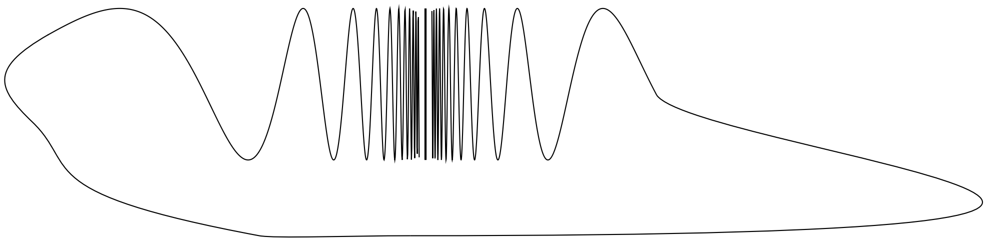
are bijective for all $n \geqslant 0$ and $x \in X$. We denote them by $f : X \xrightarrow{\sim_w} Y$.

When it is the case, we have isomorphisms of groups for $n \geqslant 1$ and a bijection between the respective sets of connected components.

So a homotopy equivalence is a weak homotopy equivalence. The reverse is not true in general: let us consider the Warsaw circle defined by

$$W \coloneqq \left\{\left(x, \sin\left(\tfrac{1}{x}\right)\right), -\tfrac{1}{2\pi} \leqslant x \leqslant \tfrac{1}{2\pi}, x \neq 0\right\} \cup \{0\} \times [-1, 1] \cup C\ ,$$

where C is a continuous arc from $\left(-\frac{1}{2\pi}, 0\right)$ to $\left(\frac{1}{2\pi}, 0\right)$ disjoint from the other sets.



It is connected but has two path components. The map $f\colon \{a, b\} \to W$ which sends a to point in $\{0\} \times [-1, 1]$ and b to a point in C is a weak homotopy equivalence. But it is not a homotopy equivalence since this would imply the existence of a map $g\colon W \to \{a, b\}$ such that fg is homotopic to the identity of W, which is impossible.

However the reverse holds true for CW-complexes.

Theorem 1.7.11 (Whitehead). *Let X and Y be two CW complexes. Any map $f\colon X \to Y$ is an homotopy equivalence if and only if it is a weak homotopy equivalence.*

This theorem shows that the Warsaw circle cannot admit any CW complex structure.

Remark. Be careful that the data of a map $f\colon X \to Y$ is mandatory. Consider for instance the two CW complexes $X \coloneqq S^2 \times \mathbb{P}^3\mathbb{R}$ and $Y \coloneqq \mathbb{P}^2\mathbb{R} \times S^3$. They are both path-connected and they share similar coverings

$$\mathbb{Z}/2\mathbb{Z} \to S^2 \times S^3 \to S^2 \times \mathbb{P}^3\mathbb{R} \quad \text{and} \quad \mathbb{Z}/2\mathbb{Z} \to S^2 \times S^3 \to \mathbb{P}^2\mathbb{R} \times S^3\ .$$

So their homotopy groups are isomorphic. But there cannot exist any homotopy equivalence between since their homology groups are different, $H_5\left(S^2 \times \mathbb{P}^3\mathbb{R}\right) \ncong H_5\left(\mathbb{P}^2\mathbb{R} \times S^3 \times\right)$ for instance. Whitehead theorem implies that there is no map $S^2 \times \mathbb{P}^3\mathbb{R} \to \mathbb{P}^2\mathbb{R} \times S^3$ which realises the isomorphisms between their homotopy groups.

The key ingredient in the proof of Whitehead theorem is the following lemma.

Lemma 1.7.12 (Compression). *Let (X, A) be a relative CW complex and (Y, B) be a pair of topological spaces, with $B \neq \varnothing$, such that, for any $n \geqslant 1$ satisfying $X^{(n-1)} \subsetneq X^{(n)}$, then $\pi_n(Y, B) \cong 0$. Any continuous map $f\colon (X, A) \to (Y, B)$ of pairs of spaces is homotopic relative to A to a map $g\colon X \to B$.*

Proof. Let us denote by $\{n_k\}_{k\in\mathbb{N}}$ the increasing sequence of the dimensions of the cells of X, that is $J_n = \varnothing$ if and only if $n \notin \{n_k\}_{k\in\mathbb{N}}$. By convention, we set $n_{-1} \coloneqq -1$ and $f^{(-1)} \coloneqq f$. Let us prove, by induction on $k \in \mathbb{N}$, i.e. on the dimension of the skeleton of X, that there exists a map $f^{(k)}\colon X \to Y$ homotopic to $f^{(k-1)}$ relative to $X^{(n_{k-1})}$ and whose restriction to $X^{(n_k)}$ has image in B, that is $f^{(k)}\left(X^{(n_k)}\right) \subset B$. We will denote such a homotopy by $H^{(k)}$.

We initiate this induction with the case n_0, which is the smallest dimension of cells in X. By definition, the following diagram commutes.

$$\begin{tikzcd}
J_{n_0} \times S^{n_0-1} \arrow[r, "\varphi_{n_0}"] \arrow[d, tail] & A \arrow[r, "f"] \arrow[d, tail] & B \arrow[d, hook] \\
J_{n_0} \times D^{n_0} \arrow[r, "\Phi_{n_0}"] \arrow[ur, phantom, "\ulcorner", very near end] & X^{(n_0)} \arrow[r, "f"] & Y
\end{tikzcd}$$

So the composite $f\Phi_{n_0}$ is a map of pairs $\left(J_{n_0} \times D^{n_0}, J_{n_0} \times S^{n_0-1}\right) \to (Y, B)$, which is equivalent to a collection $\left\{\left(D^{n_0}, S^{n_0-1}\right) \to (Y, B)\right\}_{J_{n_0}}$ of maps of pairs. By assumption, the relative homotopy group $\pi_{n_0}(Y, B) \cong 0$ is trivial; recall that it is equivalent to

$$\pi_{n_0}(Y, B) \cong [(I^{n_0}, \partial I^{n_0}), (Y, B)] \cong \left[\left(D^{n_0}, S^{n_0-1}\right), (Y, B)\right]\ .$$

This implies that the composite $f\Phi_{n_0}$ is homotopic to a map $\psi\colon J_{n_0} \times D^{n_0} \to B \subset Y$ relative to $J_{n_0} \times S^{n_0-1}$. Let us denote this homotopy by $\tilde{h}\colon J_{n_0} \times D^{n_0} \times I \to Y$, so that $\tilde{h}(-,0) = f\Phi_{n_0}$ and $\tilde{h}(-,1) = \psi$. We consider the following diagram

$$
\begin{tikzcd}
J_{n_0} \times S^{n_0-1} \times I \arrow[r, "\varphi_{n_0}\times \mathrm{id}_I"] \arrow[d, tail] & A \times I \arrow[r, "\widetilde{f}"] \arrow[d, tail] & B \arrow[dd] \\
J_{n_0} \times D^{n_0} \times I \arrow[r, "\Phi_{n_0}\times \mathrm{id}_I"] \arrow[ul, phantom, very near end, "\ulcorner"] \arrow[drr, bend right, "\tilde{h}"'] & X^{(n_0)} \times I \arrow[dr, dotted, "\exists ! h"] & \\
 & & Y
\end{tikzcd}
$$

where $\widetilde{f}(a,t) \coloneqq f(a)$. Notice that $X^{(n_0)} \times I$ is the pushout of the upper left square. Since $\tilde{h}$ is a homotopy relative to $J_{n_0} \times S^{n_0-1}$, the external square commutes and the pushout property provides us with a (unique) map $h\colon X^{(n_0)} \times I \to Y$. We claim that h is a homotopy relative to A from $f|_{X^{(n_0)}}$ to $h(-,1)\colon X^{(n_0)} \to B \subset Y$. First, the commutativity of the upper right square shows that h is a homotopy relative to A and that the image of $h(-,1)$ lives in B. Then, the commutativity of the lower left triangle gives $\tilde{h}(-,0) = f\Phi_{n_0} = h(-,0)\Phi_{n_0}$ and the pushout property defining $X^{(n_0)}$ shows that $h(-,0) = f|_{X^{(n_0)}}$.

Finally, we consider the following diagram

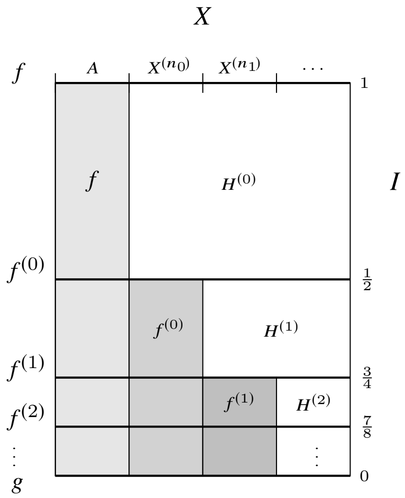
$$
\begin{tikzcd}
X^{(n_0)} \arrow[r, "i_0"] \arrow[d, tail] \arrow[dr] & X^{(n_0)} \times I \arrow[d] \arrow[ddr, bend left, "h"] & \\
X \arrow[r, "i_0"] \arrow[drr, bend right, "f"'] & X \times I \arrow[dr, dotted, "\exists H^{(0)}"] & \\
 & & Y ,
\end{tikzcd}
$$

whose exterior square commutes since $h(-,0) = f|_{X^{(n_0)}}$. We have see in Proposition 1.7.5 that the inclusions of CW subcomplex are cofibrations. Thus, the cofibration property applied to $X^{(n_0)} \rightarrowtail X$ gives the existence of a map $H^{(0)}\colon X \times I \to Y$. We claim that this provides us with a homotopy relative to A from f to $f^{(0)} \coloneqq H^{(0)}(-,1) : X \to Y$ such that $f^{(0)}\left(X^{(n_0)}\right) \subset B$. Indeed, the commutativity of the upper right triangle shows that it is a homotopy relative to A since h is and that $H^{(0)}(-,1)$ sends elements of $X^{(n_0)}$ to B since h does. The commutativity of the lower left triangle says $H^{(0)}(-,0) = f$.

Suppose now that the result holds up to k. We prove it for $k+1$ by the exact same method replacing A by $X^{(n_k)}$, $X^{(n_k)}$ by $X^{(n_{k+1})}$, and f by $f^{(k)}$.

For finite dimensional CW complexes X, the proof is over. Otherwise, we introduce the following homotopy $\mathscr{H}$. For any $k \geqslant 0$, we define

$$
\mathscr{H}(x,t) \coloneqq H^{(k)}\left(x, 2^{k+1}\left(t - 1 + \tfrac{1}{2^k}\right)\right) , \quad \text{for} \quad t \in \left[1 - \tfrac{1}{2^k}, 1 - \tfrac{1}{2^{k+1}}\right] .
$$



For any $x \in X$, there exists $k \in \mathbb{N}$ such that $x \in X^{(n_k)}$. By construction, we have $H^{(l)}(x,s) = f^{(k)}(x)$, for any $l \geqslant k$ and any $s \in I$. The assignment $\mathscr{H}(x,1) := f^{(k)}(x)$ finishes to define a continuous map $\mathscr{H}: X \times I \to Y$, which is a homotopy relative to A from f to a map $g = \mathscr{H}(-,1)\colon X \to B \subset Y$. □

Proposition 1.7.13 (Whitehead). *Let $f\colon X \to Y$ be a weak homotopy equivalence. The pushforward map $f_*\colon [Z,B] \xrightarrow{\cong} [Z,Y]$ is an isomorphism for any CW complex Z.*

Proof. We consider the factorization $X \xhookrightarrow[i_1]{} \mathrm{Cyl}(f) \xrightarrow[P]{\sim} Y$ of f given in Theorem 1.5.8. When f is a weak homotopy equivalence, so is i_1 since P is a homotopy equivalence. As a consequence, it is enough to prove the statement for embeddings and even for inclusions $X \hookrightarrow Y$ since homeomorphisms are homotopy equivalences.

Surjectivity: We apply Theorem 1.6.2 to the inclusion $X \hookrightarrow Y$ which produces the following long exact sequence of homotopy groups:

$$\cdots \longrightarrow \pi_2(X) \xrightarrow{\cong} \pi_2(Y) \longrightarrow \pi_2(Y,X) \longrightarrow \pi_1(X) \xrightarrow{\cong} \pi_1(Y) \longrightarrow \pi_1(Y,X) \longrightarrow \pi_0(X) \xrightarrow{\cong} \pi_0(Y)\ .$$

The isomorphisms $\pi_n(X) \cong \pi_n(Y)$, for $n \geqslant 0$, show that all the relative homotopy groups $\pi_n(Y,X) \cong 0$ are trivial. The compression lemma 1.7.12 implies that any map $\varphi\colon (Z,\varnothing) \to (Y,X)$ is homotopic to a map $\psi\colon Z \to X$. This shows the surjectivity of the pushforward map f_* .

Injectivity: Let $\alpha, \beta\colon Z \to X$ be two maps from a CW complex Z such that there exists a homotopy H between $f\alpha$ and $f\beta$, where $f\colon X \hookrightarrow Y$ is the inclusion. Since I is locally compact, the product $Z \times I$ is a CW complex and $(Z \times I, Z \times \partial I)$ is a CW pair so a relative pair. We apply the compression lemma 1.7.12 to $H\colon (Z \times I, Z \times \partial I) \to (Y,X)$; this produces a map $K\colon Z \times I \to X$ which is homotopic to H relative to $Z \times \partial I$. This is a homotopy from α to β and this shows the injectivity of the pushforward map f_* .

□

With this result, we can now conclude the proof of the Whitehead theorem.

Proof of Whitehead theorem 1.7.11. Let $f : X \to Y$ be a weak homotopy equivalence between two CW complexes. We apply Whitehead proposition 1.7.13 to to $Z := Y$ to get the epimorphism $[Y,X] \twoheadrightarrow [Y,Y]$. Pulling back the identity of Y, we get a map $g\colon Y \to X$ such that $fg \sim \mathrm{id}_Y$. This implies that g is weak homotopy equivalence. We apply again Whitehead proposition 1.7.13 to g and $Z = X$. The epimorphism $[X,Y] \twoheadrightarrow [X,Y]$ provides us with a map $\varphi\colon X \to Y$ satisfying $\varphi g \sim \mathrm{id}_X$. In the end, we get $\varphi \sim fg\varphi \sim f$ and then $gf \sim g\varphi \sim \mathrm{id}_X$, which concludes the proof. □

One can check from the proof of Whitehead theorem 1.7.11 that its statement holds as well for spaces that only have the *homotopy type* of CW complexes, that is for spaces that are homotopy equivalent to CW complexes.

Remark. The notion of a homotopy equivalence defines an equivalence relation on topological spaces. The notion of weak homotopy equivalence fails to define an equivalence relation on all topological spaces as the above example of the Warsaw circle shows. However, Whitehead theorem 1.7.11 shows that weak homotopy equivalence actually defines an equivalence relation on CW complexes.

Refining the aforementioned arguments, one can establish an connected version of Whitehead theorem.

Definition 1.7.14 (n-connected space). A topological space X is *n-connected* when $\pi_k(X) \cong 0$, for any $k \leqslant n$.

Under this terminology, a 0-connected space is a path-connected space and a 1-connected space is a simply connected space.

Definition 1.7.15 (n-connected map). A continuous map $f\colon X \to Y$ is *n-connected* when $\pi_k(f)\,:\,\pi_k(X) \xrightarrow{\cong} \pi_k(Y)$ is an isomorphism for any $k < n$ and an epimorphism $\pi_n(f)\,:\,\pi_n(X) \twoheadrightarrow \pi_n(Y)$ for $k = n$.

Proposition 1.7.16 (n-connected version). *Let $n \geqslant 1$ and let $f\colon X \to Y$ be an n-connected map. The pushforward map $f_*\colon [Z, X] \to [Z, Y]$ is an isomorphism for any CW complex Z of dimension at most $n - 1$ and an epimorphism any CW complex Z of dimension at most n.*

Proof. In the proof of Proposition 1.7.13, the surjectivity of f_* holds for CW complexes Z of dimension at most n and the injectivity holds for CW complexes Z of dimension at most $n - 1$. □

Theorem 1.7.17 (n-connected version). *Let X and Y be two CW complexes of dimension at most n. Any map $f\colon X \to Y$ is an homotopy equivalence if and only if it induces an isomorphism $\pi_k(f)\,:\,\pi_k(X) \xrightarrow{\cong} \pi_k(Y)$, for any $k \leqslant n$.*

Proof. The proof of the Whitehead theorem 1.7.11 still holds here with the n-connected version 1.7.16 of Whitehead corollary. □

Remark. This statement is particularly strong: for finite dimensional CW complexes, it is enough to check that a map induces isomorphisms up between homotopy groups to the top dimension of the CW complexes to get isomorphisms in *all* dimension. This remark is far from being trivial since J.-P. Serre proved in [Ser51] that any non-contractible simply-connected finite CW-complex has infinitely many nontrivial homotopy groups.

Corollary 1.7.18. *A CW complex X is contractible if and only if all its homotopy groups are trivial: $\pi_n(X) \cong 0$, for all $n \in \mathbb{N}$.*

Proof. It is enough to apply Whitehead theorem 1.7.11 to the constant map $X \twoheadrightarrow \{*\}$. □

1.7.3. *Cellular approximations.* How far are topological spaces from CW complexes?

Theorem 1.7.19 (CW approximation). *For any topological space X, there exists a CW complex X_{CW} and a weak homotopy equivalence*

$$\omega_X \;:\; X_{CW} \xrightarrow{\sim_w} X \ .$$

For any map $f\,:\,X \to Y$ and for any CW replacements $\omega_X\colon X_{CW} \to X$ and $\omega_Y\colon Y_{CW} \to Y$, there exists a map $F\colon X_{CW} \to Y_{CW}$, unique up to homotopy, such that the diagram

$$\begin{tikzcd}
X_{CW} \arrow[r, "\omega_X"] \arrow[d, dotted, "F"] & X \arrow[d, "f"] \\
Y_{CW} \arrow[r, "\omega_Y"] & Y
\end{tikzcd}$$

is homotopy commutative.

Proof. Regarding the first point, the idea amounts to working by induction on the dimension $n \in \mathbb{N}$ and to create a CW complex $X_{CW}^{(n)}$ with the same homotopy groups as X for $k \leqslant n$. In order to do so, one considers a presentation of $\pi_n(X)$ by generators and relations. The generators give rise to characteristic maps φ_n and the relations give rise to characteristic maps φ_{n+1}. At each step, this construction is mapped $X_{CW}^{(n)} \to X$ to the representative of the homotopy groups in X.

The second point is a direct application of Proposition 1.7.13 to the weak homotopy equivalence ω_Y and to the CW complex $Z = X_{CW}$: there exists a map $F\colon X_{CW} \to Y_{CW}$, unique up to homotopy, such that $\omega_Y F \sim f\omega_X$. □

Such cellular approximations do not hold only for spaces, but also for maps.

Theorem 1.7.20 (Cellular approximation). *Any map $f\colon X \to Y$ between CW complexes is homotopic to a cellular map.*

Proof. TBC □

1.7.4. *Hurewicz theorem.* Let $n \geqslant 1$ et let $f : S^n \to X$ be a continuous map. It induces a group morphism $H_n(f)\colon H_n(S^n) = H_n(S^n, \mathbb{Z}) \to H_n(X) = H_n(X, \mathbb{Z})$. Since $H_n(S^n) \cong \mathbb{Z}$, we consider the assignment

$$\begin{array}{cccc} \mathscr{H}_n & : & \pi_n(X) & \longrightarrow & H_n(X) \\ & & [f] & \longmapsto & H_n(f)(1)\ . \end{array}$$

Lemma 1.7.21. *The map $\mathscr{H}_n$ is well defined and is a group morphism.*

Proof. TBC □

Definition 1.7.22 (Hurewicz morphism). The morphism $\mathscr{H}_n\colon \pi_n(X) \to H_n(X)$ is called the *Hurewicz morphism.*

Example. For $X = S^n$, the upshot $\mathscr{H}_n(f) \in \mathbb{Z}$ of the Hurewicz morphism is the *degree* of the map $f\colon S^n \to S^n$, that is the number of times that the n-dimensional sphere wraps around itself under the map f.

Theorem 1.7.23 (Hurewicz). *Let X be a $(n-1)$-connected topological space.*

(1) *The reduced homology groups vanish $\widetilde{H}_k(X) \cong 0$, for $0 \leqslant k < n$.*
(2) *When $n = 1$, the Hurewicz morphism is the abelianisation map*

$$\mathscr{H}_1 : \pi_1(X) \twoheadrightarrow H_1(X) \cong \frac{\pi_1(X)}{[\pi_1(X), \pi_1(X)]}$$

and, when $n \geqslant 2$, the Hurewicz morphism

$$\mathscr{H}_n : \pi_n(X) \cong H_n(X)$$

is an isomorphism of abelian groups.

Proof. TBC □

Theorem 1.7.24 (Brouwer). *For any $n \geqslant 1$, we have $\pi_n(S^n) \cong \mathbb{Z}$.*

Proof. This is a direct corollary of the Hurewicz theorem 1.7.23 applied to the $(n-1)$-connected space S^n. □

Corollary 1.7.25. *Any map $f\colon X \to Y$ between two simply connected CW complexes which induces a homology isomorphisms $H_n(X) \cong H_n(Y)$, for any $n \in \mathbb{N}$, is a homotopy equivalence.*

Proof. TBC □

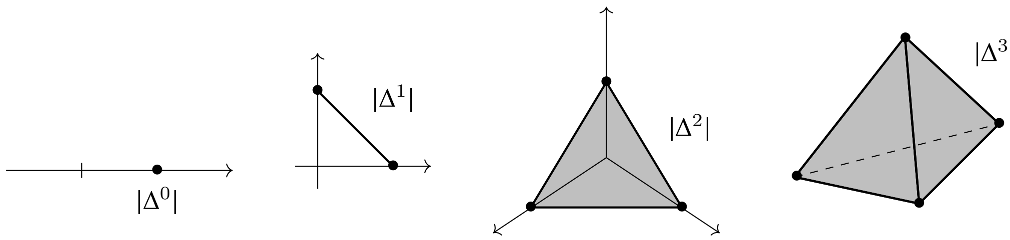
We have just seen that CW complexes provide a very large family of topological spaces (every topological space is weakly equivalent to a CW complex). Unfortunately, the data of a CW complex is not "simple" (how would one encode that in a computer?). Here, the idea will be to move from a model that uses disks (D^n, S^{n-1}) as building blocks to standard geometric n-simplices $(|\Delta^n|, \partial|\Delta^n|)$. In this way, the attaching data for cells is much simpler because it is purely combinatorial. This gives rise to the notion of simplicial sets. It is thus no coincidence that this field is often referred to as "combinatorial homotopy theory". The student in mathematics is lucky: the theory of simplicial sets admits a paradigm, an example on which (almost) all definitions and properties can be easily read: these are the standard simplices.

2.1. **Triangulated topological spaces**. Various notions from algebraic topology, such as (co)homology groups or homotopy groups, generally prove difficult to compute. As is always the case, any additional piece of information is welcome to simplify such calculations. In this section, we will consider topological spaces equipped with a suitable decomposition into cells that will take the form of points, intervals, triangles, tetrahedra, etc. More precisely, in every dimension, the basic building blocks will be the geometric simplices.

Definition 2.1.1 (Geometric simplex). A *n-dimensional geometric simplex* is the convex hull of $n+1$ affinely independent points in an affine space.

Example. The most natural example is the *standard n-dimensional geometric simplex*, which is the convex hull of the $n+1$ canonical basis vectors in $\mathbb{R}^{n+1}$:

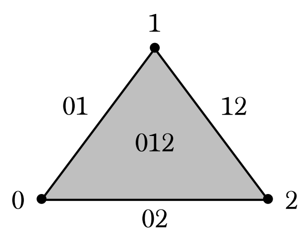
$$|\Delta^n| := \left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \;\middle|\; x_i \geqslant 0,\ \forall i \in \{0, \ldots, n\},\ \text{and}\ \sum_{i=0}^{n} x_i = 1 \right\} .$$



Since only the combinatorial data of the $n+1$ points $v_0, \ldots, v_n$ interests us, we will denote a geometric simplex by $\langle v_0, \ldots, v_n \rangle$.

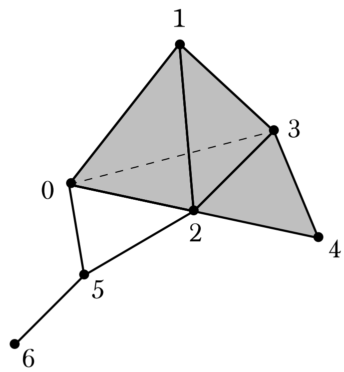
Definition 2.1.2 (Face). Let $I = \{i_0, \ldots, i_k\} \subset \{0, \ldots, n\}$. The *$I$-th face* of a geometric simplex $\langle v_0, \ldots, v_n \rangle$ is the geometric k-simplex $\langle v_{i_0}, \cdots, v_{i_k} \rangle$. The *$j$-th vertex* of a geometric simplex is its $\{j\}$-th face $\langle v_j \rangle$.

Example. The different faces of the standard 2-simplex are as follows. In this text, we will note the faces of standard geometric simplices simply by $i_0 \ldots i_k$.



A first notion of a topological space built from such elements is as follows.

Definition 2.1.3 (Simplicial polyhedron). A *simplicial polyhedron* $|\mathfrak{X}| \subset \mathbb{R}^N$ is given by a finite collection of geometric simplices of arbitrary dimension, such that any intersection of two simplices is a face of both of them.



Equivalence classes up to homeomorphism of simplicial polyhedra are faithfully represented by simple combinatorial data.

Definition 2.1.4 (Simplicial complex). A *simplicial complex* is a pair $(V, \mathfrak{X})$ where V is a set and $\mathfrak{X}$ a set of non-empty, finite subsets of V such that

- $\{v\} \in \mathfrak{X}$, for any $v \in V$;
- for every $Y \in \mathfrak{X}$, we have $Z \subset Y \Rightarrow Z \in \mathfrak{X}$.

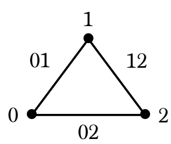
The elements of V are the *vertices*, and the elements of $\mathfrak{X}$ are the *faces*.

By a small abuse of notation, we often denote a simplicial complex simply by $\mathfrak{X}$. Equivalence classes of simplicial polyhedra up to cellular homeomorphism correspond bijectively to the equivalence classes of finite simplicial complexes up to bijection of their vertices. In the example illustrated above, the finite simplicial complex is

$$\begin{gathered}\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{0,1\}, \{0,2\}, \{0,3\}, \{0,5\}, \{1,2\}, \{1,3\}, \{2,3\}, \{2,4\}, \{2,5\},\\ \{5,6\}, \{0,1,2\}, \{0,1,3\}, \{0,2,3\}, \{1,2,3\}\{2,3,4\}, \{0,1,2,3\}\ .\end{gathered}$$

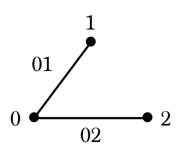
Examples.

- Every geometric simplex $\langle v_0, \ldots, v_n \rangle$ is a simplicial polyhedron. The associated combinatorial data of the simplicial complex consists of all its subsets of $\{v_0, \ldots, v_n\}$. This corresponds to the set of all its faces.
- One may also consider a geometric simplex $\langle v_0, \ldots, v_n \rangle$ with its maximal face removed; we denote it by $\partial\langle v_0, \ldots, v_n \rangle$ because it corresponds to the boundary of $\langle v_0, \ldots, v_n \rangle$. Its combinatorial data is formed by all subsets of $\{v_0, \ldots, v_n\}$ except $\{v_0, \ldots, v_n\}$ itself. This provides a model for the $(n-1)$-sphere, for $n \geqslant 1$.

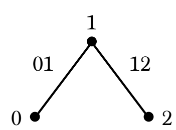


$|\partial\Delta^2| = \partial|\Delta^2|$

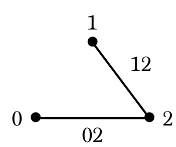
- The *k-th horn of dimension n* is the simplicial complex Λ^n_k obtained from $\partial\Delta^n$ by removing the face $0 \cdots \widehat{k} \cdots n$.



$|\Lambda^2_0|$



$|\Lambda^2_1|$



$|\Lambda^2_2|$

The n-skeleton of a simplicial complex $\mathfrak{X}$ is the simplicial complex $\mathfrak{X}^{(n)}$ consisting of the elements of $\mathfrak{X}$ with cardinality at most $n+1$. The notion of a simplicial complex makes it possible to consider "infinite-dimensional" simplicial polyhedra.

Definition 2.1.5 (Geometric realization). The *geometric realization* of a simplicial complex $\mathfrak{X}$ is the colimit

$$|\mathfrak{X}| := \operatorname*{colim}_{n \in \mathbb{N}} |\mathfrak{X}^{(n)}|$$

defined by

$$\begin{tikzcd} \mathfrak{X}_n \times \partial|\Delta^n| \arrow[r, "\varphi_n"] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near end] & |\mathfrak{X}^{(n-1)}| \arrow[d] \\ \mathfrak{X}_n \times |\Delta^n| \arrow[r, "\Phi_n"] & |\mathfrak{X}^{(n)}|\,, \end{tikzcd}$$

where $\mathfrak{X}_n := \{\, X \in \mathfrak{X} \mid |X| = n+1\}$ is the set of elements of $\mathfrak{X}$ of cardinality $n+1$, and φ_n on $\{x_0, \ldots, x_n\} \times \langle v_0, \ldots, \widehat{v_j}, \ldots, v_n\rangle$ is defined by Φ_{n-1} on $\{x_0, \ldots, \widehat{x_j}, \ldots, x_n\} \times \langle v_0, \ldots, \widehat{v_j}, \ldots, v_n\rangle$.

The geometric realization of a simplicial complex is a CW complex. The converse is provided by the following result.

Theorem 2.1.6 (Simplicial approximation). *Every CW complex is homotopy equivalent to the geometric realization of a simplicial complex.*

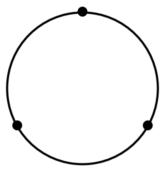
Proof. The definition of geometric realization given above is close to that of CW complexes. Hence one simply needs to rewrite the cellular attaching maps of a CW complex in a more restrictive (purely combinatorial) way of simplicial complexes. This can be done "as always" by induction on the dimension of the skeleton. The details are left to the reader, as they do not teach us anything further; see also [?, Lemma 2.2]. □

Hence, the two notions of CW complexes and simplicial complexes are homotopy equivalent.

Definition 2.1.7 (Triangulation). A *triangulation* of a topological space X is the data of a simplicial complex $\mathfrak{X}$ together with a homeomorphism $f : |\mathfrak{X}| \xrightarrow{\cong} X$.

Example.

- ⋄ Any n-simplex with its maximal face removed gives a triangulation model for the $(n-1)$-sphere.



$$|\partial\Delta^2| \xrightarrow{\cong} S^1$$

- ⋄ Every differentiable manifold is triangulable [?].

This combinatorially simple definition has one main pitfall: it is very rigid. The number of simplices used to decompose a space can be far from optimal. Indeed, two distinct faces cannot share the same set of vertices, and the vertices of a face must be all distinct. In the example of the circle, these constraints mean that it cannot be described as two segments joined at both ends, or as a single segment with identified endpoints. At least three segments are needed to obtain the circle. For the torus, one needs at least 7 vertices, 21 edges, and 14 triangles.

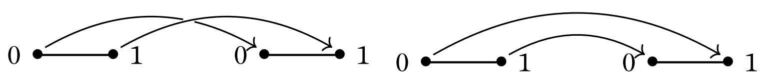
Remark. This question is far from gratuitous, for example, when one wants to compute cellular homology groups of a topological space, because the dimensions of the vector spaces involved are equal to the numbers of simplices in the triangulation. For instance, in current shape recognition methods on neuronal data using homological tools, the dimension of these spaces is on the order of 30,000.

Let us then try to find a more flexible and general notion. For that, we want to encode the manner in which different simplices glue along their faces.

Definition 2.1.8 (Simplicial map). A *simplicial map* $f : \mathfrak{X} \to \mathfrak{Y}$ between simplicial complexes is a set map between the sets of vertices of $\mathfrak{X}$ and those of $\mathfrak{Y}$ such that $\langle f(v_0), \ldots, f(v_k)\rangle$ is a simplex of $\mathfrak{Y}$ for every simplex $\langle v_0, \ldots, v_k\rangle$ of $\mathfrak{X}$.

Such data induces a unique continuous map $|\mathfrak{X}| \to |\mathfrak{Y}|$ by linear interpolation using barycentric coordinates.

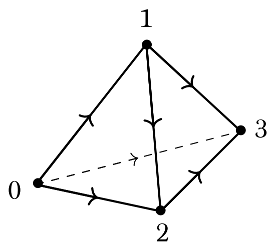
Example. There are exactly two bijective simplicial maps from one 1-simplex onto another 1-simplex:



As the above example shows, the number of simplicial maps is not optimal from a topological viewpoint, for we have too many maps. Let us then be lazy (or clever) and impose an additional condition.

Definition 2.1.9 (Ordered simplicial complex). An *ordered simplicial complex* is a simplicial complex endowed with a total order on its set of vertices. Simplicial maps between ordered simplicial complexes are those that strictly preserve these total orders on vertices.

Remark. This additional data induces an orientation of every face of a simplicial complex.



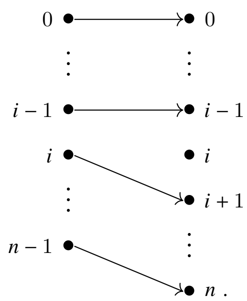
Example. There is now only one bijective simplicial map from one ordered 1-simplex onto another ordered 1-simplex.

Simplicial maps between ordered simplicial complexes are characterized by the set map that is strictly increasing between ordered sets of vertices. For instance, there are $n + 1$ injective simplicial maps from a standard $(n-1)$-simplex to a standard n-simplex. Let us examine carefully the case of standard ordered geometric simplices $|\Delta^n|$. We denote

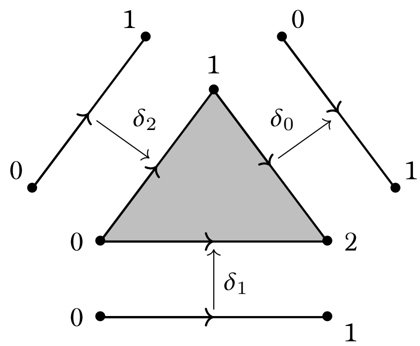
$$[n] := \{0 < \cdots < n\}$$

the totally ordered set with $n + 1$ elements, for $n \in \mathbb{N}$. There are $n + 1$ injections $\delta_i : [n-1] \to [n]$ for $0 \leqslant i \leqslant n$, which omit i:

$$\delta_i(k) := \begin{cases} k & \text{if} \quad k < i, \\ k+1 & \text{if} \quad k \geqslant i, \end{cases}$$



They are called the *cofaces* and, by a slight abuse of notation, we still denote by δ_i the corresponding simplicial maps between standard ordered geometric simplices $|\Delta^{n-1}|$ and $|\Delta^n|$.



Lemma 2.1.10. *These cofaces satisfy the relations: $\delta_j\delta_i = \delta_i\delta_{j-1}$ for $i < j$.*

Remark. Since the context is clear, we do not use an index n in the notation of the cofaces; that helps lighten the writing.

Dually, there are then $n+1$ ways to attach an ordered geometric n-simplex onto an ordered geometric $(n-1)$-simplex. We now use these properties to describe an ordered simplicial complex $\mathfrak{X}$ as a gluing of ordered standard geometric simplices. For each $n \in \mathbb{N}$, let X_n be the set of ordered geometric

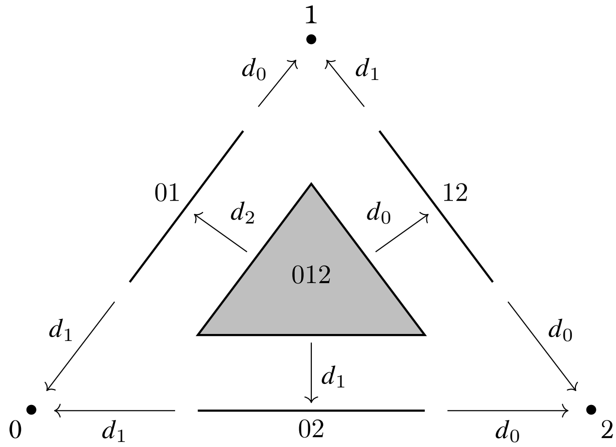
n-simplices that make up $\mathfrak{X}$. The ordered simplicial complex $\mathfrak{X}$ is then completely characterized by the set of data for how these n-simplices attach onto the $(n-1)$-simplices, which amounts to choosing $n+1$ maps $d_i : X_n \to X_{n-1}$, for $0 \leqslant i \leqslant n$, satisfying the dual relations to those of the cofaces. This yields the following definition.

Definition 2.1.11 (Δ-complex). A Δ-*complex* $\mathfrak{X}$ is the data of a collection $\{X_n\}_{n\in\mathbb{N}}$ of sets together with maps, called *faces*, $d_i : X_n \to X_{n-1}$ for $0 \leqslant i \leqslant n$, satisfying $d_i d_j = d_{j-1} d_i$ whenever $i < j$.

Example. The Δ-complex associated to the standard ordered geometric n-simplex corresponds to

$$X_0 = \{0, \ldots, n\},\ X_1 = \{01, \ldots, (n-1)n\},\ \cdots,\ X_n = \{012 \cdots n\},\ X_{n+1} = \varnothing,\ \cdots,$$

together with the face maps $d_j(i_0 \cdots i_k) = i_0 \cdots \widehat{i_j} \cdots i_k$.

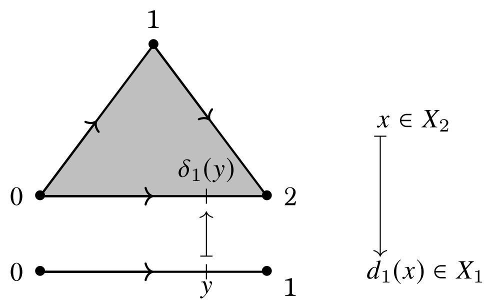


Conversely, the cellular topological space so encoded combinatorially is obtained by the following construction.

Definition 2.1.12 (Geometric realization). The *geometric realization* of a Δ-complex $\mathfrak{X}$ is the quotient topological space

$$|\mathfrak{X}|_\Delta := \left(\coprod_{n\in\mathbb{N}} X_n \times |\Delta^n| \right) / \sim,$$

by the equivalence relation generated by $(x, \delta_i(y)) \sim (d_i(x), y)$, for $x \in X_n$, $y \in |\Delta^{n-1}|$, and $(x, \sigma_i(z)) \sim \ldots$ (the higher simplicial degeneracies), where $0 \leqslant i \leqslant n$.



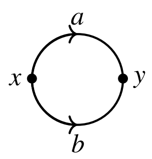
Every ordered simplicial complex induces a Δ-complex whose geometric realization is homeomorphic to the original $|\mathfrak{X}|_\Delta \cong |\mathfrak{X}|$. Conversely, the geometric realization of a Δ-complex $\mathfrak{X}$ does not necessarily form a simplicial complex for the same combinatorial data $\mathfrak{X}$. However, there always exists a simplicial complex homeomorphic to $|\mathfrak{X}|_\Delta$, possibly after considering barycentric subdivisions. In sum, the simple combinatorial notion of Δ-complex is broader and more flexible than that of a simplicial complex. For instance, we can use fewer cells than before, as the following examples show.

Example.

- The Δ-complex

$$X_0 = \{x, y\},\ X_1 = \{a, b\},\ X_2 = \varnothing,\ \cdots; \quad d_1(a) = d_1(b) = x,\ d_0(a) = d_0(b) = y$$

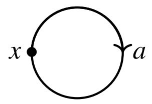
realizes the circle as the gluing of two segments.



◇ The Δ-complex

$$X_0 = \{x\},\ X_1 = \{a\},\ X_2 = \varnothing,\ \cdots; \quad d_0(a) = d_1(a) = x,$$

realizes the circle by gluing one segment onto its two endpoints.



Two questions arise: how can we obtain the definition of a Δ-complex in a simpler manner, and how should we define maps between Δ-complexes? The answers to these two questions come from using category theory. The elements of a Δ-complex are indexed by the natural numbers, or equivalently, by the ordered sets $[n], n \in \mathbb{N}$, and the maps between consecutive elements correspond to the "elementary" strictly increasing set maps. Hence, we consider the following category.

Definition 2.1.13 (Category $\bar{\Delta}$). The category $\bar{\Delta}$ has as objects the totally ordered sets $[n] = \{0 < \cdots < n\}$ for $n \in \mathbb{N}$, and as morphisms the strictly increasing maps.

Remark. The notation $\bar{\Delta}$ for this category comes from the fact that it can be defined equivalently as the category whose objects are the standard ordered geometric simplices, and whose morphisms are the strictly increasing simplicial maps.

Proposition 2.1.14. *The notion of a Δ-complex is equivalent to that of a contravariant functor from the category $\bar{\Delta}$ to the category of sets: $\bar{\Delta}^{\mathsf{op}} \to \mathsf{Ens}$.*

Proof. The proof relies exclusively on the fact that $\bar{\Delta}$ has a presentation whose generators are the cofaces δ_i satisfying the relations of Lemma 2.1.10. We will detail all the arguments in the next section when we consider the bigger category Δ. □

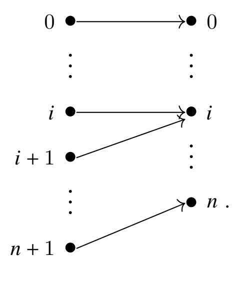
In the language of categories, we then speak of set-valued presheaves on $\bar{\Delta}$. Thanks to that description of Δ-complexes as functors, we get a notion of morphism by taking natural transformations, leading to the definition below. We denote the category of Δ-complexes by $\Delta\mathsf{Cx}$.

Definition 2.1.15 (Morphism of Δ-complexes). A *morphism* $f : \mathfrak{X} \to \mathfrak{Y}$ of Δ-complexes is the data of a collection of set maps $f_n : X_n \to Y_n$ that commute with their respective faces.

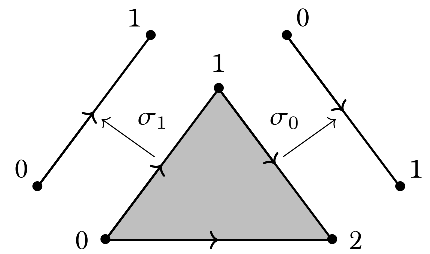
We have seen that moving from simplicial complexes to Δ-complexes gave us the ability to consider more objects. What about the maps? We will see that the count is not (yet) correct.

Consider the surjective simplicial maps from an ordered $(n+1)$-simplex to an ordered n-simplex; such maps correspond to collapsing an $(n+1)$-simplex onto one of its dimension-n faces. As before, we will describe the standard ordered geometric simplices. There are $n+1$ such simplicial surjections, given by $\sigma_i : [n+1] \to [n]$, for $0 \leqslant i \leqslant n$, which send i and $i+1$ to i:

$$\sigma_i(k) = \begin{cases} k & \text{if} \quad k \leqslant i, \\ k-1 & \text{if} \quad k > i, \end{cases}$$



These are called the *codegeneracies*, and by a slight abuse of notation, we also denote by σ_i the corresponding maps between standard ordered geometric simplices $|\Delta^{n+1}|$ and $|\Delta^n|$.



Lemma 2.1.16. *These codegeneracies satisfy the relations: $\sigma_j\sigma_i = \sigma_i\sigma_{j+1}$ for $i \leqslant j$.*

Do these simplicial maps between simplicial complexes appear at the level of the associated Δ-complexes? The answer is negative. Indeed, the Δ-complex associated to $|\Delta^{n+1}|$ has a nontrivial set in dimension $n+1$, whereas the one associated to $|\Delta^n|$ has empty sets starting from dimension n. Thus, there is no way to define a morphism from the first to the second.

We must then introduce a richer notion than that of a Δ-complex. To do so, let us consider a new category that has the same objects as $\bar{\Delta}$ but all increasing maps as morphisms.

2.2. **The simplex category.**

Definition 2.2.1 (Simplex category Δ). The *simplex category* Δ has as objects the totally ordered sets $[n] = \{0 < \cdots < n\}$ for $n \in \mathbb{N}$, and as morphisms all non-decreasing maps.

Remark. The name "simplex category" is justified by the equivalent definition: it is the category whose objects are the ordered standard geometric simplices, and whose morphisms are all increasing simplicial maps.

The factorization of a set map into a surjection followed by an injection is refined here using the cofaces and codegeneracies.

Proposition 2.2.2 (Normal form of morphisms in the simplex category). *Every morphism $\varphi : [n] \to [m]$ in the simplex category can be written uniquely in the form*

$$\varphi = \delta_{i_1} \cdots \delta_{i_r}\ \sigma_{j_1} \cdots \sigma_{j_s}, \tag{4}$$

with $i_1 \leqslant \cdots \leqslant i_r$ and $j_1 < \cdots < j_s$, where $m = n - s + r$.

Proof. Let's first show that any morphism $\varphi : [n] \to [m]$ can be written in the desired form. Let $p := |\mathrm{Im}\varphi| - 1$ be the cardinal of the image of the application φ and consider the order-preserving bijection $[p] \cong \mathrm{Im}\varphi$. The application φ factorizes into the compound $\varphi = \iota\pi$:

$$[n] \xrightarrow{\pi} [p] \xrightarrow{\iota} [m] \ ,$$

where π is surjective and ι injective. If we note $[m] = Ima\varphi \sqcup \{i_1, i_2+1, \dots, i_r+r-1\}$, with $r = m-p$ and $i_1 \leqslant \cdots \leqslant i_r$, then $\iota = \delta_{i_1} \dots \delta_{i_r}$. Now let $q_0, \dots, q_p$ be the largest antecedents of $0, \dots, p$ respectively. If we note $[n] = \{q_0, \dots, q_p\} \sqcup \{j_1, \dots, j_s\}$, with $s = n-p$ and $j_1 < \cdots < j_s$, then $\pi = sigma_{j_1} \dots \sigma_{j_s}$. In the end, we succeeded in factoring the φ application into the desired form:

$$\varphi = \delta_{i_1} \dots \delta_{i_r}\sigma_{j_1} \dots \sigma_{j_s} \ .$$

Now consider another factorization of the same form: $\varphi = \delta_{k_1} \dots \delta_{k_t}\sigma_{l_1} \dots \sigma_{l_u}$, with $k_1 \leqslant \cdots \leqslant k_t$ and $l_1 < \cdots < l_u$ and $m - t = n - u$. Note the injection $\iota' := \delta_{k_1} \dots \delta_{k_t}$ and the overjection $\pi' := \sigma_{l_1} \dots \sigma_{l_u}$. Since ι' is injective, the image of π' is in bijection with the image of φ, which imposes $p = m - r = n - s = m - t = n - u$ and therefore $t = r$ and $u = s$. The injections ι and ι' are equal because they are both increasing injections of $[p]$ into $[m]$ whose image is that of φ. The equality $\delta_{i_1} \dots \delta_{i_r} = \delta_{k_1} \dots \delta_{k_r}$ and the characterization of indices $[m] = \varphi \sqcup \{i_1, i_2+1, \dots, i_r+r-1\}$ given above show that $\{i_1, \dots, i_r\} = \{k_1, \dots, k_r\}$. In the end, as these indices are arranged in ascending order, they are equal. We then proceed in the same way with the surjections $\pi = \sigma_{j_1} \dots \sigma_{j_s} = \pi' = \sigma_{l_1} \dots \sigma_{l_s}$ using the characterization of indices given above as $[n] = \{q_0, \dots, q_p\} \sqcup \{j_1, \dots, j_s\}$. □

Lemma 2.2.3 (Relations among cofaces and codegeneracies). *The cofaces and codegeneracies satisfy:*

$$\begin{array}{ll} \delta_j\delta_i = \delta_i\delta_{j-1} & \text{for} \quad i < j, \\ \sigma_j\sigma_i = \sigma_i\sigma_{j+1} & \text{for} \quad i \leqslant j, \end{array} \qquad \sigma_j\delta_i = \begin{array}{ll} \delta_i\sigma_{j-1} & \text{for} \quad i < j, \\ \mathrm{id} & \text{for} \quad i = j \text{ and } i = j+1, \\ \delta_{i-1}\sigma_j & \text{for} \quad i > j+1. \end{array} \tag{5}$$

Proof. All verifications are straightforward. □

Theorem 2.2.4 (Presentation of the simplex category). *The simplex category Δ admits a presentation whose generators are the cofaces and codegeneracies, subject to the relations of Lemma 2.2.3.*

Proof. Lemma 2.2.3 shows that there exists a functor sending the free category generated by the δ_i and σ_i modulo relations (5) onto the Δ category of simplexes. In order to show that the latter is full and faithful, i.e. an isomorphism because identity over objects, we'll use Gröbner's base rewriting methods, see [LV12, Chapter 4] for example. To do this, we interpret the relations (5), read from left to right, as rewriting rules. It's automatic to check that all ambiguities are confluent. We now have a Gröbner basis, and the normal form of the morphisms of the free category generated by δ_i and σ_i modulo the relations expressed in Lemma 2.2.3 is $\delta_{i_1}\ldots\delta_{i_r}\sigma_{j_1}\ldots\sigma_{j_s}$ with $i_1 \leqslant \cdots \leqslant i_r$ and $j_1 < \cdots < j_s$. We conclude with the uniqueness of the writing (4) of morphisms of the simplicial category Δ established at proposition 2.2.2. □

2.3. **Simplicial sets.**

Definition 2.3.1 (Simplicial set). A *simplicial set* is a contravariant functor from the simplex category to the category of sets: $\Delta^{\mathsf{op}} \to \mathsf{Ens}$.
The category of simplicial sets, denoted $\Delta\mathsf{Ens} = \mathsf{Fon}(\Delta^{\mathsf{op}}, \mathsf{Ens})$, has as morphisms the natural transformations of functors.

Remark. We also speak of set-valued presheaves over the simplex category. Some general results on presheaf categories can be found in Appendix A.1; they are going to be used below when needed.

This definition is both conceptual and compact. However, in practice one often uses the more explicit description below.

Proposition 2.3.2. *Giving a simplicial set $\mathfrak{X}$ is equivalent to giving a family of sets X_n, for $n \in \mathbb{N}$, together with maps $d_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$, called respectively* faces *and* degeneracies, *for $0 \leqslant i \leqslant n$, satisfying the duals of the simplicial relations (5) of the simplex category:*

$$(6)\qquad \begin{array}{lll} d_i d_j = d_{j-1} d_i & \text{for} & i < j,\\ s_i s_j = s_{j+1} s_i & \text{for} & i \leqslant j, \end{array} \qquad d_i s_j = \begin{cases} s_{j-1} d_i & \text{for}\quad i < j,\\ \mathrm{id} & \text{for}\quad i = j \text{ and } i = j+1,\\ s_j d_{i-1} & \text{for}\quad i > j+1. \end{cases}$$

A morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of simplicial sets is likewise given by a collection of set maps $f_n : X_n \to Y_n$, commuting with the respective faces and degeneracies of $\mathfrak{X}$ and $\mathfrak{Y}$.

Proof. This is a direct corollary of Theorem refthm:CatSimpPres. To do this, note the image sets of a functor $\mathfrak{X} : \Delta^{\mathsf{op}} \to \mathsf{Ens}$ by $X_n := X[n]$ and the images of the opposites of cofaces and co-degrees by $d_i := \mathfrak{X}(\delta_i^{\mathrm{op}})$ and $s_i := \mathfrak{X}(\sigma_i^{\mathrm{op}})$ respectively. □

Thus one often depicts a simplicial set in the form of a diagram:

$$X_0 \;\underset{d_1}{\overset{s_0}{\underset{d_0}{\rightleftarrows}}}\; X_1 \;\; (s_0, s_1 \rightarrow;\ d_0, d_1, d_2 \leftarrow) \;\; X_2 \;\; \substack{\rightarrow\rightarrow\rightarrow\\ \leftleftarrows\leftleftarrows} \;\; X_3 \;\; \substack{\rightarrow\rightarrow\rightarrow\rightarrow\\ \leftleftarrows\leftleftarrows\leftarrow} \;\; X_4 \quad \cdots$$

Example. Let us try to associate a simplicial set to an n-dimensional ordered geometric simplex, starting from its Δ-complex. For the degeneracy maps to exist, we need to adjoin extra elements. We thus consider *all* non-decreasing sequences in $[n]$:

$$X_0 = \{0,\ldots,n\},\ X_1 = \{00, 01, \ldots, (n-1)n, nn\},\ \cdots,\ X_k = \{i_0\cdots i_k \mid 0 \leqslant i_0 \leqslant \cdots \leqslant i_k \leqslant n\},\ \cdots,$$

equipped with face and degeneracy maps

$$d_j(i_0\cdots i_k) = i_0\cdots\widehat{i_j}\cdots i_k, \quad s_j(i_0\cdots i_k) = i_0\cdots i_j i_j\cdots i_k.$$

Definition 2.3.3 (Standard n-simplex). We call this fundamental simplicial set the *standard n-simplex*, denoted Δ^n.

Remark. This example provides a quick way to derive the simplicial relations (6).

Definition 2.3.4 (n-simplex). Elements of X_n of a simplicial set $\mathfrak{X}$ are called *n-simplices.* They split into two parts: the *degenerate simplices*, i.e., those that lie in the image of at least one degeneracy map, and the rest, called *non-degenerate simplices.* We denote by $\mathrm{N}X_n$ the (possibly empty) set of non-degenerate n-simplices.

In the example of the standard simplex Δ^n, the non-degenerate k-simplices are the ones from the simplicial complex, while the degenerate ones are added anew. More generally, one may associate a simplicial set to every ordered simplicial complex by considering the n-simplexes of the form $\langle v_0, \ldots, v_0, \ldots, v_m, \ldots, v_m \rangle$, for any geometric simplex $\langle v_0, \ldots, v_m \rangle$.

Examples.

- The simplicial set $\partial\Delta^n$ associated to the simplicial complex $|\partial\Delta^n|$ has as d-simplices those of the form $\underbrace{i_0 \cdots i_0 \cdots i_m \cdots i_m}_{d+1 \text{ elements}}$, with $m < n$ and $i_0 < \cdots < i_m$ in $[n]$. It is called the *boundary* of the standard n-simplex.
- The simplicial set Λ^n_k associated to the k-th horn $|\Lambda^n_k|$ has the same d-simplices as $\partial\Delta^n$, except for those coming from $\{i_0, \ldots, i_m\} = \{0, \ldots, \widehat{k}, \ldots, n\}$.

Looking for a functor from Δ-complexes to simplicial sets is less straightforward. The next example shows there can be multiple choices for extending a Δ-complex to a simplicial set.

Example. Return to the example of a Δ-complex giving a two-cell model of the circle. Denoting the unique 0-simplex x as 0 and the 1-simplex a as 01, attempts to adjoin enough degenerate n-simplices by hand to create degeneracies lead to:

$$X_n := \{0 \cdots 0, 0 \cdots 01, 0 \cdots 011, \ldots, 01 \cdots 1\},$$

with face and degeneracy maps

$$d_j(i_0 \cdots i_k) = i_0 \cdots \widehat{i_j} \cdots i_k, \quad s_j(i_0 \cdots i_k) = i_0 \cdots i_j i_j \cdots i_k,$$

except that $d_0(01 \cdots 1) := 0 \cdots 0$. One checks that this forms a simplicial set.

The next result shows that every degenerate simplex of a simplicial set has a canonical form.

Lemma 2.3.5 (Eilenberg–Zilber). *Every degenerate simplex $x \in X_n$ of a simplicial set $\mathfrak{X}$ can be written as $x = \mathfrak{X}(\varphi^{\mathrm{op}})(y)$ for a unique pair $\varphi : [n] \twoheadrightarrow [m]$ non-decreasing and $y \in X_m$ non-degenerate.*

Proof. As for the existence of such a writing, we start by writing $x = \mathfrak{X}(\sigma_i^{\mathrm{op}})(y')$, with $y' \in X_{n-1}$, as x is degenerate. We then iterate this process until we arrive at a non-degenerate y. For uniqueness, suppose there's another pair (z, ψ) with $z \in X_k$ nondegenerate and $\psi : [n] \twoheadrightarrow [k]$ increasing such that $x = \mathfrak{X}(\psi^{\mathrm{op}})(z)$. Let's assume, without loss of generality, that $k \leqslant m$. The normal form of morphisms in the simplex category (Proposition 2.2.2) allows us to write φ as $\varphi = \sigma_{j_1} \ldots \sigma_{j_s}$ with $j_1 < \cdots < j_s$. Consider $\chi := \delta_{j_s} \ldots \delta_{j_1}$, so that $\varphi\chi = \mathrm{id}$. We then have $y = \mathfrak{X}\left((\psi\chi)^{\mathrm{op}}\right)(z)$. Since y is non-degenerate, the application $\psi\chi$ is a compound of cofaces, which implies that y is obtained from z by successive applications of faces. The fact that $k \leqslant m$ implies that $\psi\chi = \mathrm{id}$ then $k = m$ and $y = z$. As in the proof of Proposition 2.2.2, let's use the notations $q_i := \max\left(\varphi^{-1}(i)\right)$ and $r_i := \max\left(\psi^{-1}(i)\right)$, for $0 \leqslant i \leqslant m$. For $0 \leqslant i \leqslant m$, the image of i under the inclusion χ is equal to $\chi(i) = q_i$, which implies that $q_i \leqslant r_i$. Since $k = m$, we can use the same arguments again, reversing the roles of φ and ψ. This proves that $r_i = q_i$ and therefore $r_i = q_i$, for $r_i = q_i$. This shows that $\varphi = \psi$ and the proof is complete.

□

To describe a functor from Δ-complexes to simplicial sets, we can draw on the Eilenberg–Zilber lemma. We can also reason as follows. In the other sense, every simplicial set induces a Δ-complex by forgetting degeneracies. Said in the language of categories, any functor $\Delta^{\mathsf{op}} \to \mathsf{Ens}$ induces a functor $\bar{\Delta}^{\mathsf{op}} \to \mathsf{Ens}$ by drawing back along the subcategory inclusion $\Phi^{\mathrm{op}} : \bar{\Delta}^{\mathsf{op}} \hookrightarrow \Delta^{\mathsf{op}}$. One way of describing a functor in the other direction is to look for an adjoint, for example on the left.

Proposition 2.3.6. *The forgetful functor* $\mathrm{U} := (\Phi^{\mathrm{op}})^*$ *from Δ-complexes to simplicial sets admits a left adjoint* L

$$\mathrm{L} \,:\, \Delta\mathsf{Cx} \underset{\perp}{\rightleftarrows} \Delta\mathsf{Ens} \,:\, \mathrm{U}\,,$$

given explicitly by

$$(\mathrm{L}\mathfrak{X})_n = \left\{(\varphi, x) \,\middle|\, \varphi : [n] \twoheadrightarrow [m] \text{ non-decreasing}, \ x \in X_m\right\},$$

with faces given by $d_i(\varphi, x) = (\varphi\delta_i, x)$ if $\varphi\delta_i$ is surjective, etc. Faces are given by $d_i(\varphi, x) = (\varphi\delta_i, x)$, if $\varphi\delta_i$ is surjective, otherwise $\varphi\delta_i$ is uniquely written $\delta_j\psi$ with ψ surjective and then $d_i(\varphi, x) = (\psi, d_j(x))$; degeneracies are given by $s_i(\varphi, x) = (\varphi\sigma_i, x)$.

Proof. Let's start by noting that the uniqueness of $\varphi\delta_i = \delta_j\psi$, with ψ surjective, comes from the normal form of morphisms in the category of simplexes (Proposition refprop:UnicCatSim). First, we need to show that $\mathrm{L}\mathfrak{X}$ is a simplicial set. The case of degeneracies is straightforward: for $i \leqslant j$, we have

$$s_i s_j(\varphi, x) = s_i(\varphi\sigma_j, x) = (\varphi\sigma_j\sigma_i, x) = (\varphi\sigma_i\sigma_{j+1}, x) = s_{j+1}(\varphi\sigma_i, x) = s_{j+1}s_i(\varphi, x)\ ,$$

grâce au Lemma 2.1.16. Le cas des faces est assez similaire mais il faut prendre en compte les différents cas de figures; ceci est assez automatique mais la longueur fait qu'on laisse les détails aux lecteur-trices. Pour ce qui est de la structure fonctorielle, à tout morphisme $f\colon \mathfrak{X} \to \mathfrak{Y}$ d'ensembles simpliciaux, on associe l'application

$$(\varphi, x) \mapsto \big(\varphi, f_m(x)\big)$$

dont il est automatique de vérifier qu'elle commute aux faces et dégénérescences respectives. Enfin, pour montrer que le foncteur L est adjoint à gauche de U, on considère les deux transformations naturelles suivantes

$$\begin{array}{rcrcl} \upsilon(\mathfrak{X}) & : & \mathfrak{X} & \mapsto & \mathrm{UL}\,\mathfrak{X} \\ & & x \in X_n & \mapsto & \big(\mathrm{id}_{[n]}, x\big) \end{array} \qquad \text{et} \qquad \begin{array}{rcrcl} \varepsilon(\mathfrak{Y}) & : & \mathrm{LU}\,\mathfrak{Y} & \to & \mathfrak{Y} \\ & & (\varphi, y) & \mapsto & \mathfrak{Y}\,(\varphi^{\mathrm{op}})\,(y) \end{array} .$$

La première est bien définie par des morphismes de Δ-complexes et la seconde par des morphismes d'ensembles simpliciaux. On vérifie ensuite que les deux composées suivantes sont égales aux identités respectives.

$$\begin{tikzcd} \mathrm{L} \arrow[r, "\mathrm{id}_{\mathrm{L}}\circ\upsilon"'] \arrow[rr, bend left, "\mathrm{id}_{\mathrm{L}}"] & \mathrm{LUL} \arrow[r, "\varepsilon\circ\mathrm{id}_{\mathrm{L}}"'] & \mathrm{L} \end{tikzcd} \qquad \text{et} \qquad \begin{tikzcd} \mathrm{U} \arrow[r, "\upsilon\circ\mathrm{id}_{\mathrm{U}}"'] \arrow[rr, bend left, "\mathrm{id}_{\mathrm{U}}"] & \mathrm{ULU} \arrow[r, "\mathrm{id}_{\mathrm{U}}\circ\varepsilon"'] & \mathrm{U} \end{tikzcd} .$$

□

Remark. As it happens, this result is a direct corollary of a more general theory: that of presheaves and Kan extensions, see Appendix A. In this language, the functor L is equal to the left Kan extension along Φ, i.e. $\mathrm{L} = \mathrm{Lan}_{\Phi^{\mathrm{op}}}$. Indeed, we can apply the general results of Kan extensions to the left here (corollary A.2.6 and proposition A.2.1) because the category $\bar{\Delta}^{\mathsf{op}}$ is small, the category Δ^{op} is locally small and the category Ens is cocomplete. This would show readers how the author prefers to work: find an object with a particular property by a universal construction and then make it explicit, rather than the other way round, i.e. describe an object and spend hours showing that it verifies the desired property.

Exercise. Using this proposition, show that the example above of the simplicial set modeling the circle from the two-segment Δ-complex is given precisely by this functor L applied to that Δ-complex.

Remark. For any simplicial set, the n-simplices inject into the $(n+1)$-simplices by $s_0 : X_n \hookrightarrow X_{n+1}$, and the $(n+1)$-simplices map onto X_n by $d_0 : X_{n+1} \twoheadrightarrow X_n$, because $d_0 s_0 = \mathrm{id}_{X_n}$.

Definition 2.3.7 (Simplicial subobject, quotient simplicial set). A *sub-simplicial set* $\mathfrak{Y}$ of a simplicial set $\mathfrak{X}$ is a simplicial map $\mathfrak{Y} \hookrightarrow \mathfrak{X}$ that is injective in each degree $Y_n \hookrightarrow X_n$. A *quotient simplicial set* $\mathfrak{Y}$ of $\mathfrak{X}$ is a simplicial map $\mathfrak{X} \twoheadrightarrow \mathfrak{Y}$ that is surjective in each degree $X_n \twoheadrightarrow Y_n$.

Examples.

- The simplicial sets given above embed into each other:

$$\Lambda^n_k \hookrightarrow \partial\Delta^n \hookrightarrow \Delta^n .$$

- Consider the quotient simplicial set $\Delta^n/\partial\Delta^n$ for $n \geqslant 1$, whose d-simplices admit the representatives

$$\Big\{\underbrace{i_0\cdots i_0\cdots i_n\cdots i_n}_{d+1 \text{ elements}} \mid i_0 = \cdots = i_n = 0 \text{ or } i_0 = 0, i_1 = 1, \ldots, i_n = n\Big\},$$

  with faces and degeneracies given by deletion and duplication, except when faces do not make sense (removing a number that appears only once), in which case they become $0\cdots 0$. The only non-degenerate simplices are 0 and $01\cdots n$. We thus get the most economical model, with two cells, for the n-sphere (for $n \geqslant 1$). (For $n = 1$, we recover the example from above.) The obvious surjective map $\Delta^n \twoheadrightarrow \Delta^n/\partial\Delta^n$ shows it is a quotient of the standard n-simplex.

Remark. This example is instructive in that it displays a simplicial set that does not arise from a Δ-complex: for instance, it is not in the image of the functor from Proposition 2.3.6. It shows that degeneracies are essential and not gratuitous: this sphere model would not exist without the degeneracies placed in dimensions 1 up to $n-1$.

The examples above show that we often want to "generate" simplicial sets from small amounts of data. The next proposition tells us how the smallest sub-simplicial set containing all simplices of dimension at most n looks.

Proposition 2.3.8. *Given a simplicial set $\mathfrak{X}$, the following simplices*

$$(\mathrm{sq}_n\mathfrak{X})_m := \left\{x \in X_m \mid \exists\, k \leqslant n,\ \exists\, \varphi : [m] \twoheadrightarrow [k] \text{ non-decreasing, } \exists\, y \in X_k \text{ with } x = \mathfrak{X}(\varphi^{\mathrm{op}})(y)\right\}$$

form the smallest sub-simplicial set of $\mathfrak{X}$ containing all its k-simplices for $k \leqslant n$.

Proof. It's easy to check that this is a simplicial subset; for example, stability for faces comes from :

$$d_i(x) = d_i\left(\mathfrak{X}\left(\varphi^{\mathrm{op}}\right)(y)\right) = \mathfrak{X}\left((\varphi \circ \delta_i)^{\mathrm{op}}\right)(y)\ .$$

This simplicial subset contains all k-simplexes for $k \leqslant n$: it suffices to consider $k = m$, $\varphi = \mathrm{id}$ and $y = x$. The normal form of morphisms in the simplex category (Proposition refprop:UnicCatSim) shows that, for $m > n$:

$$(\mathrm{sq}_n\mathfrak{X})_m = \left\{xinX_m\ |\exists\, k \leqslant n,\ \exists\, \varphi\ :\ [m] \twoheadrightarrow [k] \text{ creasing and surjective, } \exists, y \in X_k \text{ t.q. } x = \mathfrak{X}(\varphi^{\mathrm{op}})(y)\right\}\ .$$

Let there now be a simplicial subset $\mathfrak{Y} \subset \mathfrak{X}$ which contains all k-simplexes for $k \leqslant n$, then it contains the images by the degeneracy composites of these k-simplexes. It therefore contains all $\mathrm{sq}_m\mathfrak{X}$, for $m > n$, by the previous characterization. ☐

This sub-simplicial set is formed by the k-simplices for $k \leqslant n$ and all images under compositions of degeneracies for $k > n$.

Definition 2.3.9 (Skeleton). The sub-simplicial set $\mathrm{sq}_n\mathfrak{X}$ is called the *n-skeleton* of the simplicial set $\mathfrak{X}$.

Example. The $(n-1)$-skeleton of the standard n-simplex is its boundary: $\mathrm{sq}_{n-1}\Delta^n = \partial\Delta^n$.

Exercise. Show that the 0-skeleton of a simplicial set $\mathfrak{X}$ is the constant simplicial set

$$(\mathrm{sq}_0\mathfrak{X})_m = X_0\ , \quad \text{for } m \in \mathbb{N}\ ,$$

where the faces and degeneracies are all equal to the identity.

As with CW-complexes, the notion of the skeleton of a simplicial set can be used to demonstrate by recurrence, see ??? for example.

Definition 2.3.10 (Dimension). A simplicial set $\mathfrak{X}$ has *dimension* n if

$$\mathrm{sq}_{n-1}\mathfrak{X} \subsetneq \mathrm{sq}_n\mathfrak{X} = \mathfrak{X},$$

that is, it has at least one non-degenerate simplex in dimension n and none in higher dimensions.

Examples. The earlier examples have dimensions:

$$\dim \Delta^n = n, \quad \dim \partial\Delta^n = n-1, \quad \dim \Lambda^n_k = n-1, \quad \text{and} \quad \dim \Delta^n/\partial\Delta^n = n.$$

Definition 2.3.11 ((Co)simplicial object). An *object simplicial in a category* C is a contravariant functor $\Delta^{\mathsf{op}} \to \mathsf{C}$ from the simplex category to C. We denote the corresponding category by $\Delta\mathsf{C}$. An *object cosimplicial in a category* C is a covariant functor $\Delta \to \mathsf{C}$.

Examples.

- One can consider group objects in simplicial sets, ring objects in simplicial sets, etc. For example, a simplicial group is the data of groups G_n, for $n \in \mathbb{N}$, together with face and degeneracy group morphisms $d_i : G_n \to G_{n-1}$ and $s_i : G_n \to G_{n+1}$ satisfying the same simplicial relations.
- The collection of standard ordered geometric simplices $|\Delta^n|$ together with their increasing simplicial maps form a cosimplicial topological space.

2.4. **Geometric realization.**

Definition 2.4.1 (Geometric realization). The *geometric realization* of a simplicial set $\mathfrak{X}$ is the quotient topological space

$$|\mathfrak{X}| := \left(\coprod_{n\in\mathbb{N}} X_n \times |\Delta^n|\right) / \sim,$$

by the equivalence relation generated by

$$(x, \delta_i(y)) \sim (d_i(x), y), \quad (x, \sigma_i(z)) \sim (s_i(x), z),$$

for $x \in X_n,\ \ y \in |\Delta^{n-1}|,\ z \in |\Delta^{n+1}|,\ 0 \leqslant i \leqslant n.$

At first glance, this construction may not look elementary, yet it is quite straightforward: we take the geometric standard n-simplices indexed by non-degenerate n-simplices and glue them together via the face identifications (the first type of relation). The degenerate simplices end up "not contributing new cells" thanks to the second type of relation. Moreover, there are no additional identifications. Before demonstrating this general result, we can try our hand at the previous examples.

Examples. The notations chosen from the outset find their coherence here.

- The geometric realization $|\Delta^n|$ of the n-standard geometric Δ^n is the n-standard geometric $|\Delta^n|$.
- The $|\partial\Delta^n|$ geometric realization of the $\partial\Delta^n$ edge of the n-standard Δ^n-simplex is the $|\partial\Delta^n| \cong S^{n-1}$ topological model for the $n-1$-dimensional sphere.
- The geometric realization $|\Lambda^n_k|$ of the k^{e}-cornet simplicial Λ^n_k is the k^{e}-cornet $|\Lambda^n_k|$.

Remark. Note that we have lost some information in this case: we had started from simplicial complexes, i.e. topological spaces canonically provided with a triangulation, and the passage to the geometrical realization of the associated simplicial set caused us to lose the latter. (The notations chosen here are therefore not absolutely perfect, but they have the advantage of simplicity).

Theorem 2.4.2. *For any simplicial set $\mathfrak{X}$, the following map is a continuous bijection:*

$$\coprod_{n\in\mathbb{N}} \mathrm{N}X_n \times |\mathring{\Delta}^n| \longrightarrow |\mathfrak{X}|\ .$$

Proof. The idea of the demonstration is as follows. Existence in Eilenberg–Zilber's lemma 2.3.5 shows that any point associated with a degenerate simplex can be identified with a point associated with a nondegenerate simplex. Uniqueness in Eilenberg–Zilber's lemma 2.3.5 shows that these identifications do not induce any further identification between points indexed by nondegenerate simplexes. In the end, the continuous bijection is established as in the case of CW-complexes,see Exercise 2 on sheet 3. For further details, please refer the reader to [?, Proposition I.2.10]. ☐

Note that this result is the analogue at the level of simplicial sets of for CW-complexes, see Exercise 2 on Sheet 3.

Corollary 2.4.3. *The geometric realization of the simplicial set associated to a simplicial complex is homeomorphic to that simplicial complex itself.*

Proof. Since the non-degenerate simplices of the simplicial set associated with a simplicial complex are in bijection with the faces of that complex, this follows directly from Theorem 2.4.2 (and a similar statement for realizations of simplicial complexes). ☐

Proposition 2.4.4. *The geometric realization of a simplicial set is a CW complex with exactly one n-cell for each non-degenerate n-simplex.*

Proof. This is a direct corollary of Theorem 2.4.2. ☐

2.5. **The category of simplicial sets.**

The category of simplicial sets is precisely the category of presheaves on the simplex category. This conceptually explains why it possesses excellent categorical properties. In this section, we show in detail how these standard results follow from the fact that it is a presheaf category.

Proposition 2.5.1. *The category $\Delta\mathsf{Ens}$ of simplicial sets is complete and cocomplete. For any functor $F : \mathsf{D} \to \Delta\mathsf{Ens}$, let*

$$\begin{array}{rccc} F_n & : & \mathsf{D} & \to & \mathsf{Ens} \\ & & d & \mapsto & F(d)_n \end{array}$$

be the associated functor for each $n \in \mathbb{N}$. Then $\lim_{\mathsf{D}} F$ *and* $\operatorname{colim}_{\mathsf{D}} F$ *are computed degreewise by*

$$\left(\lim_{\mathsf{D}} F\right)_n = \lim_{\mathsf{D}} F_n \qquad \textit{and} \qquad \left(\operatorname*{colim}_{\mathsf{D}} F\right)_n = \operatorname*{colim}_{\mathsf{D}} F_n .$$

Proof. For any $n \in \mathcal{N}$, consider the limit $\lim_{\mathsf{D}} F_n$ whose universal property induces the existence of face and degeneracy applications that raise those present at the level of simplicial sets $F(d)$, for any object d of D. They verify the simplicial relations (6) because the latter are verified at the level of simplicial sets $F(d)$, for any object d of D. The case of colimites is treated in the same way. □

The terminal object in $\Delta\mathsf{Ens}$ is $\Delta^0 = *$, consisting of a single point in each dimension, and the initial object is the empty set, $\varnothing$, in each dimension.

We next turn to representable presheaves. In our case, those are presheaves of the form $\mathrm{Hom}_\Delta(-, [n])$.

Proposition 2.5.2. *The representable presheaf* $\mathrm{Hom}_\Delta(-, [n])$ *is isomorphic to the standard n-simplex* Δ^n.

Proof. This result can be shown directly. The increasing applications $i_j =: [k] \to [n]$ are in bijection with the increasing sequences $i_0 \leqslant \cdots \leqslant i_k$ of elements of $[n]$ by posing $i_j := \psi(j)$. This bijection preserves faces and degeneracies well:

$$d_j(\varphi) = \varphi \circ \delta_j \text{ corresponds to } i_0 \cdots \widehat{i_j} \cdots i_k \quad \text{and} \quad s_j(\varphi) = \varphi \circ \sigma_j \text{ corresponds to } i_0 \cdots i_j i_j \cdots i_k .$$

□

This identification provides the collection of standard simplexes with a cosimplicial simplicial set structure; no, there's no mistake in that sentence, it means that there's a functor $\Delta \to \Delta\mathsf{Ens}$ whose image of $[n]$ is Δ^n. Indeed, for any increasing application $\varphi \colon [n] \to [m]$, consider the morphism of simplicial sets

$$\varphi_* : \Delta^n \cong \mathrm{Hom}_\Delta(-, [n]) \longrightarrow \mathrm{Hom}_\Delta(-, [m]) \cong \Delta^m .$$

For example, the cofaces $\left(\delta_j\right)_* : \Delta^{n-1} \to \Delta^n$ and co-degeneracies $\left(\sigma_j\right)_* : \Delta^{n+1} \to \Delta^n$ are explicitly given by

$$\left(\delta_j\right)_* (i_0 \cdots i_k) = \delta_j(i_0) \cdots \delta_j(i_k) \quad \text{and} \quad \left(\sigma_j\right)_* (i_0 \cdots i_k) = \sigma_j(i_0) \cdots \sigma_j(i_k) ,$$

for $0 \leqslant j \leqslant n$. (In the following, we will drop the notation $(\,)_*$ as the context will allow). It's a matter of raising to the level of simplicial sets via geometric realization of the cosimplicial object structure of standard ordered geometric simplexes $|\Delta^n|$. A first application is given by the following lemma.

Lemma 2.5.3.

- *The boundary $\partial\Delta^n$ of the standard n-simplex is the coequalizer*

$$\coprod_{0 \leqslant i < j \leqslant n} \Delta^{n-2} \rightrightarrows \coprod_{0 \leqslant l \leqslant n} \Delta^{n-1} \longrightarrow\!\!\!\!\rightarrow \partial\Delta^n,$$

  where the top arrow sends the copy indexed by $i < j$ onto the copy indexed by j via δ_i, and onto the copy indexed by i via δ_{j-1}; the arrow on the right sends the copy indexed by l to $\partial\Delta^n$ via δ_l.
- *The k-th horn of dimension n, Λ^n_k, is the coequalizer*

$$\coprod_{0 \leqslant i < j \leqslant n} \Delta^{n-2} \rightrightarrows \coprod_{\substack{0 \leqslant l \leqslant n \\ l \neq k}} \Delta^{n-1} \longrightarrow\!\!\!\!\rightarrow \Lambda^n_k,$$

  with the same face identifications.

Proof. In the other direction, consider the collection of applications $f_x : \Delta^n \to \mathfrak{X}$ which send $01 \cdots n \leftrightarrow \mathrm{id}_{[n]}$ on x and, in general terms $\varphi \in \mathrm{Hom}_\Delta([k], [n]) \cong (\Delta^n)_k$ on $\mathfrak{X}\,(\varphi^{\mathrm{op}})\,(x)$. This is indeed a morphism of simplicial sets. The application defined by $x \mapsto f_x$ is the inverse of the application in the statement. Naturalness on the right with respect to morphisms of simplicial sets is obvious. Naturalness on the left comes from the fact that we have

$$(f \circ \varphi_*)\,(01 \cdots m) = \mathfrak{X}\,(\varphi^{\mathrm{op}})\,(01 \cdots n) ,$$

for any increasing application $\varphi : [m] \to [n]$ and for any morphism $f \colon \Delta^n \to \mathfrak{X}$ of simplicial sets. □

Proposition 2.5.4 (Simplicial Yoneda lemma). *The map*

$$\mathrm{Hom}_{\Delta\mathsf{Ens}}(\Delta^n, \mathfrak{X}) \;\cong\; X_n \quad , \quad f \;\mapsto\; f_n(01 \cdots n)$$

is a natural bijection in $[n] \in \Delta$ and in the simplicial set $\mathfrak{X} \in \Delta\mathsf{Ens}$.

Proof. In the other direction, consider the collection of applications $f_x : \Delta^n \to \mathfrak{X}$ which send $01\cdots n \leftrightarrow \mathrm{id}_{[n]}$ on x and, in general terms $\varphi \in \mathrm{Hom}_\Delta([k],[n]) \cong (\Delta^n)_k$ on $\mathfrak{X}(\varphi^{\mathrm{op}})(x)$. This is indeed a morphism of simplicial sets. The application defined by $x \mapsto f_x$ is the inverse of the application in the statement. Naturalness on the right with respect to morphisms of simplicial sets is obvious. Naturalness on the left comes from the fact that we have

$$(f \circ \varphi_*)(01\cdots m) = \mathfrak{X}(\varphi^{\mathrm{op}})(01\cdots n)\,,$$

for any increasing application $\varphi : [m] \to [n]$ and for any morphism $f\colon \Delta^n \to \mathfrak{X}$ of simplicial sets. □

This shows that we can think of Δ^n as "the free simplicial set on a generator in dimension n". More formally, the forgetful functor $\Delta\mathsf{Ens} \to \mathsf{Ens}$ that records just the set of n-simplices has a left adjoint sending a single element to Δ^n.

Remark. The naturality also implies $f_{d_i(x)} = f_x\delta_i$ and $f_{s_i(x)} = f_x\sigma_i$; so a simplicial set can be seen as a right module over the simplex category Δ.

Corollary 2.5.5.

- *A map $\partial\Delta^n \to \mathfrak{X}$ of simplicial sets is equivalent to a collection of $n+1$ simplices $x_0,\ldots,x_n \in X_{n-1}$ satisfying $d_i(x_j) = d_{j-1}(x_i)$ for all $i < j$.*
- *A map $\Lambda^n_k \to \mathfrak{X}$ of simplicial sets is equivalent to a collection of n simplices $x_0,\ldots,x_{k-1},x_{k+1},\ldots,x_n \in X_{n-1}$ satisfying $d_i(x_j) = d_{j-1}(x_i)$ for all $i < j$ different from k.*

Proof. Directly from Lemma 2.5.3 and the Yoneda lemma 2.5.4. □

Corollary 2.5.6. *The simplex category Δ embeds as a full subcategory of $\Delta\mathsf{Ens}$, the category of simplicial sets, via the Yoneda embedding:*

$$\begin{array}{rcccl} \mathrm{Y} & : & \Delta & \hookrightarrow & \Delta\mathsf{Ens} \\ & & [n] & \mapsto & \Delta^n, \end{array}$$

and this embedding is fully faithful.

Proof. Combine Proposition 2.5.2 with the Yoneda lemma 2.5.4:

$$\mathrm{Hom}_{\Delta\mathsf{Ens}}(\Delta^n,\Delta^m) \;\cong\; \left(\Delta^m\right)_n \;\cong\; \mathrm{Hom}_\Delta([n],[m]).$$

□

Hence $\Delta\mathsf{Ens}$ is the cocompletion of Δ: the smallest cocomplete category containing it.

Remark. This further justifies the name and notation "Δ" for the simplex category.

We have seen that one of the aims of simplicial sets is to describe the combinatorics of triangulations in a topological space. Now, every triangulation is obtained by gluing geometric simplexes together. The question now is: can we do the same for simplicial sets themselves, i.e. can we write any simplicial set as a certain colimit made of standard Δ^n simplexes? The formula for the geometric realization of simplicial sets prompts us to consider the following category.

Definition 2.5.7 (Category of elements $\mathsf{E}(\mathfrak{X})$ of a simplicial set). The *category of elements* $\mathsf{E}(\mathfrak{X})$ of a simplicial set $\mathfrak{X}$ is the category whose objects are the simplices $\coprod_{n\in\mathbb{N}} X_n$, and whose morphisms $\mathrm{Hom}_{\mathsf{E}(\mathfrak{X})}(x,y)$ are those $\varphi : [n] \to [m]$ in Δ such that $\mathfrak{X}(\varphi^{\mathrm{op}})(y) = x$.

This provides a functor $\mathsf{E} : \Delta\mathsf{Ens} \to \mathsf{Cat}$ from simplicial sets to (small) categories. We have the canonical projection $\Pi : \mathsf{E}(\mathfrak{X}) \to \Delta$ sending $x \in X_n$ to $[n]$.

Theorem 2.5.8 (Density theorem for simplicial sets). *Every simplicial set $\mathfrak{X}$ is the following colimit:*

$$\mathfrak{X} \;\cong\; \operatorname*{colim}_{\mathsf{E}(\mathfrak{X})} \mathrm{Y}\,\Pi.$$

Proof. The proof is automatic, but we do it because it reveals an essential formula. Let's start by describing the category $\mathsf{Cocone}(\mathrm{Y}\,\Pi)$ of cocones on the functor functor $\mathrm{Y}\,\Pi$. Each of its elements consists in giving a simplicial set $\mathfrak{Y}$ equipped with morphisms of simplicial sets $g(x)\colon \Delta^n \to \mathfrak{Y}$, for each simplex $x \in X_n$, such that for any increasing application $\varphi\colon [n] \to [m]$ verifying $\mathfrak{X}(\varphi^{\mathrm{op}})(y) = x$, the following diagram is obtained

$$(*)\qquad \begin{tikzcd} & \mathfrak{Y} & \\ \Delta^n \arrow[ur, "g(x)"] \arrow[rr, "\varphi_*"'] & & \Delta^m \arrow[ul, "g(y)"'] \end{tikzcd}$$

is commutative. The theorem 2.2.4 asserting that the category of simplexes is generated by cofaces and co-degeneracies shows that it is necessary and sufficient for the $(*)$ diagrams associated with the latter to be commutative. The commutativity of $(*)$ for the cofaces $\delta_i : \Delta^{n-1} \to \Delta^n$ and for co-degeneracies $\sigma_i : \Delta^n \to \Delta^{n+1}$ are respectively equivalent to

$$(**) \quad g\left(d_i^{\mathfrak{X}}(x)\right)(01\cdots n-1) = d_i^{\mathfrak{Y}}\big(g(x)(01\cdots n)\big) \quad \text{and} \quad g\left(s_i^{\mathfrak{X}}(x)\right)(01\cdots n+1) = s_i^{\mathfrak{Y}}\big(g(x)(01\cdots n)\big)\ .$$

Let's now show that we can equip the simplicial set $\mathfrak{X}$ with an initial cocone structure on the functor $\mathrm{Y}\circ\Pi$. For any simplex $x \in X_n$, consider the canonical morphism $f_x\colon \Delta^n \to \mathfrak{X}$ provided by Yoneda's lemma 2.5.4. The naturalness of the latter shows that f_x morphisms verify the equations $(**)$. Finally, for any other cocone $(\mathfrak{Y}, \{g(x)\})$, there exists a unique morphism $G\colon \mathfrak{X} \to \mathfrak{Y}$ of cocones: this is the one given by $G(x) = g(x)(01\cdots n)$. ☐

Corollary 2.5.9. *Every simplicial set $\mathfrak{X}$ is the following coequalizer:*

$$\coprod_{\substack{\varphi:[n]\to[m]\\ \varphi\in\{\delta_i,\sigma_i\}}} X_m\times\Delta^n \underset{\mathrm{id}\times\varphi_*}{\overset{\mathfrak{X}(\varphi^{\mathrm{op}})\times\mathrm{id}}{\rightrightarrows}} \coprod_{n\in\mathbb{N}} X_n\times\Delta^n \twoheadrightarrow \mathfrak{X}.$$

Proof. Apply the colimit description to the small category diagram from $\mathsf{E}(\mathfrak{X})$, plus face and degeneracy relations. ☐

Hence we recover, from a category-theoretic viewpoint, the usual formula for the geometric realization of simplicial sets.

Having understood the form that any simplicial set can take, we can now ask how to obtain functors from or to simplicial sets. As it happens, we can characterize all functor pairs involving the category of simplicial sets. All examples are obtained in the following way, see below and Section ??.

Definition 2.5.10 (Representation by a cosimplicial object). Suppose $\mathfrak{C}$ is a cosimplicial object in a locally small category C, i.e., a functor $\mathfrak{C} : \Delta \to \mathsf{C}$. Then its *associated simplicial representation* is

$$\mathrm{R}_{\mathfrak{C}} : \mathsf{C} \to \Delta\mathsf{Ens}, \quad c \mapsto \mathrm{Hom}_{\mathsf{C}}(\mathfrak{C}, c).$$

Theorem 2.5.11. *Let C be a locally small and cocomplete category. Giving a pair of adjoint functors*

$$\mathrm{L} \ : \ \Delta\mathsf{Ens} \underset{\perp}{\rightleftarrows} \mathsf{C} \ : \ \mathrm{R}$$

is equivalent to giving a cosimplicial object $\mathfrak{C} : \Delta \to \mathsf{C}$ by restriction of L to $\mathrm{Y} : \Delta \hookrightarrow \Delta\mathsf{Ens}$:

$$\mathfrak{C} = \mathrm{L}\,\mathrm{Y} \ : \ \Delta \xrightarrow{\mathrm{Y}} \Delta\mathsf{Ens} \xrightarrow{\mathrm{L}} \mathsf{C}.$$

In that case, the right adjoint is the associated simplicial representation $\mathrm{R}_{\mathfrak{C}}$, and the left adjoint is given by the coequalizer

$$(7) \qquad \coprod_{\substack{\varphi:[n]\to[m]\\ \varphi\in\{\delta_i,\sigma_i\}}} X_m\times C^n \underset{\mathrm{id}\times\varphi_*}{\overset{\mathfrak{X}(\varphi^{\mathrm{op}})\times\mathrm{id}}{\rightrightarrows}} \coprod_{n\in\mathbb{N}} X_n\times C^n \twoheadrightarrow \mathrm{L}(\mathfrak{X}),$$

where $C^n = \mathfrak{C}([n])$ and $X_n \times C^n \coloneqq \coprod_{X_n} C^n$.

Proof. Let $\mathfrak{C}$ be a cosimplicial object of C. Consider the simplicial representation functor $\mathrm{R}_{\mathfrak{C}}\colon \mathsf{C} \to \Delta\mathsf{Ens}$. In the other direction, we pose

$$\mathrm{L}(\mathfrak{X}) \coloneqq \operatorname*{colim}_{\mathsf{E}(\mathfrak{X})} \mathfrak{C}\,\Pi\ .$$

Since $\mathsf{E}\colon \Delta\mathsf{Ens} \to \mathsf{Cat}$ is a functor, we see that L is also a functor. It remains to check that L is left adjoint of $\mathrm{R}_{\mathfrak{C}}$:

$$\mathrm{Hom}_{\mathsf{C}}\big(\mathrm{L}(\mathfrak{X}), c\big) \cong \mathrm{Hom}_{\Delta\mathsf{Ens}}\big(\mathfrak{X}, \mathrm{R}_{\mathfrak{C}}(c)\big)\ .$$

As $\mathrm{L}(\mathfrak{X}) = \mathrm{colim}_{\mathsf{E}(\mathfrak{X})}\,\mathfrak{C}\,\Pi$, any morphism in C to c is equivalent to giving morphisms $g(x)\colon C^n \to c$ of C, for any simplex $xinX_n$, such that $g(x) = g(y)\circ\mathfrak{C}(\varphi)$ when $\mathfrak{X}(\varphi^{\mathrm{op}})(y) = x$. Such a datum is equivalent to a morphism of simplicial sets $\mathfrak{X} \to \mathrm{R}_{\mathfrak{C}}(c)$ defined by $x \in X_n \mapsto g(x) \in \mathrm{Hom}_{\mathsf{C}}(C^n, c)$. It's easy to see that this bijection is natural on both sides.

In the other direction, the compound $\mathrm{LY}\colon \Delta \to \mathsf{C}$ defines a cosimplicial object $\mathfrak{C}$ of C. Since the functor L is left adjoint, it preserves colimites, so we have that it is of the form $\mathrm{L}(\mathfrak{X}) \coloneqq \mathrm{colim}_{\mathsf{E}(\mathfrak{X})}\,\mathfrak{C}\,\Pi$. The arguments given above show that $\mathrm{R}_{\mathfrak{C}}$ is its right-hand adjoint and is therefore equal to the functor R.

Since the functor L preserves colimites, we can also write it with the co-equalizer given in corollary 2.5.9. ☐

This last formula has the same aroma as the geometric realization; this is no coincidence, as we'll see in proposition 2.6.5 that the latter can be obtained in this way.

Remark. In the language of appendix A.2, the left adjoint functor L is the left Kan extension of $\mathfrak{C}$ along the Yoneda fold:

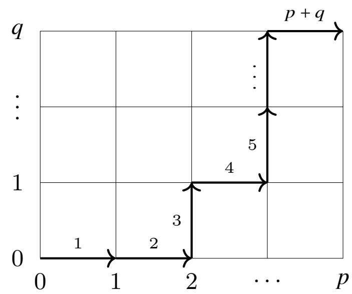
$$\begin{tikzcd} \Delta \arrow[rr, "\mathfrak{C}"] \arrow[dr, hook, "\mathrm{Y}"'] & & \mathsf{C} \\ & \Delta\mathsf{Ens} \arrow[ur, "\mathrm{L}=\mathrm{Lan}_{\mathrm{Y}}\mathfrak{C}"'] & \end{tikzcd} .$$

Definition 2.5.12 (Product of simplicial sets). The *product* $\mathfrak{X} \times \mathfrak{Y}$ of two simplicial sets $\mathfrak{X}$ and $\mathfrak{Y}$ is defined by

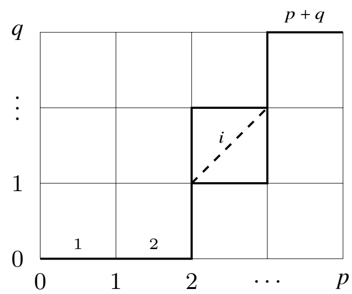
$$(\mathfrak{X} \times \mathfrak{Y})_n \ = \ X_n \times Y_n$$

with face maps $(d_i^{\mathfrak{X}} \times d_i^{\mathfrak{Y}})$ and degeneracy maps $(s_i^{\mathfrak{X}} \times s_i^{\mathfrak{Y}})$.

Example (Prismatic decomposition). To understand the effect of the product of two simplicial sets, we recommend the following exercise: show that the simplicial set $\Delta^p \times \Delta^q$ is generated by $\binom{p+q}{p}$ copies of Δ^{p+q} indexed by increasing paths on a grid $[p] \times [q]$.



or by (p, q)-battages, i.e. permutations $\sigma \in \mathbb{S}_{p+q}$ verifying $\sigma(1) < cdots < \sigma(p)$ and $sigma(p + 1) < cdots < \sigma(p + q)$. In the example above, the $(4, 3)$-shuffle is $\sigma = [1247356]$. These standard $(p + q)$-simplexes fit together as follows: for any ascending path with a diagonal that is its ith segment, the images by δ_i of Δ^{p+q-1} in the copies of Δ^{p+q} indexed by the same path but passing through the top-left corner and the bottom-right corner of the diagonal square are equal.



Finally, we obtain $\Delta^p \times \Delta^q$ as the following co-equalizer

$$\coprod \Delta^{p+q-1} \overset{\delta_i}{\underset{\delta_i}{\rightrightarrows}} \coprod \Delta^{p+q} \twoheadrightarrow \Delta^p \times \Delta^q \ ,$$

where the second coproduct covers increasing paths and the first covers increasing paths with one diagonal.

Proposition 2.5.13. *The category* $(\Delta\mathsf{Ens}, \times, *)$ *of simplicial sets equipped with their product is a symmetric monoidal cartesian category, with unit the simplicial set constant at a single point.*

Proof. Checks are automatic. Remember that "cartesian" means that the monoidal product is the categorical product and the unit is the terminal object. ☐

Let's show that this symmetrical monoidal category is closed, i.e. it has an internal hom. Suppose we have such a bifunctor $\mathfrak{Hom}$; it must then be equipped with a natural bijection

$$\mathrm{Hom}_{\Delta\mathsf{Ens}}(\mathfrak{Z}\times\mathfrak{X},\mathfrak{Y}) \cong \mathrm{Hom}_{\Delta\mathsf{Ens}}(\mathfrak{Z},\mathfrak{Hom}(\mathfrak{X},\mathfrak{Y}))\ .$$

Considering the special case of the standard simplex $\mathfrak{Z}=\Delta^n$, Yoneda's simplicial lemma imposes the following form

$$\mathfrak{Hom}(\mathfrak{X},\mathfrak{Y})_n \cong \mathrm{Hom}_{\Delta\mathsf{Ens}}(\Delta^n\times\mathfrak{X},\mathfrak{Y})\ .$$

We have here a first example of the application of Definition 2.5.10: to any simplicial set $\mathfrak{X}$, we associate the functor

$$\begin{array}{cccccc}\mathfrak{X}\times\Delta^\bullet & : & \Delta & \to & \Delta\mathsf{Ens} \\ & & [n] & \mapsto & \mathfrak{X}\times\Delta^n\ ,\end{array}$$

which is in fact the compound of the Yoneda plunge with the product with $\mathfrak{X}$. The simplicial representation associated with this cosimplicial simplicial set $\mathfrak{X}\times\Delta^\bullet$ is

$$\begin{array}{cccccc}\mathrm{R}_{\mathfrak{X}\times\Delta^\bullet} & : & \Delta\mathsf{Ens} & \to & \Delta\mathsf{Ens} \\ & & \mathfrak{Y} & \mapsto & \mathrm{Hom}_{\Delta\mathsf{Ens}}(\mathfrak{X}\times\Delta^\bullet,\mathfrak{Y})\ .\end{array}$$

This interpretation introduces the simplicial set we're looking for.

Definition 2.5.14 (Morphisms space). For any pair $\mathfrak{X},\mathfrak{Y}$ of simplicial sets, we call *space of morphisms from $\mathfrak{X}$ to $\mathfrak{Y}$* the simplicial set

$$\mathfrak{Hom}(\mathfrak{X},\mathfrak{Y}) := \mathrm{Hom}_{\Delta\mathsf{Ens}}(\mathfrak{X}\times\Delta^\bullet,\mathfrak{Y})\ .$$

The 0-simplexes of the space of morphisms is the set of morphisms of simplicial sets:

$$\mathfrak{Hom}(\mathfrak{X},\mathfrak{Y})_0 = \mathrm{Hom}_{\Delta\mathsf{Ens}}(\mathfrak{X}\times\Delta^0,\mathfrak{Y}) = \mathrm{Hom}_{\Delta\mathsf{Ens}}(\mathfrak{X},\mathfrak{Y})\ .$$

The simplicial Yoneda lemma (Proposition refprop:FullYonedaEnsS) shows that the space of morphisms from the "point" $*$ to a simplicial set $\mathfrak{Y}$ is equal to $\mathfrak{Y}$ as a whole:

$$\mathfrak{Hom}(*,\mathfrak{Y}) = \mathfrak{Y}\ .$$

Remark. The space of morphisms is sometimes also called *exponential object* and denoted $\mathfrak{Y}^{\mathfrak{X}}$.

Proposition 2.5.15. *This internal hom endows* $(\Delta\mathsf{Ens},\times,*)$ *with the structure of a closed symmetric monoidal category.*

Proof. The aim is to find a natural bijection of the form

$$\mathrm{Hom}_{\Delta\mathsf{Ens}}(\mathfrak{Z}\times\mathfrak{X},\mathfrak{Y}) \cong \mathrm{Hom}_{\Delta\mathsf{Ens}}(\mathfrak{Z},\mathfrak{Hom}(\mathfrak{X},\mathfrak{Y}))\ .$$

To do this, consider the simplicial representation functor associated with $\mathfrak{X}\times\Delta^\bullet$ which gives $\mathrm{R}_{\mathfrak{X}\times\Delta^\bullet}(\mathfrak{Y}) = \mathrm{hom}(\mathfrak{X},\mathfrak{Y})$. Theorem 2.5.11 provides a left adjoint functor L characterized by its values on standard simplexes: $\mathrm{L}(\Delta^n) = \mathfrak{X}\times\Delta^n \cong \Delta^n\times\mathfrak{X}$. This functor L is therefore the functor "product with $\mathfrak{X}$", i.e. $\mathrm{L}(\mathfrak{Y}) = \mathfrak{X}\times\mathfrak{Y}$, since the latter preserves colimites. This concludes the demonstration. ☐

The counity of this adjunction provides a natural morphism *of evaluation* :

$$\mathrm{ev}_{\mathfrak{X},\mathfrak{Y}}\ :\ \mathfrak{Hom}(\mathfrak{X},\mathfrak{Y})\times\mathfrak{X}\to\mathfrak{Y}\ .$$

We define a composition at the level of morphism spaces

$$\mathfrak{Hom}(\mathfrak{Y},\mathfrak{Z})\times\mathfrak{Hom}(\mathfrak{X},\mathfrak{Y})\to\mathfrak{Hom}(\mathfrak{X},\mathfrak{Z})$$

by considering the morphism of simplicial sets adjoint to the following double evaluation:

$$\mathfrak{Hom}(\mathfrak{Y},\mathfrak{Z})\times\mathfrak{Hom}(\mathfrak{X},\mathfrak{Y})\times\mathfrak{X} \xrightarrow{\mathrm{ev}_{\mathfrak{X},\mathfrak{Y}}} \mathfrak{Hom}(\mathfrak{Y},\mathfrak{Z})\times\mathfrak{Y} \xrightarrow{\mathrm{ev}_{\mathfrak{Y},\mathfrak{Z}}} \mathfrak{Z}\ .$$

Corollary 2.5.16. *The category of simplicial sets with the space of morphisms is enriched on itself.*

Proof. With the general composition described above, any closed symmetrical monoidal category is enriched on itself, see for example [?, Section 1.6]. ☐

Corollary 2.5.17. *There is a natural bijection*

$$\mathfrak{Hom}(\mathfrak{X}\times\mathfrak{Y},\mathfrak{Z}) \cong \mathfrak{Hom}(\mathfrak{X},\mathfrak{Hom}(\mathfrak{Y},\mathfrak{Z}))\ .$$

Proof. This is a special case of the general theory of closed symmetric monoidal categories, see for example [?, Section 1.5]. ☐

2.6. **Conceptual examples**. We continue here in the same vein as that which motivated the definition of morphism spaces: *all* the examples of functors with values in simplicial sets given below are of the form described in definition 2.5.10 and theorem 2.5.11.

2.6.1. *Constant simplicial sets.*

Definition 2.6.1 (Constant simplicial set). For any set E, we can associate the *constant simplicial set* $\mathfrak{c}(E)$ defined by

$$X_n := E, \quad d_i := \mathrm{id}_E, \;\; s_i := \mathrm{id}_E.$$

This first, particularly simple family of examples of simplicial sets is in the form of those obtained by Theorem 2.5.11; for this, we need only consider the category $\mathsf{C} = \mathsf{Ens}$ of sets and the constant cosimplicial set $\mathfrak{C} : [n] \mapsto \{*\}$ whose image is the one-element set. Using the notations of this theorem, we have $\mathrm{R}_{\mathfrak{C}}(E) = \mathfrak{c}(E)$.

Proposition 2.6.2. *The construction of constant simplicial sets is a fully faithful functor* $\mathsf{Ens} \to \Delta\mathsf{Ens}$*, whose image is the subcategory of dimension-*0 *simplicial sets.*

Proof. Checks are automatic. ☐

Proposition 2.6.3. *The "constant simplicial set" functor* $\mathfrak{c}$ *has a right-hand adjoint given by the truncation functor*

$$\mathrm{T} : \Delta\mathsf{Ens} \to \mathsf{Ens}\,, \quad \mathfrak{X} \mapsto X_0$$

and a left adjoint given by the truncation functor modulo the images of the first faces

$$\widetilde{\mathrm{T}} : \Delta\mathsf{Ens} \to \mathsf{Ens}\,, \quad \mathfrak{X} \mapsto X_0/\sim\,,$$

where $d_0(x) \sim d_1(x)$*, for* $x \in X_1$.

$$\mathfrak{c} : \mathsf{Ens} \underset{\perp}{\rightleftarrows} \Delta\mathsf{Ens} : \mathrm{T} \qquad\qquad \widetilde{\mathrm{T}} : \Delta\mathsf{Ens} \underset{\perp}{\rightleftarrows} \mathsf{Ens} : \mathfrak{c}\,.$$

Proof. Checks are automatic. Note, however, that the second case is produced by the theorem reftheo:ExConc applied to the set cosimplicial constant $\mathfrak{C} : [n] \mapsto \{*\}$ whose image is the one-element set. ☐

The addition $\mathfrak{c} \dashv \mathrm{T}$ provides the category equivalence between the category of sets and that of simplicial sets of dimension 0 established at proposition 2.6.2.

2.6.2. *Singular simplicial set.* Let's use Theorem 2.5.11 to functorially associate a simplicial set with any topological space. We have already seen that the collection of standard geometric simplexes with their cofaces and co-degeneracies form a cosimplicial topological space:

$$\begin{array}{rccc} |\Delta^\bullet| & : & \Delta & \to & \mathsf{Top} \\ & & [n] & \mapsto & |\Delta^n|\,. \end{array}$$

Definition 2.6.4 (Singular simplicial set). The *singular simplicial set* of a topological space X is defined by

$$\mathrm{Sing}\, X \;:=\; \mathrm{Hom}_{\mathsf{Top}}(|\Delta^\bullet|, X).$$

Its elements are continuous maps $f : |\Delta^n| \to X$, called *singular simplices*; the face and degeneracy maps come from post-composition with δ_i and σ_i on $|\Delta^n|$.

Proposition 2.6.5. *The application that associates the singular simplicial set with a topological space is a functor*

$$\begin{array}{rcccc} \mathrm{Sing} & : & \mathsf{Top} & \to & \Delta\mathsf{Ens} \\ & & X & \mapsto & \mathrm{Sing}\, X \end{array}$$

whose left-hand adjoint is the geometric realization

$$|\text{-}| : \Delta\mathsf{Ens} \underset{\perp}{\rightleftarrows} \mathsf{Top} : \mathrm{Sing}\,.$$

Proof. This result is a special case of the theorem 2.5.11 : the functor of singular simplicial sets is represented by the cosimplicial topological space $|\Delta^\bullet|$. It admits a left adjoint given by the coequalizer

$$\coprod_{\substack{\varphi:[n]\to[m] \\ \varphi\in\{\delta_i,\sigma_i\}}} X_m \times |\Delta^n| \underset{\mathrm{id}\times\varphi}{\overset{\mathfrak{X}(\varphi^{\mathrm{op}})\times\mathrm{id}}{\rightrightarrows}} \coprod_{n\in\mathbb{N}} X_n \times |\Delta^n| \twoheadrightarrow |\mathfrak{X}|$$

which is the formula used to define the geometric realization. ☐

This series of results therefore explains conceptually and retrospectively the formula chosen for the geometric realization.

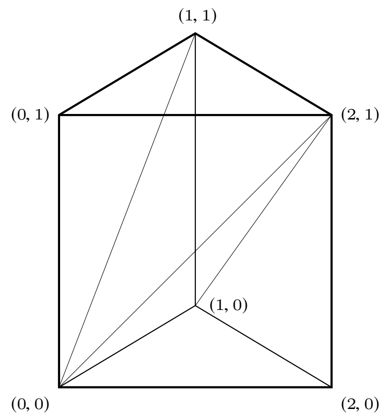
Proposition 2.6.6. *Let $\mathfrak{X}, \mathfrak{Y}$ be a pair of simplicial sets such that either both have a countable number of simplexes or at least one of the two geometric realizations is locally finite, i.e. any point is inside a simplicial polyhedron. In this case, we have a homeomorphism*

$$|\mathfrak{X} \times \mathfrak{Y}| \ \cong \ |\mathfrak{X}| \times |\mathfrak{Y}|.$$

Proof. We refer the reader to [?, Theorem 14.3]. □

Remark. The problem raised by the technical assumptions is always the same: it has to do with the topology considered on the product spaces. The above-mentioned property is therefore true if we're working in the category of compactly generated, weakly Hausdorff topological spaces. Otherwise, it suffices to require that $|\mathfrak{X} \times \mathfrak{Y}|$ is a CW-complex, which is implied by the assumptions used here.

Note that this homeomorphism, seen from left to right, is not cellular. This is in fact positive: the left-hand member provides a canonical triangulation for the product of topological spaces on the right. In the case of $\Delta^p \times \Delta^q$, we obtain the prismatic decomposition given by increasing paths or (p, q)-beats. The drawing below shows the case of $\Delta^2 \times \Delta^1$.



2.6.3. *Nerve of a category.* Consider the category Cat of small categories. The following functor provides a cosimplicial object in the category of categories:

$$\begin{array}{rcl} \mathfrak{C} \ : \quad \Delta & \to & \mathsf{Cat} \\ {[n]} & \mapsto & \mathsf{Cat}[n] := \{0 \to 1 \to \cdots \to n\} , \end{array}$$

where $\mathsf{Cat}[n]$ is the category associated with the totally ordered set $[n]$.

Definition 2.6.7 (Nerve of a category). The *nerve* of a small category C is the simplicial set

$$\mathfrak{N}\mathsf{C} \ := \ \mathrm{Hom}_{\mathsf{Cat}}(\mathfrak{C}, \mathsf{C}).$$

Exercise. Show that the nerve of the category $\mathsf{Cat}[n]$ is the standard n-simplex: $\mathfrak{N}\mathsf{Cat}[n] = \Delta^n$.

Proposition 2.6.8. *The n-simplices of $\mathfrak{N}\mathsf{C}$ are the composable chains of n morphisms in C:*

$$\mathfrak{N}\mathsf{C}_n = \left\{ \ c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} c_n \ \right\} ,$$

with the convention that $\mathfrak{N}\mathsf{C}_0$ is made up of C objects. The faces are given by

$$d_i(f_1, \ldots, f_n) = (f_1, \ldots, f_{i+1}f_i, \ldots, f_n), \quad \textit{for} \quad 1 \leqslant i \leqslant n-1$$

and $d_0(f_1, \ldots, f_n) = (f_2, \ldots, f_n)$, $d_n(f_1, \ldots, f_n) = (f_1, \ldots, f_{n-1})$. *The degenerations are given by*

$$s_i(f_1, \ldots, f_n) = (f_1, \ldots, f_i, \mathrm{id}, f_{i+1}, \ldots, f_n), \quad \textit{for} \quad 0 \leqslant i \leqslant n .$$

Proof. The checks are automatic. □

Example. The *nerve* $\mathrm{B}G$ *of a group* is the nerve of the associated single-object category. Explicitly, this simplicial set is formed by the n-simplexes $\mathrm{B}G_n = G^n$, with $\mathrm{B}G_0 = \{1\}$, provided with the following faces and degeneracies

$$d_i(g_1, \ldots, g_n) \quad = \quad \left\{ \begin{array}{ll} (g_2, \ldots, g_n) & \text{for } i = 0 , \\ (g_1, \ldots, g_{i+1}g_i, \ldots, g_n) & \text{pour } 1 \leqslant i \leqslant n-1 , \\ (g_1, \ldots, g_{n-1}) & \text{for } i = n , \end{array} \right.$$

$$s_i(g_1,\ldots,g_n) \quad = \quad (g_1,\ldots,g_i,1,g_{i+1},\ldots,g_n)\,.$$

Its geometric realization $|\mathrm{B}G|$ is the *classifying space* of the group G.

A useful feature of this conceptual approach, rather than the explicit definition, is the full application of Theorem 2.5.11, which shows that this construction is functorial and admits a computable left adjoint.

Definition 2.6.9 (Fundamental category $\tau_1(\mathfrak{X})$). To any simplicial set $\mathfrak{X}$, we associate the *fundamental category* $\tau_1(\mathfrak{X})$ whose objects are the 0-simplexes X_0 and whose morphisms are given by the free graph on arrows indexed by the 1-simplexes

$$d_1(x) \xrightarrow{x\in X_1} d_0(x)$$

quotient by the relations given by the 2-simplexes

$$\begin{tikzcd} & \bullet \arrow[dr, "d_0(y)"] & \\ \bullet \arrow[ur, "d_2(y)"] \arrow[rr, "d_1(y)"'] & y\in X_2 & \bullet\,. \end{tikzcd}$$

In this category, the images of the first degeneracy $s_0 : X_0 \to X_1$ give the identity morphisms. The second two degeneracies $s_0 : X_1 \to X_2$ and $s_1 : X_1 \to X_2$ show respectively that right- and left-hand compounding by these identities leaves the morphisms invariant.

Proposition 2.6.10. *The assignment that sends a small category to its nerve defines a functor*

$$\begin{array}{ccc} \mathsf{Cat} & \longrightarrow & \Delta\mathsf{Ens} \\ \mathsf{C} & \longmapsto & \mathfrak{N}\mathsf{C} \end{array}$$

which has as its left adjoint the fundamental-category functor τ_1

$$\tau_1 : \Delta\mathsf{Ens} \underset{\perp}{\rightleftarrows} \mathsf{Cat} : \mathfrak{N}\,.$$

Proof. This follows directly from Theorem 2.5.11. One need only show that the fundamental category $\tau_1(\mathfrak{X})$ is isomorphic to the coequalizer (7). We begin by noting that the fundamental category associated to the standard simplex is the category associated to the poset $[n]$, i.e. $\tau_1(\Delta^n) \cong \mathsf{Cat}[n]$. Then, (Proof details to be expanded.) □

Remark. Recall that a groupoid is a category in which all morphisms are invertible. The forgetful functor from the category of small groupoids to that of small categories admits a left adjoint. Applying that left adjoint to $\tau_1(\mathfrak{X})$ then produces a groupoid $\pi_1(\mathfrak{X})$ called the *fundamental groupoid* of the simplicial set $\mathfrak{X}$.

2.6.4. *Dold–Kan correspondence.* This example provides an opportunity to introduce some constructions that link simplicial sets to chain complexes. We work here over $\mathbb{Z}$, but one could equally have worked over another base ring. Linear maps between free modules that arise from set maps will be written with a straight roman font, for instance d_i.

Definition 2.6.11 (Moore complex). The *Moore complex* $\mathrm{C}\mathfrak{X}$ of a simplicial set $\mathfrak{X}$ has in degree n the free module on the n-simplices, and its differential is the alternating sum of the faces:

$$\mathrm{d} := \sum_{i=0}^{n} (-1)^i \mathrm{d}_i \ : \ \mathbb{Z}X_n \longrightarrow \mathbb{Z}X_{n-1}\,.$$

The simplicial relations (6) quickly show that the operator d is nilpotent (i.e., $\mathrm{d}^2 = 0$). One may consider the sub-chain-complex $\mathrm{D}\mathfrak{X}$ of the Moore complex generated by the degenerate simplices.

Definition 2.6.12 (Normalized complex). The *normalized complex* $\mathrm{N}\mathfrak{X}$ of a simplicial set $\mathfrak{X}$ is the quotient chain complex of the Moore complex by the degeneracy subcomplex:

$$\mathrm{N}\mathfrak{X} \ := \ \big(\mathrm{C}\mathfrak{X}\,/\,\mathrm{D}\mathfrak{X}, \mathrm{d}\big)\,.$$

Proposition 2.6.13. *For any simplicial set* $\mathfrak{X}$*, the canonical projection* $\mathrm{C}\mathfrak{X} \twoheadrightarrow \mathrm{N}\mathfrak{X}$ *is a homotopy equivalence.*

Proof. The idea is to use degeneracies to define a contracting homotopy. It makes a good exercise, hence we leave it to the reader. For more details, see [?, Theorem VIII.6.1]. □

Hence, the normalized complex indeed has for basis the non-degenerate simplices. The Moore complex and the normalized complex define functors from the category of simplicial sets.

Example. The normalized complex $\mathrm{N}\Delta^n$ associated to the standard n-simplex has as basis the elements of the form $i_0 \cdots i_m$ of degree m, where $i_0 < \cdots < i_m$ in $[n]$, and its boundary is given by

$$\mathrm{d}(i_0 \cdots i_m) = \sum_{j=0}^{m} (-1)^j \, i_0 \cdots \widehat{i_j} \cdots i_m \,.$$

Via the Yoneda embedding (Corollary 2.5.6), the cosimplicial structure on standard simplices yields a cosimplicial chain-complex structure on these normalized complexes:

$$\mathrm{N}\Delta : \Delta \to \mathsf{Ch}.$$

Proposition 2.6.14. *The normalized chain functor admits a right adjoint*

$$\mathrm{N} \ : \ \Delta\mathsf{Ens} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathsf{Ch} \ : \ \mathrm{R}_{\mathrm{N}\Delta}$$

given by

$$\mathrm{R}_{\mathrm{N}\Delta}(C) \cong \bigoplus_{0 \leqslant i_0 < \cdots < i_k \leqslant n} C_k \,.$$

Proof. This is a direct corollary of Theorem 2.5.11. □

The normalized chain functor restricts to a functor $\overline{\mathrm{N}} \colon \mathsf{sAb} \to \mathsf{Ch}_{\geqslant 0}$ from simplicial abelian groups to nonnegatively graded chain complexes by setting $(\overline{\mathrm{N}}\mathfrak{A})_n := A_n$. The other adjoint $\overline{\mathrm{R}}_{\mathrm{N}\Delta} : \mathsf{Ch}_{\geqslant 0} \to \mathsf{sAb}$ is the corresponding restriction of $\mathrm{R}_{\mathrm{N}\Delta}$.

Theorem 2.6.15 (Dold,ÄìKan equivalence). *The adjunction*

$$\overline{\mathrm{N}} \ : \ \mathsf{sAb} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathsf{Ch}_{\geqslant 0} \ : \ \overline{\mathrm{R}}_{\mathrm{N}\Delta}$$

is an equivalence between the category of simplicial abelian groups and that of nonnegatively graded chain complexes.

Proof. The same formulas as in Proposition 2.6.14 define the adjunction in question. It then follows automatically that the unit and counit of the adjunction are isomorphisms. □

2.7. **Kan complexes and ∞-categories.** We are now sufficiently equipped to uncover a simple and powerful notion of higher category. The requirements demand that ordinary categories be among its examples.

Lemma 2.7.1. *The nerve functor* $\mathfrak{N} : \mathsf{Cat} \hookrightarrow \Delta\mathsf{Ens}$ *is fully faithful.*

Proof. We begin by noting that the counit $\varepsilon_{\mathsf{C}} : \tau_1\mathfrak{N}\mathsf{C} \xrightarrow{\cong} \mathsf{C}$ of the adjunction $\tau_1 \dashv \mathfrak{N}$ is a natural isomorphism. Pulling back along it yields the desired natural bijection:

$$\mathrm{Hom}_{\mathsf{Cat}}(\mathsf{C}, \mathsf{D}) \cong \mathrm{Hom}_{\Delta\mathsf{Ens}}(\tau_1\mathfrak{N}\mathsf{C}, \mathsf{D}) \cong \mathrm{Hom}_{\Delta\mathsf{Ens}}(\mathfrak{N}\mathsf{C}, \mathfrak{N}\mathsf{D}).$$

□

Thus simplicial sets form a sufficiently large framework in which one may hope to define a notion of higher category. Let us start by characterizing which simplicial sets are nerves of categories, i.e. describing the essential image of the nerve functor.

The previous result uses the crucial fact that the counit of the fundamental-category,Äìnerve adjunction is a natural isomorphism. The same is not true for the unit of the adjunction, $\upsilon_{\mathfrak{X}} : \mathfrak{X} \to \mathfrak{N}\tau_1(\mathfrak{X})$. This already fails for 1-simplices: any element $x \in X_1$ yields a morphism $d_2(x) \to d_0(x)$, yet the morphisms of $\mathfrak{N}\tau_1(\mathfrak{X})$ are free composites of such morphisms. For the horn $\mathfrak{X} = \Lambda^2_1$, the unit of adjunction is the inclusion

$$\upsilon_{\Lambda^2_1} : \Lambda^2_1 \hookrightarrow \Delta^2 \cong \mathfrak{N}\,\tau_1(\Lambda^2_1),$$

which is not an isomorphism: we are "missing" in Λ^2_1 everything lying in Δ^2 but outside that horn. To see to which family of simplicial sets we must restrict, we continue and consider the 2-simplices

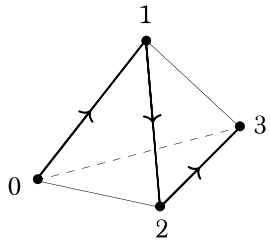
$x \in X_2$ whose image under the unit of the adjunction is $\upsilon(x) = (d_2(x), d_0(x))$. Graphically, we keep only the two "composable faces" of the 2-simplex:

$$
\begin{tikzcd}
& \bullet \arrow[dr, "d_0(x)"] \arrow[d, phantom, "x\in X_2"] & \\
\bullet \arrow[ur, "d_2(x)"] \arrow[rr, dotted] & {} & \bullet
\end{tikzcd}
.
$$

For the unit of adjunction to be an isomorphism, that lone piece of data must specify all of the 2-simplices of $\mathfrak{X}$. First, we note that this data corresponds to a simplicial map $\Lambda^2_1 \to \mathfrak{X}$. Next, it fully determines the 2-simplex if and only if the following diagram admits a unique extension:

(*)
$$
\begin{tikzcd}
\Lambda^2_1 \arrow[r] \arrow[d, hook] & \mathfrak{X} \\
\Delta^2 \arrow[ur, dashed, "\exists!"'] &
\end{tikzcd}
.
$$

In general, the unit of adjunction retains only the "longest composable chain of edges" of an n-simplex



that is, more explicitly

$$\upsilon(x) \;=\; \big(d_2^{n-1}(x),\, d_2^{n-2}d_0(x),\, \ldots,\, d_2 d_0^{n-2}(x),\, d_0^{n-1}(x)\big),$$

where $x \in X_n$ (mildly abusing notation for the faces). For a 3-simplex $x \in X_3$, we can uniquely recover "the face 012," namely $d_3(x)$, via (*), and likewise "the face 123," namely $d_0(x)$, and finally "the face 013," namely $d_2(x)$. This corresponds to a simplicial map $\Lambda^3_1 \to \mathfrak{X}$. Consequently, we need the following diagram to admit a unique extension:

(*)
$$
\begin{tikzcd}
\Lambda^3_1 \arrow[r] \arrow[d, hook] & \mathfrak{X} \\
\Delta^3 \arrow[ur, dashed, "\exists!"'] &
\end{tikzcd}
.
$$

We could equally well have finished by using "the face 023," i.e. $d_1(x)$, requiring a unique extension from the horn $\Lambda^3_2 \to \mathfrak{X}$. In higher dimensions, one is led to the following definitions and conditions.

Definition 2.7.2 (Internal and external horns). *Internal horns* are those of the form Λ^n_k with $1 \le k \le n-1$. *External horns* are those of the form Λ^n_0 or Λ^n_n.

Proposition 2.7.3. *The category of small categories is equivalent to the full subcategory of simplicial sets whose objects satisfy the unique horn-filling property for internal horns:*

$$
\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d, hook] & \mathfrak{X} \\
\Delta^n \arrow[ur, dashed, "\exists!"'] &
\end{tikzcd}
\qquad \textit{for } n \ge 2 \textit{ and } 1 \le k \le n-1.
$$

Proof. We note first that the nerve of any category satisfies that unique extension property along internal horns. Then, using the arguments above by induction on n, we show that the unit of the nerve "fundamental category" adjunction is a natural isomorphism precisely on those simplicial sets with unique fillings of internal horns. □

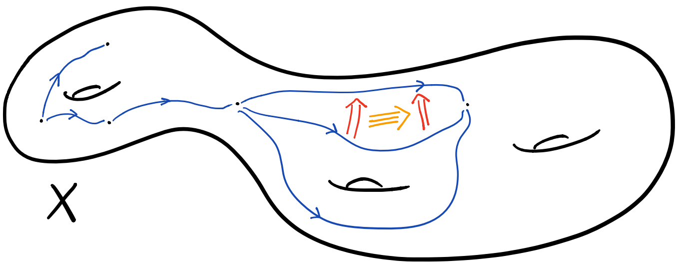
Remark. The nerve of a category generally does *not* satisfy a filling extension property for the external horns. For instance, if one considers the external 2-horns

$$\begin{tikzcd} & y \\ x \arrow[ur, "g"] \arrow[rr, "\mathrm{id}_x"'] & & x \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & x \arrow[dr, "f"] & \\ y \arrow[rr, "\mathrm{id}_y"'] & & y \end{tikzcd} ,$$

the extension property would imply the existence of left and right inverses respectively.

Exercise. Show that the category of small groupoids, i.e. small categories in which all morphisms are invertible, is equivalent to the full subcategory of simplicial sets having unique extension of *every* horn Λ^n_k, for $n \geq 2$ and $0 \leq k \leq n$.

How could higher-dimensional morphisms naturally appear on a mathematical object? We already considered this question in Section 1.1.3. One may also address it by trying to construct a *fundamental ∞-groupoid* attached to a topological space X. Recall that the fundamental groupoid of X is the category whose objects are the points of X and whose morphisms are homotopy classes of paths between two points. Because of the parameterization of paths, passing to the homotopy quotient enforces that composition be associative. The problem is that such a construction does not see the higher homotopy data. Hence we want to consider a fundamental ∞-groupoid of X whose objects are the points, whose 1-morphisms are the paths between points, whose 2-morphisms are homotopies between paths, whose 3-morphisms are homotopies between homotopies, and so on.



A first obstacle is that path-composition is then *not* strictly associative, but one can control the failure of associativity with 2-morphisms. Moreover, *Grothendieck homotopy hypothesis* would have ∞-groupoids be "the same as" topological spaces, suitably interpreted. The version we alluded to above is the globular version of a prospective fundamental ∞-groupoid. Instead, here we prefer the simplicial model given by the singular simplicial set.

Hence we want topological spaces, via their singular simplicial sets, to serve as examples for the sought-after notion of higher category. We would thus redo the analysis above but with the singular simplicial set functor in place of the nerve. However, the first result of the preceding part no longer fully holds.

Lemma 2.7.4. *The singular simplicial set functor is faithful but not full.*

Proof. Let $f : X \to Y$ be a continuous map between topological spaces. The induced simplicial map $\mathrm{Sing}(f) : \mathrm{Sing}X \to \mathrm{Sing}Y$ sends a singular simplex $\varphi : |\Delta^n| \to X$ to $f \circ \varphi$. The set of 0-singular simplices is in bijection with the points of the topological space, and under that identification, $\mathrm{Sing}(f)_0 : X \to Y$ is exactly f. Therefore, Sing is faithful.
Next, consider a totally disconnected topological space, e.g. $\mathbb{Q} \subset \mathbb{R}$, and the discrete topological space on the same underlying set, denoted $\mathbb{Q}^{\mathrm{dis}}$. In both cases, the singular simplices are constant maps, so the singular simplicial set is the constant simplicial set $\mathbb{Q}$. But the identity simplicial map $\mathrm{Sing}\mathbb{Q} \to \mathrm{Sing}\mathbb{Q}^{\mathrm{dis}}$ cannot come from the continuous identity $\mathrm{id} : \mathbb{Q} \to \mathbb{Q}^{\mathrm{dis}}$, because that identity is not continuous. □

It is in fact futile to try restricting the category of simplicial sets so as to make the adjunction between singular simplicial sets and geometric realization into an equivalence of categories. Nevertheless, that adjunction does induce an equivalence at the level of homotopy categories (see Theorem 2.8.19). In this context, the simplicial notion "equivalent" to topological spaces is the following.

Definition 2.7.5 (Kan complex). A *Kan complex* is a simplicial set $\mathfrak{X}$ satisfying the extension property for all horns:

(8)
$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & \mathfrak{X} \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd} \qquad \text{for } n \geq 2 \text{ and } 0 \leq k \leq n.$$

Remark. Observe the similarity with the homotopy extension property for fibrations in topology, Definition 1.5.2.

Proposition 2.7.6. *For any topological space X, its singular simplicial set* $\mathrm{Sing}X$ *is a Kan complex.*

Proof. From Proposition 2.6.5, we see that the Kan extension property (8) is equivalent to the analogous property of topological extension

$$\begin{tikzcd} |\Lambda^n_k| \arrow[r] \arrow[d, hook] & X \\ |\Delta^n| \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

which one proves by realizing the k-th horn $|\Lambda^n_k|$ as a retract of the standard geometric n-simplex $|\Delta^n|$. □

The horn-filling extension property for Kan complexes has a purely combinatorial description:

Lemma 2.7.7. *A simplicial set $\mathfrak{X}$ is a Kan complex if and only if, for all $n \geq 2$ and $0 \leq k \leq n$, any collection of n simplices $x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n \in X_{n-1}$ satisfying $d_i(x_j) = d_{j-1}(x_i)$ for $i < j$, $i \neq k$, admits an n-simplex $x \in X_n$ such that $d_i(x) = x_i$ for all $0 \leq i \leq n$, $i \neq k$.*

Proof. This follows directly from Corollary 2.5.5 and the Yoneda Lemma 2.5.4. □

Exercise.

- Show that the nerve of a group is a Kan complex and that, if the group is nontrivial, it cannot be isomorphic to a singular simplicial set. Hence Kan complexes do not form the essential image of the singular simplicial-set functor.
- Show that the simplicial set underlying a simplicial group is a Kan complex.
- Show that the standard n-simplices Δ^n are not Kan complexes for $n \geq 2$.

We have arrived at the crux: to obtain a good definition of higher category, one need only consider those simplicial sets that satisfy a property shared by category nerves and singular simplicial sets: the (not necessarily unique) extension property for *internal* horns.

Definition 2.7.8 (∞-category). An *∞-category* is a simplicial set $\mathfrak{X}$ that satisfies the extension property along internal horns:

(9)
$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & \mathfrak{X} \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd} \qquad \text{for } n \geq 2 \text{ and } 1 \leq k \leq n-1.$$

This notion was introduced by Boardman–Vogt in their study of homotopical properties of algebraic structures [?], under the name *weak Kan complex.* It was then intensively studied by Joyal [?] and by Lurie [Lur09].

Retrospectively, one can interpret the 0-simplices of an ∞-category as its objects, the 1-simplices as its morphisms, and so forth. The extension condition along the horn Λ^2_1

$$\begin{tikzcd} & \bullet \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow"] & \\ \bullet \arrow[ur, "f"] \arrow[rr, dashed, "\text{“}g\circ f\text{”}"'] & {} & \bullet \end{tikzcd} .$$

gives a candidate for the composite "$g \circ f$" of two composable morphisms, plus a "homotopy" from (g, f) to "$g \circ f$". Hence composition is not unique nor strict. Far from being a defect, this extra freedom is desirable and, in any case, different choices are homotopically unique.

Definition 2.7.9 (∞-groupoid)**.** An *∞-groupoid* is a simplicial set $\mathfrak{X}$ that satisfies the extension property along *all* horns Λ^n_k for $n \geq 2$ and $0 \leq k \leq n$.

Remark. The definitions of ∞-category and ∞-groupoid above are given by a *property*: the existence of fillers for horns. Sometimes one needs a more algebraic viewpoint, requiring the data of those fillers. This yields the notions of *algebraic ∞-category* and *algebraic ∞-groupoid*, which are in fact algebras over certain monads. Among other consequences, they automatically possess all limits and colimits. A key application of the latter notion is given in [?], where formulas from Lie theory (Baker–Campbell–Hausdorff) are recovered and extended by filling horns.

2.8. **Simplicial homotopy**. Simplicial homotopy is the first motivation for introducing simplicial sets. It allows one to combinatorially encode the homotopical properties of topological spaces. One must be careful, though, that the general theory applies only to those simplicial sets which are "spaces," i.e. Kan complexes.

Definition 2.8.1 (Path)**.** A *path* in a simplicial set $\mathfrak{X}$ is a map of simplicial sets $p : \Delta^1 \to \mathfrak{X}$.

Such data corresponds to

$$0 \xrightarrow{01} 1 \quad \overset{p}{\longmapsto} \quad p(0) = d_1(x) \xrightarrow{x} p(1) = d_0(x)$$

with $p(01) = x \in X_1$.

Definition 2.8.2 (Homotopic elements)**.** Two 0-simplices $a, b \in X_0$ of a simplicial set $\mathfrak{X}$ are *homotopic*, written $a \sim b$, if there is a path $p : \Delta^1 \to \mathfrak{X}$ such that $p(0) = a$ and $p(1) = b$, i.e. if there exists $x \in X_1$ with $d_1(x) = a$ and $d_0(x) = b$.

Equivalently, one says a and b lie in the same *connected component*.

Remark. This binary relation $\sim$ is *not* generally an equivalence relation. For example, in the 1-simplex Δ^1, 0 is homotopic to 1 but not vice versa.

That is precisely why one must consider Kan complexes:

Proposition 2.8.3. *For a Kan complex $\mathfrak{X}$, the relation $\sim$ is an equivalence relation.*

Proof.

Reflexivity: For any 0-simplex $a \in X_0$, we have $a \sim a$, witnessed by the path $s_0(a)$:

$$a = d_1 s_0(a) \xrightarrow{s_0(a)} d_0 s_0(a) = a \ .$$

Transitivity: Let $a \sim b$ and $b \sim c$ have paths x and y respectively. Such a datum is equivalent to the following 2-cornet of $\mathfrak{X}$:

$$\begin{tikzcd} & b \arrow[dr, "y"] \arrow[d, phantom, "z"] & \\ a \arrow[ur, "x"] \arrow[rr, dotted, "d_1(z)"'] & {} & c \end{tikzcd} .$$

Kan's extension property provides a 2-simplex $z \in X_2$ such that $d_2(z) = x$, $d_0(z) = y$. The 1-simplex $d_1(z)$ is a path connecting a and c.

Symmetry: Let $a \sim b$ be connected by a path $x \in X_1$. Consider the following 2-horn

$$\begin{tikzcd} & b \arrow[dr, dotted, "d_0(z)"] \arrow[d, phantom, "z"] & \\ a \arrow[ur, "x"] \arrow[rr, "s_0(a)"'] & {} & a \end{tikzcd}$$

which admits a filling by Kan's extension property. Then the 1-simplex $d_1(z)$ is a path connecting b to a.

□

Remark. We used *all* 2-horns, both internal and external.

Henceforth in this section, we consider only Kan complexes, unless noted otherwise.

Definition 2.8.4 (Connected components). For a Kan complex $\mathfrak{X}$, define

$$\pi_0(\mathfrak{X}) \ := \ X_0/{\sim}$$

to be the set of its *connected components.*

We now apply this to the simplicial mapping space $\mathfrak{Hom}(\mathfrak{X}, \mathfrak{Y})$ between two simplicial sets. Recall its 0- and 1-simplices:

$$\mathfrak{Hom}(\mathfrak{X}, \mathfrak{Y})_0 \ = \ \mathrm{Hom}_{\Delta\mathsf{Ens}}(\mathfrak{X}, \mathfrak{Y}) \quad \text{and} \quad \mathfrak{Hom}(\mathfrak{X}, \mathfrak{Y})_1 \ = \ \mathrm{Hom}_{\Delta\mathsf{Ens}}(\mathfrak{X} \times \Delta^1, \mathfrak{Y}).$$

Definition 2.8.5 (Homotopic morphisms). Two morphisms $f, g : \mathfrak{X} \to \mathfrak{Y}$ of simplicial sets are *homotopic* if they are homotopic as 0-simplices of $\mathfrak{Hom}(\mathfrak{X}, \mathfrak{Y})$; i.e. there is a simplicial map $H : \mathfrak{X} \times \Delta^1 \to \mathfrak{Y}$ with $H(-, 0) = f$ and $H(-, 1) = g$.

The similarity with continuous homotopies is striking. The following theorem is the simplicial analogue of endowing sets of continuous maps with the structure of a topological space:

Theorem 2.8.6. *If $\mathfrak{Y}$ is a Kan complex, then $\mathfrak{Hom}(\mathfrak{X}, \mathfrak{Y})$ is also a Kan complex for any simplicial set $\mathfrak{X}$.*

Proof. A purely combinatorial proof is lengthy; see [?, Theorem 6.9]. A categorical proof uses "anodyne extensions," e.g. [?, Corollary I.5.3]. □

Corollary 2.8.7. *For any simplicial set $\mathfrak{X}$ and Kan complex $\mathfrak{Y}$, homotopy is an equivalence relation on* $\mathrm{Hom}_{\Delta\mathsf{Ens}}(\mathfrak{X}, \mathfrak{Y})$.

Proof. Combine Proposition 2.8.3 with Theorem 2.8.6. □

Denote the set of homotopy classes of maps by $[\mathfrak{X}, \mathfrak{Y}] := \mathfrak{Hom}(\mathfrak{X}, \mathfrak{Y})/{\sim}$. One sees easily that if $f \sim g$, then $fk \sim gk$ and $lf \sim lg$ for any $k : \mathfrak{W} \to \mathfrak{X}$ and $l : \mathfrak{Y} \to \mathfrak{Z}$.

Definition 2.8.8 (Homotopy equivalence). A *homotopy equivalence* is a map of simplicial sets $f : \mathfrak{X} \xrightarrow{\sim} \mathfrak{Y}$ admitting $g : \mathfrak{Y} \to \mathfrak{X}$ such that $gf \sim \mathrm{id}_{\mathfrak{X}}$ and $fg \sim \mathrm{id}_{\mathfrak{Y}}$. Two Kan complexes $\mathfrak{X}, \mathfrak{Y}$ are *homotopy equivalent* if they are connected by a chain of such equivalences.

Proposition 2.8.9.

(1) *The geometric realization functor* $|-| : \Delta\mathsf{Ens} \to \mathsf{Top}$ *sends simplicial homotopies to topological homotopies.*

(2) *Any topological homotopy $H : X \times I \to Y$ between continuous maps $f, g : X \to Y$ induces a simplicial homotopy between* $\mathrm{Sing}(f), \mathrm{Sing}(g) : \mathrm{Sing}(X) \to \mathrm{Sing}(Y)$ *by*

$$\mathrm{Sing}(X) \times \Delta^1 \ \longrightarrow \ \mathrm{Sing}(X) \times \mathrm{Sing}(I) \ \longrightarrow \ \mathrm{Sing}(X \times I) \ \xrightarrow{\mathrm{Sing}(H)} \ \mathrm{Sing}(Y),$$

where the leftmost map is given by $01 \mapsto \mathrm{id}_I$.

Hence the realization and singular simplicial set functors induce bijections between homotopy classes of $\mathfrak{X} \to$ $\mathrm{Sing}(Y)$ *and homotopy classes of continuous maps $|\mathfrak{X}| \to Y$:*

$$\big[\, |\mathfrak{X}|, Y\big] \ = \ \mathrm{Hom}_{\mathsf{Top}}\big(|\mathfrak{X}|, Y\big)/{\sim} \ \cong \ \mathrm{Hom}_{\Delta\mathsf{Ens}}\big(\mathfrak{X}, \mathrm{Sing}(Y)\big)/{\sim} \ = \ \big[\mathfrak{X}, \mathrm{Sing}(Y)\big].$$

Proof.

(1) If $H : \mathfrak{X} \times \Delta^1 \to \mathfrak{Y}$ is a simplicial homotopy between $f, g : \mathfrak{X} \to \mathfrak{Y}$, then using local finiteness of $|\Delta^1|$ and Proposition 2.6.6, $|H|$ is a continuous homotopy $|\mathfrak{X}| \times I \to |\mathfrak{Y}|$ between $|f|, |g|$.

(2) This is exactly as stated.

□

One may extend this analogy with topological spaces further; we do not delve into details here, but we briefly mention:

- A *pair of simplicial sets* $(\mathfrak{X}, \mathfrak{A})$ is a simplicial subset $\mathfrak{A} \subseteq \mathfrak{X}$.
- A *pair of Kan complexes* $(\mathfrak{X}, \mathfrak{A})$ is likewise a pair where both are Kan complexes. For instance, if $\mathfrak{X}$ is a Kan complex, then $(\mathfrak{X}, *)$ is a Kan pair.
- A *relative homotopy* $f \sim g$ rel $\mathfrak{A}$ is a homotopy H between $f, g : \mathfrak{X} \to \mathfrak{Y}$ such that $H(a, t) = f(a) = g(a)$ for $a \in \mathfrak{A}$ and $t \in \Delta^1$. In this case, we use the classic notation $f \sim g$ rel $\mathfrak{A}$.

A *pointed Kan complex* is a Kan complex $\mathfrak{X}$ with a simplicial map $* = \Delta^0 \to \mathfrak{X}$. Equivalently, it is a Kan complex plus a chosen 0-simplex $a \in X_0$. (By mild abuse, we denote this 0-simplex by $*$ in each dimension n.) One defines its homotopy groups just like in topology: first look at the relative homotopy classes

$$\pi_n(\mathfrak{X}, *) \;\coloneqq\; \big[(\Delta^n, \partial\Delta^n), (\mathfrak{X}, *)\big] \;=\; \big\{ f : \Delta^n \to \mathfrak{X} \;\big|\; f|_{\partial\Delta^n} : \partial\Delta^n \to * \big\} \big/ \sim_{\ \mathrm{rel}\,\partial\Delta^n} .$$

Remark. Proposition 2.8.9 shows that $\pi_n(X, x) \cong \pi_n\big(\mathrm{Sing}(X), \mathrm{Sing}(x)\big)$ for any topological space X and $x \in X$.

To make the combinatorial data explicit, adopt the following notation for any n-simplex $x \in X_n$:

$$\partial x \;\coloneqq\; \big(d_0(x),\, d_1(x),\, \ldots,\, d_{n-1}(x),\, d_n(x)\big).$$

Lemma 2.8.10. *Let $(\mathfrak{X}, *)$ be a pointed Kan complex. A map $f : (\Delta^n, \partial\Delta^n) \to (\mathfrak{X}, *)$ is equivalent to giving an n-simplex $x \in X_n$ such that $\partial x = (*, \ldots, *)$. Under this identification, two maps f, g are homotopic* rel $\partial\Delta^n$ *if and only if there is an $(n+1)$-simplex $w \in X_{n+1}$ connecting the two corresponding n-simplices x, y, namely $\partial w = (*, \ldots, *, y, x)$.*

Proof. The first part is a direct consequence of Yoneda's lemma 2.5.4. The second part is longer. We begin by considering, for any $n \geqslant 1$ and any $0 \leqslant i \leqslant n$, the following relation on the n-simplexes $x \in X_n$ verifying $\partial x = (*, \ldots, *)$:

$$x \sim_i y \text{ if it exists } \in X_{n+1} \text{ such that } \partial w = (*, \ldots, *, y, x, *, \ldots, *)\,,$$

where y is in the ith place and x is in the $(i+1)i$th place. These are shown to be equivalence relations. The proof is similar to that given in proposition 2.8.3; the calculations may seem complicated, but they're not if you're drawing in dimension 3.

Reflexivity: Using the s_i degeneracy, we see that $\partial s_i(x) = (*, \ldots, *, x, x, *, \ldots, *)$ and therefore $x \sim_i x$.

Transitivity: Let $x \sim_i y$ and $y \sim_i z$ with respectively $v, w \in X_{n+1}$ such that $\partial v = (*, \ldots, *, y, x, *, \ldots, *)$ and $\partial w = (*, \ldots, *, z, y, *, \ldots, *)$. We consider $W \coloneqq (*, \ldots, *, w, -, v, *, \ldots, *)$, where w is at the ith place. The corollary 2.5.5 shows that this is a $(i+1)$th horn of dimension $n+1$ of $\mathfrak{X}$. Since $\mathfrak{X}$ is a Kan complex, this horn has a filling $Z \in X_{n+2}$, i.e. $W \subset \partial Z$. We calculate $\partial d_{i+1}(Z) = (*, \ldots, *, z, x, *, \ldots, *)$ thanks to the corollary refcoro:ElemBordCornet. This shows $x \sim_i z$.

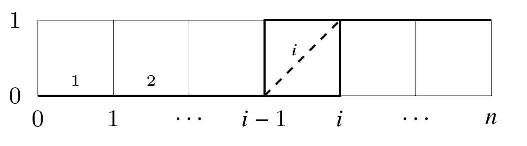
Symmetry: Let $xsim_i y$ with $w \in X_{n+1}$ such that $\partial w = (*, \ldots, *, y, x, *, \ldots, *)$. Consider $W \coloneqq (*, \ldots, *, -, s_i(x), w, *, \ldots, *)$, where w is in the $(i+2)$th place. This is a ith horn of dimension $n+1$ de $\mathfrak{X}$, which admits a filling $Z \in X_{n+2}$: $W \subset \partial Z$. We calculate $\partial d_i(Z) = (*, \ldots, *, x, y, *, \ldots, *)$ which shows $y \sim_i x$.

We now show that all these equivalence relations are equivalent. Let $0 \leqslant i \leqslant n-1$. In a way, let's give $x \sim_{i+1} y$ with $w \in X_{n+1}$ such that $\partial w = (*, \ldots, *, y, x, *, \ldots, *)$, where y is at the $(i+1)$th position. We consider $W \coloneqq (*, \ldots, *, w, s_{i+1}(y), s_i(y), -, *, \ldots, *)$, where w is at the ith place. This is a $(i+3)$th horn of the dimension $n+1$ of $\mathfrak{X}$ which admits a filling $Z \in X_{n+2}$. We calculate $\partial d_{i+3}(Z) = (*, \ldots, *, x, y, *, \ldots, *)$ where x is at the ith place, which shows $x \sim_i y$, since $\sim_i$ is reflexive. In the other direction, we give ourselves $x \sim_i y$ with $w \in X_{n+1}$ such that $\partial w = (*, \ldots, *, y, x, *, \ldots, *)$, where y is at the ith position. We consider $W \coloneqq (*, \ldots, *, -, s_{i+1}(x), s_i(x), w, *, \ldots, *)$, where w is at the $(i+3)$th position. It is a ith horn of dimension $n+1$ of $\mathfrak{X}$ which admits a filling a filling $Z \in X_{n+2}$. We calculate $\partial d_i(Z) = (*, \ldots, *, x, y, *, \ldots, *)$ where x is at the $(i+1)$th position, which shows $x \sim_{i+1} y$.

The proof of the statement is now automatic. Let $f \sim g$ rel $\partial\Delta^n$ and let $x, y \in X_n$ the two n-simplexes representing f and g respectively. Consider a morphism $H\colon \Delta^n \times \Delta^1 \to \mathfrak{X}$ such that $H(-, 0) = f$, $H(-, 1) = g$ and $H(\partial\Delta^n, -) = *$. The prismatic decomposition given in section 2.5 gives here the co-equalizer

$$\coprod_{1 \leqslant i \leqslant n} \Delta^n \;\underset{\delta_i}{\overset{\delta_i}{\rightrightarrows}}\; \coprod_{0 \leqslant j \leqslant n} \Delta^{n+1} \longrightarrow\!\!\!\!\!\rightarrow \Delta^n \times \Delta^1\,,$$

where the two morphisms on the left arrive respectively in the ith copy and the $(i-1)$th copy.



Yoneda's Lemma 2.5.4 then shows that the given morphism H is equivalent to to giving $n+1$ simplexes $z_0, z_1, \dots, z_n \in X_{n+1}$ of dimension $n+1$ satisfying $\partial z_0 = \big(y, d_1(z_1), *, \dots, *\big)$, $\partial z_n = \big(*, \dots, *, d_n(z_n), x\big)$ et $\partial z_i = \big(*, \dots, *, d_i(z_i), d_{i+1}(z_{i+1}), *, \dots, *\big)$, for all $1 \leqslant i \leqslant n-1$. This means that

$$x \sim_n d_n(z_n) \sim_{n-1} d_{n-1}(z_{n-1}) \sim_{n-2} \cdots \sim_1 d_1(z_1) \sim_0 y \ .$$

By the above, this implies $x \sim_n y$ and therefore that there exists $w \in X_{n+1}$ such that $\partial w = (*, \dots, *, y, x)$. The other way around, let $x \sim_n y$ with $w \in X_{n+1}$ such that $\partial w = (*, \dots, *, y, x)$. Consider the $n+1$ simplexes $s_0(y), s_1(y), \dots, s_{n-1}(y), w$ which define a morphism $H\colon \Delta^n \times \Delta^1 \to \mathfrak{X}$ such that $H(-,0) = f$, $H(-,1) = g$ and $H(\partial\Delta^n, -) = *$. We therefore have $f \sim g$ rel $\partial\Delta^n$. □

We denote by $[x]$ the equivalence class of $x \in X_n$ verifying $\partial x = (*, \dots, *)$ for the equivalence relation of the lemma 2.8.10. Let $x, y \in X_n$ be two n-simplices of $\mathfrak{X}$ satisfying $\partial x = \partial y = (*, \dots, *)$. The data $w \coloneqq (*, \dots, *, x, -, y)$ defines an nth horn of dimension $n+1$ of $\mathfrak{X}$ by the corollary 2.5.5. Since $\mathfrak{X}$ is a Kan complex, this horn is filled by a $(n+1)$-simplex $W \in X_{n+1}$. It quickly becomes apparent that $\partial d_n(W) = (*, \dots, *)$ and then we have

$$[x] \cdot [y] \coloneqq [d_n(W)] \ .$$

$$\begin{tikzcd} & 1 \arrow[dr, "x"] & \\ 0 \arrow[ur, "y"] \arrow[rr, dotted, "{[d_1(W)]=[x]\cdot[y]}"'] & W & 2 \end{tikzcd}$$

Lemma 2.8.11. *The product cdot is well defined.*

Proof. Let's already show that the definition does not depend on the $(n+1)$-simplex $W \in X_{n+1}$. Let $W' \in X_{n+1}$ an $(n+1)$-simplex such that $\partial W' = (*, \dots, *, x, d_n(W'), y)$. The horn $(*, \dots, *, s_n(x), -, W, W')$ admits a filler $Z \in X_{n+2}$ which can be verified as $\partial d_n(Z) = (*, \dots, *, d_n(W), d_n(W'))$. This implies $[d_n(W)] = [d_n(W')]$.
Let's now show that the definition of the product $\cdot$ does not depend on the choice of class representative for $[x]$ and $[y]$. We will only deal explicitly with the case of $[y]$, as the case of $[x]$ is similar. Let $y' \in X_n$ such that $y \sim y'$, i.e. there are $w \in X_{n+1}$ satisfying $\partial w = (*, \dots, *, y', y)$. Let $W \in X_{n+1}$ filling $(*, \dots, *, x, -, y)$ and $W' \in X_{n+1}$ filling $(*, \dots, *, x, -, y')$. We consider the horn $(*, \dots, *, s_{n-1}(x), W', -, w)$ that we fill with an $(n+2)$-simplex $Z \in X_{n+2}$. This latter ones satisfies $\partial d_{n+1}(Z) = (*, \dots, *, x, d_n(W'), y)$, which shows that $[x] \cdot [y] = [x] \cdot [y']$. □

Definition 2.8.12 (Homotopy groups of a Kan complex). For $n \geq 1$, the ***n-th homotopy group*** of a pointed Kan complex $\mathfrak{X}$ is

$$(\pi_n(\mathfrak{X}, *), \ \cdot, \ [*]).$$

Theorem 2.8.13. *Let $(\mathfrak{X}, *)$ be a pointed Kan complex.*

(1) *For each $n \geq 1$, $\big(\pi_n(\mathfrak{X}, *), \cdot, [*]\big)$ is a group.*
(2) *For $n \geq 2$, it is abelian.*
(3) *These homotopy groups are homotopy invariants.*
(4) *For all $n \geq 0$, there is a natural isomorphism (a bijection when $n = 0$)*

$$\big(\pi_n(\mathfrak{X}, *), \ \cdot, \ [*]\big) \ \cong \ \big(\pi_n(|\mathfrak{X}|, |*|), \ \cdot, \ [\mathrm{cst}]\big).$$

Proof.

(1) Unit: For any $x in X_n$ satisfying $\partial x = (*, \dots, *)$, we have $\partial s_n(x) = (*, \dots, *, x, x)$ and $\partial s_{n-1}(x) = (*, \dots, x, x, *)$, which show respectively that $[*] \cdot [x] = [x]$ and $[x] \cdot [*] = [x]$.
Inverse: For any $x \in X_n$ satisfying $\partial x = (*, \dots, *)$, we consider the $(n+1)$th horn defined by $(*, \dots, *, x, *, -)$ which is filled with a $(n+1)$-simplex $W \in X_{n+1}$. This shows that $[x] \cdot [d_{n+1}(W)] = [*]$. Similarly, the $(n-1)$th horn defined by $(*, \dots, *, -, *, x)$ is filled with a $(n+1)$-simplex $Z \in X_{n+1}$. This shows that $[d_{n-1}(Z)] \cdot [x] = [*]$ and concludes the proof of the existence of an inverse.
Associativity: Let $x, y, z \in X_n$ satisfying $\partial x = \partial y = \partial z = (*, \dots, *)$. We consider a $(n+1)$-simplex $W_{n-1} \in X_{n+1}$ filling the horn $(*, \dots, *, x, -, y)$, a $(n+1)$-simplex $W_{n+2} \in X_{n+1}$ filling the horn $(*, \dots, *, y, -, z)$ and a $(n+1)$-simplex $W_{n+1} \in X_{n+1}$ filling the horn $(*, \dots, *, d_n(W_{n-1}), -, z)$. There is a $(n+2)$-simplex $Z \in X_{n+2}$ who fills the horn $(*, \dots, *, W_{n-1}, -, W_{n+1}, W_{n+2})$. The calculation $\partial d_n(Z) = \big(*, \dots, *, x, d_n(W_{n+1}), d_n(W_{n+2})\big)$ shows that $[x] \cdot ([y] \cdot [z]) = ([x] \cdot [y]) \cdot [z]$.

(2) Abelian for $n \geq 2$ can be shown purely combinatorially or by referencing [?, Proposition 4.4]; another proof uses the forthcoming Theorem 1.3.7.
(3) It is automatic from the definitions that homotopy groups define functors taking homotopy equivalences to isomorphisms.
(4) See [?, Section 16].

□

Definition 2.8.14 (Kan fibration). A map of simplicial sets $p : \mathfrak{E} \twoheadrightarrow \mathfrak{B}$ is a *Kan fibration* if it satisfies the following extension property:

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & \mathfrak{E} \arrow[d, two heads, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dotted, "\exists"] & \mathfrak{B}, \end{tikzcd}$$

for all $n \geq 2$ and $0 \leq k \leq n$.

We depict Kan fibrations with double-headed arrows. We call $\mathfrak{B}$ the *base*, $\mathfrak{E}$ the *total space*, and that data $p : \mathfrak{E} \twoheadrightarrow \mathfrak{B}$ a *fibered space.*

Example. A simplicial set $\mathfrak{X}$ is a Kan complex if and only if the terminal map $\mathfrak{X} \to *$ is a Kan fibration.

For each 0-simplex b of $\mathfrak{B}$, consider the simplicial subset generated by b (including all its degenerate copies). Then the *fiber* $\mathfrak{F} := p^{-1}(b)$ is defined by $F_n = p_n^{-1}(b)$.

Lemma 2.8.15. *If $p : \mathfrak{E} \twoheadrightarrow \mathfrak{B}$ is a Kan fibration and $b \in \mathfrak{B}$, then the fiber $\mathfrak{F}$ over b is a Kan complex.*

Proof. Immediate from the combinatorial characterization of Kan complexes (Lemma 2.7.7). □

For completeness, we can characterize Kan fibrations themselves combinatorially:

Lemma 2.8.16. *A map of simplicial sets $p : \mathfrak{E} \twoheadrightarrow \mathfrak{B}$ is a Kan fibration iff for every $n \geq 2, 0 \leq k \leq n$, and any $x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n \in E_{n-1}$ with $d_i^{\mathfrak{E}}(x_j) = d_{j-1}^{\mathfrak{E}}(x_i)$, $i < j \neq k$, and any $y \in B_n$ with $d_i^{\mathfrak{B}}(y) = p(x_i)$, there exists $x \in E_n$ such that $p(x) = y$ and $d_i^{\mathfrak{E}}(x) = x_i$, $i \neq k$.*

Proof. Again, it follows directly from Lemma 2.7.7 plus Yoneda 2.5.4. □

Exercise. As a good exercise, verify these two properties for a Kan fibration $p : \mathfrak{E} \twoheadrightarrow \mathfrak{B}$:

- ◇ If $\mathfrak{E}$ is a Kan complex and p is surjective in every degree, then $\mathfrak{B}$ is a Kan complex.
- ◇ If $\mathfrak{B}$ is a Kan complex, then $\mathfrak{E}$ is a Kan complex.

For each 0-simplex f in the fiber, one obtains a map of pointed simplicial sets

$$(\mathfrak{F}, f) \longrightarrow (\mathfrak{E}, f) \twoheadrightarrow (\mathfrak{B}, b).$$

Theorem 2.8.17. *Any Kan fibration*

$$(\mathfrak{F}, f) \xrightarrow{i} (\mathfrak{E}, f) \overset{p}{\twoheadrightarrow} (\mathfrak{B}, b)$$

with $\mathfrak{F}, \mathfrak{E}, \mathfrak{B}$ all Kan complexes induces a long exact homotopy sequence

$$\cdots \longrightarrow \pi_2(\mathfrak{B}, b) \longrightarrow \pi_1(\mathfrak{F}, f) \longrightarrow \pi_1(\mathfrak{E}, f) \longrightarrow \pi_1(\mathfrak{B}, b) \longrightarrow \pi_0(\mathfrak{F}) \longrightarrow \pi_0(\mathfrak{E}) \longrightarrow \pi_0(\mathfrak{B}) \longrightarrow \{*\}.$$

Proof. One can give an *ad hoc* proof of about the same level of difficulty as the above arguments; see [?, Theorem 7.6] for details. □

Remark. It is straightforward to check that $\mathrm{Sing}(f) : \mathrm{Sing}(X) \twoheadrightarrow \mathrm{Sing}(Y)$ is a Kan fibration if and only if $f : X \twoheadrightarrow Y$ is a Serre fibration. The "converse," that $|p| : |\mathfrak{E}| \twoheadrightarrow |\mathfrak{B}|$ is a Serre fibration whenever $p : \mathfrak{E} \twoheadrightarrow \mathfrak{B}$ is a Kan fibration, is also true but significantly harder (see [?]). That yields another proof of the above exact sequence by passing to topological fibrations and using Theorem 1.6.3, plus the isomorphism with the homotopy groups of realizations (Theorem 2.8.13).

Definition 2.8.18 (Cofibration and weak equivalence). A map $f : \mathfrak{X} \to \mathfrak{Y}$ of simplicial sets is called

- ◇ a *cofibration* if each $f_n : X_n \hookrightarrow Y_n$ is injective for every $n \in \mathbb{N}$,
- ◇ a *weak equivalence* if all the maps $\pi_n(|f|) : \pi_n(|\mathfrak{X}|, x) \cong \pi_n(|\mathfrak{Y}|, |f|(x))$ are isomorphisms for $n \geq 1$ and a bijection for $n = 0$.

The next result directly connects the two halves of the story.

Theorem 2.8.19 ([Qui67]). *The adjunction between geometric realization and the singular simplicial set induces the following equivalence of categories:*

$$\mathsf{Top}[\mathrm{we}^{-1}] \simeq \mathsf{CW\text{-}cx} / \sim \simeq \mathsf{Kan\text{-}cx} / \sim \simeq \Delta\mathsf{Ens}[\mathrm{we}^{-1}],$$

where the two middle categories are, respectively, CW complexes and Kan complexes modulo the homotopy equivalence relation.

The beauty of this theorem is at least twofold: it establishes an equivalence between the homotopy theories of topological spaces and simplicial sets, and it provides a straightforward localized category structure (the two outermost categories). Its proof is too involved to present here; it will be the subject of the next course by Grégory Ginot.

Le but de cet appendice est de rappeler des résultats un peu plus avancés de la théorie des catégories, qui sont souvent mal connus mais dont nous nous servons à travers ce livre. On commence par la notion de préfaisceau en se focalisant sur le plongement de Yoneda qui permet de voir toute catégorie localement petite comme une sous-catégorie pleine de sa catégorie de préfaisceaux. Ce résultat ne nécessite pas d'idée nouvelle, il suffit juste de connaitre les définitions de base pour le démontrer. On traite ensuite des extensions de Kan, qui sont les objets les plus fondamentaux de la théorie des catégories : Saunders MacLane a d'ailleurs intitulé une des sections de son livre [?] : "All concepts are Kan extensions". En effet, (presque) toutes les notions de la théorie des catégories (adjonction, limites, colimites, etc.) s'expriment en ces termes. Néanmoins la notion d'extensions de Kan est très abordable et naturelle, quand elle est prise par le bon bout.

A.1. Préfaisceaux.

Definition A.1.1 (Préfaisceau). Un ***préfaisceau*** sur la catégorie C est un foncteur contravariant de C vers la catégorie des ensembles : $\mathsf{C}^{\mathsf{op}} \to \mathsf{Ens}$. La catégorie des préfaisceaux sur C est une catégorie de foncteurs : elle admet pour morphismes les transformations naturelles. On la note $\mathsf{Fon}(\mathsf{C}^{\mathsf{op}}, \mathsf{Ens})$.

EXAMPLE. Soit C une catégorie *localement petite*, c'est-à-dire que chaque classe de morphismes $\mathrm{Hom}_{\mathsf{C}}(b, a)$ est un ensemble, pour a, b dans C. À tout object a de C, on associe le foncteur représentable suivant

$$\begin{array}{rccl} \mathrm{Y}_a & : & \mathsf{C}^{\mathsf{op}} & \to & \mathsf{Ens} \\ & & b & \mapsto & \mathrm{Hom}_{\mathsf{C}}(b, a) \end{array}$$

qui est un préfaisceau sur C.

Definition A.1.2 (Plongement de Yoneda). On appelle ***plongement de Yoneda*** le foncteur

$$\begin{array}{rcccl} \mathrm{Y} & : & \mathsf{C} & \to & \mathsf{Fon}(\mathsf{C}^{\mathsf{op}}, \mathsf{Ens}) \\ & & a & \mapsto & \mathrm{Y}_a\ . \end{array}$$

Le fait que le plongement de Yoneda soit un foncteur signifie en particulier que tout morphisme $f : a \to a'$ dans la catégorie C induit une transformation naturelle $f_* : \mathrm{Y}_a \Rightarrow \mathrm{Y}_{a'}$, obtenue en composant par f. Le théorème suivant montre notamment qu'il n'y en a pas d'autres.

Theorem A.1.3 (Lemme de Yoneda). *Soit C une catégorie localement petite. Il existe une bijection*

$$\mathrm{Nat}(\mathrm{Y}_a, X) \cong X(a)$$

naturelle en a dans C et en X dans $\mathsf{Fon}(\mathsf{C}^{\mathsf{op}}, \mathsf{Ens})$.

PROOF. Considérons une transformation naturelle $\psi : \mathrm{Y}_a \Rightarrow X$. Appliquée en a, elle donne une application ensembliste $\psi_a : \mathrm{Hom}_{\mathsf{C}}(a, a) \to X(a)$ qui envoie l'identité id_a sur un élément x_ψ de $X(a)$. Cet élément caractérise complètement la transformation naturelle ψ :

$$\psi_b(g) = X(g)(x_\psi)\ ,$$

pour tout $g \in \mathrm{Hom}_{\mathsf{C}}(b, a)$. Il reste à montrer que cette bijection est naturelle, ce qui est automatique et donc laissé au lecteur-trice. □

Si on applique ce résultat au préfaisceau $\mathrm{Y}_{a'}$, la bijection $\mathrm{Nat}(\mathrm{Y}_a, \mathrm{Y}_{a'}) \cong \mathrm{Y}_{a'}(a) = \mathrm{Hom}_{\mathsf{C}}(a, a')$, décrite dans cette démonstration, associe $f^* \mapsto f$, pour tout morphisme $f : a \to a'$ dans C. Ceci montre que les seules transformations naturelles entre foncteurs représentables sont celles issues de morphismes de la catégorie C par tirage en arrière. En d'autres termes, cela donne le résultat suivant.

Corollary A.1.4. *Le plongement de Yoneda est plein et fidèle.*

PROOF. Cela signifie que l'application

$$\begin{array}{ccc} \mathrm{Hom}_{\mathsf{C}}(a, a') & \to & \mathrm{Nat}(\mathrm{Y}_a, \mathrm{Y}_{a'}) \\ f & \mapsto & f^* \end{array}$$

est bijective, ce qui est une conséquence directe du lemme de Yoneda (Théorème A.1.3) par les arguments donnés ci-dessus. □

REMARK. Ce corollaire nous dit que deux préfaisceaux représentables Y_a et $\mathrm{Y}_{a'}$ sont isomorphes si et seulement si a et a' sont isomorphes dans la catégorie C.

Ce résultat est une forme de mise an abîme : tout catégorie localement petite est une sous-catégorie pleine de sa catégorie de préfaisceaux. Plus précisément, elle est identifiée avec la sous-catégorie des préfaisceaux représentables.

Proposition A.1.5. *Pour toute catégorie* C*, sa catégorie des préfaisceaux* $\mathsf{Fon}(\mathsf{C}^{\mathsf{op}}, \mathsf{Ens})$ *est complète et cocomplète.*

Proof. Les limites et colimites dans la catégorie des préfaisceaux sont données point-par-point par celles de la catégorie des ensembles. Soit $F : \mathsf{D} \to \mathsf{Fon}(\mathsf{C}^{\mathsf{op}}, \mathsf{Ens})$ un foncteur; on note

$$\begin{array}{rccc} F_c & : & \mathsf{D} & \to & \mathsf{Ens} \\ & & d & \mapsto & F(d)(c) \end{array}$$

le foncteur associé pour tout c dans C. La limite et la colimite du foncteur F sont données par

$$\left(\lim_{\mathsf{D}} F\right)(c) = \lim_{\mathsf{D}} F_c \qquad \text{et} \qquad \left(\operatorname*{colim}_{\mathsf{D}} F\right)(c) = \operatorname*{colim}_{\mathsf{D}} F_c \ .$$

□

Le plongement de Yoneda permet donc de voir une catégorie localement petite dans une catégorie complète et cocomplète. La catégorie des préfaisceaux satisfait même la propriété universelle pour cette dernière propriété : pour tout foncteur $F : \mathsf{C} \to \mathsf{D}$ vers une catégorie cocomplète, il existe un foncteur G *cocontinu*, c'est-à-dire qui préserve les colimites, unique à unique isomorphisme qui factorise F par Y:

$$\begin{tikzcd} \mathsf{C} \arrow[r, "\mathrm{Y}"] \arrow[dr, "F"'] & \mathsf{Fon}(\mathsf{C}^{\mathsf{op}}, \mathsf{Ens}) \arrow[d, dashed, "\exists! G"] \\ & \mathsf{D} \end{tikzcd}$$

On peut alors parler de "complétion cocomplète" de la catégorie initiale.

Exercise. Avec le lemme de Yoneda, montrer le théorème de Cayley : tout groupe fini est un sous-groupe d'un groupe de permutations.

L'omniprésence des préfaisceaux représentables dans la catégorie des préfaisceaux est encore plus forte que cela. Le résultat suivant montre qu'ils forment une sous-catégorie *dense*, c'est-à-dire que tout préfaisceau peut s'écrire canoniquement comme une colimite de préfaisceaux représentables. Tout l'enjeu est alors de trouver la catégorie qui indice cette colimite.

Soit $X \in \mathsf{Fon}(\mathsf{C}^{\mathsf{op}}, \mathsf{Ens})$ un préfaisceau. On cherche donc un catégorie E munie d'un foncteur $\Pi : \mathsf{E} \to \mathsf{C}$ telle que la colimite de $\mathrm{Y} \circ \Pi$ sur E donne le foncteur X. Comme le foncteur colimite est adjoint à gauche du foncteur constant Δ, qui à tout préfaisceau Z associe le foncteur $\Delta_Z : e \in \mathsf{E} \mapsto Z \in \mathsf{Fon}(\mathsf{C}^{\mathsf{op}}, \mathsf{Ens})$, on doit avoir une bijection naturelle

$$\mathrm{Nat}(X, Z) \cong \mathrm{Nat}(\mathrm{Y} \circ \Pi, \Delta_Z) \ .$$

Une transformation naturelle $\alpha : X \Rightarrow Z$ est équivalente à la donnée d'un élément $z_{(c,x)} \in Z(c)$ pour tout $c \in \mathsf{C}$ et tout $x \in X(c)$ vérifiant $Z(f^{\mathrm{op}})(z_{(d,y)}) = z_{(c,x)}$ pour tout morphisme $f : c \to d$ de C et tout $y \in X(d)$ tel que $X(f^{\mathrm{op}})(y) = x$. On est donc amené à considérer cette catégorie d'indices issue du préfaisceau X.

Definition A.1.6 (Catégorie des éléments d'un préfaisceau). La *catégorie des éléments d'un préfaisceau* X admet pour objets les paires (c, x), avec $c \in \mathsf{C}$ et $x \in X(c)$, et pour morphismes entre (c, x) et (d, y) les morphismes $f : c \to d$ de la catégorie C qui vérifient $X(f^{\mathrm{op}})(y) = x$. On la note $\mathsf{E}(X)$.

La catégorie des éléments d'un préfaisceau X est munie d'un foncteur oubli canonique

$$\Pi \ : \ \mathsf{E}(X) \to \mathsf{C} \ , \quad (c, x) \mapsto c \ .$$

Theorem A.1.7 (Théorème de densité). *Tout préfaisceau* $X \in \mathsf{Fon}(\mathsf{C}^{\mathsf{op}}, \mathsf{Ens})$ *sur une catégorie* C *localement petite est la colimite de la composée du foncteur oubli avec le plongement de Yoneda sur la catégorie de ses éléments :*

$$X \cong \operatorname*{colim}_{\mathsf{E}(X)} \mathrm{Y} \circ \Pi \ .$$

Proof. Reprenons l'analyse entamée ci-dessus. Le lemme de Yoneda (Théorème A.1.3) fournit une transformation naturelle $\psi_{(c,x)} : \text{Y}_c \Rightarrow Z$ associée à tout élément $z_{(c,x)}$. La condition de compatibilité vérifiée par les $z_{(c,x)}$ est équivalente au fait que les $\psi_{(c,x)}$ forment une transformation naturelle ψ : $\text{Y} \circ \Pi \Rightarrow \Delta_Z$. L'application $\alpha \mapsto \psi$ est bijective, par le lemme de Yoneda, et elle est naturelle en $Z \in \mathsf{Fon}(\mathsf{C}^{\mathsf{op}}, \mathsf{Ens})$, ce qui conclut la démonstration. □

A.2. **Extensions de Kan**. L'idée à la base de la notion d'extensions de Kan est celle d'extension des foncteurs. On sait que toute application ensembliste peut s'étendre à un domaine plus gros, mais de façon non-canonique. Dans le contexte des catégories, la situation est différente : les extensions de Kan montrent que l'on peut étendre les foncteurs à un autre domaine de façon canonique et optimale. Dans cette section, nous ne traiterons en détail que la notion d'extension de Kan à gauche, la notion duale à droite étant l'exacte opposée. Dans le corps de ce texte, nous n'utilisons que les extensions de Kan à gauche et la théorie des extensions de Kan à droite s'obtient automatiquement en changeant le sens des transformations naturelles, en considérant des limites à la place des colimites pour des limites, etc.

Soit $\Phi : \mathsf{C} \to \mathsf{C}'$ un foncteur. En tirant en arrière par Φ, tout foncteur $G : \mathsf{C}' \to \mathsf{D}$ induit un foncteur $\Phi^*(G) = G \circ \Phi : \mathsf{C} \to \mathsf{D}$. Cette construction induit un foncteur entre catégories de foncteurs :

$$\Phi^* \ : \ \mathsf{Fon}(\mathsf{C}', \mathsf{D}) \to \mathsf{Fon}(\mathsf{C}, \mathsf{D}) \ .$$

On se pose alors la question de l'existence d'un adjoint à gauche et à droite.

Proposition A.2.1. *Le foncteur Φ^* admet un adjoint à gauche si et seulement si, pour tout foncteur F : $\mathsf{C} \to \mathsf{D}$, il existe un foncteur $L : \mathsf{C}' \to \mathsf{D}$ et une transformation naturelle $\alpha : F \Rightarrow L \circ \Phi$ telle que, pour tout foncteur $G : \mathsf{C}' \to \mathsf{D}$ équipé aussi d'une transformation naturelle $\beta : F \Rightarrow G \circ \Phi$, il existe une unique transformation naturelle $\gamma : L \Rightarrow G$ factorisant β, c'est-à-dire $\beta = (\gamma\Phi) \circ \alpha$.*

$$\begin{tikzcd}
\mathsf{C} \arrow[rr, bend left, "F"] \arrow[dr, "\Phi"'] & \arrow[d, phantom, "\Downarrow\alpha \quad \Downarrow\beta"] & \mathsf{D} \\
& \mathsf{C}' \arrow[ur, bend left, "L"] \arrow[ur, bend right, "G"'] \arrow[ur, phantom, "\Downarrow\exists\gamma"] &
\end{tikzcd}$$

Dualement, le foncteur Φ^* admet un adjoint à droite si et seulement s'il admet une caractérisation similaire obtenue en changeant le sens des transformations naturelles.

Proof. Ce résultat est l'application à un cas particulier d'un théorème général sur les adjonctions. Il se démontre néanmoins sans surprise de la manière suivante.
Pour montrer que la condition est nécessaire, on se donne un adjoint à gauche $\mathscr{L}$: $\mathsf{Fon}(\mathsf{C}, \mathsf{D}) \to \mathsf{Fon}(\mathsf{C}', \mathsf{D})$ et on pose

$$\chi_{F,G} \ : \ \mathrm{Nat}\,(\mathscr{L}(F), G) \cong \mathrm{Nat}\,(F, G \circ \Phi)$$

la bijection naturelle en $F : \mathsf{C} \to \mathsf{D}$ et en $G : \mathsf{C}' \to \mathsf{D}$ de cette adjonction. Pour $G = \mathscr{L}(F)$, on obtient une transformation naturelle

$$\alpha := \chi_{F, \mathscr{L}(F)}(\mathbb{1}_{\mathscr{L}(F)}) \ : \ F \Rightarrow \mathscr{L}(F) \circ \Phi \,,$$

qui vérifie

(*) $$\chi_{F,G}(\gamma) = (\Phi\gamma) \circ \alpha \,,$$

pour toute transformation naturelle $\gamma : \mathscr{L}(F) \Rightarrow G$. Pour obtenir la condition nécessaire de l'énoncé, il suffit de poser $L := \mathscr{L}(F)$ et de considérer la transformation naturelle α. Pour toute transformation naturelle $\beta : F \Rightarrow G \circ \Phi$, il existe une unique transformation naturelle $\gamma : L \Rightarrow G$ vérifiant $\beta = (\gamma\Phi) \circ \alpha$ par bijectivité de $\chi_{F,G}$ et l'équation (*).
La condition de l'énoncé est suffisante. Posons $\mathscr{L}(F) := L$, pour tout foncteur $F : \mathsf{C} \to \mathsf{D}$. Soit $\varphi : F \Rightarrow F'$ une transformation naturelle. La propriété universelle vérifiée par α implique qu'il existe une unique transformation naturelle $\mathscr{L}(\varphi) : \mathscr{L}(F) \Rightarrow \mathscr{L}(F')$ telle que $(\mathscr{L}(\varphi)\Phi) \circ \alpha = \alpha' \circ \varphi$. Cette propriété universelle montre à nouveau que $\mathscr{L}$ définit bien un foncteur. On définit alors l'application $\chi_{F,G}$ par la formule (*). Elle est naturelle par la condition définissant $\mathscr{L}(\varphi)$ et elle est bijective par la propriété universelle vérifiée par les α. □

Comme tout objet de la théorie des catégories, une paire (L, α) vérifiant la propriété universelle énoncée dans la proposition précédente est unique à isomorphisme près; elle a donc le droit à un petit nom.

Definition A.2.2 (Extensions de Kan). On appelle *extension de Kan à gauche du foncteur F le long du foncteur* Φ la paire (L, α) vérifiant la propriété universelle de la proposition A.2.1. On la note $(\mathrm{Lan}_\Phi F, \alpha)$. La paire vérifiant la condition duale est appelée *extension de Kan à droite du foncteur F le long du foncteur* Φ et notée $(\mathrm{Ran}_\Phi F, \alpha)$.

Les extensions de Kan ne factorisent en général pas le foncteur initial F (chose impossible à moins que d'avoir une sous-catégorie par exemple), mais leur composée avec le foncteur Φ fournit est la meilleure approximation de F. Les notions d'extensions de Kan sont omniprésentes en théorie des catégories, elles supplantent par exemple celles de (co)limite et d'adjonction.

Example. Soit $\mathsf{C}' = 1$ la catégorie terminale à un seul objet $*$ (et un seul morphisme) et soit $\Pi : \mathsf{C} \to 1$ l'unique foncteur de la catégorie C vers cette dernière. Dans ce cas, l'extension de Kan à gauche d'un foncteur $F : \mathsf{C} \to \mathsf{D}$ est équivalente à sa colimite

$$\mathrm{Lan}_\Pi(F)(*) = \operatorname*{colim}_{\mathsf{C}} F$$

et son extension de Kan à droite est équivalente à sa limite

$$\mathrm{Ran}_\Pi(F)(*) = \lim_{\mathsf{C}} F \ .$$

Exercise.

(1) Montrer qu'un foncteur $F : \mathsf{A} \to \mathsf{B}$ admet un adjoint à droite si et seulement si le foncteur $1_\mathsf{A} : \mathsf{A} \to \mathsf{A}$ admet une extension de Kan à gauche $(\mathrm{Lan}_F 1_\mathsf{A}, \alpha)$ le long de F vérifiant la propriété que $(F \circ \mathrm{Lan}_F 1_\mathsf{A}, F\alpha)$ est une extension de Kan à gauche de F le long de lui-même.

(2) Lorsque c'est le cas, Montrer que l'extension de Kan à gauche $\mathrm{Lan}_F 1_\mathsf{A}$ est l'adjoint à droite de F,
$$F \dashv \mathrm{Lan}_F 1_\mathsf{A}$$
et que l'unité d'ajonction est donnée par $F\alpha$.

(3) Écrire le résultat dual pour l'existence d'adjoints à gauche en terme d'extensions de Kan à droite.

Pour une nouvelle mise en abîme, rappelons que les notions d'extensions de Kan resolvent un problème d'existence d'adjonction : la proposition A.2.1 affirme que le foncteur Φ^* admet un adjoint à gauche (respectivement à droite) si et seulement si tout foncteur $F : \mathsf{C} \to \mathsf{D}$ admet une extension de Kan à gauche (respectivement à droite). Dans ce cas de figure, on a

$$\mathrm{Lan}_\Phi \dashv \Phi^* \dashv \mathrm{Ran}_\Phi \ .$$

Il reste donc à montrer que les extensions de Kan existent. Analysons le cas où le foncteur $\Phi : \mathsf{C} \hookrightarrow \mathsf{C}'$ est une inclusion d'une sous-catégorie pleine. Supposons que l'on dispose d'une extension $L : \mathsf{C}' \to \mathsf{D}$ de F, c'est-à-dire $F = L \circ \Phi$. le long d'un foncteur

$$\begin{tikzcd} \mathsf{C} \arrow[rr, "F"] \arrow[dr, hook, "\Phi"'] & & \mathsf{D} \\ & \mathsf{C}' \arrow[ur, "L"'] & \end{tikzcd}$$

Soit x un object de C' et cherchons à caractériser l'image de x par le foncteur L. Pour toute paire $(a, f : a \to x)$ avec a un objet de C et f un morphisme de C', on dispose d'un morphisme $L(f) : L(a) = F(a) \to L(x)$ de la catégorie D. Tout triangle commutatif

$$\begin{tikzcd} a \arrow[rr, "h"] \arrow[dr, "f"'] & & b \arrow[dl, "g"] \\ & x & \end{tikzcd} ,$$

où h est un morphisme de C, induit un triangle commutatif

$$\begin{tikzcd} F(a) \arrow[rr, "F(h)"] \arrow[dr, "L(f)"'] & & F(b) \arrow[dl, "L(g)"] \\ & L(x) & \end{tikzcd} .$$

Il est alors naturel d'introduire la catégorie $\mathsf{C}\downarrow x$ dont les objets sont les paires $(a, f : a \to x)$ comme ci-dessus et dont les morphismes sont ceux de C qui forment un triangle commutatif. On pose $\Pi : \mathsf{C}\downarrow x \to \mathsf{C}, (a, f) \mapsto a$ le foncteur de projection. En ces termes, l'image $L(x)$ est un cocône pour le foncteur $F \circ \Pi$ sur la catégorie $\mathsf{C}\downarrow x$.

Dans le cas général, on considère la catégorie suivante pour tout objet x de C'.

Definition A.2.3 (Catégorie $\Phi\downarrow x$). Les objets de la catégorie $\Phi\downarrow x$ sont les paires $(a, f : \Phi(a) \to x)$, où a un objet de C et f un morphisme de C'. Les morphismes de $(a, f : \Phi(a) \to x)$ vers $(b, g : \Phi(b) \to x)$ sont les morphismes $h : a \to b$ de la catégorie C vérifiant $f = g \circ \Phi(h)$.

Exercise. Écrire la catégorie des éléments d'un préfaisceau X (Définition A.1.6) comme une catégorie $x\downarrow X$ définie de manière similaire mais par des objets de la forme $(a, f : x \to X(a))$.

On considère la composée $F\circ\Pi : \Phi\downarrow x \to \mathsf{D}$ du foncteur F avec la projection canonique $\Pi : \Phi\downarrow x \to \mathsf{C}$. Les études faites ci-dessus des colimites comme extensions de Kan à gauche le long du foncteur $\mathsf{C} \to 1$ et des extensions de foncteurs suggèrent que la colimite du foncteur $F \circ \Pi$, cocône initial, doit être intimement reliée à l'image de l'extension de Kan de l'objet x. Le théorème suivant va nous donner raison.

Theorem A.2.4. *Soient $\Phi : \mathsf{C} \to \mathsf{C}'$ et $F : \mathsf{C} \to \mathsf{D}$ deux foncteurs tels que, pour tout objet x de la catégorie C', le foncteur $F \circ \Pi : \Phi\downarrow x \to \mathsf{D}$ admet une colimite dans la catégorie D. Dans ce cas, l'extension de Kan à gauche du foncteur F le long du foncteur Φ existe et elle est donnée ponctuellement par cette colimite:*

$$(\mathrm{Lan}_\Phi F)(x) = \operatorname*{colim}_{\Phi\downarrow x} F \circ \Pi \ .$$

Proof. Les arguments sont automatiques une fois que l'on a bien compris la construction ci-dessus. Pour plus de précision, nous noterons ici le foncteur de projection par $\Pi_x : \Phi\downarrow x \to \mathsf{C}$. Posons

$$L(x) := \operatorname*{colim}_{\Phi\downarrow x} F \circ \Pi \ ,$$

pour tout objet x de la catégorie C'.
Montrons d'abord que L définit bien un foncteur $L : \mathsf{C}' \to \mathsf{D}$. Soit $k : x \to y$ un morphisme de la catégorie C'. Il induit un foncteur $k_* : \Phi\downarrow x \to \Phi\downarrow y\ ,\ (a, f) \mapsto (a, k\circ f)$ qui vérifie $F\circ\Pi_y = F\circ\Pi_x\circ k_*$. Ceci montre que $L(y)$ est un cocône pour le foncteur $F \circ \Pi_x$ sur la catégorie $\Phi\downarrow x$. Comme $L(x)$ est la colimite de ce foncteur, on définit $L(k) : L(x) \to L(y)$ par sa propriété universelle. En utilisant les mêmes arguments, on voit facilement que toute paire de morphismes $k : x \to y$ et $k : y \to z$ de C' vérifient $L(l \circ k) = L(l) \circ L(k)$.
Définissons maintenant une transformation naturelle $\alpha : F \Rightarrow L \circ \Phi$. Soit c un objet de la catégorie C; on note λ_f les morphismes de structures du cocône $L(\Phi(c))$:

$$\begin{tikzcd}
F \circ \Pi_{\Phi(c)}(a, f) = F(a) \arrow[rr, "F(h)"] \arrow[dr, bend right, "\lambda_f"'] & & F(b) = F \circ \Pi_{\Phi(c)}(b, g) \arrow[dl, bend left, "\lambda_g"] \\
& L(\Phi(c)) &
\end{tikzcd}$$

On pose alors

$$\alpha(c) := \lambda_{\mathrm{id}} \ : \ F(c) = F \circ \Pi_{\Phi(c)}(a, \mathrm{id} : \Phi(c) \to \Phi(c)) \to L(\Phi(c)) \ .$$

Pour montrer que cela définit bien une transformation naturelle, on considère le diagramme suivant associé à tout morphisme $i : c \to d$ de la catégorie C :

$$\begin{tikzcd}
F(c) \arrow[r, "F(i)"] \arrow[d, "\alpha(c)=\lambda_{\mathrm{id}}"'] \arrow[dr, "\mu_{\Phi(i)}"] & F(d) \arrow[d, "\alpha(d)=\mu_{\mathrm{id}}"] \\
L(\Phi(c)) \arrow[r, "L(\Phi(i))"'] & L(\Phi(d))
\end{tikzcd}$$

qui est commutatif : le triangle supérieur droit est commutatif par la définition de $L(\Phi(d))$ comme un cocône et le triangle inférieur gauche est commutatif par la définition de $L(\Phi(i))$.
Il reste à montrer que la transformation naturelle α est universelle. Soit $G : \mathsf{C}' \to \mathsf{D}$ un foncteur muni d'une transformation naturelle $\beta : F \Rightarrow G \circ \Phi$. Soit x un objet de la catégorie C'. Pour tout

morphisme h d'un objet $(a, f : \Phi(a) \to x)$ vers un objet $(b, g : \Phi(b) \to x)$ de la catégorie $\Phi \downarrow x$, c'est-à-dire $h : a \to b$ morphisme de C vérifiant $f = g \circ \Phi(h)$, le digramme suivant est commutatif :

$$\begin{tikzcd}
F(a) = F \circ \Pi_x(a, f) \arrow[rr, "F(h)"] \arrow[d, "\beta(a)"'] & & F \circ \Pi_x(b, g) = F(b) \arrow[d, "\beta(b)"] \\
G(\Phi(a)) \arrow[rr, "G(\Phi(h))"] \arrow[dr, bend right, "G(f)"'] & & G(\Phi(b)) \arrow[dl, bend left, "G(g)"] \\
& G(x) &
\end{tikzcd} .$$

Le carré du haut l'est par définition de la transformation naturelle β et le triangle du bas l'est par définition de h et du foncteur G. Ceci montre que $G(x)$ est un cocône pour le foncteur $F \circ \Pi_x$ sur la catégorie $\Phi \downarrow x$; comme $L(x)$ est la colimite de ce foncteur, on définit $\gamma(x) : L(x) \to G(x)$ par sa propriété universelle.
Montrons maintenant que γ est une transformation naturelle. Soit $k : x \to y$ un morphisme de la catégorie C'. Pour tout objet $(a, f : \Phi(a) \to x)$ de la catégorie $\Phi \downarrow x$, on considère le diagramme suivant :

$$\begin{tikzcd}
F \circ \Pi_y(a, kf) = F(a) = F \circ \Pi_x(a, f) \arrow[r, "\lambda_f"] \arrow[d, "\beta(a)"] \arrow[rr, bend left, "\mu_{kf}"] & L(x) \arrow[r, "L(k)"] \arrow[d, "\gamma(x)"] & L(y) \arrow[d, "\gamma(y)"] \\
G(\Phi(a)) \arrow[r, "G(f)"] \arrow[rr, bend right, "G(kf)"'] & G(x) \arrow[r, "G(k)"] & G(y)
\end{tikzcd} ,$$

où λ et μ dénotent respectivement les morphismes de structure des cocônes $L(x)$ et $L(y)$. Le carré de gauche est commutatif par définition de $\gamma(x)$, le triangle du haut est commutatif par défintion de $L(k)$, celui du bas l'est par définition du foncteur G, enfin le carré extérieur est commutatif par définition de $L(y)$. Le carré de droite est donc toujours commutatif une fois précomposé par λ_f, et ce pour tout objet (a, f) de la catégorie $\Phi \downarrow x$. Par définition de $L(x)$ comme colimite, on obtient que ce carré est en fait commutatif.
Enfin, on montre l'universalité de la transformation naturelle γ. Par définition de α et de γ, on a $\beta(a) = \gamma(\Phi(a)) \circ \alpha(a)$. Il reste à établir l'unicité de la transformation naturelle γ. Soit donc $\gamma' : L \Rightarrow G$ une transformation naturelle vérifiant $\beta = (\gamma'\Phi) \circ \alpha$. Pour les autres éléments y de C', on applique le diagramme ci-dessus à $x = \Phi(a)$, $f = \mathrm{id}_{\Phi(a)}$ et $k : \Phi(a) \to y$ morphisme de C'. On obtient alors $\gamma'(y) \circ \mu_k = \gamma(y) \circ \mu_k$, ce qui implique $\gamma'(y) = \gamma(y)$ par la propriété de colimite de $L(y)$. □

Corollary A.2.5. *Sous les hypothèses d'existence données au théorème A.2.4 et dans le cas où le foncteur Φ est plein et fidèle, il existe une extension de Kan à gauche du foncteur F qui en soit une extension :*

$$F = (\mathrm{Lan}_\Phi F) \circ \Phi \quad \textit{et} \quad \alpha = \mathrm{id} .$$

Proof. Lorsque le foncteur Φ est plein et fidèle, la catégorie $\Phi \downarrow \Phi(c)$ est isomorphe à la catégorie $\mathsf{C} \downarrow c$ des objets au-dessus de c. Cette dernière admet pour objet terminal (c, id_c) et donc la colimite $L(\Phi(c))$ est donnée simplement par $F(c)$.

□

C'est par exemple le cas lorsque C est une sous-catégorie pleine de C'.

Plutôt que d'exiger l'existence point-par-point de certaines colimites, les conditions générales suivantes assurent l'existence d'extensions de Kan.

Corollary A.2.6. *Si la catégorie C est petite, la catégorie C' localement petite et la catégorie D cocomplète, alors tout foncteur $F : \mathsf{C} \to \mathsf{D}$ admet une extension de Kan le long de tout foncteur $\Phi : \mathsf{C} \to \mathsf{C}'$.*

Proof. On rappelle à toutes fins utiles qu'une catégorie est petite lorsque ses objets forment un ensemble et que tous ses classes de morphismes forment un ensemble. Une catégorie est cocomplète lorsque tout foncteur depuis une petite catégorie vers cette dernière admet une colimite. Lorsque la catégorie C est petite et la catégorie C' localement petite, alors toute catégorie $\Phi{\downarrow}x$ est petite, pour x de C'. Et comme la catégorie D est cocomplète, la colimite du foncteur $F \circ \Pi$ existe. On peut alors appliquer le théorème A.2.4. □

Les conditions d'existence de ce corollaire ne sont pas très restrictives; elles sont toujours vérifiées dans les exemples qui forment le corps de ce texte. Dans ce qui suit, on les suppose vérifiées. Par contre, la colimite donnée au théorème A.2.4 peut être difficile à calculer. Néanmoins, on peut calculer plus efficacement l'extension de Kan à gauche à l'aide d'un autre type de colimite.

Soient a, b deux objets de la catégorie C et soit x un objet de la catégorie C'. On considère l'ensemble $\mathrm{Hom}_{\mathsf{C}'}(\Phi(b), x)$ puis le coproduit de l'objet constant $F(a)$ indicé par ce dernier; on le note traditionnellement comme un *tenseur* :

$$\mathrm{Hom}_{\mathsf{C}'}(\Phi(b), x) \cdot F(a) := \coprod_{\mathrm{Hom}_{\mathsf{C}'}(\Phi(b),x)} F(a) \ .$$

Tout morphisme $h : a \to b$ de la catégorie C induit les deux morphismes suivants dans la catégorie D :

$$\begin{tikzcd}
& \mathrm{Hom}_{\mathsf{C}'}(\Phi(b), x) \cdot F(a) \arrow[dl, "\Phi(h)^* \cdot \mathrm{id}"'] \arrow[dr, "\mathrm{id} \cdot F(h)"] & \\
\mathrm{Hom}_{\mathsf{C}'}(\Phi(a), x) \cdot F(a) & & \mathrm{Hom}_{\mathsf{C}'}(\Phi(b), x) \cdot F(b) \ .
\end{tikzcd}$$

En considérant tous les morphismes de la catégorie C, on obtient ainsi un diagramme $\mathcal{D}$ de la catégorie D.

Definition A.2.7 (Cofin). On appelle *bouquet* pour le diagramme $\mathcal{D}$ tout élément d de D muni d'applications $\lambda_a : \mathrm{Hom}_{\mathsf{C}'}(\Phi(a), x) \cdot F(a) \to d$ qui rendent le diagramme associé commutatif, voir ci-dessous. La *cofin* du diagramme $\mathcal{D}$, notée

$$\int^a \mathrm{Hom}_{\mathsf{C}'}(\Phi(a), x) \cdot F(a) \ ,$$

est son bouquet universel :

$$\begin{tikzcd}
& \mathrm{Hom}_{\mathsf{C}'}(\Phi(b), x) \cdot F(a) \arrow[dl, "\Phi(h)^* \cdot \mathrm{id}"'] \arrow[dr, "\mathrm{id} \cdot F(h)"] & \\
\mathrm{Hom}_{\mathsf{C}'}(\Phi(a), x) \cdot F(a) \arrow[dr] \arrow[ddr, bend right, "\lambda_a"'] & & \mathrm{Hom}_{\mathsf{C}'}(\Phi(b), x) \cdot F(b) \arrow[dl] \arrow[ddl, bend left, "\lambda_b"] \\
& \int^a \mathrm{Hom}_{\mathsf{C}'}(\Phi(a), x) \cdot F(a) \arrow[d, dashed, "\exists!"] & \\
& d &
\end{tikzcd}$$

Remark. Même si elle en a la même saveur, la notion de bouquet n'est pas la notion de cocône; il faudrait pour cela plus de morphismes de structure. Et donc la notion de cofin n'est pas la colimite du diagramme $\mathcal{D}$.

Proposition A.2.8. *Soient C une petite catégorie, C' une catégorie localement petite et D une catégorie cocomplète. L'extension de Kan à gauche d'un foncteur $F : \mathsf{C} \to \mathsf{D}$ le long d'un foncteur $\Phi : \mathsf{C} \to \mathsf{C}'$ est donnée ponctuellement par la cofin*

$$(\mathrm{Lan}_\Phi F)(x) = \int^a \mathrm{Hom}_{\mathsf{C}'}(\Phi(a), x) \cdot F(a) \ . \qquad (10)$$

Proof. La démonstration de cette propriété ne recèle pas d'idée originale, mais utilise les propriétés générales des cofins. Par soucis de concision, nous ne la reproduisons pas ici, mais nous renvoyons à [?, Section X.4]. □

Ce résultat montre que l'on peut écrire la cofin comme une colimite mais sur un diagramme différent.

Corollary A.2.9. *Sous les hypothèses de la proposition A.2.8, l'extension de Kan à gauche est donnée ponctuellement par le coégalisateur suivant*

$$\coprod_{\substack{h:a\to b \\ \text{dans } \mathsf{C}}} \mathrm{Hom}_{\mathsf{C}'}(\Phi(b), x) \cdot F(a) \underset{F(h)}{\overset{\Phi(h)^*}{\rightrightarrows}} \coprod_{a\in\mathsf{C}} \mathrm{Hom}_{\mathsf{C}'}(\Phi(a), x) \cdot F(a) \dashrightarrow \int^a \mathrm{Hom}_{\mathsf{C}'}(\Phi(a), x) \cdot F(a)\,.$$

PROOF. La démonstration est immédiate. □

REMARK. Lorsque la catégorie D est la catégorie Ens des ensembles, les extensions de Kan correspondent à l'image directe et réciproque des préfaisceaux.

References

[EM45] Samuel Eilenberg and Saunders MacLane, *Relations between homology and homotopy groups of spaces*, Ann. of Math. (2) **46** (1945), 480–509. 2

[GH81] Marvin J. Greenberg and John R. Harper, *Algebraic topology*, Mathematics Lecture Note Series, vol. 58, Benjamin/Cummings Publishing Co., Inc., Advanced Book Program, Reading, Mass., 1981, A first course. 38

[Hat02] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. 3, 37

[Lur09] Jacob Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. 2, 72

[LV12] Jean-Louis Loday and Bruno Vallette, *Algebraic operads*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 346, Springer-Verlag, Berlin, 2012. 2, 56

[Man06] Michael A. Mandell, *Cochains and homotopy type*, Publ. Math. Inst. Hautes Études Sci. (2006), no. 103, 213–246. 2

[May99] Jon Peter May, *A concise course in algebraic topology*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1999. 3, 13, 37

[Qui67] Daniel G. Quillen, *Homotopical algebra*, Lecture Notes in Mathematics, No. 43, Springer-Verlag, Berlin, 1967. 2, 31, 78

[Ser51] J.-P. Serre, *Homologie singulière des espaces fibrés. Applications*, Ann. of Math. (2) **54** (1951), 425–505. 47

[Sm68] Arne Strøm, *Note on cofibrations. II*, Math. Scand. **22** (1968), 130–142 (1969). 31, 35

[tD08] Tammo tom Dieck, *Algebraic topology*, EMS Textbooks in Mathematics, European Mathematical Society (EMS), Zürich, 2008. 3, 43

Laboratoire Analyse, Géométrie et Applications, Université Paris 13, Sorbonne Paris Cité, CNRS, UMR 7539, 93430 Villetaneuse, France.

E-mail address: vallette@math.univ-paris13.fr