Cochain models of topological spaces

by

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The purpose of my lectures is to explain applications of operads to the definition of cochain models in algebraic topology.

The rational homotopy category of topological spaces has two standard faithful models [FHT]: the category of differential graded Lie algebras (Quillen) and the category of differential graded commutative algebras (Sullivan). The equivalence between these models can be interpreted in terms of the Koszul duality betwen the Lie operad $\mathbb{L}$ and the commutative operad $\mathbb{C}$.

In positive characteristic, the category of commutative algebras in differential graded modules (for short, dg-modules) is not well behaved. To avoid homological difficulties, one has to replace the commutative operad $\mathbb{C}$ by a Σ_*-cofibrant resolution, an operad $\mathbb{E}$ together with an operad equivalence $\mathbb{E} \xrightarrow{\sim} \mathbb{C}$ such that each $\mathbb{E}(r)$ forms a projective resolution of the trivial representation of the symmetric group Σ_r. The name E_∞-operad refers to any such operad and an E_∞-algebra refers to an algebra over some E_∞-operad. In a sense, an E_∞-algebra is a dg-module together with a product and a complete set of homotopies that make this product fully commutative in homotopy. The notion of an E_n-algebra, defined by operads equivalent to the chain operads of little n-cubes (also called E_n-operads), encodes a structure which is homotopy commutative up to some degree.

The Barratt-Eccles operad [BF], defined by the standard acyclic bar constructions of the symmetric groups, is an instance of an E_∞-operad.

The cochain complex of a space $C^*(X, \mathbb{K})$, with coefficients in any ring $\mathbb{K}$, inherits a natural E_∞-algebra structure (see [BF,HS]) and this structure gives a faithful algebraic model of the homotopy type of X, at least under reasonable finiteness and completeness assumptions on X (see [M]). These results will be surveyed in the first part of my lectures.

The issue is to define effective E_∞-models of usual topological constructions, like iterated loop spaces $\Omega^n X$. In a second part, I will explain relations:

$$H^*_{prof}(\Sigma^m \Omega^m X) \simeq H_*(\Sigma^{-m} B^m \bar{C}^*(X)) \tag{1}$$

$$\simeq \operatorname{Tor}^{\mathbb{E}}_*(\operatorname{End}_{\bar{C}^*(S^m)}, \bar{C}^*(X)) \tag{2}$$

$$\simeq H^{E_m}_*(\bar{C}^*(X)), \tag{3}$$

where we introduce the profinite cohomology H^*_{prof} to settle convergence problems, the notation B^m refers to an extension of the standard m-fold bar construction of commutative algebras to E_∞-algebras, the notation $\bar{C}^*(X)$ refers to the reduced cochain complex of a space X, the notation End_C refers to the endomorphism operad of any dg-module C, and $H^{E_m}_*$ refers to the homology theory of E_m-algebras. In the case $m = \infty$, the homology theory $H^{E_\infty}_*$ is equivalent to the Γ-homology of [R] and gives an obstruction theory for the realization of commutative cohomology theories by E_∞-ring spectra.

Identities (1-3) relate the computation of $H^*_{prof}(\Sigma^m \Omega^m X)$ to the Koszul duality of E_m-operads. I will explain that E_m-operads are in a sense Koszul self dual, for any choice of the ground ring $\mathbb{K}$. If time permits, I will explain conjectural connections of [K] between the Grothendieck-Teichmüller group and the groups of homotopy automorphisms of E_n-operads and possible applications of these conjectures in E_m-homology.

The references for results of the second part are [F3-5]. The arguments rely on the Koszul duality of operads over a ring [F1] and the modelization of functors on algebras over operads by modules over operads [F2]. These ideas will be reviewed. The other prerequisites will be addressed in previous courses.

References

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* References [BF] and [F1-5] are available from the web page:
`http://math.univ-lille1.fr/~fresse/Articles.html`