Operads and Cochain Models in Algebraic Topology

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23 April 2009

Prologue: Conventions and Recollections

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  such that $\mathbb{E} = \mathbb{E}_\infty$ is an E_∞-operad, but we have no simple intrinsic characterization of the notion of an E_n-operad when $1 < n < \infty$.

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The associative operad in sets is identified with the 0-skeleton of $\mathbb{W}$ so that $\mathbb{W}$ fits in a factorization

$$\begin{tikzcd} \mathbb{A} \arrow[rr, "\alpha"] \arrow[dr, dotted] & & \mathbb{C} \\ & \mathbb{W} \arrow[ur, dotted, "\sim"'] & \end{tikzcd}$$

in the category of simplicial operads.

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when we have $x_i = *$, the base point of X.

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From this definition, we see that the layer $\mathbb{W}_1$ is reduced to the vertices of $\mathbb{W}$, and we obtain a nested sequence such that:

$$\mathbb{W}_1 \hookrightarrow \mathbb{W}_2 \hookrightarrow \cdots \hookrightarrow \mathbb{W}_n \hookrightarrow \cdots \hookrightarrow \operatorname*{colim}_n \mathbb{W}_n = \mathbb{W}\,.$$

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Define $S_*(\mathbb{W}_n, X)$ by the same construction as $S_*(\mathbb{W}, X)$.

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by a chain of weak-equivalences of operads.
Remark: P. May + C. Berger' theorems imply J. Smith's result.

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Issue: *Understand the structure of cofibrant models of E_∞-operads.*
In the A_∞ case, we have a good cofibrant model given by the Stasheff operad. What about cases $n = 2, 3, \ldots, \infty$?