Operads and Cochain Models in Algebraic Topology

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Issue: this map AW is unital, associative, but not symmetric though it becomes apparently symmetric in homology.

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References

1.

    - "Homology of partition posets and Koszul duality of operads", in Contemporary Mathematics **346**, American Mathematical Society.
    - "Koszul duality of E_n-operads".

2.

    - "Modules over operads and functors", Lecture Notes in Mathematics, Springer-Verlag.
    - "The bar complex series".

http://math.univ-lille1.fr/~fresse/Articles.html