Higher Algebra via Rewriting of Trees

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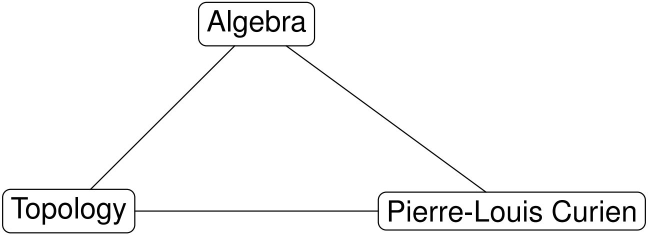
Meeting in honor of Pierre-Louis Curien
(Venezia, September 10, 2013)

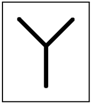


1. Algebra+Homotopy=Higher structures
2. Operadic homotopical algebra
3. Rewriting method

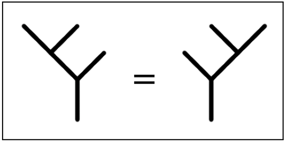
Plan

1. Algebra+Homotopy=Higher structures
2. Operadic homotopical algebra
3. Rewriting method

- **Algebraic structure:** an associative product on A



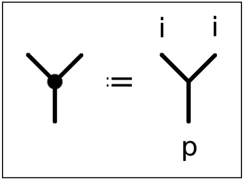
$$\nu : A^{\otimes 2} \to A, \quad \text{s.t.} \quad \nu(\nu(a,b),c) = \nu(a,\nu(b,c))\ .$$



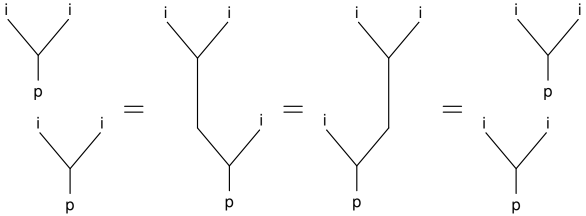
- **Isomorphic vector space:**

$$\boxed{A \underset{i}{\overset{p}{\rightleftarrows}} H} \qquad ip = \mathrm{id}_A, \quad pi = \mathrm{id}_H .$$

- **Transported structure:** $\mu_2 := p\, \nu\, i^{\otimes 2} \ : \ H^{\otimes 2} \to H$



is associative.

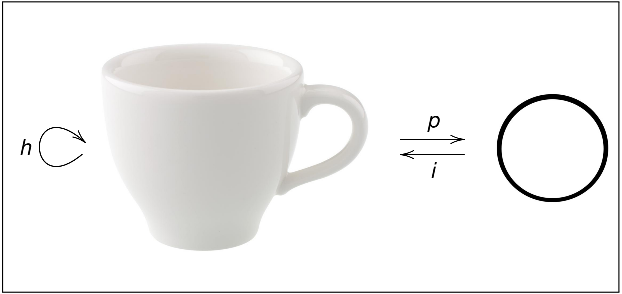


- **Homotopy equivalent space:**

Deformation retract of chain complexes

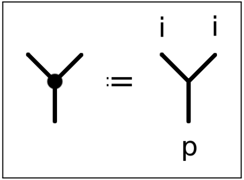
$$\boxed{\begin{tikzcd} (A, d_A) \arrow[loop left, "h"] \arrow[r, shift left, "p"] & (H, d_H) \arrow[l, shift left, "i"] \end{tikzcd}} \qquad \mathrm{id}_A - ip = d_A h + h d_A \ .$$

= algebraic version of topological deformation

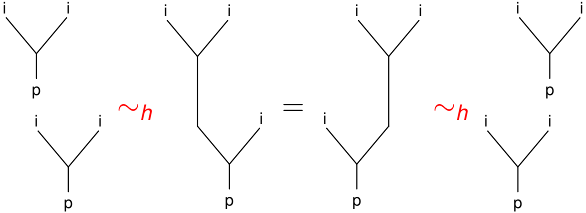


$$\boxed{h \circlearrowright (A, d_A) \underset{i}{\overset{p}{\rightleftarrows}} (H, d_H)} \qquad \mathrm{id}_A - ip = d_A h + h d_A \,.$$

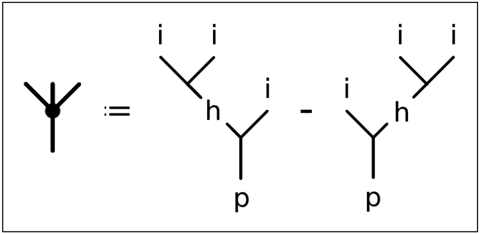
- **Transferred structure:** $\mu_2 := p\, \nu\, i^{\otimes 2} \ : \ H^{\otimes 2} \to H$



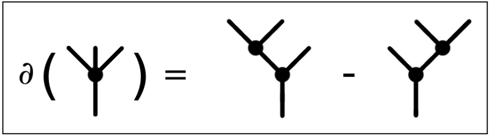
is not associative.



- **Idea:** Introduce μ_3

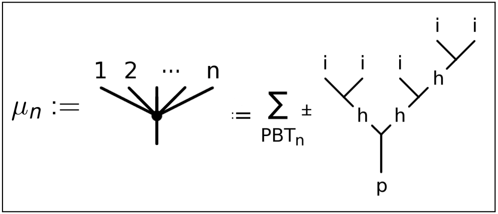


- In $\mathrm{Hom}(H^{\otimes 3}, H)$, it satisfies



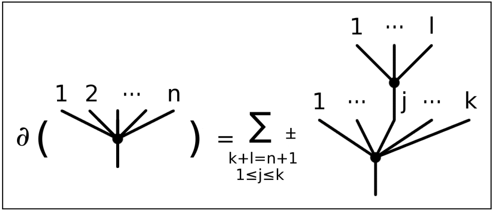
$\Longrightarrow$ μ_3 is a homotopy for the associativity relation of μ_2.

- **Higher up:** in $\mathrm{Hom}(H^{\otimes n}, H)$, we consider



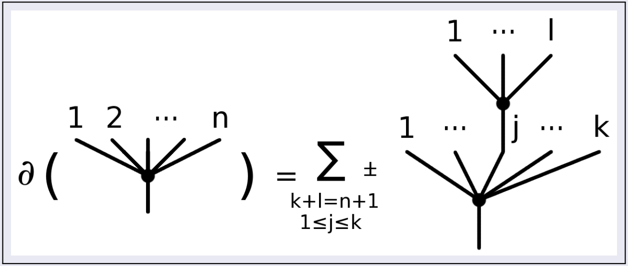
for $n \geq 2$.

- In $\mathrm{Hom}(H^{\otimes n}, H)$, it satisfies



Definition (A_∞-algebra, Stasheff '63)

An A_∞-algebra is a chain complex $(H, d_H, \mu_2, \mu_3, \ldots)$ endowed with a family of multilinear maps of degree $|\mu_n| = n - 2$ satisfying



- **Associative algebras $\subset A_\infty$-algebras**

  Associative algebra = A_∞-algebra s.t. $\mu_n = 0$, for $n \geq 3$.

Theorem (HTT for A_∞-algebras, Kadeshvili '82)

For any deformation retract

$$h \circlearrowright (A, d_A) \underset{i}{\overset{p}{\rightleftarrows}} (H, d_H) \qquad \mathrm{id}_A - ip = d_A h + h d_A$$

and any associative algebra structure on A, there exists an A_∞-algebra structure on H which is "homotopy equivalent".

Application: $A = (C^\bullet_{\mathrm{Sing}}(X), \cup)$, transferred A_∞-algebra on $H^\bullet_{\mathrm{Sing}}(X)$
= lifting of the higher Massey products.

Borromean rings : non-trivial linking

- **Next goals:**

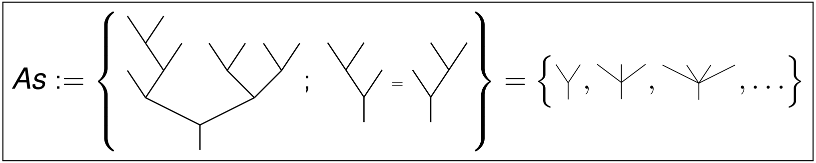
  ◇ Understand conceptually why the notion A_∞-algebra.

  $$\Downarrow$$

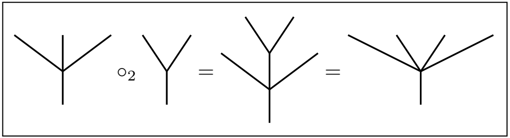
  ◇ Generalise the HTT to other types of algebras.

Plan

- **Idea:** Consider all the iterated compositions of operations of type $\mathcal{P}$ modulo their relations.



- **Compositions:**



Definition (Operad)

A collection of operations $\mathcal{P}$ with compositions $\{\circ_i\}$ is an operad (="Operations+Monad").

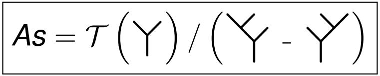
Rings $\rightsquigarrow$ Operads

- **Free operad:** answers the universal property

$$\begin{tikzcd} V \arrow[r] \arrow[dr, "\varphi"'] & \mathcal{T}(V) \arrow[d, dotted, "\exists ! \Phi\ :\ \text{morphism of operads}"] \\ & \mathcal{P} \end{tikzcd}$$

- Example: 

- **Notions:** Ideal, Quotient Operad, etc.

- Example: 

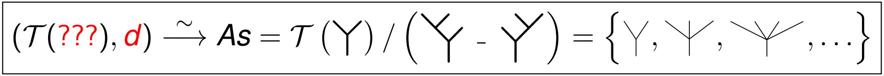
Method:

$$\begin{tikzcd}
\boxed{\text{an operad } \mathcal{P}} \arrow[d, dotted] & \boxed{\mathcal{P}_\infty\text{: quasi-free resolution (cofibrant)}} \arrow[l, "\sim"'] \arrow[d, dotted] \\
\boxed{\begin{array}{c}\text{category of algebras}\\ \text{of type } \mathcal{P}\end{array}} \arrow[r, "\subset"] & \boxed{\begin{array}{c}\text{category of } \mathcal{P}_\infty\text{-algebra} =\\ \text{category of homotopy } \mathcal{P}\text{-algebras}\end{array}}
\end{tikzcd}$$

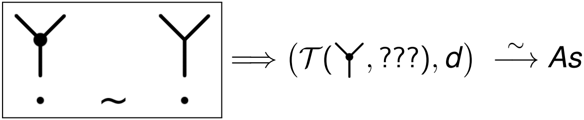
Example:

$$\begin{tikzcd}
\boxed{\text{the operad } \mathit{As}} \arrow[d, dotted] & \boxed{\text{???: quasi-free resolution}} \arrow[l, "\sim"'] \arrow[d, dotted] \\
\boxed{\text{Associative Algebras}} \arrow[r, "\subset"] & \boxed{A_\infty\text{-algebras (?)}}
\end{tikzcd}$$

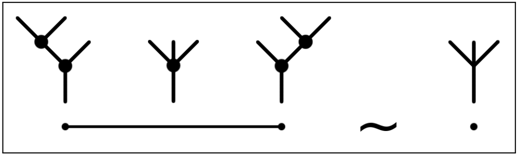
Example:



- **Arity 2/Dimension 0:**

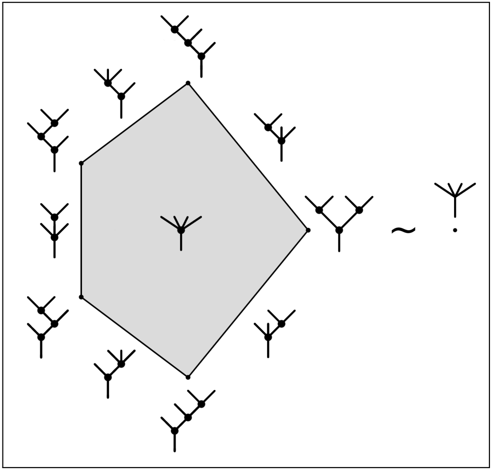


- **Arity 3/Dimension 1:**



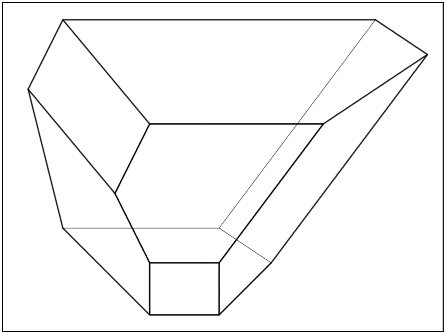
$$\Longrightarrow \boxed{(\mathcal{T}(\curlyvee, \curlyvee, ???), d) \xrightarrow{\sim} As}$$

- **Arity 4/Dimension 2:**



$$\Longrightarrow \boxed{(\mathcal{T}(\curlyvee, \curlyvee, \curlyvee, ???), d) \xrightarrow{\sim} As}$$

- **Arity 5/Dimension 3:**

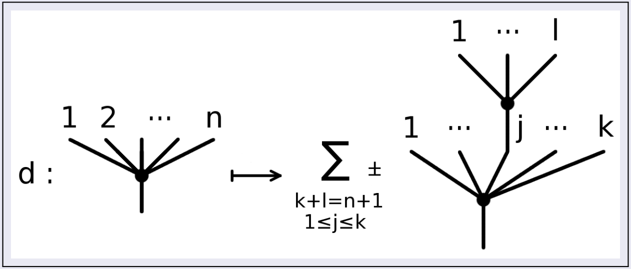


- **Arity n/Dimension n-2:** add one generator.

- **Final solution:** $\boxed{A_\infty = \left(\mathcal{T}\left(\curlyvee, \curlyvee, \curlyvee, \ldots\right), d\right) \xrightarrow{\sim} As}$

Theorem (Resolution A_∞)

With the boundary map



the operad A_∞ is a quasi-free resolution of As.

- **Step-by-step resolution:** "Koszul-Tate resolution" (=operadic syzygies).

Koszul Duality Theory

- **Extra data:** Presentation of the operad

$$\boxed{\mathcal{P} = \mathcal{T}(V)/(R)}$$

with generators V and relations R.

- **Upshot:** creates infinitely many higher operations under a universal property

$$\boxed{\mathcal{P}^{¡} := \mathcal{C}(V, R)}\ .$$

- **Koszul operad**: when this gives a quasi-free resolution

$$\boxed{(\mathcal{T}(\mathcal{P}^{¡}), d) \xrightarrow{\sim} \mathcal{P}}\ .$$

Plan

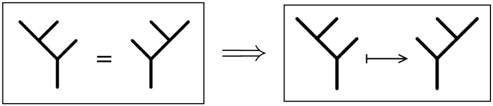
Rewriting method

Computational method to prove $\mathcal{P} = \mathcal{T}(V)/(R)$ is Koszul.

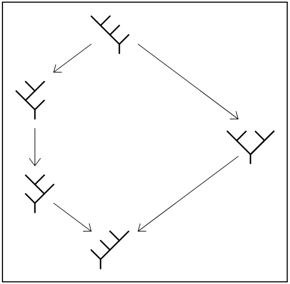
- **Dimension 0:** add a total order on the generators.

- **Dimension 1:** Consider a “good” order on trees,

Relations $\Longrightarrow$ Rewriting rules



- **Dimension 2:**



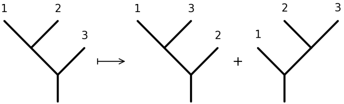
Theorem (Hoffbeck, Dotsenko–Khoroshkin '10)

$\forall$ *critical pairs* *confluent* $\Longrightarrow \mathcal{P}$ *Koszul.*

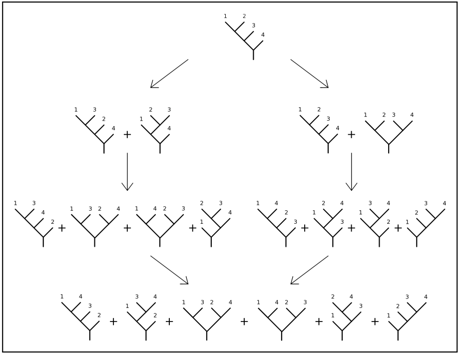
Proof.

Diamond Lemma, Gröbner bases for operads. $\square$

- **Oriented Jacobi identity:**



- **Diamond for the operad *Lie*:**



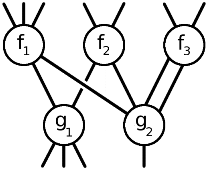
$\Longleftrightarrow$ Zamolodchikov tetrahedron equation (Lie 2-algebras).

State of Art

- **Inhomogneous Koszul duality theory:**
Confluence $\Longrightarrow$ minimality of generators and maximality of relations.

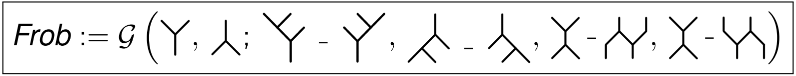
- **Reduction by filtration method:**
see the book "Algebraic Operads" [Loday–Vallette '12].
(free at http://math.unice.fr/∼brunov/Operads.pdf)

- **Koszul duality for properads (props):** [Vallette '03]

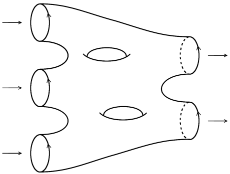


For the next 60 years

- **Main issue for the rewriting method for properads:** find a “good” order graphs of any genus. (????????)
- **Most exciting example:** The properad



of Frobenius (bi)algebras. $\Longrightarrow$ TQFT, Poincaré duality, String topology, Symplectic geometry, Floer homology, etc.



- **Relation Properads–Clones:** [Pierre-Louis Curien ’06]

Joyeux Anniversaire, Pierre-Louis !