Deformation theory of morphisms

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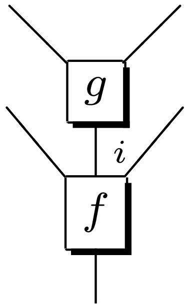
Trends in Noncommutative Geometry

1 Paradigm : Associative algebras

• Let V be a $\mathbb{K}$-module, consider $\mathrm{End}V := \{\mathrm{Hom}(V^{\otimes n}, V)\}_{n\geq 0}$.
For $f \in \mathrm{Hom}(V^{\otimes n}, V)$ and $g \in \mathrm{Hom}(V^{\otimes m}, V)$, binary product

$$f \star g := \sum_{i=1}^{n} \pm$$  $$= \sum_{i=1}^{n} \pm f \circ_i g.$$

Degree convention :
$|f| = n-1$, $|g| = m-1$, so $|f \star g| = |f| + |g|$, that is $|\ \star\ | = 0$.

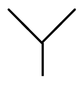
Theorem (Gerstenhaber).

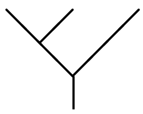
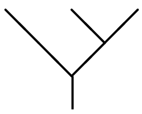
$$(f \star g) \star h - f \star (g \star h) = (f \star h) \star g - f \star (h \star g)$$

$$\textit{Assoc}\,(f, g, h) = \textit{Assoc}\,(f, h, g)$$

$(\mathrm{End}V, \star)$ *is a preLie algebra.*
$\Longrightarrow$ *with* $[f, g] := f \star g - (-1)^{|f|.|g|} g \star f$,
$(\mathrm{End}V, [\,])$ *is a Lie algebra.*

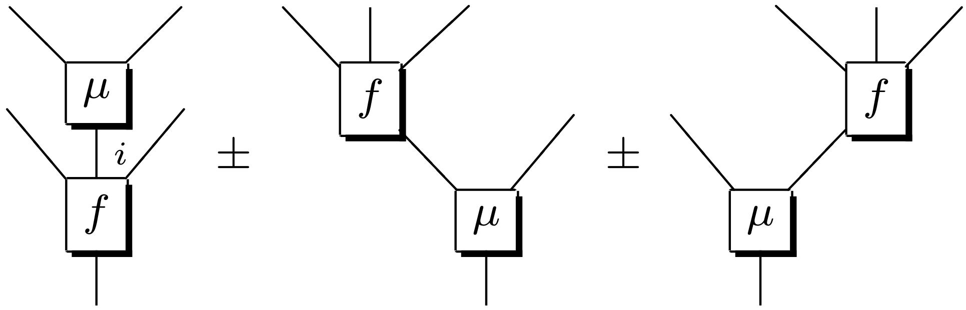
• Associative algebra structure on V :

$$\mu\ :\ V^{\otimes 2} \to V \quad ,$$ 

 $-$  $= 0$ in $\mathrm{Hom}(V^{\otimes 3}, V)$

$$\Longleftrightarrow \boxed{\mu \star \mu = 0} \Longleftrightarrow \boxed{[\mu, \mu] = 0}$$

In this case,
$$d_\mu(f) := [\mu, f] \quad \text{verifies} \quad d_\mu(f)^2 = 0.$$
Explicitly, for $f \in \mathrm{Hom}(V^{\otimes n}, V)$

$$d_\mu(f) = \sum_{i=1}^{n} \pm$$


$\in \mathrm{Hom}(V^{\otimes n+1}, V)$

$$\mathrm{Hom}(\mathbb{K}, V) \xrightarrow{d_\mu} \mathrm{Hom}(V, V) \xrightarrow{d_\mu} \mathrm{Hom}(V^{\otimes 2}, V) \xrightarrow{d_\mu} \cdots$$

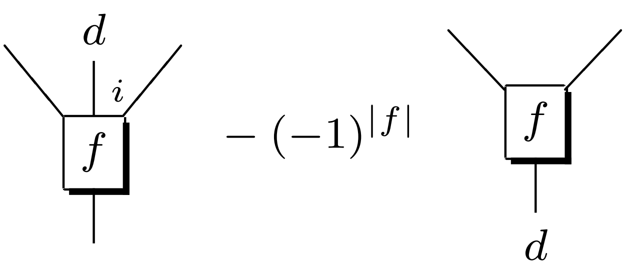
Hochschild cohomology of the associative algebra V “with coefficients into itself”
$(C^\bullet(\mathcal{A}ss, V), d_\mu, [\ ,\ ])$ dg Lie algebra (twisted by μ).

Deformation complex of the associative structure μ
(Interpretation of $\mathrm{H}^0, \mathrm{H}^1, \mathrm{H}^2$ in terms of formal deformations : see Konstevich)

Operations on $C^\bullet(\mathcal{A}ss, V)$:

- ▷ Cup product $\cup$: associative operation
- ▷ Deligne Conjecture

• (V, d) dg module, $\mathrm{End}V$ is a dg module

$$D(f) := \sum_{i=1}^{n}$$


$(\mathrm{End}V, D, \star)$ is a dg preLie algebra and $(\mathrm{End}V, D, [\,])$ is a dg Lie algebra.

(V, d, μ) is a dg associative algebra $\Longleftrightarrow$

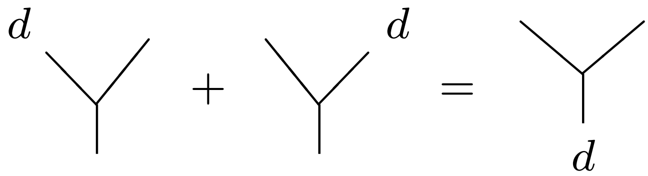
$$\boxed{D\mu + \mu \star \mu = 0} \Longleftrightarrow \boxed{D\mu + \frac{1}{2}[\mu, \mu] = 0}$$: Maurer-Cartan equation

General solutions :

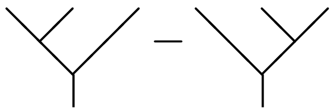
$\mu \in \{\mathrm{Hom}(V^{\otimes n}, V)\}_{n \geq 1}, \quad \mu_n : V^{\otimes n} \to V$ with $\mu_1 = d$.

$D\mu + \mu \star \mu = 0 \Longleftrightarrow$

$\underline{n = 2}$:



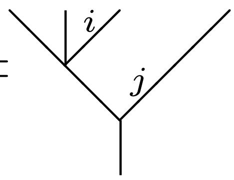
$\underline{n = 3}$:



$= D(\mu_3)$

μ_2 is associative up to the homotopy μ_3

$\underline{n}$: $\sum_{\substack{i+j=n+1 \\ i,j \geq 2}} \pm$



$= D(\mu_n)$

Definition (Stasheff).

A Maurer-Cartan element μ is an *associative algebra up to homotopy* or A_∞*-algebra* structure on $(V, d, \mu = \{\mu_n\}_n)$.

Viewpoint : An associative algebra $=$ very particular A_∞-algebra.

Once again, $d_\mu(f) := D(f) + [\mu, f]$ verifies $d_\mu^2 = 0$.
$(C^\bullet(\mathcal{A}ss, V), d_\mu, [\,,\,])$ dg Lie algebra (twisted by μ)
defines the cohomology of an A_∞-algebra.

Same interpretation of **all** the $H^\bullet$ in terms of deformations of μ.

- <u>Homological perturbation lemma</u>

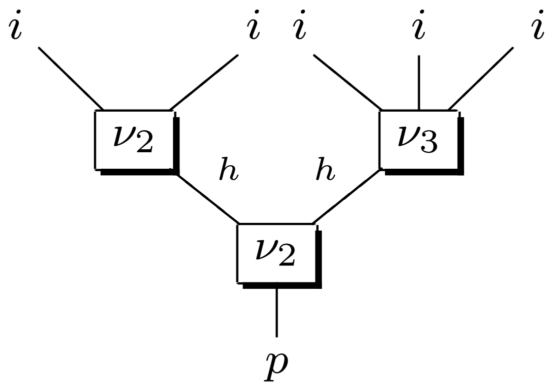
$$(V, d_V) \underset{p}{\overset{i}{\rightleftarrows}} (W, d_W) \circlearrowright h$$

$$p \circ i = \mathrm{Id}_V \quad \text{and} \quad i \circ p - \mathrm{Id}_W = d_W \circ h + h \circ d_W$$

V is a deformation retract of W.

Theorem (Kadeishvili, Merkulov, Kontsevich-Soibelman, Markl).
If $\nu = \{\nu_n\}_n$ is an A_∞-algebra structure on W, than

$$\mu_n = \sum_{\text{planar trees with } n \text{ leaves}}$$



defines an A_∞-algebra on V such that i, p and h extends to morphisms and homotopy in the category of A_∞-algebras.

- Other kind of algebraic structures :
  - ▷ Lie, commutative, Poisson, Gerstenhaber, PreLie, BV algebras.
  - ▷ Lie bialgebras, associative bialgebras.

For any type of (bi)algebras

dg module V

$\downarrow$

dg Lie algebra

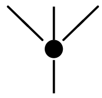
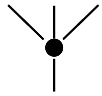
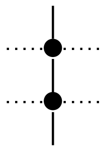
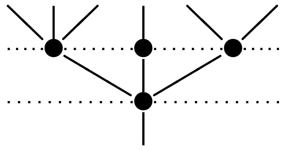
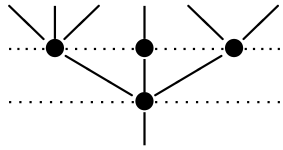
$\downarrow$

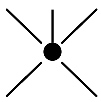
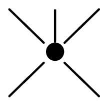
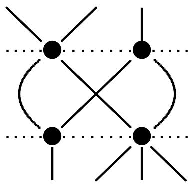
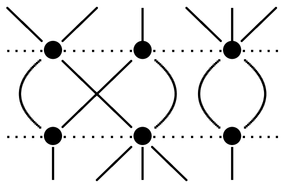
Maurer-Cartan elements $=$ (bi)algebra$_\infty$ structure on V $\longrightarrow$ Homological perturbation lemma

$\downarrow$

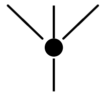
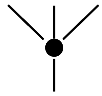
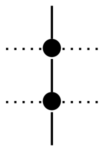
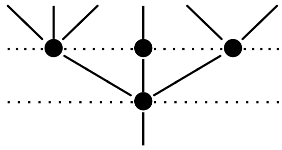
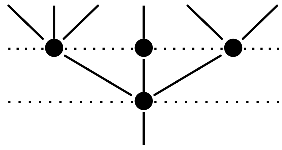
twisted dg Lie : deformation complex (cohomology)

2 ...., ..., ...

Operations		 no symmetry	
Composition		 planar	 non-planar
Monoidal category	(Vect, $\otimes$)	(gVect, $\circ$)	($\mathbb{S}$-Mod, $\circ$)
Monoid	$A \otimes A \to A$	$\mathcal{P} \circ \mathcal{P} \to \mathcal{P}$	$\mathcal{P} \circ \mathcal{P} \to \mathcal{P}$
Modules	Modules	Non-symmetric algebras	Algebras
Examples		associative algebras	Lie, commutative, Gerstenhaber algebras
Free monoid	Ladders (Tensor module)	Planar trees	Trees

Operations		
Composition		
Monoidal category	($\mathbb{S}$-biMod, $\boxtimes_c$)	($\mathbb{S}$-biMod, $\boxtimes$)
Monoid	$\mathcal{P} \boxtimes_c \mathcal{P} \to \mathcal{P}$	$\mathcal{P} \boxtimes \mathcal{P} \to \mathcal{P}$
Modules	(Bial)gebras	(Bial)gebras
Examples	Lie, associative bialgebras	
Free monoid	Connected graphs	Graphs

2 Operads, properads, props

Operations		 no symmetry	
Composition		 planar	 non-planar
Monoidal category	$(\text{Vect}, \otimes)$	$(\text{gVect}, \circ)$	$(\mathbb{S}\text{-Mod}, \circ)$
Monoid	Associative algebras	Non-symmetric operads	Operads
Modules	Modules	Non-symmetric algebras	Algebras
Examples		associative algebras	Lie, commutative, Gerstenhaber algebras
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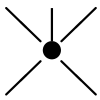
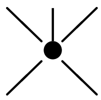
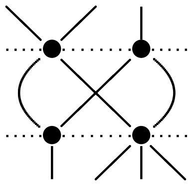
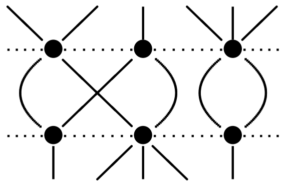
Operations		
Composition		
Monoidal category	$(\mathbb{S}\text{-biMod}, \boxtimes_c)$	$(\mathbb{S}\text{-biMod}, \boxtimes)$
Monoid	Properads	Props
Modules	(Bial)gebras	(Bial)gebras
Examples	Lie, associative bialgebras	
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3 Homological algebra for prop(erad)s

● Recall for associative (co)algebras [Cartan, Eilenberg, MacLane, Moore, ...].

Pair of adjoint functors :

bar construction $B \; : \; \{\text{dg algebras}\} \rightleftharpoons \{\text{dg coalgebras}\} \; : \; \Omega$ cobar construction

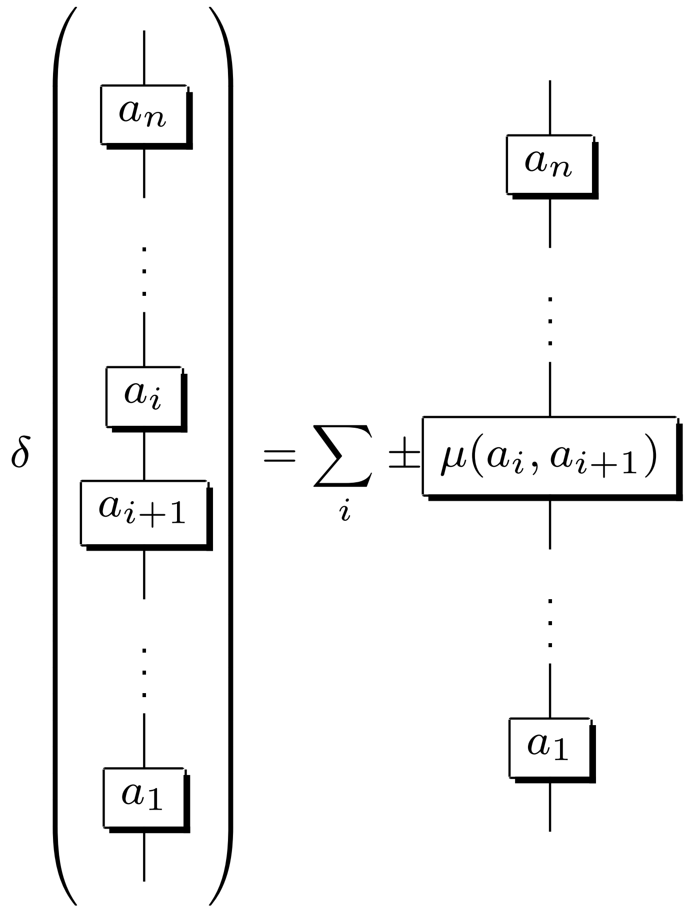
$B(A) := (T^c(s\bar{A}), \delta)$, where

- T^c : cofree connected coalgebra (tensor module)
- s homological suspension
- $\bar{A}$ augmentation ideal
- δ unique coderivation which extends the product of A

$$T^c(s\bar{A}) \twoheadrightarrow (s\bar{A})^{\otimes 2} \xrightarrow{s^{-1}} s(\bar{A} \otimes \bar{A}) \xrightarrow{s\mu} s\bar{A}$$

Explicitly,

$$\delta(a_1 \otimes \cdots a_n) = \sum_i \pm a_1 \otimes \cdots \otimes \mu(a_i, a_{i+1}) \otimes \cdots \otimes a_n.$$



Contracting internal edges : Graph homology à la Kontsevich

- For operads, pair of adjoint functors [Ginzburg-Kapranov, Getzler-Jones]

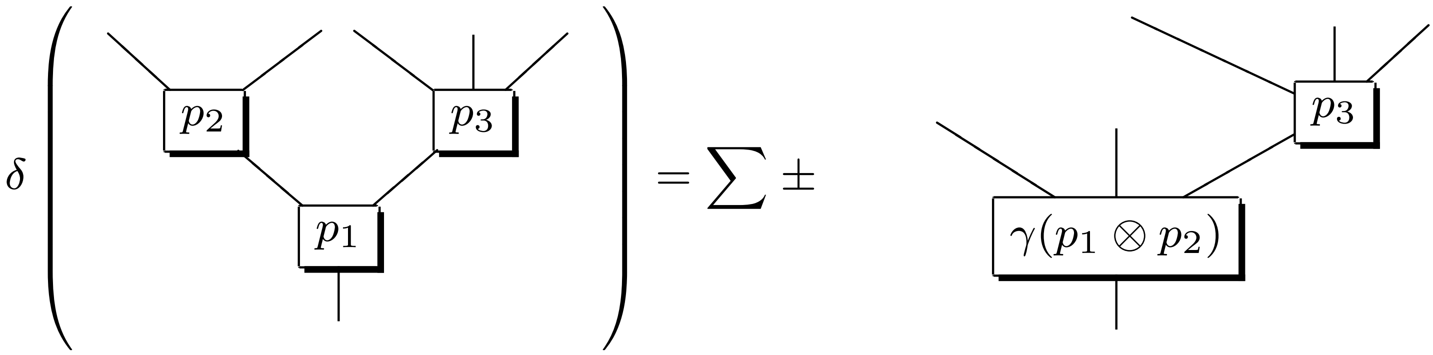
<u>bar construction</u> $B \; : \; \{\text{dg operads}\} \rightleftharpoons \{\text{dg cooperads}\} \; : \; \Omega$ <u>cobar construction</u>

$B(\mathcal{P}) := (\mathcal{F}^c(s\bar{\mathcal{P}}), \delta)$, where

- $\mathcal{F}^c$: cofree connected cooperad (trees)
- s homological suspension
- $\bar{\mathcal{P}}$ augmentation ideal
- δ unique coderivation which extends the partial product of $\mathcal{P}$ (composition of two operations)

$$\mathcal{F}^c(s\bar{\mathcal{P}}) \twoheadrightarrow (s\bar{\mathcal{P}})^{\otimes 2} \xrightarrow{s^{-1}} s(\bar{\mathcal{P}} \otimes \bar{\mathcal{P}}) \xrightarrow{s\gamma} s\bar{\mathcal{P}}$$

Explicitely,



Contracting internal edges : Graph homology à la Kontsevich

- Where do these constructions come from conceptually ?

(C, Δ) coalgebra, (A, μ) algebra; $f, g \ : \ C \to A$

$$f \star g := \quad C \xrightarrow{\Delta} \begin{array}{c} C \xrightarrow{g} A \\ \otimes \qquad\quad \otimes \\ C \xrightarrow{f} A \end{array} \xrightarrow{\mu} A$$

$(\mathrm{Hom}(C, A), \star)$ associative *convolution* algebra.

Theorem (Merkulov-V.).
For $\mathcal{C}$ a dg coprop(erad) and $\mathcal{P}$ a dg prop(erad),
$\mathrm{Hom}(\mathcal{C}, \mathcal{P})$ *is a dg prop(erad) called the* convolution prop(erad)*.*

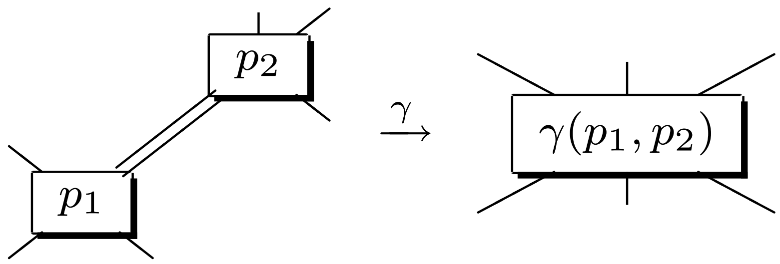
Corollary (Merkulov-V.). $(\mathrm{Hom}(\mathcal{C}, \mathcal{P}), [\ ,\ ])$ *is a dg Lie algebra.*

Tw$(\mathcal{C}, \mathcal{P})$:= set of Maurer-Cartan elements in $(\mathrm{Hom}(\mathcal{C}, \mathcal{P}), [\,,\,])$:
set of *Twisting morphisms (cochains)*.

Tw$(-, -)$ is a bifunctor, try to represent it.

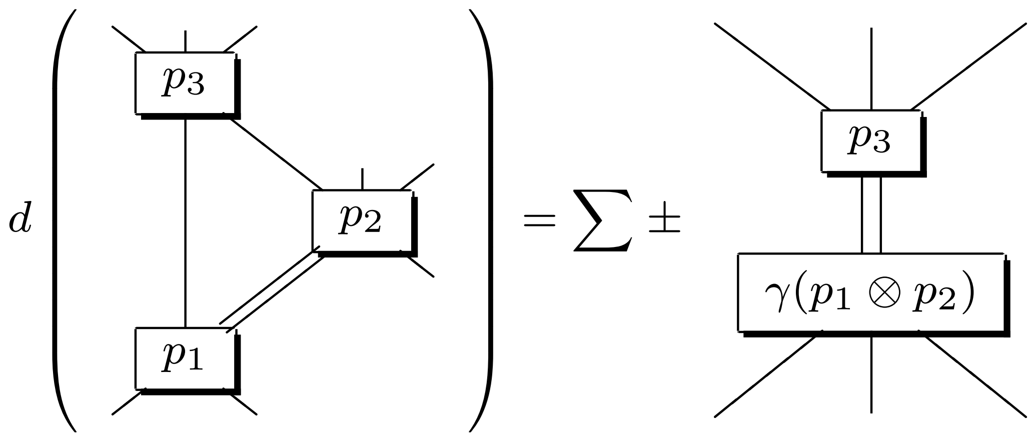
Definition. *Bar construction of a prop(erad)* : $B(\mathcal{P}) := (\mathcal{F}^c(s\bar{\mathcal{P}}), \delta)$, where

- $\mathcal{F}^c$: cofree connected coprop(erad) (graphs)
- s homological suspension
- $\bar{\mathcal{P}}$ augmentation ideal
- δ unique coderivation which extends the partial product of $\mathcal{P}$, composition of two operations



Remark : **The number of internal edges is not relevant.**

Explicitly,



Recover particular cases : Associative algebras, operads.

Cobar construction $\Omega(\mathcal{C})$ is dual.

Theorem (Merkulov, V.).

$$\mathrm{Hom}_{\textit{dg prop(erad)s}}(\Omega(\mathcal{C}), \mathcal{P}) \cong \mathit{Tw}(\mathcal{C}, \mathcal{P}) \cong \mathrm{Hom}_{\textit{dg coprop(erad)s}}(\mathcal{C}, B(\mathcal{P}))$$

Representation of Tw$(-, -)$ and adjunction.

PROOF.

$$\begin{aligned} \mathrm{Hom}_{\text{prop(erad)s}}(\Omega(\mathcal{C}), \mathcal{P}) &= \mathrm{Hom}_{\text{prop(erad)s}}(\mathcal{F}(s^{-1}\bar{\mathcal{C}}), \mathcal{P}) \\ &\cong \mathrm{Hom}^{\mathbb{S}}_{-1}(\bar{\mathcal{C}}, \mathcal{P}) \subset \mathrm{Hom}^{\mathbb{S}}_{-1}(\mathcal{C}, \mathcal{P}) \end{aligned}$$

$$\mathrm{Hom}_{\textbf{dg}\ \text{prop(erad)s}}(\Omega(\mathcal{C}), \mathcal{P}) = \mathrm{MC}(\mathrm{Hom}^{\mathbb{S}}(\mathcal{C}, \mathcal{P})) = \mathrm{Tw}(\mathcal{C}, \mathcal{P})$$

□

Theorem (V.). *Canonical bar-cobar resolution*

$$\Omega(B(\mathcal{P})) \xrightarrow{\sim} \mathcal{P}$$

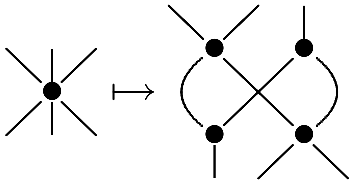
● <u>Minimal models</u>

Definition. *Minimal model* for $\mathcal{P}$

$$(\mathcal{F}(X), \partial) \xrightarrow{\sim} \mathcal{P}$$

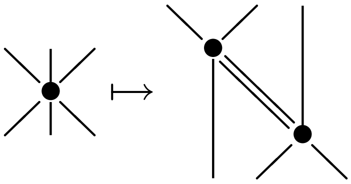
where

- ▷ $\mathcal{F}(X)$ is a (quasi)-free properad
- ▷ ∂ is a derivation $\longleftrightarrow \partial_{|X}$: $X \to \mathcal{F}(X)$

$\partial_{|X}$:  : Vertex expansion

Definition. A *quadratic model* is a *minimal model* $(\mathcal{F}(X), \partial) \xrightarrow{\sim} \mathcal{P}$ such that

$\partial_{|X} : X \to \mathcal{F}(X)^{(2)}$: graphs with 2 vertices



The number of vertices is relevant, not the number of internal edges.

If $\mathcal{P}$ has a quadratic model, it is called a *Koszul* properad.
In this case, $X = \mathcal{C}$ is a coproperad and $(\mathcal{F}(X), \partial) = \Omega(\mathcal{C})$

- Koszul duality theory

  (associative algebras [Priddy], operads [Ginzburg-Kapranov], properads [V.]) provides a method to

  - ▷ compute $X = \mathcal{P}^{¡}$: Koszul dual
  - ▷ make ∂ explicit
  - ▷ criterion to prove the quasi-isomorphism $F(X) = \Omega(\mathcal{P}^{¡}) \xrightarrow{\sim} \mathcal{P}$.
    (acyclicity of a small chain complex : the Koszul complex)

$\Longrightarrow$ Graph homology [Kontsevich, Markl-Voronov]

4 Deformation complex of a morphism

$\mathcal{P} \xrightarrow{f} \mathcal{Q}$ morphism of prop(erad)s ($\mathcal{Q}$ is a *representation* of $\mathcal{P}$).

Example.
V dg module, $\mathrm{End}V := \{\mathrm{Hom}(V^{\otimes n}, V^{\otimes m})\}_{n,m}$ is a dg prop(erad) (composition of multilinear functions)

Definition.
A structure of $\mathcal{P}$*-gebra* on V is a morphism of prop(erad)s $\mathcal{P} \to \mathrm{End}V$.

<u>Recall</u> Quillen : "(commutative) algebraic geometry"
deformation complex of morphisms of commutative algebras (cotangent complex, model category structure).

comm. algebras $\longrightarrow$ ass. algebras $\longrightarrow$ operads $\longrightarrow$ prop(erad)s
$\longmapsto$ Noncommutative geometry **[Nonlinear]**

Theorem (Merkulov-V.).

- *The category of dg prop(erad)s is a cofibrantly generated model category structure*
- *Quasi-free prop(erad)s are cofibrant*

PROOF.

$$\mathcal{F} \ : \ \text{dg } \mathbb{S}\text{-bimodules} \ \rightleftharpoons \ \text{dg prop(erad)s} \ : \ U$$

□

Definition (Deformation complex).

$$\text{cofibrant resolution} \;:\; \begin{array}{ccc} (\mathcal{R}, \partial) & \xrightarrow{\sim} & \mathcal{P} \\ & \searrow & \downarrow f \\ & & \mathcal{Q} \end{array}$$

$$C^\bullet(\mathcal{P}, \mathcal{Q}) := (\mathrm{Der}(\mathcal{R}, \mathcal{Q}), \partial^*)$$

- <u>Well defined :</u>

Extension of Quillen theory of commutative rings to prop(erad)s

To

$$\begin{array}{ccccc} & & \mathcal{O} & & \\ & & \downarrow & & \\ I & \longrightarrow & \mathcal{P} & \longrightarrow & \mathcal{Q} \end{array}$$

$$\begin{aligned} \mathrm{Der}_I(\mathcal{O}, \mathcal{Q}) &\cong \mathrm{Hom}_{\text{dg prop(erad)s } / \mathcal{P}}(\mathcal{O}, \underbrace{\mathcal{P} \ltimes \mathcal{Q}}_{\text{Eilenberg-MacLane space}}) \\ &\cong \mathrm{Hom}_{\mathcal{P}-\text{bimodules}}(\underbrace{\mathcal{P} \boxtimes_{\mathcal{O}} \Omega_{\mathcal{O}/I} \boxtimes_{\mathcal{O}} \mathcal{P}}_{\text{Cotangent complex}}, \mathcal{Q}), \end{aligned}$$

where $\Omega_{\mathcal{O}/I}$ is the module of Kähler differentials of the prop(erad) $\mathcal{O}$.

(Read the properties of the morphism $\mathcal{P} \xrightarrow{f} \mathcal{Q}$ on the cotangent complex)

Quillen adjunction $\Longrightarrow$ Total derived functor (non-additive)

$\Longrightarrow$ well defined in the homotopy categories

- Explicitly,

When $\mathcal{P}$ is Koszul : $\mathcal{R} = \Omega(\mathcal{P}^{\text{¡}}) = \mathcal{F}(s^{-1}\bar{\mathcal{P}}^{\text{¡}}) \xrightarrow{\sim} \mathcal{P}$ quadratic model

In this case,

$$C^\bullet(\mathcal{P}, \mathcal{Q}) = \mathrm{Der}(\mathcal{F}(s^{-1}\bar{\mathcal{P}}^{\text{¡}}), \mathcal{Q}) = \mathrm{Hom}^{\mathbb{S}}_{\bullet-1}(\bar{\mathcal{P}}^{\text{¡}}, \mathcal{Q}) \subset \mathrm{Hom}^{\mathbb{S}}(\mathcal{P}^{\text{¡}}, \mathcal{Q})$$

Theorem (Merkulov-V.).

$\mathrm{Hom}^{\mathbb{S}}(\mathcal{P}^{i}, \mathcal{Q})$ *is a non-symmetric dg prop(erad)*

$\Longrightarrow$ *it is a dg Lie algebra.*

Proposition (Merkulov-V.). $\mathcal{P} \xrightarrow{f} \mathcal{Q}$ *is a morphism of dg prop(erad)s iff* $\bar{f} : \mathcal{P}^{i} \to \mathcal{P} \xrightarrow{f} \mathcal{Q}$ *is a Maurer-Cartan element in* $\mathrm{Hom}^{\mathbb{S}}(\mathcal{P}^{i}, \mathcal{Q})$

In this case,

$$(C^{\bullet}(\mathcal{P}, \mathcal{Q}), d) \subset \underbrace{(\mathrm{Hom}^{\mathbb{S}}(\mathcal{P}^{i}, \mathcal{Q}), [\bar{f}, -])}_{\text{twisted dg Lie algebra}}$$

● Examples

$\mathcal{P} = \mathcal{A}ss$, $\mathcal{P}^{i} = \mathcal{A}ss^{\vee}$ and $C^{\bullet}(\mathcal{A}ss, \mathrm{End}V) = \mathrm{End}V$:

Hochschild cohomology of associative algebras

$\mathcal{P} = \mathcal{A}ss$, $\mathcal{P}^{i} = \mathcal{A}ss^{\vee}$ and $C^{\bullet}(\mathcal{A}ss, \mathcal{P}oisson)$:

Invariant of knots [Vassiliev, Turchine]

$\mathcal{P} = \mathcal{L}ie$, $\mathcal{P}^{i} = \mathcal{C}om^{\vee}$ and $C^{\bullet}(\mathcal{L}ie, \mathrm{End}V)$:

Chevalley-Eilenberg cohomology of Lie algebras

$\mathcal{P} = \mathcal{C}om$, $\mathcal{P}^{i} = \mathcal{L}ie^{\vee}$ and $C^{\bullet}(\mathcal{C}om, \mathrm{End}V)$:

Harrison cohomology of commutative algebras

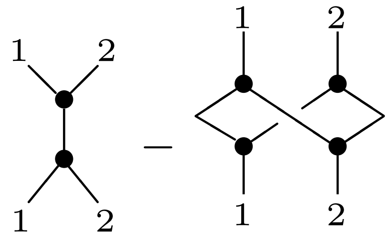
$\mathcal{P} = \mathcal{B}i\mathcal{L}ie$, $\mathcal{P}^{i} = \mathcal{F}rob_{\diamond}^{\vee}$ and $C^{\bullet}(\mathcal{B}i\mathcal{L}ie, \mathrm{End}V)$:

Ciccoli-Guerra cohomology of Lie bialgebras

$\mathcal{P} = \mathcal{B}i\mathcal{A}ss$, **not Koszul** and $C^\bullet(\mathcal{B}i\mathcal{A}ss, \mathrm{End}V)$:

Gerstenhaber-Shack bicomplex ???

$\mathcal{P}$ Koszul $\Longrightarrow \mathcal{P}$ quadratic but $\mathcal{B}i\mathcal{A}ss$ not quadratic



Interpretation of H^0, H^1, H^2 in terms of formal deformations

Definition. A $\mathcal{P}_\infty$*-gebra* (or *homotopy* $\mathcal{P}$*-gebra structure*) on V is a Maurer-Cartan element in $\mathrm{Hom}^{\mathbb{S}}(\mathcal{P}^{¡}, \mathrm{End}V)$

- Examples

$\mathcal{P} = \mathcal{A}ss$, homotopy associative algebra [Stasheff]

$\mathcal{P} = \mathcal{L}ie$, homotopy Lie algebra [Stasheff, Hinich-Schetchman, Kontsevich]

$\mathcal{P} = \mathcal{C}om$, $\mathcal{C}_\infty$-algebra [Stasheff]

$\mathcal{P} = \mathcal{G}erstenhaber$, $\mathcal{G}_\infty$-algebra [Getzler-Jones]

$\mathcal{P} = \mathcal{B}i\mathcal{L}ie$, homotopy Lie bialgebra [Gan]

$\mathcal{P} = \mathcal{B}i\mathcal{A}ss$ ...

$(\mathrm{Hom}^{\mathbb{S}}(\mathcal{P}^{¡}, \mathrm{End}V), [f, -])$: cohomology of $\mathcal{P}_\infty$-algebras

Interpretation of $\mathrm{H}^\bullet$ in terms of deformations

- Operations on the deformation complex of $\mathcal{P} \xrightarrow{f} \mathcal{Q}$

When $\mathcal{P}$ is a Koszul operad,

$$\mathrm{Hom}(\mathcal{P}^{¡}, \mathcal{Q}) \leftarrow \mathrm{Hom}(\mathcal{P}^{¡}, \mathcal{P}) \cong \underbrace{\mathcal{P}^! \otimes \mathcal{P}}_{\substack{\text{Manin complex} \\ \text{tangent complex}}} \leftarrow \underbrace{\mathcal{P}^! \circ \mathcal{P}}_{\text{Manin products}}$$

Example : $\mathcal{A}ss \circ \mathcal{A}ss = \mathcal{A}ss \Longrightarrow$ cup product

Theorem (V.). $\mathcal{P}$ *finitely generated binary non-symmetric Koszul operad*

$$\textit{Little disk operad} \longleftarrow \bullet \longrightarrow C^\bullet(\mathcal{P}, \mathcal{Q})$$

Generalized Deligne conjecture ($\mathcal{P} = \mathcal{A}ss$, $\mathcal{Q} = \mathrm{End}V$)

PROOF.

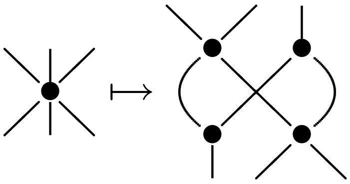
- ▷ $C^\bullet(\mathcal{P}, \mathcal{Q}) = \mathrm{Hom}(\mathcal{P}^{¡}, \mathcal{Q})$ non-symmetric operad $\Longrightarrow$ braces operations
- ▷ $\mathcal{A}ss \to \mathcal{P}^! \circ \mathcal{P} \Longrightarrow$ cup product $\cup$
- ▷ $\partial = [\cup, -]$
- ▷ (McClure-Smith)

□

Examples : 4 infinite families of operads (Koszul by poset method)

5 Beyond the Koszul case

Minimal model for $\mathcal{P}$: $(\mathcal{F}(X), \partial) \xrightarrow{\sim} \mathcal{P}$
where $\partial_{|X}$: $X \to \mathcal{F}(X)$

$\partial_{|X}$:  : not necessarily quadratic

$\partial^2 = 0 \Longrightarrow X$ is a **homotopy** coprop(erad).
(coassociative of the coprop(erad) holds 'up to homotopy')

Theorem (Merkulov-V.).
*For $\mathcal{C}$ a **homotopy** coprop(erad) and $\mathcal{P}$ a dg prop(erad),*
*$\mathrm{Hom}(\mathcal{C}, \mathcal{P})$ is a **homotopy** prop(erad)*
*$\Longrightarrow (\mathrm{Hom}(\mathcal{C}, \mathcal{P}), [\ ,\ ])$ is a **homotopy** Lie algebra.*

Proposition (Merkulov-V.).
In this case, $\mathrm{Hom}(\mathcal{C}, \mathcal{P})$ is a filtered homotopy Lie algebra
Maurer-Cartan elements : $\sum_{n\geq 0} \frac{1}{n!} l_n(f, \ldots, f) = 0$

$\mathcal{P} \xrightarrow{f} \mathcal{Q}$ is a morphism of dg prop(erad)s iff
$\bar{f}$: $\mathcal{C} \to \mathcal{P} \xrightarrow{f} \mathcal{Q}$ is a Maurer-Cartan element in $\mathrm{Hom}^{\mathbb{S}}(\mathcal{P}^{i}, \mathcal{Q})$

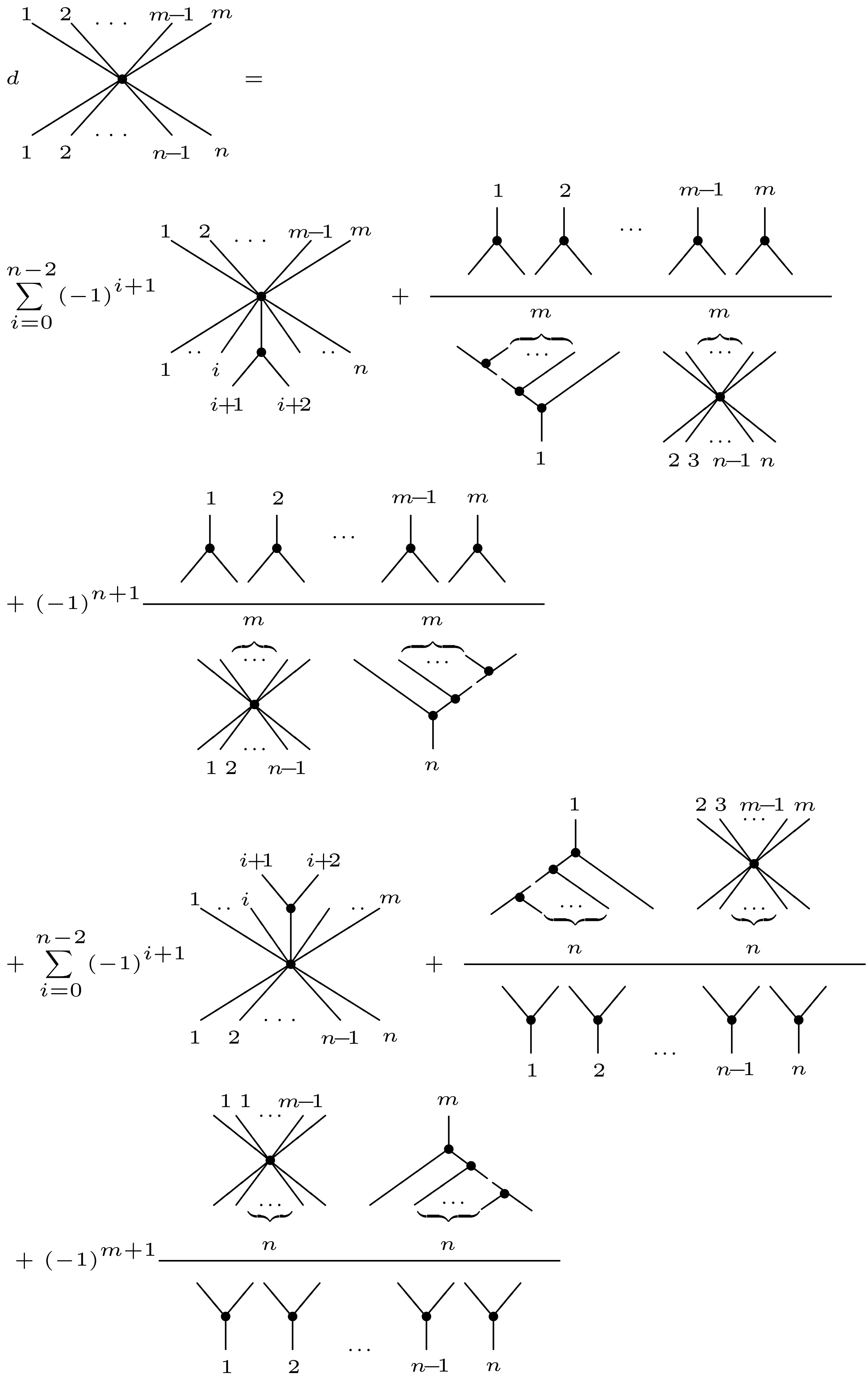
In this case,

$$(C^{\bullet}(\mathcal{P}, \mathcal{Q}), d) \subset (\underbrace{\mathrm{Hom}^{\mathbb{S}}(\mathcal{P}^{i}, \mathcal{Q}), l^{\bar{f}}}_{\textit{twisted homotopy Lie algebra}})$$

Theorem (Merkulov-V.). *For every minimal model of $\mathcal{B}i\mathcal{A}ss$, the deformation complex*

$$C^{\bullet}(\mathcal{B}i\mathcal{A}ss, \mathcal{Q}) \cong \mathcal{Q}$$

and


d
=
$\sum_{i=0}^{n-2}(-1)^{i+1}$
+
$+\ (-1)^{n+1}$
$+\ \sum_{i=0}^{n-2}(-1)^{i+1}$
+
$+\ (-1)^{m+1}$
1 2 . . . m–1 m
1 2 . . . n–1 n
1 .. i i+1 i+2 .. n
m
n

Corollary (Merkulov-V.).

$C^{\bullet}(\mathcal{B}i\mathcal{A}ss, \mathrm{End}V) =$ *Gerstenhaber-Shack bicomplex*

There is a homotopy Lie agebra structure on the Gerstenhaber-Shack bicomplex which measures the deformation of structures of bialgebras

6 Homological perturbation lemma

Theorem (V.).

$$(V, d_V) \underset{p}{\overset{i}{\rightleftarrows}} (W, d_W) \circlearrowleft h$$

If W is a homotopy $\mathcal{P}$-gebra, than there is an induced homotopy $\mathcal{P}$-gebra structure on V.

- Examples

$\mathcal{P} = \mathcal{A}ss$ (Kontsevich-Soibelman)
$\mathcal{P} = \mathcal{L}ie$ (Costello, Goncharov, Mnev, ...)
$\mathcal{P} = \mathcal{C}om$ (Cheng-Getzler)
$\mathcal{P} = \mathcal{G}erstenhaber$ **'new'**
$\mathcal{P} = \mathcal{B}i\mathcal{L}ie$ **'new'**
$\mathcal{P} = \mathcal{B}i\mathcal{A}ss$ **TBC ...**

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