



WORKSHEET 1

HOMOTOPY THEORY OF TOPOLOGICAL SPACES I

Definition (Homotopy commutative diagram). A diagram like

$$\begin{array}{ccc}
 U & \xrightarrow{f} & V \\
 \downarrow h & & \downarrow g \\
 X & \xrightarrow{k} & Y
 \end{array}$$

is *homotopy commutative* if the two composites are homotopic, i.e. here $gf \sim kh$.

Exercise 1 (Homotopy categories). Show that the naive homotopy theory hoTop satisfies the universal property defining the modern homotopy category, that is any functor $F : \text{Top} \rightarrow \mathcal{C}$, which sends homotopy equivalences to isomorphisms, factors uniquely through the canonical functor $P : \text{Top} \rightarrow \text{hoTop}$:

$$\begin{array}{ccc}
 \text{Top} & \xrightarrow{P} & \text{hoTop} \\
 & \searrow F & \downarrow \exists! \tilde{F} \\
 & & \mathcal{C} .
 \end{array}$$

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Exercise 2 (Compact-open topology and exponential law). Let us recall that a space is *locally compact* if each neighbourhood of a point x contains a compact neighbourhood. (Notice that the product of two locally compact spaces is again locally compact.)

- (1) Let X be a locally compact space. Show that the evaluation map $e : Y^X \times X \rightarrow Y$ defined by the assignment $(f, x) \mapsto f(x)$ is continuous.
- (2) Let $f : X \times Y \rightarrow Z$ be a continuous map. Show that its set-theoretical adjoint map $\tilde{f} : X \rightarrow Z^Y$ is continuous.
- (3) Give a sufficient condition (C) under which the set-theoretical adjunction

$$- \times Y : \text{Set} \xrightarrow{\perp} \text{Set} : -^Y$$

given, for any $Y \in \text{Top}$, by $\text{Set}(X \times Y, Z) \cong \text{Set}(X, Z^Y)$ induces the adjunction

$$- \times Y : \text{Top} \xrightarrow{\perp} \text{Top} : -^Y .$$

- (4) Show that under Condition (C), the aforementioned adjunction induces an adjunction on the level of the homotopy category

$$- \times Y : \text{hoTop} \xrightarrow[\perp]{} \text{hoTop} : -^Y .$$

(5) Show the exponential law, that is the natural bijection

$$\text{Top}(X \times Y, Z) \cong \text{Top}(X, Z^Y)$$

of the adjunction of Question (3) is an homeomorphism when X and Y are locally compact.

(6) Let $c_a : Z \rightarrow A$ be the constant map with value a . Show that the map

$$\psi : X^Z \times A \rightarrow (X \times A)^Z, \quad (\varphi, a) \mapsto (\varphi, c_a)$$

is continuous.

Recall that the *pushout* of a map $f : X \rightarrow Y$ is the map $f_* : X^Z \rightarrow Y^Z$ defined by $f_*(g) := fg$ and that its *pullback* is the map $f^* : Z^Y \rightarrow Z^X$ defined by $f^*(g) := gf$.

(7) Let $H : X \times I \rightarrow Y$ be a homotopy from $f : X \rightarrow Y$ to $g : X \rightarrow Y$. We denote by $H_t : X \rightarrow Y$ the map defined by $H_t(x) := H(x, t)$. Show that the map $H^Z : X^Z \times I \rightarrow Y^Z$ defined by $H^Z(-, t) := (H_t)_* : X^Z \rightarrow Y^Z$ is a homotopy from $f_* : X^Z \rightarrow Y^Z$ to $g_* : X^Z \rightarrow Y^Z$. Show that the map $Z^H : Z^Y \times I \rightarrow Z^X$ defined by $Z^H(-, t) := (H_t)^* : Z^Y \rightarrow Z^X$ is a homotopy from $f^* : Z^Y \rightarrow Z^X$ to $g^* : Z^Y \rightarrow Z^X$.

(8) Let $f : X \xrightarrow{\sim} Y$ be a homotopy equivalence. Show that its pullback $f^* : \text{Top}(Y, Z) \xrightarrow{\sim} \text{Top}(X, Z)$ and its pushout $f_* : \text{Top}(Z, X) \xrightarrow{\sim} \text{Top}(Z, Y)$ are homotopy equivalences for any $Z \in \text{Top}$.



Exercise 3 (Cofibre sequence). The goal of this exercise is to prove that the cofibre sequence

$$X \xrightarrow{f} Y \xrightarrow{f_1} \text{Cone}(f) \xrightarrow{p(f)} \Sigma X \xrightarrow{\Sigma f} \Sigma Y \xrightarrow{\Sigma f_1} \Sigma \text{Cone}(f) \xrightarrow{\Sigma p(f)} \Sigma^2 X \xrightarrow{\Sigma^2 f} \Sigma^2 Y \xrightarrow{\Sigma^2 f_1} \Sigma^2 \text{Cone}(f) \xrightarrow{\Sigma^2 p(f)} \dots$$

is h-coexact.

(1) Show that the sequence

$$X \xrightarrow{f} Y \xrightarrow{f_1} \text{Cone}(f) \xrightarrow{f_2} \text{Cone}(f_1) \xrightarrow{f_3} \text{Cone}(f_2)$$

is h-coexact.

(2) Show that all the bottom maps of the following commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{i_1} & \text{Cone}(Y) \\ \downarrow i_1 & & \downarrow f_1 & & \downarrow j_1 \\ \text{Cone}(X) & \xrightarrow{j} & \text{Cone}(f) & \xrightarrow{f_2} & \text{Cone}(f_1) \\ \downarrow p & & \downarrow p(f) & & \downarrow \sim q(f) \\ \text{Cone}(X)/i_1(X) & \xrightarrow{\cong} & \text{Cone}(f)/f_1(Y) & \xrightarrow{\cong} & \text{Cone}(f_1)/j_1(\text{Cone}(Y)) \\ \parallel & & \parallel & & \parallel \\ \Sigma X & & \Sigma X & & \Sigma X \end{array}$$

are homeomorphisms with the suspension ΣX of the space X . (The above two squares are the pushouts defining the respective mapping cones).

(3) Show that $q(f)$ is a homotopy equivalence.

We denote by $\tau : \Sigma X \rightarrow \Sigma X$ the orientation reversing homeomorphism defined by $(x, t) \mapsto (x, 1 - t)$.

(4) We consider the following diagram

$$\begin{array}{ccccc}
 \text{Cone}(f) & \xrightarrow{f_2} & \text{Cone}(f_1) & \xrightarrow{f_3} & \text{Cone}(f_2) \\
 & \searrow p(f) & \downarrow q(f) & \searrow p(f_1) & \downarrow q(f_1) \\
 & & \Sigma X & \xrightarrow{\Sigma f \circ \tau} & \Sigma Y
 \end{array}$$

Show that the triangles of the left-hand side and on the right-hand side are commutative and that the middle triangle is homotopy commutative, that is $\Sigma f \circ \tau \circ q(f) \sim p(f_1)$.

(5) Conclude that the sequence

$$X \xrightarrow{f} Y \xrightarrow{f_1} \text{Cone}(f) \xrightarrow{p(f)} \Sigma X \xrightarrow{\Sigma f} \Sigma Y$$

is h-coexact.

(6) Show that there exists an homeomorphism $\chi : \text{Cone}(\Sigma f) \xrightarrow{\cong} \Sigma \text{Cone}(f)$, which satisfies

$$\chi \circ (\Sigma f)_1 = \Sigma f_1 .$$

(7) Conclude.

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Exercise 4 (Abelian group). Prove directly that the product $+_1$ on $[\Sigma^2 X, Y]_*$ is abelian.

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Exercise 5 (Compatibility between the fiber sequence and the cofiber sequence).

(1) Describe the unit $\eta : X \rightarrow \Omega \Sigma X$ and the counit $\varepsilon : \Sigma \Omega X \rightarrow X$ of the Σ - Ω adjunction.

(2) Let $f : X \rightarrow Y$ be a pointed map. Show that the assignment

$$(x, \varphi) \mapsto \begin{cases} \varphi(2t) & \text{for } 0 \leq t \leq \frac{1}{2}, \\ (x, 2(1-t)) & \text{for } \frac{1}{2} \leq t \leq 1, \end{cases}$$

defines a pointed map

$$\tilde{\eta} : \text{Path}(f) \rightarrow \Omega \text{Cone}(f) .$$

(3) Describe the adjoint map

$$\tilde{\varepsilon} : \Sigma \text{Path}(f) \rightarrow \text{Cone}(f) .$$

(4) Show that the following diagram

$$\begin{array}{ccccccccccc}
 & & \Sigma \Omega \text{Path}(f) & \xrightarrow{\Sigma \Omega f^1} & \Sigma \Omega X & \xrightarrow{\Sigma \Omega f} & \Sigma \Omega Y & \xrightarrow{\Sigma i(f)} & \Sigma \text{Path}(f) & \xrightarrow{\Sigma f_1} & \Sigma X \\
 & & \downarrow \varepsilon & & \downarrow \varepsilon & & \downarrow \varepsilon & & \downarrow \tilde{\varepsilon} & & \parallel \\
 \Omega Y & \xrightarrow{i(f)} & \text{Path}(f) & \xrightarrow{f^1} & X & \xrightarrow{f} & Y & \xrightarrow{f_1} & \text{Cone}(f) & \xrightarrow{p(f)} & \Sigma X \\
 \parallel & & \downarrow \tilde{\eta} & & \downarrow \eta & & \downarrow \eta & & \downarrow \eta & & \\
 \Omega Y & \xrightarrow{i(f)} & \Omega \text{Cone}(f) & \xrightarrow{f^1} & \Omega \Sigma X & \xrightarrow{f} & \Omega \Sigma Y & \xrightarrow{f_1} & \Omega \Sigma \text{Cone}(f) & &
 \end{array}$$

is homotopy commutative.

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