

WORKSHEET 1

HOMOTOPY THEORY OF TOPOLOGICAL SPACES I

Definition (Homotopy commutative diagram). A diagram like

$$\begin{array}{ccc}
U & \xrightarrow{f} & V \\
\downarrow h & & \downarrow g \\
V & & \downarrow X & \xrightarrow{k} & Y
\end{array}$$

is homotopy commutative if the two composites are homotopic, i.e. here $gf \sim kh$.

Exercise 1 (Homeomorphism). Show that the quotient space $I^n/\partial I^n$ is homeomorphic to the *n*-dimensional sphere S^n .

Exercise 2 (Deformation retract). Show that the circle S^1 is a strong deformation retract of the complement $S^3 \setminus S^1$.

Exercise 3 (Compact-open topology and exponential law). Let us recall that a space is *locally compact* if each neighbourhood of a point *x* contains a compact neighbourhood. (Notice that the product of two locally compact spaces is again locally compact.)

- (1) Let X be a locally compact space. Show that the evaluation map $e: Y^X \times X \to Y$ defined by the assignment $(f, x) \mapsto f(x)$ is continuous.
- (2) Let $f: X \times Y \to Z$ be a continuous map, where Y is locally compact. Show that its settheoretical adjoint map $\tilde{f}: X \to Z^Y$ is continuous.
- (3) Give a suffisant condition (C) under which the set-theoretical adjunction

$$-\times Y$$
 : Set \longrightarrow Set : $-^Y$

given, for any $Y \in \text{Top}$, by $\text{Set}(X \times Y, Z) \cong \text{Set}(X, Z^Y)$ induces the adjunction

$$-\times Y$$
 : Top $\xrightarrow{\perp}$ Top : $-^Y$.

(4) Show that under Condition (C), the aforementioned adjunction induces an adjunction on the level of the homotopy category

$$-\times Y$$
 : hoTop $\stackrel{\longrightarrow}{}$ hoTop : $-^Y$.

(5) Show the exponential law, that is the natural bijection

$$\mathsf{Top}(X \times Y, Z) \cong \mathsf{Top}(X, Z^Y)$$

of the adjunction of Question (3) is an homeomorphism when X and Y are locally compact.

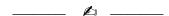
(6) Let $cst_a: Z \to A$ be the constant map with value a. Show that the map

$$\psi: X^Z \times A \to (X \times A)^Z$$
, $(\varphi, a) \mapsto (\varphi, \operatorname{cst}_a)$

is continuous.

Recall that the *push-forward* of a map $f: X \to Y$ is the map $f_*: X^Z \to Y^Z$ defined by $f_*(g) \coloneqq fg$ and that its *pullback* is the map $f^*: Z^Y \to Z^X$ defined by $f^*(g) \coloneqq gf$.

- (7) Let $H: X \times I \to Y$ be a homotopy from $f: X \to Y$ to $g: X \to Y$. We denote by $H_t: X \to Y$ the map defined by $H_t(x) := H(x,t)$. Show that the map $H^Z: X^Z \times I \to Y^Z$ defined by $H^Z(-,t) := (H_t)_*: X^Z \to Y^Z$ is a homotopy from $f_*: X^Z \to Y^Z$ to $g_*: X^Z \to Y^Z$. Show that the map $Z^H: Z^Y \times I \to Z^X$ defined by $Z^H(-,t) := (H_t)^*: Z^Y \to Z^X$ is a homotopy from $f_*: Z^Y \to Z^X$ to $g_*: Z^Y \to Z^X$.
- (8) Let $f: X \xrightarrow{\sim} Y$ be a homotopy equivalence. Show that its pullback $f^*: \operatorname{Top}(Y, Z) \xrightarrow{\sim} \operatorname{Top}(X, Z)$ and its push-forward $f_*: \operatorname{Top}(Z, X) \xrightarrow{\sim} \operatorname{Top}(Z, Y)$ are homotopy equivalences for any $Z \in \operatorname{Top}$.



Exercise 4 (Cofibre sequence). The goal of this exercise is to prove that the cofibre sequence

$$X \xrightarrow{f} Y \xrightarrow{f_1} \operatorname{Cone}(f) \xrightarrow{p(f)} \Sigma X \xrightarrow{\Sigma f} \Sigma Y \xrightarrow{\Sigma f_1} \Sigma \operatorname{Cone}(f) \xrightarrow{\Sigma p(f)} \Sigma^2 X \xrightarrow{\Sigma^2 f} \Sigma^2 Y \xrightarrow{\Sigma^2 f_1} \Sigma^2 \operatorname{Cone}(f) \xrightarrow{\Sigma^2 p(f)} \cdots$$

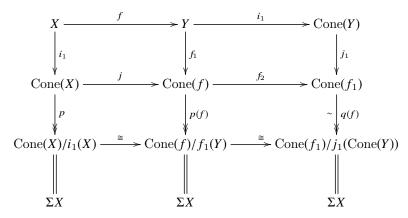
is h-coexact.

(1) Show that the sequence

$$X \xrightarrow{f} Y \xrightarrow{f_1} \operatorname{Cone}(f) \xrightarrow{f_2} \operatorname{Cone}(f_1) \xrightarrow{f_3} \operatorname{Cone}(f_2)$$

is h-coexact.

(2) Show that all the bottom maps of the following commutative diagram

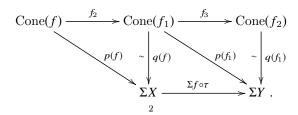


are homeomorphisms with the suspension ΣX of the space X. (The above two squares are the pushouts defining the respective mapping cones).

(3) Show that q(f) is a homotopy equivalence.

We denote by $\tau: \Sigma X \to \Sigma X$ the orientation reversing homeomorphism defined by $(x, t) \mapsto (x, 1 - t)$.

(4) We consider the following diagram



Show that the triangles of the left-hand side and on the right-hand side are commutative and that the middle triangle is homotopy commutative, that is $\Sigma f \circ \tau \circ q(f) \sim p(f_1)$.

(5) Conclude that the sequence

$$X \xrightarrow{f} Y \xrightarrow{f_1} \text{Cone}(f) \xrightarrow{p(f)} \Sigma X \xrightarrow{\Sigma f} \Sigma Y$$

is h-coexact.

(6) Show that there exists an homeomorphism $\chi : \operatorname{Cone}(\Sigma f) \xrightarrow{\cong} \Sigma \operatorname{Cone}(f)$, which satisfies

$$\chi \circ (\Sigma f)_1 = \Sigma f_1$$
.

(7) Conclude.

Exercise 5 (Abelian group). Prove directly that the product $+_1$ on $[\Sigma^2 X, Y]_*$ is abelian.

Exercise 6 (Compatibility between the fiber sequence and the cofiber sequence).

- (1) Describe the unit $\eta: X \to \Omega \Sigma X$ and the counit $\varepsilon: \Sigma \Omega X \to X$ of the $\Sigma \Omega$ adjunction.
- (2) Let $f: X \to Y$ be a pointed map. Show that the assignment

$$(x,\varphi) \mapsto \left\{ \begin{array}{ll} \varphi(2t) & \text{for } 0 \leqslant t \leqslant \frac{1}{2} \,, \\ (x,2(1-t)) & \text{for } \frac{1}{2} \leqslant t \leqslant 1 \,, \end{array} \right.$$

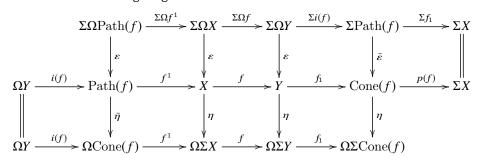
defines a pointed map

$$\tilde{\eta}: \operatorname{Path}(f) \to \Omega \operatorname{Cone}(f)$$
.

(3) Describe the adjoint map

$$\tilde{\varepsilon}: \Sigma \operatorname{Path}(f) \to \operatorname{Cone}(f)$$
.

(4) Show that the following diagram



is homotopy commutative.



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