WORKSHEET 1

HOMOTOPY THEORY OF TOPOLOGICAL SPACES I

Definition (Homotopy commutative diagram). A diagram like
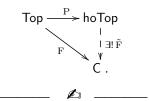
$$
\begin{array}{ccc}
U & \xrightarrow{\ f\ } & V \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} \\
X & \xrightarrow{\ k\ } & Y
\end{array}
$$

is *homotopy commutative* if the two composites are homotopic, i.e. here $gf \sim kh$.

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Exercise 1 (Deformation retract). Show that the circle S^1 is a strong deformation retract of the complement $S^3 \backslash S^1$.

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Exercise 2 (Homotopy categories). Show that the homotopy theory hoTop satisfies the universal homotopy property, that is any functor $\mathrm{F} : \mathsf{Top} \to \mathsf{C}$, which sends homotopy equivalences to isomorphisms, factors uniquely through the canonical functor $\mathrm{P} : \mathsf{Top} \to \mathsf{hoTop}$:

$$
\begin{array}{ccc}
\mathsf{Top} & \xrightarrow{\ \mathrm{P}\ } & \mathsf{hoTop} \\
 & \searrow{\scriptstyle \mathrm{F}} & \downarrow{\scriptstyle \exists!\,\tilde{\mathrm{F}}} \\
 & & \mathsf{C}\,.
\end{array}
$$

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Exercise 3 (Compact-open topology and exponential law). Let us recall that a space is *locally compact* if each neighbourhood of a point x contains a compact neighbourhood. (Notice that the product of two locally compact spaces is again locally compact.)

(1) Let X be a locally compact space. Show that the evaluation map $\mathrm{e} : Y^X \times X \to Y$ defined by the assignment $(f, x) \mapsto f(x)$ is continuous.

(2) Let $f : X \times Y \to Z$ be a continuous map. Show that its set-theoretical adjoint map $\tilde{f} : X \to Z^Y$ is continuous.

(3) Give a suffisant condition (C) under which the set-theoretical adjunction

$$
- \times Y \ : \ \mathsf{Set} \ \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \ \mathsf{Set} \ : \ -^Y
$$

given, for any $Y \in \mathsf{Top}$, by $\mathsf{Set}(X \times Y, Z) \cong \mathsf{Set}(X, Z^Y)$ induces the adjunction

$$
- \times Y \ : \ \mathsf{Top} \ \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \ \mathsf{Top} \ : \ -^Y \,.
$$

(4) Show that under Condition (C), the aforementioned adjunction induces an adjunction on the level of the homotopy category

$$
- \times Y \ : \ \mathsf{hoTop} \ \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \ \mathsf{hoTop} \ : \ -^Y \,.
$$

(5) Show the exponential law, that is the natural bijection

$$\mathsf{Top}(X \times Y, Z) \cong \mathsf{Top}(X, Z^Y)$$

of the adjunction of Question (3) is an homeomorphism when X and Y are locally compact.

(6) Let $c_a : Z \to A$ be the constant map with value a. Show that the map

$$\psi : X^Z \times A \to (X \times A)^Z, \quad (\varphi, a) \mapsto (\varphi, c_a)$$

is continuous.

Recall that the *pushout* of a map $f : X \to Y$ is the map $f_* : X^Z \to Y^Z$ defined by $f_*(g) := fg$ and that its *pullback* is the map $f^* : Z^Y \to Z^X$ defined by $f^*(g) := gf$.

(7) Let $H : X \times I \to Y$ be a homotopy from $f : X \to Y$ to $g : X \to Y$. We denote by $H_t : X \to Y$ the map defined by $H_t(x) := H(x,t)$. Show that the map $H^Z : X^Z \times I \to Y^Z$ defined by $H^Z(-,t) := (H_t)_* : X^Z \to Y^Z$ is a homotopy from $f_* : X^Z \to Y^Z$ to $g_* : X^Z \to Y^Z$. Show that the map $Z^H : Z^Y \times I \to Z^X$ defined by $Z^H(-,t) := (H_t)^* : Z^Y \to Z^X$ is a homotopy from $f^* : Z^Y \to Z^X$ to $g^* : Z^Y \to Z^X$.

(8) Let $f : X \xrightarrow{\sim} Y$ be a homotopy equivalence. Show that its pullback $f^* : \mathsf{Top}(Y, Z) \xrightarrow{\sim} \mathsf{Top}(X, Z)$ and its pushout $f_* : \mathsf{Top}(Z, X) \xrightarrow{\sim} \mathsf{Top}(Z, Y)$ are homotopy equivalences for any $Z \in \mathsf{Top}$.

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Exercise 4 (Cofibre sequence). The goal of this exercise is to prove that the cofibre sequence

$$X \xrightarrow{f} Y \xrightarrow{f_1} \mathrm{Cone}(f) \xrightarrow{p(f)} \Sigma X \xrightarrow{\Sigma f} \Sigma Y \xrightarrow{\Sigma f_1} \Sigma\mathrm{Cone}(f) \xrightarrow{\Sigma p(f)} \Sigma^2 X \xrightarrow{\Sigma^2 f} \Sigma^2 Y \xrightarrow{\Sigma^2 f_1} \Sigma^2\mathrm{Cone}(f) \xrightarrow{\Sigma^2 p(f)} \cdots$$

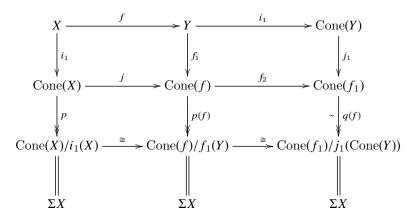is h-coexact.

(1) Show that the sequence

$$X \xrightarrow{f} Y \xrightarrow{f_1} \mathrm{Cone}(f) \xrightarrow{f_2} \mathrm{Cone}(f_1) \xrightarrow{f_3} \mathrm{Cone}(f_2)$$

is h-coexact.

(2) Show that all the bottom maps of the following commutative diagram

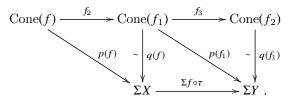are homeomorphisms with the suspension ΣX of the space X. (The above two squares are the pushouts defining the respective mapping cones).

(3) Show that $q(f)$ is a homotopy equivalence.

We denote by $\tau : \Sigma X \to \Sigma X$ the orientation reversing homeomorphism defined by $(x,t) \mapsto (x, 1-t)$.

(4) We consider the following diagram

$$\operatorname{Cone}(f) \xrightarrow{\ f_2\ } \operatorname{Cone}(f_1) \xrightarrow{\ f_3\ } \operatorname{Cone}(f_2)$$

with maps $p(f)$, $\sim \ \big\downarrow q(f)$, $p(f_1)$, $\sim \ \big\downarrow q(f_1)$

$$\Sigma X \xrightarrow{\ \Sigma f \circ \tau\ } \Sigma Y \ .$$

Show that the triangles of the left-hand side and on the right-hand side are commutative and that the middle triangle is homotopy commutative, that is $\Sigma f \circ \tau \circ q(f) \sim p(f_1)$.

(5) Conclude that the sequence

$$X \xrightarrow{\ f\ } Y \xrightarrow{\ f_1\ } \operatorname{Cone}(f) \xrightarrow{\ p(f)\ } \Sigma X \xrightarrow{\ \Sigma f\ } \Sigma Y$$

is h-coexact.

(6) Show that there exists an homeomorphism $\chi : \operatorname{Cone}(\Sigma f) \xrightarrow{\ \cong\ } \Sigma \operatorname{Cone}(f)$, which satisfies

$$\chi \circ (\Sigma f)_1 = \Sigma f_1 \ .$$

(7) Conclude.

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Exercise 5 (Abelian group). Prove directly that the product $+_1$ on $[\Sigma^2 X, Y]_*$ is abelian.

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Exercise 6 (Compatibility between the fiber sequence and the cofiber sequence).

(1) Describe the unit $\eta : X \to \Omega\Sigma X$ and the counit $\varepsilon : \Sigma\Omega X \to X$ of the Σ-Ω adjunction.

(2) Let $f : X \to Y$ be a pointed map. Show that the assignment

$$(x, \varphi) \mapsto \begin{cases} \varphi(2t) & \text{for } 0 \leqslant t \leqslant \frac{1}{2}\,, \\ (x, 2(1-t) & \text{for } \frac{1}{2} \leqslant t \leqslant 1\,, \end{cases}$$

defines a pointed map

$$\tilde{\eta} : \operatorname{Path}(f) \to \Omega\operatorname{Cone}(f) \ .$$

(3) Describe the adjoint map

$$\tilde{\varepsilon} : \Sigma\operatorname{Path}(f) \to \operatorname{Cone}(f) \ .$$

(4) Show that the following diagram

$$\begin{array}{ccccccccc}
\Sigma\Omega\operatorname{Path}(f) & \xrightarrow{\Sigma\Omega f^1} & \Sigma\Omega X & \xrightarrow{\Sigma\Omega f} & \Sigma\Omega Y & \xrightarrow{\Sigma i(f)} & \Sigma\operatorname{Path}(f) & \xrightarrow{\Sigma f_1} & \Sigma X \\
\downarrow{\varepsilon} & & \downarrow{\varepsilon} & & \downarrow{\varepsilon} & & \downarrow{\tilde{\varepsilon}} & & \| \\
\Omega Y \xrightarrow{i(f)} \operatorname{Path}(f) & \xrightarrow{f^1} & X & \xrightarrow{f} & Y & \xrightarrow{f_1} & \operatorname{Cone}(f) & \xrightarrow{p(f)} & \Sigma X \\
\| & & \downarrow{\tilde{\eta}} & & \downarrow{\eta} & & \downarrow{\eta} & & \downarrow{\eta} \\
\Omega Y \xrightarrow{i(f)} \Omega\operatorname{Cone}(f) & \xrightarrow{f^1} & \Omega\Sigma X & \xrightarrow{f} & \Omega\Sigma Y & \xrightarrow{f_1} & \Omega\Sigma\operatorname{Cone}(f)
\end{array}$$

is homotopy commutative.

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