Master 2: Homotopy Theories 2021-2022

Worksheet 1

HOMOTOPY THEORY OF TOPOLOGICAL SPACES I

Definition (Homotopy commutative diagram). A diagram like

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{k} & Y \end{array}$$

is *homotopy commutative* if the two composites are homotopic, i.e. here $gf \sim kh$.

Exercise 1 (Homotopy categories). Show that the homotopy theory hoTop satisfies the universal homotopy property, that is any functor $\mathrm{F} : \mathsf{Top} \to \mathsf{C}$, which sends homotopy equivalences to isomorphisms, factors uniquely through the canonical functor $\mathrm{P} : \mathsf{Top} \to \mathsf{hoTop}$:

$$\begin{array}{ccc} \mathsf{Top} & \xrightarrow{\mathrm{P}} & \mathsf{hoTop} \\ & {\scriptstyle \mathrm{F}}\searrow & \downarrow{\scriptstyle \exists!\, \tilde{\mathrm{F}}} \\ & & \mathsf{C}\ . \end{array}$$

✍

Exercise 2 (Compact-open topology and exponential law). Let us recall that a space is *locally compact* if each neighbourhood of a point x contains a compact neighbourhood. (Notice that the product of two locally compact spaces is again locally compact.)

(1) Let X be a locally compact space. Show that the evaluation map $\mathrm{e} : Y^X \times X \to Y$ defined by the assignment $(f, x) \mapsto f(x)$ is continuous.

(2) Let $f : X \times Y \to Z$ be a continuous map. Show that its set-theoretical adjoint map $\tilde{f} : X \to Z^Y$ is continuous.

(3) Give a suffisant condition (C) under which the set-theoretical adjunction

$$- \times Y \ : \ \mathsf{Set} \underset{\perp}{\rightleftharpoons} \mathsf{Set} \ : \ -^Y$$

given, for any $Y \in \mathsf{Top}$, by $\mathsf{Set}(X \times Y, Z) \cong \mathsf{Set}(X, Z^Y)$ induces the adjunction

$$- \times Y \ : \ \mathsf{Top} \underset{\perp}{\rightleftharpoons} \mathsf{Top} \ : \ -^Y\ .$$

(4) Show that under Condition (C), the aforementioned adjunction induces an adjunction on the level of the homotopy category

$$- \times Y \ : \ \mathsf{hoTop} \underset{\perp}{\rightleftharpoons} \mathsf{hoTop} \ : \ -^Y\ .$$

(5) Show the exponential law, that is the natural bijection

$$\mathsf{Top}(X \times Y, Z) \cong \mathsf{Top}(X, Z^Y)$$

of the adjunction of Question (3) is an homeomorphism when X and Y are locally compact.

(6) Let $c_a : Z \to A$ be the constant map with value a. Show that the map

$$\psi : X^Z \times A \to (X \times A)^Z\ , \quad (\varphi, a) \mapsto (\varphi, c_a)$$

is continuous.

Recall that the *pushout* of a map $f : X \to Y$ is the map $f_* : X^Z \to Y^Z$ defined by $f_*(g) := fg$ and that its *pullback* is the map $f^* : Z^Y \to Z^X$ defined by $f^*(g) := gf$.

(7) Let $H : X \times I \to Y$ be a homotopy from $f : X \to Y$ to $g : X \to Y$. We denote by $H_t : X \to Y$ the map defined by $H_t(x) := H(x,t)$. Show that the map $H^Z : X^Z \times I \to Y^Z$ defined by $H^Z(-,t) := (H_t)_* : X^Z \to Y^Z$ is a homotopy from $f_* : X^Z \to Y^Z$ to $g_* : X^Z \to Y^Z$. Show that the map $Z^H : Z^Y \times I \to Z^X$ defined by $Z^H(-,t) := (H_t)^* : Z^Y \to Z^X$ is a homotopy from $f^* : Z^Y \to Z^X$ to $g^* : Z^Y \to Z^X$.

(8) Let $f : X \xrightarrow{\sim} Y$ be a homotopy equivalence. Show that its pullback $f^* : \mathsf{Top}(Y,Z) \xrightarrow{\sim} \mathsf{Top}(X,Z)$ and its pushout $f_* : \mathsf{Top}(Z,X) \xrightarrow{\sim} \mathsf{Top}(Z,Y)$ are homotopy equivalences for any $Z \in \mathsf{Top}$.

✍

Exercise 3 (Cofibre sequence). The goal of this exercise is to prove that the cofibre sequence

$$X \xrightarrow{f} Y \xrightarrow{f_1} \mathrm{Cone}(f) \xrightarrow{p(f)} \Sigma X \xrightarrow{\Sigma f} \Sigma Y \xrightarrow{\Sigma f_1} \Sigma\mathrm{Cone}(f) \xrightarrow{\Sigma p(f)} \Sigma^2 X \xrightarrow{\Sigma^2 f} \Sigma^2 Y \xrightarrow{\Sigma^2 f_1} \Sigma^2\mathrm{Cone}(f) \xrightarrow{\Sigma^2 p(f)} \cdots$$

is h-coexact.

(1) Show that the sequence

$$X \xrightarrow{f} Y \xrightarrow{f_1} \mathrm{Cone}(f) \xrightarrow{f_2} \mathrm{Cone}(f_1) \xrightarrow{f_3} \mathrm{Cone}(f_2)$$

is h-coexact.

(2) Show that all the bottom maps of the following commutative diagram

$$\begin{array}{ccccc}
X & \xrightarrow{f} & Y & \xrightarrow{i_1} & \mathrm{Cone}(Y) \\
\downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle j_1} \\
\mathrm{Cone}(X) & \xrightarrow{j} & \mathrm{Cone}(f) & \xrightarrow{f_2} & \mathrm{Cone}(f_1) \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p(f)} & & {\scriptstyle \sim}\downarrow{\scriptstyle q(f)} \\
\mathrm{Cone}(X)/i_1(X) & \xrightarrow{\cong} & \mathrm{Cone}(f)/f_1(Y) & \xrightarrow{\cong} & \mathrm{Cone}(f_1)/j_1(\mathrm{Cone}(Y)) \\
\| & & \| & & \| \\
\Sigma X & & \Sigma X & & \Sigma X
\end{array}$$

are homeomorphisms with the suspension ΣX of the space X. (The above two squares are the pushouts defining the respective mapping cones).

(3) Show that $q(f)$ is a homotopy equivalence.

We denote by $\tau : \Sigma X \to \Sigma X$ the orientation reversing homeomorphism defined by $(x,t) \mapsto (x, 1-t)$.

(4) We consider the following diagram

$$\begin{array}{ccccc}
\mathrm{Cone}(f) & \xrightarrow{f_2} & \mathrm{Cone}(f_1) & \xrightarrow{f_3} & \mathrm{Cone}(f_2) \\
& \searrow^{p(f)} & {\scriptstyle \sim}\downarrow{\scriptstyle q(f)} & \searrow^{p(f_1)} & {\scriptstyle \sim}\downarrow{\scriptstyle q(f_1)} \\
& & \Sigma X & \xrightarrow{\Sigma f \circ \tau} & \Sigma Y\ .
\end{array}$$

Show that the triangles of the left-hand side and on the right-hand side are commutative and that the middle triangle is homotopy commutative, that is $\Sigma f \circ \tau \circ q(f) \sim p(f_1)$.

(5) Conclude that the sequence

$$X \xrightarrow{f} Y \xrightarrow{f_1} \mathrm{Cone}(f) \xrightarrow{p(f)} \Sigma X \xrightarrow{\Sigma f} \Sigma Y$$

is h-coexact.

(6) Show that there exists an homeomorphism $\chi : \mathrm{Cone}(\Sigma f) \xrightarrow{\cong} \Sigma\mathrm{Cone}(f)$, which satisfies

$$\chi \circ (\Sigma f)_1 = \Sigma f_1 \ .$$

(7) Conclude.

———— ✍ ————

Exercise 4 (Abelian group). Prove directly that the product $+_1$ on $[\Sigma^2 X, Y]_*$ is abelian.

———— ✍ ————

Exercise 5 (Compatibility between the fiber sequence and the cofiber sequence).

(1) Describe the unit $\eta : X \to \Omega\Sigma X$ and the counit $\varepsilon : \Sigma\Omega X \to X$ of the Σ-Ω adjunction.

(2) Let $f : X \to Y$ be a pointed map. Show that the assignment

$$(x, \varphi) \mapsto \begin{cases} \varphi(2t) & \text{for } \ 0 \leqslant t \leqslant \frac{1}{2} \ , \\ (x, 2(1-t) & \text{for } \ \frac{1}{2} \leqslant t \leqslant 1 \ , \end{cases}$$

defines a pointed map

$$\tilde{\eta} : \mathrm{Path}(f) \to \Omega\mathrm{Cone}(f) \ .$$

(3) Describe the adjoint map

$$\tilde{\varepsilon} : \Sigma\mathrm{Path}(f) \to \mathrm{Cone}(f) \ .$$

(4) Show that the following diagram

$$\begin{array}{ccccccccccc}
 & & \Sigma\Omega\mathrm{Path}(f) & \xrightarrow{\Sigma\Omega f^1} & \Sigma\Omega X & \xrightarrow{\Sigma\Omega f} & \Sigma\Omega Y & \xrightarrow{\Sigma i(f)} & \Sigma\mathrm{Path}(f) & \xrightarrow{\Sigma f_1} & \Sigma X \\
 & & \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \tilde{\varepsilon}} & & \| \\
\Omega Y & \xrightarrow{i(f)} & \mathrm{Path}(f) & \xrightarrow{f^1} & X & \xrightarrow{f} & Y & \xrightarrow{f_1} & \mathrm{Cone}(f) & \xrightarrow{p(f)} & \Sigma X \\
\| & & \downarrow{\scriptstyle \tilde{\eta}} & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta} & & \\
\Omega Y & \xrightarrow{i(f)} & \Omega\mathrm{Cone}(f) & \xrightarrow{f^1} & \Omega\Sigma X & \xrightarrow{f} & \Omega\Sigma Y & \xrightarrow{f_1} & \Omega\Sigma\mathrm{Cone}(f) & &
\end{array}$$

is homotopy commutative.

———— ✍ ————

✉ Bruno Vallette: vallette@math.univ-paris13.fr .
☞ Web page: www.math.univ-paris13.fr/~vallette/ .