WORKSHEET 2

HOMOTOPY THEORY OF TOPOLOGICAL SPACES II

Exercise 1 (Homotopy invariance).

(1) Is the property of being a cofibration a homotopy invariant notion: if $i : A \rightarrowtail X$ is a cofibration and if $j : A \to X$ a map homotopy equivalent to i, is j a cofibration?
(2) Is the property of being a fibration a homotopy invariant notion: if $p : E \twoheadrightarrow B$ is a fibration and if $q : E \to B$ a map homotopy equivalent to p, is q a fibration?

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Exercise 2 (Fibrations and cofibrations).

(1) Show that cofibrations are stable under composition: if $i : A \rightarrowtail B$ and $j : B \rightarrowtail C$ are two cofibrations then $ji : A \rightarrowtail C$ is a cofibration.
(2) Show that cofibrations are stable under coproduct: if $i : A \rightarrowtail X$ and $j : B \rightarrowtail Y$ are two cofibrations then $i \sqcup j : A \sqcup B \rightarrowtail X \sqcup Y$ is a cofibration.
(3) Show that cofibrations are stable under pushout: if $i : A \rightarrowtail X$ is a cofibration then

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ X & \longrightarrow & X \underset{f}{\sqcup} B \end{array}$$

the map $j : B \rightarrowtail X \underset{f}{\sqcup} B$ is a cofibration.
(4) Show the map $E \twoheadrightarrow \{*\}$ is a fibration.
(5) Show that fibrations are stable under composition: if $p : C \twoheadrightarrow D$ and $q : D \twoheadrightarrow E$ are two fibrations then $qp : C \twoheadrightarrow E$ is a fibration.
(6) Show that fibrations are stable under product: if $p : D \twoheadrightarrow A$ and $q : E \twoheadrightarrow B$ are two fibrations then $p \times q : D \times E \twoheadrightarrow A \times B$ is a fibration.
(7) Show that fibrations are stable under pullback: if $p : E \twoheadrightarrow B$ is a fibration then

$$\begin{array}{ccc} X \underset{f}{\times} E & \longrightarrow & E \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & B \end{array}$$

the map $q : X \underset{f}{\times} E \twoheadrightarrow X$ is a fibration.

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Exercise 3 (Uniqueness of factorisation). We work in the category Top^A of maps under A: its objects are maps $i : A \to X$ with domain A and its morphisms between two maps $i : A \to X$ and $j : A \to Y$ are maps $f : X \to Y$, such that $fi = j$:

$$\begin{array}{ccc} & A & \\ {\scriptstyle i}\swarrow & & \searrow{\scriptstyle j} \\ X & \xrightarrow{\quad f \quad} & Y \,. \end{array}$$

A *homotopy* $H : X \times I \to Y$ *under* A between two such maps $f \sim g$ is a homotopy such that every map $H(-,t)$ lives in $\mathsf{Top}^A(i,j)$, for $t \in I$, i.e. $H(-,t)i = j$. This induces an equivalence relation called *homotopy equivalence under* A. We will admit the following (seminal) theorem.

Theorem. *Let $i : A \to X$ and $j : A \to Y$ be two cofibrations and let $f : X \to Y$ such that $fi = j$. If f is a homotopy equivalence, then f is a homotopy equivalence under A.*

(1) Show that if a map is a cofibration and a homotopy equivalence, then it is a deformation retract.
(2) Show the following uniqueness statement for the factorisation of a map into the composite of a cofibration with a homotopy equivalence: let

$$X \underset{j}{\rightarrowtail} Z \xrightarrow[q]{\sim} Y \quad \text{and} \quad X \underset{j'}{\rightarrowtail} Z' \xrightarrow[q']{\sim} Y$$

be two such factorisations, then there exists a homotopy equivalence $k : Z \xrightarrow{\sim} Z'$ such that the following diagram is commutative on the left-hand side and homotopy commutative on the right-hand side

$$\begin{array}{ccccc} & & Z & & \\ & {\scriptstyle j}\nearrow & \Big\downarrow{\scriptstyle h} & \searrow{\scriptstyle q} & \\ X & & & & Y \\ & {\scriptstyle j'}\searrow & & \nearrow{\scriptstyle q'} & \\ & & Z' \,. & & \end{array}$$

Exercise 4 (Fibrations). Let $p : E \to B$ be a fibration with B path connected.

(1) Show that p is surjective.
(2) Show that two fibers $p^{-1}(b)$ and $p^{-1}(b')$, for $b, b' \in B$, are homotopy equivalent.

Exercise 5 (Hopf fibration).

(1) Show that there exists a fiber bundle of the form $S^1 \longrightarrow S^{2n+1} \longrightarrow \mathbb{P}^n\mathbb{C}$.
(2) What can you say about the homotopy groups $\pi_n(\mathbb{P}^n\mathbb{C})$, for $n \geqslant 0$, of the complex projective spaces?
(3) Compute $\pi_2(S^2)$ and prove that $\pi_n(S^3) \cong \pi_n(S^2)$, for $n \geqslant 3$.

Exercise 6 (Real projective spaces).

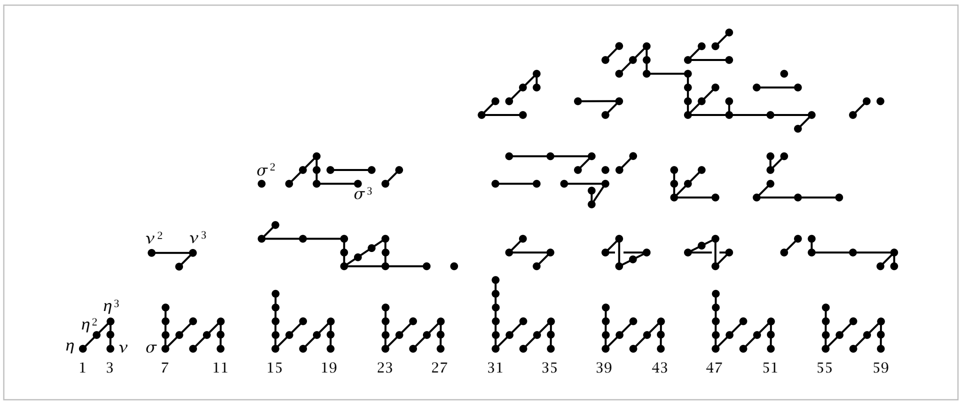
(1) What can you say about $\pi_n(\mathbb{P}^1\mathbb{R})$, for $n \geqslant 0$?
(2) Show that the embedding $\mathbb{P}^k\mathbb{R} \hookrightarrow \mathbb{P}^n\mathbb{R}$ does not admit a retraction when $0 < k < n$.
(3) Let $d \geqslant 2$. Compute the homotopy groups $\pi_n(\mathbb{P}^d\mathbb{R})$ for $0 \leqslant n \leqslant d$.
(4) What can you say about $\pi_n(\mathbb{P}^d\mathbb{R})$, for $n \geqslant d + 1$?

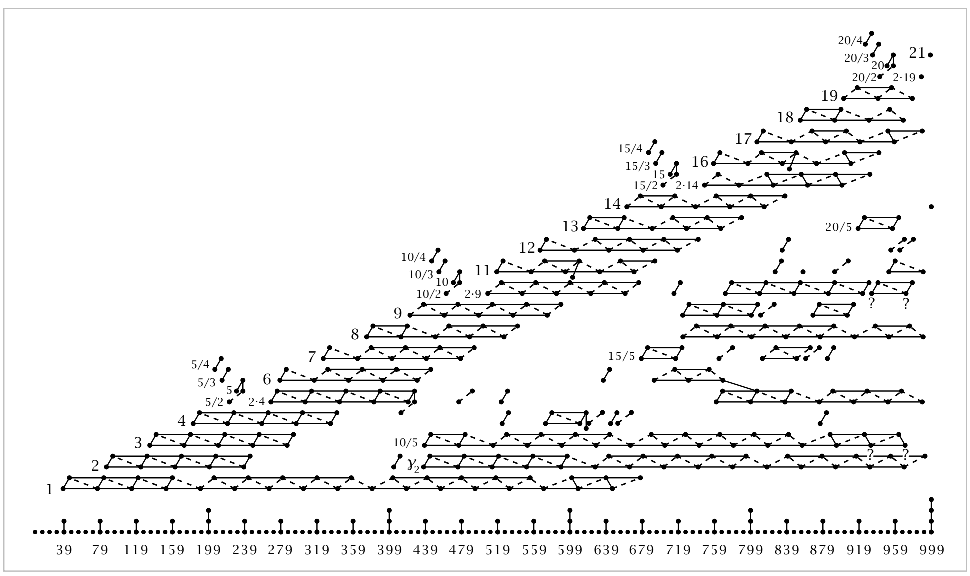
FIGURE 1. The first homotopy groups of spheres

$\pi_i(S^n)$

$n \downarrow$ \ $i \rightarrow$	1	2	3	4	5	6	7	8	9	10	11	12
1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0
2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2$
3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2$
4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{12}$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_{24}\times\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$
5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$
6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$
7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0
8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0

FIGURE 2. Examples of tables in stable homotopy theory





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