



WORKSHEET 2

HOMOTOPY THEORY OF TOPOLOGICAL SPACES II

Exercise 1 (Homotopy invariance).

- (1) Is the property of being a cofibration a homotopy invariant notion: if $i : A \rightarrow X$ is a cofibration and if $j : A \rightarrow X$ a map homotopy equivalent to i , is j a cofibration?
- (2) Is the property of being a fibration a homotopy invariant notion: if $p : E \rightarrow B$ is a fibration and if $q : E \rightarrow B$ a map homotopy equivalent to p , is q a fibration?

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Exercise 2 (Fibrations and cofibrations).

- (1) Show that cofibrations are stable under composition: if $i : A \rightarrow B$ and $j : B \rightarrow C$ are two cofibrations then $ji : A \rightarrow C$ is a cofibration.
- (2) Show that cofibrations are stable under coproduct: if $i : A \rightarrow X$ and $j : B \rightarrow Y$ are two cofibrations then $i \sqcup j : A \sqcup B \rightarrow X \sqcup Y$ is a cofibration.
- (3) Show that cofibrations are stable under pushout: if $i : A \rightarrow X$ is a cofibration then

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \downarrow i & & \downarrow j \\
 X & \xrightarrow{\quad} & X \sqcup B \\
 & & \downarrow f
 \end{array}$$

the map $j : B \rightarrow X \sqcup B$ is a cofibration.

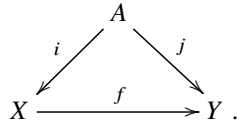
- (4) Show the map $E \rightarrow \{*\}$ is a fibration.
- (5) Show that fibrations are stable under composition: if $p : C \rightarrow D$ and $q : D \rightarrow E$ are two fibrations then $qp : C \rightarrow E$ is a fibration.
- (6) Show that fibrations are stable under product: if $p : D \rightarrow A$ and $q : E \rightarrow B$ are two fibrations then $p \times q : D \times E \rightarrow A \times B$ is a fibration.
- (7) Show that fibrations are stable under pullback: if $p : E \rightarrow B$ is a fibration then

$$\begin{array}{ccc}
 X \times E & \xrightarrow{\quad} & E \\
 \downarrow q & \lrcorner & \downarrow p \\
 X & \xrightarrow{f} & X \sqcup B \\
 & & \downarrow f
 \end{array}$$

the map $q : X \times E \rightarrow X$ is a fibration.

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Exercise 3 (Uniqueness of factorisation). We work in the category Top^A of maps under A : its objects are maps $i : A \rightarrow X$ with domain A and its morphisms between two maps $i : A \rightarrow X$ and $j : A \rightarrow Y$ are maps $f : X \rightarrow Y$, such that $fi = j$:



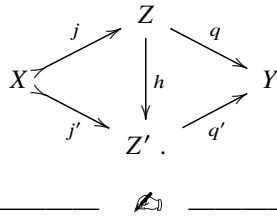
A *homotopy* $H : X \times I \rightarrow Y$ under A between two such maps $f \sim g$ is a homotopy such that every map $H(-, t)$ lives in $\text{Top}^A(i, j)$, for $t \in I$, i.e. $H(-, t)i = j$. This induces an equivalence relation called *homotopy equivalence under A* . We will admit the following (seminal) theorem.

THEOREM. *Let $i : A \rightarrow X$ and $j : A \rightarrow Y$ be two cofibrations and let $f : X \rightarrow Y$ such that $fi = j$. If f is a homotopy equivalence, then f is a homotopy equivalence under A .*

- (1) Show that if a map is a cofibration and a homotopy equivalence, then it is a deformation retract.
- (2) Show the following uniqueness statement for the factorisation of a map into the composite of a cofibration with a homotopy equivalence: let

$$X \xrightarrow{j} Z \xrightarrow{q} Y \quad \text{and} \quad X \xrightarrow{j'} Z' \xrightarrow{q'} Y$$

be two such factorisations, then there exists a homotopy equivalence $k : Z \xrightarrow{\sim} Z'$ such that the following is commutative on the left-hand side and homotopy commutative on the right-hand side



Exercise 4 (Fibrations). Let $p : E \rightarrow B$ be a fibration with B path connected.

- (1) Show that p is surjective.
- (2) Show that two fibers $p^{-1}(b)$ and $p^{-1}(b')$, for $b, b' \in B$, are homotopy equivalent.

Exercise 5 (Hopf fibration).

- (1) Show that there exists a fiber bundle of the form $S^1 \rightarrow S^{2n+1} \rightarrow \mathbb{P}^n \mathbb{C}$.
- (2) What can you say about the homotopy groups $\pi_n(\mathbb{P}^n \mathbb{C})$, for $n \geq 0$, of the complex projective spaces?
- (3) Prove that $\pi_2(S^2)$ and that $\pi_n(S^3) \cong \pi_n(S^2)$, for $n \geq 3$.

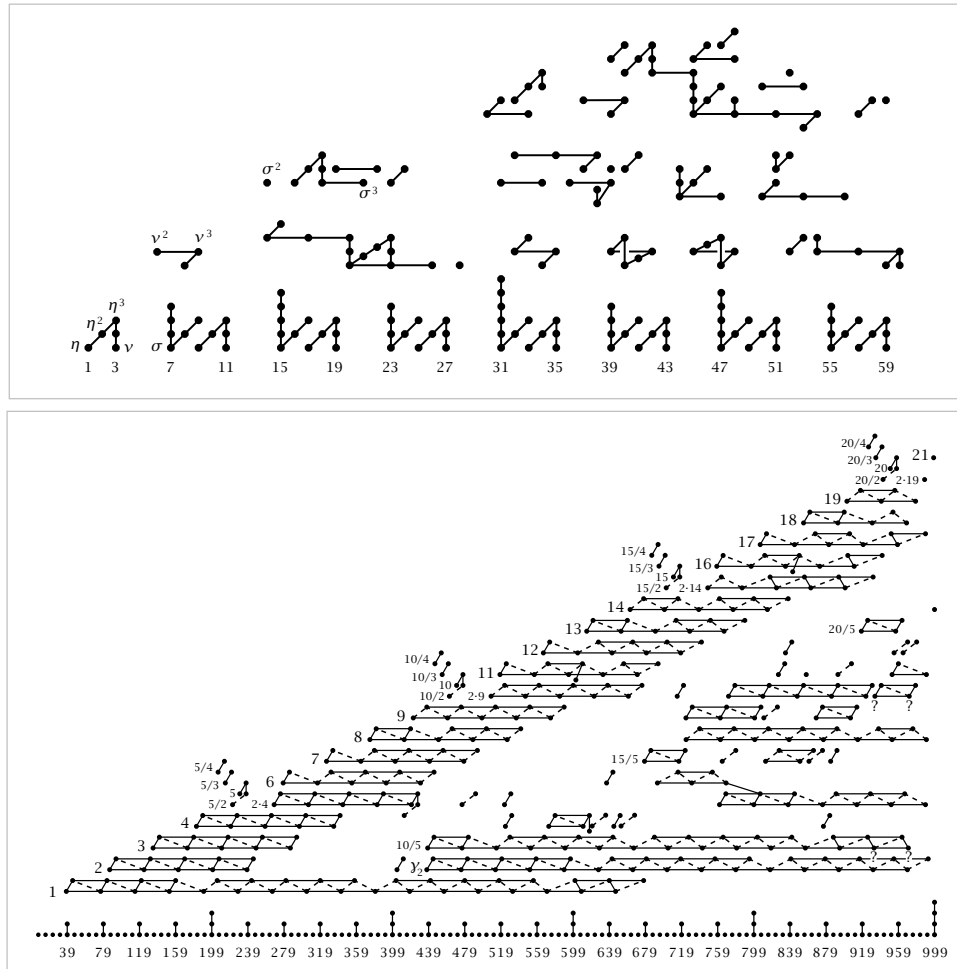
Exercise 6 (Real projective spaces).

- (1) What can you say about $\pi_n(\mathbb{P}^1 \mathbb{R})$, for $n \geq 0$?
- (2) Show that the embedding $\mathbb{P}^k \mathbb{R} \hookrightarrow \mathbb{P}^n \mathbb{R}$ does not admit a retraction when $0 < k < n$.
- (3) Let $d \geq 2$. Compute the homotopy groups $\pi_n(\mathbb{P}^d \mathbb{R})$ for $0 \leq n \leq d$.
- (4) What can you say about $\pi_n(\mathbb{P}^d \mathbb{R})$, for $n \geq d + 1$?

FIGURE 1. The first homotopy groups of spheres

		$\pi_i(S^n)$											
		$i \rightarrow$											
		1	2	3	4	5	6	7	8	9	10	11	12
$n \downarrow$	1	\mathbb{Z}	0	0	0	0	0	0	0	0	0	0	0
	2	0	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{12}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_3	\mathbb{Z}_{15}	\mathbb{Z}_2	$\mathbb{Z}_2 \times \mathbb{Z}_2$
	3	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{12}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_3	\mathbb{Z}_{15}	\mathbb{Z}_2	$\mathbb{Z}_2 \times \mathbb{Z}_2$
	4	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	$\mathbb{Z} \times \mathbb{Z}_{12}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_{24} \times \mathbb{Z}_3$	\mathbb{Z}_{15}	\mathbb{Z}_2
	5	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{30}
	6	0	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	0	\mathbb{Z}	\mathbb{Z}_2
	7	0	0	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	0	0
	8	0	0	0	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	0

FIGURE 2. Examples of tables in stable homotopy theory



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