

WORKSHEET 3

CW COMPLEXES

Exercise 1 (Examples of CW complex).

- (1) For $n \in \mathbb{N}$, is \mathbb{R}^n a CW complex? It is a finite CW complex?
- (2) Show that the product $X \times Y$ of two finite CW complexes is again a CW complex.
- (3) Let (X, A) be a relative CW complex. Show that X/A is also a CW complex.
- (4) Show that the *n*-times iterated (unpointed) suspension $\Sigma^n X$ of a CW complex is again a CW complex, where

$$\Sigma X \coloneqq \frac{X \times I}{X \times \partial I}$$

(5) Let $f: X \to Y$ be a cellular map between CW complexes. Show that the factorisation of f

$$X \longrightarrow \operatorname{Cyl}(f) \xrightarrow{\sim} Y$$

is the composite of two cellular maps.

Exercise 2 (C in CW complex).

(1) Show that every CW complex X is in set-theoretical bijection with

$$X \cong \bigsqcup_{n \in \mathbb{N}, j \in J_n} \mathring{D^n} ,$$

where $D^n = D^n \setminus \partial D^n$ is the interior of D^n , with $D^0 = D^0 = \{*\}$ by convention.

(2) Show that any compact subspace of a CW complex intersects only a finite number of cells.

Exercise 3 (Infinite CW complexes).

Show that the *n*-dimension sphere Sⁿ, for any n ∈ N, admits a CW complex structure with two k-dimensional cells for any 0 ≤ k ≤ n.
By passing to the colimit

$$S^0 \subset S^1 \subset S^2 \subset \cdots \subset S^\infty = \bigcup_{n \ge 0} S^n$$
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this defines a new CW complex called the *infinite dimensional sphere* and denoted by S^{∞} .

(2) Let X be a CW complex which is the union (colimit) of an increasing sequence of CW subcomplexes

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$$X_1 \subset X_2 \subset X_3 \subset \cdots \subset X = \bigcup_{n \ge 1} X_n$$

such that each inclusion $X_n \rightarrow X_{n+1}$ is nullhomotopic. Show that X is contractible.

(3) Show that S^{∞} is contractible.

Exercise 4 (Extension Lemma).

(1) Let (X, A) be a relative CW complex and a map $f : A \to Y$ with Y path connected. Show that f extends to a map $F : X \to Y$



if $\pi_{n-1}(Y) = 0$ for any *n* such that $X \setminus A$ has cells of dimension *n*. (2) Let (X, A) be a CW pair such that A is contractible. Show that X retracts to A.

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Exercise 5 (Cellular approximation).

- (1) Show that every map f: X → Y between CW complexes is homotopic to a cellular map. HINT: We will assume the following result, see [Hatcher, Lemma 4.10]. Let f: (X, A) → (Z, Y) be a map of CW-complexes such that the dimension of A is less than n, X \A is made up of only one n-dimensional cell, the dimension of Y is less than m, Z \Y is made up of only one cell of dimension N ≥ m. The map f is homotopic relative to A to a map g: (X, A) → (Z, Y) such that g(X \A) misses a point of Z \Y.
- (2) Prove that $\pi_k(S^n) = 0$ for k < n.
- (3) Show that every map $f: (X, A) \to (Y, B)$ between CW pairs is homotopic to a cellular map.
- (4) Let (X, A) be a CW pair such that $X \setminus A$ has only cells in dimension greater than n. Show that (X, A) is *n*-connected, that is $\pi_k(X, A) = 0$, for $k \leq n$.
- (5) Let X be a CW complex. Show that the CW pair $(X, X^{(n-1)})$ is *n*-connected and that the inclusion $X^{(n-1)} \rightarrow X$ induces isomorphisms

$$\pi_k(X^{(n-1)}) \xrightarrow{\cong} \pi_k(X),$$

for k < n, and a surjection $\pi_n(X^{(n-1)}) \twoheadrightarrow \pi_n(X)$.

Exercise 6 (Postnikov tower).

(1) Let X be a CW complex. For $n \ge 1$, show that there exists a CW complex X_n for which $X \subset X_n$ is a CW subcomplex and which satisfies

$$\begin{cases} \pi_k(X_n) \cong \pi_k(X), & \text{for } k \le n, \\ \pi_k(X_n) \cong 0, & \text{for } k > n. \end{cases}$$

(2) Show that there exists maps $X_{n+1} \rightarrow X_n$ which make the following diagram commutative



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