

WORKSHEET 4

SIMPLICIAL SETS

Exercise 1 (Simplicial approximation).

- (1) Show that every CW complex X is homotopy equivalent to the geometric realization simplicial complex \mathfrak{X} .
- (2) Show that every topological space X admits a weakly homotopy equivalent simplicial complex \mathfrak{X} .

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Exercise 2 (From Δ -complexes to simplicial sets).

Let $\Phi : \overline{\Delta} \to \Delta$ be the embedding of the reduced simplicial category in the simplicial category.

- Describe the pullback functor (Φ^{op})* : sSet → ΔC× from the category of simplicial sets to the category of Δ-complexes, defined by (Φ^{op})* (𝔅) := 𝔅 Φ^{op}, for any 𝔅 : Δ^{op} → Set.
- (2) Is the geometric realisation $|\mathfrak{X}|$ of a simplicial set \mathfrak{X} in continuous bijection with the geometric realisation $|(\Phi^{\text{op}})^*(\mathfrak{X})|_{\Delta}$ of its associated Δ -complex.
- (3) Show that the functor $(\Phi^{op})^*$ admits a left adjoint functor L given by

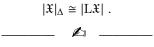
 $(L\mathfrak{X})_n = \{(\varphi, x) | \varphi : [n] \twoheadrightarrow [m] \text{ non-decreasing, } x \in X_m\},\$

where the faces are given by $d_i(\varphi, x) \coloneqq (\varphi \delta_i, x)$, if $\varphi \delta_i$ is surjective, otherwise $\varphi \delta_i$ can be written in a unique way $\delta_j \psi$ with ψ surjective and then $d_i(\varphi, x) = (\psi, d_j(x))$. The degeneracies are given by $s_i(\varphi, x) \coloneqq (\varphi \sigma_i, x)$.

(4) Describe the image under L of the following ∆-complex X, which is a model for the circle with two cells:

 $X_0 = \{0\}, \quad X_1 = \{01\}, \quad X_2 = X_3 = \dots = \emptyset \quad \text{with} \quad d_0(01) = d_1(01) = 0.$

(5) Let \mathfrak{X} be a Δ -complex. Show that its geometrical realisation is in continuous bijection with the geometrical realisation of the associated simplicial set, i.e.



Exercise 3 (*n*-skeleton). We consider the full sub-category Δ_n of the simplex category Δ made up of only the objects $[0], \ldots, [n]$; we denote this embedding of categories by $\Upsilon : \Delta_n \to \Delta$. Presheaves $\Delta_n^{\text{op}} \to \text{Set}$ over the category Δ_n are called *n*-truncated simplicial sets; we denote their category by Δ_n Set.

- (1) Describe the pullback functor $(\Upsilon^{\text{op}})^*$: sSet $\rightarrow \Delta_n$ Set from the category of simplicial sets to the category of *n*-truncated simplicial sets.
- (2) Show that the functor $(\Upsilon^{\text{op}})^*$ admits a left adjoint functor L given by $(L\mathfrak{X})_m = X_m$, for $m \leq n$ and

 $(\mathrm{L}\mathfrak{X})_m = \left\{(\varphi, x) \, | \, \varphi: [m] \twoheadrightarrow [n] \text{ non-decreasing}, x \in X_n \right\} \;, \quad \text{for} \quad m > n \;.$

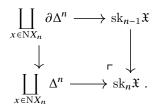
- (3) Show that $L(\Upsilon^{op})^*(\mathfrak{X}) \cong \operatorname{sk}_n \mathfrak{X}$ for every simplicial set \mathfrak{X} .
- (4) Describe the unity of this adjunction.
- (5) Using the identification of Question (3), show that the counit of adjunction amounts to the following inclusion of simplicial sets

$$L\Upsilon^*(\mathfrak{X}) \cong \mathrm{sk}_n\mathfrak{X} \to \mathfrak{X}$$
.

(6) Conclude that the restriction of this adjunction to the full sub-category of simplicial sets of dimension less than n is an equivalence of categories.

Exercise 4 ((Co)limits).

- (1) Show that the category sSet of simplicial sets admit all limits and all colimits. HINT: Consider first the point-wise set-theoretical limits and colimits.
- (2) Describe the initial simplicial set, the terminal simplicial set, the coproduct of simplicial sets, and the product of simplicial sets.
- (3) Show that the *n*-skeleton of a simplicial set can be written as a pushout of the following type:



(4) Show that the boundary ∂∆ⁿ of the standard *n*-simplex can be written as a coequalizer of the following type:

$$\coprod_{0 \leqslant i < j \leqslant n} \Delta^{n-2} \longrightarrow \coprod_{0 \leqslant l \leqslant n} \Delta^{n-1} \longrightarrow \partial \Delta^n \ .$$

(5) Show that the *k*-th horn Λ_k^n can be written as a coequalizer of the following type:

$$\coprod_{0 \leqslant i < j \leqslant n} \Delta^{n-2} \xrightarrow{\longrightarrow} \coprod_{0 \leqslant l \leqslant n \atop l \neq k} \Delta^{n-1} \longrightarrow \Lambda^n_k \ .$$

(6) Why the above question shows that the geometrical k-th horn $|\Lambda_k^n|$ can be written as a coequalizer of the following type:

$$\coprod_{0 \leqslant i < j \leqslant n} |\Delta^{n-2}| \longrightarrow \coprod_{0 \leqslant l \leqslant n \atop l \neq k} |\Delta^{n-1}| \longrightarrow |\Lambda^n_k| ?$$

Exercise 5 (Prismatic decomposition).

The product $\mathfrak{X} \times \mathfrak{Y}$ of two simplicial sets \mathfrak{X} et \mathfrak{Y} is defined by

$$(\mathfrak{X} \times \mathfrak{Y})_n := X_n \times Y_n$$

endowed by the faces $d_i^{\mathfrak{X}} \times d_i^{\mathfrak{Y}}$ and degeneracies $s_i^{\mathfrak{X}} \times s_i^{\mathfrak{Y}}$.

- (1) Describe the simplicial set $\Delta^1 \times \Delta^1$, notably its non-degenerate simplicies, and its geometric realisation $|\Delta^1 \times \Delta^1|$.
- (2) Describe the non-degenerate simplicies of the simplicial set $\Delta^p \times \Delta^q$ and insist on the case of the non-degenerate p + q simplicies. Show that $|\Delta^p \times \Delta^q| \cong |\Delta^p| \times |\Delta^q|$.
- (3) Describe the simplicial set $\Delta^p \times \Delta^1$ and draw the associate triangulation of $|\Delta^2 \times \Delta^1|$.
- Bruno Vallette: vallette@math.univ-paris13.fr.
- Web page: www.math.univ-paris13.fr/~vallette/.