

WORKSHEET 4

SIMPLICIAL SETS

Exercise 1 (Simplicial approximation).

- (1) Show that every CW complex X is homotopy equivalent to the geometric realization simplicial complex \mathfrak{X} .
- (2) Show that every topological space X admits a weakly homotopy equivalent simplicial complex \mathfrak{X} .



Exercise 2 (From Δ -complexes to simplicial sets).

Let $\Phi : \bar{\Delta} \hookrightarrow \Delta$ be the embedding of the reduced simplicial category in the simplicial category.

- (1) Describe the pullback functor $(\Phi^{\text{op}})^* : \text{sSet} \rightarrow \Delta\text{Cx}$ from the category of simplicial sets to the category of Δ -complexes, defined by $(\Phi^{\text{op}})^*(\mathfrak{X}) := \mathfrak{X} \circ \Phi^{\text{op}}$, for any $\mathfrak{X} : \Delta^{\text{op}} \rightarrow \text{Set}$.
- (2) Is the geometric realisation $|\mathfrak{X}|$ of a simplicial set \mathfrak{X} in continuous bijection with the geometric realisation $|(\Phi^{\text{op}})^*(\mathfrak{X})|_{\Delta}$ of its associated Δ -complex.
- (3) Show that the functor $(\Phi^{\text{op}})^*$ admits a left adjoint functor L given by

$$(L\mathfrak{X})_n = \{(\varphi, x) \mid \varphi : [n] \twoheadrightarrow [m] \text{ non-decreasing}, x \in X_m\},$$

where the faces are given by $d_i(\varphi, x) := (\varphi\delta_i, x)$, if $\varphi\delta_i$ is surjective, otherwise $\varphi\delta_i$ can be written in a unique way $\delta_j\psi$ with ψ surjective and then $d_i(\varphi, x) = (\psi, d_j(x))$. The degeneracies are given by $s_i(\varphi, x) := (\varphi\sigma_i, x)$.

- (4) Describe the image under L of the following Δ -complex \mathfrak{X} , which is a model for the circle with two cells:

$$X_0 = \{0\}, \quad X_1 = \{01\}, \quad X_2 = X_3 = \dots = \emptyset \quad \text{with} \quad d_0(01) = d_1(01) = 0.$$

- (5) Let \mathfrak{X} be a Δ -complex. Show that its geometrical realisation is in continuous bijection with the geometrical realisation of the associated simplicial set, i.e.

$$|\mathfrak{X}|_{\Delta} \cong |L\mathfrak{X}|.$$



Exercise 3 (n -skeleton). We consider the full sub-category Δ_n of the simplex category Δ made up of only the objects $[0], \dots, [n]$; we denote this embedding of categories by $\Upsilon : \Delta_n \hookrightarrow \Delta$. Presheaves $\Delta_n^{\text{op}} \rightarrow \text{Set}$ over the category Δ_n are called *n -truncated simplicial sets*; we denote their category by $\Delta_n\text{Set}$.

- (1) Describe the pullback functor $(\Upsilon^{\text{op}})^* : \text{sSet} \rightarrow \Delta_n\text{Set}$ from the category of simplicial sets to the category of n -truncated simplicial sets.
- (2) Show that the functor $(\Upsilon^{\text{op}})^*$ admits a left adjoint functor L given by $(L\mathfrak{X})_m = X_m$, for $m \leq n$ and

$$(L\mathfrak{X})_m = \{(\varphi, x) \mid \varphi : [m] \twoheadrightarrow [n] \text{ non-decreasing}, x \in X_n\}, \quad \text{for } m > n.$$

- (3) Show that $L(\Upsilon^{\text{op}})^*(\mathfrak{X}) \cong \text{sk}_n \mathfrak{X}$ for every simplicial set \mathfrak{X} .
- (4) Describe the unity of this adjunction.
- (5) Using the identification of Question (3), show that the counit of adjunction amounts to the following inclusion of simplicial sets

$$L\Upsilon^*(\mathfrak{X}) \cong \text{sk}_n \mathfrak{X} \hookrightarrow \mathfrak{X}.$$

- (6) Conclude that the restriction of this adjunction to the full sub-category of simplicial sets of dimension less than n is an equivalence of categories.



Exercise 4 ((Co)limits).

- (1) Show that the category \mathbf{sSet} of simplicial sets admit all limits and all colimits.
HINT: Consider first the point-wise set-theoretical limits and colimits.
- (2) Describe the initial simplicial set, the terminal simplicial set, the coproduct of simplicial sets, and the product of simplicial sets.
- (3) Show that the n -skeleton of a simplicial set can be written as a pushout of the following type:

$$\begin{array}{ccc} \coprod_{x \in NX_n} \partial\Delta^n & \longrightarrow & \mathrm{sk}_{n-1}\mathfrak{X} \\ \downarrow & & \downarrow \\ \coprod_{x \in NX_n} \Delta^n & \longrightarrow & \mathrm{sk}_n\mathfrak{X} . \end{array}$$

- (4) Show that the boundary $\partial\Delta^n$ of the standard n -simplex can be written as a coequalizer of the following type:

$$\coprod_{0 \leq i < j \leq n} \Delta^{n-2} \rightrightarrows \coprod_{0 \leq l \leq n} \Delta^{n-1} \longrightarrow \partial\Delta^n .$$

- (5) Show that the k -th horn Λ_k^n can be written as a coequalizer of the following type:

$$\coprod_{0 \leq i < j \leq n} \Delta^{n-2} \rightrightarrows \coprod_{\substack{0 \leq l \leq n \\ l \neq k}} \Delta^{n-1} \longrightarrow \Lambda_k^n .$$

- (6) Why the above question shows that the geometrical k -th horn $|\Lambda_k^n|$ can be written as a coequalizer of the following type:

$$\coprod_{0 \leq i < j \leq n} |\Delta^{n-2}| \rightrightarrows \coprod_{\substack{0 \leq l \leq n \\ l \neq k}} |\Delta^{n-1}| \longrightarrow |\Lambda_k^n| ?$$



Exercise 5 (Prismatic decomposition).

The product $\mathfrak{X} \times \mathfrak{Y}$ of two simplicial sets \mathfrak{X} et \mathfrak{Y} is defined by

$$(\mathfrak{X} \times \mathfrak{Y})_n := X_n \times Y_n$$

endowed by the faces $d_i^{\mathfrak{X}} \times d_i^{\mathfrak{Y}}$ and degeneracies $s_i^{\mathfrak{X}} \times s_i^{\mathfrak{Y}}$.

- (1) Describe the simplicial set $\Delta^1 \times \Delta^1$, notably its non-degenerate simplices, and its geometric realisation $|\Delta^1 \times \Delta^1|$.
- (2) Describe the non-degenerate simplices of the simplicial set $\Delta^p \times \Delta^q$ and insist on the case of the non-degenerate $p + q$ simplices. Show that $|\Delta^p \times \Delta^q| \cong |\Delta^p| \times |\Delta^q|$.
- (3) Describe the simplicial set $\Delta^p \times \Delta^1$ and draw the associate triangulation of $|\Delta^2 \times \Delta^1|$.

