WORKSHEET 4

SIMPLICIAL SETS

Exercise 1 (Simplicial approximation).

(1) Show that every CW complex X admits a homotopy equivalent simplicial complex $\mathfrak{X}$.
(2) Show that every topological space X admits a weakly homotopy equivalent simplicial complex $\mathfrak{X}$.

Exercise 2 (From Δ-complexes to simplicial sets).
Let $\Phi : \bar{\Delta} \rightarrowtail \Delta$ be the embedding of the reduced simplicial category in the simplicial category.

(1) Describe the pullback functor $(\Phi^{\mathrm{op}})^* : \mathsf{sSet} \to \Delta\mathsf{Cx}$ from the category of simplicial sets to the category of Δ-complexes, defined by $(\Phi^{\mathrm{op}})^*(\mathfrak{X}) := \mathfrak{X} \circ \Phi^{\mathrm{op}}$, for any $\mathfrak{X} : \Delta^{\mathrm{op}} \to \mathsf{Set}$.
(2) Is the geometric realisation $|\mathfrak{X}|$ of a simplicial set $\mathfrak{X}$ in continuous bijection with the geometric realisation $|(\Phi^{\mathrm{op}})^*(\mathfrak{X})|_\Delta$ of its associated Δ-complex.
(3) Show that the functor $(\Phi^{\mathrm{op}})^*$ admits a left adjoint functor L given by

$$(\mathrm{L}\mathfrak{X})_n = \left\{(\varphi, x) \,|\, \varphi : [n] \twoheadrightarrow [m] \text{ non-decreasing}, x \in X_m\right\},$$

where the faces are given by $d_i(\varphi, x) := (\varphi\delta_i, x)$, if $\varphi\delta_i$ is surjective, otherwise $\varphi\delta_i$ can be written in a unique way $\delta_j\psi$ with ψ surjective and then $d_i(\varphi, x) = (\psi, d_j(x))$. The degeneracies are given by $s_i(\varphi, x) := (\varphi\sigma_i, x)$.
(4) Describe the image under L of the following Δ-complex $\mathfrak{X}$, which is a model for the circle with two cells:

$$X_0 = \{0\}, \quad X_1 = \{01\}, \quad X_2 = X_3 = \cdots = \emptyset \qquad \text{with} \qquad d_0(01) = d_1(01) = 0 .$$

(5) Let $\mathfrak{X}$ be a Δ-complex. Show that its geometrical realisation is in continuous bijection with the geometrical realisation of the associated simplicial set, i.e.

$$|\mathfrak{X}|_\Delta \cong |\mathrm{L}\mathfrak{X}| .$$

Exercise 3 (n-skeleton). We consider the full sub-category Δ_n of the simplex category Δ made up of only the objects $[0], \ldots, [n]$; we denote this embedding of categories by $\Upsilon : \Delta_n \rightarrowtail \Delta$. Presheaves $\Delta_n^{\mathrm{op}} \to \mathsf{Set}$ over the category Δ_n are called *n-truncated simplicial sets*; we denote their category by $\Delta_n\mathsf{Set}$.

(1) Describe the pullback functor $\Upsilon^* : \mathsf{sSet} \to \Delta_n\mathsf{Set}$ from the category of simplicial sets to the category of n-truncated simplicial sets.
(2) Show that the functor Υ^* admits a left adjoint functor L given by $(\mathrm{L}\mathfrak{X})_m = X_m$, for $m \leqslant n$ and

$$(\mathrm{L}\mathfrak{X})_m = \left\{(\varphi, x) \,|\, \varphi : [m] \twoheadrightarrow [n] \text{ non-decreasing}, x \in X_n\right\}, \quad \text{for} \quad m > n .$$

(3) Show that $\mathrm{L}\Upsilon^*(\mathfrak{X}) \cong \mathrm{sk}_n\mathfrak{X}$ for every simplicial set $\mathfrak{X}$.
HINT: Use the Eilenberg–Zilber lemma.
(4) Describe the unity of this adjunction.

(5) Using the identification of Question (3), show that the counit of adjunction amounts to the following inclusion of simplicial sets

$$\mathrm{L}\Upsilon^*(\mathfrak{X}) \cong \mathrm{sk}_n\mathfrak{X} \rightarrowtail \mathfrak{X} \ .$$

(6) Conclude that the restriction of this adjunction to the full sub-category of simplicial sets of dimension less than n is an equivalence of categories.

Exercise 4 ((Co)limits).

(1) Show that the category sSet of simplicial sets admit all limits and colimits.
HINT: Consider first the point-wise set-theoretical limits and colimits.

(2) Describe the initial simplicial set, the terminal simplicial set, the coproduct of simplicial sets, and the product of simplicial sets.

(3) Show that the n-skeleton of a simplicial set can be written as a pushout of the following type:

$$\begin{tikzcd} \coprod\limits_{x\in \mathrm{N}X_n} \partial\Delta^n \arrow[r] \arrow[d] & \mathrm{sk}_{n-1}\mathfrak{X} \arrow[d] \\ \coprod\limits_{x\in \mathrm{N}X_n} \Delta^n \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathrm{sk}_n\mathfrak{X} \ . \end{tikzcd}$$

(4) Show that the boundary $\partial\Delta^n$ of the standard n-simplex can be written as a coequalizer of the following type:

$$\coprod_{0\leqslant i<j\leqslant n} \Delta^{n-2} \rightrightarrows \coprod_{0\leqslant l\leqslant n} \Delta^{n-1} \longrightarrow \partial\Delta^n \ .$$

(5) Show that the k-th horn Λ^n_k can be written as a coequalizer of the following type:

$$\coprod_{0\leqslant i<j\leqslant n} \Delta^{n-2} \rightrightarrows \coprod_{\substack{0\leqslant k\leqslant n \\ l\neq k}} \Delta^{n-1} \longrightarrow \Lambda^n_k \ .$$

(6) Why the above question shows that the geometrical k-th horn $|\Lambda^n_k|$ can be written as a coequalizer of the following type:

$$\coprod_{0\leqslant i<j\leqslant n} |\Delta^{n-2}| \rightrightarrows \coprod_{\substack{0\leqslant k\leqslant n \\ l\neq k}} |\Delta^{n-1}| \longrightarrow |\Lambda^n_k| \ .$$

Exercise 5 (Prismatic decomposition).

(1) Describe the simplicial set $\Delta^1 \times \Delta^1$, notably its non-degenerate simplicies, and its geometric realisation $|\Delta^1 \times \Delta^1|$.

(2) Assuming that $|\Delta^p \times \Delta^q| \cong |\Delta^p| \times |\Delta^q|$, describe the non-degenerate simplicies of the simplicial set $\Delta^p \times \Delta^q$.

(3) Describe the simplicial set $\Delta^p \times \Delta^1$ and draw the associate triangulation of $|\Delta^2 \times \Delta^1|$.

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