Worksheet 5

HOMOTOPY THEORY OF SIMPLICIAL SETS

Exercise 1 (Sets vs simplicial sets).
We consider the functor $\mathrm{cst} : \mathsf{Set} \to \mathsf{sSet}$ which assigns the constant simplicial set

$$X_n \coloneqq X\ , \quad d_i = \mathrm{id}\ , \quad \text{and} \quad s_i = \mathrm{id}$$

to any set X.

(1) Show that this functor is full, faithful, and representable.
We denote by $\mathrm{C}^\bullet$ the cosimplicial set which represents this constant functor.
(2) Show that the constant functor cst admits a left adjoint functor L and describe it.
(3) Show that the constant functor cst admits a right adjoint functor R and describe it.
(4) Show that the category Set of sets is equivalent to the category sSet_0 of simplicial sets of dimension 0.

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Exercise 2 (Dold–Kan correspondence).
The *Moore complex* $\mathrm{C}_*\mathfrak{X}$ of a simplicial set $\mathfrak{X}$ admits for chains in degree n the free $\mathbb{Z}$-module on the set of n-simplicies, i.e. $\mathrm{C}_n\mathfrak{X} \coloneqq \mathbb{Z}\, X_n$, and for differential the alternated sum of the faces:

$$\mathrm{d} \coloneqq \sum_{i=0}^{n} (-1)^i \mathrm{d}_i \ : \ \mathbb{Z}\, X_n \to \mathbb{Z}\, X_{n-1}\ .$$

(1) Show that $\mathrm{d}^2 = 0$.
(2) Show that the graded module $\mathrm{D}_*\mathfrak{X}$ whose chains in degree n are made up of the free $\mathbb{Z}$-module on the set of degenerate n-simplicies $\mathrm{D}_n\mathfrak{X} \coloneqq \mathbb{Z}\, X_n \backslash \mathrm{N}X_n$ is a chain sub-complex of the Moore complex, i.e. $\mathrm{D}_*\mathfrak{X} \subset \mathrm{C}_*\mathfrak{X}$.

The *normalised Moore complex* $\mathrm{N}_*\mathfrak{X}$ of a simplicial set $\mathfrak{X}$ is the quotient chain complex of $\mathrm{C}_*\mathfrak{X}$ by $\mathrm{D}_*\mathfrak{X}$, i.e.

$$\mathrm{N}_*\mathfrak{X} := (\mathrm{C}_*\mathfrak{X}/\mathrm{D}_*\mathfrak{X}, \mathrm{d})\ .$$

(3) Show that both the Moore chain complex C_* and the normalised Moore chain complex N_* form a functor.
(4) Show that the normalised Moore chain complex N_* admit a right adjoint functor and describe it.

Remark. One can prove that the canonical projection $\mathrm{C}_*\mathfrak{X} \twoheadrightarrow \mathrm{N}_*\mathfrak{X}$ is a homotopy equivalence. The normalised Moore functor N_* is part of an equivalence of categories between the category of simplicial abelian groups and non-negatively graded chain complexes over $\mathbb{Z}$.

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Exercise 3 (Kan complexes).
A *Kan complex* is a simplicial set $\mathfrak{X}$ which satisfies the following extension property along every horn: for every morphism of simplicial sets $f : \Lambda^n_k \to \mathfrak{X}$, there exists a morphism of simplicial sets

$F : \Delta^n \to \mathfrak{X}$ such that $f = iF$, where $i : \Lambda^n_k \hookrightarrow \Delta^n$ is the canonical inclusion

(*)
$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & \mathfrak{X} \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd} \qquad \text{for } n \geqslant 2 \text{ and } 0 \leqslant k \leqslant n \ .$$

(1) Prove that a simplicial set $\mathfrak{X}$ is a Kan complex if and only if, for any $n \geqslant 2$, $0 \leqslant k \leqslant n$, and any collection $x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n \in X_{n-1}$ such that $d_{j-1}(x_i) = d_i(x_j)$, for $i < j$, $i, j \neq k$, there exists a $x \in X_n$ such that $d_i(x) = x_i$, for any $i \neq k$.
(2) Prove that the singular chains $\mathrm{Sing}(X)$ of any topological space is a Kan complex.
(3) Show that the nerve of a group, viewed as a one-point category, is a Kan complex.
(4) Do Kan complexes form the essential image of the singular chain functor?
(5) Show that the underlying simplicial set of a simplicial group, i.e. after forgetting the group structures, forms a Kan complex.
(6) Show that the standard n-simplicies Δ^n are not Kan complexes for $n \geq 1$.

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Exercise 4 (Kan fibration). A *Kan fibration* is a morphism of simplicial sets $p : \mathfrak{X} \twoheadrightarrow \mathfrak{Y}$ which satisfies the following extension property: for every diagram as below, there exists a diagonal map $\Delta^n \to \mathfrak{X}$ which factors it

(**)
$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & \mathfrak{X} \arrow[d, two heads] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "\exists"'] & \mathfrak{Y} \end{tikzcd} \qquad \text{for } n \geqslant 2 \text{ and } 0 \leqslant k \leqslant n \ .$$

(1) Give a characterisation of a Kan complex in terms of Kan fibrations.
(2) Show that a continuous map $f : X \to Y$ is a Serre fibration if and only if the induced morphism of simplicial set $\mathrm{Sing}(f) : \mathrm{Sing}(X) \to \mathrm{Sing}(Y)$ is a Kan fibration.
(3) Give a combinatorial characterisation of a Kan fibration similar to that of a Kan complex given in Question (1) of Exercise 3.
(4) $p : \mathfrak{X} \twoheadrightarrow \mathfrak{Y}$ be a Kan fibration and let $y \in Y_0$ be a vertex of $\mathfrak{Y}$. We denote by $*$ the simplicial subset of $\mathfrak{Y}$ generated by y_0, that is its smallest simplicial subset containing y_0. Show that (the *fiber*) $p^{-1}(*)$ is simplicial subset of $\mathfrak{X}$ and a Kan complex.

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Exercise 5 (Mapping simplicial sets).
Let $i : \mathfrak{K} \hookrightarrow \mathfrak{L}$ be a monomorphism of simplicial sets and let $p : \mathfrak{X} \twoheadrightarrow \mathfrak{Y}$ be a Kan fibration. They induce the following commutative diagram

$$\begin{tikzcd} \mathfrak{Hom}(\mathfrak{L}, \mathfrak{X}) \arrow[r, "p_*"] \arrow[d, "i^*"] & \mathfrak{Hom}(\mathfrak{L}, \mathfrak{Y}) \arrow[d, "i^*"] \\ \mathfrak{Hom}(\mathfrak{K}, \mathfrak{X}) \arrow[r, "p_*"] & \mathfrak{Hom}(\mathfrak{K}, \mathfrak{Y}) \ . \end{tikzcd}$$

(4) Show that the induced map

$$\mathfrak{Hom}(\mathfrak{L}, \mathfrak{X}) \xrightarrow{(i^*, p_*)} \mathfrak{Hom}(\mathfrak{K}, \mathfrak{X}) \times_{\mathfrak{Hom}(\mathfrak{K}, \mathfrak{Y})} \mathfrak{Hom}(\mathfrak{L}, \mathfrak{Y})$$

is a Kan fibration.

HINT. Use the fact that p satisfies the following right lifting property:

$$\begin{tikzcd}
(\Lambda^n_k \times \mathfrak{L}) \cup_{(\Lambda^n_k \times \mathfrak{K})} (\Delta^n \times \mathfrak{K}) \arrow[r] \arrow[d, hook] & \mathfrak{X} \arrow[d, "p"] \\
\Delta^n \times \mathfrak{K} \arrow[r] \arrow[ur, dashed, "\exists"] & \mathfrak{Y} .
\end{tikzcd}$$

(5) Show that the pushforward $p_* : \mathfrak{Hom}(\mathfrak{K}, \mathfrak{X}) \twoheadrightarrow \mathfrak{Hom}(\mathfrak{K}, \mathfrak{Y})$ is a Kan fibration.
(6) Show that the pullback $i^* : \mathfrak{Hom}(\mathfrak{L}, \mathfrak{X}) \twoheadrightarrow \mathfrak{Hom}(\mathfrak{K}, \mathfrak{X})$ is a Kan fibration when $\mathfrak{X}$ is a Kan complex.

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Exercise 6 (Topological homotopy vs simplicial homotopy).

(1) Show that the geometric realisation functor $|\ | : \Delta\mathsf{Ens} \to \mathsf{Top}$ sends simplicial homotopies to a topological homotopies.
(2) Show that the singular chain functor $\mathrm{Sing} : \mathsf{Top} \to \Delta\mathsf{Ens}$ sends topological homotopies to simplicial homotopies.

REMARK. These two functors induce an equivalence of categories between the homotopy category of CW-complexes and the homotopy category of Kan complexes.

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Exercise ☢ **7** (Homotopy groups).

(1) Can one find a Kan complex $\mathfrak{X}$, such that each n-simplex X_n is finite, which models the circle?
(2) Compute all the rational homotopy groups of spheres.
HINT. We admit that the spheres are rationally formal.
(3) Compute all the homotopy groups of spheres.

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✉ Bruno Vallette: `vallette@math.univ-paris13.fr` .
☞ Web page: `www.math.univ-paris13.fr/~vallette/` .