WORKSHEET 6

HOMOTOPY THEORY OF SIMPLICIAL SETS

Exercise 1 (Kan complexes).

(1) Prove that a simplicial set $\mathfrak{X}$ is a Kan complex if and only if, for any $n \geqslant 2$, $0 \leqslant k \leqslant n$, and any collection $x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n \in X_{n-1}$ such that $d_{j-1}(x_i) = d_i(x_j)$, for $i < j$, $i, j \neq k$, there exists a $x \in X_n$ such that $d_i(x) = x_i$, for any $i \neq k$.
(2) Show that the nerve of a group, viewed as a one-point category, is a Kan complex.
(3) Do Kan complexes form the essential image of the singular functor?
(4) Show that the underlying simplicial set of a simplicial group, i.e. after forgetting the group structures, forms a Kan complex.
(5) Show that the standard n-simplicies Δ^n are not Kan complexes for $n \geqslant 1$.

Exercise 2 (Kan fibration). A *Kan fibration* is a morphism of simplicial sets $p : \mathfrak{X} \twoheadrightarrow \mathfrak{Y}$ which satisfies the following extension property: for every diagram as below, there exists a diagonal map $\Delta^n \to \mathfrak{X}$ which factors it

$$(\star) \qquad \begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & \mathfrak{X} \arrow[d, two heads] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "\exists"] & \mathfrak{Y} \end{tikzcd} \qquad \text{for } n \geqslant 2 \text{ and } 0 \leqslant k \leqslant n\ .$$

(1) Give a characterisation of a Kan complex in terms of Kan fibrations.
(2) Show that a continuous map $f : X \to Y$ is a Serre fibration if and only if the induced morphism of simplicial set $\mathrm{Sing}(f) : \mathrm{Sing}(X) \to \mathrm{Sing}(Y)$ is a Kan fibration.
(3) Give a combinatorial characterisation of a Kan fibration similar to that of a Kan complex given in Question (1) of Exercise 1.
(4) Let $p : \mathfrak{X} \twoheadrightarrow \mathfrak{Y}$ be a Kan fibration and let $y \in Y_0$ be a vertex of $\mathfrak{Y}$. We denote by $*$ the simplicial subset of $\mathfrak{Y}$ generated by y_0, that is its smallest simplicial subset containing y_0. Show that (the *fiber*) $p^{-1}(*)$ is simplicial subset of $\mathfrak{X}$ and a Kan complex.

Exercise ☢ 3 (Mapping space).
Let $i : \mathfrak{K} \hookrightarrow \mathfrak{L}$ be a monomorphism of simplicial sets and let $p : \mathfrak{X} \twoheadrightarrow \mathfrak{Y}$ be a Kan fibration. They induce the following commutative diagram

$$\begin{tikzcd} \mathfrak{Hom}(\mathfrak{L}, \mathfrak{X}) \arrow[r, "p_*"] \arrow[d, "i^*"] & \mathfrak{Hom}(\mathfrak{L}, \mathfrak{Y}) \arrow[d, "i^*"] \\ \mathfrak{Hom}(\mathfrak{K}, \mathfrak{X}) \arrow[r, "p_*"] & \mathfrak{Hom}(\mathfrak{K}, \mathfrak{Y})\ . \end{tikzcd}$$

(4) Show that the induced map

$$\mathfrak{Hom}(\mathfrak{L}, \mathfrak{X}) \xrightarrow{(i^*, p_*)} \mathfrak{Hom}(\mathfrak{K}, \mathfrak{X}) \times_{\mathfrak{Hom}(\mathfrak{K}, \mathfrak{Y})} \mathfrak{Hom}(\mathfrak{L}, \mathfrak{Y})$$

is a Kan fibration.

Hint. Use the fact that p satisfies the following right lifting property:

$$\begin{tikzcd}
(\Lambda^n_k \times \mathfrak{L}) \cup_{(\Lambda^n_k \times \mathfrak{K})} (\Delta^n \times \mathfrak{K}) \arrow[r] \arrow[d, hook] & \mathfrak{X} \arrow[d, "p"] \\
\Delta^n \times \mathfrak{L} \arrow[r] \arrow[ur, dashed, "\exists"] & \mathfrak{Y}\ .
\end{tikzcd}$$

(5) Show that the pushforward $p_* : \mathfrak{Hom}(\mathfrak{K}, \mathfrak{X}) \twoheadrightarrow \mathfrak{Hom}(\mathfrak{K}, \mathfrak{Y})$ is a Kan fibration.
(6) Show that the pullback $i^* : \mathfrak{Hom}(\mathfrak{L}, \mathfrak{X}) \twoheadrightarrow \mathfrak{Hom}(\mathfrak{K}, \mathfrak{X})$ is a Kan fibration when $\mathfrak{X}$ is a Kan complex.

Exercise 4 (Topological homotopy vs simplicial homotopy).

(1) Show that the geometric realisation functor $|\ | : \Delta\mathsf{Ens} \to \mathsf{Top}$ sends simplicial homotopies to a topological homotopies.
(2) Show that the singular chain functor $\mathrm{Sing} : \mathsf{Top} \to \Delta\mathsf{Ens}$ sends topological homotopies to simplicial homotopies.

Remark. These two functors induce an equivalence of categories between the homotopy category of CW-complexes and the homotopy category of Kan complexes.

Exercise 5 (Classifying space again). We use the same objects and notations as in Exercise 4 of Worksheet 5.

(1) Compute the simplicial homotopy groups of the nerve $\mathrm{B}G$ of a group.
(2) Show that $\mathrm{E}G$ is contractible: there exist two morphisms of simplicial sets $f : \mathrm{E}G \to * = \Delta^0$ and $g : * \to \mathrm{E}G$ such that $fg \sim \mathrm{id}_*$ and $gf \sim \mathrm{id}_{\mathrm{E}G}$.
(3) Show that the morphism of simplicial sets $\mathrm{E}G \twoheadrightarrow \mathrm{E}G/G \cong \mathrm{B}G$ is a Kan fibration.

Exercise 6 (Homotopy groups).

(1) Can one find a Kan complex $\mathfrak{X}$, such that each n-simplex X_n is finite, which models the circle?
(2) Compute all the homotopy groups of spheres.

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