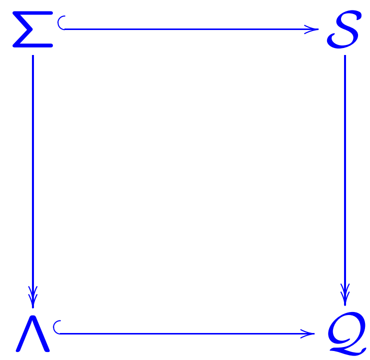
The smash product

of symmetric functions

Marcelo Aguiar
Walter Ferrer
Walter Moreira

Symmetric functions and related algebras



Λ: Symmetric functions.
$\dim \Lambda_n = p(n)$ (partitions).

Σ: Non-commutative symmetric functions.
$\dim \Sigma_n = 2^{n-1}$ (compositions).

$\mathcal{Q}$: Quasi-symmetric functions.
$\dim \mathcal{Q}_n = 2^{n-1}$ (compositions).

$\mathcal{S}$: Algebra of Malvenuto and Reutenauer.
$\dim \mathcal{S}_n = n!$ (permutations).

(All are graded Hopf algebras.)

Symmetric functions

Let $\Lambda_n := \mathrm{Rep}_{\mathbb{Q}}(S_n)$ be the *Grothendieck ring* of S_n. Given $V \in \Lambda_n$ and $W \in \Lambda_n$, define

$$V \circ W := V \otimes W \in \Lambda_n .$$

Let $\Lambda := \oplus_{n \geq 0} \Lambda_n$. Given $V \in \Lambda_p$ and $W \in \Lambda_q$, define

$$V * W := \mathrm{Ind}_{S_p \times S_q}^{S_{p+q}}(V \otimes W) \in \Lambda_{p+q} .$$

Theorem (Frobenius).

$$(\Lambda, *) \cong \varprojlim \mathbb{Q}[x_1, \ldots, x_m]^{S_m}$$

via

$$V \mapsto \sum_{\lambda \vdash n} \frac{\chi_V(\lambda)}{z_\lambda} \sum_{i_1, \ldots, i_k} x_{i_1}^{\ell_1} x_{i_2}^{\ell_2} \cdots x_{i_k}^{\ell_k} ,$$

$\lambda = (\ell_1 \geq \ell_2 \geq \cdots \geq \ell_k)$ (partition of n).

Terminology:

$\Lambda =$ algebra of symmetric functions

$* =$ external product

$\circ =$ internal product

Solomon's descent algebra I

The *descent set* of a permutation $\sigma \in S_n$ is

$$\mathrm{Des}(\sigma) := \{i \in [n-1] \mid \sigma(i) > \sigma(i+1)\}.$$

For instance, $\mathrm{Des}(31542) = \{1,3,4\}$.

Given $J \subseteq [n-1]$, let $\quad X_J := \sum_{\mathrm{Des}(\sigma)\subseteq J} \sigma \quad \in \quad \mathbb{Q}S_n$.

Let $\Sigma_n := \mathrm{Span}\{X_J \mid J \subseteq [n-1]\}$.

Theorem (Solomon, 1976).
Σ_n is a subalgebra of the group algebra $\mathbb{Q}S_n$.

Subsets of $[n-1] \leftrightarrow$ compositions of n.
For instance, if $n = 9$,

$$J = \{1,3,7\} \quad \leftrightarrow \quad \alpha = (1,2,4,2)\,.$$

Write X_α for X_J.

Solomon's descent algebra II

Theorem (Solomon, Garsia-Reutenauer).
$X_\alpha \circ X_\beta = \sum_{\gamma \models n} c_{\alpha\beta}^{\gamma} X_\gamma$ where $c_{\alpha\beta}^{\gamma}$ is the number of matrices M with non-negative entries such that

$$\Sigma_{row}(M) = \alpha,\ \Sigma_{col}(M) = \beta,\ \|M\| = \gamma\,.$$

Example. $\alpha = (1,1,2)$, $\beta = (3,1)$.
The matrices M are

1	0	2	3
0	1	0	1
1	1	2	

0	1	2	3
1	0	0	1
1	1	2	

and

1	1	1	3
0	0	1	1
1	1	2	

Therefore,

$$X_{(1,1,2)} \circ X_{(3,1)} = 2X_{(1,2,1)} + X_{(1,1,1,1)}\,.$$

Solomon's descent algebra III

Given $\alpha = (a_1, \ldots, a_k) \vDash n$, let

$$S_\alpha := S_{a_1} \times \cdots \times S_{a_k} \hookrightarrow S_n \,.$$

Theorem (Solomon, 1976). The map

$$\phi_n : (\Sigma_n, \circ) \to (\Lambda_n, \circ)\,, \quad X_\alpha \mapsto \mathrm{Ind}_{S_\alpha}^{S_n}(\mathbb{1})$$

is a surjective morphism of algebras.

Let $\Sigma := \oplus_{n\geq 0}\Sigma_n$ and $\phi := \oplus_{n\geq 0}\phi_n$.
Given $(a_1, \ldots, a_k) \vDash p$ and $(b_1, \ldots, b_h) \vDash q$ define

$$X_{(a_1,\ldots,a_k)} * X_{(b_1,\ldots,b_h)} := X_{(a_1,\ldots,a_k,b_1,\ldots,b_h)} \,.$$

Then $\phi : (\Sigma, *) \twoheadrightarrow (\Lambda, *)$ is a morphism of algebras.

Terminology:

$\Sigma =$ algebra of non-commutative symmetric functions
$* =$ external product
$\circ =$ internal product

(Reutenauer, Gelfand–Krob–Lascoux–Leclerc–Retakh–Thibon)

The algebra of permutations

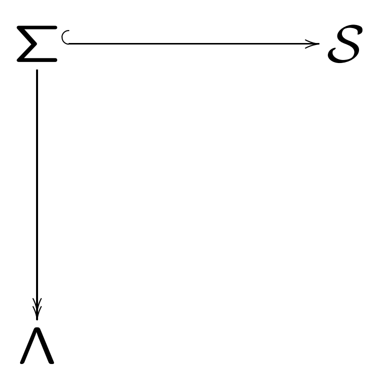
Let $\mathcal{S} := \oplus_{n\geq 0}\mathbb{Q}S_n$. For $p+q=n$ let

$\mathsf{Sh}(p,q) = \{\zeta \in S_n : \zeta_1 < \cdots < \zeta_p,\ \zeta_{p+1} < \cdots < \zeta_{p+q}\}$.

Given $\sigma \in \mathcal{S}_p$ and $\tau \in \mathcal{S}_q$ define

$$\sigma * \tau := \sum_{\zeta \in \mathsf{Sh}(p,q)} \zeta \circ (\sigma \times \tau)\,.$$

Theorem (Malvenuto-Reutenauer, 1993).
$(\mathcal{S}, *)$ is an algebra and $(\Sigma, *)$ is a subalgebra.



Internal and external products everywhere.

One more step: endomorphisms

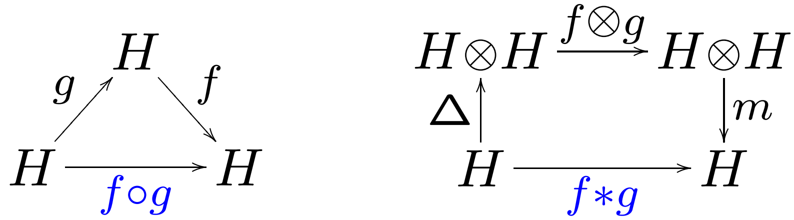
Let V be a vector space and $T(V) = \oplus_{n\geq 0} V^{\otimes n}$ its tensor algebra.

S_n acts on $V^{\otimes n}$ by permuting the tensor factors

$$\Rightarrow \mathbb{Q}S_n \hookrightarrow \mathrm{End}(V^{\otimes n}), \quad \mathcal{S} \hookrightarrow \mathrm{End}(T(V)).$$

Remark (Reutenauer).
Internal product in $\mathcal{S}$ $\leftrightarrow$ composition in $\mathrm{End}(T(V))$.
External product in $\mathcal{S}$ $\leftrightarrow$ *convolution* in $\mathrm{End}(T(V))$.

Let H be a Hopf algebra. There are two products in $\mathrm{End}(H)$: composition and *convolution:*



$T(V)$ is a Hopf algebra with $\Delta(v) = 1 \otimes v + v \otimes 1$,

$$\Delta(v_1 \dots v_n) = \sum_{\substack{p+q=n \\ \zeta \in Sh(p,q)}} v_{\zeta(1)} \cdots v_{\zeta(p)} \otimes v_{\zeta(p+1)} \cdots v_{\zeta(p+q)} \,.$$

Going back: $\mathrm{End}(T(V)) \supset \mathcal{S} \supset \Sigma$.

$GL(V)$ acts on V and diagonally on $V^{\otimes n}$.

Theorem (Schur–Weyl duality).

$$\mathbb{Q}S_n = \mathrm{End}_{GL(V)}(V^{\otimes n}) .$$

This implies that $\mathcal{S}$ is closed under composition and convolution products.

Let $L(V)$ be the smallest subspace of $T(V)$ containing V and closed under $[x, y] := xy - yx$.

$T(V)$ is the free associative algebra on V,
$L(V)$ is the free Lie algebra on V.

Theorem (Garsia-Reutenauer).
Let $\varphi \in \mathcal{S}$. Then $\varphi \in \Sigma$ if and only if for every $P_1, \ldots, P_k \in L(V)$, the subspace

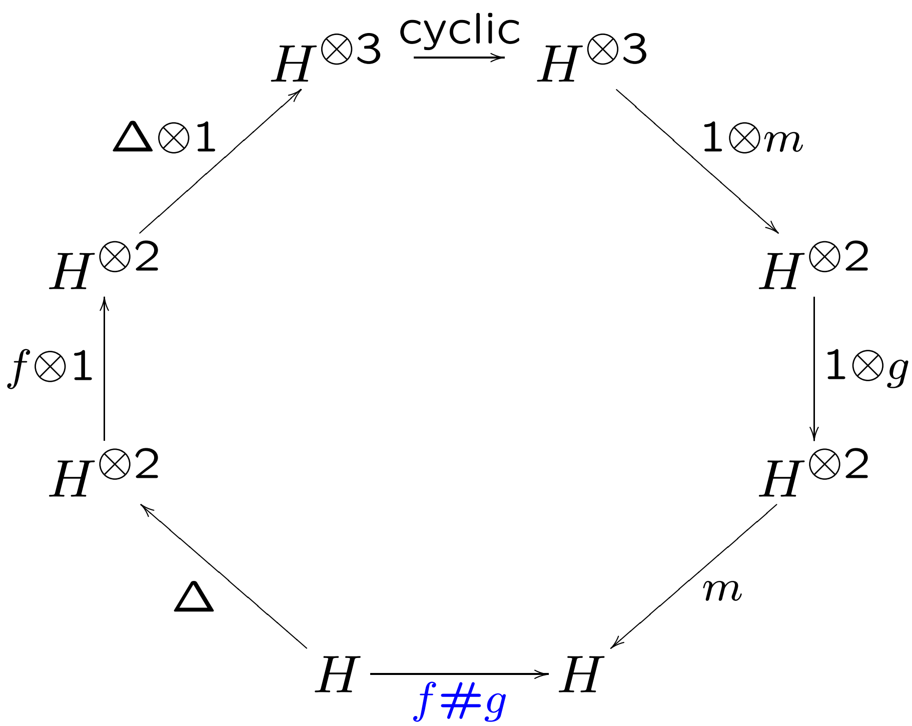
$$\mathrm{Span}\{P_{\tau(1)} \cdots P_{\tau(k)} : \tau \in S_k\}$$

is invariant under φ.

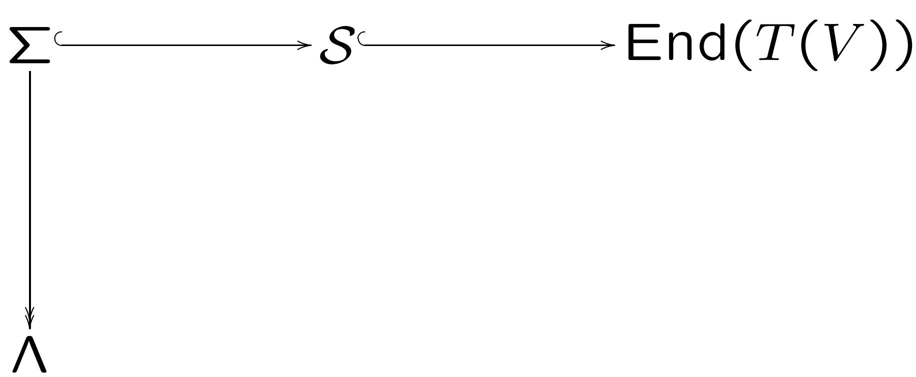
This implies Σ is closed under composition and convolution products.

The smash product

Let H be a Hopf algebra. There is another associative product in End(H): the *smash* product.



Theorem (A.-Ferrer-Moreira, 2004).
The smash product of End($T(V)$) restricts to $\mathcal{S}$ and to Σ and descends to Λ:



The same holds for a fourth product in End(H), *Drinfeld's* product.

Proofs of closure of Σ under smash product

1. Via characterization in terms of action on Lie monomials.

2. Via description of descent classes as equivalence classes.

3. Via explicit matrix rule.

4. Via larger algebra of Solomon-Tits.

For the internal product of Σ_n (descent algebra), the corresponding proofs are due to:

1. Garsia-Reutenauer.

2. Blessenohl-Laue.

3. Solomon, Garsia-Reutenauer.

4. Bidigare, K. Brown.

Remarks on the smash product

Let G be a group acting by automorphisms on a group N. The *semidirect product* is $N \rtimes G := N \times G$ with

$$(m, f) \cdot (n, g) := \Big(m(f \cdot n),\ fg \Big) .$$

Let H be a Hopf algebra "acting by automorphisms" on an algebra A. The *smash product* is $A \# H := A \otimes H$ with

$$(a \otimes f) \cdot (b \otimes g) := \sum a(f_1 \cdot b) \otimes f_2 \, g \, .$$

H acts on itself by translation and also by conjugation, hence also on H^*. Get two products on $H^* \otimes H$.

These products exist on (the larger space) End(H).

Translation smash product $\longrightarrow$ our smash product.

Conjugation smash product $\longrightarrow$ Drinfeld product (if H is cocommutative).

Interpolation

Let H be a graded connected Hopf algebra, $f \in \text{End}(H_p)$, $g \in \text{End}(H_q)$. Then

$$f\#g \;\in \bigoplus_{n=\max(p,q)}^{p+q} \text{End}(H_n)\,.$$

Moreover,

$$(f\#g)_{p+q} = f * g \;\text{ and, if } p = q,\; (f\#g)_p = g \circ f\,.$$

Corollary. The smash product interpolates between the internal and external products of $\mathcal{S}$, Σ and Λ.

Theorem. The smash product on $\mathcal{S}$:

$$\sigma\#\tau = \sum_{\substack{\max(p,q)\leq n\leq p+q \\ \zeta\in \text{Sh}(p,n-p) \\ \eta\in \text{Sh}(p+q-n,n-q)}} \zeta\circ\Big((\sigma\circ\eta)\times 1_{n-p}\Big)\circ\beta_{2n-p-q,p+q-n}\circ(1_{n-q}\times\tau)\,.$$

The smash product on Σ

Theorem.
Let $\alpha = (a_1, \dots, a_k) \vDash p$, $\beta = (b_1, \dots, b_h) \vDash q$. Then

$$X_\alpha \# X_\beta = \sum_{\substack{n \\ \gamma \vDash n}} c^\gamma_{\alpha\beta} X_\gamma$$

where $c^\gamma_{\alpha\beta}$ is the number of matrices $M \in \mathcal{M}_{k+1,h+1}(\mathbb{N})$ such that:

$$\begin{array}{c|ccc|c}
0 & m_{01} & \cdots & m_{0k} & n-q \\
\hline
m_{10} & m_{11} & \cdots & m_{1k} & b_1 \\
\vdots & \vdots & & \vdots & \vdots \\
m_{h0} & m_{h1} & \cdots & m_{hk} & b_h \\
\hline
n-p & a_1 & \cdots & a_k &
\end{array}$$

and $\gamma = (m_{01}, \dots, m_{0k}, \dots, m_{hk})$.

Remark. When $n = p = q$ this reduces to Garsia-Reutenauer rule for $X_\alpha \circ X_\beta$. When $n = p + q$ this reduces to Reutenauer rule for $X_\alpha * X_\beta$.

The matrix rule

When $n = p = q$:

$$
\begin{array}{c|c|c}
0 & 0 \cdots 0 & 0 \\ \hline
0 & m_{11} \cdots m_{1k} & b_1 \\
\vdots & \vdots \qquad\quad \vdots & \vdots \\
0 & m_{h1} \cdots m_{hk} & b_h \\ \hline
0 & a_1 \cdots a_k &
\end{array}
$$

When $n = p + q$:

$$
\begin{array}{c|c|c}
0 & m_{01} \cdots m_{0k} & p \\ \hline
m_{10} & 0 \ \cdots \ 0 & b_1 \\
\vdots & \vdots \qquad \vdots & \vdots \\
m_{h0} & 0 \ \cdots \ 0 & b_h \\ \hline
q & a_1 \cdots a_k &
\end{array}
$$

An example:

$$
\begin{array}{c|c|c}
0 & m_{01} \cdots m_{0p} & n - q \\ \hline
m_{10} & m_{11} \cdots m_{1p} & 1 \\
\vdots & \vdots \qquad\quad \vdots & \vdots \\
m_{q0} & m_{q1} \cdots m_{qp} & 1 \\ \hline
n - p & 1 \ \cdots \ 1 &
\end{array}
$$

$$X_{(1^p)} \# X_{(1^q)} = \sum_{n=\max(p,q)}^{p+q} \binom{p}{n-q} \binom{q}{n-p} (p+q-n)! X_{(1^n)} \, .$$

The smash product of representations

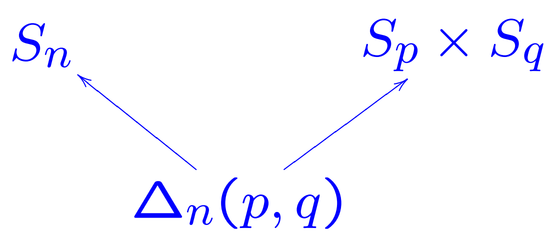
Let $\max(p,q) \le n \le p+q$. Define

$$\Delta_n(p,q) := S_{n-q} \times S_{p+q-n} \times S_{n-p}\,.$$

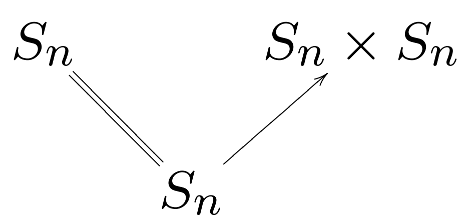
Two embeddings:

$$\Delta_n(p,q) \hookrightarrow S_p \times S_q, \quad (\sigma,\rho,\tau) \mapsto (\sigma \times \rho,\ \rho \times \tau)\,,$$

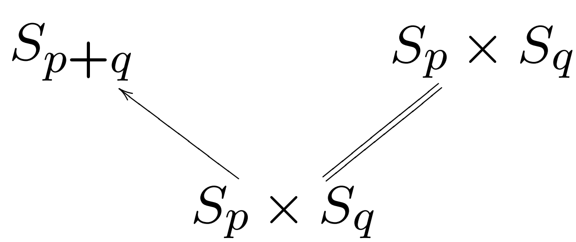
$$\Delta_n(p,q) \hookrightarrow S_n, \quad (\sigma,\rho,\tau) \mapsto \sigma \times \rho \times \tau\,.$$



When $p = q = n$:



When $n = p + q$:



The smash product of representations II

Let $V \in \Lambda_p = \mathrm{Rep}_{\mathbb{Q}}(S_p)$, $W \in \Lambda_q = \mathrm{Rep}_{\mathbb{Q}}(S_q)$.

Then $V \otimes W \in \mathrm{Rep}_{\mathbb{Q}}(S_p \times S_q)$.

Theorem.

$$V \# W = \sum_{n=\max(p,q)}^{p+q} \mathrm{Ind}_{\Delta_n(p,q)}^{S_n} \mathrm{Res}_{\Delta_n(p,q)}^{S_p \times S_q} (V \otimes W)\,.$$

Remark. $\Delta_p(p,p) = S_p$ and we recover

$$V \circ W = \mathrm{Res}_{S_p}^{S_p \times S_p} (V \otimes W)\,.$$

$\Delta_{p+q}(p,q) = S_p \times S_q$ and we recover

$$V * W = \mathrm{Ind}_{S_p \times S_q}^{S_n} (V \otimes W)\,.$$

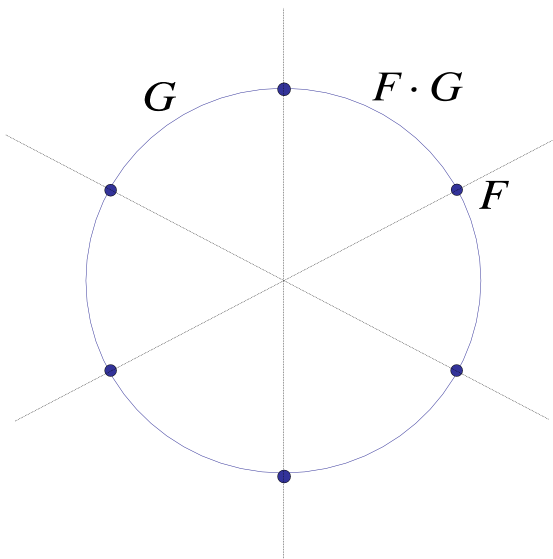
Solomon-Tits algebra

Let Γ_n be the *Coxeter complex* of S_n.

- Γ_n is a simplicial complex
- Γ_n is a monoid under

$$F \circ G := \text{ the face of Star}(F) \text{ that is closest to } G$$

- S_n acts on Γ_n and $\sigma(F \circ G) = \sigma(F) \circ \sigma(G)$.



Theorem (Bidigare, 1997). $\Sigma_n^{op} \cong (\mathbb{Q}\Gamma_n)^{S_n}$

The elements of Γ_n are *set-compositions*:

$$(A_1, \ldots, A_k) \text{ such that } \bigsqcup_{i=1}^{k} A_i = [n] \,.$$

The embedding is

$$X_{(a_1,\ldots,a_k)} \mapsto \sum \{(A_1, \ldots, A_k) : |A_i| = a_i\} \,.$$

(See also recent work by K.Brown.)

The smash product on $\Gamma := \oplus_{n\geq 0}\mathbb{Q}\Gamma_n$.

Theorem (A.-Mahajan, Bergeron et al, Hivert, Patras-Schocker).
The external product of Σ can be extended to Γ.

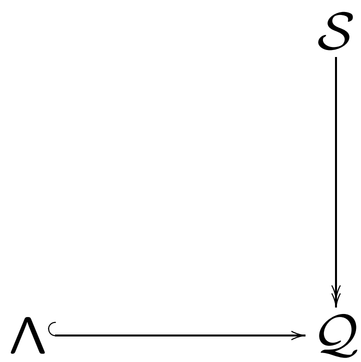
Theorem (A.-Ferrer-Moreira, 2004).
The smash product can be extended to Γ as well.

Let $\max(p,q) \leq n \leq p+q$. An n-quasishuffle of p and q is a map $\zeta : [p] \sqcup [q] \to [n]$ such that $\zeta|_{[p]}$ and $\zeta|_{[q]}$ are increasing and $\zeta([p]) \cup \zeta([q]) = [n]$.
Let $\mathrm{Sh}_n(p,q)$ be the set of quasishuffles.

Given $\mathcal{A} = (A_1, \ldots, A_k) \vDash [p]$, $\mathcal{B} = (B_1, \ldots, B_h) \vDash [q]$, define

$$\mathcal{A}\#\mathcal{B} = \sum_{\substack{n=\max(p,q)\\ \zeta\in\mathrm{Sh}_n(p,q)}}^{p+q} \Big([n]\setminus\zeta([p]), \zeta(A_1), \ldots, \zeta(A_k)\Big) \circ \Big([n]\setminus\zeta([q]), \zeta(B_1), \ldots, \zeta(B_h)\Big).$$

Smash coproduct



Internal and external coproducts everywhere.

Let $\mathbf{X} = \{x_1, x_2, \ldots\}$ be a countable set.
Recall $\Lambda \hookrightarrow \mathbb{Q}[[\mathbf{X}]]$.

Theorem (Grothendieck, Lascoux-Schützenberger,...).
The dual of the internal product on Λ is

$$\Delta_i\Big(f(\mathbf{X})\Big) = f(\mathbf{X} \times \mathbf{Y})\,.$$

The dual of the external product on Λ is

$$\Delta_e\Big(f(\mathbf{X})\Big) = f(\mathbf{X} + \mathbf{Y})\,.$$

The smash coproduct on quasi-symmetric functions

Let $\mathbf{X} = \{x_1, x_2, \ldots\}$ be an *alphabet*.
For $\alpha = (a_1, \ldots, a_k) \models n$ define

$$M_\alpha(\mathbf{X}) := \sum_{i_1 < \cdots < i_k} x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k} \in \mathbb{Q}[[\mathbf{X}]]\,,$$

$$\mathcal{Q}_n := \text{Span}\{M_\alpha \mid \alpha \models n\},\ \ \mathcal{Q} := \bigoplus_{n \geq 0} \mathcal{Q}_n\,.$$

View $\mathcal{Q} = \Sigma^*$ via $\langle M_\alpha, X_\beta \rangle = \delta_{\alpha,\beta}$.

Theorem (Gessel, Reutenauer, Thibon et al).
The dual of the internal product on Σ is

$$\Delta_i\Big(f(\mathbf{X})\Big) = f(\mathbf{X} \times \mathbf{Y})\,.$$

The dual of the external product on Σ is

$$\Delta_e\Big(f(\mathbf{X})\Big) = f(\mathbf{X} + \mathbf{Y})\,.$$

Theorem (A.-Ferrer-Moreira, Ram).
The dual of the smash product on Σ is

$$\Delta_s\Big(f(\mathbf{X})\Big) = f(\mathbf{X} + \mathbf{XY} + \mathbf{Y})\,.$$

Formal group laws for alphabets

Let $\mathbf{X} = \{x_1, x_2, \ldots\}$, $\mathbf{Y} = \{y_1, y_2, \ldots\}$ be alphabets. Define

(1) $\mathbf{X} + \mathbf{Y} := \mathbf{X} \sqcup \mathbf{Y}$ with $x_i < y_j \ \forall\, i, j$;

(2) $\mathbf{X} \times \mathbf{Y} := \{(x_i, y_j) \mid i, j \geq 1\}$ with revlex order;

(3) $\mathbf{X} + \mathbf{XY} + \mathbf{Y} := \{(x_i, y_j) \mid i, j \geq 0,\ (i, j) \neq (0, 0)\}$ with $x_0 < x_i$, $y_0 < y_j$, and revlex order.

Note $\mathbf{X} + \mathbf{XY} + \mathbf{Y} = (\mathbf{1} + \mathbf{X}) \times (\mathbf{1} + \mathbf{Y}) - \mathbf{1}$.

The antipode of $(\mathcal{Q}, \Delta_e)$ is

$$S_e\Big(f(\mathbf{X})\Big) = f(-\mathbf{X})\,,$$

where

$$M_\alpha(-\mathbf{X}) := (-1)^k \sum_{i_1 \geq \cdots \geq i_k} x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}\,.$$

Theorem The antipode of $(\mathcal{Q}, \Delta_s)$ is

$$S_s\Big(f(\mathbf{X})\Big) = f\Big(-\frac{\mathbf{X}}{\mathbf{1} + \mathbf{X}}\Big)\,,$$

where

$$\frac{\mathbf{X}}{\mathbf{1} + \mathbf{X}} := \mathbf{X} - \mathbf{X}^2 + \mathbf{X}^3 - \mathbf{X}^4 + \ldots$$