M2 Seminar on Infinity Categories

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1 Introduction

In classical category theory, a category is a structure consisting of objects with a set of morphisms between any pair of objects. These can be composed uniquely and are subject to the usual axioms governing functions and their composition. But why stop at just morphisms between objects, what about morphisms between morphisms *ad infinitum*. ∞-categories provide a robust framework to deal with categories with 'higher morphisms' by combining category theory with homotopy theory. ∞-categories are already widely used in algebraic topology, K-theory, algebraic geometry, mathematical physics, and more.

In the seminar we will learn the basics of $\infty-$categories, finding analogues to the known notions and theorems from classical category theory. The last three talk will provide an introduction to stable homotopy theory, first invented to investigate stable homotopy groups of spheres, but can be show to underlie topics such as the classification of smooth structures on manifolds, and homological algebra. The goal of this seminar is for students to obtain a working knowledge of ∞-categories, understanding the main concept and be ready to further specialise in its branches. After reflecting on the content of last year's edition of this seminar, we will focus less on technical details and more on how to 'think ∞-categorically'.

Each participant is **expected to prepare a talk** with our help, and present it to the class. We expect **each talk to be roughly 90 minutes long**, accounting for questions and comments, and students can pair-up to divide each talk in half. In addition all should attend the each of the talks and participate actively.

It is expected that you have prior knowledge in classical category theory, acquaintance with simplicial sets and classical homotopy theory. We review basics on simplicial sets on the first week. More concretely, we encourage the students to be familiar with the M2 courses (or equivalent to) from

Mathématiques fondamentales '**Homologie, cohomologie et faisceaux**' and '**Homotopie I**'.
There will be some later talks describing the Synthetic Category Theory [RS17, GWB24] approach which is based on (homotopy) type theory. We recommend taking the course on Homotopy Type Theory (HoTT) at the Master Parisien de Recherche en Informatique (MPRI).

2 Preliminary Plan

First a brief note on the sources we will follow. Based on the practical approach we will take, we will use a variety of recent lecture notes. The courses that these notes compliment adopt a similar philosophy to ours and so will provide a nice template on which to base talks.

For a rigorous treatment, speakers are also encouraged to consult additional sources. The book of Markus Land [Lan21] provides a relatively gentle albeit rigorous introduction to the theory of ∞-categories. This is complimented nicely by the set of notes written by Charles Rezk [Rez22] which comes much of the same material in a similar style. Finally, no course on ∞-categories can be conducted without mentioning the seminal book *Higher Topos Theory* by Jacob Lurie [Lur09] which provides a nearby encyclopedic treatment of the theory. Though Higher Topos Theory is long and at times difficult, it contains a wealth of material, including expository notes, that the shorter texts omit. Finally, Lurie is in the process of compiling an online resource, *Kerodon* [Lur18], which contains many more details than even Higher Topos Theory.

1. **Motivation for ∞-categories, Basic Definitions and Constructions:**

   The first lecture will serve as an introduction to the homotopy theory with many concepts already defined, or to be defined in Homotopie I& II classes. We suggest to go quickly through the construction of homotopy colimits and limits of spaces, suspension and loop as adjoints. Then, moving to generalised cohomology theories, and spectra, giving examples and defining the corresponding properties that correspond to the Eilenberg-Steenrod axioms. Finish with stating the Brown representability theorem and sketch its proof.

   References: [Cno24, Chapter 1], [Hau].

2. **Definition of ∞-categories:**

   Give the definition of an ∞-category, explain its terms recalling basic constructions in simplicial sets and explain its objects, morphisms, and compositions. Discuss some of the basic philosophy and informal 'rules' that govern the manipulation of ∞-categories. Discuss the geometric realisation in simplicial sets. Construct the nerve functor and and explain how discrete categories are an example of ∞-categories. Then, using the singular simplicial complex show that spaces are also ∞-categories. Define Kan complexes and ∞-groupoids and state their equivalence.

References: [Gal23, Chapter 1], [Dav24, Section 1.1-1.3 until Definition 1.3.11].

3. **Functors and Mapping Spaces:**

   Define functors, functor categories, and natural transformations. State that the functor category is an ∞-category and without proving it, illustrate the technical ideas going into the proof, namely fibrations, anodyne maps and Kan fibrations. Then, define the mapping spaces, their construction and examples. Give definitions of fully faithfulness and essential surjectivity and outline the proof of functors being equivalent $\iff$ they are fully faithful and essentially surjective.

   References: [Gal23, Section 2.1-2.2]

4. **Spc and $\mathbf{Cat}_\infty$, and Limits and Colimits:**

   Construct the ∞-category of spaces using the homotopy coherent nerve. Define the (large) ∞-category of small ∞-categories, without dwelling too much on the size issues. Then define limits and colimits via the usual (co)limit cone construction and give explicit description of (co)products, (co)equalizers, mapping telescopes, initial/terminal objects, pushouts and pullbacks. Give the definition of (co)completeness and state that ∞-categories has finite (co)limits $\iff$ has (co)equalizers and finite (co)products $\iff$ has (pullbacks)pushout and finite coproducts, and give examples of such ∞-categories.

   References: [Gal23, Section 2.3, 3.1 and 3.2]

5. **Slice categories, alternative description of (co)limits and Adjuntions:**

   Give description of slice categories via joins, and thus give an alternative description of limits and colimits. Then give definition of adjunctions, show that the inclusion $\mathrm{Spc} \hookrightarrow \mathrm{Cat}_\infty$ admits left and right adjoint. Define the ∞-category BG in Spc, and show that homotopy orbits and homotopy fixed points are adjoint functors, motivating the example of I-shaped (co)limits and adjunction between constant functor and I-shaped colimit. Finally, mention that Kan extensions can be made sense of for ∞-categories. And if time permits show that left adjoints preserve colimits by first showing that left adjoint preserve left kan extensions.

References: [Gal23, Section 3.3, and 3.4, Appendix A.1, and A.2], [Dav24, Section 1.6-1.9].

6. **Straightening-Unstraightening and Yoneda's lemma:**

   Building on the theory of adjunctions from before, state and sketch the proof that the left (resp. right) functor preserve all colimits (resp limits) that exist in the source ∞-category. Then, define Yoneda embedding, the ∞-category of presheaves and sketch the proof for Yoneda's lemma. Then introduce the ideas and definitions of the Grothendieck construction and state the Straightening-Unstraightening theorem, explain the concepts. Motivate this result by defining the twisted arrow category and providing an alternative definition of the Yoneda embedding.

   References: [Gal23, Section 2.4, 3.4, and Appendix A.3]

7. **Presentable Categories:**

   Define an accessible and a (locally) presentable ∞-category and show that Spc is presentable. Prove the adjoint functor theorem for (locally) presentable ∞-categories.

   References: [Gal23, Section 4.1-4.4]

8. **Informal Homotopy Type Theory** This lecture will lay the foundation for arguing within type theory. For develop the basic language of the theory: Judements, Terms, Types, Dependent Types and introduction/elimination rules. This part should be prefaced with "why" we would want to argue within a type theory: Mechanized Proofs, Formalization, Synthetic Arguments.

   After that introduce Inductive Types by talking about the usual induction principles we are used to in mathematics and compare these to universal properties in category theory. Lastly introduce Higher Inductive Types and how they make sense in a homotopy coherent setting. Give the example of pushouts in HoTT naturally corresponding to homotopy pushouts in Algebraic Topology.

   References: [Uni13, Chapter 1, 5 and 6], [Rij22], [Rie25]

9. **Homotopy Theory in HoTT** The first half of this lecture should introduce the basic constructions in homotopy theory such as suspensions, loop spaces and truncations. Then define S^n as a suspension of the type booleans $\mathbb{B}$ and prove that $\pi_1(S^1) = \mathbb{Z}$ in informal HoTT. And if time permits mention the definitions Eilenberg-Maclane spaces and Cohomology in HoTT.

   The second half will be about models of HoTT. For this define ∞-topoi and give the equivalent characterization via the Lurie-Giraud

Axioms. Then informally discuss how they form models of Homotopy Type Theory, making the theory "as consistent as ZFC".

References: [Uni13, Chapter 8], [LF14, Wä23], [Shu19], [Lur09, Chapter 6]

10. **Simplicial Type Theory and Synthetic Category Theory** The last lecture of the synthetic approach will follow the approach by Rhiel and Shulman [RS17] to define a type theory where we formally manipulate simplicial anima and use it to define Segal Types to represent higher categories.

    The plan for this is to introduce the axioms of Simplicial Type Theory, that is axiomatize the interval type and define simplicies, then use those to define Segal Types and then finish up by proving basic statements in higher category theory such as the Yoneda Lemma.

    References: [RS17, Rie23], [GWB24], [GWB25]

11. **Stable Categories and ∞-category of Spectra:**

    Motivate the need for stable categories using homological algebra - the derived categories of rings - and stable homotopy theory - categories of cohomology theories. Define stable ∞-categories and prove alternative characterizations. Provide intuition for what it means for a category to be stable. Define the functor $Sp(-)$ and construct the ∞-category of spectra. Demonstrate that $Sp(-)$ participates in a variety of adjunctions and in particular allows us to lift the Yoneda embedding to spectra. Discuss how $calD(-)$ is stable.

    References: [Cno24, Section 3.1-3.3]

12. **Spectra and Outlook**
    In this final very informal talk we will look at the analogy of spectra with Abelian groups. Discuss the tensor product of spectra and (very briefly) rings and modules. We shall continue to state some examples of ring spectra and finish by determining the chromatic levels in ∞-category of spectra, a very useful tool to determine the homotopy groups of spheres.

References

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