Diffusive transport of partially quantized particles : $L \log L$ solutions.

Nicolas Vauchelet

Institut de Mathématiques de Toulouse (UMR 5219), Université Paul Sabatier,
118, route de Narbonne, 31062 Toulouse cedex 4, France

vauchel@mip.ups-tlse.fr

Abstract

The paper is devoted to the analysis of a drift-diffusion-Schrödinger-Poisson (DDSP) system. From the physical point of view, it describes the transport of a quasi-bidimensional electron gas confined in a nanostructure. Existence, uniqueness and long time behaviour of a weak solution were already obtained in [*N. Ben Abdallah, F. Méhats, N. Vauchelet, Proc. Edinb. (2006) 49, 513–549*] for constant scalar diffusion matrices. In the present contribution, we develop an $L \log L$ existence theory for the DDSP system for a general class of smooth diffusion matrices. Our argument relies on a Trudinger estimate for the entropy functional and a sharp bound on the Hamiltonian's spectrum.

Key words: Schrödinger equation, drift-diffusion system, Poisson equation, free energy, Aubin-Lions compactness method, long-time behavior, convex minimization, Trudinger inequality, Young inequality.

AMS Subject Classification: 35Q40, 76R99, 49S05, 49K20

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1 Introduction and main results

1.1 Presentation of the model

In this work, we consider the drift-diffusion-Schrödinger-Poisson (DDSP) system whose derivation has been described in [8, 9]. This coupled quantum-classical system models the transport of an electron gas partially confined in nanoscale semiconductor devices [31, 32]. The coupling occurs in the momentum variable : electrons are assumed to behave like point particles in the directions x parallel to the gas (classical transport) while they are described by wavefunctions in the transversal direction z (quantum description). Let ω be a bounded regular domain of $\mathbb{R}^2$, for $(x, z) \in \Omega = \omega \times (0, 1)$, the DDSP system is defined by :

$$
(DDSP) \quad \begin{cases} (DD) & \begin{cases} \partial_t N_s - \text{div}_x(\mathbb{D}(\nabla_x N_s + N_s \nabla_x V_s)) = 0, \quad t > 0, \\ N_s(0, x) = N_s^0(x), \quad x \in \omega, \end{cases} \\ (S) & \begin{cases} -\frac{1}{2}\partial_z^2 \chi_k + V\chi_k = \epsilon_k \chi_k \qquad (k \geq 1), \\ \chi_k[V](0) = \chi_k[V](1) = 0, \qquad \int_0^1 |\chi_k[V]|^2 \, dz = 1 \,, \end{cases} \\ (P) & -\Delta_{x,z} V = N = N_s \sum_{k=1}^{+\infty} \frac{e^{-\epsilon_k}}{\sum_\ell e^{-\epsilon_\ell}} |\chi_k|^2 \,, \end{cases}
$$

where $\mathbb{D}$ is the given diffusion matrix and the effective potential V_s is defined by :

$$
V_s(t, x) = -\log\left(\mathcal{Z}(t, x)\right), \qquad \mathcal{Z}(t, x) = \sum_{k=1}^{+\infty} e^{-\epsilon_k(t,x)}. \tag{1.1}
$$

$\mathcal{Z}$ is the repartition function. We complete this system with the following conservative boundary conditions :

$$
\partial_\nu V(t, x, z) = 0 \text{ on } \partial\omega \times (0, 1), \qquad V(t, x, 0) = V(t, x, 1) = 0 \text{ for } x \in \omega, \tag{1.2}
$$

$$
\partial_\nu N_s(t, x) = 0 \text{ on } \partial\omega \times (0, 1), \tag{1.3}
$$

where $\partial\omega$ is the boundary of ω and $\nu(x)$ denotes the outward unit normal vector at $x \in \partial\omega$. The unknowns of the system are the surface density $N_s(t, x)$, the electrostatic potential $V(t, x, z)$, the eigenenergies $\epsilon_k(t, x)$ and the eigenvectors $\chi_k(t, x, z)$ of the 1D Schrödinger operator in the z variable.

The drift-diffusion equation (DD), governing the evolution of the surface density N_s, describes the transport of the gas in the x direction in a diffusive regime [25, 26]. Such equations are widely used in semiconductors simulations (see e.g. [13, 16, 20, 35]). In the transversal direction z, electrons are assumed to be strongly confined. This leads to a partial quantization of the energy in subband ϵ_k. The system is at equilibrium in the confined direction and is Boltzmann distributed. We shall use the notation ρ_k for the occupation numbers of the states, namely

$$
\rho_k(t, x) = \frac{N_s(t, x)}{\mathcal{Z}(t, x)} e^{-\epsilon_k(t,x)} = u(t, x) e^{-\epsilon_k(t,x)},
$$

where u is the so-called Slotboom variable. The Fermi level ϵ_F is defined by $\epsilon_F = \log u$. We have then $\rho_k(t,x) = e^{\epsilon_F(t,x)-\epsilon_k(t,x)}$ which is the exponential form of the Boltzmann statistics. Finally, the electrons generate an electric potential V in the device calculated from the total density N through the resolution of the Poisson equation (P).

It is well known that the drift-diffusion equation can be obtained from kinetic theory when the mean free path is small compared to the system length-scale [18, 29]. Similarly, the DDSP system can be derived from a kinetic system for partially quantized particles [34]. We refer the reader to this derivation for explanations of the expressions of the density and the effective energy (1.1). This diffusion limit is done in the natural framework given by the entropy functional. The estimate on the entropy furnishes a $L \log L$ bound on the distribution function and a H^1 bound on the electrostatic potential (see [34]). For the DDSP system, we can define similarly the free energy by :

$$W = \sum_{k\geq 1} \int_\omega \rho_k \log \rho_k \, dx + \frac{1}{2} \iint_\Omega |\nabla_{x,z} V|^2 \, dx dz + \frac{1}{2} \sum_k \int_\omega \rho_k \langle |\partial_z \chi_k|^2 \rangle \, dx. \qquad (1.4)$$

In this expression and in the rest of the paper, we use the notation $\langle f \rangle = \int_0^1 f(z)\, dz$. Therefore the $L \log L$ bound on the surface density and the H^1 bound on the electrostatic potential are the natural estimates for this model.

1.2 Main results

In [8] we have analyzed the DDSP system and proved an existence and uniqueness result and studied the long time behaviour. However we have obtained this result only in the case of a constant and scalar diffusion matrix which allows to obtain a L^2 estimate on the surface density N_s. If we want to consider a general diffusion matrix we have to work in the natural framework given by the physical estimate on the free energy. Therefore, we will develop a $L \log L$ theory in order to extend the results of [8] under the following assumptions :

Assumption 1.1 *The function $\mathbb{D}$ is assumed to be a C^1 function on $\overline{\Omega}$ into the set of 2×2 symmetric positive definite matrix such that for all $x \in \Omega$ we have $\mathbb{D}(x) \geq \alpha I$, where $\alpha > 0$ is given.*

Assumption 1.2 *The initial condition satisfies $N_s^0 \geq 0$ a.e. and $N_s^0 \log N_s^0 \in L^1(\omega)$. And we denote*

$$\mathcal{N}_I = \int_\omega N_s^0 \, dx.$$

The first result of this work concerns the existence of solutions for the (DDSP) system.

Theorem 1.3 *Let $T > 0$. Under Assumptions 1.1 and 1.2 the $(DDSP)$ system coupled with the conditions* (1.2)–(1.3) *admits a weak solution such that*

$$N_s \log N_s \in L^\infty([0,T], L^1(\omega)) \text{ and } \sqrt{N_s} \in L^2([0,T], H^1(\omega)),$$
$$V \in L^\infty([0,T], H^1(\omega)).$$

In order to present our second result, we introduce the steady state (N_s^∞, V^∞) of our system. The stationary drift-diffusion equation with the boundary condition (1.3) implies that the Slotboom variable u^∞ is constant. The conservation of the total mass of the system implies

$$u^\infty = \frac{\mathcal{N}_I}{\sum_k \int_\Omega e^{-\epsilon_k[V^\infty](x)}\,dx}.$$

And V^∞ is the solution of the (S)–(P) system with the boundary conditions (1.2) for the constant given u^∞. We can prove that this system admits a unique solution $V^\infty \in C^2(\overline{\Omega})$ (see [15, 27]).

Theorem 1.4 *Under Assumptions 1.1 and 1.2, if N_s, V is a solution of $(DDSP)$ coupled to* (1.2)–(1.3) *defined in Theorem 1.3 and if N_s^∞, V^∞ is the solution of the stationary system. Then there exist two constants $\kappa > 0$ and $C > 0$ such that for all $t \geq 0$*

$$\|N_s - N_s^\infty\|_{L^1(\omega)}(t) + \|V - V^\infty\|_{H^1(\Omega)}(t) \leq Ce^{-\kappa t}.$$

We present our results only in the case of conservative boundary conditions (1.2)-(1.3). But following [8], we can extend them to Dirichlet boundary conditions. The reason of this simple choice will appear clearly in the study of the long time behaviour where the techniques used need the conservation of the mass.

1.3 Strategy of the proofs

Regarding the whole system, we shall take advantage of its structure : an evolution drift-diffusion equation (DD) coupled to a quasistatic Schrödinger-Poisson system (S)-(P). Therefore following [6, 8], this structure in two blocks suggests us to use a fixed point procedure to construct solutions : the first block furnishes the surface density N_s with the knowledge of the electrostatic potential V by solving (DD), whereas for a given surface density the Schrödinger-Poisson system (S)-(P) allows us to compute the electrostatic potential V. It is then important to analyze carefully each block of this system.

In the first block, containing the transport equation (DD), the difference with [8] is due to the diffusion matrix $\mathbb{D}$. As a consequence, we do not conserve the L^2 estimate on N_s. In fact, multiplying the drift-diffusion equation (DD) by N_s and integrating by parts lead to

$$\frac{1}{2}\frac{d}{dt}\int_\omega N_s^2\,dx + \int_\omega (\nabla N_s)^\top \mathbb{D}\,\nabla N_s\,dx + \frac{1}{2}\int_\omega (\nabla(N_s^2))^\top \mathbb{D}\,\nabla V_s\,dx = 0.$$

In the case of a scalar and constant diffusion matrix, the third term of this identity can be controlled by the first two terms after an integration by parts and a straightforward study of the quantity $-\Delta_x V_s$ (in Section 2.3 of [8] we use strongly the fact that most of the terms which appear in the expression of this quantity are nonnegative). Otherwise, we have to deal with the quantity $-\mathrm{div}\,_x(\mathbb{D}\nabla_x V_s)$. Therefore, the space dependence of $\mathbb{D}$ and the structure of the matrix (not scalar) involve new terms which are not nonnegative and can not be controlled. We do not hope to recover a L^2 estimate on the surface density. Thus, we work in the framework given by the entropy estimate : it is proved in Section 2 that under Assumptions 1.1 and 1.2 the free energy (1.4) is bounded and that this bound furnishes a $L\log L$ born on the occupation factor $(\rho_k)_{k\geq 1}$.

Then the "quasistatic" Schrödinger-Poisson block (S)-(P) has to be studied in this new functional framework. Spectral properties of the Hamiltonian (detailed in Appendix) show that for an electrostatic potential $V \in H^1(\Omega)$, we can not hope to have a better estimate than $\chi_k \in H^1(\Omega)$. Regarding the right hand side of the Poisson equation (P), we have to give a sense to the product of ρ_k in $L\log L$ with $|\chi_k|^2$. When the surface density stays in L^2, the Cauchy-Schwarz inequality suffices. Here we need a sharper estimate. An idea to overcome this difficulty is to use the Trudinger inequality and to improve the estimates on the eigenvectors of the Hamiltonian $(\chi_k)_k$ with respect to V obtained in the Appendix of [8]. First Lemma A.2 shows that it suffices to prove that the product $N_s\|V\|_{L^2_z(0,1)}$ has a sense.

To this aim, we use the Young and Trudinger inequalities. Let a and b be two real numbers with $a>0$, then we have the Young inequality :

$$ab \leq a\log a - a + e^b. \tag{1.5}$$

If $\omega \subset \mathbb{R}^2$ is a regular bounded domain, the Trudinger inequality [33, 17] insures the existence of a constant $\gamma>0$ depending only on ω such that, for all $u \in H^1(\omega)$, we have

$$\int_\omega exp\left(\gamma\frac{u(x)^2}{\|u\|^2_{H^1(\omega)}}\right)\,dx < \infty. \tag{1.6}$$

Thus, applying the Young inequality gives

$$\begin{aligned}\int_\omega N_s\|V\|_{L^2_z(0,1)}\,dx &= \frac{1}{\gamma}\left(\int_\omega N_s\frac{\gamma\|V\|_{L^2_z(0,1)}}{\|V\|_{H^1(\Omega)}}\,dx\right)\|V\|_{H^1(\Omega)}\\ &\leq \frac{1}{\gamma}\|V\|_{H^1(\Omega)}\left(\int_\omega (N_s\log N_s - N_s)\,dx + \int_\omega e^{\frac{\gamma\|V\|_{L^2_z(0,1)}}{\|V\|_{H^1(\Omega)}}}\,dx\right).\end{aligned}$$

Since the entropy estimate gives a bound of V in $H^1(\Omega)$ and of N_s in $L\log L(\omega)$, the fact that $\|V\|_{L^2_z(0,1)} \in H^1(\omega)$ and the Trudinger inequality (1.6) insure that this last term is finite. Moreover, it proves too that for $N_s \in L\log L(\omega)$ and for all $0<\beta\leq 2$, there exists a nonnegative constant C such that

$$\int_\omega N_s\|V\|^\beta_{L^2_z(0,1)}\,dx \leq C\|V\|^\beta_{H^1(\Omega)}. \tag{1.7}$$

This use of the Trudinger inequality appears in some recent works dealing with systems of gas dynamics too (see e.g. [19, 24]). It allows to deal with the study of the Schrödinger-Poisson system, presented in Section 3, for a given surface density N_s in $L \log L(\omega)$.

Finally with a non-scalar and non-constant diffusion matrix $\mathbb{D}$, we can not use the fixed point procedure presented in [8] which relies on the L^2 estimate on N_s. Thus we regularize the (DDSP) system with a linear operator R^ε for a small parameter $\varepsilon \in (0,1)$ such that $R^\varepsilon \to Id$ as $\varepsilon \to 0$. Usual techniques [5, 6, 11, 30] consist in finding solutions of the regularized problem and letting $\varepsilon \to 0$ in these solutions. Nevertheless, these techniques does not give uniqueness of solutions.

The outline of the paper is as follows. In section 2, we present the linear regularization operator and a priori estimates obtained for the regularized system. In section 3 we analyze the regularized Schrödinger-Poisson system in the general framework given by our estimates. And in section 4 we detail the proof of Theorem 1.3 which is decomposed into several steps : existence of solutions for the regularized system, using the uniform estimates to pass to the limit $\varepsilon \to 0$ in the solutions of the regularized system to obtain solutions of the unregularized system. In the last section, we present the proof of Theorem 1.4. The conservation of the mass allows us to use a general method based on the logarithmic Sobolev inequalities [2, 3, 10, 14, 21]. This method consists in proving the exponential decay as t grows to $+\infty$ of the relative entropy.

2 Free energy and a priori estimates

We recall the regularization strategy developed in [6, 7]. For a parameter $\varepsilon \in [0,1]$, the linear regularization operator is

$$\begin{array}{rcl} R^\varepsilon : L^1(\Omega) & \to & C^\infty(\overline{\Omega}) \\ V & \to & R^\varepsilon[V](x,z) = (\overline{V} *_x \xi_{\varepsilon,x} *_z \xi_{\varepsilon,z})|_{\overline{\Omega}}, \end{array} \tag{2.8}$$

where $\overline{V}$ is the extension of V by zero outside Ω and $\xi_{\varepsilon,x}$ and $\xi_{\varepsilon,z}$ are C^∞ nonnegative compactly supported even approximations of the unity, respectively on $\mathbb{R}^2$ and $\mathbb{R}$. We can prove straightforwardly from convolution results the following properties :

Lemma 2.1 *(i) R^ε is a bounded operator on $L^p_x L^q_z(\Omega)$ for $1 \le p, q \le +\infty$ and satisfies for all $V \in L^p_x L^q_z(\Omega)$, where $L^p_x L^q_z(\Omega) = \{u \in L^1_{loc}(\Omega)$ s.t. $\|u\|_{L^p_x L^q_z(\Omega)} = (\int_\omega \|u(x,\cdot)\|^p_{L^q(0,1)}\, dx)^{1/p} < +\infty\}$,*

$$\|R^\varepsilon[V]\|_{L^p_x L^q_z(\Omega)} \le \|V\|_{L^p_x L^q_z(\Omega)} \quad \text{and} \quad \lim_{\varepsilon \to 0} \|R^\varepsilon[V] - V\|_{L^p_x L^q_z(\Omega)} = 0.$$

(ii) R^ε is self-adjoint on $L^2(\Omega)$ and for all $V \in W^{1,2}(\Omega)$,

$$\nabla_x R^\varepsilon[V] = R^\varepsilon[\nabla_x V]\,; \qquad \lim_{\varepsilon \to 0} \|\nabla_x R^\varepsilon[V] - \nabla_x V\|_{L^2(\Omega)} = 0.$$

Then the regularized system $(DDSP_\varepsilon)$ is defined for $\varepsilon \in [0,1]$ by :

$$(DD_\varepsilon)\begin{cases} \partial_t N_s^\varepsilon - \text{div}_x(\mathbb{D}(\nabla_x N_s^\varepsilon + N_s^\varepsilon \nabla_x V_s^\varepsilon)) = 0, \quad t>0, \\ N_s^\varepsilon(0,x) = N_s^{\varepsilon,0}(x) := \min\{N_s^0, \varepsilon^{-1}\}, \quad x \in \omega \\ \partial_\nu N_s(t,x) = 0 \text{ on } \partial\omega \times (0,1), \end{cases}$$

$$(SP_\varepsilon)\begin{cases} \begin{cases} -\dfrac{1}{2}\,\partial_z^2 \chi_k^\varepsilon + R^\varepsilon[V^\varepsilon]\chi_k^\varepsilon = \epsilon_k^\varepsilon \chi_k^\varepsilon & (k \geq 1), \\ \chi_k^\varepsilon(t,x,\cdot) \in H_0^1(0,1), & \displaystyle\int_0^1 \chi_k^\varepsilon\, \chi_\ell^\varepsilon \, dz = \delta_{k\ell}\,, \end{cases} \\ -\Delta_{x,z} V^\varepsilon = R^\varepsilon \left[\displaystyle\sum_k N_s^\varepsilon \frac{e^{-\epsilon_k^\varepsilon}}{\mathcal{Z}^\varepsilon} |\chi_k^\varepsilon|^2 \right], \\ \partial_\nu V(t,x,z) = 0 \text{ on } \partial\omega \times (0,1), \quad V(t,x,0) = V(t,x,1) = 0 \text{ for } x \in \omega. \end{cases}$$

Remark 2.2 *When $\varepsilon = 0$, we have $R^0 = Id$ and the regularized problem reduces to the unregularized system.*

Therefore the solutions of the overall problem are obtained by the passage to the limit $\varepsilon \to 0$ on solutions of the regularized problem $(DDSP_\varepsilon)$. The key points for the passage at the limit is then to establish uniform estimates independents of ε. For the sake of clarity we skip all exponents ε in the following.

Lemma 2.3 *Let $\varepsilon \in [0,1]$. Consider any weak solution (N_s, V) of the regularized system $(DDSP_\varepsilon)$ such that $N_s \log N_s \in L^\infty(\mathbb{R}^+, L^1(\omega))$, $V \in L^\infty(\mathbb{R}^+, H^1(\Omega))$ and $\sqrt{N_s} \in L^2(\mathbb{R}^+, H^1(\omega))$. Then the total free energy of the system defined in* (1.4) *satisfies :*

$$\frac{d}{dt}W(t) = -D(t) := -\int_\omega \mathcal{Z}\, \frac{(\nabla_x u)^\top \mathbb{D} \nabla_x u}{u}\, dx.$$

Proof. First, we notice that our boundary condition (1.3) implies the conservation of the total mass of the system. Since the regularization operator R^ε is linear we obtain, thanks to the drift-diffusion equation (DD_ε), the identity

$$\begin{aligned} \frac{d}{dt}\sum_{k\geq 1}\int_\omega (\rho_k \log \rho_k - \rho_k)\, dx \; &= \int_\omega \text{div}_x \left(\mathbb{D} \left(\sum_k e^{-\epsilon_k} \nabla_x u \right) \right) \log u \, dx \\ &\quad + \sum_k \iint_\Omega \rho_k |\chi_k|^2 R^\varepsilon[\partial_t V]\, dxdz - \frac{d}{dt}\sum_k \int_\omega \rho_k \epsilon_k \, dx. \end{aligned} \tag{2.9}$$

Moreover, the selfadjointness of R^ε on $L^2(\Omega)$ and the Poisson equation imply

$$\sum_k \iint_\Omega \rho_k |\chi_k|^2 R^\varepsilon[\partial_t V]\, dxdz = \frac{d}{dt}\sum_k \int_\omega \rho_k \langle |\chi_k|^2 R^\varepsilon[V] \rangle \, dx - \frac{1}{2}\frac{d}{dt}\iint_\Omega |\nabla_{x,z} V|^2\, dxdz.$$

Hence, after an integration by parts on the first term of the right hand side of (2.9), we obtain

$$\frac{d}{dt}\sum_k \int_\omega (\rho_k \log \rho_k - \rho_k)\,dx + \frac{1}{2}\frac{d}{dt}\iint_\Omega |\nabla_{x,z}V|^2\,dxdz$$
$$+\frac{d}{dt}\sum_k \int_\omega \rho_k(\epsilon_k - \langle|\chi_k|^2 R^\varepsilon[V]\rangle)\,dx = -\int_\omega \left(\sum_k e^{-\epsilon_k}\right)\frac{(\nabla_x u)^\top \mathbb{D}\nabla_x u}{u}\,dx.$$

To conclude the proof it suffices to notice that, from the Schrödinger equation, we have

$$\epsilon_k - \langle|\chi_k|^2 R^\varepsilon[V]\rangle = \frac{1}{2}\langle|\partial_z \chi_k|^2\rangle.$$

□

The coercivity of the matrix $\mathbb{D}$ insures the non-negativity of the dissipation rate D and therefore the decay of the free energy. This allows us to prove some a priori estimates stated in the following Corollary whose proof is postponed in the Appendix :

Corollary 2.4 *Let $\varepsilon \in [0,1]$ and (N_s, V) such as in Lemma 2.3 and satisfying Assumptions 1.1 and 1.2. Then the following estimates holds :*

(i) mass :

$$\forall\, t \in \mathbb{R}^+, \int_\omega N_s\,dx = \mathcal{N}_I$$

(ii) entropy : there exist nonnegative constants C_1, C_2 and C_3 independent of ε, such that

$$\forall\, t \in \mathbb{R}^+, \quad \sum_{k\geq 1}\int_\omega \rho_k(1 + k^2 + |\log \rho_k|)\,dx \leq C_1, \tag{2.10}$$

$$\forall\, t \in \mathbb{R}^+, \quad \int_\omega N_s|\log N_s|\,dx \leq C_2, \tag{2.11}$$

$$\forall\, t \in \mathbb{R}^+, \quad \iint_\Omega |\nabla_{x,z}V|^2\,dxdz \leq C_3. \tag{2.12}$$

(iii) dissipation : there exist nonnegative constants C_4 and C_5 independent of ε such that

$$\forall\, t \in \mathbb{R}^+, \quad \int_0^t \int_\omega |\nabla_x \sqrt{N_s}|^2\,dxds \leq C_4 \tag{2.13}$$

$$\forall\, t \in \mathbb{R}^+,\ \forall\, p \in [1, +\infty) \quad \int_0^t \int_\omega \|N_s\|_{L^p(\omega)}(s)\,ds \leq C_5. \tag{2.14}$$

Remark 2.5 *By considering the relative entropy of (ρ_k, V) with respect to the extensions on ω of the boundary data $(\underline{\rho_k}, \underline{V})$ (see [8]), we can obtain similar a priori estimates for Dirichlet boundary conditions. In fact the entropy estimate furnishes a bound on the distance between (ρ_k, V) and $(\underline{\rho_k}, \underline{V})$. Then for regular enough boundary data, the a priori estimates announced in Corollary 2.4 still remain the same with constants C_i depending on boundary data too.*

3 The regularized Schrödinger-Poisson system

In this section the surface density N_s is assumed to be given and we only consider the resolution of the "quasi-static" regularized Schrödinger-Poisson system (SP_ε) for $\varepsilon \in [0,1]$. We will solve this system in the framework given by the a priori estimates of Corollary 2.4. We assume then that N_s satisfies the following assumption :

H1 : $N_s \geq 0$ a.e. and there exists a nonnegative constant C_T such that:

$$\forall t \in [0,T], \qquad \int_\omega (N_s |\log N_s| + 1)\, dx \leq C_T, \tag{3.15}$$

In the sequel we will use the functional space $H^1_{01} = \{V \in L^\infty((0,T), H^1(\Omega)) : V(t,x,0) = V(t,x,1) = 0\}$.

Following an idea of Nier [27] which has been developed in [6, 8], we can establish the following existence and uniqueness result in this new framework.

Proposition 3.1 (Existence and uniqueness) *Let $\varepsilon \in [0,1]$ and $T > 0$ and assume $N_s \in L^\infty(0,T; L^1(\omega))$ satisfy* **H1**.

Then the regularized Schrödinger-Poisson system (SP_ε) admits a unique solution $(V^\varepsilon, (\epsilon_k[V^\varepsilon], \chi_k[V^\varepsilon]))$ such that $V^\varepsilon \in H^1_{01}$ with a bound independent of ε.

Proof. This existence result is obtained like in [8] thanks to a minimization of a convex functional. This functional is defined on H^1_{01} by

$$J(V) = \frac{1}{2} \iint_\Omega |\nabla V|^2 \, dxdz + \int_\omega N_s \log \sum_k e^{-\epsilon_k[R^\varepsilon[V]]}\, dx = J_0(V) + J_1(V, N_s). \tag{3.16}$$

We have proved (see Section 2.4 of [8]) that this functional is strongly convex on H^1_{01} and that each minimizer defines a solution of (SP_ε). The relevant point here compared to [8] resides in the proof of the continuity and the coercivity of the functional. In fact, due to the only $L \log L$ estimate on the surface density, we need to use the Trudinger inequality (1.6) and sharper estimates on the eigenvalues of the Schrödinger operator $\epsilon_k[V]$ stated in (A.1). Then

$$\begin{aligned} |J_1(V, N_s) - J_1(\widetilde{V}, N_s)| &\leq \int_\omega N_s(x) \sup_k (|\epsilon_k[R^\varepsilon[V]] - \epsilon_k[R^\varepsilon[\widetilde{V}]]|(x))\, dx \\ &\leq C \int_\omega N_s (1 + \|R^\varepsilon[V]\|^{1/2}_{L^2_z(0,1)} + \|R^\varepsilon[\widetilde{V}]\|^{1/2}_{L^2_z(0,1)}) \|R^\varepsilon[V - \widetilde{V}]\|_{L^2_z(0,1)}\, dx. \end{aligned} \tag{3.17}$$

In the following C and C_T will stand for nonnegative constants depending only on the data and not on ε. We have seen in (1.7) that the Trudinger inequality and **H1** imply

$$\int_\omega N_s \|R^\varepsilon[V - \widetilde{V}]\|_{L^2_z(0,1)}\, dx \leq C_T \|R^\varepsilon[V - \widetilde{V}]\|_{H^1(\Omega)} \leq C_T \|V - \widetilde{V}\|_{H^1(\Omega)}, \tag{3.18}$$

thanks to the properties of the regularization R^ε (Lemma 2.1). Doing the same for the others term, we have

$$\int_\omega N_s \|R^\varepsilon[V]\|_{L^2_z(0,1)}^{1/2} \|R^\varepsilon[V-\widetilde{V}]\|_{L^2_z(0,1)}\, dx \le C\|V\|_{H^1(\Omega)}^{1/2}\|V-\widetilde{V}\|_{H^1(\Omega)} \times \int_\omega \left(N_s \log N_s - N_s + \exp \frac{\gamma \|R^\varepsilon[V]\|_{L^2_z(0,1)}^{1/2}\|R^\varepsilon[V-\widetilde{V}]\|_{L^2_z(0,1)}}{\|R^\varepsilon[V]\|_{H^1(\Omega)}^{1/2}\|R^\varepsilon[V-\widetilde{V}]\|_{H^1(\Omega)}} \right) dx,$$

where γ is the constant defined in the Trudinger inequality (1.6). Applying the Cauchy-Schwarz inequality, we obtain

$$\int_\omega \exp \frac{\gamma \|R^\varepsilon[V]\|_{L^2_z(0,1)}^{1/2}\|R^\varepsilon[V-\widetilde{V}]\|_{L^2_z(0,1)}}{\|R^\varepsilon[V]\|_{H^1(\Omega)}^{1/2}\|R^\varepsilon[V-\widetilde{V}]\|_{H^1(\Omega)}}\, dx \le$$
$$\le \left(\int_\omega \exp \frac{\gamma\|R^\varepsilon[V]\|_{L^2_z(0,1)}}{\|R^\varepsilon[V]\|_{H^1(\Omega)}}\, dx \right)^{1/2} \left(\int_\omega \exp \frac{\gamma\|R^\varepsilon[V-\widetilde{V}]\|^2_{L^2_z(0,1)}}{\|R^\varepsilon[V-\widetilde{V}]\|^2_{H^1(\Omega)}}\, dx \right)^{1/2} \le C,$$

thanks to the Trudinger inequality (1.6). Thus, with **H1**, we have

$$\int_\omega N_s \|R^\varepsilon[V]\|_{L^2_z(0,1)}^{1/2} \|R^\varepsilon[V-\widetilde{V}]\|_{L^2_z(0,1)}\, dx \le C_T\|V\|_{H^1(\Omega)}^{1/2}\|V-\widetilde{V}\|_{H^1(\Omega)}. \tag{3.19}$$

Obviously we have the same estimate (3.19) by exchanging the role of V and $\widetilde{V}$. Thus (3.18) and (3.19) injected in (3.17) prove

$$|J_1(V,N_s) - J_1(\widetilde{V},N_s)| \le C_T(1 + \|V\|_{H^1(\Omega)}^{1/2} + \|\widetilde{V}\|_{H^1(\Omega)}^{1/2})\|V-\widetilde{V}\|_{H^1(\Omega)}. \tag{3.20}$$

Hence $J_1(\cdot, N_s)$ is Lipschitz continuous on H^1_{01}. Now if we take $\widetilde{V}=0$ in (3.20), from **H1** we have that $0 \ge J_1(0,N_s) \ge -C_T$. Thus, there exist two nonnegative constants C_1 and C_2 such that

$$J(V) \ge \frac{1}{2}\|\nabla V\|^2_{L^2(\Omega)} - C_1(1 + \|V\|^{1/2}_{H^1(\Omega)})\|V\|_{H^1(\Omega)} - C_2.$$

If we apply the Poincaré inequality in H^1_{01}, we find

$$J(V) \ge C_3\|V\|^2_{H^1(\Omega)} - C_4\|V\|^{3/2}_{H^1(\Omega)} - C_5,$$

which proves the coercivity of J. □

It is important yet to study the behaviour of the obtained solution V with respect to the given N_s. The following result establishes a Lipschitz dependence. Despite this result is not new compared to [8], the proof is new. In fact in Proposition 2.1 of [8] the elliptic regularity on the Poisson equation is needed and allows to obtain the Lipschitz dependence of V in $H^2(\Omega)$ with respect to N_s in $L^2(\Omega)$. Here the result is stated in $H^1(\Omega)$.

Proposition 3.2 (Continuity) *Let $\varepsilon \in [0,1]$ and $T > 0$. Assume N_s and $\widetilde{N_s}$ are given in $L^\infty(0,T;L^1(\omega))$ and satisfy* **H1**.

Then the corresponding solutions V^ε and $\widetilde{V}^\varepsilon$ of the Schrödinger-Poisson system (SP_ε) verify

$$\forall\, t \in [0,T], \quad \|V^\varepsilon - \widetilde{V}^\varepsilon\|_{H^1(\Omega)} \le C_T \|N_s - \widetilde{N_s}\|_{L^1(\omega)}^{1/8}, \tag{3.21}$$

for a nonnegative constant C_T depending only on the data and not on ε.

Moreover, as $\varepsilon \to 0$, the solution V^ε of the regularized problem (SP_ε) converges to the solution V of the unregularized problem (S)–(P) in $L^\infty(0,T;H^1(\Omega))$ and uniformly with respect to $N_s \in L^\infty(0,T;L^1(\omega))$ and satisfying **H1**.

Proof. In the proof all the constants are nonnegative and depend only on the data and not on ε. If we multiply the Poisson equation by $(V^\varepsilon - \widetilde{V}^\varepsilon)$ and integrate :

$$\begin{aligned}\iint_\Omega |\nabla(V^\varepsilon - \widetilde{V}^\varepsilon)|^2\, dxdz &= \sum_k \iint_\Omega (N_s - \widetilde{N_s}) \frac{e^{-\epsilon_k^\varepsilon}}{\mathcal{Z}^\varepsilon} |\chi_k^\varepsilon|^2 R^\varepsilon[V^\varepsilon - \widetilde{V}^\varepsilon]\, dxdz \\ &+ \sum_k \iint_\Omega \widetilde{N_s} \left(\frac{e^{-\epsilon_k^\varepsilon}}{\mathcal{Z}^\varepsilon} |\chi_k^\varepsilon|^2 - \frac{e^{-\widetilde{\epsilon_k^\varepsilon}}}{\mathcal{Z}[\widetilde{V}^\varepsilon]} |\widetilde{\chi_k^\varepsilon}|^2 \right) R^\varepsilon[V^\varepsilon - \widetilde{V}^\varepsilon]\, dxdz,\end{aligned} \tag{3.22}$$

where we use the selfadjointness of the linear operator R^ε on $L^2(\Omega)$.

We use the notation of the Appendix : $\epsilon_k^s = \epsilon_k[sR^\varepsilon[V^\varepsilon] + (1-s)R^\varepsilon[\widetilde{V}^\varepsilon]]$ and $\chi_k^s = \chi_k[sR^\varepsilon[V^\varepsilon] + (1-s)R^\varepsilon[\widetilde{V}^\varepsilon]]$. Then we have,

$$\frac{e^{-\epsilon_k[V]}}{\mathcal{Z}[V]} |\chi_k[V]|^2 - \frac{e^{-\epsilon_k^\varepsilon}}{\mathcal{Z}^\varepsilon} |\widetilde{\chi_k^\varepsilon}|^2 = \int_0^1 \frac{d}{ds} \left(\frac{e^{-\epsilon_k^s}}{\mathcal{Z}^s} |\chi_k^s|^2 \right) ds.$$

The expressions of the derivatives of the eigenfunctions and eigenvalues, given in Lemma A.1, allow us to write

$$\begin{aligned}&\sum_k \int_0^1 \frac{d}{ds} \left(\frac{e^{-\epsilon_k^s}}{\mathcal{Z}^s} |\chi_k^s|^2 \right) R^\varepsilon[V^\varepsilon - \widetilde{V}^\varepsilon]\, dz = -\sum_k \frac{e^{-\epsilon_k^s}}{\mathcal{Z}^s} \langle |\chi_k^s|^2\, R^\varepsilon[V^\varepsilon - \widetilde{V}^\varepsilon] \rangle^2 + \\ &\left(\frac{\sum_k e^{-\epsilon_k^s} \langle |\chi_k^s|^2\, R^\varepsilon[V^\varepsilon - \widetilde{V}^\varepsilon] \rangle}{\mathcal{Z}^s} \right)^2 + \frac{1}{\mathcal{Z}^s} \sum_k \sum_{\ell \ne k} \frac{e^{-\epsilon_k^s} - e^{-\epsilon_\ell^s}}{\epsilon_k^s - \epsilon_\ell^s} \left\langle \chi_k^s \chi_\ell^s\, R^\varepsilon[V^\varepsilon - \widetilde{V}^\varepsilon] \right\rangle^2.\end{aligned}$$

Thanks to the Cauchy-Schwarz inequality, this last term is non-positive. Thus the second term of the right hand side of (3.22) is non-positive.

For the first term of the right hand side of (3.22), we use the bound of $\|\chi_k\|_{L_z^\infty}$ given in Lemma A.2. We deduce

$$\begin{aligned}&\sum_k \iint_\Omega (N_s - \widetilde{N_s}) \frac{e^{-\epsilon_k^\varepsilon}}{\mathcal{Z}^\varepsilon} |\chi_k^\varepsilon|^2 R^\varepsilon[V^\varepsilon - \widetilde{V}^\varepsilon]\, dxdz \le \\ &\qquad \le C_1 \int_\omega |N_s - \widetilde{N_s}| (1 + \|R^\varepsilon[V^\varepsilon]\|_{L_z^2}^{1/2}) \|R^\varepsilon[V^\varepsilon - \widetilde{V}^\varepsilon]\|_{L_z^2}\, dx.\end{aligned} \tag{3.23}$$

By the Hölder inequality,

$$\int_\omega |N_s - \widetilde{N_s}|(1 + \|R^\varepsilon[V^\varepsilon]\|_{L^2_z}^{1/2})\|R^\varepsilon[V^\varepsilon - \widetilde{V}^\varepsilon]\|_{L^2_z}\,dxdz \leq$$
$$\leq \|N_s - \widetilde{N_s}\|_{L^1(\omega)}^{1/4}\left(\int_\omega (N_s + \widetilde{N_s})(1 + \|R^\varepsilon[V^\varepsilon]\|_{L^2_z}^{2/3})\|R^\varepsilon[V^\varepsilon - \widetilde{V}^\varepsilon]\|_{L^2_z}^{4/3}\,dx\right)^{3/4}. \tag{3.24}$$

The assumption **H1** implies a bound in $L\log L$ of the densities N_s and $\widetilde{N_s}$. Thus we can follow the idea of the proof of Proposition 3.1 and prove after straightforward calculations using the Young inequality (1.5) and the Trudinger inequality (1.6), that

$$\int_\omega (N_s+\widetilde{N_s})(1+\|R^\varepsilon[V^\varepsilon]\|_{L^2_z}^{2/3})\|R^\varepsilon[V^\varepsilon-\widetilde{V}^\varepsilon]\|_{L^2_z}^{4/3}\,dx \leq C_2(1+\|V^\varepsilon\|_{H^1(\Omega)}^{2/3})\|V^\varepsilon-\widetilde{V}^\varepsilon\|_{H^1(\Omega)}^{4/3},$$

which is bounded since V^ε and $\widetilde{V}^\varepsilon$ are bounded in $H^1(\Omega)$ with Proposition 3.1. Thus looking at (3.23), we have proved that there exists a nonnegative constant C_3 such that

$$\|V^\varepsilon - \widetilde{V}^\varepsilon\|_{H^1(\Omega)}^2 \leq C_3\|N_s - \widetilde{N_s}\|_{L^1(\omega)}^{1/4}.$$

This leads to (3.21).

For the convergence as $\varepsilon \to 0$, we consider V^ε and V the solutions of the unregularized and respectively regularized problem for $N_s \in L^\infty(0,T;L^1(\omega))$ satisfying **H1**. We have

$$-\Delta(V^\varepsilon - V) = (R^\varepsilon - Id)\left[N_s\frac{e^{-\epsilon_k^\varepsilon}}{\mathcal{Z}^\varepsilon}|\chi_k^\varepsilon|^2\right] + N_s\sum_k\left(\frac{e^{-\epsilon_k^\varepsilon}}{\mathcal{Z}^\varepsilon}|\chi_k^\varepsilon|^2 - \frac{e^{-\epsilon_k}}{\mathcal{Z}}|\chi_k|^2\right).$$

Multiplying by $(V^\varepsilon - V)$ and integrate, we obtain after an integration by parts and thanks to the selfadjointness of R^ε in $L^2(\Omega)$ that

$$\iint_\Omega |\nabla(V^\varepsilon - V)|^2\,dxdz = I + II + III, \tag{3.25}$$

where

$$I = \iint_\Omega N_s\sum_k\frac{e^{-\epsilon_k^\varepsilon}}{\mathcal{Z}^\varepsilon}|\chi_k^\varepsilon|^2(R^\varepsilon - Id)[V^\varepsilon - V]\,dxdz,$$
$$II = \iint_\Omega N_s\sum_k\left(\frac{e^{-\epsilon_k^\varepsilon}}{\mathcal{Z}^\varepsilon}|\chi_k^\varepsilon|^2 - \frac{e^{-\epsilon_k}}{\mathcal{Z}}|\chi_k|^2\right)(V^\varepsilon - R^\varepsilon[V^\varepsilon])\,dxdz,$$
$$III = \iint_\Omega N_s\sum_k\left(\frac{e^{-\epsilon_k^\varepsilon}}{\mathcal{Z}^\varepsilon}|\chi_k^\varepsilon|^2 - \frac{e^{-\epsilon_k}}{\mathcal{Z}}|\chi_k|^2\right)(R^\varepsilon[V^\varepsilon] - V)\,dxdz.$$

For the first two terms, we can obtain with the Trudinger inequality (1.6) and Lemma A.2 that

$$|I| \leq C_1\|(R^\varepsilon - Id)[V^\varepsilon - V]\|_{H^1(\Omega)},$$

and

$$|II| \leq C_2(1 + \|V^\varepsilon\|_{H^1(\Omega)} + \|V\|_{H^1(\Omega)})\|(R^\varepsilon - Id)[V^\varepsilon]\|_{H^1(\Omega)},$$

From the properties of the regularization R^ε (see Lemma 2.1), we deduce easily that, since V^ε is bounded in $H^1(\Omega)$ independently of ε, we have $\lim_{\varepsilon\to 0} |I| = 0 = \lim_{\varepsilon\to 0} |II|$. Finally, as we have shown for the second term of the right hand side of (3.22), the term III is non-positive. Thus a Poincaré inequality in (3.25) implies that

$$\lim_{\varepsilon\to 0} \|V^\varepsilon - V\|_{L^\infty(0,T;H^1(\Omega))} = 0.$$

□

4 Existence of solutions

4.1 Existence of solutions for the regularized system

Proposition 4.1 *Let $T > 0$ and $\varepsilon \in (0,1)$ be fixed. Then the regularized problem $(DDSP_\varepsilon)$ with the boundary condition* (1.2)–(1.3) *admits a unique solution $(N_s^\varepsilon, V^\varepsilon)$ with $N_s^\varepsilon \in C(0,T;L^2(\mathbb{R}^2)) \cap L^2(0,T;H^1(\omega))$.*

Proof. We do not give in detail the proof which can be straightforwardly adapted from the proof of the existence result in [8]. The proof relies on a fixed point argument on the map $F : N_s \mapsto \widehat{N_s}$ defined on the set

$$\mathcal{S}_T = \{n : \left(\max_{0\leq t\leq T} \|n(t)\|^2_{L^2(\omega)} + \int_0^T \|n(t)\|^2_{H^1(\omega)}\, dt\right) < +\infty\}$$

by

1. For a given $N_s \geq 0$, we solve the regularized Schrödinger-Poisson system (SP_ε) and obtained $V \in L^\infty(0,T;C^\infty(\overline{\Omega}))$.

2. We construct $V_s = -\log \sum_k e^{-\epsilon_k[V]}$. From the spectral properties of the Hamiltonian, we have $V_s \in L^\infty(0,T;C^\infty(\overline{\omega}))$.

3. For this effective potential V_s, we solve the following parabolic equation for the unknown $\widehat{N_s}$:

$$\partial_t \widehat{N_s} - \text{div}_{\,x}(\mathbb{D}(\nabla_x \widehat{N_s} + \widehat{N_s}\nabla_x V_s)) = 0,$$

   with the initial condition N_s^0 and the boundary condition $\partial_\nu N_s = 0$ for $x \in \partial\omega$.

We can prove as in [8] that F is a contraction on the set $\mathcal{S}_{T_0}$ for T_0 small enough. Thus we have a solution on $[0,T_0]$ that we can extend to $[0,T]$. □

4.2 Passing to the limit $\varepsilon \to 0$

We have now all matters to construct solutions of the unregularized problem and to prove Theorem 1.3. We will use the Aubin-Lions compactness method to show that the solution of the regularized system converges when the parameter of regularization goes to 0 towards solutions of the unregularized system, up to an extraction of a subsequence. However this method does not give any uniqueness result. We first recall a simple statement of an Aubin-Lions lemma [4, 23] :

Lemma 4.2 (Aubin Lemma) *Take $T > 0$, $q \in (1, +\infty)$ and let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence of functions in $L^q(0,T;H)$ where H is a Banach space. If $(f_n)_{n\in\mathbb{N}}$ is bounded in $L^q(0,T;V)$ where V is compactly embedded in H and $\partial f_n/\partial t$ is bounded in $L^q(0,T;V')$ uniformly with respect to $n \in \mathbb{N}$, then $(f_n)_{n\in\mathbb{N}}$ is relatively compact in $L^q(0,T;H)$.*

Proof of Theorem 1.3. We fix T>0. From Proposition 4.1, there exist N_s^ε and V^ε solution of the regularized problem $(DDSP_\varepsilon)$ with the initial data $N_s^{0,\varepsilon}$. Assertion (iii) of Corollary 2.4 proves that we have $\sqrt{N_s^\varepsilon} \in L^2(0,T;H^1(\omega))$ and with (i), $\sqrt{N_s^\varepsilon} \in L^\infty(0,T;L^2(\omega))$. Thus since we have $\nabla_x N_s^\varepsilon = 2\sqrt{N_s^\varepsilon}\nabla_x\sqrt{N_s^\varepsilon}$, we deduce that $N_s^\varepsilon \in L^2(0,T;W^{1,1}(\omega))$ with a uniform bound with respect to ε. Moreover, we have after a Cauchy-Schwarz inequality :

$$\begin{aligned}\int_0^T \left(\int_\omega |\nabla_x N_s^\varepsilon + N_s^\varepsilon \nabla_x V_s^\varepsilon|\, dx\right)^2 dt \;&\le \mathcal{N}_I \int_0^T \int_\omega \frac{|\nabla_x N_s^\varepsilon + N_s^\varepsilon \nabla_x V_s^\varepsilon|^2}{N_s^\varepsilon}\, dxdt \\ &\le \frac{\mathcal{N}_I}{\alpha}\int_0^T D^\varepsilon(t)\, dt,\end{aligned}$$

where D^ε is defined in Lemma 2.3 and is bounded in $L^1(0,T)$ uniformly with respect to ε. Therefore we conclude with (DD_ε) that $\partial_t N_s^\varepsilon$ is bounded in $L^2(0,T;W^{-1,1}(\omega))$ uniformly with respect to ε.

Hence we can apply Lemma 4.2 for $q = 2$, $H = L^1(\omega)$ and $V = W^{1,1}(\omega)$. There exists a subsequence (still denoted abusively N_s^ε) such that $N_s^\varepsilon \to N_s$ strongly in $L^2(0,T;L^1(\omega))$. From the weak continuity we have that for a.e. $t \in [0,T]$,

$$\int_\omega N_s(1 + |\log N_s|)\, dx \le C_T,$$

for a nonnegative constant C_T. For this function N_s, we can solve the unregularized Schrödinger-Poisson system (S)–(P) and construct $(V, (\epsilon_k, \chi_k)_{k\ge1})$ with $V \in L^\infty(0,T;H^1(\Omega))$ as proved in Proposition 3.1. Thanks to Proposition 3.2, we have that

$$\|V^\varepsilon - V\|_{L^2(0,T;H^1(\Omega))} \to 0 \quad \text{as } \varepsilon \to 0.$$

We want now to pass to the limit $\varepsilon \to 0$ in the drift-diffusion equation. Thanks to Lemma A.2, we have

$$\begin{aligned}\iint_{[0,T]\times\omega} N_s^\varepsilon \nabla_x V_s^\varepsilon \, dxdt &= \iint_{[0,T]\times\omega} N_s^\varepsilon \sum_k \frac{e^{-\epsilon_k^\varepsilon}}{\mathcal{Z}^\varepsilon} \langle |\chi_k^\varepsilon|^2 \nabla_x V^\varepsilon \rangle \, dxdt \\ &\leq \iint_{[0,T]\times\omega} N_s^\varepsilon (1 + \|R^\varepsilon[V^\varepsilon]\|_{L_z^2}^{1/2}) \|\nabla_x V^\varepsilon\|_{L_z^2} \, dxdt.\end{aligned}$$

By the Cauchy-Schwarz inequality and the Sobolev embedding $H^1(\Omega) \hookrightarrow L_x^p L_z^2(\Omega)$ for all $p \in [1, +\infty)$, we deduce :

$$\iint_{[0,T]\times\omega} N_s^\varepsilon \nabla_x V_s^\varepsilon \, dxdt \leq C_T \|V^\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} (1 + \|V^\varepsilon\|_{L^\infty(0,T;H^1(\Omega))}) \|N_s^\varepsilon\|_{L^1(0,T;L^4(\omega))}.$$

But assertion (iii) of Corollary 2.4 shows that $\|N_s^\varepsilon\|_{L^1(0,T;L^4(\omega))}$ is bounded independently of ε. Thus there exists a nonnegative constant C_T independent of ε such that :

$$\iint_{[0,T]\times\omega} N_s^\varepsilon \nabla_x V_s^\varepsilon \, dxdt \leq C_T.$$

Hence we can give a sense to the drift-diffusion equation as $\varepsilon \to 0$. From Lemma A.1, we have

$$\nabla_x \epsilon_k^\varepsilon[V^\varepsilon] = \langle |\chi_k^\varepsilon|^2 \nabla_x V^\varepsilon \rangle.$$

The convergence of V^ε in $L^2(0,T;H^1(\Omega))$ and the local Lipschitz dependence of the eigenvectors of the Hamiltonian with respect to the potential (see Lemma A.4 of Appendix of [8]) allow us to conclude that $\nabla_x \epsilon_k^\varepsilon[V^\varepsilon] \to \nabla_x \epsilon_k[V]$ in $L^p((0,T)\times\omega)$ for all $p \in [1,2)$. From Corollary A.3, we deduce that for all $q \in [1, +\infty)$, we can extract a subsequence such that $\epsilon_k^\varepsilon[V^\varepsilon] \to \epsilon_k[V]$ in $L^2(0,T;L^q(\omega))$. Thus, since we have

$$|e^{-\epsilon_k^\varepsilon[V^\varepsilon]} - e^{-\epsilon_k[V]}| \leq e^{-\frac{1}{2}\pi^2 k^2} |\epsilon_k^\varepsilon[V^\varepsilon] - \epsilon_k[V]|,$$

we deduce that for any $r \in [1, p)$

$$\nabla_x V_s^\varepsilon = \frac{\sum_k \nabla_x \epsilon_k^\varepsilon e^{-\epsilon_k^\varepsilon}}{\mathcal{Z}^\varepsilon}$$

converges in $L^r((0,T)\times\omega)$. This is enough to prove

$$N_s^\varepsilon \nabla_x V_s^\varepsilon \rightharpoonup N_s \nabla_x V_s \quad \text{in } \mathcal{D}'([0,T]\times\omega).$$

Thus up to an extraction, (N_s, V) is a solution of $(DDSP)$ in the distribution sense. By the weak semi-continuity we get moreover

$$\iint_{[0,T]\times\omega} N_s \nabla_x V_s \, dxdt \leq \liminf_{\varepsilon\to 0} \iint_{[0,T]\times\omega} N_s^\varepsilon \nabla_x V_s^\varepsilon \, dxdt.$$

The semi-continuity of the L^2 norm gives

$$\iint_{[0,T]\times\omega} |\nabla_x \sqrt{N_s}|^2 \, dxdt \leq \liminf_{\varepsilon\to 0} \iint_{[0,T]\times\omega} |\nabla_x \sqrt{N_s^\varepsilon}|^2 \, dxdt.$$

And for a.e. $t \in [0,T]$,

$$W(t) + \int_0^t D(s)\,ds \le \liminf_{\varepsilon\to 0} \,(W^\varepsilon(t) + \int_0^t D^\varepsilon(s)\,ds).$$

□

5 Long time behaviour

This section is devoted to the proof of Theorem 1.4. We use an entropy method relying on the Logarithmic Sobolev inequalities and on the Csiszàr-Kullback inequalities. This method has been widely used by several authors (see for instance [2, 3, 10, 12, 14]). We introduce the relative entropy of (ρ_k, V) with respect to the stationary solutions $(\rho_k^\infty, V^\infty)$:

$$\begin{aligned} W^\infty(t) = & \sum_k \int_\omega (\rho_k \log(\rho_k/\rho_k^\infty) - \rho_k + \rho_k^\infty)\,dx + \frac{1}{2}\iint_\Omega |\nabla_{x,z}(V - V^\infty)|^2\,dxdz \\ & + \sum_k \int_\omega \rho_k \left(\epsilon_k[V] - \epsilon_k[V^\infty] - \int_0^1 |\chi_k|^2 (V - V^\infty)\,dz\right)\,dx. \end{aligned} \tag{5.26}$$

We recall that the decay of ρ_k with respect to k insures that the last term is nonnegative. We can prove with the same calculations as in Lemma 2.3 that

$$\frac{d}{dt} W^\infty(t) = -\int_\omega N_s \mathbb{D} \frac{\nabla_x u \cdot \nabla_x u}{u^2}\,dx \le -\alpha \int_\omega N_s |\nabla_x (\log u)|^2\,dx,$$

where we use Assumption 1.1 for the last inequality. We denote $\overline{u} = \mathcal{N}_I / \int_\omega \mathcal{Z}(x)\,dx$. Therefore we have

$$\frac{d}{dt} W^\infty(t) \le -\alpha \int_\omega N_s |\nabla_x \log \frac{u}{\overline{u}}|^2\,dx. \tag{5.27}$$

We recall the Gross logarithmic Sobolev inequality [2, 14, 21] : for two nonnegative functions f and g such that $\int_\omega f\,dx = \int_\omega g\,dx$ then there exists a nonnegative constant λ such that

$$\int_\omega f \log \frac{f}{g} \le \frac{1}{\lambda} \int_\omega f |\nabla \log(f/g)|^2\,dx.$$

Hence if we apply this inequality in (5.26) with $f = N_s$ and $g = \overline{u}\mathcal{Z}$, we obtain

$$\frac{d}{dt} W^\infty(t) \le -\alpha\lambda \int_\omega N_s \log \frac{u}{\overline{u}}\,dx.$$

We can then decompose

$$\log \frac{u}{\overline{u}} = \log \frac{\rho_k}{\rho_k^\infty} + \epsilon_k - \epsilon_k^\infty + \log \frac{\rho_k^\infty}{\overline{u} e^{-\epsilon_k^\infty}} = \log \frac{\rho_k}{\rho_k^\infty} + \epsilon_k - \epsilon_k^\infty + \log \frac{u^\infty}{\overline{u}}.$$

Moreover, the Slotboom variable for the stationary problem u^∞ is a constant. Thus

$$\frac{u^\infty}{\overline{u}} = \left(\frac{u^\infty}{\mathcal{N}_I}\int_\omega \mathcal{Z}(x)\,dx\right) = \sum_k \int_\omega u^\infty e^{-\epsilon_k}\,\frac{dx}{\mathcal{N}_I}$$

Therefore with the identity $u^\infty = \rho_k^\infty e^{-\epsilon_k^\infty}$, by a Jensen inequality, we have

$$\log\frac{u^\infty}{\overline{u}} = \log\sum_k\int_\omega \rho_k^\infty e^{\epsilon_k^\infty-\epsilon_k}\,\frac{dx}{\mathcal{N}_I} \geq \sum_k\int_\omega (\epsilon_k^\infty-\epsilon_k)\frac{\rho_k^\infty}{\mathcal{N}_I}\,dx.$$

(We recall that $\sum_k\int_\omega \rho_k^\infty = \mathcal{N}_I$). Finally we have obtained

$$\begin{aligned}
\frac{d}{dt}W^\infty(t) \;&\leq\; -\alpha\lambda(\sum_k\int_\omega \rho_k\log\frac{\rho_k}{\rho_k^\infty}\,dx + \sum_k\int_\omega(\rho_k-\rho_k^\infty)(\epsilon_k-\epsilon_k^\infty)\,dx)\\
&\leq\; -\alpha\lambda(\sum_k\int_\omega \rho_k\log\frac{\rho_k}{\rho_k^\infty}\,dx + \iint_\Omega (N-N^\infty)(V-V^\infty)\,dxdz\\
&\quad +\sum_k\int_\omega \rho_k(\epsilon_k-\epsilon_k^\infty-\langle|\chi_k|^2(V-V^\infty)\rangle)\,dx\\
&\quad +\sum_k\int_\omega \rho_k^\infty(\epsilon_k^\infty-\epsilon_k-\langle|\chi_k^\infty|^2(V^\infty-V)\rangle\,dx).
\end{aligned}$$

The last term of the right hand side which is similar to the last term of W^∞ is nonnegative. According to the Poisson equation, the second term of the right hand side is equal to

$$\iint |\nabla_{x,z}(V-V^\infty)|^2\,dxdz.$$

Thus, we have shown that

$$\frac{d}{dt}W^\infty(t) \leq -\alpha\lambda W^\infty(t),$$

and a Gronwall type argument yields the exponential convergence of the relative entropy : $W^\infty(t) \leq W^\infty(0)\,e^{-\alpha\lambda t}$. Next we shall use the Csiszàr-Kullback inequality [1] : for all f and g in $L^1_+(\omega)$ with $\int_\omega f\,dx = \int_\omega g\,dx = N_I$, we have

$$\|f-g\|^2_{L^1(\omega)} \leq 2N_I\int_\omega f\log\frac{f}{g}\,dx.$$

Therefore we deduce that

$$\|\rho_k-\rho_k^\infty\|_{\ell^1(L^1(\omega))} \leq \sqrt{2\mathcal{N}_I W^\infty(0)}\,e^{-\frac{\alpha\lambda}{2}t}.$$

□

Appendix

1 Sharp estimates on the spectrum of the Hamiltonian

The aim of this Appendix is to present sharp estimates on the eigenvalues and eigenfunctions of the one-dimensional Schrödinger operator, needed in the framework of this paper. For a given real valued function V in $L^2(0,1)$, the Schrödinger operator

$$H[V] := -\frac{1}{2}\frac{d^2}{dz^2} + V(z)$$

defined on $H^2(0,1) \cap H_0^1(0,1)$ admits a strictly increasing sequence of real eigenvalues $(\epsilon_k[V])_{k\geq 1}$ going to $+\infty$. The corresponding eigenvectors, denoted by $(\chi_k[V](z))_{k\geq 1}$ (chosen such that $\chi_k'(0) > 0$ and $\int_0^1 |\chi_k[V]|^2\,dz = 1$), form an orthonormal basis of $L^2(0,1)$ and are in $H_0^1(0,1)$. For $V = 0$, we have $\epsilon_k[0] = \frac{1}{2}\pi^2 k^2$ and $\chi_k[0](z) = \sqrt{2}\sin(\pi k z)$.

In the sequel we will use the standard notation $\langle f \rangle = \int_0^1 f(z)\,dz$ and when there is no confusion possible ϵ_k will stand for $\epsilon_k[V]$ and χ_k for $\chi_k[V]$.

By adapting the proofs of [28], we can prove :

Lemma A.1 *Let $V = V(\lambda, z) \in L^\infty_{loc}(0,\Lambda; L^2_z(0,1))$ with $\lambda \in (0,\Lambda)$ (typically $\lambda = t$ or $\lambda = x_i$). If $\partial_\lambda V \in L^1_{loc}(\lambda, L^2_z(0,1))$, then $\partial_\lambda \epsilon_k \in L^1_{loc}$, $\partial_\lambda \chi_k \in L^1_{loc}(\lambda, L^\infty_z(0,1))$ and we have*

$$\partial_\lambda \epsilon_k = \langle |\chi_k|^2 \partial_\lambda V\rangle \qquad \textit{and} \qquad \partial_\lambda \chi_k = \sum_{\ell\neq k} \frac{\langle \chi_k\,\chi_\ell\,\partial_\lambda V\rangle}{\epsilon_k - \epsilon_\ell}\chi_\ell.$$

Thanks to this Lemma, we can improve the L^∞ estimate on the eigenvectors presented in Lemmata A.2 and A.4 of [8].

Lemma A.2 *Let $V \in L^2(0,1)$ such that $V \geq 0$, then the eigenvectors of the Schödinger operator satisfy*

$$\|\chi_k[V]\|_{L^\infty(0,1)} \leq C(1 + \|V\|^{1/2}_{L^2(0,1)}).$$

Corollary A.3 *Let V and $\widetilde{V}$ be two given nonnegative potentials in $L^2(0,1)$. Then there exists a nonnegative constant C such that*

$$|\epsilon_k[V] - \epsilon_k[\widetilde{V}]| \leq C(1 + \|V\|^{1/2}_{L^2_z(0,1)} + \|\widetilde{V}\|^{1/2}_{L^2_z(0,1)})\|V - \widetilde{V}\|_{L^2_z(0,1)}. \tag{A.1}$$

Remark A.4 *Compared with the Appendix of [8], the dependence of the eigenfunctions on the potential is sub-linear whereas we had an exponential dependence in the previous work.*

Proof. The result of Lemma 1 Chapter 1 of [28] provides :

$$\chi_k(z) = A_k \sin(\sqrt{2\epsilon_k} z) + 2\int_0^z \frac{\sin(\sqrt{2\epsilon_k}(z-t))}{\sqrt{2\epsilon_k}} V(t)\chi_k(t)\,dt, \tag{A.2}$$

where A_k is a nonnegative constant to be determined. Thanks to a Cauchy-Schwarz inequality, we deduced

$$\left|\int_0^z \frac{\sin(\sqrt{2\epsilon_k}(z-t))}{\sqrt{2\epsilon_k}} V(t)\chi_k(t)\,dt\right| \leq \frac{\int_0^1 V(t)|\chi_k(t)|\,dt}{\sqrt{2\epsilon_k}} \leq \frac{\langle|\chi_k|^2 V\rangle^{1/2}}{\sqrt{2\epsilon_k}}\|V\|_{L^2(0,1)}^{1/2}.$$

Moreover, from the Schrödinger equation we obtain

$$\epsilon_k = \frac{1}{2}\langle|\partial_z\chi_k|^2\rangle + \langle|\chi_k|^2 V\rangle \geq \langle|\chi_k|^2 V\rangle$$

Thus,

$$\left|\int_0^z \frac{\sin(\sqrt{2\epsilon_k}(z-t))}{\sqrt{2\epsilon_k}} V(t)\chi_k(t)\,dt\right| \leq \frac{1}{\sqrt{2}}\,\|V\|_{L^2(0,1)}^{1/2}. \tag{A.3}$$

Thus from (A.2) we have for all $z \in [0,1]$

$$|\chi_k(z)| \leq A_k + \sqrt{2}\,\|V\|_{L^2(0,1)}^{1/2}. \tag{A.4}$$

Now, we will use the condition $\|\chi_k\|_{L^2(0,1)} = 1$ to bound A_k. If we use the expression of χ_k (A.2) in the identity $\int_0^1 \chi_k^2\,dz = 1$, we obtain

$$1 \geq A_k^2 \int_0^1 \sin(\sqrt{2\epsilon_k}z)^2\,dz + 4A_k \int_0^1 \sin(\sqrt{2\epsilon_k}z) \int_0^z \frac{\sin(\sqrt{2\epsilon_k}(z-t))}{\sqrt{2\epsilon_k}} V(t)\chi_k(t)\,dtdz. \tag{A.5}$$

For the second term we have from (A.3)

$$\left|\int_0^1 \sin(\sqrt{2\epsilon_k}z) \int_0^z \frac{\sin(\sqrt{2\epsilon_k}(z-t))}{\sqrt{2\epsilon_k}} V(t)\chi_k(t)\,dtdz\right| \leq \frac{1}{\sqrt{2}}\|V\|_{L^2(0,1)}^{1/2}.$$

And we can calculate

$$\int_0^1 [\sin(\sqrt{2\epsilon_k}z)]^2\,dz = \frac{1}{2} - \frac{\sin(2\sqrt{2\epsilon_k})}{4\sqrt{2\epsilon_k}}.$$

Since $V \geq 0$, the Min-Max formula [22] implies $\epsilon_k[V] \geq \epsilon[0] = \frac{1}{2}\pi^2 k^2$, for all $k \geq 1$. Thus we can inject these remarks in (A.5), it leads to

$$1 \geq A_k^2\left(\frac{1}{2} - \frac{1}{4\pi}\right) - 2\sqrt{2}\,A_k\|V\|_{L^2(0,1)}^{1/2}.$$

This implies that there exists a nonnegative constant C such that

$$A_k \leq C(1 + \|V\|_{L^2(0,1)}^{1/2}), \quad \forall\, k \geq 1.$$

It remains to inject this last estimate in (A.4) to prove Lemma A.2.

Corollary A.3 is an easy consequence of the Hölder estimates and Lemma A.2 with the remark that

$$\epsilon_k[V] - \epsilon_k[\widetilde{V}] = \int_0^1 \partial_\lambda\epsilon_k(\lambda)\,d\lambda = \int_0^1 \langle|\chi_k[W(\lambda,\cdot)](z)|^2(V-\widetilde{V})\rangle\,d\lambda,$$

where we denote for $\lambda \in [0,1]$, $W(\lambda,z) = \widetilde{V} + \lambda(V-\widetilde{V})$ and $\epsilon_k(\lambda) = \epsilon_k[W(\lambda,\cdot)]$. □

2 Proof of Corollary 2.4

The first point is an easy consequence of our choice of boundary conditions. From Lemma 2.3 we deduce after an integration in time that

$$\forall\, t \in \mathbb{R}^+, \quad W(t) = W(0) - \int_0^t D(s)\, ds \leq W(0). \tag{A.6}$$

Assumption 1.2 on initial data insures that $W(0)$ is bounded independently on ε. In fact, solving the Schrödinger-Poisson system for a given N_s^0 in $L\log L(\omega)$ furnishes $V(t=0,\cdot,\cdot) \in H^1(\Omega)$ (see Proposition 3.1) and the Trudinger inequality implies that the product of a $L\log L(\omega)$ term with the square of a $H^1(\omega)$ term is bounded in $L^1(\omega)$.

We have proved in Section 2.2.1 of [8] that the decay of ρ_k with respect to k implies

$$\sum_k \rho_k(\epsilon_k - \epsilon_k[0] - \langle |\chi_k|^2 R^\varepsilon[V]\rangle) \geq 0. \tag{A.7}$$

And $\epsilon_k[0] = \pi^2 k^2/2$. Thus (A.6) implies that

$$\sum_k \int_\omega \rho_k(\frac{\pi^2 k^2}{2} + \log \rho_k)\, dx + \frac{1}{2}\iint_\Omega |\nabla_{x,z} V|^2\, dxdz \leq W(0). \tag{A.8}$$

Taking $K = \sum_k e^{-k^2}$, by the Jensen inequality we have

$$\begin{aligned} W(0) \geq \sum_k \int_\omega \rho_k(k^2 + \log\rho_k)\, dx \quad &= K \sum_k \int_\omega \frac{\rho_k}{e^{-k^2}}(\log \frac{\rho_k}{e^{-k^2}})\,\frac{e^{-k^2}}{K}\, dx \\ &\geq \int_\omega N_s \log(N_s/K)\, dx. \end{aligned}$$

Thus from (A.8),

$$\forall\, t \in \mathbb{R}^+, \quad \int_\omega N_s \log N_s\, dx \leq W(0) + \mathcal{N}_I \log K.$$

Then estimate (2.11) is a direct consequence of the remark that $\forall\, a > 0$, $a|\log a| \leq a\log a + 2/e$. In fact,

$$\begin{aligned} \sum_k \int_\omega \rho_k |\log \rho_k|\, dx \quad &\leq \sum_k \int_\omega \rho_k(|\log(\rho_k/e^{-k^2})| + k^2)\, dx \\ &\leq \sum_k \int_\omega \rho_k(\log(\rho_k/e^{-k^2}) + k^2)\, dx + \frac{2}{e}\sum_k \int_\omega e^{-k^2}\, dx \\ &\leq \sum_k \int_\omega \rho_k(\log\rho_k + 2k^2)\, dx + \frac{2}{e}|\omega| K, \end{aligned}$$

where $|\omega|$ denotes the Lebesgue measure of ω. Thus, we deduce from (A.8) that

$$\forall\, t \in \mathbb{R}^+, \quad \sum_k \int_\omega \rho_k |\log\rho_k|\, dx \leq W(0) + \frac{2}{e}|\omega| K. \tag{A.9}$$

Hence from (A.8), we obtain

$$\sum_k \int_\omega \rho_k k^2 \, dx + \frac{1}{2} \iint_\Omega |\nabla_{x,z} V|^2 \, dxdz \le 2W(0) + \frac{2}{e}|\omega| K.$$

For the point (iii), we have obtained in the proof of Proposition 3.1 of [8] the estimate

$$\forall\, t \in \mathbb{R}^+, \qquad \int_\omega |\nabla_x \sqrt{N_s}|^2 \, dx \le C(1 + D(t)),$$

for a nonnegative constant C. Since we have from Lemma 2.3 that $\int_0^t D(s)\, ds \le W(0)$, for all $t \in \mathbb{R}^+$, we obtain (2.13) and from the Gagliardo-Nirenberg inequality we have (2.14). □

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