THE BERKOVICH REALIZATION FOR RIGID ANALYTIC MOTIVES

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ABSTRACT. We prove that the functor associating to a rigid analytic variety the singular complex of the underlying Berkovich topological space is motivic, and defines the maximal Artin quotient of a motive. We use this to generalize Berkovich’s results on the weight-zero part of the étale cohomology of a variety defined over a non-archimedean valued field.

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1. Introduction

One of the key features of any motivic theory Mot(\k) over a field \k is the existence of realization functors, that is functors from Mot(\k) to some category of vector spaces (with further structure, if possible) that would produce and generalise regulator maps, comparison theorems and periods. In this paper, we will adopt the language of Voevodsky-Ayoub (mixed, derived, with coefficients in \Lambda) motives DA(\k, \Lambda) and we will mostly be interested to the case \Lambda \supset \Q so that adding transfers (and hence considering the categories DM(\k, \Lambda)) makes no difference in the theory (see [1] and [27]).

Whenever \k is a subfield of C one can consider the Betti realization (see [4]), the ℓ-adic realizations [1] or the de Rham realization [5] (possibly enriched, see [22]). The well-known comparison theorems show that they are all equivalent, up to a change of coefficients. Among other things, these functors can be used to define motivic Galois groups [5], and some formal properties of them (say, being conservative on compact objects) reflect some deep geometrical facts of the theory of algebraic varieties (see [2]). We remark that the Betti and the de Rham realizations can be extended to, and defined by means of the category of complex analytic motives AnDA(\C, \Lambda) (equivalent to D(\Lambda), see [4, Theorem 1.8]).

Whenever the characteristic of \k is positive, the array of possible realizations is more limited. There are ℓ-adic realizations (but comparison theorems are not present in full generality) constructed in [1]. For p-adic realizations, the classical technique would consist in associating to a variety (more generally, a motive) over \k a rigid analytic “variety” (better saying, a rigid analytic motive) over complete valued field \K of characteristic 0 and residue equal to \k, and then using realization functors for such objects. The problem is hence transferred into producing realization functors for rigid analytic motives RigDA(\K, \Lambda). In [28] we constructed a de Rham-like realization, giving rise to the rigid realization on DA(\k, \Lambda). In [8] we constuct
the \(\ell\)-adic realizations compatible with the ones of \(DA(k, \Lambda)\). In this article, we deal with a Betti-like realization functor.

The most naive approach, in analogy with the complex Betti realization, consists in considering the singular homology of the Berkovich space \(|X(C)|_{\text{Berk}}\) underlying the base change of rigid analytic variety \(X/K\) to a complete algebraically closed field \(C\). The first result of this paper is to show that this approach works, at least on the category of effective motives \(\text{RigDA}^{\text{eff}}(K, \Lambda)\) (see Theorem 2.14).

**Theorem 1.1.** There is a realization functor

\[ \mathbb{L}B^* : \text{RigDA}^{\text{eff}}(K, \Lambda) \to D(\Lambda) \]

such that, for any rigid analytic variety \(X\) and any \(n \in \mathbb{Z}\)

\[ H_n(\mathbb{L}B^* \Lambda(X)) \cong H_n^{\text{Sing}}(|X(C)|_{\text{Berk}}, \Lambda). \]

On the other hand, this (co)homology theory is unsatisfying for many aspects. Indeed, Berkovich spaces are “too much contractible”. For example, \(|G_m(C)|_{\text{Berk}}\) is (strongly) homotopically equivalent to a point, destroying therefore any information linked to monodromy and any hope to extend this realization to stable motives. On the other hand, some results of Berkovich [12] analogous to Tate’s conjecture, hint to the fact that this cohomology theory captures the weight-zero part of the other realizations, as he proves that for any algebraic variety \(X\) over a discretely valued \(K\) we have

\[ H_i^{\text{Sing}}(|X(C)|_{\text{Berk}}, \mathbb{Q}_\ell) \cong H_i^{\text{et}}(X, \mathbb{Q}_\ell)_0 \]

where the right hand side is the maximal sub-representation of \(H_i^{\text{et}}(X, \mathbb{Q}_\ell)\) on which (a lift of) Frobenius acts by roots of unity. The main result of this paper is to provide the following motivic interpretation/generalization of these formulas (see Theorem 3.10).

**Theorem 1.2.** Let \(K\) be a complete non-archimedean field and \(\Lambda\) be a \(\mathbb{Q}\)-algebra. The functor \(\mathbb{L}B^*\) can be enriched with a Galois action, so that \(\mathbb{L}B^* M\) is an Artin motive. Also, for any motive \(M \in \text{RigDA}^{\text{eff}}(K, \Lambda)\) there exists a canonical map

\[ M \to \mathbb{L}B^* M \]

which is universal among maps from \(M\) to an Artin motive.

As an application, we obtain in particular that the formula (1) holds true for any rigid analytic variety (or more generally, any compact rigid analytic motive) \(X\) and without any assumption on \(K\). We can also answer positively to a conjecture of Ivorra and Sebag [23].

In Section 2 we prove Theorem 1.1 also in its simplicial variant (without coefficients) and in Section 3 we prove Theorem 1.2. In Section 4 we show that the previous results are compatible with the motivic tilting equivalence of [26] defined whenever \(K\) is perfectoid, while in Section 5 we deduce the formulas (1) via the étale realization functors.

2. The Berkovich Realization

The aim of this first section is to define a functor from the category of additive motives of rigid analytic varieties \(\text{RigDA}^{\text{eff}}(K, \Lambda)\) to the derived category of \(\Lambda\)-modules, such that the complex associated to the motive of a variety computes its singular (co-)homology with coefficients in \(\Lambda\).

In order to define our functor, we will simply use the universal property of the categories of (effective, without transfers) motives, which we will now briefly recall in a more general setting for the convenience of the reader. All the results appear in [15] (for the simplicial case) and in [13] (for the general case) and we refer to these sources for definitions and proofs.
Definition 2.1. Let C be any small category. We can endow the category sPsh(C) of simplicial presheaves on C [resp. the category Ch Psh(C, Λ) of complexes of presheaves of Λ-modules on C] with the projective model structure, for which cofibrations and weak equivalences are defined point-wise. This defines a model category UC [resp. a Λ-enriched model category UdγC = UCh(Λ) C].

The Yoneda embedding C → Psh(C) can be composed with the functor Set → sSet sending any set to the constant simplicial set [resp. the functor Set → Λ-Mod sending a set to the free Λ-module attached to it]. This defines a Yoneda-like embedding y: C → UC [resp. y: C → UdγC] which is universal in the following sense.

Proposition 2.2 ([13]). Let γ: C → D be any functor, and suppose D is endowed with a [Λ-enriched] model category structure. There exists a Quillen functor L: UC → D [resp. a Quillen functor L: UdγC → D of Λ-enriched model categories] such that the induced triangle

\[ \begin{array}{ccc} C & \xrightarrow{\gamma} & D \\ \downarrow y & & \downarrow L \\ UC & & \end{array} \]

is commutative up to a weak equivalence L ∘ y ⇒ γ. Moreover L is unique up to a contractible choice.

Suppose now that C is endowed with a Grothendieck topology τ and a choice of an object I. Under some hypotheses, we can consider the Bousfield localization of UC and UdγC with respect to τ-hypercovers and I-homotopy, in the following sense.

Definition 2.3. Let T be the a dense set of τ-hypercovers. Consider the set of arrows S in UC [resp. UdγC] given by

\[ S = \{ \text{hocolim } h(U_\bullet)[i] \to h(X)[i]: (U_\bullet \to X) \in T, i \in \mathbb{Z} \} \cup \{ h(X \times I)[i] \to h(X)[i]: X \in C, i \in \mathbb{Z} \}. \]

The Bousfield localization of UC with respect to S will be denoted it by UC/(τ, I) [resp. UdγC/(τ, I)].

The model categories above still enjoy a universal property, by composing Proposition 2.2 with the universal property of localizations.

Proposition 2.4 ([13] Corollary 5.14). Let γ: C → D be any functor, and suppose D is endowed with a [Λ-enriched] model category structure. The Quillen functor L of Proposition 2.2 factors over UC/(τ, I) [resp. UdγC/(τ, I)] whenever γ(X × I) → γ(X) is a weak equivalence and hocolim γ(U_\bullet) = γ(X) for each τ-hypercover U_\bullet → X in C.

Remark 2.5. The construction of UC/(τ, I) is functorial on the triples (C, τ, I) in some suitable sense.

Remark 2.6. One may omit the choice of the object I and consider only the localization with respect to τ. In this case, the category Udγ(C)/(τ) [resp. Udγ(C)/(τ)] is Quillen equivalent to the categories of simplicial [resp. complexes of] sheaves over C endowed with its local model structure. In particular, one can replace C by a τ-dense full subcategory without changing the homotopy category. We remark then that the homotopy category HoUdγ(C)/(τ) is then equivalent to the (unbounded) derived category D(SH_S(C, Λ)) of sheaves of Λ-modules.

Example 2.7. Suppose we take C = Sm/k the category of smooth varieties over a field k. We can endow it with the étale topology and we can select I to be the affine line A^1_k. The
homotopy category \( \text{Ho}(U_d(Sm/k)/(\text{ét}, A^1)) \) is the category of unstable Voevodsky motives without transfers \( \text{DA}^{\text{eff}}(k, \Lambda) \).

**Example 2.8.** Suppose we take \( C = \text{RigSm}/K \) the category of smooth rigid analytic varieties over a non-archimedean field \( K \). We can endow it with the étale topology and we can select \( I \) to be the closed disc \( \mathbb{B}_K = \text{Spa} K(T) \). The homotopy category \( \text{Ho}(U_d(\text{RigSm}/K)/(\text{ét}, \mathbb{B}_K)) \) is the category of rigid analytic Ayoub motives (unstable, without transfers) \( \text{RigDA}^{\text{eff}}(K, \Lambda) \). Whenever \( K \) is perfect, we can also consider the \( \text{Frob}_\text{ét} \)-topology, that is the one generated by étale covers and the relative Frobenius maps. In this case we obtain the category \( \text{RigDA}^{\text{eff}}_{\text{Frob}_\text{ét}}(K, \Lambda) \).

**Example 2.9.** Consider \( C \) resp. \( C' \) to be the category of finite étale extensions resp. finite Galois extensions of a field \( K \) and endow them with the étale topology. The two homotopy categories are canonically equivalent to \( D \text{Sh}(\text{Et}/K, \Lambda) \) which we will denote by \( D_{\text{ét}}(K, \Lambda) \) following [6].

We recall the following classical statement.

**Proposition 2.10** ([6] Remark 1.21]). The category \( \text{Sh}_{\text{ét}}(K, \Lambda) \) is equivalent to the category of continuous \( \text{Gal}(C/K) \)-representations by means of the functor

\[
\sigma^*: \mathcal{F} \mapsto \lim_{L \subset C, L/K \text{ finite Galois}} \mathcal{F}(L).
\]

In particular, the category \( D_{\text{ét}}(K, \Lambda) \) is equivalent to the derived category of the (semi-simple) category of continuous \( \Lambda \)-representations of \( \text{Gal}(K^{\text{sep}}/K) \).

We now prove the existence of the simplicial version of the Berkovich realization. Here, we endow the category of topological spaces \( \text{Top} \) with the classical Quillen model structure, see [20].

**Proposition 2.11.** The functor \( X \mapsto |X|_{\text{Berk}} \) induces a Quillen adjunction \( \mathcal{U} \text{Psh}(\text{RigSm}/K)/(\text{Frob}_\text{ét}, I) \rightleftharpoons \text{Top} \).

**Proof.** We first prove that \( B^* \) sends the maps \( U_* \rightarrow X \) to weak equivalences, for any étale hypercover \( U_* \) of \( X \). To this aim, by [16] Theorem 8.6] it suffices to show that \( \bigsqcup_i U_i|_{\text{Berk}} \cong \bigsqcup_i |U_i|_{\text{Berk}} \) for any finite set of rigid varieties \( \{U_i\}_{i \in I} \) (which is obvious) and that \( \mathbb{L}B^*(\text{hocolim} U_*) \cong \mathbb{L}B^*X \) where \( U_* \) is a split basal étale hypercover of \( X \). This means in particular that:

(i) \( U_0 \) is representable, and \( U_0 \rightarrow X \) is étale surjective.

(ii) There exists a representable presheaf \( N_k \) for each \( k \) such that \( U_i = \bigsqcup_\sigma N_\sigma \) where \( \sigma \) runs among surjections \( [i] \rightarrow [k] \) in the simplex category \( \Delta \) and \( N_\sigma \) is a copy of \( N_k \).

We warn the reader that we constantly make abuse of notation by indicating with \( U \) both a space and the presheaf it represents.

As we already remarked, the functor \( B^* \) preserves coproducts. In particular, the simplical object \( |U_*|_{\text{Berk}} \) is also a split simplicial topological space, that is, it enjoys property (ii) with “topological space” in place of “representable presheaf”.

The homotopy colimit functor is the left derived Quillen functor of the \( \text{colim} \) functor, targeting the category \( \text{Top} \), endowed with usual Quillen model structure, from the diagram category \( s\text{Top} \), endowed with the induced Reedy model structure. By the ”topological trick” of Dugger-Isaksen [17] Theorem A.7] this homotopy colimit is weakly equivalent to the one computed with respect to the Strøm model category structure on \( \text{Top} \) (for which all objects are fibrant and cofibrant, and weak equivalences are \textit{strong} homotopy equivalences) and the induced Reedy model structure on \( s\text{Top} \).
Since \( |U_\bullet|_{\text{Berk}} \) is split, we deduce that it is Strøm-cofibrant in \( s\text{Top} \). In particular, its homotopy colimit coincides with its colimit, which is in turn isomorphic to

\[
\text{coeq} \left( |U_1|_{\text{Berk}} \rightrightarrows |U_0|_{\text{Berk}} \right)
\]

On the other hand, by definition of a hypercover, the map \( |U_1|_{\text{Berk}} \to |U_0 \times_X U_0|_{\text{Berk}} \) is surjective. The same is true for the map \( |U_0 \times_X U_0|_{\text{Berk}} \to |U_0|_{\text{Berk}} \times |X|_{\text{Berk}} \to |U_0|_{\text{Berk}} \). We deduce that the maps of the diagram above factor over \( E := |U_0|_{\text{Berk}} \times |X|_{\text{Berk}} \to |U_0|_{\text{Berk}} \). The map \( U_0 \to X \) is an étale cover, and hence \( |U_0|_{\text{Berk}} \to |X|_{\text{Berk}} \) is a quotient map of topological spaces (see [11, Lemma 5.11]), with respect to the equivalence relation \( E \). The coequalizer above is then simply \( |U_0|_{\text{Berk}} / E = |X|_{\text{Berk}} \) as wanted.

We now prove that \( B^* \) sends the relative Frobenius maps \( X^{(1)} \to X \) to weak equivalences. This follows at once since the relative Frobenius map induces actually a homeomorphism \( |X^{(1)}| \cong |X| \). This implies that the functor \( B^* \) factors over the \( \text{Frob}^\bullet \)-localization.

We now claim that the left derived functor \( \mathbb{L}B^* \) sends the maps \( \pi_X : X \times \mathbb{B}^1 \to X \) to isomorphisms, i.e. that \( |X \times \mathbb{B}^1|_{\text{Berk}} \to |X|_{\text{Berk}} \) is a weak equivalence in \( \text{Top} \). This follows from Berkovich’s results about the contractibility of the disc [10, Theorem 6.1.4] and [11, Corollary 8.7(ii)].

**Remark 2.12.** We denote by \( |X| \) the topological space underlying a rigid analytic variety (following Huber). If \( U \to X \) is an étale cover, then the map \( |U| \to |X| \) is open and surjective, and hence a quotient map. This shows that the functor \( |\cdot| \) induces a Quillen functor from \( s\text{Psh}(\text{RigSm}/K) \) to \( \text{Top} \) factoring over the étale realization. This fact, together with the Quillen equivalence induced by the weak-Hausdorff quotient functor on Kelley spaces, can be used as an alternative of the result of Berkovich [11, Lemma 5.11] in the proof of Proposition 2.11. On the other hand, the topological space \( |\mathbb{B}^1| \) is not weakly contractible, so that the singular homology of such spaces may not have the same good properties as the associated Berkovich spaces, which are their (weak) Hausdorffification.

**Remark 2.13.** Any Nisnevich square induces a cover of Huber spaces which admits locally a section (see [7, Remark 1.2.4]). The proof of the Nisnevich descent is therefore simpler, and it substantially coincides with the archimedean version of Proposition 2.11 proven in [17].

**Theorem 2.14.** Let \( \Lambda \) be a ring. There is a Quillen adjunction

\[
B^*: (\text{Ch Psh}(\text{RigSm}/K, \Lambda))/(\text{Frob}^\bullet, \mathbb{B}^1) \rightleftarrows \text{Ch}(\Lambda) : B_*
\]

inducing an adjunction:

\[
\mathbb{L}B^*: \text{RigDA}_{\text{Frob}^\bullet}(K, \Lambda) \rightleftarrows \text{D}(\Lambda) : \mathbb{R}B_* .
\]

Moreover, for any rigid analytic variety \( X \) and any \( n \in \mathbb{Z} \) we have

\[
\mathbb{L}B^* \Lambda(X) \cong C_{\text{Sing}}(|X|_{\text{Berk}}, \Lambda)
\]

where \( C_{\text{Sing}} \) denotes the singular complex. In particular

\[
H_n(\mathbb{L}B^* \Lambda(X)) \cong H^n_{\text{Sing}}(|X|_{\text{Berk}}, \Lambda).
\]

**Proof.** We consider the functor \( B : \text{RigSm}/K \to \text{Ch}(\Lambda) \) given by \( X \mapsto C_{\text{Sing}}(|X|_{\text{Berk}}, \Lambda) \). It induces a Quillen adjunction from \( \mathcal{U}_{dg} \text{RigSm}/K \to \text{Ch}(\Lambda) \) and we want to show that it factors over the (\( \text{Frob}^\bullet, \mathbb{B}^1 \))-localization.

The functor \( \text{Sing} : \text{Top} \to s\text{Set} \) mapping each topological space to its singular simplicial complex is an part of an exact Quillen equivalence of model categories. We then deduce from Proposition 2.11 that the functor \( \hat{B} : X \mapsto \text{Sing}(|X|_{\text{Berk}}) \) induces a Quillen adjunction \( \mathcal{U} \text{RigSm}/K \rightleftarrows s\text{Set} \) factoring over the (\( \text{Frob}^\bullet, \mathbb{B}^1 \))-localization.
We now consider the left Quillen functor $N\Lambda: s\text{Set} \to \text{Ch}(\Lambda)$ induced by the composition of the $\Lambda$-enrichment $s\text{Set} \to s\Lambda\text{-Mod}$ followed by the Dold-Kan functor $s\Lambda\text{-Mod} \to \text{Ch}(\Lambda)$.

It gives rise to the following commutative diagram of left Quillen functors:

$$
\begin{array}{ccc}
U \text{RigSm}/K & \xrightarrow{\tilde{B}} & s\text{Set} \\
\downarrow & & \downarrow_{N\Lambda} \\
U_{dg} \text{RigSm}/K & \xrightarrow{B} & \text{Ch}(\Lambda)
\end{array}
$$

Since the functor on top factors over the (Frobéret, $B^1$)-localization, we deduce that the bottom functor does as well. On the other hand, we remark that this functor is the one induced by mapping an object $X$ of RigSm/$K$ to $N\Lambda(\text{Sing}(|X|_{\text{Berk}}))$ which is canonically isomorphic to $B(X)$ therefore proving our claim. □

Remark 2.15. By replacing simplicial sets with spectra in the construction of $UC$ one can deduce from the previous result the existence of the Berkovich realization

$$LB^*: \text{RigSH}^{\text{eff}}(K) \rightleftharpoons \text{SH}:\mathbb{R}B,$$

for the category of effective (anabelian) motives $\text{RigSH}^{\text{eff}}(K)$ (see [7, Definition 1.3.2]) with values in the stable homotopy category $\text{SH}$ (see [18]).

Having defined a realization from a category of motives, it is natural to see what sort of cohomology theory arises from it. As a matter of fact, this cohomology theory turns out to be quite pathological, as the two following remarks explain.

Remark 2.16. The Quillen pair of Theorem 2.14 does not descend to the stable category of motives, the one constructed by inverting the Tate twist in a universal way. Indeed, the object defining the Tate twist $[T^1*/*]$ is mapped to the zero complex, since $* \to |T^1|_{\text{Berk}}$ is a homotopy equivalence.

Remark 2.17. By means of [7] we can define the category $\text{FormSH}^{\text{eff}}_0(K^\circ)$ of motives of formal schemes over $K^\circ$. The special fiber functor and the generic fiber functor defines Quillen pairs

$DA^{\text{eff}}(k, \Lambda) \xleftarrow{\sim} \text{FormDA}^{\text{eff}}(K^\circ, \Lambda) \to \text{RigDA}^{\text{eff}}(K, \Lambda)$

the first one being an equivalence by [7]. By composition, we obtain in particular a cohomological realization

$$DA^{\text{eff}}(k, \Lambda) \xrightarrow{\sim} \text{FormDA}^{\text{eff}}(K^\circ, \Lambda) \to \text{RigDA}^{\text{eff}}(K, \Lambda) \xrightarrow{LB^*} \text{D}(\Lambda)$$

which is surprising at first sight: it looks as if it defined a cohomology with $\mathbb{Z}$-coefficients for varieties in positive characteristic (by taking $\Lambda = \mathbb{Z}$). This is not quite the case. Indeed, since rigid analytic varieties with good reduction are contractible, the composite realization (2) coincides with the one induced by the functor mapping a connected smooth variety $\bar{X}$ to the trivial topological space. In particular, the homology theory on connected smooth algebraic varieties over $k$ obtained through the composite realization (2) is

$$H_*(X) = \begin{cases} 
\Lambda & \text{if } i = 0 \\
0 & \text{if } i > 0
\end{cases}$$

3. THE BERKOVICH REALIZATION AS THE MAXIMAL ARTIN QUOTIENT

From now on, we make the following assumption:

Assumption 3.1. We suppose that $\Lambda$ is a $\mathbb{Q}$-algebra. Fix a separable closure $K^{\text{sep}}$ of $K$ and let $C$ be its completion.
The first aim of this section is to enrich the realization constructed in Theorem 2.14 into a functor taking values in Galois representations.

We recall some crucial results of Berkovich on the singular cohomology of Berkovich spaces that we list below.

**Proposition 3.2.** Let $X$ be a smooth quasi-compact rigid analytic variety over $K$.

1. $H^i_{\text{Sing}}(|X|_{\text{Berk}}, \Lambda) \cong H^i(|X|_{\text{Berk}}, \Lambda)$ and they have finite dimension.
2. There is a finite Galois extension $L$ such that $H^i(|X_L|_{\text{Berk}}, \Lambda) \cong H^i(|X_L'|_{\text{Berk}}, \Lambda)$ for each field extension $L'|L$.
3. If $L/K$ is a finite Galois extension then $|X|_{\text{Berk}} \cong \text{Gal}(L/K)\backslash |X_L|_{\text{Berk}}$ and $H^i(|X|_{\text{Berk}}, \Lambda) \cong H^i(|X_L|_{\text{Berk}}, \Lambda)^{\text{Gal}(L/K)}$.
4. $|X|_{\text{Berk}} \cong \text{Gal}(C/K)\backslash |X_L|_{\text{Berk}}$ and $\text{Gal}(C/K)$ acts continuously on the $\Lambda$-module $H^i(|X_C|_{\text{Berk}}, \Lambda)$.
5. If $X$ has good reduction, then $|X|_{\text{Berk}}$ is contractible. In particular $H^i_{\text{Sing}}(|X|_{\text{Berk}}, \Lambda) = 0$ if $i > 0$.

**Proof.** The first statement follows from [11 Corollary 9.6]. If $X$ is smooth, it is locally isomorphic to $\text{Spa} K<\tau_1, \ldots, \tau_n>/ \langle p_1, \ldots, p_n \rangle$ with $p_i$ polynomials in $K[\tau_i]$. In particular, it is an open subvariety of the analytification of $\text{Spec} \langle \tau_1, \ldots, \tau_n>/ \langle p_1, \ldots, p_n \rangle$. We then apply [11 Theorem 10.1] to get the second point. The third point follows from [10, Proposition 1.3.5] and [19, Paragraph 5.3] while the fourth statement follows from [10, Corollary 1.3.6] and the previous points. The fifth point is proved in [11, Section 5].

**Definition 3.3.** For any smooth affinoid variety $X$ and any $F$ finite étale over $K$, we remark that the complex of $\Lambda$-modules $C_{\text{Sing}}(|X_F|_{\text{Berk}}, \Lambda)$ comes equipped with a canonical continuous action of $\text{Gal}(C/K)$. Since $\mathbb{Q} \subset \Lambda$ it is a complex of acyclic Galois representations. We denote with $B_F(X)$ the induced object of $\text{Ch Sh}_{\text{ét}}(\text{Et}/K, \Lambda)$:

$$B_F(X) : \text{Spa} L \mapsto C_{\text{Sing}}(|X_F|_{\text{Berk}}, \Lambda)^{\text{Gal}(C/L)}.$$ We finally define $B_C(X)$ to be $\text{holim}_{F \in \text{Et}/K} B_F(X)$ in $\text{Ch Psh}(\text{Gal}/K, \Lambda)$. We denote with the same symbol the corresponding object in $\text{Ch Sh}_{\text{ét}}(\text{Gal}/K, \Lambda)$.

**Remark 3.4.** In the definition of $B_C(X)$ we use a homotopy limit over the complexes obtained with finite Galois extensions rather than taking the singular complex of $|X_C|$. This is akin to the situation considered by Quick [25, Section 3].

**Remark 3.5.** The object $B_C(X)$ is a homotopy limit of the étale-fibrant complexes $B_F(X)$ (they are levelwise Galois-acyclic) hence it is also étale fibrant. We deduce that it can also be computed directly in $\text{Ch Sh}_{\text{ét}}(\text{Gal}/K, \Lambda)$ as a homotopy limit of the complexes of sheaves $B_F(X)$.

**Remark 3.6.** Let $L$ be in $\text{Gal}/K$. The functor $\text{ev}_L : \mathcal{F} \mapsto \mathcal{F}(L)$ from $\text{Ch Psh}(\text{Gal}/K, \Lambda)$ to $\text{Ch}(\Lambda\text{-Mod})$ is exact and preserves homotopy limits. Moreover, the collection of functors $\{\text{ev}_L\}_{L \in \text{Gal}/K}$ reflects the weak equivalences (that is, a map $\mathcal{F} \to \mathcal{G}$ is a weak equivalence if and only if all maps $\mathcal{F}(L) \to \mathcal{G}(L)$ are quasi-isomorphisms). This follows from the very definition of the projective module structure that we put on $\text{Ch Psh}(\text{Gal}/K, \Lambda)$.

**Proposition 3.7.** The sheaf $B_{\text{Gal}(K)}(X)$ corresponds to the continuous Galois representations $H^i_{\text{Sing}}(|X_C|_{\text{Berk}}, \Lambda)$ by means of the equivalence given in Proposition 2.10.

**Proof.** By construction we have

$$B_C(X)(L) = (\text{holim}_F B_F(X))(L) \cong \text{holim}_F(B_F(X)(L)) \cong \text{holim}_F(C_{\text{Sing}}(|X_F|)^{\text{Gal}(L/K)}).$$
whose homology is by [24 Theorem 3.15] and isomorphic to \( \lim_F H_i(|X_F|)^{\text{Gal}(C/L)} \cong H_i(|X_L|) \cong H_i(|X_C|)^{\text{Gal}(C/L)} \).

We are now ready to enrich the Berkovich realization with a Galois action.

**Proposition 3.8.** The functor \( B_{\text{Gal}(K)} : \text{Aff Sm}/K \to \text{Ch Sh}_{\text{et}}(K, \Lambda) \) induces a Quillen adjunction

\[
B_{\text{Gal}(K)}^* : \left( \text{Ch Psh}(\text{RigSm}/K, \Lambda) \big/ (\text{Frobét, } \mathbb{B}^1) \right) \rightleftarrows \text{Ch Sh}_{\text{et}}(K, \Lambda) : B_{\text{Gal}(K)^*}
\]

and hence an adjunction:

\[
\mathbb{L} B_{\text{Gal}(K)}^* : \text{RigDA}_{\text{eff}}(K, \Lambda) \rightleftarrows D_{\text{et}}(K, \Lambda) : \mathbb{R} B_{\text{Gal}(K)^*}.
\]

**Proof.** By Proposition 2.4, it suffices to prove that the functor

\[
B_{\text{Gal}(K)}^* : \mathcal{U}_{dg} \text{RigSm}/K \to \text{Ch Psh}(\text{Gal}/K, \Lambda) \cong \mathcal{U}_{dg}(\text{Gal}/K)
\]

sends the maps in the set \( S \) of Definition 2.3 to weak equivalences. By Remark 3.6 we can fix a Galois extension \( L/K \) and check that the composite functor

\[
\mathcal{U}_{dg} \text{RigSm}/K \to \text{Ch Psh}(\text{Gal}/K, \Lambda) \xrightarrow{\text{ev}} \text{Ch}(\Lambda)
\]

sends the maps in the set \( S \) to weak equivalences. On the other hand, we remark that by definition the functor above is the one induced by \( X \mapsto \text{holim}_F C_{\text{Sing}}(|X|, \Lambda)^{\text{Gal}(C/L)} \). This last complex is canonically quasi-isomorphic to \( C_{\text{Sing}}(|X|, \Lambda) \) as the following sequence of isomorphisms shows, where we let for simplicity \( F \supset L \) (we repeatedly use the hypothesis \( \mathbb{Q} \subset \Lambda \) and Proposition 3.2):

\[
H_i(C_{\text{Sing}}(|X|, \Lambda)^{\text{Gal}(C/L)}) \cong H_{i}^{\text{Sing}}(|X|, \Lambda)^{\text{Gal}(C/L)} \cong H_{i}^{\text{Sing}}(|X|, \Lambda)^{\text{Frobét, } \mathbb{B}^1} \cong H_{i}^{\text{Sing}}(|X|, \Lambda)^{C/L} \cong H_{i}^{\text{Sing}}(|X|, \Lambda).
\]

We already proved that the functor \( X \mapsto C_{\text{Sing}}(|X|, \Lambda) \) factors over the \( (\text{Frobét, } \mathbb{B}^1) \)-localization in Theorem 2.14 hence the set \( S \) is sent to weak equivalences, as claimed.

We recall that there is another adjunction between the categories above but defined in the opposite direction: it is the pair induced by the inclusion of the small site into the big site \( \iota : \text{Et}/K \to \text{Aff Sm}/K \) giving rise to:

\[
\mathbb{L} \iota^* : D_{\text{et}}(K, \Lambda) \rightleftarrows \text{RigDA}^{\text{eff}}(K, \Lambda) : \mathbb{R} \iota^*
\]

**Definition 3.9.** The objects in the essential image of \( \mathbb{L} \iota^* \) are called Artin motives, and the full subcategory they form is denoted by \( \text{RigDA}^{\text{eff}}(K, \Lambda)_0 \).

**Theorem 3.10.** Let \( K \) be a complete non-archimedean field and \( \Lambda \) be a \( \mathbb{Q} \)-algebra. The inclusion of Artin motives in effective rigid analytic motives over \( K \)

\[
\text{RigDA}^{\text{eff}}(K, \Lambda)_0 \subset \text{RigDA}^{\text{eff}}(K, \Lambda)
\]

admits a left adjoint \( \omega_0 := \mathbb{L} \iota^* \circ \mathbb{L} B_{\text{Gal}(K)^*} \). In particular, for any motive \( M \) the map

\[
M \to \omega_0 M
\]

is universal among maps from \( M \) to an Artin motive.

Differently put, we want to prove the following result.

**Theorem 3.11.** Let \( K \) be a complete non-archimedean field and \( \Lambda \) be a \( \mathbb{Q} \)-algebra. The functor \( \mathbb{L} B_{\text{Gal}(K)^*} \) is a left adjoint to the functor \( \mathbb{L} \iota^* \) and the unit map \( \mathbb{L} B_{\text{Gal}(K)} \mathbb{L} \iota^* \Rightarrow \text{id} \) is invertible.
Proof of Theorem 3.10 from Theorem 3.11. By definition, the category $\text{RigDA}^{\text{eff}}(K, \Lambda)_0$ is the essential image of $\mathbb{L}\iota^*$ which is fully faithful given that $\mathbb{L}\mathbb{B}_{\text{Gal}(K)}^* \mathbb{L}\iota^* \cong \text{id}$. The two categories are then equivalent, and the adjunction pair of Theorem 3.10 can be deduced from the one of Theorem 3.11. □

Remark 3.12. The content of the previous results does not lie in the existence of a left adjoint functor $\omega_0$ which could be proved with purely categorical methods (see [9, Chapter 5]) but rather in its explicit description through Berkovich spaces. This produces interesting applications, see Section 5.

We prove Theorem 3.11 in several steps. We start by checking the last claim.

Proposition 3.13. There is an invertible natural transformation $\mathbb{L}\mathbb{B}_{\text{Gal}(K)}^* \circ \mathbb{L}\iota^* \cong \text{id}$.

Proof. Let $L$ be a fixed finite Galois extension of $K$. The object $\mathbb{L}(\mathbb{B}_{\text{Gal}(K)}^* \circ \iota)^*$ is the following complex:

$$\text{holim}_F C_{\text{Sing}}(|\text{Spa}(L \otimes_K F)|, \Lambda) \cong \text{holim}_F C_{\text{Sing}}(\bigsqcup_{\text{Hom}(L, F)} *, \Lambda) \cong \bigoplus_{\text{Hom}(L, C)} \Lambda$$

which is canonically isomorphic, as a Galois representation, to $\lim_{\rightarrow_F} \Lambda(\text{Spec } L)(F)$ hence the claim. □

We recall that an object $X$ of a triangulated category is compact if $\text{Hom}(X, -)$ commutes with direct sums.

Proposition 3.14. Let $F: T \to T'$ and $G: T' \to T$ be triangulated functors commuting with direct sums between triangulated categories $T$ and $T'$ generated (as triangulated categories with small sums) by a set of compact objects $K$ and $K'$ respectively. Suppose that $F(X)$ is compact for each $X \in K$ and that there is an invertible transformation $F \circ G \cong \text{id}$. In order to prove that $F$ is a left adjoint to $G$ it suffices to prove

(3) $\text{Hom}(X[n], Y) \cong \text{Hom}(FX[n], Y)$

where $n$ varies in $\mathbb{Z}$ and where $X$ and $Y$ vary in $K$ and $K'$ respectively.

Proof. The invertible transformation gives rise to a bi-functorial map

$$\text{Hom}(X, Y') \to \text{Hom}(FX, FGY') \cong \text{Hom}(FX, Y).$$

We want to show it is invertible for all $X$ and $Y$ by knowing it is invertible for a set of compact generators of the two categories, and their shifts.

Fix an object $X$ in the chosen class of compact generators $K$ and let $C$ be the full subcategory of $T'$ whose objects $Y$ are such that (3) is invertible for all $n$. Let $Y_1$ and $Y_2$ be in $C$ and pick a distinguished triangle

$$Y_1 \to Y_2 \to C \to$$

By the map of long exact sequences

$$\text{Hom}(X, Y_1) \to \text{Hom}(X, Y_2) \to \text{Hom}(X, GC) \to \text{Hom}(X, GY_1[1]) \to$$

$$\text{Hom}(FX, Y_1) \to \text{Hom}(FX, Y_2) \to \text{Hom}(FX, C) \to \text{Hom}(FX, Y_1[1]) \to$$
we deduce that \( C \) is also in \( C \). Let now \( \{ Y_i \}_{i \in I} \) be a class of objects in \( C \). As \( F \) maps compact objects to compact objects, and both \( F \) and \( G \) commute with direct sums we deduce:

\[
\operatorname{Hom}(X, G \bigoplus Y_i) \cong \operatorname{Hom}(X, \bigoplus G Y_i) \cong \bigoplus \operatorname{Hom}(X, G Y_i)
\]

\[
\cong \bigoplus \operatorname{Hom}(FX, Y_i) \cong \operatorname{Hom}(FX, \bigoplus Y_i).
\]

We have then showed that \( C \) is closed both under direct sums and under cones, and it contains a family of generators for \( T' \) and hence it coincides with it.

We have then showed that for a class of compact generators \( X \), the functor \( Y \mapsto \operatorname{Hom}(X, G Y) \) is corepresentable by \( FX \). It suffices to invoke \([26, \text{Lemma 5.6}]\) to conclude.

\[\square\]

**Proposition 3.15.** Suppose \( \mathbb{Q} \subset \Lambda \). The object \( \Lambda[n] \) is \( \mathbb{B}_1 \)-local in \( \operatorname{ChSh}_\text{rig}(\text{RigSm}/K, \Lambda) \) and for any motive \( \Lambda(X) \) of a smooth rigid analytic variety \( X \), we have \( \operatorname{Hom}(\Lambda(X), \Lambda[n]) \cong H^n_{\text{Sing}}(|X|_{\text{Berk}}, \Lambda) \).

**Proof.** The fact that \( H^n_{\text{et}}(X, \Lambda) = H^n(|X|, \Lambda) \) follows from \([14, \text{Remark 4.2.6-1}]\). By overconvergence \([21, \text{Proposition 8.2.6}]\) we obtain \( H^n(|X|, \Lambda) \cong H^n(|X|_{\text{Berk}}, \Lambda) \) which coincides with its singular cohomology. We already proved the homotopy invariance of singular cohomology in Proposition \(2.11\).

\[\square\]

We are finally ready to prove Theorem 3.11.

**Proof of Theorem 3.11.** The functors \( \mathbb{L} \ell^* \) and \( \mathbb{L}B^*_{\text{Gal}(K)} \) send compact objects to compact objects and commute with direct sums. By means of Propositions \(3.13, 3.14\) and \([7, \text{Theorem 2.5.34}]\) it suffices to show that

\[
\operatorname{Hom}(X[n], \mathbb{L} \ell^* Y) \cong \operatorname{Hom}(\mathbb{L}B^*_{\text{Gal}(K)} X[n], Y)
\]

whenever \( X \) is a connected rigid analytic variety of potentially good reduction and \( Y \) is \( \operatorname{Spec} K' \) is Galois over \( \text{Spa} \ K \). We can consider the Quillen adjunction

\[
\mathbb{L}e^*_{K'/K'} : \text{RigDA}_{\text{eff}}(K, \Lambda) \rightleftarrows \text{RigDA}_{\text{eff}}(K', \Lambda) : \mathbb{R}e_{K'/K}'
\]

arising from the base change functor \( \text{RigSm}/K \rightarrow \text{RigSm}/K' \). From the equivalences

\[
\operatorname{Hom}(X[n], \mathbb{L} \ell^* Y) \cong \operatorname{Hom}(\mathbb{L}e_{K'/K} X[n], \mathbb{L}e_{K'/K}' \mathbb{L} \ell^* Y)^{\text{Gal}(K'/K)}
\]

\[
\cong \operatorname{Hom}(\mathbb{L}e_{K'/K} X[n], \mathbb{L} \ell^* \mathbb{L}e_{K'/K}' Y)^{\text{Gal}(K'/K)}
\]

and

\[
\operatorname{Hom}(\mathbb{L}B^*_{\text{Gal}(K)} X[n], Y) \cong \operatorname{Hom}(\mathbb{L}e_{K'/K}' \mathbb{L}B^*_{\text{Gal}(K)} X[n], \mathbb{L}e_{K'/K}' Y)^{\text{Gal}(K'/K)}
\]

\[
\cong \operatorname{Hom}(\mathbb{L}B^*_{\text{Gal}(K')} \mathbb{L}e_{K'/K}' X[n], \mathbb{L}e_{K'/K}' Y)^{\text{Gal}(K'/K)}
\]

we then deduce that we can prove \( (3) \) up to a finite Galois extension of the base field. In particular, we can assume \( X \) has good reduction and that \( Y \cong \Lambda^\oplus N \) or even \( Y \cong \Lambda \).

By \([11, \text{Section 5}]\) we know that \( |X|_{\text{Berk}} \) is contractible, hence \( \mathbb{L}B^* \Lambda(X_{\bar{K}}) = \Lambda[0] \). We are then left to prove that \( \operatorname{Hom}(\Lambda(X)[n], \Lambda) \) is \( \Lambda \) if \( n = 0 \) and \( 0 \) otherwise. This follows from Proposition 3.15.

\[\square\]

We recall once more that an object \( X \) of a triangulated category is compact if \( \operatorname{Hom}(X, -) \) commutes with direct sums. Examples of compact objects in \( \text{RigDA}_{\text{eff}}(K, \Lambda) \) are motives of quasi-compact smooth rigid analytic varieties over \( K \) and motives attached to the analytification of smooth algebraic varieties over \( K \). The full subcategory of compact objects in a category \( T \) will be denoted by \( T^{\text{rep}} \).
Proposition 3.16. The adjunction of Theorem 3.10 restricts to compact objects defining a left adjoint functor

$$\omega_0 : \text{RigDA}^{\text{eff}}(K, \Lambda)^{cp} \to \text{RigDA}_{\text{Frob}^\text{et}}^{\text{eff}}(K, \Lambda)_0^{cp}$$

to the inclusion functor.

Proof. It suffices to show that the functors $$\mathbb{L}^*$$ and $$\mathbb{L}B_{\text{Gal}(K)}^*$$ send a set of compact generators of the two categories to compact objects. For $$\mathbb{L}^*$$ this is immediate. For $$\mathbb{L}B_{\text{Gal}(K)}^*$$ this follows from Propositions 3.2 and 3.7. □

Remark 3.17. The functors $$\mathbb{L}B_{\text{Gal}(K)}^*$$ and $$\omega_0$$ defined above are tensorial, with respect to the monoidal structure on rigid analytic motives (see [3, Propositions 4.2.76 and 4.4.63]). Indeed, it suffices to check that for two rigid analytic varieties $$X$$ and $$Y$$ over $$K$$ the singular complex with $$\mathbb{Q}$$-coefficients $$C_{\text{Sing}}(|X \times Y|_C)$$ is quasi-isomorphic to $$C_{\text{Sing}}(|X|_C) \otimes C_{\text{Sing}}(|Y|_C)$$. This follows from [11, Corollary 8.7] and the usual Künneth formula for singular homology.

4. Compatibility with the Tilting Equivalence

Suppose now that $$K$$ is a perfectoid field of characteristic 0 and $$\mathbb{Q} \subset \Lambda$$. Under such hypothesis, we can define a perfect non-archimedean complete field $$K^b$$ (the tilt of $$K$$) and construct a ”motivic tilting equivalence” (see [26]):

$$\text{RigDA}^{\text{eff}}_{\text{ét}}(K, \Lambda) \cong \text{PerfDA}^{\text{eff}}_{\text{ét}}(K, \Lambda) \cong \text{PerfDA}^{\text{eff}}_{\text{ét}}(K^b, \Lambda) \cong \text{RigDA}^{\text{eff}}_{\text{Frob}^\text{et}}(K^b, \Lambda)$$

On the other hand, the category $$\text{D}_{\text{ét}}(K, \Lambda)$$ is equivalent to $$\text{D}_{\text{ét}}(K^b, \Lambda)$$ by means of the functor that associates to a (perfectoid) finite étale extension $$L/K$$ the extension $$L^b/K^b$$ (this is the classic Fontaine and Wintenberger theorem in its motivic form). We now specify that the two equivalences above are compatible with each other, and also to the Berkovich realization defined above.

Proposition 4.1. Let $$K$$ be a perfectoid field and let $$\Lambda$$ be a $$\mathbb{Q}$$-algebra. The functor $$\omega_0$$ commutes with the tilting equivalence.

Proof. By means of the adjunction property, we can alternatively prove that the following diagram is commutative

$$
\begin{array}{ccc}
\text{D}_{\text{ét}}(K, \Lambda) & \xrightarrow{\mathbb{L}^*} & \text{RigDA}^{\text{eff}}(K, \Lambda) \\
\sim & & \sim \\
\text{D}_{\text{ét}}(K^b, \Lambda) & \xrightarrow{\mathbb{L}^*} & \text{RigDA}^{\text{eff}}_{\text{Frob}^\text{et}}(K^b, \Lambda)
\end{array}
$$

for a perfectoid field $$K$$ of characteristic zero with tilt $$K^b$$.

We will now decompose this diagram in some sub-squares. We recall that the equivalence $$\text{PerfDA}^{\text{eff}}(K, \Lambda) \cong \text{RigDA}^{\text{eff}}_{\text{Frob}^\text{et}}(K, \Lambda)$$ is obtained as the composite of the two functors

$$\text{PerfDA}^{\text{eff}}(K, \Lambda) \xrightarrow{\mathbb{L}j^*} \text{RigDA}^{\text{eff}}_{\text{ét}}(K, \Lambda) \xrightarrow{\mathbb{L}i^*} \text{RigDA}^{\text{eff}}(K, \Lambda)$$

where the category in the middle is the category of semi-perfectoid motives (see [26]) the functor $$\mathbb{L}j^*$$ is induced by the inclusion of smooth perfectoid spaces inside smooth semi-perfectoid spaces, while $$\mathbb{L}i^*$$ is the left adjoint of the functor $$\mathbb{L}i^*$$ induced by the inclusion of smooth rigid analytic varieties inside smooth semi-perfectoid spaces.
First, we consider the diagram

\[
\begin{array}{ccc}
\text{RigDA}^{\text{eff}}(K, \Lambda) & \xrightarrow{L^*_t} & \text{D}_{\text{ét}}(K, \Lambda) \\
& & \downarrow \sim \\
\text{PerfDA}^{\text{eff}}(K, \Lambda) & \xrightarrow{L^*_t} & \text{RigDA}^{\text{eff}}_{\text{ét}}(K, \Lambda)
\end{array}
\]

where we indicate with \( \iota_1, \iota_2, \iota_3 \) the inclusion of the small étale site over \( K \) in the big étale site of rigid analytic varieties resp. smooth semi-perfectoid spaces resp. smooth perfectoid spaces.

The lower square commutes by the equivalence \( \iota_2 \cong j \circ \iota_3 \). Similarly, we have an equivalence \( \iota_2 \cong i \circ \iota_1 \) which implies \( L_{\iota_2}^* \cong L_i^* \circ L_{\iota_1}^* \). Since \( L_{\iota_1}^* \cong L_i^* \) is equivalent to the identity, this yields \( L_{\iota_1}^* \cong L_i^* \circ L_{\iota_2}^* \) hence the commutativity of the upper triangle.

We now consider the following square

\[
\begin{array}{ccc}
\text{D}_{\text{ét}}(K, \Lambda) & \xrightarrow{L^*_t} & \text{PerfDA}^{\text{eff}}(K, \Lambda) \\
& & \downarrow \sim \\
\text{D}_{\text{ét}}(K^\flat, \Lambda) & \xrightarrow{L^*_t} & \text{PerfDA}^{\text{eff}}(K^\flat, \Lambda)
\end{array}
\]

which commutes by definition of the tilting equivalence on both sides.

We are left to consider the triangle

\[
\begin{array}{ccc}
\text{RigDA}^{\text{eff}}_{\text{Frob} \text{\text{ét}}}(K^\flat, \Lambda) & \xrightarrow{L^*_t} & \text{D}_{\text{ét}}(K^\flat, \Lambda) \\
& & \downarrow \sim \\
\text{PerfDA}^{\text{eff}}(K^\flat, \Lambda) & \xrightarrow{L^*_t} & \text{D}_{\text{ét}}(K^\flat, \Lambda)
\end{array}
\]

where now the equivalence on the right is induced simply by means of the (completed) perfection functor \( \text{Perf} \). It is then immediate to prove it commutes (a finite étale extension of \( K^\flat \) is already perfect).

\[ \square \]

5. Compatibility with the étale realization

We show in this section that our main theorem in Section 3 can be interpreted as a motivic version of the results of Berkovich [12] showing that the singular cohomology \( H^*_\text{Sing}(|X|^\text{an}|_{\text{Berk}}, \mathbb{Q}_\ell) \) of the Berkovich space associated to the analytification of an algebraic variety over \( K \) are canonically isomorphic to the weight-zero part of the étale cohomology \( H^*_\text{ét}(X_C, \mathbb{Q}_\ell) \) for \( \ell \neq p \).

In particular, we show how to obtain these equivalence via our theorem and the étale realization. This allows us to generalize them further to arbitrary analytic varieties.

From now on, we assume that the residue characteristic of \( K \) is \( p > 0 \) and we pick a prime \( \ell \neq p \). We consider the étale realization for rigid analytic motives

\[ \mathcal{R}_\text{ét} : \text{RigDA}^{\text{eff}}_{\text{Frob} \text{\text{ét}}}(K^\flat, \mathbb{Q}) \rightarrow \widehat{D}^{\text{cp}}_{\text{ét}}(K, \mathbb{Q}_\ell) \]
which is constructed in [8] following the algebraic construction of [1]. Above, the category \( \hat{D}_{\text{et}}^{\text{cp}}(K, \mathbb{Q}_\ell) \) is the derived category of constructible \( \ell \)-adic sheaves following Ekedhal (see [1, Definition 9.3]). We first recall some of its basic properties.

**Proposition 5.1.** Fix a prime \( \ell \) coprime to the residue characteristic \( p \) of \( K \). The functor \( \mathcal{R}_{\text{et}} \) has the following properties:

1. It is tensorial and triangulated.
2. The composition \( \mathcal{R}_{\text{et}} \circ \mathbb{L}^* \) is canonically isomorphic to the functor \( \nu^* : D_{\text{et}}(K, \mathbb{Q})^{\text{cp}} \to \hat{D}_{\text{et}}^{\text{cp}}(K, \mathbb{Q}_\ell) \) induced by extending coefficients.
3. For any smooth rigid analytic variety \( X \), the Galois representation attached to \( \langle H_i^{\mathcal{R}_{\text{et}}}(\mathbb{Q}_\ell(X)) \rangle^\vee \) is the étale representation \( H^i_{\text{et}}((X_C, \mathbb{Q}_\ell)) \).

**Remark 5.2.** In [8] it is shown that the functor \( \mathcal{R}_{\text{et}} \) extends to the stable category of motives \( \text{RigDA}_{\text{Frobet}}(K, \mathbb{Q})^{\text{cp}} \).

If we want to relate the functor \( \mathcal{R}_{\text{et}} \) with Berkovich’s version of Tate’s conjecture [12] we need to introduce weights of Weil numbers appearing as eigenvalues of a lift of Frobenius. We then consider the functor \( H_\ell : \hat{D}_{\text{et}}^{\text{cp}}(K, \mathbb{Q}_\ell) \to \bigoplus \text{Rep}_{\text{ct}}(\text{Gal}(K), \mathbb{Q}_\ell) \) associating to a complex its homology sheaves, which are \( \mathbb{Q}_\ell \)-vector spaces endowed with a continuous action of \( \text{Gal}(K) \) (with respect to the \( \ell \)-adic topology on \( \mathbb{Q}_\ell \)). We use the following notation of Berkovich.

**Definition 5.3.** Let \( V \) be a continuous \( \ell \)-adic representation of \( \hat{\mathbb{Z}} \) and let \( F \) be a topological generator of \( \mathbb{Z} \). We say \( V \) has weight zero if the eigenvalues of \( F \) are Weil numbers of weight equal to 0 and the subcategory of representations \( \text{Rep}_{\text{ct}}(\hat{\mathbb{Z}}, \mathbb{Q}_\ell) \) they form will be denoted by \( \text{Rep}_{\text{ct}}(\hat{\mathbb{Z}}, \mathbb{Q}_\ell)_0 \). For any representation \( V \) we let \( V_0 \) be the maximal sub-representation of \( V \) such that the eigenvalues of \( F \) are Weil numbers of weight equal to 0.

**Proposition 5.4.** We let \( \omega_0 \) be the functor \( \text{Rep}_{\text{ct}}(\hat{\mathbb{Z}}, \mathbb{Q}_\ell)^{\text{cp}} \to \text{Rep}_{\text{ct}}(\hat{\mathbb{Z}}, \mathbb{Q}_\ell)^{\text{cp}} \) mapping \( V \) to \( (V^\vee)_0^\vee \). It is a left adjoint functor to the canonical inclusion.

**Proof.** The functor \( (\cdot)^\vee \) is a self-anti-equivalence on the category of compact continuous representations. It then suffices to show that \( V \mapsto V_0 \) is the right adjoint to the inclusion, and this follows from its very definition.

**Remark 5.5.** We need to introduce an unnatural double dual operation in order to transform a canonical right adjoint into a left adjoint. This is done to create a parallel with the (left adjoint) functor \( \omega_0 \) constructed in the motivic level. In that case, we can not argue by taking duals, since the functor \( \omega_0 \) is only defined at the level of effective motives, and here compact objects are not necessarily dualizable. The price we have to pay for this lack of symmetry is the restriction to the categories of compact objects.

If \( V \) be a continuous \( \ell \)-adic Galois representation, we can consider \( F \in \text{Gal}(K) \) to be a lift of the geometric Frobenius and restrict \( V \) to a representation of \( \langle F \rangle \cong \hat{\mathbb{Z}} \). This defines a functor

\[
\bigoplus \text{Rep}_{\text{cont}}(\text{Gal}(K), \mathbb{Q}_\ell) \to \bigoplus \text{Rep}_{\text{cont}}(\hat{\mathbb{Z}}, \mathbb{Q}_\ell).
\]

By composition, we have then constructed a functor

\[
H^\ell_\cdot \mathcal{R}_{\text{et}} : \text{RigDA}_{\text{Frobet}}^\text{eff}(K, \mathbb{Q}_\ell)^{\text{cp}} \to \bigoplus \text{Rep}_{\text{cont}}(\hat{\mathbb{Z}}, \mathbb{Q}_\ell)^{\text{cp}}
\]

and we now show that the functors \( \omega_0 \)'s are compatible.
Proposition 5.6. The following diagram is commutative:

\[
\begin{array}{ccc}
\text{RigDA}^\text{eff}_{\text{Frobét}}(K, \mathbb{Q}_\ell)^{\text{gp}} & \xrightarrow{H^F \mathbb{R} \mathfrak{f}} & \bigoplus \text{Rep}_\text{cont}(\hat{\mathbb{Z}}, \mathbb{Q}_\ell)^{\text{gp}} \\
\omega_0 & & \omega_0 \\
\end{array}
\]

Proof. We denote by \( \iota \) be the right adjoint functors of \( \omega_0 \) and \( \bar{\omega}_0 \). From the commutativity \( \iota \circ H^F \mathbb{R} \mathfrak{f} \cong H^F \mathbb{R} \mathfrak{f} \circ \iota \) and the unit of the adjunction we deduce the existence of a natural transformation

\[
H^F \mathbb{R} \mathfrak{f} \circ \omega_0 \cong \iota \circ H^F \mathbb{R} \mathfrak{f} \circ \iota \omega_0
\]

which induces a natural transformation \( \eta : \bar{\omega}_0 \circ H^F \mathbb{R} \mathfrak{f} \Rightarrow H^F \mathbb{R} \mathfrak{f} \circ \omega_0 \). We let \( T \) be the full subcategory of \( \text{RigDA}^\text{eff}_{\text{Frobét}}(K, \mathbb{Q}_\ell)^{\text{gp}} \) of those objects \( C \) such that \( \eta(C) \) is invertible. All the functors involved commute with finite sums and shifts, so we deduce that \( T \) is closed under these operations. We now let

\[
X \to Y \to C \to
\]

be a distinguished triangle of \( \text{RigDA}^\text{eff}_{\text{Frobét}}(K, \mathbb{Q}_\ell)^{\text{gp}} \) with \( X \) and \( Y \) inside \( T \). Since the functors \( \omega_0 \) and \( \mathfrak{R}_\text{et} \) are triangulated, we obtain the following long exact sequence

\[
H_i(\mathfrak{R}_\text{et}, \omega_0(X)) \to H_i(\mathfrak{R}_\text{et}, \omega_0(Y)) \to H_i(\mathfrak{R}_\text{et}, \omega_0(C)) \to H_{i-1}(\mathfrak{R}_\text{et}, \omega_0(X))
\]

On the other hand, since \( \mathfrak{R}_\text{et} \) is triangulated and the functors \( V \to V^\vee, V \to V_0 \) are exact, we also deduce the following long exact sequence

\[
\bar{\omega}_0 H_i(\mathfrak{R}_\text{et}(X)) \to \bar{\omega}_0 H_i(\mathfrak{R}_\text{et}(Y)) \to \bar{\omega}_0 H_i(\mathfrak{R}_\text{et}(C)) \to \bar{\omega}_0 H_{i-1}(\mathfrak{R}_\text{et}(X))
\]

The transformation \( \eta \) induces a morphism between the two long exact sequences above. By the five-lemma and the isomorphisms \( H_i(\mathfrak{R}_\text{et}, \omega_0(X)) \cong \bar{\omega}_0 H_i(\mathfrak{R}_\text{et}(X)) \) and \( H_i(\mathfrak{R}_\text{et}, \omega_0(Y)) \cong \bar{\omega}_0 H_i(\mathfrak{R}_\text{et}(Y)) \) we then deduce \( H_i(\mathfrak{R}_\text{et}, \omega_0(C)) \cong \bar{\omega}_0 H_i(\mathfrak{R}_\text{et}(C)) \) proving that \( \eta(C) \) is invertible as well. We have therefore proved that \( T \) is closed under cones.

In order to show \( T = \text{RigDA}^\text{eff}_{\text{Frobét}}(K, \mathbb{Q}_\ell)^{\text{gp}} \) it then suffices to prove that a set of generators of \( \text{RigDA}^\text{eff}_{\text{Frobét}}(K, \mathbb{Q}_\ell)^{\text{gp}} \) (as a triangulated category) lie in \( T \). For example, we can take motives of the form \( \mathbb{Q}_\ell(X) \) for \( X \) a rigid analytic variety of potentially good reduction by [7, Theorem 2.5.34]. We fix then \( X \) and we suppose that for some finite extension \( K'/K \) there is a smooth formal model \( \mathcal{X} \) over \( \mathcal{O}_{K'} \) whose generic fiber is \( X_{K'} \). In particular, \( |X_{K'}| \) is weakly contractible. As \( \omega_0 \cong \iota \circ \mathbb{L}B^*_{\text{Gal}(K)} \) we obtain that

\[
(H^i_\mathfrak{R} \circ \omega_0(\mathcal{X}_C))(\mathbb{Q}_\ell)^{\vee} \cong \bigoplus \text{H} \text{Sing}(|X_C|_{\text{Berk}}, \mathbb{Q}_\ell) \cong H^i_{\text{Sing}}(|X_C|_{\text{Berk}}, \mathbb{Q}_\ell) \cong \bigoplus_{\pi_0(X_C)} \mathbb{Q}_\ell.
\]

By smooth base change, if we let \( \bar{k} \) be the residue field of \( C \), we obtain on the other hand that \( H^i_\mathfrak{R}(X_C, \mathbb{Q}_\ell) \cong H^i_\mathfrak{R}(\bar{k}, \mathbb{Q}_\ell) \) which is zero unless \( i = 0 \), and equal to \( \bigoplus_{\pi_0(X_{\bar{k}})} \mathbb{Q}_\ell \) otherwise (it is of weight 0). Also, since \( \pi_0(X_C) \cong \pi_0(X_{\bar{k}}) \) as \( \text{Gal}(K) \)-sets, we deduce \( H^i_\mathfrak{R}(X_C, \mathbb{Q}_\ell) \cong (H^i_\mathfrak{R} \circ \omega_0(\mathcal{X}_C))(\mathbb{Q}_\ell)^{\vee} \) which entails \( \omega_0 \tilde{\mathfrak{R}}_\mathfrak{f}(X_\ell) \cong \mathfrak{R}_\mathfrak{f}(\mathfrak{Q}_\mathfrak{f}(X)) \) as wanted. \( \square \)

We can finally generalize Berkovich’s formulas [12] to arbitrary rigid analytic compact motives and for any non-archimedean field \( K \).

Corollary 5.7. Let \( M \) be in \( \text{RigDA}^\text{eff}(K, \mathbb{Q}_\ell)^{\text{et}} \) and \( i \) be in \( \mathbb{Z} \). Then \( H^i(\mathbb{L}B^*_{\text{Gal}(K)} M) \otimes \mathbb{Q}_\ell \) coincides with \( H^i_\mathfrak{R}(M_C, \mathbb{Q}_\ell) \). In particular, if \( X \) is a smooth quasi-compact rigid variety or an analytification of an algebraic variety, we have \( H^i_{\text{Sing}}(|X_C|_{\text{Berk}}, \mathbb{Q}_\ell) \cong H^i_\mathfrak{R}(X_C, \mathbb{Q}_\ell) \).
**Proof.** We take for simplicity $M = \Lambda(X)$ and we refer to the previous proposition: in its proof, we showed that $\bigoplus H^i_{\text{Sing}}(|X_C|_{\text{Berk}}, \mathcal{Q}_\ell)^\vee \cong (H^*_{\text{ét}} \circ \omega_0)(\Lambda(X))$ and $\bigoplus (H^i_{\text{ét}}(X_C, \mathcal{Q}_\ell)_0)^\vee \cong (\omega_0 \circ H^*_{\text{ét}})(\Lambda(X))$ so the claim follows from the commutativity of the diagram in the statement of the proposition. □

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**REFERENCES**


