RIGID COHOMOLOGY VIA THE TILTING EQUIVALENCE

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ABSTRACT. We define a de Rham cohomology theory for analytic varieties over a valued field $K^\flat$ of equal characteristic p with coefficients in a chosen untilt of the perfection of $K^\flat$ by means of the motivic version of Scholze's tilting equivalence. We show that this definition generalizes the usual rigid cohomology in case the variety has good reduction. We also prove a conjecture of Ayoub yielding an equivalence between rigid analytic motives with good reduction and unipotent algebraic motives over the residue field, also in mixed characteristic.

CONTENTS

1. INTRODUCTION

Rigid cohomology can be considered as a substitute of the pathological (at least for non-proper varieties) de Rham cohomology over a field k of positive characteristic. Its definition is based on the idea of associating to a variety $\bar{X}/k$ another variety X over a field K of characteristic 0 and then considering its well-behaved de Rham cohomology.

The classic path to reach this goal, elaborating on the work of Monsky and Washnitzer, is to find a smooth formal model $\mathfrak{X}$ of $\bar{X}$ over a valuation ring K° of mixed characteristic, then to consider its (rigid analytic) generic fiber X.

From Fontaine's and Scholze's work in p-adic Hodge theory, we are now accustomed with another strategy to change characteristic: the so-called *tilting equivalence*. This equivalence is built between perfectoid spaces over a perfect, non-archimedean field $K^\flat$ of characteristic p and perfectoid spaces over a fixed untilt K of $K^\flat$ which is a complete non-archimedean field of mixed characteristic.

An alternative method for making $\bar{X}$ "change characteristic" is then at reach: we can base change $\bar{X}$ to a perfectoid field $K^\flat$ then we can take its analytification, followed by its perfection. This defines a perfectoid space $\widehat{X}^\flat$ over $K^\flat$ that we can finally switch (by Scholze's equivalence) to a perfectoid space $\widehat{X}$ over a chosen untilt K of $K^\flat$ having characteristic 0.

The initial problem, namely the definition of a de Rham cohomology for $\bar{X}$, is not yet solved with the above procedures. Indeed, for rigid analytic varieties as well as for perfectoid spaces, the de Rham complex is still problematic (its cohomology groups can be oddly infinite-dimensional for smooth, affinoid rigid analyitic varieties). Nonetheless, the results of [12] and [17] show that some natural de Rham cohomology groups for both smooth rigid analytic varieties as well as smooth perfectoid spaces can truly be defined.

One aim of this paper is that the two recipes mentioned above are actually equivalent. We now rephrase this statement in terms of motives. This allows for a more precise result, and further corollaries that we examine afterwards.

Out of the categories of smooth varieties over a field, smooth rigid analytic varieties over a non-archimedean field, smooth perfectoid spaces over a perfectoid field, or smooth formal schemes over a valuation ring, one can construct the associated category of motives, written as $\mathbf{DM}(K)$ (or $\mathbf{RigDM}(K)$, $\mathbf{PerfDM}(K)$ and $\mathbf{FormDM}(\mathcal{O}_K)$ in the various cases). We consider here derived, effective motives with rational coefficients following Voevodsky. In particular, these categories are Verdier quotients of the classical derived categories of étale sheaves with $\mathbb{Q}$-coefficients defined on each big site.

The special fiber and the generic fiber functors on varieties induce some functors also at the level of motives:

$$\xi\colon \mathbf{DM}(k) \xleftarrow{\sim} \mathbf{FormDM}(\mathcal{O}_K) \to \mathbf{RigDM}(K)$$

(the first one has a natural quasi-inverse as proven in [6]). The composition from left to right is then the motivic version of the first recipe (Monsky-Washnitzer's) sketched above.

Suppose now K is perfectoid. It is possible to rephrase Scholze's tilting equivalence in motivic terms by saying that the categories $\mathbf{PerfDA}(K)$ and $\mathbf{PerfDA}(K^\flat)$ are equivalent. In [17] we descended this result to the rigid analytic situation, by proving the following:

Theorem ([17])**.** *Let K be a perfectoid field. There is a canonical monoidal, triangulated equivalence*

$$\mathfrak{G}\colon \mathbf{RigDM}(K) \cong \mathbf{PerfDM}(K).$$

In particular, for any perfectoid field K of mixed characteristic, we obtain a canonical monoidal, triangulated equivalence
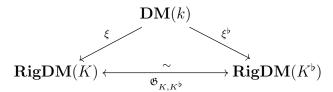
$$\mathfrak{G}_{K,K^\flat}\colon \mathbf{RigDM}(K) \cong \mathbf{RigDM}(K^\flat).$$

We can then consider the following functors

$$\xi^\flat\colon \mathbf{DM}(k) \to \mathbf{DM}(K^\flat) \xrightarrow{\mathrm{Rig}^*} \mathbf{RigDM}(K^\flat) \cong \mathbf{RigDM}(K)$$

and their composition corresponds to the second recipe that we sketched above. We then prove (see Theorem 3.2 and Proposition 3.5):

Theorem. *Let K be a perfectoid field. The following diagram commutes, up to an invertible natural transformation.*

$$
\begin{array}{ccc}
 & \mathbf{DM}(k) & \\
{\scriptstyle \xi}\swarrow & & \searrow{\scriptstyle \xi^\flat} \\
\mathbf{RigDM}(K) & \xleftarrow[\mathfrak{G}_{K,K^\flat}]{\sim} & \mathbf{RigDM}(K^\flat)
\end{array}
$$

This tells us in particular that the method based on Scholze's tilting functor is just as good as Monsky-Washnitzer's (and Berthelot's) for defining rigid cohomology, up to enlarging the coefficient ring to a perfectoid field K (see Corollary 4.12).

Incidentally, we point out that this second method is directly generalizable to (algebraic, perfectoid or) analytic varieties over $K^\flat$ giving rise to a cohomology theory "à la de Rham" that takes values in K-vector spaces (see Section 4). To our knowledge, this is the first time that such a definition appears in the literature, the ring of coefficients K being "smaller" than the one used for rigid cohomology over $K^\flat$ and different from the fields of coefficients considered by Lazda and Pal [**?**] (but relations between these approaches are envisaged). This definition can be applied in particular to families $\{\bar{X}_t\}$ of varieties over a field of characteristic p which

are generically smooth, and coincides with rigid cohomology of the special fiber $\bar{X}_0$ in case the family is smooth everywhere. We therefore obtain a generalization of the rigid cohomology of a log-scheme (in the case of semi-stable reduction) without the need of introducing a log-structure.

The technical difficulty in the proof is the construction of a natural transformation between the two functors. Due to the intricate definition of $\mathfrak{G}$ the way we obtain it is indirect, via the introduction of some auxiliary categories (of semi-perfectoid spaces over K° and over k) and the generalization of some results in [17]. Once the transformation is defined, one can deduce it is invertible from some explicit computations on very special motives generating $\mathbf{DM}(k)$ (see Theorem 3.6).

Our main theorem has further, non-obvious consequences that are not necessarily correlated with rigid cohomology. We briefly describe some of them.

(1) In general, there is not an explicit, geometric formula giving the image of the motive of a variety X by the motivic tilting equivalence $\mathfrak{G}_{K,K^\flat}\colon \mathbf{RigDM}(K) \cong \mathbf{RigDM}(K^\flat)$. From the theorem above, we find such a description for varieties X with good reduction, i.e. having a smooth model $\mathfrak{X}$ over $\mathcal{O}_K$. Their tilt is simply the motive of (the base change of the analytification of) their reduction $\bar{X}$ i.e. the special fiber $\mathfrak{X}_\sigma$ (see Proposition 5.18).

(2) Ayoub [6] has proved that in the equal-characteristic zero case, the category $\mathbf{RigDM}(K)$ has a convenient description as a full, triangulated subcategory of motives above $\mathbb{G}_{m,k}$. The same is true for the equal-characteristic p case, at the cost of considering only motives with good reduction, and introducing some technical hypotheses on the field. The mixed-characteristic counterpart was conjectured in [6] and we are able to prove it here (see Theorem 5.25).

(3) The weak tannakian formalism [4], [5] allows to use the Betti realization functor $\mathbf{DM}(K) \to \mathbf{D}(\mathbb{Q})$ to define a motivic Galois group for subfields K of $\mathbb{C}$. Moreover, the equivalence of the previous point also gives rise to a Galois group for a non-archimedean field of equi-characteristic zero. This induces relations between the absolute motivic Galois group of a field, the motivic Galois group of its completion over a non-archimedean place of equal characteristic 0 and the motivic Galois group of the residue field, mimicking the relations between the absolute Galois groups and the decomposition group associated to a non-archimedean place. We can use the de Rham realization to obtain similar statements in any characteristic. This will be studied in detail in a following paper.

We warn the reader that we mostly use here the variant of motives *without transfers*, denoted typically with $\mathbf{DA}$ rather than $\mathbf{DM}$. These two approaches are canonically equivalent whenever the base field is perfect and the ring of coefficients contains $\mathbb{Q}$ (see [19]): the one without transfers has the quality of being more direct for definitions.

This article is organized as follows. In Section 2 we introduce some accessory geometric categories over a valued field K, its ring of valuation $\mathcal{O}_K$ as well as its residue field k and we define the associated categories of motives. In Section 3 we prove our main theorem. In Section 4 we describe how to define a de Rham cohomology over a field of equi-characteristic p while in Section 5 we solve Ayoub's conjecture on rigid analytic motives of good reduction in mixed characteristic.

2. MOTIVES OF SEMI-PERFECTOID SPACES OVER K° AND k

In this section we make the following hypotheses.

Assumption 2.1. We let K be a perfectoid field, we denote its tilt by $K^\flat$ and we fix a pseudo-uniformizer $\pi \in K$.

For the notions of perfectoid fieds, perfectoid spaces over them and their properties we refer to [16]. We fix here some notations and examples.

Example 2.2.   (1) The space $\widehat{\mathbb{B}}^1 := \mathrm{Spa}(K\langle v^{1/p^\infty}\rangle)$ is perfectoid. The global function v induces a map $\widehat{\mathbb{B}}^1 \to \mathbb{B}^1$ where $\mathbb{B}^1 = \mathrm{Spa}\,K\langle v\rangle$ is the usual closed rigid disc. This map also induces a map of formal schemes $\widehat{\mathfrak{B}}^1 := \mathrm{Spf}\,\mathcal{O}_K\langle v^{1/p^\infty}\rangle \to \mathfrak{B}^1 := \mathrm{Spf}\,\mathcal{O}_K\langle v\rangle$ with special fiber $\widehat{\mathbb{A}}^1 := \mathrm{Spec}\,k[v^{1/p^\infty}] \to \mathbb{A}^1 = \mathrm{Spec}\,k[v]$.

   (2) The maps above restrict to the rational subspaces defined by the equation $|v| = 1$ that is, we have maps $\widehat{\mathbb{T}}^1 := \mathrm{Spa}\,K\langle v^{\pm 1/p^\infty}\rangle \to \mathbb{T}^1 := \mathrm{Spa}\,K\langle v^{\pm 1}\rangle$ and $\widehat{\mathfrak{T}}^1 := \mathrm{Spf}\,\mathcal{O}_K\langle v^{\pm 1/p^\infty}\rangle \to \mathfrak{T}^1 := \mathrm{Spf}\,\mathcal{O}_K\langle v^{\pm 1}\rangle$ with special fiber $\widehat{\mathbb{G}}_m^1 := \mathrm{Spec}\,k[v^{\pm 1/p^\infty}] \to \mathbb{G}_m = \mathrm{Spec}\,k[v^{\pm 1}]$.

   (3) For any positive integer N we denote by X^N the N-fold product of X therefore defining the spaces $\mathbb{T}^N$, $\widehat{\mathbb{T}}^N$, $\mathfrak{T}^N$ etc.

In [17] we defined the category of semi-perfectoid spaces over K. These objects form a convenient category of sheafy adic spaces containing both smooth rigid analytic spaces and small smooth perfectoid spaces. We recall briefly their definition.

Definition 2.3. The category of *smooth semi-perfectoid spaces over K* is the full subcategory of adic spaces formed by spaces which are locally isomorphic to $X_0 \times_{\mathbb{T}^N} \widehat{\mathbb{T}}^N$ for some étale morphism or rigid analytic varieties $X_0 \to \mathbb{T}^N \times \mathbb{T}^M$. We remark that $X_0 \times_{\mathbb{T}^N} \widehat{\mathbb{T}}^N \sim \varprojlim X_h$ where $X_h := X_0 \times_{\mathbb{T}^N, \varphi_h} \mathbb{T}^N$ and the map φ_h is induced by raising the coordinates v to the p^h-th power (for the precise meaning of $\sim$ we refer to [15, Section 2.4]).

If one imposes $N = 0$ in the definition above, one recovers the usual full subcategory category $\mathrm{RigSm}\,/K$ of smooth rigid analytic varieties over K while if one imposes $M = 0$, one defines the full subcategory $\mathrm{PerfSm}\,/K$ of *smooth perfectoid spaces* over K.

As remarked in [17], for all the categories above we can define the étale topology by considering Scholze's definition (see [16, Definition 7.1]).

We now make the analogous definition in the case of smooth formal schemes over K° and of smooth algebraic varieties over k. Such geometrical categories will only play an accessory role in our main theorem. On the other hand, we remark that the category $\mathrm{PerfSm}\,/K^\circ$ is at the core of the constructions made in [9].

Definition 2.4. A *smooth semi-perfectoid space over k* [resp. K°] is a [formal] scheme which is Zariski locally isomorphic to $X_0 \times_{\mathbb{G}_m^N} \widehat{\mathbb{G}}_m^N$ [resp. $\mathfrak{X}_0 \times_{\mathfrak{T}^N} \widehat{\mathfrak{T}}^N$] for some [formally] étale morphism $X_0 \to \mathbb{G}_m^N \times \mathbb{G}_m^M$ [resp. $\mathfrak{X}_0 \to \mathfrak{T}^N \times \mathfrak{T}^M$]. The full subcategory of schemes they form will be denoted by $\mathrm{sPerfSm}\,/k$ [resp. $\mathrm{sPerfSm}\,/K^\circ$]. Locally, a smooth semi-perfectoid space is the inverse limit of the system $X_h := X_0 \times_{\mathbb{G}_m^N, \varphi_h} \mathbb{G}_m^N$ [resp. $\mathfrak{X}_h := \mathfrak{X}_0 \times_{\mathfrak{T}_m^N, \varphi_h} \mathfrak{T}_m^N$] where φ_h is induced by raising the coordinates v to the p^h-th power.

If one imposes $N = 0$ in the definitions above, one recovers the usual full subcategory category $\mathrm{Sm}\,/k$ of smooth schemes over k [resp. the category $\mathrm{FormSm}\,/K^\circ$ of smooth formal schemes topologically of finite type over K°]. If one imposes $M = 0$ in the definitions above, one defines the full subcategory $\mathrm{PerfSm}\,/k$ of *smooth perfectoid spaces over k* [resp. $\mathrm{PerfSm}\,/K^\circ$ of *smooth formal perfectoid spaces over K°*].

For all the categories above, we can define the étale topology by specializing the definition of étale maps and coverings for schemes and formal schemes.

Proposition 2.5. *Let* $\mathrm{Spf}\,A$ *be étale over* $\widehat{\mathfrak{T}}^N \times \mathfrak{T}^M$. *The generic fiber functor* $(-)_\eta$ *induced by* $\mathrm{Spf}\,A \mapsto \mathrm{Spa}(A[1/\pi], A)$ *and the special fiber functor* $(-)_\sigma$ *induced by* $\mathrm{Spf}\,A \mapsto \mathrm{Spec}(A/\mathfrak{m})$

define well-defined functors from to the étale topos of sPerfSm $/K°$ *from the one of* sPerfSm $/K$ *and of* sPerfSm $/k$ *respectively. These functors restrict to functors to the topos of formal schemes of tft from the one of smooth rigid varieties over* K *and of smooth schemes over* k *respectively. They also restrict to maps on the associated étale topoi of perfectoid spaces.*

Proof. All the adic spaces involved in the statement are sheafy, so the claim follows from [15, Lemma 3.5.1(1)]. □

Out of these geometric categories, we can immediately define the associated motives. We use here a triangulated, more down-to-earth definition rather the model-categorical one we used in [17] (recalled in Remark 2.7 and that we will freely use in proofs). Motives will be here defined as quotients of triangulated categories, namely the derived categories of Λ-sheaves over a site. The quotient is taken in order to impose homotopy-invariance on the (co-)homological functors, i.e. invariance with respect to the projections $X \times I \to X$ induced by a chosen "interval object" I in the geometric category.

Definition 2.6. Fix a commutative ring Λ. Let κ be in $\{k, K°, K\}$ and fix an object I in sPerfSm $/\kappa$.

- We denote by $\Lambda(X)$ the presheaf with values in Λ-modules represented by X i.e. the presheaf associating Y to the free Λ-module associated to the set $\mathrm{Hom}(Y, X)$.
- We denote by Frobét the topology on sPerfSm $/\kappa$ generated by étale covers and relative Frobenius morphisms (in case the base ring has positive characteristic) and let $a_{\mathrm{Frobét}}$ be the associated sheafification functor.
- The category of *effective semi-perfectoid motives* $\mathbf{sPerfDA}^{\mathrm{eff}}_{\mathrm{Frobét},I}(\kappa, \Lambda)$ is the Verdier quotient of the derived category of Frobét-sheaves on smooth semi-perfectoid spaces with values in Λ-modules $\mathbf{D}(\mathbf{Sh}_{\mathrm{Frobét}}(\mathrm{sPerfSm}\,/\kappa, \Lambda))$ over its sub-triangulated category with small sums generated by the cones of the projection maps $a_{\mathrm{Frobét}}\Lambda(X \times I) \to a_{\mathrm{Frobét}}\Lambda(X)$ by letting X vary in sPerfSm $/\kappa$.
- We make an analogous definition for small perfectoid spaces over κ in which case the categories of motives will be denoted by $\mathbf{PerfDA}^{\mathrm{eff}}_{\mathrm{Frobét},I}(\kappa, \Lambda)$.
- We can also make the analogous definitions for smooth schemes over k, smooth formal schemes of tft over $K°$ and smooth rigid varieties over K and the associated motives are denoted by $\mathbf{DA}^{\mathrm{eff}}_{\mathrm{Frobét},I}(k, \Lambda)$, $\mathbf{FormDA}^{\mathrm{eff}}_{\mathrm{Frobét},I}(K°, \Lambda)$ and $\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frobét},I}(K, \Lambda)$ respectively.

Remark 2.7. Fix a site $(\mathbf{C}, \tau)$ and an object $I \in \mathbf{C}$ (for example, we can take the Frob-étale site on smooth rigid analytic varieties over K and I to be the ball $\mathbb{B}^1$). The category of the associated motives (in the example, the category $\mathbf{RigDA}^{\mathrm{eff}}(K, \Lambda)$) admits the following presentation as a homotopy category of a model category structure.

One can first consider the projective model structure on (unbounded) complexes of presheaves on $\mathbf{C}$ with coefficients in Λ. It contains representable presheaves $\Lambda(X)$ as well as the complexes $\Lambda(\mathcal{U})$ associated to any simplicial $\mathcal{U}$ of $\mathbf{C}$. Its homotopy category is the (unbounded) derived category $\mathbf{D}(\mathbf{Psh}(\mathbf{C}, \Lambda))$. Let S be the class of maps containing all shifts of the the canonical arrows $\Lambda(X \times I) \to \Lambda(X)$, $\Lambda(\mathcal{U}) \to \Lambda(X)$ for any object X and any τ-hypercover $\mathcal{U}$ of it. The left Bousfield localisation of the projective model structure over S is well defined, and its homotopy category coincides with the triangulated category of effective motives over the datum $(\mathbf{C}, \tau, I)$. The interested reader can find all the details in [3] (briefly resumed in [17]).

This presentation of motives as homotopy categories of a Quillen model structure equip them with a natural structure of a tensor DG-category, not merely of a a triangulated category. We remark that the tensor product $\Lambda(X) \otimes \Lambda(Y)$ is isomorphic to $\Lambda(X \times Y)$. The language of

model categories is also very convenient to define natural adjoint functors of motivic categories out of functors at the level of the underlying sites $(\mathbf{C}, \tau)$.

Example 2.8. We now list the objects I that we will consider.
 (1) The disc of radius one $\mathbb{B}^1 = \operatorname{Spa} K\langle v \rangle$ is an object of RigSm/K. It has a smooth formal model $\mathfrak{B}^1 = \operatorname{Spf} K\langle v \rangle$ which is an object of $\operatorname{FormSm}/K^\circ$. Its special fiber is the affine line $\mathbb{A}^1_k$ in Sm/k.
 (2) The perfectoid disc of radius one $\widehat{\mathbb{B}}^1 = \operatorname{Spa} K\langle v^{1/p^\infty} \rangle$ is an object of PerfSm/K. It has a smooth formal model $\widehat{\mathfrak{B}}^1 = \operatorname{Spf} K\langle v^{1/p^\infty} \rangle$ which is an object of $\operatorname{PerfSm}/K^\circ$ since it is covered by the two subspaces $\operatorname{Spf} K\langle v^{\pm 1/p^\infty} \rangle$ and $\operatorname{Spf} K\langle (v+1)^{\pm 1/p^\infty} \rangle$ both isomorphic to $\widehat{\mathfrak{T}}^1$. Its special fiber is $\widehat{\mathbb{A}}^1_k := \operatorname{Spec} k[v^{1/p^\infty}]$ which lies in PerfSm/k.

Remark 2.9. From now on, we adopt the following notations regarding the categories of motives:
 (a) In case the characteristic of the base ring is 0 we will drop the pedex Frobét from the notation.
 (b) In case I is $\mathbb{A}^1$ resp. $\mathfrak{B}^1$ resp. $\mathbb{B}^1$ for smooth semi-perfectoid spaces over k resp. K° resp. K we will drop the pedex I from the notation.
 (c) In case I is $\mathbb{A}^1$ resp. $\mathfrak{B}^1$ resp. $\mathbb{B}^1$ for smooth varieties over k resp. smooth formal schemes of tft over K° resp. smooth rigid analytic varieties over K we will drop the pedex I from the notation.
 (d) In case I is $\widehat{\mathbb{A}}^1$ resp. $\widehat{\mathfrak{B}}^1$ resp. $\widehat{\mathbb{B}}^1$ for perfectoid varieties over k resp. K° resp. K we will drop the pedex I from the notation.
 (e) We will assume $\mathbb{Q} \subset \Lambda$ and we drop Λ from the notation, if the context allows it.

Remark 2.10. Suppose that $\operatorname{char} K = p$. If one considers the étale topology instead of the finer Frobét topology, one gets the category of étale motives $\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{ét}}(K, \Lambda)$ and a natural functor $\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{ét}}(K, \Lambda) \to \mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frobét}}(K, \Lambda)$. This functor coincides with the Verdier quotient over the subcategory generated by cones of the relative Frobenius maps $X \to X \times_{\mathrm{Frob}} K$ (see [19]).

The knowledgeable reader will notice that we use here the version of (effective, derived) motives *without transfers* rather than the version *with transfers*, typically denoted with $\mathbf{DM}^{\mathrm{eff}}$ and mentioned in the introduction. We recall here briefly their definition, which makes use of correspondences. For more properties of such categories, we refer to [6, Chapter 2].

Definition 2.11. Ler K' be any complete valued field (eventually trivially valued).
 • We define the category RigCor/K' as the category whose objects are those of RigSm/K' and whose morphisms $\operatorname{Hom}(X, Y)$ are the free Λ modules generated to the set of integral, closed subvarieties Z of $X \times Y$ which are finite and surjective over a connected component of X and where composition is induced by the intersection formula (see [6, Remark 2.2.21]).
 • The category $\mathbf{Psh}(\operatorname{RigCor}/K')$ will be denoted by $\mathbf{PST}(\operatorname{RigSm}/K')$. Its full subcategory of those presheaves $\mathcal{F}$ that are sheaves when restricted to RigSm/K' will be denoted by $\mathbf{Sh}_{\mathrm{tr}}(\operatorname{RigSm}/K')$. The sheaf represented by a smooth variety X is denoted by $\Lambda_{\mathrm{tr}}(X)$.
 • The category of effective rigid analytic motives with transfers $\mathbf{RigDM}^{\mathrm{eff}}(K', \Lambda)$ is the Verdier quotient of the derived category of $\mathbf{Sh}_{\mathrm{tr}}(\operatorname{RigSm}/K')$ over its sub-triangulated category with small sums generated by the cones of the projection maps $\Lambda_{\mathrm{tr}}(X \times \mathbb{B}^1) \to \Lambda_{\mathrm{tr}}(X)$ by letting X vary in RigSm/K'. In case K' is trivially valued, it will simply be denoted with $\mathbf{DM}^{\mathrm{eff}}(K', \Lambda)$

Remark 2.12. Also the category $\mathbf{RigDM}^{\mathrm{eff}}(K', \Lambda)$ admits a presentation as a homotopy category of a model category structure, analogous to the case of motives without transfers (Remark 2.7).

One can first consider the projective model structure on complexes $\mathbf{Ch}\,\mathbf{PST}(K', \Lambda)$. Its homotopy category is the (unbounded) derived category $\mathbf{D}(\mathbf{Psh}(\mathbf{C}, \Lambda))$. Let S be the class of maps containing all shifts of the the canonical arrows $\Lambda_{\mathrm{tr}}(X \times I) \to \Lambda_{\mathrm{tr}}(X)$, $\Lambda_{\mathrm{tr}}(\mathcal{U}) \to \Lambda_{\mathrm{tr}}(X)$ for any smooth variety X and any ét-hypercover $\mathcal{U}$ of it. The left Bousfield localisation of the projective model structure over S is well defined, and its homotopy category coincides with $\mathbf{RigDM}^{\mathrm{eff}}(K', \Lambda)$. The interested reader can find all the details in [6].

This presentation of motives as homotopy categories of a Quillen model structure equip them with a natural structure of a tensor DG-category, not merely of a a triangulated category. We remark that the tensor product $\Lambda_{\mathrm{tr}}(X) \otimes \Lambda_{\mathrm{tr}}(Y)$ is isomorphic to $\Lambda_{\mathrm{tr}}(X \times Y)$.

Adding correspondences or not makes no difference in our context aso soon as $\mathbb{Q} \subset \Lambda$ thanks to the following fact (which is classical in a vast list of situations, see [4, Appendix B] and the other references of [19]).

Theorem 2.13 ([19])**.** *Under Assumption 2.1 and the hypothesis that Λ is a $\mathbb{Q}$-algebra, there are canonical triangulated, monoidal (Quillen) equivalences*

$$\mathbf{DA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(k, \Lambda) \cong \mathbf{DM}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(k, \Lambda) \qquad \mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(K, \Lambda) \cong \mathbf{RigDM}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(K, \Lambda)$$

Remark 2.14. The canonical functor appearing in the theorem above is the derived functor $\mathbb{L}a_{\mathrm{tr}}$ induced by the canonical inclusion $\mathrm{Sm}/K \to \mathrm{RigCor}/K$.

We now list the adjoint pairs that link the various categories of motives we introduced above. They are all defined thanks to the functoriality of the very definition of motives.

Theorem 2.15. *(1) The inclusions $\iota\colon \mathrm{Sm}/k \to \mathrm{sPerfSm}/k$ resp. $\iota\colon \mathrm{FormSm}/K^\circ \to \mathrm{sPerfSm}/K^\circ$ resp. $\iota\colon \mathrm{RigSm}/K \to \mathrm{sPerfSm}/K$ induce the following (Quillen) triangulated adjunctions, whose left adjoint is monoidal:*

$$\mathbb{L}\iota^*\colon \mathbf{DA}^{\mathrm{eff}}(k, \Lambda) \rightleftarrows \mathbf{sPerfDA}^{\mathrm{eff}}(k, \Lambda) :\mathbb{R}\iota_*$$

$$\mathbb{L}\iota^*\colon \mathbf{FormDA}^{\mathrm{eff}}(K^\circ, \Lambda) \rightleftarrows \mathbf{sPerfDA}^{\mathrm{eff}}(K^\circ, \Lambda) :\mathbb{R}\iota_*$$

$$\mathbb{L}\iota^*\colon \mathbf{RigDA}^{\mathrm{eff}}(K, \Lambda) \rightleftarrows \mathbf{sPerfDA}^{\mathrm{eff}}(K, \Lambda) :\mathbb{R}\iota_*$$

(2) Let κ be in $\{k, K^\circ, K\}$. We let $\widehat{I}$ be $\widehat{\mathbb{A}}^1_k$, $\widehat{\mathfrak{B}}^1$ or $\widehat{\mathbb{B}}^1$ accordingly. The canonical inclusion $j\colon \mathrm{PerfSm}/\kappa \to \mathrm{sPerfSm}/\kappa$ induces the following (Quillen) triangulated adjunction, whose left adjoint is monoidal:

$$\mathbb{L}j^*\colon \mathbf{PerfDA}^{\mathrm{eff}}(\kappa, \Lambda) \rightleftarrows \mathbf{sPerfDA}^{\mathrm{eff}}_{\widehat{I}}(\kappa, \Lambda) :\mathbb{R}j_*$$

Proof. We use the definition of motives as homotopy categories given in Remark 2.7. The fact that the maps of topoi induce such Quillen adjunctions follows formally from the functoriality of the construction of motives, and the isomorphisms $\iota\mathbb{A}^1 \cong \mathbb{A}^1$, $\iota\mathfrak{B}^1 \cong \mathfrak{B}^1$, $\iota\mathcal{B}^1 \cong \mathbb{B}^1$, $j\widehat{I} \cong \widehat{I}$. Monoidality follows from the formulas $\iota(X \times Y) \cong \iota X \times \iota Y$ and $j(X \times Y) \cong jX \times jY$. $\quad\square$

Theorem 2.16. *(1) The maps of topoi $(-)_\sigma$ considered in Proposition 2.5 induce the following (Quillen) monoidal, triangulated equivalences:*

$$\mathbb{L}(\cdot)^*_\sigma\colon \mathbf{sPerfDA}^{\mathrm{eff}}(K^\circ, \Lambda) \rightleftarrows \mathbf{sPerfDA}^{\mathrm{eff}}(k, \Lambda) :\mathbb{R}(\cdot)_{\sigma*}$$

$$\mathbb{L}(\cdot)^*_\sigma\colon \mathbf{PerfDA}^{\mathrm{eff}}(K^\circ, \Lambda) \rightleftarrows \mathbf{PerfDA}^{\mathrm{eff}}(k, \Lambda) :\mathbb{R}(\cdot)_{\sigma*}$$

$$\mathbb{L}(\cdot)^*_\sigma\colon \mathbf{FormDA}^{\mathrm{eff}}(K^\circ, \Lambda) \rightleftarrows \mathbf{DA}^{\mathrm{eff}}(k, \Lambda) :\mathbb{R}(\cdot)_{\sigma*}$$

(2) The map of topoi $(-)_\eta$ induce the following (Quillen) triangulated adjunctions, whose left adjoint is monoidal:

$$\mathbb{L}(\cdot)_\eta^*\colon \mathbf{sPerfDA}^{\mathrm{eff}}(K^\circ, \Lambda) \rightleftarrows \mathbf{sPerfDA}^{\mathrm{eff}}(K, \Lambda) \colon\! \mathbb{R}(\cdot)_{\eta*}$$

$$\mathbb{L}(\cdot)_\eta^*\colon \mathbf{PerfDA}^{\mathrm{eff}}(K^\circ, \Lambda) \rightleftarrows \mathbf{PerfDA}^{\mathrm{eff}}(K, \Lambda) \colon\! \mathbb{R}(\cdot)_{\eta*}$$

$$\mathbb{L}(\cdot)_\eta^*\colon \mathbf{FormDA}^{\mathrm{eff}}(K^\circ, \Lambda) \rightleftarrows \mathbf{RigDA}^{\mathrm{eff}}(K, \Lambda) \colon\! \mathbb{R}(\cdot)_{\eta*}$$

Proof. We again use the definition of motives as homotopy categories given in Remark 2.7. The fact that the maps of topoi induce such Quillen adjunctions follows formally from the functoriality of the construction of motives, and the isomorphisms $(\mathfrak{B}^1)_\eta \cong \mathbb{B}^1$, $(\widehat{\mathfrak{B}}^1)_\eta \cong \widehat{\mathbb{B}}^1$, $(\mathfrak{B}^1)_\sigma \cong \mathbb{A}^1$, $(\widehat{\mathfrak{B}}^1)_\sigma \cong \widehat{\mathbb{A}}^1$. Monoidality follows formally from the formulas $(\mathfrak{X} \times \mathfrak{Y})_\eta \cong \mathfrak{X}_\eta \times \mathfrak{Y}_\eta$ and $(\mathfrak{X} \times \mathfrak{Y})_\sigma \cong \mathfrak{X}_\sigma \times \mathfrak{Y}_\sigma$. We are left to prove that the each adjunction of the first set is actually an equivalence.

To this aim, it suffices to adapt the proof of [6, Proposition 1.4.21(2)] to the (semi-)perfectoid setting, and then use the argument given in [6, Corollary 1.4.24]. In our situation, following the notations of [6, Proposition 1.4.21(2)], it is sufficient to consider the case where $\mathfrak{X} = \varprojlim \mathfrak{X}_h$ is an affinoid semi-perfectoid space over K° (see the notations in Definition 2.4) and $\check{\mathcal{Z}}$ is its special fiber (when considering a closed immersion $\mathcal{Z} = \mathfrak{X}_\sigma \to \mathfrak{X}'_\sigma = \varprojlim \mathfrak{X}'_{h\sigma}$ for Step 3 of the proof of [6, Proposition 1.4.21] we can take the normal bundle of the associated closed immersion $\mathcal{Z}_0 \to \mathfrak{X}'_{0\sigma}$ as each $\mathfrak{X}'_{h+1\sigma} \to \mathfrak{X}'_{h\sigma}$ is a universal homeomorphism). $\square$

Remark 2.17. In particular, we obtain the following monoidal, triangulated left adjoint functors

$$\xi := \mathbb{L}(\cdot)_\eta^* \mathbb{R}(\cdot)_{\sigma*}\colon \mathbf{sPerfDA}^{\mathrm{eff}}(k, \Lambda) \longrightarrow \mathbf{sPerfDA}^{\mathrm{eff}}(K, \Lambda)$$

$$\xi := \mathbb{L}(\cdot)_\eta^* \mathbb{R}(\cdot)_{\sigma*}\colon \mathbf{PerfDA}^{\mathrm{eff}}(k, \Lambda) \longrightarrow \mathbf{PerfDA}^{\mathrm{eff}}(K, \Lambda)$$

$$\xi := \mathbb{L}(\cdot)_\eta^* \mathbb{R}(\cdot)_{\sigma*}\colon \mathbf{DA}^{\mathrm{eff}}(k, \Lambda) \longrightarrow \mathbf{RigDA}^{\mathrm{eff}}(K, \Lambda)$$

whose right adjoints $\mathbb{L}(\cdot)_\sigma^* \mathbb{R}(\cdot)_{\eta*}$ will be typically denoted by χ.

Corollary 2.18. *Let K be a perfectoid field of positive characteristic. The functor $\mathbb{R}(\cdot)_{\sigma*}$ is canonically isomorphic to $\mathbb{L}(\cdot \widehat{\times}_{\mathrm{Spec}\, k} \mathrm{Spf}\, K^\circ)$. In particular, the motive $\Lambda(\mathfrak{X})$ is canonically isomorphic to $\Lambda(\bar{\mathfrak{X}} \widehat{\times}_{\mathrm{Spec}\, k} \mathrm{Spf}\, K^\circ)$.*

Proof. Since $\mathbb{R}(-)_{\sigma*}$ is an inverse of $\mathbb{L}(-)_\sigma^*$ it suffices to prove that for any object $\bar{X}$ in $\mathrm{sPerfSm}\,/k$ the motive $\mathbb{L}(-)_\sigma^* \Lambda(\bar{X} \widehat{\times}_{\mathrm{Spec}\, k} \mathrm{Spf}\, K^\circ)$ is isomorphic to $\Lambda(\bar{X})$ which is clear. $\square$

We complete the list of natural functors of motives with the ones induced by the analytification procedure considered in [6].

Theorem 2.19 ([6, Proposition 1.4.14]). *The functor associating to an algebraic variety X its analytification X^{an} (see [6, Section 1.1.3]) induces the following (Quillen) monoidal, triangulated adjunction:*

$$\mathbb{L}\,\mathrm{Rig}^*\colon \mathbf{DA}^{\mathrm{eff}}(K, \Lambda) \rightleftarrows \mathbf{RigDA}^{\mathrm{eff}}(K, \Lambda) \colon\! \mathbb{R}\,\mathrm{Rig}_*$$

3. THE COMMUTATIVITY

In this section we still assume that K is a perfectoid field with residue k (see Assumpion 2.1). From now on, we also fix a section $k \to K^{\flat\circ}$ and we assume that $\mathbb{Q} \subset \Lambda$.

We can rephrase Scholze's tilting equivalence in motivic terms by saying that the categories $\mathbf{PerfDA}^{\mathrm{eff}}(K, \Lambda)$ and $\mathbf{PerfDA}^{\mathrm{eff}}(K^\flat, \Lambda)$ are equivalent. In [17] we descended this result to the rigid analytic situation, by proving the following:

Theorem 3.1 ([17]). *Let K be a perfectoid field. There is a canonical monoidal, triangulated equivalence*

$$\mathfrak{G} = \mathbb{R}j_* \circ \mathbb{L}\iota^* \colon \mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(K, \Lambda) \cong \mathbf{PerfDA}^{\mathrm{eff}}(K, \Lambda).$$

In particular, for any perfectoid field K of mixed characteristic, we obtain a canonical monoidal, triangulated equivalence

$$\mathfrak{G}_{K, K^\flat} \colon \mathbf{RigDA}^{\mathrm{eff}}(K, \Lambda) \cong \mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(K^\flat, \Lambda).$$

Since the residue field k of K coincides with the one of $K^\flat$ we get by Remark 2.17 two functors
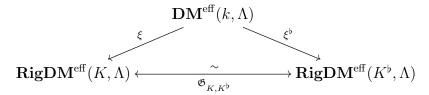
$$\xi \colon \mathbf{DA}^{\mathrm{eff}}(k, \Lambda) \xrightarrow[\sim]{\mathbb{R}(\cdot)_{\sigma*}} \mathbf{FormDA}^{\mathrm{eff}}(K^\circ, \Lambda) \xrightarrow{\mathbb{L}(\cdot)^*_\eta} \mathbf{RigDA}^{\mathrm{eff}}(K, \Lambda)$$

$$\xi^\flat \colon \mathbf{DA}^{\mathrm{eff}}(k, \Lambda) \xrightarrow[\sim]{\mathbb{R}(\cdot)_{\sigma*}} \mathbf{FormDA}^{\mathrm{eff}}(K^{\flat\circ}, \Lambda) \xrightarrow{\mathbb{L}(\cdot)^*_\eta} \mathbf{RigDA}^{\mathrm{eff}}(K^\flat, \Lambda)$$

whose right adjoints will be denoted by χ and $\chi^\flat$ respectively.

The aim of this section is to prove that they are compatible with the tilting equivalence. In other words, we want to prove the following result.

Theorem 3.2. *The following diagram is commutative, up to an invertible natural transformation.*

Remark 3.3. The theorem is equivalent to the commutation of the adjoint diagram, whose sides are χ and $\chi^\flat$. It is also equivalent to the version with the categories without transfers (see Theorem 2.13). We will then consider the analogous diagram with $\mathbf{DA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(k, \Lambda)$, $\mathbf{RigDA}^{\mathrm{eff}}(K, \Lambda)$ and $\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(K^\flat, \Lambda)$ in the proof.

We point out that the functor $\xi^\flat \colon \mathbf{DA}^{\mathrm{eff}}(k, \Lambda) \to \mathbf{RigDA}^{\mathrm{eff}}(K^\flat, \Lambda)$ takes a particularly explicit form. Recall that we have chosen a section $k \to K^{\flat\circ}$

Definition 3.4 ([6, Section 2.5]). Let $\bar{X} = \mathrm{Spec}\,\bar{A}$ be a smooth affine scheme over k. We let $Q^{\mathrm{rig}}(\bar{X})$ be the rigid variety $\mathfrak{X}_\eta$ where $\mathfrak{X}$ is the formal scheme associated to the completion of $\bar{A} \times_k K^{\flat\circ}$.

Proposition 3.5. *The functor $\xi^\flat \colon \mathbf{DA}^{\mathrm{eff}}(k, \Lambda) \to \mathbf{RigDA}^{\mathrm{eff}}(K^\flat, \Lambda)$ is canonically isomorphic to the functor $\mathbb{L}(\mathrm{Rig} \circ (\cdot \times K^\flat))^*$ and to the (Quillen) functor induced by $Q^{\mathrm{rig}}(\cdot)$.*

Proof. The two last functors are canonically equivalent, as shown in [6, Lemma 2.5.59].

From the canonical isomorphism $(\bar{X} \times_k K^{\flat\circ})_\sigma \cong \bar{X}$ we deduce that $\mathbb{L}(\cdot \times_k K^{\flat\circ})$ is a right inverse for $\mathbb{L}(\cdot)^*_\sigma$ and therefore coincides canonically with $\mathbb{R}(\cdot)_{\sigma*}$ by Theorem 2.16, and we obtain the claim. $\square$

The commutativity stated in Theorem 3.2 will be reached by combining the two following propositions.

Theorem 3.6. *Let K be a perfectoid field. The functors ι and j induce a commutative diagram of left adjoint functors*

$$
\begin{array}{ccccc}
\mathbf{DA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(k, \Lambda) & \xrightarrow{\mathbb{L}\iota^*} & \mathbf{sPerfDA}^{\mathrm{eff}}_{\widehat{\mathbb{A}}^1}(k, \Lambda) & \xleftarrow{\mathbb{L}j^*} & \mathbf{PerfDA}^{\mathrm{eff}}(k, \Lambda) \\
\downarrow{\scriptstyle\xi} & & \downarrow{\scriptstyle\xi} & & \downarrow{\scriptstyle\xi} \\
\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(K, \Lambda) & \xrightarrow{\mathbb{L}\iota^*} & \mathbf{sPerfDA}^{\mathrm{eff}}_{\widehat{\mathbb{B}}^1}(K, \Lambda) & \xleftarrow{\mathbb{L}j^*} & \mathbf{PerfDA}^{\mathrm{eff}}(K, \Lambda)
\end{array}
$$

which has the following extra properties:

(1) The compositions $\mathbb{R}j_*\mathbb{L}\iota^*$ defined on the two lines are monoidal, triangulated equivalences of categories. On the first line, the functor $\mathbb{R}j_*\mathbb{L}\iota^*$ is canonically isomorphic to $\mathbb{L}\operatorname{Perf}^*$. If $\operatorname{char}K = p$ the functor $\mathfrak{G} = \mathbb{R}j_*\mathbb{L}\iota^*$ on the second line is canonically isomorphic to $\mathbb{L}\operatorname{Perf}^*$ as well.

(2) There is a natural transformation $\xi \circ \mathbb{R}j_* \Rightarrow \mathbb{R}j_* \circ \xi$ and an invertible transformation $\xi \circ \mathbb{L}\iota^* \overset{\sim}{\Rightarrow} \mathbb{L}\iota^* \circ \xi$.

(3) The natural transformation $\xi \circ \mathbb{L}\operatorname{Perf}^* \cong \xi \circ \mathbb{R}j_* \circ \mathbb{L}\iota^* \Rightarrow \mathbb{R}j_* \circ \mathbb{L}\iota^* \circ \xi$ is invertible.

Proof. We can reproduce the proofs of [17, Proposition 6.8 and Theorem 6.9] for the case of a trivial valuation (in which case the proofs are simplified) in order to obtain the following facts:

(i) The functor $\mathbb{L}\iota^*$ restricts to a functor $\mathbf{DA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(k,\Lambda) \to \mathbf{sPerfDA}^{\mathrm{eff}}_{\widehat{\mathbb{A}}^1}(k,\Lambda)$.

(ii) The composition $\mathbb{R}j_*\mathbb{L}\iota^*$ is canonically equivalent to $\mathbb{L}\operatorname{Perf}^*$ and restricts to a monoidal, triangulated equivalence of categories $\mathbf{DA}_{\mathrm{Frob\acute{e}t}}(k,\Lambda) \cong \mathbf{PerfDA}(k,\Lambda)$.

The facts above prove the first claim. The commutativity of the diagram in the statement follows easily from the following commutative diagram, where we omit Λ for brevity.

$$
\begin{array}{ccccccc}
\mathbf{DA}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(k) & \xrightarrow{\mathbb{L}\iota^*} & \mathbf{sPerfDA}^{\mathrm{eff}}(k) & \longrightarrow & \mathbf{sPerfDA}^{\mathrm{eff}}_{\widehat{\mathbb{A}}^1}(k) & \xleftarrow{\mathbb{L}j^*} & \mathbf{PerfDA}^{\mathrm{eff}}(k) \\
\sim \uparrow & & \sim \uparrow & & \sim \uparrow & & \sim \uparrow \\
\mathbf{FormDA}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(K^\circ) & \xrightarrow{\mathbb{L}\iota^*} & \mathbf{sPerfDA}^{\mathrm{eff}}(K^\circ) & \longrightarrow & \mathbf{sPerfDA}^{\mathrm{eff}}_{\widehat{\mathfrak{B}}^1}(K^\circ) & \xleftarrow{\mathbb{L}j^*} & \mathbf{PerfDA}^{\mathrm{eff}}(K^\circ) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(K) & \xrightarrow{\mathbb{L}\iota^*} & \mathbf{sPerfDA}^{\mathrm{eff}}(K) & \longrightarrow & \mathbf{sPerfDA}^{\mathrm{eff}}_{\widehat{\mathbb{B}}^1}(K) & \xleftarrow{\mathbb{L}j^*} & \mathbf{PerfDA}^{\mathrm{eff}}(K)
\end{array}
$$

From the diagram above, we also deduce that for the second claim, it is sufficient to provide a natural transformation $\mathbb{L}(\cdot)^*_\eta \mathbb{R}j_* \Rightarrow \mathbb{R}j_*\mathbb{L}(\cdot)^*_\eta$ and an invertible natural transformation $\mathbb{L}(\cdot)^*_\eta \mathbb{L}\iota^* \cong \mathbb{L}\iota^*\mathbb{L}(\cdot)^*_\eta$. The latter follows from the commutation between the functors $(\cdot)_\eta$ and ι. The former can be obtained by adjunction from the transformation:

$$
\mathbb{L}j^* \circ \mathbb{L}(\cdot)^*_\eta \circ \mathbb{R}j_* \cong \mathbb{L}(\cdot)^*_\eta \circ \mathbb{L}j^* \circ \mathbb{R}j_* \Rightarrow \mathbb{L}(\cdot)^*_\eta.
$$

We now prove the last point. In case $\operatorname{char}K = p$ it boils down to the commutation between Perf and the base change thanks to assertion in (1).

We then assume $\operatorname{char}K = 0$. In order to prove that a natural transformation is invertible, it suffices to test it on a specific set of generators of the category $\mathbf{DA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(k,\Lambda)$ such as motives $M = \Lambda(\mathfrak{X}_\sigma)$ associated to special fibers of formal schemes $\mathfrak{X}$ endowed with an étale map over $\mathfrak{T}^N$. We now fix such an $\mathfrak{X}$.

By means of Theorem 2.16 we have that $\mathbb{R}(\cdot)_{\sigma*}M \cong \Lambda(\mathfrak{X})$. Similarly, since $(\mathfrak{X} \times_{\mathfrak{T}^N} \widehat{\mathfrak{T}}^N)_\sigma \cong \mathfrak{X}^{\mathrm{Perf}}_\sigma$ we deduce that $\mathbb{R}(\cdot)_{\sigma*}\Lambda(\mathfrak{X}^{\mathrm{Perf}}_\sigma) \cong \Lambda(\mathfrak{X} \times_{\mathfrak{T}^N} \widehat{\mathfrak{T}}^N))$. We then conclude the following sequence of isomorphisms:

$$
\xi^*\mathbb{L}\operatorname{Perf}^* M \cong \mathbb{L}(\cdot)^*_\eta \mathbb{R}(\cdot)_{\sigma*}\Lambda(\mathfrak{X}^{\mathrm{Perf}}_\sigma) \cong \mathbb{L}(\cdot)^*_\eta \Lambda(\mathfrak{X} \times_{\mathfrak{T}^N} \widehat{\mathfrak{T}}^N)) \cong \Lambda(\mathfrak{X}_\eta \times_{\mathbb{T}^N} \widehat{\mathbb{T}}^N).
$$

This object is isomorphic, by means of the transformation above, to

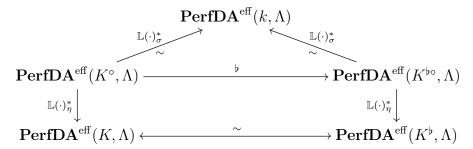$$
\mathbb{R}j_*\mathbb{L}\iota^*\xi M \cong \mathbb{R}j_*\mathbb{L}\iota^*\mathbb{L}(\cdot)^*_\eta \mathbb{R}(\cdot)_{\sigma*}M \cong \mathfrak{G}\Lambda(\mathfrak{X}_\eta)
$$

as shown in [17, Proposition 5.4 and Corollary 7.15]. $\qquad\square$

Proposition 3.7. *The following diagram is commutative, up to a canonical transformation.*

$$\mathbf{PerfDA}^{\mathrm{eff}}(k,\Lambda)$$

$$\xi \swarrow \qquad \searrow \xi^{\flat}$$

$$\mathbf{PerfDA}^{\mathrm{eff}}(K,\Lambda) \xleftarrow{\quad\sim\quad} \mathbf{PerfDA}^{\mathrm{eff}}(K^{\flat},\Lambda)$$

Proof. We consider the following diagram

$$\mathbf{PerfDA}^{\mathrm{eff}}(k,\Lambda)$$

$$\mathbb{L}(\cdot)^{*}_{\sigma}\nearrow \sim \qquad \sim \nwarrow \mathbb{L}(\cdot)^{*}_{\sigma}$$

$$\mathbf{PerfDA}^{\mathrm{eff}}(K^{\circ},\Lambda) \xrightarrow{\quad\flat\quad} \mathbf{PerfDA}^{\mathrm{eff}}(K^{\flat\circ},\Lambda)$$

$$\mathbb{L}(\cdot)^{*}_{\eta}\Big\downarrow \qquad\qquad \mathbb{L}(\cdot)^{*}_{\eta}\Big\downarrow$$

$$\mathbf{PerfDA}^{\mathrm{eff}}(K,\Lambda) \xleftarrow{\quad\sim\quad} \mathbf{PerfDA}^{\mathrm{eff}}(K^{\flat},\Lambda)$$

where the functor $\flat$ is induced by $A \mapsto A^{\flat} = \varprojlim A/p$. It sends smooth perfectoid formal affinoid spaces over K° to the same objects over $K^{\flat\circ}$ and preserves the étale topology.

By definition, this functor makes the triangle as well as the rectangle of the diagram commute. On the other hand, since the two sides of the triangle are equivalences, we deduce that it also is.

We conclude that there is an invertible natural transformation from ξ to $\xi^{\flat}$ as wanted. □

Proof of Theorem 3.2. We combine the diagrams of Theorem 3.6 for K and $K^{\flat}$ with the diagram of Proposition 3.7 obtaining the following (we can assume $\mathrm{char}\,K = 0$ and we omit Λ):

$$\overset{\mathrm{id}}{\overgroup{\qquad\qquad\qquad\qquad\qquad\qquad\qquad}}$$

$$\mathbf{DA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(k) \xrightarrow[\sim]{\mathbb{L}\,\mathrm{Perf}^{*}} \mathbf{PerfDA}^{\mathrm{eff}}(k) =\!=\!= \mathbf{PerfDA}^{\mathrm{eff}}(k) \xleftarrow[\sim]{\mathbb{L}\,\mathrm{Perf}^{*}} \mathbf{DA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(k)$$

$$\Big\downarrow \qquad\qquad \Big\downarrow \qquad\qquad \Big\downarrow \qquad\qquad \Big\downarrow$$

$$\mathbf{RigDA}^{\mathrm{eff}}(K) \xrightarrow{\ \sim\ } \mathbf{PerfDA}^{\mathrm{eff}}(K) \xleftarrow{\ \sim\ } \mathbf{PerfDA}^{\mathrm{eff}}(K^{\flat}) \xleftarrow{\ \sim\ } \mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(K^{\flat})$$

$$\underset{\mathfrak{G}_{K,K^{\flat}}}{\undergroup{\qquad\qquad\qquad\qquad\qquad\qquad\qquad}}$$

□

As stated in the introduction, this commutativity gives an explicit formula for tilting motives of varieties of good reduction.

Corollary 3.8. *Let M be in $\mathbf{FormDA}^{\mathrm{eff}}(K^{\circ},\Lambda)$. The tilt of the motive $\mathbb{L}(\cdot)^{*}_{\eta}M$ is the motive $\mathbb{L}(\cdot\widehat{\times}_{\mathrm{Spec}\,k}\mathrm{Spa}\,K^{\flat})\mathbb{L}(\cdot)^{*}_{\sigma}M$. In particular, the tilt of the motive $\Lambda(\mathfrak{X}_{\eta})$ of a rigid analytic variety of good reduction is the motive associated to $\mathfrak{X}_{\sigma} \times_{k} \mathrm{Spa}\,K^{\flat}$ that is, the analytification of the special fiber base-changed to $K^{\flat}$. It is also isomorphic to the motive of $Q_{\mathrm{rig}}(\mathfrak{X}_{\sigma})$.*
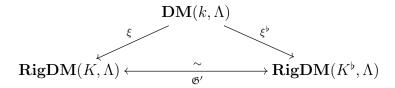
Proof. By Theorem 2.16 the motive $\mathbb{L}(\cdot)^{*}_{\eta}M$ is canonically isomorphic to $\xi\mathbb{L}(\cdot)^{*}_{\sigma}M$ and hence its tilt is isomorphic to $\xi^{\flat}\mathbb{L}(\cdot)^{*}_{\sigma}M$ by Theorem 3.2. The statement follows from the explicit description of $\xi^{\flat}$ given in Proposition 3.5. □

We conclude this section by mentioning the stable version of Theorem 3.2. By formally inverting the Tate shift $\Lambda(1)$ it is possible to define the categories $\mathbf{DM}(k,\Lambda)$ and $\mathbf{RigDM}(K,\Lambda)$. We refer to [6, Definition 2.5.27] for their formal definition. Under the assumption $\mathbb{Q} \subset \Lambda$ these categories fully faithfully contain the effective categories of motives $\mathbf{DM}^{\mathrm{eff}}(k,\Lambda)$ and

$\mathbf{RigDM}^{\mathrm{eff}}(K, \Lambda)$ (see [6, 2.5.49]) and are generated (as triangulated categories with small sums) by the Tate twists of effective motives.

Also in the stable setting, adding transfers or not makes no difference [1, Appendix B] as long as we consider the Frobét-topology and the ring of coefficients Λ contains $\mathbb{Q}$ in the rigid setting [19]. We can therefore write alternatively $\mathbf{DA}(K, \Lambda)$ or $\mathbf{DM}(K, \Lambda)$, $\mathbf{RigDA}_{\mathrm{Frobét}}(K, \Lambda)$ or $\mathbf{RigDM}(K, \Lambda)$.

Corollary 3.9. *The following diagram is commutative, up to a natural transformation, and extends the triangle of Theorem 3.2.*

$$
\begin{array}{ccc}
 & \mathbf{DM}(k, \Lambda) & \\
 \xi \swarrow & & \searrow \xi^{\flat} \\
\mathbf{RigDM}(K, \Lambda) & \xleftarrow[\mathfrak{G}']{\sim} & \mathbf{RigDM}(K^{\flat}, \Lambda)
\end{array}
$$

Proof. The diagrams and the natural transformations of Theorems 3.6 and 3.7 can be constructed also for stable motives, since the left Quillen functors which appear there preserve the Tate shift. We can then adopt the same strategy as in the proof of Theorem 3.2: in order to prove that a natural transformation of functors between stable motives is invertible, it is sufficient to test it on Tate twists of effective ones. As the functors involved preserve the Tate twist, the result follows from the check made in Theorem 3.2. $\qquad\square$

4. DE RHAM COHOMOLOGY OVER LOCAL FIELDS OF POSITIVE CHARACTERISTIC

Suppose that K is a complete non-archimedean field of characteristic 0. In [18] we proved that the overconvergent de Rham cohomology of rigid analytic varieties over K is represented by a motive of $\mathbf{RigDA}^{\mathrm{eff}}(K, \Lambda)$. We briefly recall here its construction. We can consider the complex of overconvergent differential forms $\Omega^{\dagger}$ which is a complex of presheaves on smooth dagger varieties. It induces an object of the category of dagger motives $\mathbf{RigDA}^{\dagger\,\mathrm{eff}}(K, \Lambda)$ when $K \subset \Lambda$ and hence a motive of $\mathbf{RigDM}^{\mathrm{eff}}(K, \Lambda)$ by means of the canonical equivalence

$$\mathbb{L}l^*\colon \mathbf{RigDA}^{\dagger\,\mathrm{eff}}(K, \Lambda) \cong \mathbf{RigDA}^{\mathrm{eff}}(K, \Lambda) \cong \mathbf{RigDM}^{\mathrm{eff}}(K, \Lambda).$$

By abuse of notation, we will still denote the motive $\mathbb{L}l^*\Omega^{\dagger}$ by $\Omega^{\dagger}$.

Remark 4.1. We show in [18] that the induced cohomology theory defined on $\mathbf{DM}(k, \Lambda)$ via the functor $\xi\colon \mathbf{DM}(k, \Lambda) \to \mathbf{RigDM}(K, \Lambda)$ and the complex $\Omega^{\dagger}$ is the usual rigid cohomology. We remark that by functoriality of this construction and the fact that the relative Frobenius is invertible in $\mathbf{DM}(k, \Lambda)$ (see [19] for example) this cohomology theory gains automatically a canonical action of Frobenius in case k is a finite field. The same fact holds for formal motives, and formal perfectoid motives, beacuse of the equivalences $\mathbf{FormDA}(K^{\circ}, \Lambda) \cong \mathbf{DA}(k, \Lambda) \cong \mathbf{PerfDA}(K^{\circ}, \Lambda)$.

We now suppose that K is perfectoid. By using the rigid analytic tilting equivalence, the motive $\Omega^{\dagger}$ represents a cohomology theory also on rigid analytic varieties over $K^{\flat}$ as well as on (locally small) smooth perfectoid spaces over K (or $K^{\flat}$). We now list the formal properties of such cohomology theories, and their relationship with rigid cohomology of the special fibers, in light of the results of Section 3.

Definition 4.2. Let K' be a perfectoid field of any characteristic, and let K be an untilt of $K'^{\flat}$ of mixed characteristic. Put $\Lambda = K$. Let

$$\mathfrak{G}_{K',K}\colon \mathbf{RigDM}^{\mathrm{eff}}(K', \Lambda) \xrightarrow{\sim} \mathbf{RigDM}^{\mathrm{eff}}(K'^{\flat}, \Lambda) \xleftarrow{\sim} \mathbf{RigDM}^{\mathrm{eff}}(K, \Lambda)$$

be the canonical motivic tilting equivalence from rigid motives over K' to the ones over K. The K-*de Rham (overconvergent) cohomology* $H_{\mathrm{dR}}^i(M, K)$ of a rigid analytic motive M over K' is the overconvergent de Rham cohomology of the motive $\mathfrak{G}_{K',K} M$.

Let now M be an algebraic motive in $\mathbf{DM}^{\mathrm{eff}}(K', \Lambda)$. The K-*de Rham (overconvergent) cohomology* $H_{\mathrm{dR}}^i(M, K)$ of M is the K-de Rham cohomology of the motive $\mathbb{L} \operatorname{Rig}^* M$.

Remark 4.3. The above definition specializes obviously to the following objects:

(1) Smooth rigid analytic varieties X over (any valued subfield of) $K'^\flat$: it suffices to consider $M = \Lambda(X)$.
(2) Arbitrary quasi-projective algebraic varieties X over (any subfield of) $K'^\flat$: it suffices to consider $M = \mathbb{L} \operatorname{Rig}^* \Lambda(X)$. If X is smooth, this motive is $\Lambda(X^{\mathrm{an}})$.

Remark 4.4. If K has mixed characteristic, the K-de Rham cohomology over $\mathbf{RigDM}^{\mathrm{eff}}(K, \Lambda)$ coincides with the usual overconvergent de Rham cohomology, and hence the the K-de Rham cohomology over $\mathbf{DM}^{\mathrm{eff}}(K, \Lambda)$ is the usual algebraic de Rham cohomology with coefficients in K.

Remark 4.5. In general, the K-de Rham overconvergent cohomology of algebraic varieties over $K^\flat$ does not coincide with rigid cohomology: indeed, its coefficients lie in K which is way "smaller" than a non-archimedean field with residue $K^\flat$. We also point out that the topology of $K^\flat$ plays a crucial role in this definition, as it is not considered merely as an abstract field (the tilting equivalence is used crucially in the construction).

Definition 4.6. An object X of a triangulated category with small sums $\mathbf{T}$ is *compact* if for any small collection $\{Y_i\}$ of objects in $\mathbf{T}$ one has

$$\operatorname{Hom}(X, \bigoplus Y_i) \cong \bigoplus \operatorname{Hom}(X, Y_i).$$

Example 4.7. The motive $\Lambda(X)$ of a quasi-compact smooth variety X over K is compact in $\mathbf{RigDA}^{\mathrm{eff}}(K, \Lambda)$ (see [6, Proposition 1.2.34]).

Proposition 4.8. *The K-de Rham cohomology $H_{\mathrm{dR}}^i(M, K)$ satisfies the following properties:*

(1) *It is represented by the motive $\mathfrak{G}_{K,K'} \Omega^\dagger$ i.e. $H_{\mathrm{dR}}^i(M, K) \cong H_i \operatorname{Hom}_\bullet(M, \mathfrak{G}_{K.K'} \Omega^\dagger)$.*
(2) *If M is compact, then it is finite dimensional and equal to 0 for $|i| \gg 0$.*
(3) *It satisfies étale descent.*
(4) *It is homotopy-invariant.*
(5) *It is compatible with field extensions L'/K'.*
(6) *It satisfies the Künneth formula on compact motives i.e.*

$$H_{\mathrm{dR}}^n(M \otimes N, K) \cong \bigoplus_{i+j=n} H_{\mathrm{dR}}^i(M, K) \otimes H_{\mathrm{dR}}^j(N, K)$$

for M, N compact.

Proof. By construction, it suffices to prove the statement for the K-de Rham cohomology of analytic K-varieties, that is the overconvergent de Rham cohomology with coefficients in K. All the properties above are proved in [18, Section 5]. $\square$

Remark 4.9. By means of the equivalence $\mathbf{PerfDA}(K, \Lambda) \cong \mathbf{RigDA}(K, \Lambda)$ we can define a K'-de Rham cohomology for locally small perfectoid spaces (that is, perfectoid spaces which are locally étale over the perfectoid ball) enjoying all the properties liste in Proposition 4.8.

The following corollary specifies how this cohomology theory is an extension of rigid cohomology.

Corollary 4.10. *The K-de Rham cohomology of a rigid analytic variety X over $K^\flat$ with good reduction coincides with the rigid cohomology of its reduction $\bar{X}$ with coefficients in K.*

Proof. Let $\mathfrak{X}$ be a smooth formal model of X. From Theorems 2.16 and 3.2 we obtain the following isomrophisms:

$$\mathfrak{G}_{K^\flat,K}\Lambda(X) \cong \mathfrak{G}_{K^\flat,K}\mathbb{L}(\cdot)_\eta^*\Lambda(\mathfrak{X}) \cong \mathfrak{G}_{K^\flat,K}\xi\mathbb{L}(\cdot)_\sigma^*\Lambda(\mathfrak{X}) \cong \xi\Lambda(\mathfrak{X}_\sigma).$$

We already showed in [18] that the overconvergent de Rham cohomology of a motive of the form ξM coincides with the rigid cohomology of M hence the claim. $\square$

Remark 4.11. More generally, we proved that the K-de Rham cohomology of a rigid analytic motive over $K^\flat$ of the form $\mathbb{L}(\cdot)_\eta^*M$ coincides with the rigid cohomology of its reduction $\mathbb{L}(\cdot)_\sigma^*M$ with coefficients in K.

The following corollary provides an alternative method to define and compute rigid cohomology. In order to pass from the characteristic p side to the characteristic 0 side, we can either find a formal model over a valuation ring of mixed characteristic and then consider its rigid analytic generic fiber (this is the classic recipe of Monsky and Washnitzer) we can alternatively take the perfection, and then use Scholze's tilting, at the cost of enlarging the ring of coefficients. We recall that the chosen section $k \to K^{\flat\circ}$ induces a map $k \to K^\circ/p$ and hence a map $W(k) \to K^\circ$.

Corollary 4.12. *Let k be the residue field of a perfectoid field K. Up to a base change along $W(k)[1/p] \to K$, we can compute the rigid cohomology of a smooth variety $\bar{X}$ over k as the K-de Rham cohomology of $\bar{X}_{K^\flat}$.*

Proof. It suffices to read the equivalence of Remark 4.11 backwards. $\square$

Remark 4.13. In the above definition of a de Rham cohomology for $X/K^\flat$ a choice of an untilt of $K^\flat$ is necessary. It is expected to "globalize" this definition and to associate to a variety $X/K^\flat$ a vector bundle over the Fargues-Fontaine curve related to $K^\flat$ such that its stalk at a point x_K corresponding to an untilt K of $K^\flat$ coincides with $H_{dR}^i(X,K)$.

5. On a conjecture of Ayoub

Along this section, we make the following hypotheses. In particular, we no longer require K to be perfectoid.

Assumption 5.1. We let K be a complete valued field with respect to a non-archimedean valuation of rank 1. We let K° be its ring of integers and k its residue field, which we assume to have characteristic $p > 0$. We also suppose that Λ is a $\mathbb{Q}$-algebra.

In the final sections of [6] Ayoub proves a series of results on the relations between the categories of rigid motives $\mathbf{RigDM}(K,\Lambda)$ and some categories of motives defined over the residue field k. The most striking result holds for $K \cong \mathbb{C}((t))$ where $\mathbf{RigDM}(K,\Lambda)$ is shown to embed in $\mathbf{DM}(\mathbb{G}_{m,\mathbb{C}},\Lambda)$. This fact allowed Ayoub to define a motivic version of the decomposition Galois group as well as some important non-obvious applications ([5], [7] and [8]).

The case of positive residue characteristic is more delicate. An analogous, yet weaker, statement for fields of the form $k((t))$ with $\operatorname{char} k = p$ is still proved in [6] and deals with rigid analytic motives *of good reduction*. We now recall it, and prove its analogue in the mixed characteristic case, which was conjectured by Ayoub. It is surely hoped to adapt the motivic Galois group-theoretical machinery to this situation in order to produce some generalizations of the consequences obtained in the equal characteristic 0 case.

Remark 5.2. In this section, we will consider complete valued fields which may not be perfect. Therefore, we can not refer to Theorem 2.13. All our statements will be given in terms of motives with transfers, even though in proofs, motives without transfers will also be used.

Definition 5.3. We denote by $\mathbf{RigDM}^{\mathrm{eff,gr}}(K, \Lambda)$ [resp. $\mathbf{RigDM}^{\mathrm{gr}}(K, \Lambda)$] the triangulated subcategory of $\mathbf{RigDM}^{\mathrm{eff}}(K, \Lambda)$ [resp. $\mathbf{RigDM}(K, \Lambda)$] closed by small sums generated by [Tate twists of] motives of the form $\Lambda_{\mathrm{tr}}(\mathfrak{X}_\eta)$ with $\mathfrak{X}$ a smooth affine formal scheme, topologically of finite type over K°. Their elements are called *motives of good reduction*.

Remark 5.4. Since the motives of the form $\Lambda_{\mathrm{tr}}(\mathfrak{X})$ with $\mathfrak{X}$ affine and smooth generate $\mathbf{FormDA}^{\mathrm{eff}}(K^\circ, \Lambda)$ as a triangulated category with small sums, we deduce that the category $\mathbf{RigDM}^{\mathrm{eff,gr}}(K, \Lambda)$ [resp. $\mathbf{RigDM}^{\mathrm{gr}}(K, \Lambda)$] is equally generated by the motives of the form $\mathbb{L}a_{\mathrm{tr}} \circ \mathbb{L}(\cdot)_\eta M$ with M in $\mathbf{FormDA}^{\mathrm{eff}}(K^\circ, \Lambda)$ [resp. in $\mathbf{FormDA}(K^\circ, \Lambda)$] where $\mathbb{L}a_{\mathrm{tr}} \colon \mathbf{RigDA}^{\mathrm{eff}}(K, \Lambda) \to \mathbf{RigDM}^{\mathrm{eff}}(K, \Lambda)$ is the "adding transfers" functor, that is the (left Quillen derived) functor induced by the inclusion $\mathrm{RigSm}\,/K \to \mathrm{RigCor}\,/K$.

Remark 5.5. The category $\mathbf{RigDM}^{\mathrm{gr}}(K, \Lambda)$ is denoted by $\mathbf{RigDM}^{\mathrm{br}}(K, \Lambda)$ in [6] according to the French terminology *bonne reduction*.

Remark 5.6. Having a description of the category $\mathbf{RigDM}^{\mathrm{eff,gr}}(K, \Lambda)$ has important consequences also for arbitrary compact motives in $\mathbf{RigDM}^{\mathrm{eff}}(K, \Lambda)$. Indeed, as shown in [6, Theorem 2.5.34] *any* compact motive M is of good reduction, up to a finite extension K'/K of the base field.

Proposition 5.7. *An object M of $\mathbf{RigDM}^{\mathrm{eff,gr}}(K, \Lambda)$ [resp. $\mathbf{RigDM}^{\mathrm{gr}}(K, \Lambda)$] is compact if and only if it is compact in the bigger category $\mathbf{RigDM}^{\mathrm{eff}}(K, \Lambda)$ [resp. $\mathbf{RigDM}(K, \Lambda)$].*

Proof. By [2, Theorem 2.1.24] we obtain that the triangulated subcategory of compact objects in $\mathbf{RigDM}^{\mathrm{eff,gr}}(K, \Lambda)$ [resp. $\mathbf{RigDM}^{\mathrm{gr}}(K, \Lambda)$] is generated by the [twists of] direct factors of motives of the form $\Lambda_{\mathrm{tr}}(\mathfrak{X}_\eta)$ with $\mathfrak{X}$ an affine smooth formal scheme. Such motives are also compact in $\mathbf{RigDM}^{\mathrm{eff}}(K, \Lambda)$ hence the claim. $\qquad\square$

Remark 5.8. Thanks to the previous proposition, the class of compact motives of $\mathbf{RigDM}(K, \Lambda)$ which are also of good reduction coincide with the class of compact objects in the subcategory $\mathbf{RigDM}^{\mathrm{gr}}(K, \Lambda)$ of motives of good reduction. We can then unambiguously refer to them as "compact motives of good reduction".

The following theorem by Ayoub amounts to a description of rigid motives of good reduction in terms of algebraic motives over the residue field. We recall that Λ is supposed to be a $\mathbb{Q}$-algebra.

Definition 5.9. Let S be an algebraic variety over a field k and let Λ be a ring. We denote by $\mathbf{DM}(S, \Lambda)$ [resp. $\mathbf{DM}^{\mathrm{eff}}(S, \Lambda)$] the category of [effective] motives over S following Voevodsky. We denote by $\mathbf{U^n DM}(k, \Lambda)$ [resp. $\mathbf{U^n DM}^{\mathrm{eff}}(k, \Lambda)$] the triangulated subcategory with small sums of $\mathbf{DM}(\mathbb{G}_{m,k}^n, \Lambda)$ [resp. $\mathbf{DM}^{\mathrm{eff}}(\mathbb{G}_{m,k}^n, \Lambda)$] generated by [twists of] motives of the varieties of the form $\mathbb{G}_{m,X}^n$ with X smooth over k.

Theorem 5.10 ([6, Theorem 2.5.57]). *Let K be a complete valued field of equal characteristic, with residue field k and a value group which is free of rank n over $\mathbb{Z}$. Fix a section $k \to K^\circ$ of the reduction and some elements $\pi_1, \ldots, \pi_n$ of K° whose values generate $|K^\times|$. If the characteristic of K is positive, we also suppose that $k[\pi_1^{\pm 1}, \ldots, \pi_n^{\pm 1}]$ is dense in K. The composite functors*

$$\mathbf{U^n DM}^{\mathrm{eff}}(k, \Lambda) \hookrightarrow \mathbf{DM}^{\mathrm{eff}}(\mathbb{G}_{m,k}^n, \Lambda) \xrightarrow{(\pi_1, \ldots, \pi_n)^*} \mathbf{DM}^{\mathrm{eff}}(K, \Lambda) \xrightarrow{\mathrm{Rig}^*} \mathbf{RigDM}^{\mathrm{eff}}(K, \Lambda)$$

$$\mathbf{U^n DM}(k, \Lambda) \hookrightarrow \mathbf{DM}(\mathbb{G}_{m,k}^n, \Lambda) \xrightarrow{(\pi_1, \ldots, \pi_n)^*} \mathbf{DM}(K, \Lambda) \xrightarrow{\mathrm{Rig}^*} \mathbf{RigDM}(K, \Lambda)$$

take values in the subcategories of motives of good reduction, and induce monoidal, triangulated equivalences

$$\mathbf{U^n DM}^{\mathrm{eff}}(k, \Lambda) \cong \mathbf{RigDM}^{\mathrm{eff,gr}}(K, \Lambda) \quad \mathbf{U^n DM}(k, \Lambda) \cong \mathbf{RigDM}^{\mathrm{gr}}(K, \Lambda)$$

Remark 5.11. We point out that the fact above has important applications also for arbitrary compact motives (not necessarily of good reduction). Indeed, following Remark 5.6 we deduce that for any pair of compact motives M, M' we can calculate $\mathrm{Hom}_{\mathbf{RigDM}(K,\Lambda)}(M, M')$ in terms of the Galois-invariants in a Hom-group of $\mathbf{DM}(\mathbb{G}_{m,k'}^n, \Lambda)$ for some finite Galois extension K'/K (here k' is the residue field of K') see [6, Remark 2.5.71].

Remark 5.12. The previous result allows to compute the functor $\chi\colon \mathbf{RigDM}(K,\Lambda) \to \mathbf{DM}(k, \Lambda)$ on motives of good resuction in an alternative way, as the following composition:

$$\mathbf{RigDM}^{\mathrm{gr}}(K, \Lambda) \cong \mathbf{U^n}\mathbf{DM}(k, \Lambda) \subset \mathbf{DM}(\mathbb{G}_{m,k}, \Lambda) \xrightarrow{P_\sharp} \mathbf{DM}(k, \Lambda)$$

with $P\colon \mathbb{G}_{m,k} \to \mathrm{Spec}\, k$ being the structure morphism. In particular $\chi\Lambda \cong \Lambda \oplus \Lambda(-1)[-2]$.

Ayoub conjectured that the last two equivalences of the theorem above still hold true in the mixed characteristic case, under some technical hypotheses on K (see Assumption 5.14).

Our strategy is simple, and we first sketch it for the case of $\mathbb{Q}_p$ (the general case will be proved in Corollary 5.25). In this case, the conjecture takes the following form.

Theorem 5.13. *There are triangulated, monoidal equivalences of categories*

$$\mathbf{U^1}\mathbf{DM}^{\mathrm{eff}}(\mathbb{F}_p, \Lambda) \cong \mathbf{RigDM}^{\mathrm{eff,gr}}(\mathbb{Q}_p, \Lambda) \quad \mathbf{U^1}\mathbf{DA}(\mathbb{F}_p, \Lambda) \cong \mathbf{RigDM}^{\mathrm{gr}}(\mathbb{Q}_p, \Lambda)$$

Let $\widehat{K}$ be the completion of the field $\mathbb{Q}_p(\mu_{p^\infty})$. It is a perfectoid field with tilt $\widehat{K}^\flat$ isomorphic to the completion of $\mathbb{F}_p((t))(t^{1/p^\infty})$. This field is the perfection of $K^\flat := \mathbb{F}_p((t))$ which is a field that satisfies the assumptions of Theorem 5.10 and has the same residue field $\mathbb{F}_p$ as $\mathbb{Q}_p$. The analogue of the theorem above for $K = \mathbb{Q}_p$ will then be reached via the following diagram of equivalences:

$$(1) \qquad \begin{array}{ccc} \mathbf{RigDM}^{\mathrm{gr}}(K, \Lambda) & & \mathbf{RigDM}^{\mathrm{gr}}(K^\flat, \Lambda) \\ \sim\downarrow & & \downarrow\sim \\ \mathbf{RigDM}^{\mathrm{gr}}(\widehat{K}, \Lambda) & \xleftarrow{\ \sim\ } & \mathbf{RigDM}^{\mathrm{gr}}(\widehat{K}^\flat, \Lambda) \end{array}$$

We now introduce Ayoub's conjecture for a general field K of mixed characteristic (not just $\mathbb{Q}_p$). We recall that in the case of equal characteristic p we assumed that $k[\pi_1^{\pm 1}, \ldots, \pi_n^{\pm 1}]$ is dense in K. Similarly, in the case of a mixed characteristic field we need to consider some special assumptions on the valuation, that we list below:

Assumption 5.14. From now on, we fix a complete, non-archimedean field K over $\mathbb{Q}_p$ with the following extra properties:

(i) There are n elements $\pi_1, \ldots, \pi_n$ of K° with π_1 algebraic over $\mathbb{Q}_p$ such that the group $|K^\times|$ is free of rank n generated by the valuations of π_i.
(ii) There is a complete discrete valued subfield K_0 having π_1 as uniformizer and the same residue field k as K.
(iii) The ring $K_0[\pi_1^{\pm 1}, \ldots, \pi_n^{\pm 1}]$ is dense in K.

Remark 5.15. Any finite field extension of $\mathbb{Q}_p$ obviously satisfies Assumption 5.14.

In the case of $\mathbb{Q}_p$ we introduced the intermediate, auxiliary fields $\mathbb{Q}_p(p^{1/p^\infty})$ and $\mathbb{F}_p((t))$. We now try to generalize this situation.

Definition 5.16. If $\mathrm{char}\, K = 0$ we say $\widehat{K}$ is a [Galois] *completed perfection* of K if it is a perfectoid field obtained as the completion of a [Galois] algebraic extension of K inducing purely inseparable extensions of residue fields.

Proposition 5.17. *Let K satisfy Assumption 5.14. There exists a Galois completed perfection $\widehat{K}$ of K and a complete valued field $K^\flat$ of positive characteristic with the following properties:*

(i) $\widehat{K}$ has a value group which is a free module over $\mathbb{Z}[1/p]$ of rank n.
(ii) $\widehat{K}^\flat$ is the completed perfection of $K^\flat$.
(iii) $K^\flat$ has a free value group of rank n over $\mathbb{Z}$ and residue field k.
(iv) We can choose some elements $\pi_1^\flat, \ldots, \pi_n^\flat$ in $K^\flat$ such that their values $|\pi_i^\flat|$ generate the value group, and such that the ring $k[(\pi_i^\flat)^{\pm 1}]$ is dense in $K^\flat$.

Proof. Choose some representatives of a finite subset of $k = K_0^\circ/\pi_1$ in K_0 and some p-th roots of them in a complete algebraic closure C of K. The extensions of K and K_0 generated by these elements satisfy Assumption 5.14 with respect to the same elements π_i and the same value group, and they induce a finite purely inseparable map on the residue fields. With a direct limit process, we obtain a pair (F, F_0) which is algebraic over (K, K_0) and whose completion satisfies Assumption 5.14 with the same value group of K and residue field isomorphic to k^{Perf}.

We now take L to be the completion of $F(\mu_{p^\infty})$. It is the completion of a Galois extension over K and its subfield $L_0 := K_0(\mu_{p^\infty})^\wedge$ is perfectoid, with tilt isomorphic to $k^{\mathrm{perf}}((t))(t^{1/p^\infty})^\wedge$ and value group isomorphic to $\mathbb{Z}[1/p]$ generated by the valuation of $\widetilde{\pi}_1 := t^\sharp$.

We finally consider $\widehat{K}$ to be the completion of $L(\pi_2^{1/p^\infty} \ldots, \pi_n^{1/p^\infty})$ for a choice of compatible p-th roots of each π_i with $i > 1$. Its value group is free over $\mathbb{Z}[1/p]$ generated by the values of $\widetilde{\pi}_1$ and the π_i's with $i > 1$. In order to prove that it is perfectoid, it suffices to show that the Frobenius is surjective on the ring $\widehat{K}^\circ/\pi_1$ which by density coincides with $L_0[\pi_i^{\pm 1/p^\infty}]^\circ/\widetilde{\pi}_1$. This follows from the fact that $L_0^\circ/\widetilde{\pi}_1$ is perfect, and any element of the ring $L_0[\pi_i^{\pm 1/p^\infty}]^\circ$ is a sum of terms of the form $u\widetilde{\pi}_1^{q_1}\pi_2^{q_2} \cdot \ldots \cdot \pi_n^{q_n}$ with $u \in L_0^{\circ\times}$ and $q_i \in \mathbb{Z}[1/p]$. Also the field $\widehat{K}$ is the completion of a Galois extension over K.

We now put $K^\flat$ to be the completion of the subfield $k(\pi_1^\flat, \ldots, \pi_n^\flat)$ of $\widehat{K}^\flat$ where $\pi_1^\flat = t$ and the other elements $\pi_i^\flat$ are chosen to satisfy $|\pi_i^{\flat\sharp}| = |\pi_i|$. In particular, the values $|\pi_i^\flat|$ generate the value group of $K^\flat$ which is a free $\mathbb{Z}$-module of rank n and the residue field of $K^\flat$ is k.

In order to see that the completed perfection of $K^\flat$ coincides with $\widehat{K}^\flat$ it suffices to show that $k^{\mathrm{Perf}}[(\pi_i^\flat)^{\pm 1/p^\infty}]^\circ$ is dense in $\widehat{K}^{\flat\circ}$ i.e. that for any n and any $\lambda \in \widehat{K}^{\flat\circ}$ there exists an element $\xi \in k^{\mathrm{Perf}}[(\pi_i^\flat)^{\pm 1/p^\infty}]^\circ$ such that $\lambda - \xi \in \pi_1^{\flat n}\widehat{K}^{\flat\circ}$. By induction on n it suffices to prove this statement for $n = 1$. We obtain the following isomorphisms:

$$k^{\mathrm{Perf}}[(\pi_i^\flat)^{\pm 1/p^\infty}]_{i \geq 1}^\circ/\pi_1^\flat \cong L_0[\pi_i^{\pm 1/p^\infty}]_{i \geq 2}^\circ/\widetilde{\pi}_1 \cong \widehat{K}^\circ/\widetilde{\pi}_1 \cong \widehat{K}^{\flat\circ}/\pi_1^\flat$$

proving our claim. $\qquad\square$

We start by proving that the tilting equivalence respect motives of good reduction: that is, we prove the bottom horizontal equivalence of the diagram (1). In order to do this, we use the results of the previous sections.

Proposition 5.18. *Let $\widehat{K}$ be a perfectoid field. The motivic tilting equivalence of Theorem 3.1 restricts to equivalences*

$$\mathbf{RigDM}^{\mathrm{eff},\mathrm{gr}}(\widehat{K}, \Lambda) \cong \mathbf{RigDM}^{\mathrm{eff},\mathrm{gr}}(\widehat{K}^\flat, \Lambda) \quad \mathbf{RigDM}^{\mathrm{gr}}(\widehat{K}, \Lambda) \cong \mathbf{RigDM}^{\mathrm{gr}}(\widehat{K}^\flat, \Lambda)$$

Proof. It suffices to show that motives of the form $\mathbb{L}(\cdot)_\eta M$ are sent by $\mathfrak{G}$ to motives of the same form, in the two directions.

This follows from the following commutative diagram (up to a natural transformation) which appears in Theorem 3.2.

$$\mathbf{DA}^{\mathrm{eff}}(k,\Lambda) \xleftarrow{\;\sim\;} \mathbf{FormDA}^{\mathrm{eff}}(K^{\circ},\Lambda) \xrightarrow{\;\mathbb{L}(\cdot)_{\eta}\;} \mathbf{RigDA}^{\mathrm{eff}}(K,\Lambda) \cong \mathbf{RigDM}^{\mathrm{eff}}(K,\Lambda)$$

$$\mathbf{DA}^{\mathrm{eff}}(k,\Lambda) \xleftarrow{\;\sim\;} \mathbf{FormDA}^{\mathrm{eff}}(K^{\flat\circ},\Lambda) \xrightarrow{\;\mathbb{L}(\cdot)_{\eta}\;} \mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(K^{\flat},\Lambda) \cong \mathbf{RigDM}^{\mathrm{eff}}(K^{\flat},\Lambda)$$

with a left vertical $\|$ and a right vertical $\sim$.

$\square$

We now show how to obtain the right vertical equivalence of the diagram (1).

Proposition 5.19. *Suppose* $\mathrm{char}\, K = p$ *and let* $\widehat{K}$ *be its completed perfection. The base-change functor* $(\widehat{K}/K)^{*}$ *induces triangulated, monoidal equivalences:*

$$\mathbf{RigDM}^{\mathrm{eff}}(K,\Lambda) \cong \mathbf{RigDM}^{\mathrm{eff}}(\hat{K},\Lambda) \qquad \mathbf{RigDM}(K,\Lambda) \cong \mathbf{RigDM}(\widehat{K},\Lambda).$$

These equivalences restrict to the subcategories of motives of good reduction.

Proof. Write $\widehat{K}$ as a completion of the direct limit of some finite, purely inseparable extensions of K. It suffices to show the equivalence for compact motives, which follows from Lemmas 5.21 and 5.23 and the remark that any finite, purely inseparable extension K'/K induces an equivalence $\mathbf{RigDM}^{\mathrm{eff}}(K',\Lambda) \cong \mathbf{RigDM}^{\mathrm{eff}}(K,\Lambda)$ (see [6, Proposition 2.2.22]). This equivalence preserves motives of good reduction as one can see from the following commutative diagram, where we set k' to be the residue field of K' which is totally unramified over k.

$$(2) \quad \mathbf{DM}(k) \cong \mathbf{DA}(k) \xleftarrow{\;\sim\;} \mathbf{FormDA}(K^{\circ}) \xrightarrow{\;\mathbb{L}(\cdot)_{\eta}\;} \mathbf{RigDA}(K) \xrightarrow{\;\mathbb{L}a_{\mathrm{tr}}\;} \mathbf{RigDM}(K)$$

with vertical maps $\sim$, $(K'^{\circ}/K^{\circ})^{*}$, $(K'/K)^{*}$

$$\mathbf{DM}(k') \cong \mathbf{DA}(k') \xleftarrow{\;\sim\;} \mathbf{FormDA}(K'^{\circ}) \xrightarrow{\;\mathbb{L}(\cdot)_{\eta}\;} \mathbf{RigDA}(K') \xrightarrow{\;\mathbb{L}a_{\mathrm{tr}}\;} \mathbf{RigDM}(K')$$

$\square$

Remark 5.20. We point out that Proposition 5.19 is the motivic version of the well known isomorphism between the absolute Galois group of a field of characteristic p and the one of its completed perfection.

The lemmas below were used in the previous proof.

Lemma 5.21. *If* $\mathrm{char}\, K = p$ *we let* $\hat{K}$ *be its completed perfection obtained by completing the directed set of Frobenius maps* $K_i \subset K_{i+1}$. *If* $\mathrm{char}\, K = 0$ *we fix a completion* C *of some algebraic closure of* K *and we let* $\widehat{K} \subset C$ *be the completion of the union* $\varinjlim K_h$ *of a small directed system of inclusions* $K_h \subset K_{h'}$ *of complete subfields.*
We obtain the following canonical triangulated, monoidal equivalences:

$$\mathbf{RigDM}^{\mathrm{eff}}(\widehat{K},\Lambda) \cong \mathbf{RigDM}^{\mathrm{eff}}(2-\varinjlim_{i} \mathrm{RigSm}\,/K_i,\Lambda)$$

$$\mathbf{RigDM}(\widehat{K},\Lambda) \cong \mathbf{RigDM}(2-\varinjlim_{i} \mathrm{RigSm}\,/K_i,\Lambda)$$

where the categories on the right are constructed by $(\mathrm{\acute{e}t}, \mathbb{B}^1)$*-localization out of the étale site on the limit category defined by the base change functors* $\mathrm{RigSm}\,/K_i \to \mathrm{RigSm}\,/K_{i+1}$.

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Proof. We only prove the statement for the effective category, for brevity. First, we remark that motives $\Lambda(X)$ associated to varieties defined over some K_h form a set of compact generators of $\mathbf{RigDM}^{\mathrm{eff}}(\hat{K}, \Lambda)$. Indeed, every smooth variety X is locally étale over the ball $\mathbb{B}_{\hat{K}}^n$ and therefore it locally admits a model over some K_h by density and [6, Lemma 1.1.52].

For the same reason, any ét-covering of such an X can be refined with a covering coming from $2 - \varinjlim \mathrm{RigSm}\,/K_h$. We deduce that the functor ι_* coming from the following Quillen adjunction

$$\iota^* \colon \mathbf{Ch}\,\mathbf{Psh}_{\mathrm{ét},\mathbb{B}^1}(2 - \varinjlim \mathrm{RigSm}\,/K_i, \Lambda) \leftrightarrows \mathbf{Ch}\,\mathbf{Psh}_{\mathrm{ét},\mathbb{B}^1}(\mathrm{RigSm}\,/\hat{K}, \Lambda) \colon \iota_*$$

commutes with ét-sheafification and hence preserves $\tau_{\mathrm{ét}}$-weak equivalences. Since it also commutes with $\mathrm{Sing}^{\mathbb{B}^1}$ we deduce it derives trivially.

We now prove that $\mathbb{L}\iota^*$ is fully faithful on a set of generators, say varieties X with an étale map over a poly-disc $\mathbb{B}^n$ defined over some K_h. From Lemma 5.22 and following exactly the same proof as [17, Proposition 4.2] or [18, Proposition 4.22] we obtain that for any smooth affinoid varieties X, Y which are étale over some poly-discs over K the canonical map

$$N\Lambda \operatorname{Hom}_{2-\varinjlim \mathrm{RigSm}\,/K_i}(Y_{K_i} \times \square_{K_i}^\bullet, X_{K_i}) \to N\Lambda \operatorname{Hom}_{\mathrm{RigSm}\,/\hat{K}}(Y_{\hat{K}} \times \square_{\hat{K}}^\bullet, X_{\hat{K}})$$

is a quasi-isomorphism. In other words, we obtain an isomorphism in the derived category $\mathrm{Sing}^{\mathbb{B}^1} \iota_* \mathbb{L}\iota^* \Lambda(X) \cong \mathrm{Sing}^{\mathbb{B}^1} \Lambda(X)$ which implies $\iota_* \mathbb{L}\iota^* \Lambda(X) \cong \Lambda(X)$. It follows that (ι^*, ι_*) is a Quillen equivalence. We conclude that $\mathbf{RigDA}_{\mathrm{ét}}^{\mathrm{eff}}(2 - \varinjlim \mathrm{RigSm}\,/K_i, \Lambda) \cong \mathbf{RigDA}_{\mathrm{ét}}^{\mathrm{eff}}(\hat{K}, \Lambda)$.

To promote the equivalence to motives with transfers $\mathbf{RigDM}$ we use the adjunctions

$$\mathbb{L}a_{\mathrm{tr}} \colon \mathbf{RigDA}(K, \Lambda) \rightleftarrows \mathbf{RigDM}(K, \Lambda) \colon \mathbb{R}o_{\mathrm{tr}}$$

$$\mathbb{L}a_{\mathrm{tr}} \colon \mathbf{RigDA}(2 - \varinjlim \mathrm{RigSm}\,/K_i, \Lambda) \rightleftarrows \mathbf{RigDM}(2 - \varinjlim \mathrm{RigSm}\,/K_i, \Lambda) \colon \mathbb{R}o_{\mathrm{tr}}$$

induced by the "adding transfers" operation. By the results of [19] these adjunctions can also be interpreted as Bousfield localizations over the relative Frobenius maps, and therefore preserve the Quillen equivalence of the statement ([14, Proposition 2.3]). $\qquad\square$

Lemma 5.22. *Under the same hypotheses of Lemma 5.21, we let $X = \operatorname{Spa} R$ be a smooth affinoid variety over K and we denote $R_h := R \otimes K_h$. Any element ξ of $R \widehat{\otimes} \hat{K}$ which is algebraic separable over each generic point of $\operatorname{Spec} \varinjlim R_h$ lies in $\varinjlim R_h$.*

Proof. The proof is a simplified version of [17, Proposition A.3] which we reproduce here briefly for the convenience of the reader.

Step 1. We prove that we can restrict to an arbitrary non-empty rational subset $U(f_i|g)$ of $\operatorname{Spa} R$. Fix a countable set I of linearly independent elements in R generating a dense subset, and complete it to a countable set $I \sqcup J$ of linearly independent elements in $R\langle f_i|g\rangle$ generating a dense subset. By [10, Proposition 2.7.1/3] we can choose I and J in a way that the projection maps induce K-linear homeomorphisms $R \cong \widehat{\bigoplus}_I K$ and $R\langle f_i|g\rangle \cong \widehat{\bigoplus}_{I \sqcup J} K$. Suppose now that ξ lies in the intersection of $R\langle f_i|g\rangle \cong \widehat{\bigoplus}_{I \sqcup J} K$ and $R \widehat{\otimes} \hat{K} \cong \widehat{\bigoplus}_I \hat{K}$ inside $R\langle f_i|g\rangle \widehat{\otimes} \hat{K} \cong \widehat{\bigoplus}_{I \sqcup J} \hat{K}$. By the explicit description of the complete direct sum in terms of the direct product [10, Proposition 2.1.5/7] we deduce that ξ lies in $\widehat{\bigoplus}_I K \cong R$ as wanted.

Step 2. By the previous step we can make the following assumptions. The details are made explicit in [17].

  (1) ξ is algebraic over R.
  (2) The sup-norm is multiplicative on R, $\varinjlim R_h$, $R \widehat{\otimes} \hat{K}$, $R[\xi]$.
  (3) The sup-norm on $R[\xi]$ coincides with the norm induced by its inclusion in $R \widehat{\otimes} \hat{K}$.

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Step 3. Let $\{\xi_i\}$ be the conjugates of ξ different from ξ in the separable closure S of the completion of $\mathrm{Frac}\, R \widehat{\otimes} \hat{K}$ with respect to the sup-norm. We want to prove that this set is empty and we argue by contradiction. By a density argument and by the identity of norms of the previous step, we can fix an element β in $\varinjlim R_h$ such that $|\xi - \beta| < \epsilon := \min\{|\xi_i - \xi|\}$ where we consider the sup-norm in S. It restricts to the sup-norm on R by [10, Lemma 3.8.1/6]. Up to rescaling indexes, we can assume $\beta \in R$. Any choice of an element ξ_i induces a R-linear isomorphism $\tau_i \colon R[\xi] \cong R[\xi_i]$ which is an isometry with respect to the sup-norm. Therefore one has

$$|\xi - \xi_i| \le \max\{|\xi - \beta|, |\xi_i - \beta|\} = \max\{|\xi - \beta|, |\tau_i(\xi - \beta)|\} = |\xi - \beta| < \epsilon$$

leading to a contradiction. $\qquad\square$

Lemma 5.23. *Fix a directed system of strict inclusions of complete valued fields K_i. The canonical functors induce equivalences*

$$\mathbf{RigDM}^{\mathrm{eff,cp}}(2 - \varinjlim \mathrm{RigSm}\,/K_i, \Lambda) \cong 2 - \varinjlim \mathbf{RigDM}^{\mathrm{eff,cp}}(K_i, \Lambda)$$

$$\mathbf{RigDM}^{\mathrm{cp}}(2 - \varinjlim \mathrm{RigSm}\,/K_i, \Lambda) \cong 2 - \varinjlim \mathbf{RigDM}^{\mathrm{cp}}(K_i, \Lambda)$$

Proof. We let ι_{ij} be the base change functors $\mathrm{RigSm}\,/K_i \to \mathrm{RigSm}\,/K_j$ induced by the arrows of the directed system, and ι_i be the one induced by $\mathrm{RigSm}\,/K_i \to \varinjlim \mathrm{RigSm}\,/K_i$.

Fix an index 0 of the directed system. It suffices to prove that for any two motives M, N in a set of generators for $\mathbf{RigDM}(K_0, \Lambda)$ the following isomorphism holds

$$\varinjlim \mathrm{Hom}_{\mathbf{RigDM}(K_i, \Lambda)}(\mathbb{L}\iota_{0i}^* M, \mathbb{L}\iota_{0i}^* N) \cong \mathrm{Hom}_{\mathbf{RigDM}(\varinjlim \mathrm{RigSm}\,/K_i, \Lambda)}(\mathbb{L}\iota_0^* M, \mathbb{L}\iota_0^* N).$$

We can finally conclude by means of [6, Proposition 1.A.1] to be adapted to the rigid analytic setting with transfers (see [6, Proof of Proposition 2.5.52]). $\qquad\square$

We finally show how to obtain the left vertical equivalence of the diagram (1). We will again use the main result of the previous sections.

Proposition 5.24. *Suppose K satisfies Assumption 5.14 and let $\widehat{K}$ be a Galois completed perfection of K constructed in Proposition 5.17. The base-change functor $(\widehat{K}/K)^*$ induces triangulated, monoidal equivalences:*

$$\mathbf{RigDM}^{\mathrm{eff,gr}}(K, \Lambda) \cong \mathbf{RigDM}^{\mathrm{eff,gr}}(\widehat{K}, \Lambda) \quad \mathbf{RigDM}^{\mathrm{gr}}(K, \Lambda) \cong \mathbf{RigDM}^{\mathrm{gr}}(\widehat{K}, \Lambda).$$

Proof. We only prove the stable version of the statement for brevity. The category $\mathbf{RigDM}^{\mathrm{gr}}(K, \Lambda)$ is generated, as a triangulated category with small sums, by the motives associated to smooth affinoid varieties of good reduction, which are compact. In particular, we deduce that it suffices to show that the subcategories of compact objects (see Proposition 5.7) $\mathbf{RigDM}^{\mathrm{cp,gr}}(\widehat{K}, \Lambda)$ and $\mathbf{RigDM}^{\mathrm{cp,gr}}(K, \Lambda)$ are equivalent.

Suppose that $\widehat{K}$ is the completion of a Galois extension L/K inducing a purely inseparable extension of residue fields. We can write L as a union $\varinjlim K_h$ of finite algebraic extensions K_h/K in a fixed algebraic closure containing $\widehat{K}$. We denote by $\widehat{k}$ the residue field of $\widehat{K}$ which is the union of the residue fields k_h of each K_h.

Let $\mathfrak{X}$ be a formal scheme over $\widehat{K}^\circ$ which is étale over $\mathfrak{B}^n$. From [13, Theorem 18.1.2] and the equality $\widehat{k} \cong \varinjlim k_h$ we deduce the following equivalences of categories:

$$\mathrm{Et}\,\mathrm{Aff}\,/\mathfrak{B}^n_{\widehat{K}^\circ} \cong \mathrm{Et}\,\mathrm{Aff}\,/\mathbb{A}^n_{\widehat{k}} \cong 2 - \varinjlim_h \mathrm{Et}\,\mathrm{Aff}\,/\mathbb{A}^n_{k_h} \cong 2 - \varinjlim_h \mathrm{Et}\,\mathrm{Aff}\,/\mathfrak{B}^n_{K_h^\circ}$$

so that $\mathfrak{X}$ has a model over some K_h°. In particular, we obtain that $\Lambda(\mathfrak{X}_\eta)(n)$ lies in the triangulated category generated by the image of $2 - \varinjlim_h \mathbf{RigDM}^{\mathrm{cp,gr}}(K_h, \Lambda)$. On the other

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hand, the set of motives of the form $\Lambda(\mathfrak{X}_\eta)(n)$ generate the category $\mathbf{RigDM}^{\mathrm{cp,gr}}(\widehat{K}, \Lambda)$. We then deduce from Lemmas 5.21 and 5.23 the following equivalence:

$$2 - \varinjlim_h \mathbf{RigDM}^{\mathrm{cp,gr}}(K_h, \Lambda) \cong \mathbf{RigDM}^{\mathrm{cp,gr}}(\widehat{K}, \Lambda).$$

From the diagram (2) we deduce that each functor $\mathbf{RigDM}^{\mathrm{cp,gr}}(K, \Lambda) \to \mathbf{RigDM}^{\mathrm{cp,gr}}(K_h, \Lambda)$ sends a set of generators to a set of generators. From the previous equivalence, we then conclude that the functor $\mathbf{RigDM}^{\mathrm{cp,gr}}(K, \Lambda) \to \mathbf{RigDM}^{\mathrm{cp,gr}}(\widehat{K}, \Lambda)$ also sends a set of generators to a set of generators. We now prove it is also fully faithful. To this aim, since all compact objects in $\mathbf{RigDM}^{\mathrm{gr}}(K, \Lambda)$ are dualizable, it suffices to prove that

$$\mathrm{Hom}_{\mathbf{RigDM}(K)}(\mathbb{L}(\cdot)_\eta^* M, \Lambda) \cong \mathrm{Hom}_{\mathbf{RigDM}(\widehat{K})}(\mathbb{L}(\cdot)_\eta^* M_{\widehat{K}^\circ}, \Lambda)$$

for each M in $\mathbf{FormDA}^{\mathrm{cp}}(K^\circ)$. From Lemma 5.21 the right hand side can be computed in the category $\mathbf{RigDM}(\varinjlim \mathrm{RigSm}\,/K_h)$ and by Galois descent we deduce

$$\mathrm{Hom}_{\mathbf{RigDM}(K)}(\mathbb{L}(\cdot)_\eta^* M, \Lambda) \cong \mathrm{Hom}_{\mathbf{RigDM}(\widehat{K})}(\mathbb{L}(\cdot)_\eta^* M_{\widehat{K}^\circ}, \Lambda)^{\mathrm{Gal}(\widehat{K}/K)}.$$

We now prove that $\mathrm{Gal}(\widehat{K}/K)$ acts trivially on

$$\mathrm{Hom}_{\mathbf{RigDM}(\widehat{K})}(\mathbb{L}(\cdot)_\eta^* M_{\widehat{K}^\circ}, \Lambda) \cong \mathrm{Hom}_{\mathbf{DM}(\widehat{k})}(\mathbb{L}(\cdot)_\sigma^* M_{\widehat{K}^\circ}, \chi\Lambda).$$

Let $K^\flat$ be the field constructed in Proposition 5.17. By Propositions 5.18 and Remarks 3.3 and 5.12 we deduce that $\chi\Lambda \cong \Lambda \oplus \Lambda(-1)[-2]$ so that

$$\mathrm{Hom}_{\mathbf{DM}(\widehat{k})}(\mathbb{L}(\cdot)_\sigma^* M_{\widehat{K}}, \chi\Lambda) \cong \mathrm{Hom}_{\mathbf{DM}(\widehat{k})}(M_{\widehat{k}}, \Lambda \oplus \Lambda(-1)[-2])$$

with Galois action induced by the action on $M_{\widehat{k}}$. Since the extension $\widehat{k}/k$ is unseparable, we have $\mathbf{DM}(k) \cong \mathbf{DM}(\widehat{k})$ and the Galois action is therefore trivial, proving our claim. $\qquad\square$

We can then finally state and prove Ayoub's conjecture in the following form.

Theorem 5.25. *Let K be a field satisfying Assumption 5.14 and Λ be a $\mathbb{Q}$-algebra.*

(1) Let $\widehat{K}$ and $K^\flat$ be as in the statement of Proposition 5.17. The canonical functors induce triangulated, monoidal equivalences:

$$\mathbf{RigDM}^{\mathrm{eff,gr}}(K, \Lambda) \cong \mathbf{RigDM}^{\mathrm{eff,gr}}(\widehat{K}, \Lambda) \cong \mathbf{RigDM}^{\mathrm{eff,gr}}(K^\flat, \Lambda)$$

$$\mathbf{RigDM}^{\mathrm{gr}}(K, \Lambda) \cong \mathbf{RigDM}^{\mathrm{gr}}(\widehat{K}, \Lambda) \cong \mathbf{RigDM}^{\mathrm{gr}}(K^\flat, \Lambda)$$

(2) Let k be the residue field of K. There are equivalences of categories:

$$\mathbf{U^n DM}^{\mathrm{eff}}(k, \Lambda) \cong \mathbf{RigDM}^{\mathrm{eff,gr}}(K, \Lambda) \qquad \mathbf{U^n DM}(k, \Lambda) \cong \mathbf{RigDM}^{\mathrm{gr}}(K, \Lambda)$$

induced by a choice of elements $\pi_i^\flat$ as in Proposition 5.17.

Proof. The effective and the stable case are analogous, and we only consider here the latter. From our assumptions on the fields $\widehat{K}$ and $K^\flat$ (see Proposition 5.17) Propositions 5.19 and 5.24 show that there exist canonical equivalences $\mathbf{RigDM}^{\mathrm{gr}}(K, \Lambda) \cong \mathbf{RigDM}^{\mathrm{gr}}(\widehat{K}, \Lambda)$ and $\mathbf{RigDM}^{\mathrm{gr}}(K^\flat, \Lambda) \cong \mathbf{RigDM}^{\mathrm{gr}}(\widehat{K}^\flat, \Lambda)$. If we combine them with the tilting equivalence of Proposition 5.18 we obtain the first claim.

The equivalence of the second claim follows by combining the first point with the equivalence of Ayoub $\mathbf{U^n DM}(k, \Lambda) \cong \mathbf{RigDM}^{\mathrm{gr}}(K^\flat, \Lambda)$ (see Theorem 5.10) induced by a choice of elements in $K^\flat$ generating its value group. $\qquad\square$

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