

SOME ASPECTS OF RATIONAL POINTS AND RATIONAL CURVES

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ABSTRACT. Various methods have been used to construct rational points and rational curves on rationally connected algebraic varieties. We survey recent advances in two of them, the descent and the fibration method, in a number-theoretical context (rational points over number fields) and in an algebro-geometric one (rational curves on real varieties), and discuss the question of rational points over function fields of p -adic curves.

1. INTRODUCTION

Let X denote an algebraic variety over a field k and $X(k)$ the set of its rational points.

The search for explicit descriptions of the set $X(k)$ when k is a number field is one of the oldest themes of number theory. A modern point of view on this problem consists in embedding $X(k)$ diagonally into the topological space $X(\mathbf{A}_k)$ of adelic points of X and attempting to identify its topological closure. By general principles that were formulated by Lang after the works of Mordell, Weil and Siegel, the answer is expected to depend in a crucial manner on the geometry of X . For instance, assuming that X is smooth and projective and that an embedding $k \hookrightarrow \mathbf{C}$ is given, the set $X(k)$ is conjectured to be finite if the complex variety $X_{\mathbf{C}}$ is *hyperbolic* (see [Lan74]). One may then seek to count, list or bound its elements. At the other end of the spectrum, if $X_{\mathbf{C}}$ is a *rationally connected* smooth projective variety in the sense of Campana [Cam92] and Kollár–Miyaoaka–Mori [KMM92], then one expects that the set $X(k)$ is Zariski dense in X whenever it is nonempty; more precisely, by a conjecture of Colliot-Thélène, the closure of $X(k)$ in $X(\mathbf{A}_k)$ should coincide in this case with the Brauer–Manin set $X(\mathbf{A}_k)^{\text{Br}(X)}$ defined by Manin [Man71]. This far-reaching conjecture encompasses in particular the inverse Galois problem, and its refinement the Grunwald problem (see [Eke90], [Ser08, §3.5], [Har07], [DLAN17]).

Criteria for the existence of rational points on X are also of relevance outside of number theory, when k is no longer assumed to be a number field. For instance, the Graber–Harris–Starr theorem [GHS03], a central result in the theory of rational curves on complex algebraic varieties, is equivalent to the statement that $X(k) \neq \emptyset$ if k is the function field of a complex curve and X is a rationally connected variety. (We say that X is *rationally connected* to mean that for any algebraically closed field extension K of k , the variety X_K over K is rationally connected in the sense of [Cam92, KMM92].) As another example, if X is a real algebraic variety with no real point and k denotes the function field of the real conic given by $x^2 + y^2 = -1$, the existence of a geometrically rational curve on X —a

property conjectured by Kollár to hold whenever X is a positive-dimensional rationally connected variety—is equivalent to the statement that $X(k) \neq \emptyset$.

The results we discuss in this expository article concern the existence of rational points in two very distinct contexts, leading to the following two concrete theorems. As we shall see, their proofs roughly follow, perhaps somewhat surprisingly, a common general strategy.

Theorem A (see [HW20]). *Let G be a finite nilpotent group. Let k be a number field.*

- (1) *There exist Galois extensions K/k with Galois group G .*
- (2) *If v_1, \dots, v_n are pairwise distinct places of k none of which is a finite place dividing the order of G , and w_1, \dots, w_n are places of K above v_1, \dots, v_n , then in (1), one can require that the extensions K_{w_i}/k_{v_i} be isomorphic to any prescribed collection of Galois extensions of k_{v_1}, \dots, k_{v_n} whose Galois groups are subgroups of G .*

Theorem B (see [BW21]). *Let X be a smooth, proper variety over \mathbf{R} . Let $\varepsilon : \mathbf{S}^1 \rightarrow X(\mathbf{R})$ be a continuous map. Assume that X is birationally equivalent to a homogeneous space of a linear algebraic group over \mathbf{R} . Then there exist morphisms of algebraic varieties $\mathbf{P}_{\mathbf{R}}^1 \rightarrow X$ that induce maps $\mathbf{P}^1(\mathbf{R}) = \mathbf{S}^1 \rightarrow X(\mathbf{R})$ arbitrarily close to ε in the compact-open topology.*

Theorem A (1) was first proved by Shafarevich in his seminal work on the inverse Galois problem for solvable groups (see [NSW08, Chapter IX, §6]; it should be noted that nilpotent groups form the most difficult case in his proof); the proof given in [HW20] is independent from his and has a geometric flavour. Theorem A (2), on the other hand, was new in [HW20] and was not accessible with Shafarevich’s methods.

As far as we know, Theorem B might hold under the sole assumption that X is rationally connected. This is a question we raise in [BW21]. Theorem B provides the first examples of a positive answer to it for varieties that are not \mathbf{R} -rational (indeed, not even \mathbf{C} -rational). For \mathbf{R} -rational varieties, the conclusion of Theorem B was previously shown, by Bochnak and Kucharz [BK99], to follow from the Stone–Weierstrass theorem.

The first step in the proofs of Theorem A and Theorem B consists in strengthening and reformulating the desired conclusion in terms of the existence of suitable rational points on suitable varieties over suitable fields. In the case of Theorem A, the varieties in question are homogeneous spaces of SL_n over number fields; for the proof, though not for the statement, it is crucial to *not* restrict to homogeneous spaces that have rational points (i.e. to homogeneous spaces of the form SL_n/G). In the case of Theorem B, the varieties in question are homogeneous spaces of linear algebraic groups, over the rational function field $\mathbf{R}(t)$; for the proof, though not for the statement, it is crucial to *not* restrict to homogeneous spaces or algebraic groups that are defined over \mathbf{R} . In the remainder of the proofs of Theorems A and B, one establishes the validity of these strengthened formulations by combining geometric *déviages* of the underlying algebraic varieties with two general tools: the descent method and the fibration method. The fibration method, whose first instance can be found in the work of Hasse on the local-global principle for quadratic forms, consists in reducing the desired property for a variety V endowed with a morphism $p : V \rightarrow B$ with geometrically irreducible generic fibre to the same property for B and for a collection of smooth fibres of p . The descent method, which goes back to Fermat, attempts to reduce the desired property for a variety V endowed with a torsor $p : W \rightarrow V$

under a (possibly disconnected) linear algebraic group over k to the same property for W and for all of its twists. It was developed in the context of elliptic curves, for torsors under finite abelian groups, by Mordell, Cassels and Tate, and the set-up was later extended to torsors under positive-dimensional linear algebraic groups by Colliot-Thélène and Sansuc, Skorobogatov, Harari.

We take Theorems A and B as excuses leading us to the general study of rational points on rationally connected varieties defined over number fields or over function fields of real curves. We discuss recent advances in the fibration and descent methods in these two contexts in §2 and in §3, stating along the way the main open questions that surround Theorems A and B and their proofs. We then turn, in §4, to function fields of p -adic curves, and speculate about the existence of a p -adic analogue of the “tight approximation” property discussed in §3 that would enable one to exploit fibration and descent methods in the study of rational curves over p -adic fields and more generally of rational points over function fields of p -adic curves.

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2. SOLVABLE GROUPS AND THE GRUNWALD PROBLEM IN INVERSE GALOIS THEORY

2.1. Homogeneous spaces. It is the following general theorem about the arithmetic of homogeneous spaces of linear algebraic groups that underlies Theorem A.

Theorem 2.1. *Let V be a homogeneous space of a connected linear algebraic group L over a number field k . Let X be a smooth compactification of V . Let $\bar{v} \in V(\bar{k})$. Assume that the group of connected components G of the stabiliser of \bar{v} is supersolvable, in the sense that it possesses a normal series $1 = G_0 \triangleleft \cdots \triangleleft G_m = G$ such that the quotients G_{i+1}/G_i are cyclic while the subgroups G_i are normal in G and are stable under the natural outer action of $\text{Gal}(\bar{k}/k)$ on G . Then the subset $X(k)$ is dense in $X(\mathbf{A}_k)^{\text{Br}(X)}$.*

Here and elsewhere, by “compactification of V ”, we mean a proper variety over k that contains V as a dense open subset; we do not require that the algebraic group L act on the compactification. Examples of supersolvable groups with respect to the trivial outer action of $\text{Gal}(\bar{k}/k)$ include finite nilpotent groups and dihedral groups. With a nontrivial outer action of $\text{Gal}(\bar{k}/k)$, however, even abelian groups need not be supersolvable. Previous work of Borovoi [Bor96] nevertheless establishes the conclusion of Theorem 2.1 in many cases where the stabiliser of \bar{v} is abelian but not necessarily supersolvable.

Theorem 2.1 can be found in [HW20, Théorème B] in the particular case where L is semi-simple simply connected and the stabiliser of \bar{v} is finite, and in [HW21, Corollary 4.5] in general. To deduce Theorem A from it, embed G into $\text{SL}_n(k)$ for some n , take $L = \text{SL}_n$ and $V = \text{SL}_n/G$ and let H denote the set of points of V above which the fibre of the étale cover $\pi : L \rightarrow V$ is irreducible. The function field of the fibre of π above any rational point contained in H is a Galois extension of k with Galois group G . On the other hand, by a

theorem of Ekedahl [Eke90], the density of $X(k)$ in $X(\mathbf{A}_k)^{\mathrm{Br}(X)}$ implies that of $X(k) \cap H$ in $X(\mathbf{A}_k)^{\mathrm{Br}(X)}$. Thus, Theorem 2.1 ensures the existence of Galois extensions K/k with Galois group G having a local behaviour prescribed by any element of the Brauer–Manin set $X(\mathbf{A}_k)^{\mathrm{Br}(X)}$; that is, one may freely prescribe the completions of K at any finite set of places of k , as long as these prescriptions satisfy a certain global reciprocity condition determined by $\mathrm{Br}(X)$. By a theorem of Lucchini Arteche [LA19, §6], this reciprocity condition imposes, in fact, no restriction at the places indicated in Theorem A (2).

2.2. Geometry. In the special case where $L = \mathrm{SL}_n$ and the stabiliser of \bar{v} is a finite group G , the geometry behind the proof of Theorem 2.1 can be summarised with the following assertion: there exist an algebraic torus T over k and a torsor $\bar{Y} \rightarrow X_{\bar{k}}$ under $T_{\bar{k}}$ whose isomorphism class is invariant under $\mathrm{Gal}(\bar{k}/k)$, such that for any torsor $Y \rightarrow X$ under T whose base change to $X_{\bar{k}}$ is isomorphic to \bar{Y} , there exist a dense open subset $W \subseteq Y$ and a smooth morphism $p : W \rightarrow Q$ to a quasi-trivial torus Q (i.e. a torus of the form $R_{E/k} \mathbf{G}_m$ for a nonzero étale k -algebra E) whose fibres are homogeneous spaces of SL_n with geometric stabiliser isomorphic to G_{m-1} . In addition, the morphism p admits a rational section over \bar{k} .

This geometry is the key to a proof of Theorem 2.1 by an induction on m , at each step of which one applies the descent method and the fibration method, in the form of Theorem 2.2 and Theorem 2.3 below. It should be noted that even if G is embedded into $\mathrm{SL}_n(k)$ and $V = \mathrm{SL}_n/G$, the homogeneous spaces of SL_n that arise as fibres of p need not possess rational points. Thus, for the induction to be possible, one cannot restrict the statement of Theorem 2.1 to homogeneous spaces of the form SL_n/G , even though only homogeneous spaces of this form are relevant for Theorem A.

2.3. Descent. The following theorem, which was established in [HW20] and can also be deduced from [Cao18], is the definitive statement of descent theory in the case of smooth and proper rationally connected varieties over number fields. For geometrically rational X , this theorem is due to Colliot-Thélène and Sansuc [CTS87]. The homogeneous spaces of Theorem 2.1 are not geometrically rational in general (Saltman, Bogomolov; see [CTS07]).

Theorem 2.2. *Let X be a smooth and proper rationally connected variety over a number field k . Let T be a torus over k and $\bar{Y} \rightarrow X_{\bar{k}}$ a torsor under $T_{\bar{k}}$ whose isomorphism class is invariant under $\mathrm{Gal}(\bar{k}/k)$. Then*

$$X(\mathbf{A}_k)^{\mathrm{Br}(X)} = \bigcup_{f:Y \rightarrow X} f'(Y'(\mathbf{A}_k)^{\mathrm{Br}(Y')}),$$

where the union ranges over the torsors $f : Y \rightarrow X$ under T whose base change to $X_{\bar{k}}$ is isomorphic to \bar{Y} , and Y' denotes a smooth compactification of Y such that f extends to a morphism $f' : Y' \rightarrow X$. In particular, if $Y'(k)$ is dense in $Y'(\mathbf{A}_k)^{\mathrm{Br}(Y')}$ for every such f , then $X(k)$ is dense in $X(\mathbf{A}_k)^{\mathrm{Br}(X)}$.

(To bridge the gap between Theorem 2.2 and [HW20, Théorème 2.1], one needs to know that $X(\mathbf{A}_k)^{\mathrm{Br}(X)} \neq \emptyset$ implies the existence of at least one f . This goes back to [CTS87] and follows from [Wit08, Theorem 3.3.1], [CTS87, Proposition 2.2.5], [Wit18, (3.3)].)

2.4. Fibration. The following fibration theorem suffices for the proof of Theorem 2.1. It results from combining a descent with the work of Harari [Har94] on the fibration method.

Theorem 2.3. *Let $p : Z \rightarrow B$ be a dominant morphism between irreducible, smooth and proper varieties over a number field k , with rationally connected generic fibre. Assume that*

- (1) *there exist dense open subsets $W \subset Z$ and $Q \subset B$ such that Q is a quasi-trivial torus over k and p induces a smooth morphism $W \rightarrow Q$ with geometrically irreducible fibres;*
- (2) *the morphism p admits a rational section over \bar{k} ;*
- (3) *for all $b \in B(k)$ in a dense open subset of B , the set $Z_b(k)$ is dense in $Z_b(\mathbf{A}_k)^{\text{Br}(Z_b)}$.*

Then $Z(k)$ is dense in $Z(\mathbf{A}_k)^{\text{Br}(Z)}$.

The assumptions of Theorem 2.3 imply that B is k -rational. Under the condition that B is k -rational, the first two assumptions of Theorem 2.3 are expected to be superfluous (even under weaker hypotheses on the generic fibre of p than rational connectedness, see [HW16, Corollary 9.23 (1)–(2)]), but removing them altogether is a wide open problem, well connected with analytic number theory (see [HW16, §9], [HWW21]). Removing (2) while keeping (1) might be within reach, though:

Question 2.4. In the statement of Theorem 2.3, can one dispense with the assumption that p admit a rational section over \bar{k} ?

This would allow one to replace “supersolvable” with “solvable” in the statement of Theorem 2.1. Indeed, in §2.2, the cyclicity of the quotient G_m/G_{m-1} plays a rôle only to ensure the existence of a rational section of p over \bar{k} (see [HW20, Proposition 3.3 (ii)]).

2.5. An application to Massey products. Theorem 2.1 has concrete applications, over number fields, beyond the inverse Galois problem: for the homogeneous spaces that appear in its statement, it turns the problem of deciding the existence of a rational point into the much more approachable question of deciding the non-vacuity of the Brauer–Manin set. In this way, Theorem 2.1 can be used to confirm, in the case of number fields, the conjecture of Mináč and Tân on the vanishing of Massey products in Galois cohomology (see [HW19]). Indeed, this conjecture—which posits that for any field k , any prime number p , any integer $m \geq 3$ and any classes $a_1, \dots, a_m \in H^1(k, \mathbf{Z}/p\mathbf{Z})$, the m -fold Massey product of a_1, \dots, a_m vanishes if it is defined (see [MT17, MT16])—can be reinterpreted, according to Pál and Schläpfl [PS16], in terms of the existence of rational points on appropriate homogeneous spaces of SL_n over k (with $n \gg 0$), and it so happens that the geometric stabilisers of these homogeneous spaces are finite and supersolvable.

3. RATIONAL CURVES ON REAL ALGEBRAIC VARIETIES

3.1. A few questions. Let X be a smooth variety over \mathbf{R} . The interplay between the topology of the \mathcal{C}^∞ manifold $X(\mathbf{R})$ and the geometry of the algebraic variety X lies at the core of classical real algebraic geometry. One of the fundamental problems in this area consists in investigating which submanifolds of $X(\mathbf{R})$ can be approximated, in the Euclidean topology, by Zariski closed submanifolds. Even for 1-dimensional submanifolds, i.e. disjoint unions of \mathcal{C}^∞ loops, various phenomena—of a topological, Hodge-theoretic,

or yet more subtle nature—can obstruct the existence of algebraic approximations (see [BW20a, §4]). In the case of 1-dimensional submanifolds, however, all known obstructions vanish when X is rationally connected. One can thus raise the following questions, in which $H_1^{\text{alg}}(X(\mathbf{R}), \mathbf{Z}/2\mathbf{Z})$ denotes the image of the cycle class map $\text{CH}_1(X) \rightarrow H_1(X(\mathbf{R}), \mathbf{Z}/2\mathbf{Z})$ defined by Borel and Haefliger [BH61].

Questions 3.1. Let X be a smooth, proper, rationally connected variety, over \mathbf{R} .

- (1) Can all \mathcal{C}^∞ loops in $X(\mathbf{R})$ be approximated, in the Euclidean topology, by real loci of algebraic curves? or even by real loci of rational algebraic curves?
- (2) Is $H_1(X(\mathbf{R}), \mathbf{Z}/2\mathbf{Z}) = H_1^{\text{alg}}(X(\mathbf{R}), \mathbf{Z}/2\mathbf{Z})$? Is $H_1(X(\mathbf{R}), \mathbf{Z}/2\mathbf{Z})$ generated by classes of rational algebraic curves on X ?

The first parts of Questions 3.1 (1) and (2) are in fact equivalent to each other, by the work of Akbulut and King (see [BW20b, Theorem 6.8]), and were studied in a systematic fashion in [BW20a, BW20b]. The second part of Question 3.1 (1) is, however, as far as we know, genuinely stronger than the second part of Question 3.1 (2). We note that in order to formulate the second part of Question 3.1 (1) precisely, it is better to work with possibly non-injective \mathcal{C}^∞ maps $\mathbf{P}^1(\mathbf{R}) \rightarrow X(\mathbf{R})$ rather than with submanifolds of $X(\mathbf{R})$. Indeed, there are examples of \mathbf{R} -rational surfaces X and of \mathcal{C}^∞ loops in $X(\mathbf{R})$ such that the desired rational algebraic curves necessarily have singular real points (see [KM16, Theorem 3]).

A specific motivation for Question 3.1 (2) is its analogy with the following questions in complex geometry raised by Voisin [Voi07] and by Kollár [Kol10]:

Questions 3.2. Let X be a smooth, proper, rationally connected variety, over \mathbf{C} . Is the group $H_2(X(\mathbf{C}), \mathbf{Z})$ generated by homology classes of algebraic curves? Is it generated by homology classes of rational algebraic curves?

The two parts of Questions 3.2 are in fact equivalent: Tian and Zong [TZ14] have shown that the homology class of any algebraic curve on a rationally connected variety over \mathbf{C} is a linear combination of homology classes of rational curves. The real analogue of their result remains unknown in general. Its validity is an interesting open problem.

The first parts of Questions 3.1 (2) and of Questions 3.2 are in fact related by more than an analogy: if X is a smooth, proper, rationally connected variety over \mathbf{R} such that $X(\mathbf{R}) \neq \emptyset$ and such that Questions 3.2 admit a positive answer for $X_{\mathbf{C}}$, then the equality $H_1(X(\mathbf{R}), \mathbf{Z}/2\mathbf{Z}) = H_1^{\text{alg}}(X(\mathbf{R}), \mathbf{Z}/2\mathbf{Z})$ is equivalent to the *real integral Hodge conjecture* for 1-cycles on X , a property formulated and studied in [BW20a, BW20b].

In a different line of investigation around the abundance of rational curves on rationally connected varieties, many authors have considered the problem of finding rational curves through a prescribed set of points, or more generally through a curvilinear 0-dimensional subscheme, on any smooth, proper, rationally connected variety X . Over the complex numbers, such curves exist unconditionally (Kollár, Miyaoka, Mori, see [Kol96, Chapter IV.3]). Over the real numbers, such curves exist under the necessary condition that all the prescribed points that are real belong to the same connected component of $X(\mathbf{R})$ (Kollár, see [Kol99], [Kol04]). This problem can be generalised to one-parameter families: given a morphism $f : \mathcal{X} \rightarrow B$ with rationally connected generic fibre between smooth

and proper varieties, where B is a curve, one looks for sections of f whose restriction to a given 0-dimensional subscheme of B is prescribed, thus leading to Questions 3.3 below. For simplicity of notation, in the statement of Question 3.3 (2), this 0-dimensional subscheme of B is assumed to be reduced; there is however no loss of generality in doing this, since jets of sections can be prescribed at any higher order by replacing \mathcal{X} with a suitable iterated blow-up (see [Has10, Proposition 1.4]).

Questions 3.3. Let B be a smooth, proper, connected curve over a field k_0 . Let \mathcal{X} be a smooth, proper variety over k_0 , endowed with a flat morphism $f : \mathcal{X} \rightarrow B$ with rationally connected generic fibre. Let $P \subset B$ be a reduced 0-dimensional subscheme. Let $s : P \rightarrow \mathcal{X}$ be a section of f over P .

- (1) If $k_0 = \mathbf{C}$, can s be extended to a section of f ?
- (2) If $k_0 = \mathbf{R}$ and the map $s|_{P(\mathbf{R})} : P(\mathbf{R}) \rightarrow \mathcal{X}(\mathbf{R})$ can be extended to a \mathcal{C}^∞ section of $f|_{\mathcal{X}(\mathbf{R})} : \mathcal{X}(\mathbf{R}) \rightarrow B(\mathbf{R})$, can then s be extended to a section of f ?

Let X be the generic fibre of f and k the function field of B . The existence of sections extending any given s as above is equivalent to the density of $X(k)$ in the topological space $X(\mathbf{A}_k) = \prod_b X(k_b)$ of adelic points of X , where the product runs over the closed points b of B and k_b denotes the completion of k at b . This is the *weak approximation* property.

The Graber–Harris–Starr theorem [GHS03] provides a positive answer to Question 3.3 (1) when $P = \emptyset$ and it is a conjecture of Hassett and Tschinkel that the answer to this question is in the affirmative in general (see [CTG04], [HT06], [Has10], [Tia15] for known results). Particular cases of Question 3.3 (2) were first studied by Colliot-Thélène [CT96], who conjectured the validity of weak approximation (i.e. a positive answer to Question 3.3 (2) even without assuming that $s|_{P(\mathbf{R})}$ can be extended to a \mathcal{C}^∞ section of $f|_{\mathcal{X}(\mathbf{R})}$) when X is birationally equivalent to a homogeneous space of a connected linear algebraic group over k , and proved his conjecture when the geometric stabilisers are trivial. Scheiderer [Sch96] then proved the same conjecture when the geometric stabilisers are connected. Ducros [Duc98b, Duc98a] stated Question 3.3 (2) in these exact terms, and gave a positive answer when X is a conic bundle surface, or more generally when there exists a dominant map $X \rightarrow \mathbf{P}_k^1$ whose generic fibre is a Severi–Brauer variety.

3.2. Tight approximation. The main insight behind the proof of Theorem B is the observation that formulating a suitable common strengthening of Questions 3.1 and Questions 3.3, through the notion of *tight approximation*, can render all of these questions fully amenable to both the descent method and the fibration method. We note that Questions 3.1 and Questions 3.3 are somewhat orthogonal in spirit, insofar as the former consider global constraints on curves lying on X , while the latter are aimed at local constraints.

The idea of establishing a descent method (resp. fibration method) for Question 3.3 (2) already appeared in [Duc98b] (resp. [PS20a]), though in *op. cit.* the implementations are subject to miscellaneous restrictions. The possibility of a descent method and a fibration method for studying Questions 3.1, however, is new and turns out to require a shift in perspective from single rationally connected varieties to one-parameter families of such.

Let us illustrate how Questions 3.1 need to be strengthened for a fibration argument to go through. We start with a dominant morphism $p : X \rightarrow Y$ with rationally connected generic fibre between smooth, proper, rationally connected varieties, over \mathbf{R} , and a \mathcal{C}^∞ loop $\gamma : \mathbf{S}^1 \rightarrow X(\mathbf{R})$ that we want to approximate, in the Euclidean topology, by a Zariski closed submanifold of X , assuming that we can solve the same problem on Y as well as on the fibres of p . By assumption, we can approximate $p \circ \gamma : \mathbf{S}^1 \rightarrow Y(\mathbf{R})$ by a \mathcal{C}^∞ map $\xi : \mathbf{S}^1 \rightarrow Y(\mathbf{R})$ with Zariski closed image. The best we can hope to find, then, is a \mathcal{C}^∞ loop $\tilde{\gamma} : \mathbf{S}^1 \rightarrow X(\mathbf{R})$ arbitrarily close to γ and such that $p \circ \tilde{\gamma} = \xi$. We draw two conclusions:

- (1) If such a $\tilde{\gamma}$ exists, the next and final step is not finding an algebraic approximation for a \mathcal{C}^∞ loop in a fibre of p , but, rather, considering the algebraic curve B underlying $\xi(\mathbf{S}^1)$, viewing $\tilde{\gamma}$ as a \mathcal{C}^∞ section of the projection $(X \times_Y B)(\mathbf{R}) \rightarrow B(\mathbf{R})$, and looking for an algebraic section of $X \times_Y B \rightarrow B$ approximating $\tilde{\gamma}$. Thus, even when we start with just two real varieties X and Y , we need to consider one-parameter algebraic families of fibres of p , rather than single fibres.
- (2) Consider the example where p is the blow-up of a surface Y at a real point b and γ meets $p^{-1}(b)(\mathbf{R})$, transversally. Then for any $\tilde{\gamma}$ sufficiently close to γ in the Euclidean topology, the loop $p \circ \tilde{\gamma}$ has to go through b . Hence ξ has to be required to go through b for a loop $\tilde{\gamma}$ as above to exist. Thus, a condition of weak approximation type must be considered in conjunction with Questions 3.1 (as was already noted by Bochnak and Kucharz [BK99]).

Let us now similarly contemplate a fibration argument in the context of Question 3.3 (2). We assume that $\mathcal{X} \xrightarrow{f} B$ can be factored as $\mathcal{X} \xrightarrow{p} \mathcal{Y} \xrightarrow{g} B$, where the variety \mathcal{Y} is smooth and proper over \mathbf{R} , the morphism p is dominant with rationally connected generic fibre, and g is flat. Starting from a section $s : P \rightarrow \mathcal{X}$ of f over P such that $s|_{P(\mathbf{R})}$ can be extended to a \mathcal{C}^∞ section s' of $f|_{\mathcal{X}(\mathbf{R})}$, a positive answer to Question 3.3 (2) for g produces for us a section τ of g that extends $p \circ s$. Let $\mathcal{Z} = p^{-1}(\tau(B))$ and let $h : \mathcal{Z} \rightarrow B$ denote the restriction of f . At this point, one would like to apply a positive answer to Question 3.3 (2) for h to obtain a section of h extending s , thus completing the argument, as $\mathcal{Z} \subseteq \mathcal{X}$. In order to do so, one needs to know that $s|_{P(\mathbf{R})} : P(\mathbf{R}) \rightarrow \mathcal{Z}(\mathbf{R})$ can be extended to a \mathcal{C}^∞ section of $h|_{\mathcal{Z}(\mathbf{R})} : \mathcal{Z}(\mathbf{R}) \rightarrow B(\mathbf{R})$. However, the map $h|_{\mathcal{Z}(\mathbf{R})}$ in general even fails to be surjective. To correct this problem, one should require, at the very least, that $\tau(B(\mathbf{R}))$ approximate, in the Euclidean topology, the image of $p \circ s' : B(\mathbf{R}) \rightarrow \mathcal{Y}(\mathbf{R})$. Thus, all in all, an approximation condition in the Euclidean topology has to be considered in conjunction with Question 3.3 (2).

The above discussion leads to the following definition. (This definition slightly differs from the one given in [BW21], which considers the more general question of approximating holomorphic maps by algebraic ones, à la Runge, and which, as a consequence, is useful also for studying complex curves on complex varieties, without reference to the reals; however, all of the statements we make below are true with respect to either of the definitions.)

Definition 3.4. Let B be a smooth, proper, connected curve over \mathbf{R} . A variety X over $k = \mathbf{R}(B)$ satisfies the *tight approximation* property if for any proper model $f : \mathcal{X} \rightarrow B$ of X over B with \mathcal{X} smooth over \mathbf{R} , any reduced 0-dimensional subscheme $P \subset B$, any

section $s' : P \rightarrow \mathcal{X}$ of f over P and any \mathcal{C}^∞ section $s : B(\mathbf{R}) \rightarrow \mathcal{X}(\mathbf{R})$ of $f|_{\mathcal{X}(\mathbf{R})}$ such that $s|_{P(\mathbf{R})} = s'|_{P(\mathbf{R})}$, there exists a section $\sigma : B \rightarrow \mathcal{X}$ of f such that $\sigma|_P = s'|_P$ and such that $\sigma|_{B(\mathbf{R})}$ lies arbitrarily close to s in the compact-open topology.

Given a smooth, proper, rationally connected variety X over \mathbf{R} , the validity of the tight approximation property for the variety obtained from X by extension of scalars from \mathbf{R} to $\mathbf{R}(t)$ implies positive answers to Questions 3.1 for X .

The tight approximation property is (tautologically) a birational invariant, and it holds for \mathbf{P}_k^n by a theorem of Bochnak and Kucharz [BK99]. (In *op. cit.*, weak approximation conditions at complex points are ignored, but they create no additional difficulty.) The next two results provide more examples of varieties satisfying tight approximation.

3.3. Descent. The following theorem implements the descent method for the tight approximation property, in full generality (including non-abelian descent, as formalised by Harari and Skorobogatov). Its proof, given in [BW21], builds on the work of Scheiderer [Sch96] and, in the case where G is finite, on an argument of Colliot-Thélène and Gille [CTG04].

Theorem 3.5. *Let k be the function field of a real curve. Let X be a smooth variety over k . Let G be a linear algebraic group over k . Let $f : Y \rightarrow X$ be a left torsor under G . Consider twists $f' : Y' \rightarrow X$ of f by right torsors under G , over k . If every such Y' satisfies the tight approximation property, then so does X .*

3.4. Fibration. The next theorem implements the fibration method for the tight approximation property, in full generality. Its proof, contained in [BW21], makes essential use of the weak toroidalisation theorem of Abramovich, Denef and Karu [ADK13] to establish a version of the Néron smoothening process (as in [BLR90, 3.1/3]) for higher-dimensional bases—the point being that in the discussion at the beginning of §3.2, the loop $\tilde{\gamma}$ is easily seen to exist once the morphism p is smooth along γ (see [BW20b, Lemma 6.11]).

Theorem 3.6. *Let k be the function field of a real curve. Let $p : Z \rightarrow B$ be a dominant morphism between smooth varieties over k . If B and the fibres of p above the rational points of a dense open subset of B satisfy the tight approximation property, then so does Z .*

3.5. Homogeneous spaces. We are now in a position to sketch the proof of the following theorem, which in the “constant case”, i.e. when the algebraic group and the homogeneous space are both defined over \mathbf{R} , immediately implies Theorem B.

Theorem 3.7. *Homogeneous spaces of connected linear algebraic groups over the function field of a real curve satisfy the tight approximation property.*

The proof of Theorem 3.7 starts by noting that quasi-trivial tori over k are k -rational, hence satisfy the tight approximation property (since so does \mathbf{P}_k^n). Any torus T can be inserted into an exact sequence $1 \rightarrow S \rightarrow Q \rightarrow T \rightarrow 1$ where S is a torus and Q is a quasi-trivial torus. As any twist of Q as a torsor remains isomorphic to Q (Hilbert’s Theorem 90) and hence satisfies the tight approximation property, we deduce, by the descent method (Theorem 3.5), that all tori over k satisfy the tight approximation property. Next, as every connected linear algebraic group over k is birationally equivalent to a relative torus over

a k -rational variety (namely over the variety of maximal tori, when the algebraic group is reductive), we deduce, by the fibration method (Theorem 3.6), that connected linear algebraic groups over k satisfy the tight approximation property. By descent (Theorem 3.5 again), it follows that homogeneous spaces of connected linear algebraic groups over k satisfy the tight approximation property when they have a rational point. Finally, it is a theorem of Scheiderer that homogeneous spaces of connected linear algebraic groups over k satisfy the Hasse principle with respect to the real closures of k , so that if X denotes such a homogeneous space, then $X(k) \neq \emptyset$ whenever a \mathcal{C}^∞ section $s : B(\mathbf{R}) \rightarrow \mathcal{X}(\mathbf{R})$ as in Definition 3.4 exists. This completes the proof of Theorem 3.7.

3.6. Further comments. Theorem 3.7 implies that homogeneous spaces of connected linear algebraic groups over the function field of a real curve satisfy weak approximation, as conjectured by Colliot-Thélène. Indeed, in the notation of Definition 3.4, if X is such a homogeneous space and P contains the locus of singular fibres of f , Scheiderer’s work implies that $f^{-1}(b)(\mathbf{R})$ is nonempty and connected for all $b \in B(\mathbf{R}) \setminus P(\mathbf{R})$, so that a \mathcal{C}^∞ section $s : B(\mathbf{R}) \rightarrow \mathcal{X}(\mathbf{R})$ with $s|_{P(\mathbf{R})} = s'|_{P(\mathbf{R})}$ always exists.

The main open problem surrounding the notion of tight approximation is the following.

Question 3.8. Let k be the function field of a real curve. Do all rationally connected varieties over k satisfy the tight approximation property?

Building on Theorem 3.5 and Theorem 3.6, the tight approximation property is shown in [BW21] to hold for various classes of rationally connected varieties beyond homogeneous spaces of connected linear algebraic groups. For instance, it holds for smooth cubic hypersurfaces of dimension ≥ 2 that are defined over \mathbf{R} , thus yielding, for such hypersurfaces, a positive answer to (the second part of) Question 3.1 (1).

Question 3.8 is open for cubic surfaces over k . Even Question 3.3 (2) is open when X is a cubic surface, although Question 3.3 (1) has an affirmative answer in this case, by a theorem of Tian [Tia15].

In another direction, Question 3.8 is open for surfaces defined over \mathbf{R} , and so is (the second part of) Question 3.1 (1). By inspecting the birational classification of geometrically rational surfaces and using the fibration method (Theorem 3.6), one can see that a positive answer to these questions for surfaces defined over \mathbf{R} would follow from a positive answer for del Pezzo surfaces of degree 1 or 2 defined over \mathbf{R} . In these cases, it would suffice, by an application of the descent method (Theorem 3.5), to know that for any real del Pezzo surface X of degree 1 or 2, the universal torsors of X , in the sense of Colliot-Thélène and Sansuc [CTS87], are \mathbf{R} -rational whenever they have a real point. This last question, unfortunately, is very much open—even the unirationality of real del Pezzo surfaces of degree 1 is unknown. In fact, not a single example of a minimal real del Pezzo surface of degree 1 is known to be unirational. For a description of these surfaces, see [Rus02, §5].

Naturally, one hopes for the answer to Question 3.8 to be in the affirmative in general. This conjecture would have a host of interesting consequences, among which: a version of the Graber–Harris–Starr theorem over the reals (i.e. a positive answer to Question 3.3 (2) when $P = \emptyset$); Lang’s widely open conjecture from [Lan53] that the function field of a real curve with no real point is C_1 (see [HX09, Corollary 1.5] for the implication); and the

existence of a geometrically rational curve on any smooth, proper, rationally connected variety of dimension ≥ 1 over \mathbf{R} .

This last consequence is a conjecture of Kollár, who showed the existence of rational curves on those real rationally connected varieties of dimension ≥ 1 that have real points (see [AK03, Remarks 20]). For real rationally connected varieties with no real point, it is interesting to consider a weaker property: the existence of a geometrically irreducible curve of even geometric genus. The latter can be reinterpreted in terms of the real integral Hodge conjecture (see [BW20a]). Using Hodge theory and a real adaptation of Green’s infinitesimal criterion for the density of Noether–Lefschetz loci, such curves of even genus can be shown to exist on all real Fano threefolds (see [BW20b]). However, even on smooth quartic hypersurfaces in $\mathbf{P}_{\mathbf{R}}^4$, the existence of geometrically rational curves remains a challenge, as well as the mere existence of an absolute bound, independent of the chosen quartic hypersurface, on the minimal geometric genus of a geometrically irreducible curve of even geometric genus lying on such a hypersurface.

4. FUNCTION FIELDS OF CURVES OVER p -ADIC FIELDS

4.1. Some motivation: rational curves over number fields. Even though the main questions about rational points of rationally connected varieties over number fields and over function fields of real curves are still wide open, the Brauer–Manin obstruction and the tight approximation property at least provide rather satisfactory conjectural answers. It would be highly desirable to obtain a similar conjectural picture for rational points over other fields, for significant classes of varieties—including, at a minimum, concrete criteria for the existence of rational points.

Over the field $\mathbf{Q}(t)$, this would encompass questions about rational curves on rationally connected varieties over \mathbf{Q} , about which very little is known. For example, it is unknown whether any rationally connected variety of dimension ≥ 1 over \mathbf{Q} that possesses a rational point also contains a rational curve defined over \mathbf{Q} . Much more ambitiously, it is unknown whether any such variety contains enough rational curves to imply the finiteness of the set of R -equivalence classes of rational points, a question asked in [CT11, Question 10.12]. (Known results on this problem are listed in *loc. cit.*) As another example, the regular inverse Galois problem over \mathbf{Q} , which asks for the construction of a regular Galois extension of $\mathbf{Q}(t)$ with specified Galois group, and which can be reinterpreted as a problem about the existence of appropriate rational curves on the homogeneous space SL_n/G over \mathbf{Q} , is open even for finite nilpotent groups G . All of these problems are currently out of reach.

As a first step towards these questions, let us replace \mathbf{Q} with its completions and turn to rational points over the field $\mathbf{Q}_p(t)$ or over its finite extensions.

4.2. Rational curves on varieties over p -adic fields. In the constant case (that is, for varieties obtained by scalar extension from varieties defined over a p -adic field, i.e. a finite extension of \mathbf{Q}_p), various existence results are known:

- (1) the regular inverse Galois problem over \mathbf{Q}_p has a positive solution (Harbater [Har87], by “formal patching”; reproved and generalised in different directions by Pop [Pop96] and by Colliot-Thélène [CT00]; see also [Liu95], [MB01], [Kol03]);

- (2) for any smooth, proper, rationally connected variety X over a p -adic field k , Kollár [Kol99, Kol04] has shown that the rational points of X fall into finitely many R -equivalence classes, and that there exist rational curves on X , defined over k , passing through any finite set of rational points of X that belong to the same R -equivalence class (with prescribed jets of any given order at these points).

This last statement concerns conditions of weak approximation type that can be imposed on rational curves on rationally connected varieties over p -adic fields. It would be interesting to formulate an analogue, in this p -adic context, of the surjectivity of the Borel–Haefliger cycle class map $\mathrm{CH}_1(X) \rightarrow H_1(X(\mathbf{R}), \mathbf{Z}/2\mathbf{Z})$ (i.e. of Questions 3.1 (2)).

We saw in §3 that in order to answer questions about homology classes of rational curves on real varieties, it can be useful to consider more generally the tight approximation property, for non-constant varieties over the function field of a real curve. By analogy, this gives incentive to investigate the possibility of a p -adic analogue of the tight approximation property for non-constant varieties over the function field of a curve over a p -adic field, the validity of which would have consequences for a likely easier to formulate *p -adic integral Hodge conjecture* for 1-cycles on varieties over p -adic fields.

4.3. Quadrics and other homogeneous spaces. In the non-constant case, even the simplest varieties over $\mathbf{Q}_p(t)$ lead to difficult problems when it comes to their rational points. For instance, it is only a relatively recent theorem of Parimala and Suresh [PS10], for $p \neq 2$, and of Leep [Lee13], based on work of Heath-Brown [HB10], for arbitrary p , that every projective quadric of dimension ≥ 7 over $\mathbf{Q}_p(t)$ possesses a rational point. (In the language of quadratic forms, “the u -invariant of $\mathbf{Q}_p(t)$ is equal to 8”.) Many other articles have been devoted to local-global principles for varieties over function fields of curves over p -adic fields (e.g. [Hu10, HH10, HHK09, CTPS12, Pre13, HHK14, Hu14, PS14, HSS15, HHK15, CTPS16, HS16, Hu17, PPS18, HHK+19, CTHH+19, HHKP21, Meh19, CTHH+20, PS20b, HKP21, Tia21, CTHH+21]).

A patching technique was developed by Harbater, Hartmann and Krashen (“patching over fields”, a successor to formal patching), and was applied to study rational points of homogeneous spaces over such fields. It was used, in [HHK09], to give another proof of the aforementioned theorem of Parimala and Suresh, and, in [CTPS12], to establish, more generally, the local-global principle for the existence of rational points on smooth projective quadrics of dimension ≥ 1 over $\mathbf{Q}_p(t)$ (or over a finite extension of $\mathbf{Q}_p(t)$), with respect to all discrete valuations on this field, when p is odd.

4.4. Reciprocity obstructions. Let k be a finite extension of $\mathbf{Q}_p(t)$. Let Ω denote the set of equivalence classes of discrete valuations (of rank 1) on k and, for $v \in \Omega$, let k_v denote the completion of k at v . Let X be an irreducible, smooth and proper variety over k . We embed $X(k)$ diagonally into the product topological space $\prod_{v \in \Omega} X(k_v)$, which we shall also denote $X(\mathbf{A}_k)$ (recall that X is proper).

We now explain how, building on the work of Bloch–Ogus and of Kato, an analogue of the Brauer–Manin obstruction can be set up in this context. These ideas, which are due to Colliot-Thélène, appear in print, and are put to use, in [CTPS16, §2.3], in a very slightly

different (equicharacteristic) situation. We refer the reader to *loc. cit.* for more details. (The “reciprocity obstructions” of [HSS15, §4] are weaker than those we discuss here.)

Our goal is thus to define, in complete generality, a closed subset $X(\mathbf{A}_k)^{\text{rec}} \subseteq X(\mathbf{A}_k)$ containing $X(k)$, using on the one hand a reciprocity law coming from k and on the other hand an analogue of the Brauer group of X .

Grothendieck’s purity theorem for the Brauer group equates $\text{Br}(X)$ with the unramified cohomology group $H_{\text{nr}}^2(X/k, \mathbf{Q}/\mathbf{Z}(1))$. We recall the definition of unramified cohomology: for any irreducible smooth variety V over a field K of characteristic 0 and any torsion Galois module M over K , the group $H_{\text{nr}}^q(V/K, M)$ is the subgroup of the Galois cohomology group $H^q(K(V), M)$ consisting of those classes whose residues along all codimension 1 points of V vanish. It is the unramified cohomology group $H_{\text{nr}}^3(X/k, \mathbf{Q}/\mathbf{Z}(2))$ that will serve as a substitute for $\text{Br}(X)$ here. (The shift in degree is explained by the fact that the field k has cohomological dimension 3 while number fields have virtual cohomological dimension 2.) For any field extension K/k , Bloch–Ogus theory provides an evaluation map $H_{\text{nr}}^3(X/k, \mathbf{Q}/\mathbf{Z}(2)) \rightarrow H^3(K, \mathbf{Q}/\mathbf{Z}(2))$, $\alpha \mapsto \alpha(x)$ along any K -point x of X (see [BO74]).

Let \mathcal{B} denote an irreducible normal proper scheme over \mathbf{Z}_p with function field k . In contrast with what happens over number fields, here it is not one reciprocity law that will play a rôle, but infinitely many of them: one for each closed point of \mathcal{B} , for each such \mathcal{B} . Namely, given any closed point $b \in \mathcal{B}$, Kato [Kat86, §1] has constructed a complex

$$(4.1) \quad H^3(k, \mathbf{Q}/\mathbf{Z}(2)) \longrightarrow \bigoplus_{\xi \in \mathcal{B}_{1,b}} \text{Br}(\kappa(\xi)) \longrightarrow \mathbf{Q}/\mathbf{Z},$$

where ξ ranges over the set $\mathcal{B}_{1,b}$ of 1-dimensional irreducible closed subsets of \mathcal{B} that contain b , and where $\kappa(\xi)$ denotes the function field of ξ (which is either a global field of characteristic p or a local field of characteristic 0). The second arrow in (4.1) is the sum of the invariant maps from local class field theory at the finitely many places of $\kappa(\xi)$ that lie over b . The first arrow of (4.1) is induced by residue maps $\partial_v : H^3(k_v, \mathbf{Q}/\mathbf{Z}(2)) \rightarrow \text{Br}(\kappa(\xi))$ constructed by Kato in *loc. cit.*, where v denotes the discrete valuation of k defined by ξ .

For any $\alpha \in H_{\text{nr}}^3(X/k, \mathbf{Q}/\mathbf{Z}(2))$, there are only finitely many 1-dimensional irreducible closed subsets ξ of \mathcal{B} such that the map $X(k_v) \rightarrow \text{Br}(\kappa(\xi))$, $x \mapsto \partial_v(\alpha(x))$ does not identically vanish, if we denote by v the discrete valuation of k defined by ξ (see [CTPS16, Proposition 2.7 (ii)] and note that for the proof given there, it is enough to assume that a dense open subset of \mathcal{B} , rather than \mathcal{B} itself, is a scheme over a field—an assumption satisfied here). As a consequence, it makes sense to define $X(\mathbf{A}_k)^{\text{rec}}$ to be the set of $(x_v)_{v \in \Omega} \in X(\mathbf{A}_k)$ such that for any irreducible normal proper scheme \mathcal{B} over \mathbf{Z}_p with function field k , for any closed point $b \in \mathcal{B}$, and for any $\alpha \in H_{\text{nr}}^3(X/k, \mathbf{Q}/\mathbf{Z}(2))$, the family $(\partial_v(\alpha(x_v)))_{\xi \in \mathcal{B}_{1,b}} \in \bigoplus_{\xi \in \mathcal{B}_{1,b}} \text{Br}(\kappa(\xi))$ belongs to the kernel of the second arrow of (4.1). The fact that (4.1) is a complex immediately implies that $X(k) \subseteq X(\mathbf{A}_k)^{\text{rec}}$.

4.5. Sufficiency of the reciprocity obstruction. Although evidence is scarce, the answer to the following question might always be in the affirmative, as far as one knows:

Question 4.1. Let k be a finite extension of $\mathbf{Q}_p(t)$. Let X be a smooth, proper, rationally connected variety over k . If $X(\mathbf{A}_k)^{\text{rec}} \neq \emptyset$, does it follow that $X(k) \neq \emptyset$?

Question 4.1 has a positive answer when X is a quadric and $p \neq 2$. Indeed, we recall from §4.3 that even $X(\mathbf{A}_k) \neq \emptyset$ then implies $X(k) \neq \emptyset$ (see [CTPS12]). It also has a positive answer when X is birationally equivalent to a torsor under a torus over k . This follows from the work of Harari, Scheiderer, Szamuely, Tian [HS16, Theorem 5.1], [Tia20, §0.3.1] (modulo the comparison between the reciprocity obstruction defined here and the reciprocity obstruction considered in these articles; the latter is weaker but turns out to suffice to detect rational points on torsors under tori). We note that there are examples of torsors under tori over k whose smooth compactifications X satisfy $X(\mathbf{A}_k)^{\text{rec}} = \emptyset$ while $X(\mathbf{A}_k) \neq \emptyset$ (see [CTPS16, Remarque 5.10]). Positive answers to Question 4.1 are known in various other cases in which X is birationally equivalent to a homogeneous space of a connected linear algebraic group over k . For specific statements, we refer the reader to the articles quoted in §4.3. Question 4.1 remains open in general for smooth compactifications of torsors under connected linear algebraic groups over k , for smooth compactifications of homogeneous spaces of SL_n with finite stabilisers, and for conic bundle surfaces over \mathbf{P}_k^1 .

Question 4.1 focuses on the existence of rational points rather than on the density of $X(k)$ in $X(\mathbf{A}_k)^{\text{rec}}$ as the latter property is only known for projective space (see [AW45, Theorem 1]) and hence for varieties that are rational as soon as they possess a rational point, such as quadrics. For smooth compactifications of tori, the density of $X(k)$ in $X(\mathbf{A}_k)^{\text{rec}}$ is known to hold off the set of discrete valuations of k whose residue field has characteristic p (see [HSS15, Theorem 5.2]; for the meaning of “off” here, see [Wit18, Definition 2.9]).

To obtain more positive answers to Question 4.1, it is natural to wish for flexible tools such as general descent theorems and fibration theorems. In the same way that introducing the tight approximation property and replacing Question 3.3 (2) with Question 3.8 was a key step to obtain a problem that behaves well with respect to fibrations into rationally connected varieties (see the discussion in §3.2), it is likely that in order to obtain compatibility with descent and fibrations, one will have to strengthen Question 4.1 by incorporating into it a p -adic analogue of the approximation condition in the Euclidean topology that appears in Definition 3.4. The main challenge, here, is to provide the correct formulation for such a p -adic tight approximation property.

We note that in any case, a general fibration theorem has to lie deep, as it would presumably give a direct route to the local-global principle for the existence of rational points on smooth projective quadrics over k (so far unknown when $p = 2$) and hence to the computation of the u -invariant of k (equal to 8; see §4.3). Indeed, in the case of conics over k , this local-global principle follows from Tate–Lichtenbaum duality [Lic69]; applying a fibration theorem to a general pencil of hyperplane sections of a fixed smooth projective quadric of dimension $n \geq 2$ would allow one to deduce the general case by induction on n .

4.6. Further questions. A good understanding of rational points of rationally connected varieties over function fields of curves over p -adic fields, be it via Question 4.1 or otherwise, should shed light on concrete test questions such as the following:

Questions 4.2. Let p be a prime number and k be a finite extension of $\mathbf{Q}_p(t)$.

- (1) Does the conjecture of Mináč and Tân on the vanishing of Massey products in Galois cohomology hold for k ? (See §2.5 and [MT17, MT16].)

- (2) Is there an algorithm that takes as input a smooth, projective, rationally connected variety X over k and decides whether X has a rational point?

One might approach the first of these questions by trying to mimic [HW19] over k , which would require making progress on the arithmetic, over k , of homogeneous spaces of SL_n with finite supersolvable geometric stabilisers.

To put the second question in perspective, let us recall what is known about algorithms for deciding the existence of rational points on arbitrary varieties (“Hilbert’s tenth problem”) over various fields of interest. Over \mathbf{Q} or $\mathbf{C}(t)$, the existence of such an algorithm is an outstanding open problem. Denef [Den78] showed that over $\mathbf{R}(t)$, such an algorithm does not exist. His method was extended to prove that there is no such algorithm over $\mathbf{Q}_p(t)$ (Kim and Roush [KR95], completed by Degroote and Demeyer [DD12]), over any finite extension of $\mathbf{R}(t)$ that possesses a real place (Moret-Bailly [MB05]), or, when $p \neq 2$, over any finite extension of $\mathbf{Q}_p(t)$ (Eisenträger [Eis07], Moret-Bailly [MB05]). In addition, over number fields, it is known that restricting from arbitrary varieties to smooth projective varieties makes no difference (see [Smo91, §II.7], [Poo09, Theorem 1.1 (i)]). Restricting to smooth, projective, rationally connected varieties, however, does make a drastic difference: Question 4.2 (2) might well have an affirmative answer for all of the fields just mentioned. Over $\mathbf{C}(t)$, this is trivially so, by the Graber–Harris–Starr theorem. Over $\mathbf{R}(t)$, a positive answer to Question 4.2 (2) would follow from a positive answer to Question 3.8. Indeed, in the notation of Definition 3.4, if X satisfies the tight approximation property, then X has a rational point if and only if $f|_{\mathcal{X}(\mathbf{R})}$ admits a \mathcal{C}^∞ section, a property that can be decided algorithmically. Over number fields, as was observed by Poonen [Poo06, Remark 5.3], a positive answer to Question 4.2 (2) would follow from the conjecture that rational points are always dense in the Brauer–Manin set. It seems likely that a positive answer to Question 4.1 would similarly imply a positive answer to Question 4.2 (2). To mimic Poonen’s argument, one runs into the difficulty that the elements of $H_{\mathrm{nr}}^3(X/k, \mathbf{Q}/\mathbf{Z}(2))$ are harder to describe than those of $H_{\mathrm{nr}}^2(X/k, \mathbf{Q}/\mathbf{Z}(1)) = \mathrm{Br}(X)$, whose interpretation in terms of Azumaya algebras is a key point in *loc. cit.*; however, this can be remedied by viewing $H_{\mathrm{nr}}^3(X/k, \mathbf{Q}/\mathbf{Z}(2))$, using Bloch–Ogus theory, as the group of global sections of the Zariski sheaf associated with the presheaf $U \mapsto H_{\mathrm{\acute{e}t}}^3(U, \mathbf{Q}/\mathbf{Z}(2))$, and describing $H_{\mathrm{\acute{e}t}}^3(U, \mathbf{Q}/\mathbf{Z}(2))$ via Čech cohomology.

4.7. Other fields. There are a number of other fields over which a better understanding of rational points of rationally connected varieties would be valuable. One of the simplest example is the fraction field $k = \mathbf{C}((x, y))$ of the ring of formal power series $\mathbf{C}[[x, y]]$, which can be seen as a first step before considering function fields of complex surfaces. This field presents both local and global features, and a reciprocity obstruction can again be defined (in terms of the unramified Brauer group—recall that k has cohomological dimension 2). This obstruction was used in [CTPS16] to produce the first example of a torsor Y under a torus, over k , such that $Y(k) = \emptyset$ but $Y(k_v) \neq \emptyset$ for every discrete valuation v on k . The analogues of Question 4.1 and of Questions 4.2 can be asked over this field too. It is not known, however, whether the reciprocity obstruction explains the absence of rational points on smooth proper varieties that are birationally equivalent to torsors under tori over k .

(though see [Izq19, Corollaire 4.4] for a closely related result involving possibly ramified Brauer classes). We refer the interested reader to [CTOP02, CTGP04, Izq19, ILA21] for the state of the art.

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