CHOW GROUPS OF ZERO-CYCLES IN FIBRATIONS

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(report on joint work with Yonatan Harpaz)

Let X be a smooth, proper, irreducible variety over a number field k. Denote by $\mathrm{CH}_0(X)$ the Chow group of zero-cycles up to rational equivalence, by $\mathrm{Br}(X)$ the cohomological Brauer group of X, by $\Omega = \Omega_f \sqcup \Omega_\infty$ the set of places of k, and let $\widehat{M} = \varprojlim_{n \geq 1} M/nM$ for any abelian group M. According to the reciprocity law of global class field theory, the local pairings

$$\langle -, - \rangle_v : \mathrm{CH}_0(X \otimes_k k_v) \times \mathrm{Br}(X \otimes_k k_v) \to \mathrm{Br}(k_v) \hookrightarrow \mathbf{Q}/\mathbf{Z}$$

for $v \in \Omega$, characterised by the property that $\langle P, \alpha \rangle_v$ is the local invariant of $\alpha(P) \in \mathrm{Br}(k_v(P))$ whenever P is a closed point of $X \otimes_k k_v$, fit together in a complex

$$\widehat{\mathrm{CH}_0(X)} \longrightarrow \prod_{v \in \Omega_f} \mathrm{CH}_0\widehat{(X \otimes}_k k_v) \times \prod_{v \in \Omega_\infty} \frac{\mathrm{CH}_0(X \otimes_k k_v)}{N_{\bar{k}_v/k_v}(\mathrm{CH}_0(X \otimes_k \bar{k}_v))} \xrightarrow{\sum_{v \in \Omega} \langle -, - \rangle_v} \mathrm{Hom}(\mathrm{Br}(X), \mathbf{Q}/\mathbf{Z}).$$

Conjecture (Colliot-Thélène [1], Kato and Saito [7, §7]). *The above complex is exact for any smooth, proper, irreducible variety X over k.*

For rational surfaces, a more precise conjecture, which also predicts the kernel of the first map, appears in [3]. Even the case of cubic surfaces over $\mathbf{Q}$ is widely open.

A notable consequence of the exactness of this complex would be that X possesses a zero-cycle of degree 1 if and only if there exists a family $(z_v)_{v \in \Omega}$ of local zero-cycles of degree 1 whose image in $\mathrm{Hom}(\mathrm{Br}(X), \mathbf{Q}/\mathbf{Z})$ vanishes.

Saito [9] proved the conjecture for curves, under the assumption that the divisible subgroup of the Tate–Shafarevich group of the Jacobian is trivial. (When the curve has a rational point, this assumption is in fact equivalent to the conjecture.) The aim of the talk was to discuss the following fibration theorem and its proof.

Theorem. *Let $f : X \to C$ be a dominant morphism with rationally connected (e.g., geometrically unirational) generic fiber, where C is a smooth, proper, irreducible curve with $\mathrm{III}(k, \mathrm{Jac}(C))_{\mathrm{div}} = 0$. If the smooth fibers of f satisfy the conjecture, then so does X.*

The oldest instance of a fibration argument establishing a particular case of the above conjecture is Hasse's proof of the Hasse–Minkowski theorem for quadratic forms in four variables with rational coefficients. It was based on Dirichlet's theorem on primes in arithmetic progressions and on the global reciprocity law. A delicate argument relying on the same two ingredients allowed Salberger [10] to settle the conjecture for conic bundle surfaces over $\mathbf{P}^1_k$. His proof was later generalised in various directions (see [5], [4], [2], [6], [11], [12], [8]). In all of these papers, Dirichlet's theorem on primes in arithmetic progressions for general number fields was used in the following form: given a finite subset $S \subset \Omega_f$ and elements $\xi_v \in k_v^*$ for $v \in S$, there exists $\xi \in k^*$ arbitrarily

close to ξ_v for $v \in S$ such that ξ is a unit outside S except at a unique (unspecified) place v_0, at which it is a uniformiser. (Strictly speaking, when k is not totally imaginary, a more general statement which incorporates approximation conditions at the real places, and which builds on results of Waldschmidt in transcendence theory, needs to be used.) Given a finite *abelian* extension L/k from which ξ_v is a local norm for $v \in S$ and which is unramified outside S, it then follows, in view of the global reciprocity law, that ξ is a local norm from L at v_0 too, and hence that v_0 splits in L.

The reciprocity argument we have just described fails when the extension L/k is not abelian. This failure has led to severe restrictions on the fibrations to which the previous methods could apply. In the proof of the above theorem, the following elementary lemma serves as a substitute for Dirichlet's theorem.

Lemma. *Let L/k be a finite Galois extension. Let S be a finite set of places of k. For each $v \in S$, let $\xi_v \in k_v^*$. If ξ_v is a local norm from L/k for each $v \in S$, there exists $\xi \in k^*$ arbitrarily close to ξ_v for $v \in S$, such that ξ is a unit outside S except at places which split in L.*

Proof. Affine space minus any codimension 2 subset satisfies strong approximation off one place. The lemma is a consequence of this fact applied to $(R_{L/k}\mathbf{A}_L^1) \setminus D$, where D denotes the singular locus of the complement of the torus $R_{L/k}\mathbf{G}_m$ in $R_{L/k}\mathbf{A}_L^1$. $\qquad\square$

Another ingredient of the proof is an arithmetic duality theorem (obtained in [12, §5]) for a variant of Rosenlicht's relative Picard group, denoted $\mathrm{Pic}_+(C)$, whose definition we briefly recall. Let $M \subset C$ be the set of points over which the fiber of f is singular. For each $m \in M$, fix a finite extension $L_m/k(m)$. Then $\mathrm{Pic}_+(C)$ is defined as the quotient of $\mathrm{Div}(C \setminus M)$ by the subgroup generated by the principal divisors $\mathrm{div}(h)$ such that for each $m \in M$, the function h is invertible at m and $h(m)$ is a norm from L_m.

In the very simple situation of a fibration over $\mathbf{P}_k^1$ with only two singular fibers, above 0 and ∞, each of which possesses an irreducible component of multiplicity 1, if we let $L_0 = L_\infty = L$ and if we are given an adelic point $(P_v)_{v\in\Omega} \in X(\mathbf{A}_k)$ supported outside of $f^{-1}(M)$, it is easy to see that the class of $(f(P_v))_{v\in\Omega}$ belongs to the image of the diagonal map $\mathrm{Pic}_+(C) \to \prod_{v\in\Omega}\mathrm{Pic}_+(C \otimes_k k_v)$ if and only if there exists $c \in k^*$ such that ct_v is a local norm from L for all $v \in \Omega$, where $t_v \in k_v^*$ denotes the coordinate of $f(P_v) \in \mathbf{P}^1(k_v)$. If this condition is satisfied, applying the lemma to $\xi_v = ct_v$ and setting $t = \xi/c$ yields a point $t \in k^* = \mathbf{G}_m(k) \subset \mathbf{P}^1(k)$ such that $X_t(\mathbf{A}_k) \neq \varnothing$, provided L and S were chosen large enough in the first place.

It is this argument, which bypasses any abelianness assumption, which forms the core of the proof of the theorem.

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