INTERMEDIATE JACOBIANS OVER NON-CLOSED FIELDS AND APPLICATIONS TO RATIONALITY

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Abstract. Clemens and Griffiths discovered in 1972 a powerful method, based on intermediate Jacobians, to establish the irrationality of smooth projective complex threefolds in certain cases. A few years ago, it was realised that this method extends to non-algebraically closed fields, with concrete consequences for the rationality of geometrically rational threefolds. We report on these developments, addressing the underlying algebraic theory of intermediate Jacobians as well as its applications to questions of rationality.

1. Introduction

The *intermediate Jacobian* of a pure integral Hodge structure H of odd weight is a complex torus—even a polarised abelian variety when H has Hodge level ≤ 1 and is itself polarised—introduced by Weil [Wei52], Lieberman [Lie68] and Griffiths [Gri68, Gri69b] as a natural extension of the classical definitions of the Jacobian (resp. Albanese and Picard) varieties of smooth projective curves (resp. varieties) over $\mathbf{C}$.

The middle cohomology of a rationally connected smooth projective complex threefold X thus determines a principally polarised complex abelian variety $J^2(X)$. In 1972, Clemens and Griffiths [CG72] had the groundbreaking insight that the *rationality* of X, that is, the existence of a birational map between X and $\mathbf{P}^3_{\mathbf{C}}$, can in certain cases be ruled out by a comparison of $J^2(X)$ with Jacobians of curves. They proved, in this way, that smooth cubic threefolds in $\mathbf{P}^4_{\mathbf{C}}$ are irrational, thus solving a long-standing problem. In the following years, the same idea was exploited by many authors in other geometric settings, leading to significant progress in the understanding of rationality for complex threefolds (see [IP99, §8.1] for some references). In addition, Murre [Mur72, Mur73] was able to adapt the arguments of Clemens and Griffiths to positive characteristic to disprove the rationality of smooth cubic threefolds over algebraically closed fields of characteristic ≥ 3.

As a result of the work of Castelnuovo, Zariski, Segre, Manin and Iskovskikh, the rationality of smooth projective surfaces has been fully elucidated since the 1970s over arbitrary fields (see [BW23, Proposition 4.16] for a precise statement). At present, a comparable understanding of the rationality of threefolds remains beyond

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reach—be it over algebraically closed fields or, for geometrically rational threefolds, over non-algebraically closed fields. There has nevertheless been some recent progress in the latter direction following the observation, made in a joint work with Olivier Benoist [BW20], that the method of Clemens and Griffiths can also be applied over perfect fields to disprove the rationality of certain geometrically rational threefolds. Shortly afterwards, Hassett and Tschinkel [HT21, §11.5] realised that instead of comparing $J^2(X)$ with Jacobians of curves as in [BW20], one gains more information by also comparing certain naturally occurring torsors under $J^2(X)$ with the natural torsors under the Jacobians of the curves in question. In this way, they characterised those smooth intersections of two quadrics in $\mathbf{P}^5_{\mathbf{R}}$ that are rational. These ideas were then extended to arbitrary fields in [BW23] and were subsequently shown by a number of authors to be effective across a range of geometric settings (see [BW23, §4], [KP23], [KP24], [FJS$^+$24b], [FJS$^+$24a], [JJ24], [FJ24], [JS25]). Notable consequences include a characterisation of those Fano threefolds of geometric Picard number 1 over fields of characteristic 0 that are rational (due to Kuznetsov and Prokhorov, in [KP23, Theorem 1.1]), or the first examples of smooth projective varieties that possess rational points, become rational over a purely inseparable extension of the ground field, but are themselves irrational (in [BW23, Theorem B]).

All of these applications to the rationality of threefolds over fields other than $\mathbf{C}$ or $\mathbf{R}$ necessarily depend on an algebraic theory of intermediate Jacobians. Over algebraically closed fields, Murre [Mur85] developed an approach based on a universal property satisfied by the abelian variety $J^2(X)$ equipped with the Abel–Jacobi map $\mathrm{CH}^2(X)_{\mathrm{hom}} \to J^2(X)$. Achter, Casalaina-Martin and Vial [ACMV17] extended Murre's work, using Galois descent, to perfect fields. The principal polarisation on $J^2(X)$ is characterised and shown to exist in [BW20] for geometrically rational threefolds. Finally, an alternative approach to intermediate Jacobians, based on a functorial point of view, is set out in [BW23], leading to a definition of $J^2(X)$ for geometrically rational smooth projective threefolds over arbitrary fields.

The purpose of this report is twofold. First, we aim to provide an account of recent advances in the construction of an algebraic theory of intermediate Jacobians, a topic of general interest in which considerable room for improvement remains, to say the least. Second, we present the aforementioned refinements of the Clemens–Griffiths method over non-algebraically closed fields, highlighting some of the method's limitations and the questions that they naturally raise.

The text is structured as follows. We start in §2 by reviewing the transcendental theory of intermediate Jacobians of smooth projective complex varieties, and include a discussion of the case of real varieties. This discussion suffices for all applications of the Clemens–Griffiths method over $\mathbf{R}$. Algebraic intermediate Jacobians are the subject of §3. Building on §3, we turn to the Clemens–Griffiths method in §4 and, in particular, show how the results on smooth cubic threefolds in characteristic $\neq 2$

and on three-dimensional smooth intersections of two quadrics can be brought under a single framework. Finally, the few open questions that did not fit within the main body of the text are gathered in §5.

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2. Transcendental intermediate Jacobians

We introduce intermediate Jacobians associated with integral Hodge structures of odd weight in §2.1 and discuss their polarisations in §2.2. Intermediate Jacobians of smooth projective complex varieties come equipped with an Abel–Jacobi map; we review some of the (expected) properties of the latter in §2.3. Intermediate Jacobians of real varieties are considered in §2.5.

The phrase "Hodge structure" will always mean "pure Hodge structure". We adopt Griffiths' definition of intermediate Jacobians (see [Lew99, Lecture 12]). In the whole of §2, we denote by X a smooth projective variety over $\mathbf{C}$, set $d = \dim(X)$ and write $h^{p,q} = \dim H^q(X, \Omega_X^p)$ for its Hodge numbers.

2.1. **A complex torus.** Recall that a *real Hodge structure H of weight k* is a pair consisting of a finite-dimensional real vector space $H_{\mathbf{R}}$ and a decomposition of the complex vector space $H_{\mathbf{C}} = H_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$ into a direct sum $H_{\mathbf{C}} = \bigoplus_{p+q=k} H^{p,q}$ (the *Hodge decomposition*), subject to the condition that $\overline{H^{p,q}} = H^{q,p}$ for all p, q, where the bar denotes the antilinear involution of $H_{\mathbf{C}}$ induced by complex conjugation. An *integral* (resp. *rational*) *Hodge structure H of weight k* is a pair consisting of a finitely generated abelian group $H_{\mathbf{Z}}$ (resp. a finite-dimensional $\mathbf{Q}$-vector space $H_{\mathbf{Q}}$) and a real Hodge structure of weight k whose underlying real vector space is $H_{\mathbf{R}} = H_{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathbf{R}$ (resp. $H_{\mathbf{R}} = H_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{R}$). We set $F^r H_{\mathbf{C}} = \bigoplus_{p \geq r} H^{p,k-p}$.

If $k = 2n-1$ is odd, then $H_{\mathbf{C}} = \overline{F^n H_{\mathbf{C}}} \oplus F^n H_{\mathbf{C}}$; if in addition H is integral, the image of the natural map $H_{\mathbf{Z}} \to H_{\mathbf{C}}/F^n H_{\mathbf{C}}$ is therefore a lattice.

Definition 2.1. Let H be an integral Hodge structure of odd weight $2n-1$. The *intermediate Jacobian* associated with H is the complex torus

$$J(H) = \operatorname{Coker}\Big(H_{\mathbf{Z}} \to H_{\mathbf{C}}/F^n H_{\mathbf{C}}\Big).$$

For $m \in \mathbf{Z}$, we let $\mathbf{Z}(m)$ denote the integral Hodge structure of weight $-2m$ with underlying abelian group $i^m \mathbf{Z} \subset \mathbf{C}$.

When X is a smooth projective variety over $\mathbf{C}$ and n is an integer, we write $J^n(X)$ for the intermediate Jacobian associated with the weight -1 integral Hodge structure

$H^{2n-1}(X(\mathbf{C}),\mathbf{Z}(n))$. Letting $d=\dim(X)$, the complex tori $J^n(X)$ and $J^{d+1-n}(X)$ are dual to each other (see [BL99, Chapter 4, Proposition 2.1]).

2.2. **Polarisations.** Writing $\mathbf{R}(m)$ for $\mathbf{Z}(m)\otimes_{\mathbf{Z}}\mathbf{R}$, a *polarisation* on a real Hodge structure H of weight k is a morphism of Hodge structures $Q: H\otimes_{\mathbf{R}} H\to \mathbf{R}(-k)$ subject to a positivity condition (see [Del71, Définition 2.1.15] or [Gri84, Definition 7]; there exist various conventions; for definiteness, we require that $i^kQ: H_{\mathbf{R}}\times H_{\mathbf{R}}\to\mathbf{R}$ should be a polarisation in the sense of [CMSP17, Definition 1.2.4(ii)]). A *polarisation* on an integral (resp. rational) Hodge structure H of weight k is a morphism of Hodge structures $Q: H\otimes_{\mathbf{Z}} H\to\mathbf{Z}(-k)$ (resp. $Q: H\otimes_{\mathbf{Q}} H\to\mathbf{Q}(-k)$) such that $Q_{\mathbf{R}}$ is a polarisation on $H_{\mathbf{R}}$.

When X is a smooth projective variety over $\mathbf{C}$, any ample divisor on X naturally induces a polarisation on the integral Hodge structure $H^k(X(\mathbf{C}),\mathbf{Z})_{\mathrm{prim}}$ for any k, where "prim" denotes the subgroup of primitive classes (primitive with respect to the chosen ample divisor; see [Voi02, §7.1.2]), and hence, thanks to the Lefschetz decomposition, also on the rational Hodge structure $H^k(X(\mathbf{C}),\mathbf{Q})$ for any k (*loc. cit.*, or see [CMSP17, Corollary 2.3.5]). Let $d=\dim(X)$. The latter polarisation, on $H^k(X(\mathbf{C}),\mathbf{Q})$, comes from a polarisation on $H^k(X(\mathbf{C}),\mathbf{Z})$ as soon as $H^k(X(\mathbf{C}),\mathbf{Q}) = H^k(X(\mathbf{C}),\mathbf{Q})_{\mathrm{prim}}$, in particular when $k\leq d$ and $H^{k-2}(X(\mathbf{C}),\mathbf{Q})=0$. When $k=d$ and $H^{d-2}(X(\mathbf{C}),\mathbf{Q})=0$, this polarisation on $H^d(X(\mathbf{C}),\mathbf{Z})$ is simply

$$(x,y)\mapsto(-1)^{\frac{1}{2}d(d-1)}\mathrm{Tr}(x\smile y), \tag{2.1}$$

where $\mathrm{Tr}: H^{2d}(X(\mathbf{C}),\mathbf{Z})\to\mathbf{Z}(-d)$ denotes the trace map (see [CMSP17, Theorem 2.3.3], [Del71, (2.2.6)]).

If $J=J(H)$ denotes the intermediate Jacobian of an integral Hodge structure H of weight -1, the canonical isomorphism of abelian groups $H^1(J(\mathbf{C}),\mathbf{Z})=\mathrm{Hom}(H_{\mathbf{Z}},\mathbf{Z})$ is not, in general, compatible with the Hodge decompositions—the Hodge structure on the right-hand side need not be effective—and as a consequence, a polarisation on H does not, in general, induce a polarisation of the complex torus J. In fact, even when $H=H^k(X(\mathbf{C}),\mathbf{Z})_{\mathrm{prim}}$ for a smooth projective variety X over $\mathbf{C}$, it can happen that J is not even an abelian variety. (An example of this is given by a very general quintic complex threefold X, with $k=3$, as one can check by computing the Mumford–Tate group of the Hodge structure $H^3(X(\mathbf{C}),\mathbf{Q})$ using [PS03, Lemma 4, Example 5, Proposition 14].) Nevertheless, the canonical isomorphism $H^1(J(\mathbf{C}),\mathbf{Z})=\mathrm{Hom}(H_{\mathbf{Z}},\mathbf{Z})$ does indeed underlie an isomorphism of integral Hodge structures if H has Hodge level ≤ 1, i.e. if $H^{p,q}=0$ whenever $|p-q|>1$; under this assumption, a polarisation on H therefore induces a polarisation of J; in particular, if H is polarisable and has Hodge level ≤ 1, then J is an abelian variety.

Example 2.2. A particularly favourable situation is that of a smooth projective variety X over $\mathbf{C}$ of odd dimension d such that $H^{d-2}(X(\mathbf{C}),\mathbf{Q})=0$ and such that $H^d(X(\mathbf{C}),\mathbf{Q})$ has Hodge level ≤ 1. (For example, any curve satisfies these conditions, as well as any threefold with $h^{1,0}=h^{3,0}=0$.) In this case, the intermediate Jacobian $J^{(d+1)/2}(X)$ is a principally polarised abelian variety. Indeed, we have seen that the formula (2.1) defines a polarisation on $H=H^d(X(\mathbf{C}),\mathbf{Z})$ and therefore also one on $J(H)$; this polarisation is principal as the cup-product pairing on H is unimodular, by Poincaré duality.

2.3. **Abel–Jacobi map.** Let X be a smooth projective variety over $\mathbf{C}$. For any integer n, the complex torus $J^n(X)$ is intimately related to the Chow group of codimension n cycles on X through the *Abel–Jacobi map*

$$\mathrm{AJ}^n:\mathrm{CH}^n(X)_{\mathrm{hom}}\to J^n(X)$$

defined by Griffiths, where $\mathrm{CH}^n(X)_{\mathrm{hom}}$ denotes the kernel of the cycle class map $\mathrm{CH}^n(X)\to H^{2n}(X(\mathbf{C}),\mathbf{Z}(n))$ (see [Voi02, §12.1.2]). Of particular importance is the subgroup $\mathrm{CH}^n(X)_{\mathrm{alg}}\subseteq \mathrm{CH}^n(X)_{\mathrm{hom}}$ of cycle classes that are algebraically equivalent to zero. The quotient $\mathrm{CH}^n(X)_{\mathrm{hom}}/\mathrm{CH}^n(X)_{\mathrm{alg}}$ is countable, by the theory of Chow varieties (see [Fri91, Proposition 1.8]).

When $n=1$, not much happens: the map AJ^1 is bijective, the complex torus $J^1(X)$ is an abelian variety, and one has $\mathrm{CH}^1(X)_{\mathrm{alg}}=\mathrm{CH}^1(X)_{\mathrm{hom}}$ (see [Lew99, Lecture 12, Remark 12.2(2)] and [Voi02, Corollaire 12.8 and §12.2.2]).

When $n>1$, however, all of these properties can fail. If X is a very general quintic complex threefold, we have already mentioned that $J^2(X)$ is not an abelian variety, and Griffiths [Gri69a, Corollary 14.2] has shown that $\mathrm{CH}^2(X)_{\mathrm{alg}}\neq\mathrm{CH}^2(X)_{\mathrm{hom}}$. Returning to an arbitrary smooth projective variety X, let us now briefly discuss the injectivity and surjectivity of AJ^n.

Surjectivity of AJ^n fails if $H^{2n-1}(X(\mathbf{C}),\mathbf{Z}(n))$ does not have Hodge level ≤ 1 (i.e. if $h^{p,2n-1-p}\neq 0$ for some $p\leq n-2$), and is expected to hold otherwise. Indeed, by a theorem of Griffiths, if $h^{p,2n-1-p}\neq 0$ for some $p\leq n-2$, then $\mathrm{AJ}^n\Big(\mathrm{CH}^n(X)_{\mathrm{alg}}\Big)$ must be contained in a strict complex subtorus of $J^n(X)$ (see [Voi02, Théorème 20.15]); the image of AJ^n must then be contained in a countable union of translates of a strict complex subtorus of $J^n(X)$, so that AJ^n cannot be surjective. Conversely, when $H^{2n-1}(X(\mathbf{C}),\mathbf{Z}(n))$ has Hodge level ≤ 1, the surjectivity of the Abel–Jacobi map AJ^n, and even of its restriction to $\mathrm{CH}^n(X)_{\mathrm{alg}}$, would follow from the Hodge conjecture (see [Voi02, §12.2.2]).

Letting $d=\dim(X)$, injectivity of AJ^d fails as soon as $h^{p,0}\neq 0$ for some $p\geq 2$ (Mumford [Mum69] when $d=2$, Roitman [Roi72] in general), and is expected to hold otherwise (by the Bloch–Beilinson–Murre conjecture combined with another theorem of Roitman—see [Voi02, Théorème 22.11, Conjecture 23.22], [Voi14, Conjecture 1.11], [Lew99, Proposition 15.3]). Similarly, the Abel–Jacobi map AJ^2 is expected to be injective if and only if $h^{2,0} = 0$ (see [Sch93, Theorem 0.3], [Voi02, Conjecture 23.22], [CTSS83, Corollaire 5]). Predicting exactly when AJ^n should be injective seems in general more delicate when $2 < n < d$ (see [Sch93], [Lew89], [Via13, Proposition 2.12], [Tot97, Theorem 7.2], [SV05, §4]; for n such that $2 < n < d$, we note that the map AJ^n has other defects anyway, see [OS20, Corollary 1.2]).

The remarkably good properties of AJ^1 can sometimes be transferred to AJ^2 by an argument known as "decomposition of the diagonal" (see [Blo10, Appendix to Lecture 1], [BS83], [Voi14]), to which the next definition is relevant.

Definition 2.3. If X is a smooth and proper variety over a field k, the group $\mathrm{CH}_0(X)_{\mathbf{Q}}$ is *supported in dimension* $\leq j$ if there exists a Zariski closed subset $Z \subset X$ of dimension $\leq j$ such that for every algebraically closed field extension Ω of k, the push-forward map $\mathrm{CH}_0(Z_\Omega)_{\mathbf{Q}} \to \mathrm{CH}_0(X_\Omega)_{\mathbf{Q}}$ is onto.

When k has characteristic 0, decomposition of the diagonal shows that if $\mathrm{CH}_0(X)_{\mathbf{Q}}$ is supported in dimension $\leq j$, then $h^{p,0} = 0$ for all $p > j$ (see [Voi14, Theorem 1.5]). This implication is in fact expected to be an equivalence (*loc. cit.*, Conjecture 1.11).

If X is a rationally connected variety, then $\mathrm{CH}_0(X)_{\mathbf{Q}}$ is supported in dimension 0.

The following theorem, which results from the proof of [BS83, Theorem 1], rests in particular on the decomposition of the diagonal argument (see the discussion in [Voi14, Theorem 6.24, Remark 6.25]).

Theorem 2.4 (Bloch, Srinivas)**.** *If X is a smooth projective variety over $\mathbf{C}$ such that $\mathrm{CH}_0(X)_{\mathbf{Q}}$ is supported in dimension ≤ 1, then $J^2(X)$ is an abelian variety, the map AJ^2 is bijective, and $\mathrm{CH}^2(X)_{\mathrm{alg}} = \mathrm{CH}^2(X)_{\mathrm{hom}}$.*

2.4. **Examples.** If X is a smooth projective curve over $\mathbf{C}$, then $J^1(X)$ is a principally polarised abelian variety (see §2.2), the *Jacobian* of X. In general, if X is a smooth projective variety of dimension d over $\mathbf{C}$, the complex tori $J^1(X)$ and $J^d(X)$ are always abelian varieties (as $H^1(X(\mathbf{C}), \mathbf{Q})$ and $H^{2d-1}(X(\mathbf{C}), \mathbf{Q})$ have Hodge level ≤ 1; see §2.2), called the *Picard* and the *Albanese* variety of X, respectively; they need not be principally polarisable.

Intermediate Jacobians of odd-dimensional smooth intersections of two quadrics were elucidated by Reid [Rei72, Theorem 4.14]. If $X \subset \mathbf{P}^{2m+3}_{\mathbf{C}}$ is the base locus of a pencil of quadrics $L \subset \mathbf{P}(H^0(\mathbf{P}^{2m+3}_{\mathbf{C}}, \mathscr{O}(2)))$, the hypotheses of Example 2.2 are satisfied and $J^{m+1}(X)$ is thus a principally polarised abelian variety. It turns out to be the Jacobian of a curve canonically associated with X, isomorphic to the double cover of $L \simeq \mathbf{P}^1_{\mathbf{C}}$ branched over the $2m + 4$ points of L that parametrise singular quadrics. In addition, the variety of m-dimensional linear spaces contained in X is canonically a torsor under the abelian variety $J^{m+1}(X)$.

Tyurin [Tyu75, Chapter 5, §4] and Beauville [Bea77, Chapitre VI] gave a similar description of intermediate Jacobians of odd-dimensional smooth intersections of three quadrics, in terms of Prym varieties. In a related vein, intermediate Jacobians of odd-dimensional smooth projective varieties fibred into quadrics over $\mathbf{P}^2_{\mathbf{C}}$ can be expressed in terms of Prym varieties (Mumford [CG72, Appendix C], Beauville [Bea77, Chapitre II]). Finally, let us mention that the intermediate Jacobian of any smooth cubic threefold $X \subset \mathbf{P}^4_{\mathbf{C}}$ is canonically isomorphic to the Albanese variety of the variety of lines contained in X (see [CG72, Theorem 11.19]).

2.5. **Over the reals.** It was observed in [BW20, Remark 2.6] that the transcendental theory of intermediate Jacobians of varieties over $\mathbf{C}$ can be adapted, with little effort, to varieties over $\mathbf{R}$, by viewing real abelian varieties as complex abelian varieties endowed with an antiholomorphic involution. (Comessatti [Com24, Com25] and Silhol [Sil82] had previously dealt, in this way, with Albanese and Picard varieties; moreover, the motivic origin of intermediate Jacobians and Abel–Jacobi maps were made clear long ago, see [Jan94, (2.7), §2.5].)

Namely, if H is an integral Hodge structure of odd weight $k = 2n-1$ and F_∞ is an involution of the abelian group $H_{\mathbf{Z}}$ such that the induced $\mathbf{C}$-linear involution of $H_{\mathbf{C}}$ maps $H^{p,q}$ to $H^{q,p}$ for all p, q, then the $\mathbf{C}$-antilinear involution of $H_{\mathbf{C}}$ defined by $z \mapsto F_\infty(\bar{z})$ descends to an antiholomorphic involution of the complex torus $J(H)$. Letting F_∞ act as the identity on the abelian group underlying $\mathbf{Z}(m)$, polarisations $H \otimes_{\mathbf{Z}} H \to \mathbf{Z}(-k)$ are required to respect the actions of F_∞ on the underlying abelian groups. With these definitions in place, all of the assertions made in §2.2 that concern varieties over $\mathbf{C}$ remain correct *verbatim* for varieties X over $\mathbf{R}$, when the abelian group $H^k(X(\mathbf{C}), \mathbf{Z}(m))$ is equipped, for all k and m, with the involution F_∞ induced by the complex conjugation involution of $X(\mathbf{C})$. In particular, if X is a smooth projective threefold over $\mathbf{R}$ with $h^{1,0} = h^{3,0} = 0$, one obtains, in this way, a principally polarised abelian variety $J^2(X)$ over $\mathbf{R}$.

For any smooth projective variety X over $\mathbf{R}$ and any integer n, the Abel–Jacobi map $\mathrm{AJ}^n : \mathrm{CH}^n(X_{\mathbf{C}})_{\mathrm{hom}} \to J^n(X_{\mathbf{C}})$ is $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$-equivariant, if $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$ acts on $J^n(X_{\mathbf{C}})$ through the antiholomorphic involution described above (as can be deduced from [Jan90, Lemma 9.2]).

The importance of the Tate twist is visible already for $n = 1$: had we defined $J^1(X)$ starting from $H^1(X(\mathbf{C}), \mathbf{Z})$ instead of $H^1(X(\mathbf{C}), \mathbf{Z}(1))$, the map AJ^1 would not have been $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$-equivariant in general, and the real abelian variety $J^1(X)$ would have been isomorphic not to the Picard variety of X, but to its *quadratic twist*.

3. Intermediate Jacobians: algebraic points of view

3.1. **Parametrising algebraic cycles.** An algebraic approach to intermediate Jacobians is required if we wish to define and use them over fields other than $\mathbf{C}$

and $\mathbf{R}$, in particular over fields of positive characteristic. Deligne [Del72] took a first step in this direction in a rather special case, that of smooth complete intersections of Hodge level 1. For more general varieties, all known algebraic approaches rely on the Abel–Jacobi map, and focus on algebraic cycles instead of complex tori.

If X is a smooth projective variety of dimension d over $\mathbf{C}$ and n is an integer such that $H^{2n-1}(X(\mathbf{C}), \mathbf{Q})$ has Hodge level 1 (so that the complex torus $J^n(X)$ is an abelian variety and the restriction of AJ^n to $\mathrm{CH}^n(X)_{\mathrm{alg}}$ is expected to be surjective, see §§2.2–2.3), the question becomes: how can we parametrise codimension n cycles on X that are algebraically equivalent to zero by the points of an abelian variety?

To answer this question, two perspectives have been explored: defining $J^n(X)$ as the solution of a universal problem (as Albanese did for $n = d$; see [LS24] for a modern treatment), and defining $J^n(X)$ by its functor of points (as Grothendieck did for $n = 1$; see [Kle05] for a detailed account). Let us discuss some results, old and new, obtained for $n = 2$.

3.2. **Universal regular homomorphism.** Let X be a smooth projective variety over an algebraically closed field k. Consider pairs (A, ψ) consisting of an abelian variety A over k and a homomorphism $\psi : \mathrm{CH}^2(X)_{\mathrm{alg}} \to A(k)$ that is *regular*, i.e. such that for any smooth irreducible variety T over k, any $t_0 \in T(k)$ and any $Z \in \mathrm{CH}^2(T \times X)$, the map $T(k) \to A(k)$, $t \mapsto \psi(Z_t - Z_{t_0})$ comes from a morphism of varieties $T \to A$. A morphism of pairs $(A, \psi) \to (A', \psi')$ is a morphism of abelian varieties $f : A \to A'$ such that $\psi' = f \circ \psi$. Murre [Mur85, Theorem A], with a correction by Kahn [Kah21], proved the existence of an initial object in the category of such pairs. We denote it $(\mathrm{Ab}^2_X, \psi^2_X)$. The map ψ^2_X is necessarily surjective.

When $k = \mathbf{C}$, Murre [Mur85, Theorem C] showed that Ab^2_X can be embedded, in a canonical way, into $J^2(X)$, and that the maps ψ^2_X and AJ^2 then agree on $\mathrm{CH}^2(X)_{\mathrm{alg}}$. In particular, if $k = \mathbf{C}$ and $\mathrm{CH}_0(X)_{\mathbf{Q}}$ is supported in dimension ≤ 1 in the sense of Definition 2.3, it follows from Theorem 2.4 that $(\mathrm{Ab}^2_X, \psi^2_X) = (J^2(X), \mathrm{AJ}^2)$: under these assumptions, we recover the intermediate Jacobian.

If X is a smooth projective variety over a perfect field k, with algebraic closure $\bar{k}$, Achter, Casalaina-Martin and Vial [ACMV17, Theorem 4.4] proved that $\mathrm{Ab}^2_{X_{\bar{k}}}$ descends to an abelian variety Ab^2_X over k such that $\psi^2_{X_{\bar{k}}} : \mathrm{CH}^2(X_{\bar{k}})_{\mathrm{alg}} \to \mathrm{Ab}^2_X(\bar{k})$ is a $\mathrm{Gal}(\bar{k}/k)$-equivariant map. As $\psi^2_{X_{\bar{k}}}$ is surjective, the resulting abelian variety Ab^2_X over k is unique, up to a unique isomorphism.

This algebraic theory of intermediate Jacobians over k, parametrising cycles of codimension 2 on smooth projective varieties X such that $\mathrm{CH}_0(X)_{\mathbf{Q}}$ is supported in dimension ≤ 1, is perfectly adequate for the applications to rationality problems that we discuss in §4, as long as the ground field is perfect. When k is imperfect with algebraic closure $\bar{k}$, however, the defining universal property of $\mathrm{Ab}^2_{X_{\bar{k}}}$ does not provide a descent datum on this abelian variety from $\bar{k}$ down to k.

3.3. **Bloch's Abel–Jacobi map.** This is a canonical homomorphism

$$\lambda^n_X : \mathrm{CH}^n(X)\{\ell\} \to H^{2n-1}_{\text{ét}}(X, \mathbf{Q}_\ell/\mathbf{Z}_\ell(n))$$

defined by Bloch [Blo79] for any smooth projective variety X over an algebraically closed field k and any prime number ℓ invertible in k, where $\mathrm{CH}^n(X)\{\ell\}$ denotes the ℓ-primary torsion subgroup of $\mathrm{CH}^n(X)$. In modern language, the construction of λ^n_X results from the exact sequence of motivic cohomology groups

$$H^{2n-1}_{\mathrm{mot}}(X, \mathbf{Q}(n)) \to H^{2n-1}_{\mathrm{mot}}(X, \mathbf{Q}_\ell/\mathbf{Z}_\ell(n)) \to \mathrm{CH}^n(X)\{\ell\} \to 0$$

(see [MVW06, Corollary 19.2]) in view of Bloch's observation [Blo79, Lemma 2.4] that a weight argument forces the natural map $H^m_{\mathrm{mot}}(X, \mathbf{Q}(j)) \to H^m_{\text{ét}}(X, \mathbf{Q}_\ell(j))$ to vanish whenever $m \neq 2j$.

The importance of this map for us stems from the next proposition, which bridges the gap between Ab^2_X and $H^3_{\text{ét}}(X, \mathbf{Q}_\ell(2))$, thus recovering an algebraic avatar of the formula $H_1(J^2(X), \mathbf{Z}) = H^3(X(\mathbf{C}), \mathbf{Z}(2))/(\mathrm{tors})$ arising from Definition 2.1.

Proposition 3.1 ([BW20, Proposition 2.3])**.** *Let X be a smooth projective variety over a perfect field k, such that $\mathrm{CH}_0(X)_{\mathbf{Q}}$ is supported in dimension ≤ 1 in the sense of Definition 2.3. For any prime number ℓ invertible in k, the map $\psi^2_{X_{\bar{k}}}$ is bijective; composing $\lambda^2_{X_{\bar{k}}}$ with the map induced by $(\psi^2_{X_{\bar{k}}})^{-1}$ on ℓ-primary torsion subgroups leads to a $\mathrm{Gal}(\bar{k}/k)$-equivariant isomorphism $T_\ell(\mathrm{Ab}^2_{X_{\bar{k}}}) \xrightarrow{\sim} H^3_{\acute{e}t}(X_{\bar{k}}, \mathbf{Z}_\ell(2))/(\mathrm{tors})$.*

3.4. **Functor of points.** Let X be a smooth projective variety over a field k. Let $\bar{k}$ be an algebraic closure of k. Grothendieck's approach to the Picard variety consists in verifying that the fppf sheaf associated with the functor $(\mathrm{Sch}/k)^{\mathrm{op}} \to (\mathrm{Ab})$, $T \mapsto \mathrm{Pic}(X \times T)$ is represented by a group scheme locally of finite type over k, the *Picard scheme* $\mathrm{Pic}_{X/k}$, whose reduced identity component $\mathrm{Pic}^0_{X/k,\mathrm{red}}$ is an abelian variety over k, and which satisfies

$$\mathrm{Pic}_{X/k}(\bar{k}) = \mathrm{Pic}(X_{\bar{k}}) = \mathrm{CH}^1(X_{\bar{k}}) \supset \mathrm{CH}^1(X_{\bar{k}})_{\mathrm{alg}} = \mathrm{Pic}^0(X_{\bar{k}}) = \mathrm{Pic}^0_{X/k,\mathrm{red}}(\bar{k})$$

(see [BLR90, Chapter 8], [Kle05]).

A similar approach in the case of codimension 2 cycles was initiated in [BW23]. The naive attempt that would start with the formula "$T \mapsto \mathrm{CH}^2(X \times T)$" faces the issue that Chow groups of singular schemes do not vary contravariantly; thus, this formula does not define a functor on $(\mathrm{Sch}/k)^{\mathrm{op}}$. Restricting to the category of (formally) smooth schemes to avoid this issue would not be a solution, as it would exclude smooth $\bar{k}$-schemes as possible choices for T whenever k is imperfect. A contravariant substitute for Chow groups is therefore required.

There exist several candidates for such a substitute, some of them susceptible to generalisation to cycles of higher codimension, as well as to varieties of arbitrary dimension. The one we chose in *op. cit.* relies on the observation, specific to cycles of codimension 2, that when $\dim(X) = 3$ and $\mathrm{CH}_0(X_{\bar{k}}) = \mathbf{Z}$, the second Chern class induces—as can be deduced from Jouanolou's Riemann–Roch theorem without denominators, see [Ful98, Example 15.3.6]—an isomorphism

$$\mathrm{Ker}\left(K_0(X_{\bar{k}}) \xrightarrow{\mathrm{rk}\times\det\times\chi} \mathbf{Z}\times\mathrm{Pic}(X_{\bar{k}})\times\mathbf{Z}\right) \xrightarrow[\sim]{c_2} \mathrm{CH}^2(X_{\bar{k}}), \tag{3.1}$$

where the left-hand side parametrises virtual vector bundles E of rank 0 on $X_{\bar{k}}$ such that $\det(E)$ is free and $\chi(X_{\bar{k}}, E) = 0$. The groups appearing in the left-hand side of (3.1) visibly enjoy contravariant functoriality even for singular schemes. We are therefore led to the following definition.

Definition 3.2. Denote by $K_{0,X/k} : (\mathrm{Sch}/k)^{\mathrm{op}} \to (\mathrm{Ab})$ the fppf sheaf associated with the functor $T \mapsto K_0(X \times T)$. When X is a threefold and $\mathrm{CH}_0(X)_{\mathbf{Q}}$ is supported in dimension 0 (see Definition 2.3), we set

$$\mathrm{CH}^2_{X/k} = \mathrm{Ker}\left(K_{0,X/k} \xrightarrow{\mathrm{rk}\times\det\times\chi} \mathbf{Z}\times\mathrm{Pic}_{X/k}\times\mathbf{Z}\right).$$

For such X, the group of $\bar{k}$-points of the sought-for intermediate Jacobian of X is expected, by Theorem 2.4, to coincide with $\mathrm{CH}^2(X_{\bar{k}})_{\mathrm{alg}} \subseteq \mathrm{CH}^2(X_{\bar{k}})$. The following theorem, taken from [BW23], confirms that Definition 3.2 works well at least for geometrically rational threefolds (which suffices for the applications discussed in §4) and in particular fulfills this expectation.

We denote by $\mathrm{NS}^2(X_{\bar{k}}) = \mathrm{CH}^2(X_{\bar{k}})/\mathrm{CH}^2(X_{\bar{k}})_{\mathrm{alg}}$ the group of codimension 2 cycles on $X_{\bar{k}}$ up to algebraic equivalence.

Theorem 3.3 ([BW23, Theorem 3.1])**.** *Let X be a smooth projective threefold over a field k, with algebraic closure $\bar{k}$. If $X_{\bar{k}}$ is rational, then $\mathrm{CH}^2_{X/k}$ is representable by a smooth group scheme over k satisfying the following properties:*

(1) this group scheme fits into an exact sequence of commutative group schemes

$$0 \longrightarrow J \longrightarrow \mathrm{CH}^2_{X/k} \longrightarrow \mathrm{NS}^2_{X/k} \longrightarrow 0 \tag{3.2}$$

where $J = (\mathrm{CH}^2_{X/k})^0$ is an abelian variety over k and $\mathrm{NS}^2_{X/k}$ is étale over k;

(2) there is a canonical isomorphism $J_{\bar{k}} = \mathrm{Ab}^2_{X_{\bar{k}}}$ of abelian varieties over $\bar{k}$ (and in fact already over the perfect closure of k);

(3) the canonical isomorphism of abelian groups $\mathrm{CH}^2_{X/k}(\bar{k}) = \mathrm{CH}^2(X_{\bar{k}})$ induces isomorphisms $J(\bar{k}) = \mathrm{CH}^2(X_{\bar{k}})_{\mathrm{alg}}$ and $\mathrm{NS}^2_{X/k}(\bar{k}) = \mathrm{NS}^2(X_{\bar{k}})$.

Property (2) implies that J coincides with $J^2(X)$ when $k = \mathbf{C}$, making it legitimate to call J the intermediate Jacobian of X in general. The group $\mathrm{NS}^2(X_{\bar{k}})$ appearing in Property (3) turns out, under the assumptions of Theorem 3.3, to be a free $\mathbf{Z}$-module of finite rank. For $\alpha \in \mathrm{NS}^2(X_{\bar{k}})^{\mathrm{Aut}(\bar{k}/k)} = \mathrm{NS}^2_{X/k}(k)$, we shall denote by $(\mathrm{CH}^2_{X/k})^\alpha$ the fibre over α of the map $\mathrm{CH}^2_{X/k} \to \mathrm{NS}^2_{X/k}$ appearing in (3.2). It is a torsor under J.

The proof of Theorem 3.3 given in [BW23] relies in an essential way on the assumption that $X_{\bar{k}}$ is rational. The representability of the fppf sheaf $\mathrm{CH}^2_{X/k}$ on $(\mathrm{Sch}/k)^{\mathrm{op}}$ follows, by flat descent, from the representability of $\mathrm{CH}^2_{X_{k'}/k'}$ for any given finite extension k'/k. The latter is verified by choosing a birational map from $\mathbf{P}^3_{\bar{k}}$ to $X_{\bar{k}}$, resolving its indeterminacies using blow-ups with smooth centres (Abhyankar [Abh98]), fixing a finite extension k'/k large enough that the chosen birational map and the chosen blow-ups are defined over k', and studying how the functor $\mathrm{CH}^2_{X_{k'}/k'}$ behaves with respect to such blow-ups.

One may consider taking advantage of Artin's criteria (see e.g. [HR19]) to relax the assumptions of Theorem 3.3. Two concrete questions in this direction were posed in [BW23, Remarks 3.2 (i)–(iii)]:

Questions 3.4. (i) Do all of the conclusions of Theorem 3.3 still hold true if we replace the assumption that $X_{\bar{k}}$ is rational with the assumption that $\mathrm{CH}_0(X)_{\mathbf{Q}}$ is supported in dimension 0 in the sense of Definition 2.3?

(ii) If X is a smooth projective variety of arbitrary dimension over a field k and if $\mathrm{CH}_0(X)_{\mathbf{Q}}$ is supported in dimension ≤ 1, can one use an alternative definition for the functor $\mathrm{CH}^2_{X/k}$—a few candidates that may work in this generality are suggested in *loc. cit.*, Remark 3.2 (ii)—and still prove the conclusions of Theorem 3.3?

3.5. **Polarisations.** Let X be a smooth projective threefold over an algebraically closed field k such that $\mathrm{CH}_0(X)_{\mathbf{Q}}$ is supported in dimension 0 in the sense of Definition 2.3. When $k = \mathbf{C}$, decomposition of the diagonal ensures that $h^{1,0} = h^{3,0} = 0$ (see [Voi14, Theorem 1.5]) and hence Example 2.2 applies: the intermediate Jacobian $J^2(X)$ is naturally a principally polarised abelian variety. For arbitrary k, can we similarly equip Ab^2_X with a natural principal polarisation?

By analogy with the transcendental definition of this polarisation when $k = \mathbf{C}$ (for which see §2.2 and especially (2.1)), what we are looking for is a divisor class $\theta_X \in \mathrm{NS}(\mathrm{Ab}^2_X)$ such that for every prime number ℓ invertible in k, the image of θ_X in $H^2_{\text{ét}}(\mathrm{Ab}^2_X, \mathbf{Z}_\ell(1))$ coincides, via the natural identifications

$$\begin{aligned} H^2_{\text{ét}}(\mathrm{Ab}^2_X, \mathbf{Z}_\ell(1)) &= \Big(\bigwedge^2 H^1_{\text{ét}}(\mathrm{Ab}^2_X, \mathbf{Z}_\ell)\Big)(1) = \Big(\bigwedge^2 \mathrm{Hom}\big(T_\ell(\mathrm{Ab}^2_X), \mathbf{Z}_\ell\big)\Big)(1) \\ &= \mathrm{Hom}\Big(\bigwedge^2 T_\ell(\mathrm{Ab}^2_X), \mathbf{Z}_\ell(1)\Big) \end{aligned}$$

and via the isomorphism $T_\ell(\mathrm{Ab}^2_X) \xrightarrow{\sim} H^3_{\text{ét}}(X, \mathbf{Z}_\ell(2))/(\text{tors})$ of Proposition 3.1, with the alternating pairing

$$H^3_{\text{ét}}(X, \mathbf{Z}_\ell(2))/(\text{tors}) \times H^3_{\text{ét}}(X, \mathbf{Z}_\ell(2))/(\text{tors}) \to \mathbf{Z}_\ell(1) \tag{3.3}$$

given by $(x, y) \mapsto -\mathrm{Tr}(x \smile y)$, where $\mathrm{Tr} : H^6_{\text{ét}}(X, \mathbf{Z}_\ell) \to \mathbf{Z}_\ell(-3)$ denotes the trace map (see [Mil25, Chapter VI, Theorem 11.1]).

The class θ_X is unique, since $\mathrm{NS}(\mathrm{Ab}^2_X)$ is torsion-free. Its existence was proved in [BW20, Proposition 2.5, Corollary 2.8] when either k has characteristic 0 or X is rational, and by Achter, Casalaina-Martin and Vial in [ACMV25, Theorem 12.12 (2)] in general. We are thus faced with two questions:

Questions 3.5. Let X be a smooth projective threefold over an algebraically closed field k, such that $\mathrm{CH}_0(X)_{\mathbf{Q}}$ is supported in dimension 0.

(i) Is $\theta_X \in \mathrm{NS}(\mathrm{Ab}^2_X)$ ample?
(ii) Is the homomorphism $\varphi_{\theta_X} : \mathrm{Ab}^2_X \to \widehat{\mathrm{Ab}}^2_X$ associated with θ_X an isomorphism?

A positive answer to both questions would mean that θ_X is a principal polarisation. By comparison with the transcendental theory, it is easy to see that these questions admit positive answers when k has characteristic 0 (see [BW20, Proposition 2.5]). When k has characteristic $p > 0$, however, both questions are open in general. One immediate observation is that the unimodularity of the pairing (3.3) for all ℓ invertible in k (see [Zar21]) forces in any case φ_{θ_X} to be an isogeny of p-power degree. (In fact, it is even purely inseparable, as was shown in [ACMV25, Theorem 12.12 (2)].) Let us review what is known beyond this.

By studying how Questions 3.5 behave under birational morphisms and in particular under blow-ups with smooth centres, a positive answer to both questions was provided in [BW20, Corollary 2.8] when X is rational.

Achter, Casalaina-Martin and Vial proved, in [ACMV25, Theorem 12.12 (2)], that Question 3.5 (ii) admits a positive answer if there exists an integer N prime to p such that $\mathrm{CH}_0(X)_{\mathbf{Z}[\frac{1}{N}]}$ is *universally trivial* (i.e. such that $\mathrm{CH}_0(X_F)_{\mathbf{Z}[\frac{1}{N}]} = \mathbf{Z}[\frac{1}{N}]$ for all fields F containing k; this property holds if X is stably rational or if there exists a rational map $\mathbf{P}^3_k \dashrightarrow X$ of degree N).

The same authors also answered Question 3.5 (i) in the affirmative when X can be lifted to a smooth projective variety Y in characteristic 0 such that $\mathrm{CH}_0(Y)_{\mathbf{Q}}$ is supported in dimension 0 (see [ACMV25, Corollary 13.3]).

It is now natural to wish for a complete resolution of Questions 3.5.

Remark 3.6. Suppose that X is a smooth projective threefold over a field k, such that $\mathrm{CH}_0(X)_{\mathbf{Q}}$ is supported in dimension 0 and such that all of the conclusions of Theorem 3.3 hold true; in particular, the abelian variety $\mathrm{Ab}^2_{X_{\bar{k}}}$, which is defined over an algebraic closure $\bar{k}$ of k, naturally descends to the abelian variety $J = (\mathrm{CH}^2_{X/k})^0$

over k. The uniqueness of the class $\theta_{X_{\bar{k}}} \in \mathrm{NS}(\mathrm{Ab}^2_{X_{\bar{k}}})$ forces it to be $\mathrm{Aut}(\bar{k}/k)$-invariant and hence forces $\varphi_{\theta_{X_{\bar{k}}}}$ to be $\mathrm{Aut}(\bar{k}/k)$-equivariant. As Hom-schemes between abelian varieties over k are étale over k (see [Bri22, Proposition 4.10]), it follows that the homomorphism $\varphi_{\theta_{X_{\bar{k}}}} : J_{\bar{k}} \to \widehat{J}_{\bar{k}}$ uniquely descends to a homomorphism $J \to \widehat{J}$. We thus obtain a polarisation (resp. a principal polarisation) on $(\mathrm{CH}^2_{X/k})^0$, in the sense of [Mil86, §13], whenever Question 3.5 (i) admits a positive answer (resp. whenever Questions 3.5 admit positive answers).

3.6. **Examples.** We illustrate the results and questions outlined in §3 with two examples in which the intermediate Jacobian $(\mathrm{CH}^2_{X/k})^0$ and the torsors $(\mathrm{CH}^2_{X/k})^\alpha$ can be described quite explicitly, even in positive characteristic.

3.6.1. *Intersections of two quadrics.* Let $X \subset \mathbf{P}^5_k$ be a three-dimensional smooth intersection of two quadrics over a field k, with algebraic closure $\bar{k}$. It is a classical fact (at least in characteristic $\neq 2$; for characteristic 2, see [BW23, Remark 4.4]) that $X_{\bar{k}}$ contains a line and that projecting from any line defines a birational map $X_{\bar{k}} \dashrightarrow \mathbf{P}^3_{\bar{k}}$. Thus, Theorem 3.3 is applicable and Questions 3.5 admit positive answers for $X_{\bar{k}}$. The next theorem fully describes $\mathrm{CH}^2_{X/k}$ in terms of two objects naturally associated with X: the variety F of lines of X, and a certain smooth, projective, geometrically irreducible curve D of genus 2 over k that parametrises rational equivalence classes of conics on X (see [BW23, Theorem 4.5 (iii)]). This theorem was discovered by Cassels [Cas93] and Wang [Wan18] when the characteristic of k is not 2. A general proof is given in [BW23, Theorem 4.5].

Theorem 3.7. *There are canonical isomorphisms*
1. $\mathrm{NS}^2(X_{\bar{k}}) = \mathbf{Z}$ *of abelian groups, given by the degree of curves in* $\mathbf{P}^5_k$*;*
2. $(\mathrm{CH}^2_{X/k})^0 = \mathrm{Pic}^0_{D/k}$ *of principally polarised abelian varieties over* k*;*
3. $(\mathrm{CH}^2_{X/k})^1 = F$ *of varieties over* k*;*
4. $(\mathrm{CH}^2_{X/k})^2 = \mathrm{Pic}^1_{D/k}$ *of torsors under the abelian varieties* $(\mathrm{CH}^2_{X/k})^0 = \mathrm{Pic}^0_{D/k}$.

The curve D equipped with the isomorphism of Theorem 3.7 (2) is unique up to a unique isomorphism, by the Torelli theorem (see [Ser01, Théorème 1]). We recall that in Theorem 3.7 (3)–(4), the notation $(\mathrm{CH}^2_{X/k})^d$ refers to the fibre of the natural map $\mathrm{CH}^2_{X/k} \to \mathrm{NS}^2_{X/k} = \mathbf{Z}$ above $d \in \mathbf{Z}$ (see the discussion after Theorem 3.3).

3.6.2. *Conic bundles.* Let X be a smooth projective threefold over a field k of characteristic $\neq 2$, let W be a geometrically rational smooth projective surface over k and let $\pi : X \to W$ be a morphism all of whose fibres are isomorphic to conics. Let $\Delta \subset W$ denote the locus of singular fibres and $\varpi : \tilde{\Delta} \to \Delta$ the double cover parametrising the irreducible components of the geometric fibres of π above Δ. Following Frei, Ji, Sankar, Viray, and Vogt [FJS+24b], who built on older work of Mumford and Beauville, let us assume that both Δ and $\tilde{\Delta}$ are smooth and geometrically irreducible. (See [BW20, §3.3 and Remark 3.5] for a detailed study of the intermediate Jacobian when $\tilde{\Delta}$ is geometrically disconnected, including the case of characteristic 2 ground fields.)

As $\varpi : \tilde{\Delta} \to \Delta$ is an étale double cover, the kernel of $\varpi_* : \mathrm{Pic}^0_{\tilde{\Delta}/k} \to \mathrm{Pic}^0_{\Delta/k}$ is an extension of $\mathbf{Z}/2\mathbf{Z}$ by a principally polarised abelian variety, the *Prym variety* $\mathrm{Prym}_{\tilde{\Delta}/\Delta}$ of ϖ (see [Mum74, §3]). The other connected component of this kernel is a torsor under $\mathrm{Prym}_{\tilde{\Delta}/\Delta}$. We denote it $\mathrm{Prym}'_{\tilde{\Delta}/\Delta}$.

The next theorem, which is essentially a reformulation of [FJS+24b, Theorem 5.1, Proposition 5.3], fully describes $\mathrm{CH}^2_{X/k}$ assuming that it is represented by a smooth group scheme.

Theorem 3.8. *There is a canonical short exact sequence of* $\mathrm{Aut}(\bar{k}/k)$*-modules*

$$0 \longrightarrow \mathbf{Z} \longrightarrow \mathrm{NS}^2(X_{\bar{k}}) \xrightarrow{\pi_*} \mathrm{Pic}(W_{\bar{k}}) \longrightarrow 0, \tag{3.4}$$

where $1 \in \mathbf{Z}$ *is sent to the class* $\gamma_0 \in \mathrm{NS}^2(X_{\bar{k}})$ *of any irreducible component of any singular fibre of* π. *If* $\mathrm{CH}^2_{X/k}$ *is represented by a smooth group scheme over* k *(e.g. if* $X_{\bar{k}}$ *is rational, by Theorem 3.3), then all of the conclusions of Theorem 3.3 hold true and Questions 3.5 both have a positive answer. In addition, in this case, there are canonical isomorphisms*

(1) $(\mathrm{CH}^2_{X/k})^0 = \mathrm{Prym}_{\tilde{\Delta}/\Delta}$ *of principally polarised abelian varieties,*
(2) $(\mathrm{CH}^2_{X/k})^{\gamma_0} = \mathrm{Prym}'_{\tilde{\Delta}/\Delta}$ *of torsors under these isomorphic abelian varieties.*

Proof. As Δ is smooth and geometrically irreducible, the group $H^3_{\text{ét}}(X_{\bar{k}}, \mathbf{Z}_2)$ is torsion-free (see [Bea77, Théorème 2.1 (i), Remarque 2.7]); hence, by Poincaré duality, so is $H^4_{\text{ét}}(X_{\bar{k}}, \mathbf{Z}_2)$. (We note that the assumption on Δ is crucial here; see [AM72, Proposition 3].) Applying [BS83, Theorem 1 (iii)], we find that the 2-torsion subgroup of $\mathrm{NS}^2(X_{\bar{k}})$ is trivial. The first assertion of Theorem 3.8 results from this and from [FJS+24b, proof of Proposition 5.3 (vi)].

Let us now assume that $\mathrm{CH}^2_{X/k}$ is represented by a smooth group scheme. Letting $\iota : \tilde{\Delta} \to \tilde{\Delta}$ denote the nontrivial involution such that $\varpi \circ \iota = \varpi$, we recall from *op. cit.*, Lemma 4.1, that $\iota_* - \mathrm{Id}$ induces an isomorphism $\mathrm{Pic}^0_{\tilde{\Delta}/k}/\varpi^*\mathrm{Pic}^0_{\Delta/k} \xrightarrow{\sim} \mathrm{Prym}_{\tilde{\Delta}/\Delta}$. Composing its inverse with the morphism $\varepsilon_* j_* p^* : \mathrm{Pic}_{\tilde{\Delta}/k} \to \mathrm{CH}^2_{X/k}$ constructed in *op. cit.*, Lemma 5.7, yields a morphism

$$\mathrm{Prym}_{\tilde{\Delta}/\Delta} \to (\mathrm{CH}^2_{X/k})^0; \tag{3.5}$$

indeed, the morphism $\mathrm{Pic}^0_{\Delta/k} \to (\mathrm{CH}^2_{X/k})^0$ induced by $\varepsilon_* j_* p^* \varpi^*$ vanishes on $\bar{k}$-points (*loc. cit.*, Theorem 5.8), hence it vanishes, being a morphism between smooth group schemes. Similarly, if $p_* j^* \varepsilon^* : \mathrm{CH}^2_{X/k} \to \mathrm{Pic}_{\tilde{\Delta}/k}$ denotes the morphism constructed in *loc. cit.*, Lemma 5.7, the morphism $(\mathrm{CH}^2_{X/k})^0 \to \mathrm{Pic}^0_{\Delta/k}$ induced by $\varpi_* p_* j^* \varepsilon^*$ vanishes as it is a morphism between smooth group schemes that vanishes on $\bar{k}$-points (*loc. cit.*, Lemma 5.4 (i), noting that $\mathrm{Pic}^0(W_{\bar{k}}) = 0$); thus $p_* j^* \varepsilon^*$ induces a morphism

$$(\mathrm{CH}^2_{X/k})^0 \to \mathrm{Prym}_{\tilde{\Delta}/\Delta}. \tag{3.6}$$

In view of *loc. cit.*, Theorem 5.8, which guarantees that (3.5) is onto on $\bar{k}$-points, and of *loc. cit.*, Lemma 5.5, the maps induced on $\bar{k}$-points by the morphisms (3.5) and (3.6) are mutually inverse bijections. As $(\mathrm{CH}^2_{X/k})^0$ and $\mathrm{Prym}_{\tilde{\Delta}/\Delta}$ are smooth group schemes, we deduce that (3.5) and (3.6) are mutually inverse isomorphisms.

The canonical isomorphism $(\mathrm{CH}^2_{X/k})^{\gamma_0} = \mathrm{Prym}'_{\tilde{\Delta}/\Delta}$ is obtained in *op. cit.*, §5.4.

The equality $\mathrm{CH}^2(X_{\bar{k}})_{\mathrm{alg}} = (\mathrm{CH}^2_{X/k})^0(\bar{k})$ of subgroups of $\mathrm{CH}^2(X_{\bar{k}}) = \mathrm{CH}^2_{X/k}(\bar{k})$ is a general fact that holds whenever the functor $\mathrm{CH}^2_{X/k}$ is representable, and which follows from [BW23, Lemma 2.12 (i)–(ii)] (applied to smooth *curves* T). According to [Bea77, Proposition 3.1.8 (ii), Proposition 3.3], the proofs of which are valid under the weaker assumption on W made here, the map induced by (3.6) on $\bar{k}$-points is a universal regular homomorphism. There results a canonical isomorphism $\alpha : \mathrm{Ab}^2_{X_{\bar{k}}} \xrightarrow{\sim} (\mathrm{Prym}_{\tilde{\Delta}/\Delta})_{\bar{k}}$ such that α^{-1} composed with (3.6) is an isomorphism $\psi : (\mathrm{CH}^2_{X/k})^0_{\bar{k}} \xrightarrow{\sim} \mathrm{Ab}^2_{X_{\bar{k}}}$ that on $\bar{k}$-points coincides with the map $\psi^2_{X_{\bar{k}}}$ from §3.2. We have now verified all of the conclusions of Theorem 3.3.

We claim that the pull-back of the Prym polarisation in $\mathrm{NS}((\mathrm{Prym}_{\tilde{\Delta}/\Delta})_{\bar{k}})$ by the isomorphism α coincides with $\theta_{X_{\bar{k}}} \in \mathrm{NS}(\mathrm{Ab}^2_{X_{\bar{k}}})$. This will imply that $\theta_{X_{\bar{k}}}$ is a principal polarisation, thus resolving Questions 3.5 in the affirmative.

To verify the claim, it suffices to show that the pull-back of the canonical polarisation of $\mathrm{Pic}^0_{\tilde{\Delta}/k}$ by $p_* j^* \varepsilon^* : (\mathrm{CH}^2_{X/k})^0 \to \mathrm{Prym}_{\tilde{\Delta}/\Delta} \subset \mathrm{Pic}^0_{\tilde{\Delta}/k}$ is $2\psi^*\theta_{X_{\bar{k}}}$. For this, in view of the definition of $\theta_{X_{\bar{k}}}$, of the compatibility of Bloch's Abel–Jacobi map with the action of correspondences—such as $p_* j^* \varepsilon^*$—and of its compatibility, in the case of codimension 1 cycles, with the Kummer map (see [Blo79, Proposition 3.5, Proposition 3.6]), it suffices to check that for all $x, y \in H^3_{\text{ét}}(X_{\bar{k}}, \mathbf{Z}_\ell(2))$, one has

$$-2\mathrm{Tr}(x \smile y) = \mathrm{Tr}(p_* j^* \varepsilon^* x \smile p_* j^* \varepsilon^* y), \tag{3.7}$$

where $p_* j^* \varepsilon^* x$ and $p_* j^* \varepsilon^* y$ live in $H^1_{\text{ét}}(\tilde{\Delta}_{\bar{k}}, \mathbf{Z}_\ell(1))$ and Tr denotes the two trace maps $H^6_{\text{ét}}(X_{\bar{k}}, \mathbf{Z}_\ell) \to \mathbf{Z}_\ell(-3)$ and $H^2_{\text{ét}}(\tilde{\Delta}_{\bar{k}}, \mathbf{Z}_\ell) \to \mathbf{Z}_\ell(-1)$ (see (3.3)).

Recall from [FJS+24b, Theorem 5.8] that $\varepsilon_* j_* p^* \varpi^*$ vanishes on $\mathrm{Pic}^0(\Delta_{\bar{k}})$, so that $\varepsilon_* j_* p^* \circ (\iota_* - \mathrm{Id}) = -2\varepsilon_* j_* p^*$ on $\mathrm{Pic}^0(\tilde{\Delta}_{\bar{k}})$, which, by Lemma 5.5 in *op. cit.*, implies that $\varepsilon_* j_* p^* p_* j^* \varepsilon^* \varepsilon_* j_* p^* = -2\varepsilon_* j_* p^*$ on $\mathrm{Pic}^0(\tilde{\Delta}_{\bar{k}})$. As $\varepsilon_* j_* p^* : \mathrm{Pic}^0(\tilde{\Delta}_{\bar{k}}) \to \mathrm{CH}^2(X_{\bar{k}})_{\mathrm{alg}}$ is onto (*loc. cit.*, Theorem 5.8), we infer that $\varepsilon_* j_* p^* p_* j^* \varepsilon^* = -2$ as endomorphisms of $\mathrm{CH}^2(X_{\bar{k}})_{\mathrm{alg}}$, hence also as endomorphisms of $H^3_{\text{ét}}(X_{\bar{k}}, \mathbf{Z}_\ell(2))$, since $T_\ell(\mathrm{CH}^2(X_{\bar{k}})_{\mathrm{alg}}) =$

$H^3_{\text{ét}}(X_{\bar{k}}, \mathbf{Z}_\ell(2))$ (see Proposition 3.1). Together with the projection formula, this finally implies (3.7). □

4. Irrationality and the Clemens–Griffiths method

For an account of the classical Clemens–Griffiths method for proving irrationality over $\mathbf{C}$, we refer the reader to [Bea16, §3]. In the whole of §4, we denote by X a smooth projective threefold over a field k, with algebraic closure $\bar{k}$, and assume that $\mathrm{CH}_0(X)_{\mathbf{Q}}$ is supported in dimension 0 in the sense of Definition 2.3.

4.1. **A necessary condition for rationality.** Theorem 4.1 below describes, in its most modern formulation, the method inaugurated by Clemens and Griffiths for disproving the rationality of threefolds via their intermediate Jacobians. This statement can be found in [BW23, Theorem 3.1 (vii)] and builds on a series of earlier variants of this method (see [CG72], [Mur73], [BW20], [HT21]).

Let us assume that $\mathrm{CH}^2_{X/k}$ is representable by a smooth group scheme over k, that all of the conclusions of Theorem 3.3 hold true and that Question 3.5 (i) admits a positive answer; the intermediate Jacobian $J = (\mathrm{CH}^2_{X/k})^0$ is then a polarised abelian variety over k (see Remark 3.6). These assumptions do not cost us anything since they are implied by the rationality of X, which we aim to contradict.

Theorem 4.1. *If X is rational, there exists a smooth projective curve B such that the group scheme $\mathrm{CH}^2_{X/k}$ is a* polarised direct factor *of* $\mathrm{Pic}_{B/k}$.

By "polarised direct factor", we mean that $\mathrm{CH}^2_{X/k}$ is a direct factor of $\mathrm{Pic}_{B/k}$ in such a way that the pull-back of the canonical polarisation of the Jacobian $\mathrm{Pic}^0_{B/k}$ coincides with the polarisation of J given by $\theta_{X_{\bar{k}}}$. We stress that the curve B is not assumed to be connected; thus $(\mathrm{Pic}^0_{B/k})_{\bar{k}}$ is really a product of Jacobians over $\bar{k}$.

The strategy for the proof of Theorem 4.1 is Clemens and Griffiths': choose a birational map from $\mathbf{P}^3_k$ to X; resolve its indeterminacies to obtain a diagram

$$\begin{array}{ccccccccc} X' = Y_N & \longrightarrow & Y_{N-1} & \longrightarrow & \cdots & \longrightarrow & Y_1 & \longrightarrow & Y_0 \\ \big\downarrow{\scriptstyle h} & & & & & & & & \| \\ X & \leftarrow & - & - & - & - & - & - & \mathbf{P}^3_k \end{array} \tag{4.1}$$

in which h is a birational morphism and Y_{j+1} is obtained from Y_j by blowing up an irreducible closed subvariety $Z_j \subset Y_j$ of dimension $d_j \in \{0, 1\}$; compute $\mathrm{CH}^2_{Y_j/k}$ for all j, starting at $j = 0$, one j at a time; finally, check that h^* and h_* realise $\mathrm{CH}^2_{X/k}$ as a polarised direct factor of $\mathrm{CH}^2_{X'/k}$. When k is perfect, the centres Z_j can be chosen to be smooth (Abhyankar [Abh98]); for an arbitrary k, one can require the Z_j to be regular (Cossart and Piltant [CP08]).

To compute $\mathrm{CH}^2_{Y_j/k}$, one establishes a canonical isomorphism $\mathrm{CH}^2_{\mathbf{P}^3_k/k} = \mathbf{Z}$ and then shows, for all j, that if $d_j = 0$, then $\mathrm{CH}^2_{Z_{j+1}/k} = \mathrm{CH}^2_{Z_j/k} \times \mathbf{Z}_{Z_j/k}$, where $\mathbf{Z}_{Z_j/k}$ denotes the Weil restriction of scalars of the constant group scheme $\mathbf{Z}$ from Z_j to k, and if $d_j = 1$, then $\mathrm{CH}^2_{Z_{j+1}/k} = \mathrm{CH}^2_{Z_j/k} \times \mathrm{Pic}_{Z_j/k}$.

Verifying all of these assertions requires some care, as what one needs to analyse is not the diagram (4.1) but its base change from k to an arbitrarily singular k-scheme T. Fortunately, the resulting morphisms $Y_{j+1} \times T \to Y_j \times T$ are blow-ups along regularly embedded centres, making them amenable to the results of Thomason [Tho93]. A theorem of Chatzistamatiou and Rülling [CR15] also plays a key role for checking the identity $h_*h^* = \mathrm{Id}$ on $K_0(X \times T)$, and hence on $\mathrm{CH}^2_{X/k}$. Finally, let us note that even though X' and the Y_j need not be smooth, they are nevertheless local complete intersection varieties, which suffices for defining the morphism h_* and for establishing the above claims.

Letting $G = \mathrm{Ker}(h_* : \mathrm{CH}^2_{X'/k} \to \mathrm{CH}^2_{X/k})$, we thus end up with isomorphisms of group schemes

$$\mathrm{CH}^2_{X/k} \times G \xrightarrow{\sim} \mathrm{CH}^2_{X'/k} \xleftarrow{\sim} \mathrm{CH}^2_{\mathbf{P}^3_k/k} \times \prod_{d_j=1} \mathrm{Pic}_{Z_j/k} \times \prod_{d_j=0} \mathbf{Z}_{Z_j/k}. \tag{4.2}$$

Let B' be the disjoint union of a copy of $\mathbf{P}^1_k$, a copy of $\mathbf{P}^1_k \times Z_j$ for each j such that $d_j = 0$ and a copy of Z_j for each j such that $d_j = 1$. The diagram (4.2) exhibits $\mathrm{CH}^2_{X/k}$ as a direct factor of $\mathrm{Pic}_{B'/k}$. When k is perfect, one sets $B = B'$ and it only remains to check that the natural polarisation on the Jacobian $\mathrm{Pic}^0_{B/k}$ induces on J, via (4.2), the polarisation defined by $\theta_{X_{\bar{k}}}$.

Two genuinely new difficulties arise when k is imperfect, both of which have to do with the fact that B' need not be smooth.

The first one is that confirming the compatibility between $\theta_{X_{\bar{k}}}$ and theta divisors of Jacobians is more subtle, as we now explain. Even formulating this compatibility requires caution: the group scheme $\mathrm{Pic}^0_{B'/k}$ does not carry a theta divisor as it need not even be an abelian variety. Starting from (4.2), one may extend the scalars from k to $\bar{k}$ and then pass to the largest abelian varieties that are quotients of the identity components of these commutative group schemes. This presents $J_{\bar{k}}$ as a direct factor of the product, over all j such that $d_j = 1$, of the Jacobians of the smooth projective curves Z'_j over $\bar{k}$ obtained by normalising the reduced curves $((Z_j)_{\bar{k}})_{\mathrm{red}}$. Only then does it make sense to ask whether $\theta_{X_{\bar{k}}}$ is induced by the canonical polarisations $\theta_j \in \mathrm{NS}(\mathrm{Pic}^0_{Z'_j/\bar{k}})$ of the Jacobians of these smooth projective curves. This turns out to be true—though with an added twist: what induces $\theta_{X_{\bar{k}}}$ is, a priori, the product not of the θ_j, but of the $m_j\theta_j$, where m_j denotes the multiplicity of the possibly non-reduced curve $(Z_j)_{\bar{k}}$ (see [BW23, Lemma 3.7]). It is only because we know, thanks to our assumption that X is rational, that Question 3.5 (ii) admits a positive answer—i.e. that the polarisation defined by $\theta_{X_{\bar{k}}}$ is principal—that we can deduce that the relevant m_j are equal to 1 and that the product of the θ_j does also induce $\theta_{X_{\bar{k}}}$.

The second difficulty is the required smoothness of the curve B appearing in the statement of Theorem 4.1, which precludes setting $B = B'$. Let Y_j denote the étale k-scheme of connected components of Z_j and B be the disjoint union of a copy of $\mathbf{P}^1_k$, a copy of $\mathbf{P}^1_k \times Y_j$ for each j such that either $d_j = 0$ or Z_j is a non-smooth curve, and a copy of Z_j for each j such that Z_j is a smooth curve. What we prove in [BW23, Corollary 1.8, Lemma 3.8] is that for any j such that Z_j is a non-smooth curve, the map $\mathrm{Pic}_{Z_j/k} \to \mathrm{CH}^2_{X/k}$ coming from (4.2) vanishes on $\mathrm{Pic}^0_{Z_j/k}$. As a consequence, quotienting out (4.2) by $\mathrm{Pic}^0_{Z_j/k}$ for all such j will realise $\mathrm{CH}^2_{X/k}$ as a (polarised) direct factor of $\mathrm{Pic}_{B/k}$.

4.2. **Reaching a contradiction—examples of applications.** There exist various ways of disproving rationality of threefolds via Theorem 4.1. We indicate some of them below and illustrate them with applications.

4.2.1. *Intermediate Jacobian.* First, as in the original work of Clemens and Griffiths [CG72], let us focus on the polarised abelian variety J and make use of the following well-known result.

Proposition 4.2. *Let (A, θ) be a principally polarised abelian variety over a field k. The set of abelian subvarieties $E \subseteq A$ such that $(E, \theta|_E)$ is an indecomposable principally polarised abelian variety is finite. The natural map from the product of these subvarieties E to A is an isomorphism.*

Proof. This was noticed by Shimura over $\mathbf{C}$ (see [CG72, Lemma 3.20]). The proof in *loc. cit.* works over algebraically closed fields (see [Mur73, Lemma 10 (ii)]). The truth of the statement for algebraically closed k can be seen to imply its truth for perfect k by Galois descent (see [BW20, §2.1]) and then for arbitrary k by noting that the scheme $\mathrm{End}(A)$ is étale over k (see e.g. [Bri22, Proposition 4.10]). □

In view of Proposition 4.2 and of the indecomposability of Jacobians of smooth projective connected curves (for which see [BW20, §2.1]), Theorem 4.1 implies:

Corollary 4.3. *If X is rational, then $J = (\mathrm{CH}^2_{X/k})^0$ is isomorphic, as a polarised abelian variety, to the Jacobian of a smooth projective curve over k.*

Over algebraically closed fields, there are many ways to prove that a principally polarised abelian variety is not a Jacobian: among others, through the geometry of the theta divisor ([CG72]), singular degenerations ([Col79, Bar84, Gwe05]), clever uses of automorphisms ([Bea12, Bea13]), or point counts over finite fields ([MR18]).

As a first application of Corollary 4.3, we show how the irrationality of smooth cubic threefolds over fields of characteristic $\neq 2$, a theorem originally due to Clemens and Griffiths [CG72] and to Murre [Mur73], fits into the framework presented here.

Corollary 4.4. *Any smooth cubic threefold over an algebraically closed field of characteristic $\neq 2$ is irrational.*

Proof. Blowing up a general line contained in the cubic threefold yields a smooth projective threefold X admitting a morphism $\pi : X \to \mathbf{P}^2_k$ all of whose fibres are isomorphic to conics, and whose associated double cover $\varpi : \tilde{\Delta} \to \Delta$ (see §3.6.2) is an irreducible étale double cover of a smooth quintic curve $\Delta \subset \mathbf{P}^2_k$ defined by an odd theta-characteristic (see [Mur73, §2.1], [Pro18, §3.4]). If X is rational, Theorem 3.8 and Corollary 4.3 imply that $\mathrm{Prym}_{\tilde{\Delta}/\Delta}$ is isomorphic, as a principally polarised abelian variety, to the Jacobian of a (possibly disconnected) smooth projective curve; this would contradict Mumford's analysis of the singular locus of the theta divisor of a Prym variety (see [CG72, p. 355]). □

Corollary 4.4 was recently generalised by Ciurca [Ciu24] to characteristic 2.

Over a non-algebraically closed field, it can happen that a principally polarised abelian variety over k becomes, over $\bar{k}$, the Jacobian of a smooth projective curve over $\bar{k}$, without being itself the Jacobian of a smooth projective curve over k. For example, if C is a non-hyperelliptic smooth projective geometrically irreducible curve of genus $g \geq 3$ over k, any non-trivial quadratic twist of $\mathrm{Pic}^0_{C/k}$ is such a principally polarised abelian variety (see [BW20, Proposition 3.2]). This observation leads to the following application, obtained in [BW20, Theorem 5.4, Example 5.5].

Corollary 4.5. *The affine threefold $Y \subset \mathbf{A}^4_{\mathbf{R}}$ defined by $x^2 + y^2 + z^4 + w^4 = 1$ is not rational, even though it is unirational and $Y_{\mathbf{C}}$ is rational.*

Proof. Performing the linear change of variables $(u, v) = (x + iy, x - iy)$ and solving for u shows that $Y_{\mathbf{C}}$ is rational. Assume that Y is rational, so that we have a polarised abelian variety J at our disposal. One verifies that Y can be compactified to a smooth projective threefold X endowed with a conic bundle structure $\pi : X \to \mathbf{P}^2_{\mathbf{R}}$ whose discriminant locus $\Delta \subset \mathbf{P}^2_{\mathbf{R}}$ is the curve $z^4 + w^4 = v^4$ (*loc. cit.*, Example 5.5), and that J is isomorphic, as a polarised abelian variety, to the non-trivial quadratic twist of $\mathrm{Pic}^0_{\Delta/\mathbf{R}}$ (see Theorem 3.3 (2) and [BW20, Proposition 3.4 (iii)]). Corollary 4.3 now yields a contradiction, in view of the observation made before Corollary 4.5. For the unirationality of Y, see [BW20, Proposition 5.2]. □

In the example of Corollary 4.5, neither the unramified cohomology groups of Y over $\mathbf{R}$ nor the diffeomorphism type of $Y(\mathbf{R})$ are enough to distinguish Y from the rational quadric $x^2 + y^2 + z^2 + w^2 = 1$ (see [BW20, Theorem 5.4, Example 5.5]).

4.2.2. *Torsors.* Hassett and Tschinkel [HT21, §11.5] were the first to realise that taking torsors into account can lead to finer obstructions to rationality than what Corollary 4.3 provides; this allowed them to settle the question of rationality for smooth intersections of two quadrics in $\mathbf{P}^5_{\mathbf{R}}$, using the approach of §2.5. Here is what can, in general, be drawn from Theorem 4.1 in this direction, under the simplifying assumption that J is the Jacobian of a curve of genus ≥ 2, so that Corollary 4.3 is of no effect. (See [BW23, Theorem 3.11] for a statement under weaker hypotheses.)

Corollary 4.6. *Assume given a smooth projective geometrically irreducible curve D of genus ≥ 2 over k and an isomorphism of polarised abelian varieties $J = \mathrm{Pic}^0_{D/k}$. If X is rational, then for any $\alpha \in \mathrm{NS}^2(X_{\bar{k}})^{\mathrm{Aut}(\bar{k}/k)}$, there exist an integer $d \in \mathbf{Z}$ and an isomorphism $(\mathrm{CH}^2_{X/k})^\alpha \simeq \mathrm{Pic}^d_{D/k}$ of torsors under J.*

Proof. By Theorem 4.1, there exist a smooth projective curve B and an injection $f : \mathrm{CH}^2_{X/k} \hookrightarrow \mathrm{Pic}_{B/k}$ inducing a morphism $J = \mathrm{Pic}^0_{D/k} \hookrightarrow \mathrm{Pic}^0_{B/k}$ of principally polarised abelian varieties. By Proposition 4.2, by the indecomposability of Jacobians of smooth projective connected curves (see [BW20, §2.1]) and by the Torelli theorem, one of the connected components of B must be isomorphic to D, in such a way that the composition $g : \mathrm{CH}^2_{X/k} \to \mathrm{Pic}_{D/k}$ of f with the projection $\mathrm{Pic}_{B/k} \to \mathrm{Pic}_{D/k}$ restricts, on the identity components, to the given isomorphism $J = \mathrm{Pic}^0_{D/k}$. Now g must induce an isomorphism of torsors from the connected component $(\mathrm{CH}^2_{X/k})^\alpha$ to some connected component of $\mathrm{Pic}_{D/k}$, as desired. □

We now explain, in the next corollary, how Corollary 4.6 can settle the question of the rationality of smooth intersections of two quadrics in $\mathbf{P}^5_k$, for any k. In 2014, Auel, Bernardara and Bolognesi formulated Corollary 4.7 as a question, when $k = k_0(t)$ for an algebraically closed field k_0; in 2019, Kuznetsov and Prokhorov conjectured its validity for any k; Hassett and Tschinkel [HT21, §11.5] proved it when $k = \mathbf{R}$; Corollary 4.7 is [BW23, Theorem A].

Corollary 4.7. *Let k be a field. A three-dimensional smooth intersection of two quadrics $X \subset \mathbf{P}^5_k$ is rational if and only if it contains a line of $\mathbf{P}^5_k$.*

Proof. If X contains a line, projecting from it shows that X is rational (see [BW23, §4.2]). Assume, conversely, that X is rational, and let F, D be as in §3.6.1; we must prove that $F(k) \neq \varnothing$. Let $P = \mathrm{Pic}^1_{D/k}$. Applying Theorem 3.7, we find that F is a torsor under $J = (\mathrm{CH}^2_{X/k})^0 = \mathrm{Pic}^0_{D/k}$ and that $2[F] = [P]$ and $2[P] = 0$ in $H^1(k, J)$ (the latter because the canonical line bundle of D has degree 2). By Corollary 4.6, there exists $d \in \mathbf{Z}$ such that $[F] = d[P]$. Multiplying $[F] = d[P]$ by 2 yields $2[F] = 0$, hence $[P] = 0$, hence $[F] = 0$, i.e. $F(k) \neq \varnothing$, as desired. □

Nearly identical lines of reasoning enabled Kuznetsov and Prokhorov [KP23] to establish analogues of Corollary 4.7 for other classes of Fano threefolds (*op. cit.*, Corollary 7.9, Corollary 7.10), thereby allowing them to give a complete solution to the rationality problem for Fano threefolds of geometric Picard number 1 over fields of characteristic 0 (*op. cit.*, Theorem 1.1). They also investigated, in [KP24], Fano threefolds with Picard number 1 and geometric Picard number > 1, and leveraged the arguments underlying the proof of Corollary 4.5 above, based on quadratic twists of Jacobians of non-hyperelliptic curves, to prove the irrationality of some of them.

Over fields k of characteristic $\neq 2$, Frei, Ji, Sankar, Viray and Vogt [FJS$^+$24b] examined conic bundle threefolds $\pi : X \to \mathbf{P}^2_k$ whose associated double cover $\varpi : \tilde{\Delta} \to \Delta$ (see §3.6.2) is a geometrically irreducible étale double cover of a smooth quartic curve $\Delta \subset \mathbf{P}^2_k$. As is well known, such threefolds are geometrically rational (*op. cit.*, Proposition 8.1 (ii)), and one naturally wishes to understand exactly when they are rational. Corollary 4.3 is of no help, as $J = (\mathrm{CH}^2_{X/k})^0$ is always the Jacobian of a smooth projective curve of genus 2 (*op. cit.*, Theorem 4.5, based on results of Bruin). The following question arises:

Question 4.8. Let k be a field of characteristic $\neq 2$ and $\pi : X \to \mathbf{P}^2_k$ be a conic bundle threefold whose associated double cover is a geometrically irreducible étale double cover of a smooth quartic curve. Does the necessary condition for rationality expressed by Corollary 4.6 actually characterise rationality?

The authors of [FJS$^+$24b] have uncovered a remarkable phenomenon. They prove, in Theorem 1.4 (i) and Theorem 1.5 of *op. cit.*:

Theorem 4.9. *Question 4.8 admits a negative answer for $k = \mathbf{R}$, and a positive answer whenever the 2-torsion subgroup of the Brauer group of k vanishes.*

5. Some open ends

We list a few open questions that remain to be addressed.

5.1. *Algebraic theory of intermediate Jacobians.* A more general and systematic algebraic theory of intermediate Jacobians would be desirable; in this direction, Questions 3.4 and Questions 3.5 above are among the most pressing first steps. Such a global algebraic theory should fit in with Bloch's formula for the tangent space to the Chow group (see [Blo73, p. 5]). One would also like to develop a relative algebraic theory over a base scheme rather than over a base field. For Albanese varieties, this question is considered in [LS24]. Intermediate Jacobians in families play a role in the work of Griffiths [Gri70], Zucker [Zuc76], Clemens [Cle83], more recently Schnell [Sch12], among many others. Even over ground fields, an algebraic point of view on the intermediate Jacobians associated in [Zuc76, §2] with singular fibres of families of odd-dimensional smooth projective varieties would be valuable.

5.2. *Scope of the method of Clemens and Griffiths.* As is well known, the scope of the method has limits imposed by the geometry of the underlying threefolds; thus, for instance, it leads to a complete characterisation of rationality for many conic bundle threefolds over $\mathbf{C}$, but not for all of them (see [Pro18, Theorem 9.6]). Over non-algebraically closed fields, a new phenomenon appears: as Theorem 4.9 clearly demonstrates, the scope of the Clemens–Griffiths method also depends on the ground field. The following example illustrates this point. One can verify—we do not do so here—that Corollary 4.6 implies the irrationality of the real threefold defined by

$$x^2 + (t^2 - 1)y^2 + (t^2 - 2)z^2 + (t^2 + 3)w^2 = 0 \tag{5.1}$$

(and in this case, no other proof of irrationality is known), while neither Corollary 4.6 nor Theorem 4.1 suffices to decide the rationality of the real threefold defined by

$$x^2 + (t^2 - 1)y^2 + (t^2 + 2)z^2 + (t^2 + 3)w^2 = 0 \tag{5.2}$$

(and it is in fact unknown whether or not this one is rational). It would be desirable to characterise rationality for the threefolds of Question 4.8 even when $\mathrm{Br}(k)$ contains nontrivial 2-torsion, and to determine whether or not (5.2) is rational. Can the scope of the method be extended so as to close these gaps?

5.3. *Stable rationality.* Effective as it may be for disproving rationality, the method of Clemens and Griffiths is blind to stable rationality. Recent breakthroughs in the study of stable rationality, from Voisin's work [Voi15] to the proof, by Engel, de Gaay Fortman, and Schreieder [EdGFS25], of the stable irrationality of very general cubic threefolds over $\mathbf{C}$, have relied on degeneration arguments. Such arguments can be exploited for geometrically rational varieties over non-archimedean real closed fields, by degenerating along a valuation with non-algebraically closed residue field (see [BP24, CTPS25]). However, they must fail for geometrically rational varieties over $\mathbf{R}$, since degenerating a variety over $\mathbf{R}$ amounts to degenerating the variety over $\mathbf{C}$ obtained by scalar extension (see [EP05, Exercise 3.5.3]). Corollary 4.5 and Corollary 4.7 thus suggest the following (open) questions. Is the threefold $Y \subset \mathbf{A}^4_{\mathbf{R}}$ defined by $x^2 + y^2 + z^4 + w^4 = 1$ stably rational? Consider irrational smooth intersections of two quadrics $X \subset \mathbf{P}^5_{\mathbf{R}}$ with $X(\mathbf{R})$ nonempty and connected. Is one of them stably irrational? Is one of them stably rational?

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