Rational points of surfaces fibered into curves of genus 1

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Some questions about rational points

Let X be a smooth projective surface over a number field k, and $f : X \to \mathbf{P}^1_k$ a morphism whose fibers are genus 1 curves.

- If f has a section, what can one say about the ranks of the elliptic curves $X_x = f^{-1}(x)$, $x \in \mathbf{P}^1(k)$?
- In general, can one decide whether $X(k) \neq \varnothing$? When does the Hasse principle hold? Density of $X(k)$ in X for the Zariski topology? (Assume f has at most two multiple fibers.)

Some answers (Zariski density)

Theorem (Várilly-Alvarado, 2011)

Let X be the degree 1 del Pezzo surface over $\mathbf{Q}$ defined by $w^2 = z^3 + ax^6 + by^6$ in $\mathbf{P}(1,1,2,3)$. Assume Tate–Shafarevich groups of elliptic curves over $\mathbf{Q}$ with j-invariant 0 are finite. If $3ab \notin \mathbf{Q}^{\star 2}$, then $X(\mathbf{Q})$ is dense in X.

(X is birationally equivalent to an isotrivial elliptic surface; study variation of root number)

Theorem (Logan, McKinnon, van Luijk, 2010)

Let $a, b, c, d \in \mathbf{Q}^\star$ such that $abcd \in \mathbf{Q}^{\star 2}$. Let $X \subset \mathbf{P}^3_{\mathbf{Q}}$ be defined by $ax^4 + by^4 + cz^4 + dw^4 = 0$. If X has a rational point outside of the 48 lines and the coordinate hyperplanes, then $X(\mathbf{Q})$ is dense in X.

(Geometric construction using the two pencils of curves of genus 1 on X.)

Some more answers (existence and Zariski density)

A series of papers

- Swinnerton–Dyer (1995),
- Colliot-Thélène, Skorobogatov and Swinnerton-Dyer (1998),
- Heath-Brown (1999),
- Bender and Swinnerton-Dyer (2001),
- Colliot-Thélène (2001),
- Swinnerton-Dyer (2000),
- Swinnerton-Dyer (2001),
- Skorobogatov and Swinnerton-Dyer (2005),
- Wittenberg (2007)

establish the existence and Zariski-density of rational points for X under various hypotheses on $f : X \to \mathbf{P}^1_k$, most of them assuming two major conjectures:

- Finiteness of 2-primary (or 3-primary) torsion subgroup of Tate–Shafarevich groups of elliptic curves;
- Schinzel's hypothesis.

Schinzel's hypothesis

Hypothesis (H)

Let $f_1, \ldots, f_s \in \mathbf{Z}[t]$ be irreducible polynomials with positive leading coefficients. Assume no integer > 1 divides $\prod_{i=1}^{s} f_i(m)$ for all $m \in \mathbf{Z}$. Then

$$\{m \in \mathbf{Z}; f_1(m), \ldots, f_s(m) \text{ are all prime}\}$$

is infinite.

Case $s = 1$, $\deg(f_1) = 1$: Dirichlet's theorem.

Homogeneous version established for binary cubic forms $ax^3 + by^3$ (Heath-Brown, Moroz).

Sample results obtained by this method

Theorem (Swinnerton-Dyer, 2001)

Assume $\#Ш\{3\} < \infty$ *for elliptic curves over number fields. Let* $a_0, \ldots, a_4 \in \mathbf{Q}^*$*. Assume the threefold* $X \subset \mathbf{P}^4_{\mathbf{Q}}$ *defined by*

$$a_0 x_0^3 + a_1 x_1^3 + a_2 x_2^3 + a_3 x_3^3 + a_4 x_4^3 = 0$$

has points everywhere locally. Then it has a rational point.

Theorem (–, 2007)

Assume Schinzel's hypothesis, and $\#Ш\{2\} < \infty$*. Let* k *be a number field and* $X \subset \mathbf{P}^5_k$ *be a smooth intersection of two quadrics. Assume* X *has points everywhere locally. Then* X *has a rational point.*

Sample results obtained by this method

Theorem (Swinnerton-Dyer, 2000)

Assume Schinzel's hypothesis, and $\#\text{Ш}\{2\} < \infty$*. Let* $a_0, \ldots, a_3 \in \mathbf{Q}^*$*, such that* $a_0 a_1 a_2 a_3 \in \mathbf{Q}^{\star 2}$*. Assume* $a_0 a_1 a_2 a_3 \notin \mathbf{Q}^{\star 4}$ *and* $\pm a_i a_j \notin \mathbf{Q}^{\star 2}$ *for any* $i \neq j$*. Then the surface* $X \subset \mathbf{P}^3_{\mathbf{Q}}$ *defined by*

$$a_0 x_0^4 + a_1 x_1^4 + a_2 x_2^4 + a_3 x_3^4 = 0$$

has a rational point if there is no algebraic Brauer–Manin obstruction.

Main theorem

Setup:

- X smooth projective surface over a number field k,
- $f : X \to \mathbf{P}^1_k$ morphism whose generic fiber X_η is a genus 1 curve, and whose singular fibers are reduced,
- E_η the Jacobian of X_η (elliptic curve over $K = k(t)$).

Theorem (Colliot-Thélène, Skorobogatov, Swinnerton-Dyer, –)

Assume $E_\eta[2](K) \simeq \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ and X_η is a 2-covering of E_η.
Assume E_η has good or multiplicative reduction at every point of $\mathbf{P}^1_k$.
Assume Schinzel's hypothesis and $\#Ш\{2\} < \infty$.
Assume Condition (D) holds.
Then $X(k) \neq \varnothing$ as soon as there is no algebraic Brauer–Manin obstruction (and $X(k)$ is dense in X if in addition f has no section).

Strategy for the proof

Setup:
- $f : X \to \mathbf{P}^1_k$ proper with reduced fibers, and generic fiber X_η a genus 1 curve over $K = k(t)$,
- the Jacobian E_η of X_η has good or multiplicative reduction at every point of $\mathbf{P}^1_k$,
- $E_\eta[2](K) = \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ and X_η is a 2-covering of E_η.

Strategy to find rational points on X:

1. Find $x \in \mathbf{P}^1(k)$ such that $X_x = f^{-1}(x)$ is a smooth genus 1 curve with points everywhere locally. Thus X_x defines a class $[X_x]$ in

$$Ш(k, E_x) = \mathrm{Ker}\Big(H^1(k, E_x) \to \prod_{v \in \Omega} H^1(k_v, E_x)\Big)$$

2. By "performing a 2-descent on E_x while letting x vary", find an $x \in \mathbf{P}^1(k)$ such that in addition $Ш(k, E_x)[2]$ is generated by $[X_x]$.
3. Because the Cassels–Tate pairing $Ш(k, E_x) \times Ш(k, E_x) \to \mathbf{Q}/\mathbf{Z}$ is alternating and non-degenerate, the order of $Ш(k, E_x)[2]$ is a square. Hence $Ш(k, E_x)[2] = 0$, hence $X_x(k) \neq \varnothing$.

More on second step: finding small Selmer groups

Setup:
- $f : X \to \mathbf{P}^1_k$ proper with reduced fibers, and generic fiber X_η a genus 1 curve over $K = k(t)$,
- the Jacobian E_η of X_η has good or multiplicative reduction at every point of $\mathbf{P}^1_k$,
- $E_\eta[2](K) = \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ and X_η is a 2-covering of E_η,
- there are $x \in \mathbf{P}^1(k)$ such that X_x is smooth and has points everywhere locally.

2. By "performing a 2-descent on E_x while letting x vary", find an $x \in \mathbf{P}^1(k)$ such that $Ш(k, E_x)[2]$ is generated by $[X_x]$.

Instead of searching for an x with small Tate–Shafarevich group $Ш(k, E_x)[2]$, find a small Selmer group $\mathrm{Sel}_2(k, E_x)$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_x(k)/2E_x(k) & \longrightarrow & \mathrm{Sel}_2(k, E_x) & \longrightarrow & Ш(k, E_x)[2] & \longrightarrow & 0 \\ & & \| & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & E_x(k)/2E_x(k) & \longrightarrow & H^1(k, E_x[2]) & \longrightarrow & H^1(k, E_x)[2] & \longrightarrow & 0 \\ & & & & \| & & & & \\ & & & & k^*/k^{*2} \times k^*/k^{*2} & & & & \end{array}$$

Admissible points

Notation:

- S a large enough finite set of places of k;
- $\mathscr{O}_S$ the ring of S-integers of k;
- $\mathscr{M} = \{M \in \mathbf{P}^1_k\,;\, E_\eta \text{ has bad reduction at } M\}$;
- for $M \in \mathbf{P}^1_k$ a closed point, $\widetilde{M} \subset \mathbf{P}^1_{\mathscr{O}_S}$ is its Zariski closure.

Definition

A point $x \in \mathbf{P}^1(k)$ is *S-admissible* if $x \in \mathbf{A}^1(\mathscr{O}_S)$ and $\widetilde{x} \cap \widetilde{M}$ is reduced and has cardinality 1 for each $M \in \mathscr{M}$.

Proposition (Serre)

Assuming Schinzel's hypothesis, S-admissible points are dense in $\prod_{v \in S \cap \Omega_f} \mathbf{P}^1(k_v)$.

First step: fibers with points everywhere locally

Setup: • $f : X \to \mathbf{P}^1_k$ proper with reduced fibers, and generic fiber X_η a genus 1 curve over $K = k(t)$.

The implication "no algebraic Brauer–Manin obstruction to the existence of a rational point $\Rightarrow$ there exists $x \in \mathbf{P}^1(k)$ such that X_x is smooth and has points everywhere locally", under Schinzel's hypothesis, is now standard and has nothing to do with curves of genus 1. We do not discuss it, and instead prove:

Theorem (Colliot-Thélène, Skorobogatov, Swinnerton-Dyer, –)

Assume $E_\eta[2](K) \simeq \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ and X_η is a 2-covering of E_η.
Assume E_η has good or multiplicative reduction at every point of $\mathbf{P}^1_k$.
Assume Schinzel's hypothesis and $\#Ш\{2\} < \infty$.
Assume Condition (D) holds.
Then $X(k) \neq \varnothing$ as soon as there is an S-admissible $x \in \mathbf{P}^1(k)$ such that X_x is smooth and has points everywhere locally (moreover $X(k)$ is dense in X if in addition f has no section).

Back to second step: lifting 2-coverings

Notation:

- $\mathscr{U}_{\mathscr{O}_S} = \mathbf{A}^1_{\mathscr{O}_S} \setminus \left(\bigcup_{M\in\mathscr{M}} \widetilde{M}\right)$; $\quad U = \mathscr{U}_{\mathscr{O}_S} \otimes_{\mathscr{O}_S} k = \mathbf{A}^1_k \setminus \mathscr{M}$;
- $E \to \mathscr{U}_{\mathscr{O}_S}$ abelian scheme extending E_η;
- w_M is the place of $k(M)$ defined by $\widetilde{x} \cap \widetilde{M}$, for $M \in \mathscr{M}$;
- v_M is the place of k below w_M; set $S(x) = S \cup \{v_M\,;\, M \in \mathscr{M}\}$.

Note the elliptic curve E_x has good reduction outside $S(x)$, hence

$$\mathrm{Sel}_2(k, E_x) \subseteq H^1(\mathscr{O}_{S(x)}, E_x[2]) \subseteq H^1(k, E_x[2]).$$

Key proposition

For any S-admissible $x \in \mathbf{P}^1(k)$, the evaluation map

$$\mathrm{ev}_x : H^1(\mathscr{U}_{\mathscr{O}_S}, E[2]) \to H^1(\mathscr{O}_{S(x)}, E_x[2])$$

is an isomorphism.

Lifting 2-coverings, II

Key proposition

For any S-admissible $x \in \mathbf{P}^1(k)$, the evaluation map

$$\mathrm{ev}_x : H^1(\mathscr{U}_{\mathscr{O}_S}, E[2]) \to H^1(\mathscr{O}_{S(x)}, E_x[2])$$

is an isomorphism.

$$\begin{array}{ccc} H^1(K, E_\eta[2]) & & \mathrm{Sel}_2(k, E_x) \\ \bigcup & & \bigcap \\ H^1(\mathscr{U}_{\mathscr{O}_S}, E[2]) & \xrightarrow[\sim]{\mathrm{ev}_x} & H^1(\mathscr{O}_{S(x)}, E_x[2]) \end{array}$$

For $\alpha \in H^1(\mathscr{U}_{\mathscr{O}_S}, E[2])$, the corresponding 2-covering $(Y_\alpha)_\eta$ of E_η extends to a smooth projective surface $Y_\alpha \to \mathbf{P}^1_k$.

Conclusion: for *any* S-admissible $x \in \mathbf{P}^1(k)$,

$$\mathrm{Sel}_2(k, E_x) = \left\{\alpha(x)\,;\, \alpha \in H^1(\mathscr{U}_{\mathscr{O}_S}, E[2]) \text{ such that } (Y_\alpha)_x(\mathbf{A}_k) \neq \varnothing\right\}.$$

Local study of the fibers of the Y_α's

Fix α. Let $Y = Y_\alpha$. For $v \notin S$ and $x_v \in \mathbf{P}^1(k_v)$, when is $Y_{x_v}(k_v) \neq \varnothing$?

For simplicity, assume the singular fibers of $f : X \to \mathbf{P}^1_k$ are of type I_2. Then the singular fibers of $Y \to \mathbf{P}^1_k$ are of type I_2 or $2\mathrm{I}_2$. For $M \in \mathscr{M}$, let $K_{M,Y}/k(M)$ be the quadratic (or trivial) extension which splits Y_M.

If $\widetilde{x_v} \cap \left(\bigcup_{M \in \mathscr{M}} \widetilde{M}\right) = \varnothing$ (in $\mathbf{P}^1_{\mathscr{O}_v}$) and $x_v \in \mathbf{A}^1(\mathscr{O}_v)$, then Y_{x_v} is a smooth curve of genus 1 with good reduction, so $Y_{x_v}(k_v) \neq \varnothing$.

If $\widetilde{x_v} \cap \widetilde{M}$ is reduced and has cardinality 1 for some $M \in \mathscr{M}$ (thus defining a place w of $k(M)$ above v), then the reduction of Y_{x_v} mod v is the reduction of Y_M mod w. Thus $Y_{x_v}(k_v) = \varnothing$ if Y_M is not reduced, and if Y_M is reduced, then

$$Y_{x_v}(k_v) \neq \varnothing \iff w \text{ splits in } K_{M,Y} \iff \mathrm{inv}_v\, A_{M,Y}(x_v) = 0$$

where $A_{M,Y} \in \mathrm{Br}(k(t))$ is $A_{M,Y} = \mathrm{Cores}_{k(M)/k} \underbrace{(K_{M,Y}/k(M), t - t_M)}_{\in \mathrm{Br}(k(M)(t))}$

($t_M \in k(M)$ is the coordinate of M in $\mathbf{A}^1_k$).

Making the Selmer group smaller

Pick an S-admissible $x \in \mathbf{P}^1(k)$ such that $X_x(\mathbf{A}_k) \neq \varnothing$. If $\mathrm{Sel}_2(k, E_x)$ is generated by $E_x(k)/2E_x(k)$ and by the class of X_x, then we are done. Otherwise $\mathrm{Sel}_2(k, E_x)$ contains another class, say $\alpha(x)$ for some $\alpha \in H^1(\mathscr{U}_{\mathscr{O}_S}, E[2])$. Let $Y = Y_\alpha$. Thus $Y_x(\mathbf{A}_k) \neq \varnothing$. It follows from the local study of Y_x at v_M that Y_M is reduced for any $M \in \mathscr{M}$.

Suppose there is an $M_0 \in \mathscr{M}$ such that $K_{M_0,Y}/k(M_0)$ is quadratic (i.e., not trivial).

Pick a place w_0 of $k(M_0)$ which is inert in $K_{M_0,Y}$, of degree 1 over a place $v_0 \notin S$ of k. Pick $x_{v_0} \in \mathbf{P}^1(k_{v_0})$ such that $\widetilde{x_{v_0}} \cap \widetilde{M}_0 = \{w_0\}$ in $\mathbf{P}^1_{\mathscr{O}_{v_0}}$.

Let $S^+ = S \cup \{v_0\}$. Let $x^+ \in \mathbf{P}^1(k)$ be S^+-admissible, close to x at $v \in S$, close to x_{v_0} at $v = v_0$. Then $Y_{x^+}(k_{v_0}) = \varnothing$.

Claim: $\mathrm{ev}_{x^+}^{-1}(\mathrm{Sel}_2(k, E_{x^+})) \subseteq \mathrm{ev}_x^{-1}(\mathrm{Sel}_2(k, E_x))$.

(If claim proved, finished: the inclusion must be strict since $Y_{x^+}(\mathbf{A}_k) = \varnothing$.)

Setup:
- $f : X \to \mathbf{P}^1_k$ proper with reduced fibers; generic fiber X_η a 2-covering of its Jacobian E_η;
- v_0 place of k below a place w_0 of $k(M_0)$ inert in $K_{M_0,Y}$, where $Y = Y_\alpha$, $\alpha \in H^1(\mathscr{U}_{\mathscr{O}_S}, E[2])$;
- $x \in \mathbf{P}^1(k)$ is S-admissible, $X_x(\mathbf{A}_k) \neq \varnothing$; $x^+ \in \mathbf{P}^1(k)$ is S^+-admissible, $S^+ = S \cup \{v_0\}$;
- $\alpha \in H^1(\mathscr{U}_{\mathscr{O}_S}, E[2])$ satisfies $\alpha(x) \in \mathrm{Sel}_2(k, E_x)$, $\alpha(x^+) \notin \mathrm{Sel}_2(k, E_{x^+})$.

Claim: $\mathrm{ev}_{x^+}^{-1}(\mathrm{Sel}_2(k, E_{x^+})) \subseteq \mathrm{ev}_x^{-1}(\mathrm{Sel}_2(k, E_x))$.

Proof.

Let $\beta \in H^1(\mathscr{U}_{\mathscr{O}_S}, E[2])$. If $(Y_\beta)_x(\mathbf{A}_k) = \varnothing$, then $(Y_\beta)_{x^+}(\mathbf{A}_k) = \varnothing$?

- If $(Y_\beta)_x(k_v) = \varnothing$ for a $v \in S$: OK (x^+ v-adically close to x).
- If $(Y_\beta)_x(k_v) = \varnothing$ for $v = v_M$, $M \neq M_0$:

$$(Y_\beta)_x(k_{v_M}) = \varnothing \iff w_M \text{ is inert in } K_{M,Y_\beta} \iff \mathrm{inv}_{v_M}\, A_{M,Y_\beta}(x) \neq 0$$

$$\iff \sum_{v \in S} \mathrm{inv}_v\, A_{M,Y_\beta}(x) \neq 0 \iff \sum_{v \in S} \mathrm{inv}_v\, A_{M,Y_\beta}(x^+) \neq 0$$

$$\iff \mathrm{inv}_{v_M^+}\, A_{M,Y_\beta}(x^+) \neq 0$$

$$\iff (Y_\beta)_{x^+}(k_{v_M^+}) = \varnothing.$$

Setup:
- $f : X \to \mathbf{P}^1_k$ proper with reduced fibers; generic fiber X_η a 2-covering of its Jacobian E_η;
- v_0 place of k below a place w_0 of $k(M_0)$ inert in $K_{M_0,Y}$, where $Y = Y_\alpha$, $\alpha \in H^1(\mathscr{U}_{\mathscr{O}_S}, E[2])$;
- $x \in \mathbf{P}^1(k)$ is S-admissible, $X_x(\mathbf{A}_k) \neq \varnothing$; $x^+ \in \mathbf{P}^1(k)$ is S^+-admissible, $S^+ = S \cup \{v_0\}$;
- $\alpha \in H^1(\mathscr{U}_{\mathscr{O}_S}, E[2])$ satisfies $\alpha(x) \in \mathrm{Sel}_2(k, E_x)$, $\alpha(x^+) \notin \mathrm{Sel}_2(k, E_{x^+})$.

Claim: $\mathrm{ev}_{x^+}^{-1}(\mathrm{Sel}_2(k, E_{x^+})) \subseteq \mathrm{ev}_x^{-1}(\mathrm{Sel}_2(k, E_x))$.

Proof.

Let $\beta \in H^1(\mathscr{U}_{\mathscr{O}_S}, E[2])$. If $(Y_\beta)_x(\mathbf{A}_k) = \varnothing$, then $(Y_\beta)_{x^+}(\mathbf{A}_k) = \varnothing$?

- If $(Y_\beta)_x(k_v) = \varnothing$ for a $v \in S$: OK (x^+ v-adically close to x).
- If $(Y_\beta)_x(k_v) = \varnothing$ for $v = v_M$, $M = M_0$:

$$(Y_\beta)_x(k_{v_M}) = \varnothing \iff w_M \text{ is inert in } K_{M,Y_\beta} \iff \mathrm{inv}_{v_M} A_{M,Y_\beta}(x) \neq 0$$

$$\iff \sum_{v \in S} \mathrm{inv}_v A_{M,Y_\beta}(x) \neq 0 \iff \sum_{v \in S} \mathrm{inv}_v A_{M,Y_\beta}(x^+) \neq 0$$

$$\iff \mathrm{inv}_{v_M^+} A_{M,Y_\beta}(x^+) + \mathrm{inv}_{v_0} A_{M,Y_\beta}(x^+) \neq 0$$

$$\implies (Y_\beta)_{x^+}(k_{v_M^+}) = \varnothing \text{ or } (Y_\beta)_{x^+}(k_{v_0}) = \varnothing.$$

Setup:
- $f : X \to \mathbf{P}^1_k$ proper with reduced fibers; generic fiber X_η a 2-covering of its Jacobian E_η;
- v_0 place of k below a place w_0 of $k(M_0)$ inert in $K_{M_0,Y}$, where $Y = Y_\alpha$, $\alpha \in H^1(\mathscr{U}_{\mathscr{O}_S}, E[2])$;
- $x \in \mathbf{P}^1(k)$ is S-admissible, $X_x(\mathbf{A}_k) \neq \varnothing$; $x^+ \in \mathbf{P}^1(k)$ is S^+-admissible, $S^+ = S \cup \{v_0\}$;
- $\alpha \in H^1(\mathscr{U}_{\mathscr{O}_S}, E[2])$ satisfies $\alpha(x) \in \mathrm{Sel}_2(k, E_x)$, $\alpha(x^+) \notin \mathrm{Sel}_2(k, E_{x^+})$.

Claim: $\mathrm{ev}_{x^+}^{-1}(\mathrm{Sel}_2(k, E_{x^+})) \subseteq \mathrm{ev}_x^{-1}(\mathrm{Sel}_2(k, E_x))$.

Proof.

Let $\beta \in H^1(\mathscr{U}_{\mathscr{O}_S}, E[2])$. If $(Y_\beta)_x(\mathbf{A}_k) = \varnothing$, then $(Y_\beta)_{x^+}(\mathbf{A}_k) = \varnothing$? Yes.
Problem: must also consider β in the larger group $H^1(\mathscr{U}_{\mathscr{O}_{S^+}}, E[2])$.

Key lemma

For $\beta \in H^1(\mathscr{U}_{\mathscr{O}_{S^+}}, E[2])$, if $\beta(x^+) \in \mathrm{Sel}_2(k, E_{x^+})$, then $\beta \in H^1(\mathscr{U}_{\mathscr{O}_S}, E[2])$.

Proof uses:
- global reciprocity law for $\alpha(x^+) \smile \beta(x^+) \in \mathrm{Br}(k)$;
- local Tate duality: $E_{x^+}(k_v)/2E_{x^+}(k_v)$ and $H^1(\mathscr{O}_v, E_{x^+}[2])$ maximal totally isotropic subspaces of $H^1(k_v, E_{x^+}[2])$;
- reciprocity arguments, to check $(Y_\alpha)_{x^+}(k_v) = \varnothing \Leftrightarrow v \in \{v_0, v_{M_0}^+\}$.

Summary

Thus the new Selmer group is strictly smaller. Recall: we started with a class in $\mathrm{Sel}_2(k, E_x)$, wrote it as $\alpha(x)$ for an $\alpha \in H^1(\mathscr{U}_{\mathscr{O}_S}, E[2])$, and made the following assumption on the singular fibers of $Y = Y_\alpha \to \mathbf{P}^1_k$:

Suppose there is an $M_0 \in \mathscr{M}$ such that $K_{M_0,Y}/k(M_0)$ is quadratic (i.e., not trivial).

To ensure $X_{x^+}(\mathbf{A}_k) \neq \varnothing$, we want w_0 to split in $K_{M_0,X}$. Thus we need:

Suppose there is an $M_0 \in \mathscr{M}$ such that $K_{M_0,Y}/k(M_0)$ does not embed into $K_{M_0,X}/k(M_0)$.

Condition (D) in the case of reduction type I_2

For any 2-covering Y_η of E_η with good reduction above U, there exists an $M \in \mathscr{M}$ such that either the fiber Y_M is double or $K_{M,Y}/k(M)$ does not embed into $K_{M,X}/k(M)$, unless the curve Y_η is isomorphic to X_η or to E_η.

Variants

- Starting with a family of elliptic curves $f : E \to \mathbf{P}^1_k$, find infinitely many fibers with small Selmer group, hence small Mordell–Weil group. Application to elliptic curves over $k(t)$ of *elevated rank*.
- Conversely, if $f : X \to \mathbf{P}^1_k$ has no section, find infinitely many $x \in \mathbf{P}^1(k)$ for which $\mathrm{rk}(E_x(k)) > \mathrm{rk}(E_\eta(K))$.
- If $f : X \to \mathbf{P}^1_k$ does not satisfy Condition (D), find many $x \in \mathbf{P}^1(k)$ such that $X_x(\mathbf{A}_k) \neq \emptyset$ and $[X_x]$ is orthogonal to $\text{Ш}(k, E_x)[2]$ for the Cassels–Tate pairing. (Uses the full Brauer group of X.) Replace Condition (D) with higher descents?