Intermediate Jacobians over non-closed fields and applications to rationality

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2025 Summer Research Institute in Algebraic Geometry

Fort Collins, August 1st, 2025

Rationality

X smooth projective variety over a field k.

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$$\Leftrightarrow k(X)/k \text{ is purely transcendental}$$

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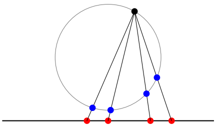

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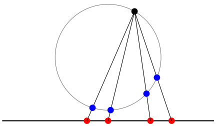

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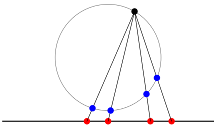

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Theorem (Castelnuovo, Zariski, Segre, Manin, Iskovskikh)

If X is a minimal surface, then
X is rational $\Leftrightarrow X(k) \neq \varnothing$ and $q(X) = P_2(X) = 0$ and $K_X^2 \geq 5$.

Rationality in dimension 3

Still mysterious. Existing methods:

- birational rigidity (Iskovskikh–Manin 1971, ...)
- unramified cohomology (Artin–Mumford 1972, ...)
- intermediate Jacobians (Clemens–Griffiths 1972, ...)
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Theorem (Murre 1973 for $p > 2$, Ciurca 2024 for $p = 2$)

Same statement, over $k = \bar{k}$ of char. $p > 0$.

Sample theorems over non-closed fields (1/2)

Theorem (Benoist–W. 2020)

The affine threefold $x^2 + y^2 + z^4 + w^4 = 1$ *over* $\mathbf{R}$ *is not rational (but is unirational and geometrically rational).*

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Corollary (Benoist–W. 2023)

There exist smooth projective varieties that have a rational point, become rational over a purely inseparable extension, but are irrational.

Sample theorems over non-closed fields (2/2)

Fano threefolds (Kuznetsov–Prokhorov 2023, 2024)

Conic bundles (Frei–Ji–Sankar–Viray–Vogt 2024, Ji–Ji 2024)

Certain Fano schemes (Ji–Suzuki 2024)

Theorem (W. 2025)

Let $X \to \mathbf{P}^1_{\mathbf{R}}$ be a quadric surface bundle with ≥ 6 singular geometric fibres, no reducible fibre, no section. If one fibre of $X(\mathbf{R}) \to \mathbf{P}^1(\mathbf{R})$ is a torus, then X is irrational.

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An algebraic theory of intermediate Jacobians?

Murre's idea: Abel–Jacobi isomorphism

$$\mathrm{CH}^2(X)_{\mathrm{alg}} \xrightarrow{\sim} J(\mathbf{C})$$

for any rationally connected threefold X over $\mathbf{C}$ (Bloch, Srinivas 1983).
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Murre (1983): "universal problem" point of view, $k = \bar{k}$.
$\rightsquigarrow$ Achter, Casalaina-Martin, Vial (2017): perfect k.

Benoist–W. (2023): "functor of points" point of view.

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Better: for any smooth projective rationally connected threefold X,

$$\mathrm{Ker}\left(K_0(X_{\bar{k}}) \xrightarrow{\mathrm{rk}\times\det\times\chi} \mathbf{Z} \times \mathrm{Pic}(X_{\bar{k}}) \times \mathbf{Z}\right) \xrightarrow[\sim]{c_2} \mathrm{CH}^2(X_{\bar{k}}).$$

Definition

$K_{0,X/k}$:= the fppf sheafification of $T \mapsto K_0(X \times T)$

$\mathrm{CH}^2_{X/k} := \mathrm{Ker}\left(K_{0,X/k} \xrightarrow{\mathrm{rk}\times\det\times\chi} \mathbf{Z} \times \mathrm{Pic}_{X/k} \times \mathbf{Z}\right)$

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$\Big\}$ functors $(\mathrm{Sch}/k)^{\mathrm{op}} \to (\mathrm{Ab})$

Theorem (Benoist–W. 2023)

Let X be a smooth projective $\bar{k}$-rational threefold over a field k.

1. $\mathrm{CH}^2_{X/k}$ *is representable and fits into an exact sequence*

$$0 \longrightarrow J \longrightarrow \mathrm{CH}^2_{X/k} \longrightarrow \mathrm{NS}^2_{X/k} \longrightarrow 0$$

   where $J = (\mathrm{CH}^2_{X/k})^0$ is a p.p.a.v. and $\mathrm{NS}^2_{X/k}$ is étale over k;

2. $\mathrm{CH}^2_{X/k}(\bar{k}) = \mathrm{CH}^2(X_{\bar{k}})$ *and* $J(\bar{k}) = \mathrm{CH}^2(X_{\bar{k}})_{\mathrm{alg}}$.

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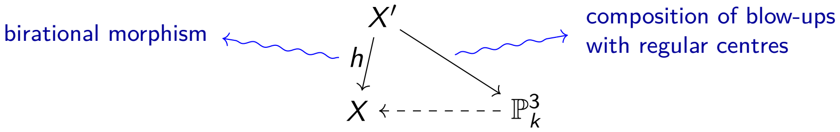
Question

Can "$\bar{k}$-rational" be weakened to "rationally connected"?

Obstruction to rationality

Idea (Clemens–Griffiths): if X is rational, then



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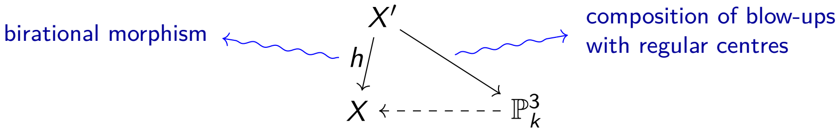

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Theorem (BW 2023; builds on BW 2020 and Hassett–Tschinkel 2021)

If X is a rational smooth projective threefold,

1. *there exist a smooth projective curve D over k and an isomorphism of p.p.a.v. $(\mathrm{CH}^2_{X/k})^0 \simeq \mathrm{Pic}^0_{D/k}$;*
2. *if D is geometrically connected, then for any $\alpha \in \mathrm{NS}^2_{X/k}(k)$, the torsor $(\mathrm{CH}^2_{X/k})^\alpha$ is isomorphic to $\mathrm{Pic}^n_{D/k}$ for some $n \in \mathbf{Z}$.*

(If D is geometrically connected of genus ≥ 2, it is unique.)

Example: intersections of two quadrics

$X \subset \mathbf{P}^5_k$ smooth intersection of two quadrics.
F the variety of lines on X.

Theorem (Cassels & Wang in char. $\neq 2$, Benoist–W. in general)

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1. $\delta^{-1}(0) = \mathrm{Pic}^0_{D/k}$ *for a unique genus 2 curve D (so $\mathrm{Pic}^2_{D/k} \simeq \mathrm{Pic}^0_{D/k}$)*,
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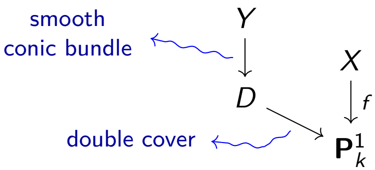
Thus: $F \simeq \mathrm{Pic}^n_{D/k} \underset{\times 2}{\Longrightarrow} \mathrm{Pic}^1_{D/k} \simeq \mathrm{Pic}^{2n}_{D/k} \simeq \mathrm{Pic}^0_{D/k} \underset{\times n}{\Longrightarrow} F(k) \neq \varnothing$.

Example: quadric surface bundles (char. $k \neq 2$)



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the fibres of f are irreducible quadrics;

then $Y = \{\text{lines in the fibres of } f\}$
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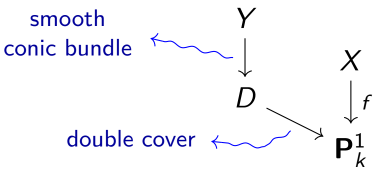

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There is a canonical exact sequence

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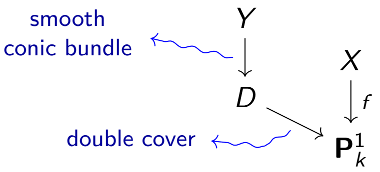

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$\Uparrow$ *X is rational and D is geometrically connected of genus ≥ 2.*

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- Decide stable rationality?
  - Challenge: is $x^2 + y^2 + z^4 + w^4 = 1$ stably rational over **R**?
  - Pick *any* smooth irrational $X = X^{2,2} \subset \mathbf{P}^5_{\mathbf{R}}$ with $X(\mathbf{R})$ connected. Challenge: is X stably rational?

(Breakthroughs over **C** (Voisin 2015, Engel–de Gaay Fortman–Schreieder 2025), via degenerations. No such argument can work over **R** for **C**-rational varieties.)