Intermediate Jacobians and the rationality of intersections of two quadrics

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Intersections of two quadrics

Focus today:

$X \subset \mathbf{P}^n_k$ smooth complete intersection of two quadrics over k, i.e. $q(x_0, \dots, x_n) = q'(x_0, \dots, x_n) = 0$.

A bit of geometry:

$n = 3$: curve of genus 1 (including all elliptic curves)
$n = 4$: del Pezzo surface of degree 4 (all of them)
$n = 5$: Fano threefold (..., Reid, Donagi, Cassels, Wang, ...).

A few articles by Peter Swinnerton-Dyer

Rational zeros of two quadratic forms, Acta Arith. **9** (1964), 261–270.

(with B. Birch) *The Hasse problem for rational surfaces*, J. reine angew. Math. **274/275** (1975), 164–174.

(with J.-L. Colliot-Thélène and J.-J. Sansuc) *Intersections of two quadrics and Châtelet surfaces, I, II*, J. reine angew. Math. **373** (1987), 37–107, **374** (1987), 72–168.

Rational points on certain intersections of two quadrics, in: Abelian varieties (Egloffstein, 1993), 273–292, de Gruyter, Berlin, 1995.

(with A. O. Bender) *Solubility of certain pencils of curves of genus* 1*, and of the intersection of two quadrics in* $\mathbf{P}^4$, Proc. London Math. Soc. (3) **83** (2001), no. 2, 299–329.

Weak approximation on del Pezzo surfaces of degree 4, in: Arithmetic of higher-dimensional algebraic varieties (Palo Alto, CA, 2002), 235–257, Progr. Math., 226, Birkhäuser, 2004.

Over number fields, existence of rational points? Weak approximation? To start with: rationality?

The birationality of cubic surfaces over a given field, Michigan Math. J. **17** (1970), 289–295.

The Brauer group of cubic surfaces, Math. Proc. Camb. Phil. Soc. **113** (1993), no. 3, 449–460.

Rationality

X smooth projective variety over a field k.

$$\begin{aligned} X \text{ is } \textit{rational} &\Leftrightarrow \exists \text{ birational map } \mathbf{P}_k^{\dim(X)} \dashrightarrow X \\ &\Leftrightarrow k(X)/k \text{ is purely transcendental} \end{aligned}$$

Proposition

If X is a curve, then X is rational $\Leftrightarrow X(k) \neq \varnothing$ and $g(X) = 0$.

Example

$X \subset \mathbf{P}_k^3$ smooth complete intersection of two quadrics
$\Rightarrow X$ curve of genus 1: not rational.

Rationality in dimension 2

Theorem (Swinnerton-Dyer, 1970; Manin, Iskovskikh, 1979)

Let k be a field and $X \subset \mathbf{P}^4_k$ a smooth intersection of two quadrics. Then:

X *is rational* $\Leftrightarrow X(k) \neq \varnothing$ *and* $X_{\bar{k}}$ *contains a line that meets none of its conjugates under* $\mathrm{Aut}(\bar{k}/k)$.

Today, rationality completely understood for smooth proper surfaces (Castelnuovo, Segre, Manin, Iskovskikh).

Rationality in higher dimension

Still very mysterious despite major progress since the 1970's.

1. Noether–Fano method
   (Manin–Iskovskikh for surfaces, quartic threefolds; ...),
2. unramified cohomology
   (Artin–Mumford, Bogomolov–Saltman, Colliot-Thélène–Ojanguren, ...)
3. intermediate Jacobians
   (Clemens–Griffiths, Beauville, Murre, ...),
4. degeneration methods
   (Kollár, Voisin, ...).

Theorem (Clemens–Griffiths, 1972)

Let $X \subset \mathbf{P}^4_{\mathbf{C}}$ be a smooth cubic threefold. Then X is not rational.

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Tool: intermediate Jacobian.

X smooth projective threefold over $\mathbf{C}$ with $h^{1,0} = h^{3,0} = 0$
$\rightsquigarrow J$ principally polarised abelian variety over $\mathbf{C}$

J is built from the Hodge structure $H^3(X(\mathbf{C}), \mathbf{Z})$:

$$J(\mathbf{C}) = \frac{H^2(X, \Omega^1)}{\mathrm{Im}\,(H^3(X(\mathbf{C}), \mathbf{Z}))}$$

Clemens and Griffiths prove:

1. X is any rational threefold $\Rightarrow J$ is a Jacobian of a curve
2. X is a cubic in $\mathbf{P}^4_{\mathbf{C}} \Rightarrow J$ is not a Jacobian of a curve

Clemens–Griffiths over non-closed fields

Method adapted to non-closed fields (BW, 2019).

Example/Theorem (BW, 2019)

The affine threefold defined by $x^2 + y^2 + z^4 + w^4 = 1$ *is*

1. *rational over* $\mathbf{C}$,
2. *unirational over* $\mathbf{R}$,
3. *but not rational over* $\mathbf{R}$.

Intersections of two quadrics in $\mathbf{P}^5_k$

Theorem (Swinnerton-Dyer, 1970; Manin, Iskovskikh, 1979)

Let k be a field and $X \subset \mathbf{P}^4_k$ a smooth intersection of two quadrics. Then:
X is rational $\Leftrightarrow$ $X(k) \neq \varnothing$ and $X_{\bar{k}}$ contains a line that meets none of its conjugates under $\mathrm{Aut}(\bar{k}/k)$.

Theorem (BW, 2019)

Let k be a field and $X \subset \mathbf{P}^5_k$ a smooth intersection of two quadrics. Then:
X is rational $\Leftrightarrow$ X contains a line of $\mathbf{P}^5_k$.

History:

- In 2014, question raised by Auel, Bernardara, Bolognesi, for special k such as $k = \mathbf{C}(t)$.
- In 2019, conjectured by Kuznetsov and Prokhorov and proved for $k = \mathbf{R}$ by Hassett and Tschinkel, building on our previous work on rationality via intermediate Jacobians and on Krasnov's topological classification of $X(\mathbf{R})$.
- Underlying tools later applied by Kuznetsov and Prokhorov to other Fano threefolds.

Theorem (BW, 2019)

Let k be a field and $X \subset \mathbf{P}^5_k$ a smooth intersection of two quadrics. Then X is rational $\Leftrightarrow$ X contains a line of $\mathbf{P}^5_k$.

Corollary (inseparable counterexamples to the Lüroth problem)

There exist a purely inseparable extension of fields k'/k and a smooth projective threefold X over k such that

- *X is unirational (over k) and $X_{k'}$ is rational (over k'), but*
- *X is not rational (over k).*

Example

Let $k = \kappa((t))$ and $k' = \kappa((\sqrt{t}))$ with κ algebraically closed of characteristic 2, and choose $a, b, c \in \kappa$ pairwise distinct.

$$X : \begin{cases} tx_0x_1 + x_2x_3 + x_4x_5 = 0 \\ t(x_0^2 + ax_0x_1 + x_1^2) + (x_2^2 + bx_2x_3 + x_3^2) + (x_4^2 + cx_4x_5 + x_5^2) = 0 \end{cases}$$

Plan for rest of the talk

Theorem (BW, 2019)

Let k be a field and $X \subset \mathbf{P}^5_k$ a smooth intersection of two quadrics. Then X is rational $\Leftrightarrow$ X contains a line of $\mathbf{P}^5_k$.

General idea:

1. X any smooth projective $\bar{k}$-rational threefold over k
   $\rightsquigarrow$ intermediate Jacobian J (p.p.a.v. over k + torsors)
2. X rational $\Rightarrow$ J can be found inside the Jacobian of curve (+ torsors)
3. $X \subset \mathbf{P}^5_k$ a smooth intersection of two quadrics
   $\Rightarrow$ J cannot be found inside the Jacobian of curve (+ torsors)

Intermediate Jacobians over k

X smooth projective $\bar{k}$-rational threefold over k.

- $k = \mathbf{C}$: Abel–Jacobi map $\mathrm{CH}^2(X)_{\mathrm{alg}} \xrightarrow{\sim} J(\mathbf{C})$ (Bloch, Srinivas)
- $k = \bar{k}$: Murre (1983)
  $\rightsquigarrow$ an abelian variety J over k, and $\mathrm{CH}^2(X_{\bar{k}})_{\mathrm{alg}} \xrightarrow{\sim} J(\bar{k})$
- descends to perfect k (Achter, Casalaina-Martin, Vial, 2017)

Murre's definition will not descend to imperfect k.

What we do: mimic $\mathrm{Pic}_{X/k}$. Define an fppf sheaf

$$\mathrm{CH}^2_{X/k} : (\mathrm{Sch}/k)^{\mathrm{op}} \to (\mathrm{Ab}),$$

prove its representability, and construct a principal polarisation.

Mimicking the Picard functor

To do (for X a smooth projective $\bar{k}$-rational threefold over k)

Define an fppf sheaf

$$\mathrm{CH}^2_{X/k} : (\mathrm{Sch}/k)^{\mathrm{op}} \to (\mathrm{Ab})$$

prove its representability, and construct a principal polarisation.

Picard functor: $T \mapsto \mathrm{Pic}(X \times T)$, then sheafify.
First idea: $T \mapsto \mathrm{CH}^2(X \times T)$. Fails: not a contravariant functor.

Better idea: K-theory. Jouanolou:

$$\mathrm{Ker}\left(K_0(X_{\bar{k}}) \xrightarrow{\mathrm{rk}\times\det\times\chi} \mathbf{Z} \times \mathrm{Pic}(X_{\bar{k}}) \times \mathbf{Z}\right) \xrightarrow[\sim]{c_2} \mathrm{CH}^2(X_{\bar{k}})$$

Definition

$K_{0,X/k} :=$ the fppf sheafification of $T \mapsto K_0(X \times T)$.
$\mathrm{CH}^2_{X/k} := \mathrm{Ker}\left(K_{0,X/k} \xrightarrow{\mathrm{rk}\times\det\times\chi} \mathbf{Z} \times \mathrm{Pic}_{X/k} \times \mathbf{Z}\right)$

Thus $\mathrm{CH}^2_{X/k}(\bar{k}) = \mathrm{CH}^2(X_{\bar{k}})$.

Representability and rationality

Theorem (representability)

Let X be a smooth projective $\bar{k}$-rational threefold over a field k.

1. $\mathrm{CH}^2_{X/k}$ *is represented by a smooth group scheme over k;*
2. $J = (\mathrm{CH}^2_{X/k})^0$ *is (canonically) a p.p.a.v.;*
3. $J(\bar{k}) = \mathrm{CH}^2(X_{\bar{k}})_{\mathrm{alg}}$, *hence* $V_\ell(J) = H^3(X_{\bar{k}}, \mathbf{Q}_\ell(2))$ *(Bloch).*

Theorem (condition for rationality)

If furthermore X is rational, then

4. *there exist a smooth projective curve B over k and a group scheme G over k such that*

$$\mathrm{CH}^2_{X/k} \times G \simeq \mathrm{Pic}_{B/k},$$

inducing an isomorphism of p.p.a.v.

$$(\mathrm{CH}^2_{X/k})^0 \times G^0 \simeq \mathrm{Pic}^0_{B/k}.$$

Beginning of the proof

To prove representability of $\mathrm{CH}^2_{X/k}$, may freely extend the scalars:

Lemma

Let $\mathscr{F}$ be an fppf sheaf on (Sch/k) and k'/k be a finite extension.
If $\mathscr{F} \times_k k'$ is representable by a smooth group scheme over k',
then $\mathscr{F}$ is representable by a smooth group scheme over k.

Hence may assume X rational: $X \dashleftarrow \mathbf{P}^3_k$ birational.

Beginning of the proof

Abhyankar, Cossart–Piltant (2009): can resolve indeterminacies.

$$\begin{tikzcd} X' = Y_N \arrow[r] \arrow[d, "h"] & Y_{N-1} \arrow[r] & \cdots \arrow[r] & Y_1 \arrow[r] & Y_0 \arrow[d, equal] \\ X & & & & \mathbf{P}^3_k \arrow[llll, dashed] \end{tikzcd}$$

h birational morphism and Y_{j+1} blow-up of $Z_j \subset Y_j$ irreducible regular. All Y_j are projective regular. For simplicity assume the Z_j are curves.

$\mathrm{CH}^2_{Y_{j+1}/k} = \mathrm{CH}^2_{Y_j/k} \times \mathrm{Pic}_{Z_j/k}$ (Thomason) ; $\mathrm{CH}^2_{\mathbf{P}^3_k/k} = \mathbf{Z}$;

$\mathrm{CH}^2_{X/k} \underset{h_*}{\overset{h^*}{\rightleftarrows}} \mathrm{CH}^2_{X'/k}$ satisfy $h_* h^* = \mathrm{Id}$ (Chatzistamatiou–Rülling).

$$\rightsquigarrow \quad \boxed{\mathrm{CH}^2_{X/k} \times \underbrace{\mathrm{Ker}(h_*)}_{G} \xrightarrow{\sim} \mathrm{CH}^2_{X'/k} \xleftarrow{\sim} \underbrace{\mathbf{Z} \times \prod \mathrm{Pic}_{Z_j/k}}_{\mathrm{Pic}_{B/k}}}$$

For rationality criterion: need to track polarisations and to prove the non-smooth Z_j do not contribute to the intermediate Jacobian.

Back to intersections of two quadrics

$X \subset \mathbf{P}^5_k$ smooth intersection of two quadrics.
F the variety of lines on X.

Theorem (classical in characteristic $\neq 2$)

The intermediate Jacobian of X is $\mathrm{Pic}^0_{D/k}$ for a unique genus 2 curve D. There is an exact sequence

$$0 \longrightarrow \mathrm{Pic}^0_{D/k} \longrightarrow \mathrm{CH}^2_{X/k} \xrightarrow{\delta} \mathbf{Z} \longrightarrow 0$$

and identifications $F = \delta^{-1}(1)$ and $\mathrm{Pic}^1_{D/k} = \delta^{-1}(2)$.

$X \subset \mathbf{P}^5_k$ smooth intersection of two quadrics $\rightsquigarrow$ intermediate Jacobian $= \mathrm{Pic}^0_{D/k}$ for a genus 2 curve D ;

$0 \to \mathrm{Pic}^0_{D/k} \to \mathrm{CH}^2_{X/k} \xrightarrow{\delta} \mathbf{Z} \to 0$; variety of lines $F = \delta^{-1}(1)$; $\mathrm{Pic}^1_{D/k} = \delta^{-1}(2)$.

Assume X is rational, so $\mathrm{CH}^2_{X/k} \hookrightarrow \prod \mathrm{Pic}_{B_j/k}$ for smooth projective connected curves $B_1, \ldots, B_m$ over k, compatibly with polarisations.

May assume $\mathrm{Pic}^0_{D/k} = \mathrm{Pic}^0_{B_1/k}$, and then $D = B_1$ (Torelli).

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Pic}^0_{D/k} \arrow[r] \arrow[d, hook] & \mathrm{CH}^2_{X/k} \arrow[r, "\delta"] \arrow[d, hook] & \mathbf{Z} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \prod \mathrm{Pic}^0_{B_j/k} \arrow[r] \arrow[d, "\mathrm{pr}_1"] & \prod \mathrm{Pic}_{B_j/k} \arrow[r] \arrow[d, "\mathrm{pr}_1"] & \prod \mathrm{NS}_{B_j/k} \arrow[r] \arrow[d, "\mathrm{pr}_1"] & 0 \\
0 \arrow[r] & \mathrm{Pic}^0_{D/k} \arrow[r] & \mathrm{Pic}_{D/k} \arrow[r] & \mathbf{Z} \arrow[r] & 0
\end{tikzcd}$$

so $[F] = n[\mathrm{Pic}^1_{D/k}]$ in $H^1(k, \mathrm{Pic}^0_{D/k})$ for some n.

As $[\mathrm{Pic}^1_{D/k}] = 2[F]$ and $2[\mathrm{Pic}^1_{D/k}] = 0$, get $[F] = 0$, hence $F(k) \neq \varnothing$.