Asymptotic properties of ranked heights in Brownian excursions

by

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Summary. Pitman and Yor [20,21] recently studied the distributions related to the ranked excursion heights of a Brownian bridge. In this paper, we study the asymptotic properties of the ranked heights of Brownian excursions. The heights of both high and low excursions are characterized by several integral tests and laws of the iterated logarithm. Our analysis relies on the distributions of the ranked excursion heights considered up to some random times.

KEY WORDS: Ranked heights, Brownian and Bessel excursions, integral test, law of the iterated logarithm.

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1. INTRODUCTION

Let $\{B(t), t \geq 0\}$ be a one-dimensional Brownian motion starting from 0 and consider the sequence

$$M_1(t) \geq M_2(t) \geq ... \geq M_n(t) \geq ... \tag{1.1}$$

the ranked heights of all the excursions of the reflected Brownian motion $|B|$ considered up to time t (including the meander height $\sup_{g_t \leq u \leq t} |B(u)|$, where g_t denotes the last zero of B before t), which gives a natural way to order the countable many Brownian excursions. In the literature, there is an another well-studied way to order excursions, namely by considering the ranked excursion lengths, see e.g. Csáki et al. [6] for random walk and Brownian excursion lengths, and some recent papers: Pitman and Yor [18,19] combined with their references for studies of laws and many further developments, Révész [23], Hu and Shi [14,15] and the references therein for the behaviors of longest and shortest excursion lengths. For the ranked excursion heights, the most recent references are Pitman and Yor [20,21] who have characterized the laws of ranked heights of excursions of a Brownian bridge, or more generally, of a bridge of a recurrent self-similar Markov process, considered up to some random times. Let us mention in particular Pitman and Yor [20] for the law of $(M_n, n \geq 1)$ taking at an independent exponential time (cf. Remark 2.2 below).

Here, we are interested in the almost sure asymptotic behaviors of $\{M_n(t), n \geq 1\}$ as $t \to \infty$. Our analysis relies on a distribution result which will be stated in Section 2 for all recurrent Bessel processes. First of all, let us observe that $M_1(t) = \sup_{0 \leq s \leq t} |B(s)|$; hence the classical Erdős-Feller-Kolmogorov-Petrowski (EFKP) and Chung's integral tests for the Brownian motion can be directly applied to $M_1(t)$. Consequently, their respective laws of the iterated logarithm (LIL) read as follows:

$$\limsup_{t \to \infty} \frac{M_1(t)}{\sqrt{2t \log \log t}} = 1, \text{ a.s.} \tag{1.2}$$

$$\liminf_{t \to \infty} \sqrt{\frac{\log \log t}{t}}\, M_1(t) = \frac{\pi}{\sqrt{8}}, \text{ a.s.} \tag{1.3}$$

(see Csörgő and Révész [8], Révész [22] for detailed accounts). Let us consider $M_n(t)$ for $n \geq 2$. Obviously, when $M_n(t)$ is very big, then $M_1(t), ..., M_{n-1}(t)$ should also be very big. This simple fact prevents $M_n(t)$ from reaching the same bound as $M_1(t)$. On the other hand, for $n \geq 2$, $M_n(t)$ could reach much smaller values than $M_1(t)$. The following result confirms this intuitive idea:

Theorem 1.1. *Let $f > 0$ be a nondecreasing function. For fixed $n \geq 2$, we have*

$$\mathbb{P}\Big(M_n(t) > \sqrt{t}\, f(t), \text{ i.o.}\Big) = \begin{cases} 0 \\ 1 \end{cases} \Longleftrightarrow \int^{\infty} \frac{dt f(t)}{t} \exp\Big(- \frac{(2n-1)^2 f^2(t)}{2}\Big) \begin{cases} < \infty \\ = \infty \end{cases}, \quad (1.4)$$

$$\mathbb{P}\Big(M_n(t) < \frac{\sqrt{t}}{f(t)}, \text{ i.o.}\Big) = \begin{cases} 0 \\ 1 \end{cases} \Longleftrightarrow \int^{\infty} \frac{dt}{t\, f(t)} \begin{cases} < \infty \\ = \infty \end{cases}, \quad (1.5)$$

where, here and in the sequel, "i.o." means "infinitely often" as the relevant index goes to infinity. Consequently, we have

$$\limsup_{t\to\infty} \frac{M_n(t)}{\sqrt{2t \log\log t}} = \frac{1}{2n-1}, \qquad \text{a.s.} \quad (1.6)$$

$$\liminf_{t\to\infty} \frac{(\log t)^a}{\sqrt{t}} M_n(t) = \begin{cases} 0 \\ \infty \end{cases} \Longleftrightarrow \begin{cases} a \leq 1 \\ a > 1 \end{cases}, \qquad \text{a.s.} \quad (1.7)$$

In the case of $n = 1$, (1.4) is just the famous EFKP's test. Now, we turn to the problem of low excursion heights, i.e. of the asymptotic behaviors of $M_{[n(t)]}(t)$ as $t \to \infty$, with $n(t)$ depending on t and $[n(t)]$ meaning the integer part of $n(t)$. We have

Theorem 1.2. *Let $n(t) \uparrow \infty$ be a nondecreasing function such that $n(t)/\sqrt{t \log\log t}$ is nonincreasing. Assume that $\lim_{t\to\infty} n(t)/\log\log t = c$ with $c \in [0, \infty]$. We have*

$$\limsup_{t\to\infty} \frac{n(t)\, M_{[n(t)]}(t)}{\sqrt{t}\, \log\log t} = r_c, \text{ a.s.} \quad (1.8)$$

where $r_0 = 1/\sqrt{2}$, $r_\infty = \sqrt{2}$, and for $0 < c < \infty$ the constant r_c is the unique $r > 0$ such that $\mu(c\, r^{-2}) = r^{-2}$, where $\mu(x) = x(\log\cosh\lambda_0 + \lambda_0 - \lambda_0^2 x/2)$, and λ_0 is the unique positive solution $\lambda = \lambda_0(x)$ of $\lambda x = 1 + \tanh\lambda$.

The rest of this paper is organized as follows: in Section 2 we give the distributions of the ranked heights of excursions of recurrent Bessel processes taken at some random times, which consist of the core of the proofs of Theorem 1.1 and 1.2. In Section 3, we prove Theorem 1.1, whereas Section 4 is devoted to the study of low excursions, and to the proof of Theorem 1.2.

Before closing this introduction, we would like to point out that Theorems 1.1 and 1.2 admit their natural generalizations to excursions of all the recurrent Bessel processes by using Corollary 2.1 and Lemma 2.2 below, and to excursions of a simple random walk on $\mathbb{Z}$ by using the Skorokhod embedding. Let us also mention the difference of the lower functions for $M_1(t)$ and for $M_2(t)$ (cf. (1.3) and (1.5)), one way to get better understanding of this difference is to consider the joint lower functions of $(M_1(t), M_2(t))$. Some further studies of this kind will be presented in [7]. We refer to [5] for the characterizations of the joint lower functions of $(\sup_{0\le s\le t} B(s), -\inf_{0\le s\le t} B(s))$ (and random walk case), where Chung-type and Hirsch-type tests are unified.

Throughout this paper, we use the notation $f(x) \sim g(x)$ as $x \to x_0 \in [0, \infty]$ (resp: $f(x) \asymp g(x)$ as $x \in I \subset \mathbb{R}_+$) meaning $\lim_{x\to x_0} f(x)/g(x) = 1$ (resp: $0 < C_1 \le f(x)/g(x) \le C_2 < \infty$, for all $x \in I$), where, here and in the sequel, $C_1, C_2, ..., C_{13}$ denote some universal positive constants.

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2. DISTRIBUTION OF BESSEL EXCURSION HEIGHTS

In this Section, we consider a recurrent Bessel process $\{R(t), t \ge 0\}$ starting from 0, of dimension $0 < \mathrm{d} < 2$. See Revuz and Yor [24, Chap. XI] for detailed accounts on Bessel processes. We only mention that in the case of $\mathrm{d} = 1$, R is in fact a reflected Brownian motion. Denote by

$$M_1^{(\mathrm{d})}(t) \ge M_2^{(\mathrm{d})}(t) \ge ... \ge M_n^{(\mathrm{d})}(t) \ge ... \tag{2.1}$$

the sequence of ranked excursion heights of R over $[0, t]$ (including the meander height $\sup_{g_R(t)\le s\le t} R(s)$ with $g_R(t)$ the last zero of R before t). Let

$$H_n^{(\mathrm{d})}(r) \stackrel{\text{def}}{=} \inf\{t > 0 : M_n^{(\mathrm{d})}(t) > r\}, \qquad r > 0. \tag{2.2}$$

We aim at characterizing the law of the sequence $\{M_i^{(\mathrm{d})}(H_n^{(\mathrm{d})}(1)), i \ge 1; H_n^{(\mathrm{d})}(1)\}$. Note that by definition $M_n^{(\mathrm{d})}(H_n^{(\mathrm{d})}(1)) = 1$, a.s. Write $\nu \equiv (2 - \mathrm{d})/2 \in (0, 1)$ for the sake of notational simplification.

Proposition 2.1. *The two sequences* $\{M_i^{(\mathrm{d})}(H_n^{(\mathrm{d})}(1)), i \le n\}$ *and* $\{M_j^{(\mathrm{d})}(H_n^{(\mathrm{d})}(1)), j \ge n + 1\}$ *are independent. Furthermore, we have*

(i) *For* $n \geq 2$,

$$\left\{0 \leq \frac{1}{M_1^{(\mathrm{d})}(H_n^{(\mathrm{d})}(1))} \leq \ldots \leq \frac{1}{M_{n-1}^{(\mathrm{d})}(H_n^{(\mathrm{d})}(1))} \leq 1\right\} \stackrel{\mathrm{law}}{=} \{0 \leq \xi_1 \leq \ldots \leq \xi_{n-1} \leq 1\}, \tag{2.3}$$

where $\{\xi_1, \ldots, \xi_{n-1}\}$ *is a rearranged nondecreasing sequence of* $n-1$ *i.i.d. variables taking values in* $[0,1]$ *with common distribution* $2\nu x^{2\nu-1} \mathbb{1}_{(0 \leq x \leq 1)} dx$;

(ii) *The law of* $\{M_j^{(\mathrm{d})}(H_n^{(\mathrm{d})}(1)), j \geq n+1\}$ *is characterized as follows: for every measurable function* $f \geq 0$, *we have*

$$\mathbb{E} \exp\Big(-\sum_{j \geq n+1} f(M_j^{(\mathrm{d})}(H_n^{(\mathrm{d})}(1)))\Big) = \Big(1 + 2\nu \int_0^1 \frac{dx}{x^{2\nu+1}} (1 - e^{-f(x)})\Big)^{-n}; \tag{2.4}$$

(iii) *The law of* $H_n^{(\mathrm{d})}(1)$ *conditioned on* $\{M_i^{(\mathrm{d})}(H_n^{(\mathrm{d})}(1)), i \geq 1\}$ *is determined as follows: for every* $\lambda > 0$, *we have*

$$\mathbb{E}\Big(e^{-\frac{\lambda^2}{2} H_n^{(\mathrm{d})}(1)} \,\Big|\, \{M_i^{(\mathrm{d})}(H_n^{(\mathrm{d})}(1)) = x_i > 0, i \neq n\}\Big) = \ell_\nu(\lambda) \prod_{i \neq n} \ell_\nu^2(\lambda x_i), \tag{2.5}$$

where here and in the sequel, $\ell_\mu(x) \stackrel{\mathrm{def}}{=} \frac{x^\mu}{2^\mu \Gamma(1+\mu)\, I_\mu(x)}$ *for all* $\mu > -1$ *and* $x > 0$, *with* I_μ *the modified Bessel function with index* μ.

Remark 2.1. By BES_r^μ we mean a Bessel process of dimension $2(1+\mu)$, starting from $r \geq 0$ (hence $R \stackrel{\mathrm{law}}{=} BES_0^{-\nu}$). Write $T_{r \to a}^{(\mu)}$ its first hitting time at $a \geq 0$ (if it is finite) by the BES_r^μ. According to Kent [16], we have

$$\mathbb{E} \exp\Big(-\frac{x^2}{2} T_{0\to 1}^{(\mu)}\Big) = \ell_\mu(x), \qquad x > 0,\ \mu > -1, \tag{2.6}$$

$$\mathbb{E} \exp\Big(-\frac{x^2}{2} T_{1\to 0}^{(\mu)}\Big) = \frac{2^{1+\mu}}{\Gamma(-\mu)} x^{-\mu} K_\mu(x), \qquad x > 0,\ \mu < 0, \tag{2.7}$$

where K_μ denotes the modified Bessel function with index μ (cf. [1] for the modified Bessel functions I_μ and K_μ). Consequently, (2.5) is in fact a decomposition of $H_n^{(\mathrm{d})}(1)$ conditioning on $\{M_i^{(\mathrm{d})}(H_n^{(\mathrm{d})}(1)), i \geq 1\}$, as the sum of independent hitting times related to BES_0^ν.

Proof of Proposition 2.1. The proof relies on Itô's excursion theory for the recurrent Bessel process R, cf. also Bertoin [2]. Here, we begin by taking one choice of local times

$\{L(t, x), t \geq 0, x \geq 0\}$ of the Bessel process R, determined by the following density formula: for every measurable function $f : \mathbb{R}_+ \to \mathbb{R}_+$, we have

$$\int_0^t f(R(s))\, ds = \frac{1}{\nu} \int_{\mathbb{R}_+} f(x) L(t, x) x^{1-2\nu}\, dx,$$

(notice that in the case d $= 1$, $\nu = 1/2$, $L(t, 0)$ is equal to the local time at 0 of a Brownian motion, which is the half of that of a reflected Brownian motion).

Denote by $(\mathbf{e}(t), t \geq 0)$ the excursion process of R associated with $\{L(t, x), t \geq 0, x \geq 0\}$, and $\mathbf{n}$ the Itô measure. Denote by $h(\epsilon) \equiv \sup\{\epsilon(u), u \geq 0\}$, the height for a generic excursion ϵ and $V(\epsilon) \equiv \inf\{t > 0 : \epsilon(t) = 0\}$ its lifetime. According to Biane and Yor [3, pp. 43–45, formula (3h)] together with our choice of local times, we have ($\mathbf{n}$ equals in fact their $\widehat{n}_\nu$, notice that in formula (3h) the constant 4 should be 2)

$$\mathbf{n}\Big(h \geq x\Big) = x^{-2\nu}, \qquad x > 0. \tag{2.8}$$

Let $\alpha_n \stackrel{\text{def}}{=} L(H_n^{(\mathrm{d})}(1), 0)$ be the local time at 0 up to time $H_n^{(\mathrm{d})}(1)$. Observe that in the excursion time scale, α_n is the first time when there are exactly n excursions whose heights are larger than 1. Remark that $\{M_1^{(\mathrm{d})}(H_n^{(\mathrm{d})}(1)) \geq ... \geq M_{n-1}^{(\mathrm{d})}(H_n^{(\mathrm{d})}(1)) \geq 1\}$ is the rearranged nonincreasing sequence of $\{h(\mathbf{e}(\alpha_1)), ..., h(\mathbf{e}(\alpha_{n-1}))\}$, and the latter $n-1$ variables are measurable with respect to the excursion process $\{\mathbf{e}(t)\, \mathbb{1}_{(h(\mathbf{e}(t))\geq 1)}, t \geq 0\}$, which is independent of $\{\mathbf{e}(t)\, \mathbb{1}_{(h(\mathbf{e}(t))<1)}, t \geq 0\}$. Hence the two sequences $\{M_i^{(\mathrm{d})}(H_n^{(\mathrm{d})}(1)), i \leq n-1\}$ and $\{M_j^{(\mathrm{d})}(H_n^{(\mathrm{d})}(1)), j \geq n+1\}$ are independent, since the latter is measurable with respect to $\{\mathbf{e}(t)\, \mathbb{1}_{(h(\mathbf{e}(t))<1)}, t \geq 0\}$ (here, we suppose that $n \geq 2$, for there is nothing to do for showing the independence if $n = 1$).

Using successively the strong Markov property at the stopping times $\alpha_{n-1}, ..., \alpha_1$ for the excursion process $\mathbf{e}$, we see that $\{h(\mathbf{e}(\alpha_1)), ..., h(\mathbf{e}(\alpha_{n-1}))\}$ are $n-1$ i.i.d. variables, with common law: For $x > 1$,

$$\begin{aligned}\mathbb{P}\Big(h(\mathbf{e}(\alpha_1)) < x\Big) &= \mathbb{P}\Big(\inf\{t > 0 : h(\mathbf{e}(t)) \in [1, x)\} < \inf\{t > 0 : h(\mathbf{e}(t)) \in [x, \infty)\}\Big) \\ &= \int_0^\infty ds (1 - x^{-2\nu}) e^{-s\,(1-x^{-2\nu})}\, e^{-s\, x^{-2\nu}} \\ &= 1 - x^{-2\nu}, \qquad x > 1, \end{aligned} \tag{2.9}$$

since by using (2.8), the two first entrance time variables in the RHS of the first equality are independent, both exponentially distributed with respective parameters $1 - x^{-2\nu}$ and $x^{-2\nu}$. Hence (2.3) follows.

To get the law of $\{M_i^{(\mathrm{d})}(H_n^{(\mathrm{d})}(1)), i \geq n+1\}$, consider $\{(\mathbf{e}(s), \alpha_{j-1} < s < \alpha_j)\}_{1 \leq j \leq n}$ (with convention $\alpha_0 \equiv 0$). Again the strong Markov property says that these n excursion processes are independent and identically distributed, with common law that of the excursion process $(\widehat{\mathbf{e}}(s), 0 \leq s < \alpha_1)$, where $\widehat{\mathbf{e}}$ is defined as the restriction of $\mathbf{e}$ on $\mathcal{A} \stackrel{\text{def}}{=} \{\epsilon : h(\epsilon) < 1\}$. Notice that $\widehat{\mathbf{e}}$ has characteristic measure $1\!\!1_{\mathcal{A}}\mathbf{n}$ and is independent of α_1, whereas α_1 is exponentially distributed with parameter $\mathbf{n}(\mathcal{A}^c) = 1$. It turns out from (2.8) that

$$\begin{aligned}
\mathbb{E}\exp\Big(-\sum_{i\geq n+1} f(M_i^{(\mathrm{d})}(H_n^{(\mathrm{d})}(1)))\Big) &= \mathbb{E}\exp\Big(-\sum_{j=1}^{n}\sum_{\alpha_{j-1}<s<\alpha_j} f(h(\mathbf{e}(s)))\Big) \\
&= \Big(\mathbb{E}\exp\Big(-\sum_{s<\alpha_1} f(h(\widehat{\mathbf{e}}(s)))\Big)\Big)^n \\
&= \Big(\int_0^\infty dt\, e^{-t}\,\mathbb{E}\Big[\exp\Big(-\sum_{s<t} f(h(\widehat{\mathbf{e}}(s)))\Big)\Big]\Big)^n \\
&= \Big(\int_0^\infty dt e^{-t}\exp\Big(-t\int_{\mathcal{A}} \mathbf{n}(d\epsilon)(1-e^{-f(h(\epsilon))})\Big)\Big)^n \\
&= \Big(1+2\nu\int_0^1 \frac{dx}{x^{2\nu+1}}(1-e^{-f(x)})\Big)^{-n}, \qquad (2.10)
\end{aligned}$$

yielding (2.4). It remains to show (2.5). By using Williams' [26] path–decomposition of R at $g_R(t)$, with $g_R(t) \stackrel{\text{def}}{=} \sup\{s \leq t : R(s) = 0\}$ being the last zero of R before t, we have that $H_n^{(\mathrm{d})}(1) - g_R(H_n^{(\mathrm{d})}(1))$ is independent of $\sigma\{R(s \wedge g_R(H_n^{(\mathrm{d})}(1))), s \geq 0\}$, and $H_n^{(\mathrm{d})}(1) - g_R(H_n^{(\mathrm{d})}(1)) \stackrel{\text{law}}{=} T_{0\to 1}^{(\nu)}$ (in fact, Williams' original path–decomposition deals with Brownian motion, but the corresponding version for recurrent Bessel processes follows from a time-change argument, cf. Biane and Yor [3, Lemma 3.1]). Now, remark that

$$g_R(H_n^{(\mathrm{d})}(1)) = \sum_{s<\alpha_n; h(\mathbf{e}(s))>0} V(\mathbf{e}(s)) = \sum_{s<\alpha_n; h(\mathbf{e}(s))>0} \Big(\frac{V(\mathbf{e}(s))}{h(\mathbf{e}(s))^2}\Big)\, h(\mathbf{e}(s))^2,$$

$$\{h(\mathbf{e}(s)) : h(\mathbf{e}(s)) > 0, s < \alpha_n\} = \{M_i^{(\mathrm{d})}(H_n^{(\mathrm{d})}(1)) : i \neq n\}.$$

According to Biane and Yor [3, pp.43], under the Itô measure $\mathbf{n}$ and conditioning on $\{h(\mathbf{e}(s)) : h(\mathbf{e}(s)) > 0, s < \alpha_n\}$, the variables $\{V(\mathbf{e}(s))/h(\mathbf{e}(s))^2 : h(\mathbf{e}(s)) > 0, s < \alpha_n\}$ are i.i.d., with common law as the sum of two independent copies of $T_{0\to 1}^{(\nu)}$. Hence, (2.5) follows from (2.6). □

The following formula shows a decomposition of the law of $H_n^{(\mathrm{d})}(1)$ conditioning on $\{M_i^{(\mathrm{d})}(H_n^{(\mathrm{d})}(1)), 1 \leq i \leq n-1\}$, as a sum of independent Bessel hitting times, and of i.i.d. variables whose Laplace transform corresponds to $\ell_{-\nu}(\lambda)/\ell_\nu(\lambda)$. See also Pitman and Yor [20] for some closely related formulas in Brownian motion case.

Corollary 2.1. *Recall* $\nu \overset{\text{def}}{=} (2-\mathrm{d})/2 \in (0,1)$ *and Remark 2.1. Let* $n \geq 2$. *For* $\lambda > 0$ *and* $x_1 \geq ... \geq x_{n-1} \geq 1$, *we have*

$$\mathbb{E}\Big[e^{-\frac{\lambda^2}{2}H_n^{(\mathrm{d})}(1)} \,\Big|\, \big(M_i^{(\mathrm{d})}(H_n^{(\mathrm{d})}(1)) = x_i\big)_{1\leq i\leq n-1}\Big] = \ell_\nu(\lambda)\Big[\frac{\ell_{-\nu}(\lambda)}{\ell_\nu(\lambda)}\Big]^n \prod_{i=1}^{n-1}\ell_\nu^2(\lambda x_i), \quad (2.11)$$

$$H_n^{(\mathrm{d})}(1) \overset{\text{law}}{=} \sum_{i=1}^{n}\sigma_i^{(\mathrm{d})} + \sum_{j=1}^{n-1}\xi_j^{(\mathrm{d})}, \quad (2.12)$$

where all the variables $\{\sigma_i^{(\mathrm{d})}, 1 \leq i \leq n; \xi_j^{(\mathrm{d})}, 1 \leq j \leq n-1\}$ *are independent, and* $\sigma_i^{(\mathrm{d})} \overset{\text{law}}{=} T_{0\to 1}^{(-\nu)}$ *for* $1 \leq i \leq n$, $\xi_j^{(\mathrm{d})} \overset{\text{law}}{=} T_{1\to 0}^{(-\nu)}$ *for* $1 \leq j \leq n-1$.

Proof of Corollary 2.1. By applying the strong Markov property of R successively at the stopping times $H_1^{(\mathrm{d})}(1), ..., H_{n-1}^{(\mathrm{d})}(1)$, (2.12) follows. We also point out that in view of (2.6)–(2.7), (2.12) is in agreement with a direct computation based on (2.11) and (2.3).

To show (2.11), using (2.5) and (2.4) to write the LHS of (2.11) as

$$\begin{aligned}&\ell_\nu(\lambda)\prod_{i=1}^{n-1}\ell_\nu^2(\lambda x_i)\,\mathbb{E}\Big[\prod_{j\geq n+1}\ell_\nu^2(\lambda M_j^{(\mathrm{d})}(H_n^{(\mathrm{d})}(1)))\Big]\\ &= \ell_\nu(\lambda)\prod_{i=1}^{n-1}\ell_\nu^2(\lambda x_i)\Big[1 + 2\nu\int_0^1 \frac{dx}{x^{1+2\nu}}\big(1-\ell_\nu^2(\lambda x)\big)\Big]^{-n},\end{aligned}$$

yielding (2.11) by using the fact that $\frac{d}{dx}\big(\frac{x^{-2\nu}\,\ell_\nu(x)}{\ell_{-\nu}(x)}\big) = -2\nu x^{-1-2\nu}\ell_\nu^2(x)$, which can be obtained from the fact that $(\frac{I_{-\nu}}{I_\nu})'(x) = -\frac{2\sin(\nu\pi)}{\pi}\frac{1}{xI_\nu^2(x)}$ (cf. Abramovitz and Stegun [1, pp. 375]). □

Remark 2.2. Taking $\mathrm{d} = 1$ (i.e. $\nu = 1/2$) in (2.12) (and in (2.6)–(2.7)), we recover Pitman and Yor [20]'s formula:

$$\mathbb{E}\exp\Big(-\frac{\lambda^2}{2}H_n^{(1)}(1)\Big) = \Big[\cosh\lambda\Big]^{-n} e^{-\lambda(n-1)}, \qquad \lambda > 0. \quad (2.13)$$

Let us end this section with two preliminary results:

Lemma 2.1. *Let* $Y_1, \ldots, Y_k$ *be independent random variables and assume that for* $i = 1, \ldots, k$, *there exist some constants* $\beta_i > 0$ *and* $\alpha_i \in \mathbb{R}$ *such that*

$$\mathbb{P}\Big(Y_i < \epsilon\Big) \asymp \epsilon^{\alpha_i}\exp\Big(-\frac{\beta_i}{\epsilon}\Big), \qquad \epsilon \to 0. \quad (2.14)$$

Then we have

$$\mathbb{P}\Big(Y_1+\ldots+Y_k<\epsilon\Big) \asymp \epsilon^{\alpha_1+\ldots+\alpha_k-(k-1)/2}\exp\Big(-\frac{(\sqrt{\beta_1}+\ldots\sqrt{\beta_k})^2}{\epsilon}\Big), \qquad \epsilon\to 0. \quad (2.15)$$

Proof of Lemma 2.1. Follows from elementary computations by using integration by parts and Laplace method. □

Lemma 2.2. *Recall* $0<\nu=(2-\mathrm{d})/2<1$. *We have (n being fixed)*

$$\mathbb{P}\Big(H_n^{(\mathrm{d})}(1)<\epsilon\Big) \asymp \epsilon^{\nu}\exp\Big(-\frac{(2n-1)^2}{2\epsilon}\Big), \qquad \epsilon\to 0,\ n\geq 1, \quad (2.16)$$

$$\mathbb{P}\Big(H_n^{(\mathrm{d})}(1)>x\Big) \sim \frac{n-1}{2^{\nu}\Gamma(1+\nu)}\,x^{-\nu}, \qquad x\to\infty,\ n\geq 2. \quad (2.17)$$

Proof of Lemma 2.2. The proof is based on (2.12). First, from (2.7), the density of $\xi_1^{(\mathrm{d})}\overset{\text{law}}{=}T_{1\to 0}^{(-\nu)}$ can be obtained by inverting the Laplace transform; here we adopt an argument of Bessel time reversal (cf. [27]), which implies that ξ_1 has the same law as the last exit time at 1 by a transient Bessel process BES_0^{ν} (of dimension $4-\mathrm{d}$). Hence, it follows from Getoor [12] that

$$\mathbb{P}\Big(\xi_1^{(\mathrm{d})}\in dt\Big)/dt=\frac{1}{2^{\nu}\Gamma(\nu)}\,t^{-(1+\nu)}\exp\Big(-\frac{1}{2t}\Big), \qquad t>0. \quad (2.18)$$

The tail of $\sigma_1^{(\mathrm{d})}\overset{\text{law}}{=}T_{0\to 1}^{(-\nu)}$ is given by Gruet and Shi [13], more generally, for all $\mu>-1$, they have obtained that:

$$\mathbb{P}\Big(T_{0\to 1}^{(\mu)}<\epsilon\Big) \sim \frac{2^{1-\mu}}{\Gamma(1+\mu)}\,\epsilon^{-\mu}\exp\Big(-\frac{1}{2\epsilon}\Big), \qquad \epsilon\to 0, \qquad \mu>-1. \quad (2.19)$$

From (2.18), we have that

$$\mathbb{P}\Big(\xi_1^{(\mathrm{d})}<\epsilon\Big)\sim\frac{2^{1-\nu}}{\Gamma(\nu)}\,\epsilon^{1-\nu}e^{-1/(2\epsilon)}, \qquad \epsilon\to 0,$$

which in view of (2.19) with $\mu=-\nu$, yields (2.16) by applying Lemma 2.1 to (2.12).

Now, applying (2.6) and (2.7) to (2.12) for the Laplace transform of $H_n^{(\mathrm{d})}(1)$, we deduce that for $n\geq 2$,

$$\int_0^{\infty} dte^{-\lambda t}\,\mathbb{P}\Big(H_n^{(\mathrm{d})}(1)>t\Big)=\frac{1}{\lambda}\Big(1-\mathbb{E}e^{-\lambda H_n^{(\mathrm{d})}(1)}\Big)\sim(n-1)\,2^{-\nu}\,\lambda^{-(1+\nu)}, \qquad \lambda\to 0,$$

which, according to a Tauberian theorem (cf. Feller [11, pp. 423]) yields (2.17). $\square$

3. PROOF OF THEOREM 1.1

From now on, let us go back to the reflected Brownian motion $|B|$. Recall (1.1) and define for $n \geq 1$,

$$H_n(r) \stackrel{\text{def}}{=} \inf\{t > 0 : M_n(t) > r\}, \qquad r \geq 0. \tag{3.1}$$

Let us fix $n \geq 2$ in this section. Taking $\nu = 1/2$ in Lemma 2.2 and using self-similarity yields that for a fixed $t > 0$, we have as $\lambda \to \infty$

$$\mathbb{P}\Big(M_n(t) \geq \lambda\sqrt{t}\Big) = \mathbb{P}\Big(H_n(1) < \lambda^{-2}\Big) \asymp \lambda^{-1} \exp\Big(-\frac{(2n-1)^2\lambda^2}{2}\Big), \tag{3.2}$$

$$\mathbb{P}\Big(M_n(t) \leq \frac{\sqrt{t}}{\lambda}\Big) = \mathbb{P}\Big(H_n(1) > \lambda^2\Big) \sim (n-1)\sqrt{\frac{2}{\pi}}\,\lambda^{-1}. \tag{3.3}$$

From (3.2), (3.3) and the monotonicity of $M_n(t)$, it is a routine to prove the convergent parts of the tests in (1.4) and (1.5), see e.g. Erdős [10]. We omit here the details and only prove the divergent parts.

Proof of Theorem 1.1: the divergent part of (1.4). Let f be a nondecreasing function such that the integral in (1.4) diverges. Without any loss of generality, we can limit our attention to the "critical" case:

$$\frac{1}{2n-1}\sqrt{\log\log t} \leq f(t) \leq \frac{2}{2n-1}\sqrt{\log\log t}, \qquad t \geq t_0, \tag{3.4}$$

for some constant $t_0 > 0$ (see e.g. Erdős [10] for a rigorous justification). Let $i \geq i_0$, where here and in the sequel, i_0 denotes some fixed but sufficiently large constant whose value may change from one line to the next. Define

$$r_i \stackrel{\text{def}}{=} \exp(i/\log i), \qquad t_i = r_i^2/\widehat{f}^2(r_i), \qquad \widehat{f}(t) \stackrel{\text{def}}{=} f(t^3), \qquad t \geq t_0.$$

Notice that $H_n(r_i) < t_i$ implies that $M_n(t_i) \geq r_i = \sqrt{t_i}\widehat{f}(r_i) = \sqrt{t_i}f(r_i^3) > \sqrt{t_i}f(t_i)$ by (3.4). Therefore, if we would have proven that

$$\mathbb{P}\Big(H_n(r_i) < t_i, \text{ i.o.}\Big) > 0, \tag{3.5}$$

we would obtain that $\mathbb{P}\Big(M_n(t) > \sqrt{t}f(t), \text{ i.o.}\Big) > 0$, which, from Kolmogorov's 0–1 law, equals in fact 1 and ends the proof.

It remains to show (3.5). The idea consists of working with "favorite" events which are given as follows:

$$E_i \stackrel{\text{def}}{=} \Big\{ H_n(r_i) < t_i \Big\} \cap \Big\{ M_1(H_n(r_i)) < r_{i+1} \Big\}, \qquad i \ge i_0. \tag{3.6}$$

It suffices to show $\mathbb{P}\Big(E_i, \text{ i.o.}\Big) > 0$. First, let us estimate $\mathbb{P}\Big(E_i\Big)$. Taking d $= 1$ in (2.11) gives the following first equality ($x_1 \ge \ldots \ge x_{n-1} > 1$)

$$\begin{aligned}
&\mathbb{E}\Big(e^{-\frac{\lambda^2}{2} H_n(1)} \Big| \{ M_i(H_n(1)) = x_i > 1, 1 \le i \le n-1 \} \Big) \\
&= \frac{\lambda}{\sinh \lambda} \Big(\frac{\tanh \lambda}{\lambda} \Big)^n \prod_{i=1}^{n-1} \Big(\frac{\lambda x_i}{\sinh(\lambda x_i)} \Big)^2 \\
&= \mathbb{E} \exp\Big(- \frac{\lambda^2}{2} (\eta_0 + \zeta^{(n)} + \sum_{i=1}^{n-1} x_i (\eta_i + \tilde{\eta}_i)) \Big),
\end{aligned} \tag{3.7}$$

where all the variables $\{\eta_0, \zeta^{(n)}, \eta_i, \tilde{\eta}_i, 1 \le i \le n-1\}$ in (3.7) are positive and independent, with $\eta_0 \stackrel{\text{law}}{=} \eta_i \stackrel{\text{law}}{=} \tilde{\eta}_i$ and

$$\mathbb{E} \exp\Big(- \frac{\lambda^2}{2} \eta_0 \Big) = \frac{\lambda}{\sinh \lambda}, \qquad \mathbb{E} \exp\Big(- \frac{\lambda^2}{2} \zeta^{(n)} \Big) = \Big(\frac{\tanh \lambda}{\lambda} \Big)^n, \qquad \lambda > 0.$$

Here we used the well-known fact that both $\lambda/(\sinh \lambda)$ and $(\tanh \lambda)/\lambda$ are Laplace transforms, at $\lambda^2/2$, of some positive random variables. From (3.7), we see that conditioning on $\big\{ M_i(H_n(1)) = x_i > 1, 1 \le i \le n-1 \big\}$, $H_n(1)$ is stochastically smaller than $\eta_0 + \zeta^{(n)} + x_1 \sum_{i=1}^{n-1} (\eta_i + \tilde{\eta}_i)$. It turns out that for $x > 1$

$$\begin{aligned}
\mathbb{P}\Big(H_n(r_i) < t_i \,\Big|\, M_1(H_n(r_i)) = x r_i \Big) &= \mathbb{P}\Big(H_n(1) < \frac{1}{\widehat{f}^2(r_i)} \,\Big|\, M_1(H_n(1)) = x \Big) \\
&\ge \mathbb{P}\Big(\eta_0 + \zeta^{(n)} + x \sum_{i=1}^{n-1} (\eta_i + \tilde{\eta}_i) < \frac{1}{\widehat{f}^2(r_i)} \Big) \\
&\ge \mathbb{P}\Big(\eta_0 + \zeta^{(n)} + \sum_{i=1}^{n-1} (\eta_i + \tilde{\eta}_i) < \frac{1}{x \widehat{f}^2(r_i)} \Big), \\
&= \mathbb{P}\Big(\Sigma^{(n)} + \sum_{i=1}^{n-1} \eta_i < \frac{1}{x \widehat{f}^2(r_i)} \Big),
\end{aligned} \tag{3.8}$$

where $\Sigma^{(n)} \stackrel{\text{def}}{=} \eta_0 + \zeta^{(n)} + \sum_{i=1}^{n-1} \tilde{\eta}_i$ is independent of $(\eta_i, 1 \le i \le n-1)$. We shall estimate the small deviation in (3.8). Observe that $\mathbb{E} e^{-\frac{\lambda^2}{2} \Sigma^{(n)}} = \cosh^{-n}(\lambda)$, it turns out that $\Sigma^{(n)} \stackrel{\text{law}}{=} \sum_{i=1}^n \sigma_i$, with σ_i being i.i.d. and $\sigma_i \stackrel{\text{law}}{=} T_{0 \to 1}^{(-1/2)}$ (recalling the notations in Remark

2.1), whereas $\eta_i \stackrel{\text{law}}{=} T^{(1/2)}_{0\to 1}$ has the same law of the first hitting time at 1 by a three-dimensional Bessel process starting from 0. By using (2.19) with $\mu = -1/2$ for the small deviation of σ_i and with $\mu = 1/2$ for that of η_i, we have

$$\mathbb{P}\Big(\sigma_i < \epsilon\Big) \asymp \epsilon^{1/2} \exp\Big(-\frac{1}{2\epsilon}\Big) \quad \text{and} \quad \mathbb{P}\Big(\eta_i < \epsilon\Big) \asymp \epsilon^{-1/2} \exp\Big(-\frac{1}{2\epsilon}\Big), \qquad \epsilon \to 0,$$

which in view of Lemma 2.1 yields that

$$\mathbb{P}\Big(\Sigma^{(n)} + \sum_{i=1}^{n-1} \eta_i < \epsilon\Big) \asymp \epsilon^{-n+3/2} \exp\Big(-\frac{(2n-1)^2}{2\epsilon}\Big), \qquad \epsilon \to 0. \tag{3.9}$$

Notice that $\widehat{f}(r_i) \asymp \sqrt{\log i}$, and $\frac{r_{i+1}}{r_i} - 1 \sim \frac{1}{\log i} \ge C_3 \widehat{f}^{-2}(r_i)$ for some universal constant $C_3 > 0$. The law of $M_1(H_n(1))$ follows from (2.3) with $\nu = 1/2$:

$$\mathbb{P}\Big(M_1(H_n(r_i)) < r_i x\Big) = \Big(1 - \frac{1}{x}\Big)^{n-1}, \qquad x \ge 1. \tag{3.10}$$

Combining (3.8)–(3.10) with $1 < x < 1 + C_3 \widehat{f}^{-2}(r_i)$, we obtain that

$$\begin{aligned}
\mathbb{P}\Big(E_i\Big) &\ge \mathbb{P}\Big(\sigma^{(n)} + \sum_{i=1}^{n-1} \eta_i < \big(1 + C_3 \widehat{f}^{-2}(r_i)\big)^{-1} \widehat{f}^{-2}(r_i)\Big)\, \mathbb{P}\Big(M_1(H_n(r_i)) < r_i\big(1 + C_3 \widehat{f}^{-2}(r_i)\big)\Big) \\
&\ge C_4 \Big(\widehat{f}(r_i)\Big)^{2n-3} \exp\Big(-\frac{(2n-1)^2 \widehat{f}^2(r_i)}{2}\Big) \Big(\widehat{f}^{-2}(r_i)\Big)^{n-1} \\
&= C_4 \widehat{f}^{-1}(r_i) \exp\Big(-\frac{(2n-1)^2 \widehat{f}^2(r_i)}{2}\Big),
\end{aligned}$$

for some constant $C_4 > 0$. In light of (3.2), we have shown

$$\mathbb{P}\Big(E_i\Big) \asymp \mathbb{P}\Big(H_n(r_i) < t_i\Big) \asymp \widehat{f}^{-1}(r_i) \exp\Big(-\frac{(2n-1)^2 \widehat{f}^2(r_i)}{2}\Big). \tag{3.11}$$

From the assumption of divergence of the integral in (1.4), it is elementary to see that

$$\sum_i \mathbb{P}\Big(E_i\Big) = \infty. \tag{3.12}$$

Let us estimate the second moment $\mathbb{P}\Big(E_i \cap E_j\Big)$. Define

$$D_i \stackrel{\text{def}}{=} \inf\{t > H_n(r_i) : B(t) = 0\},$$

the first return time to 0 of B (i.e. of $|B|$) after the stopping time $H_n(r_i)$. Let $\widehat{B}(t) \stackrel{\text{def}}{=} B(t+D_i), t \ge 0$; By the strong Markov property at the stopping time D_i, $\widehat{B}$ is a standard

Brownian motion starting from 0, and independent of $\mathcal{F}_{D_i}$, where $(\mathcal{F}_t, t \ge 0)$ denotes the natural σ-fields generated by B. Define the processes $\widehat{M}_n, \widehat{H}_n$ related to $\widehat{B}$ exactly in the same way as M_n, H_n do to B. Consider $j > i > i_0$ and $\omega \in E_i \cap E_j$, if additionally $M_1(D_i)(\omega) < r_j$, all excursions having heights larger than r_j live after the time $D_i(\omega)$, which implies that $\widehat{H}_n(r_j)(\omega) < t_j$. Therefore using (3.11) gives that

$$\begin{aligned}\mathbb{P}\Big(E_i \cap E_j \cap \big\{M_1(D_i) < r_j\big\}\Big) &\le \mathbb{P}\Big(E_i \cap \big\{\widehat{H}_n(r_j) < t_j\big\}\Big) \\ &= \mathbb{P}\Big(E_i\Big)\mathbb{P}\Big(\widehat{H}_n(r_j) < t_j\Big) \\ &\le C_5 \mathbb{P}\Big(E_i\Big)\mathbb{P}\Big(E_j\Big). \end{aligned} \tag{3.13}$$

Observe that $E_i \cap E_j \cap \{M_1(D_i) \ge r_j\} \subset E_i \cap \{r_{j+1} > M_1(D_i) \ge r_j\} \cap \{\widehat{H}_{n-1}(r_j) < t_j\}$. Furthermore $M_1(D_i)$ is independent of E_i, since $M_1(D_i)$ in fact is the maximum of $|B(H_n(r_i) + \cdot)|$ considered up to its first hitting time at 0, hence $\mathbb{P}\Big(M_1(D_i) \in [r_j, r_{j+1})\Big) = r_i/r_j - r_i/r_{j+1}$. It follows that

$$\begin{aligned}\mathbb{P}\Big(E_i \cap E_j \cap \big\{M_1(D_i) \ge r_j\big\}\Big) &\le \mathbb{P}\Big(E_i \cap \big\{r_j \le M_1(D_i) < r_{j+1}\big\} \cap \big\{\widehat{H}_{n-1}(r_j) < t_j\big\}\Big) \\ &= \mathbb{P}\Big(E_i\Big)\, \mathbb{P}\Big(r_j \le M_1(D_i) < r_{j+1}\Big)\, \mathbb{P}\Big(\widehat{H}_{n-1}(r_j) < t_j\Big) \\ &\le C_6 \mathbb{P}\Big(E_i\Big)\, \frac{r_i}{r_j}\, \widehat{f}(r_j)\, \exp\Big(- \frac{(2n-3)^2 \widehat{f}^2(r_j)}{2}\Big) \\ &\le C_7 \mathbb{P}\Big(E_i\Big)\, \exp\Big(- \frac{j-i}{\log j}\Big)\, j^{-C_8}, \end{aligned} \tag{3.14}$$

where we have used the facts that $r_i/r_j \le \exp\big(- (j-i)/\log j\big)$ and $\widehat{f}^2(r_j) \asymp \log j, j \ge i_0$. From (3.11) and (3.13)–(3.14), some elementary calculations show that

$$\liminf_{n\to\infty} \frac{\sum_{i_0 \le i,j \le n} \mathbb{P}\Big(E_i \cap E_j\Big)}{\Big(\sum_{i_0 \le i \le n} \mathbb{P}\Big(E_i\Big)\Big)^2} \le C_5,$$

which according to Kochen and Stone's version [17] of the Borel-Cantelli lemma, together with (3.12) yields that

$$\mathbb{P}\Big(E_i, \quad \text{i.o.}\Big) \ge 1/C_5 > 0,$$

resulting (3.5) and ending the proof of the divergent part of (1.4). $\square$

Proof of Theorem 1.1: the divergent part of (1.5). Since $H_n(r) \ge H_2(r)$, it suffices to prove this part for $n = 2$. Assume that the integral in (1.5) diverges and we work again with the "critical" case when

$$\sqrt{\log t} \le f(t) \le \log^2 t, \qquad t \ge t_0. \tag{3.15}$$

Let

$$\tilde{f}(t) \stackrel{\text{def}}{=} f(t^3), \qquad r_i \stackrel{\text{def}}{=} 2^i, \qquad t_i \stackrel{\text{def}}{=} r_i^2 \tilde{f}^2(r_i).$$

Our aim is to prove that

$$\mathbb{P}\Big(H_2(r_i) > t_i, \text{ i.o.}\Big) > 0, \tag{3.16}$$

which would imply that for these i.o. i such that $H_2(r_i) > t_i$, we have $M_2(t_i) < r_i = \sqrt{t_i}/\tilde{f}(r_i) \le \sqrt{t_i}/f(t_i)$, so that $\mathbb{P}\Big(M_2(t) < \sqrt{t}\, f(t), \text{ i.o.}\Big) > 0$, and the desired result follows from Kolmogorov's 0–1 law.

Now, let $d_{H_2(r)} \stackrel{\text{def}}{=} \inf\{t > H_2(r) : B(t) = 0\}$ be the first return time to 0 after $H_2(r)$. Define $m(r) \stackrel{\text{def}}{=} \sup\{|B(s)| : H_2(r) \le s \le d_{H_2(r)}\}$ the maximal height of the excursion straddling $H_2(r)$, $m(r)$ is independent of $H_2(r_i)$ by strong Markov property at $H_2(r_i)$. Consider the event

$$F_i \stackrel{\text{def}}{=} \Big\{t_{i+1} > H_2(r_i) > t_i\Big\} \cap \Big\{m(r_i) < r_{i+1}\Big\} \cap \Big\{d_{H_2(r_i)} < t_{i+2}\Big\}. \tag{3.17}$$

It follows from (3.3) and the self-similarity that

$$\begin{aligned} \mathbb{P}\Big(F_i\Big) &= \mathbb{P}\Big(2\tilde{f}^2(r_i) > H_2(1) > \tilde{f}^2(r_i)\Big)\, \mathbb{P}\Big(\big\{m(1) < 2\big\} \cap \big\{d_{H_2(1)} < 4\tilde{f}^2(r_i)\big\}\Big) \\ &\asymp \tilde{f}^{-1}(r_i), \end{aligned} \tag{3.18}$$

which, in light of the fact that $\int^\infty dt/\big(t\tilde{f}(t)\big) = \infty$, implies

$$\sum_i \mathbb{P}\Big(F_i\Big) = \infty. \tag{3.19}$$

The second moment $\mathbb{P}\Big(F_i \cap F_j\Big)$ is easy to estimate. Let $j \ge i+3$, and define $\widetilde{B}(t) \stackrel{\text{def}}{=} B(t + d_{H_2(r_j)}), t \ge 0$. B is a Brownian motion independent of F_i. Define $\widetilde{H}_1(r), \widetilde{H}_2(r)$ related to $\widetilde{B}$ in the same way as $H_1(r), H_2(r)$ do to B. Consider a path $\omega \in F_i \cap F_j$, we have $H_2(r_j)(\omega) - d_{H_2(r_i)}(\omega) > t_j - t_{i+2} \ge t_j/2$. There are only two possibilities. First consider the case $(M_1(H_2(r_i))(\omega) < r_j)$, meaning that the two highest excursions before $H_2(r_j)(\omega)$ are realized after the stopping time $d_{H_2(r_i)}(\omega)$, hence we have $H_2(r_j)(\omega) = \widetilde{H}_2(r_j)(\omega) + d_{H_2(r_i)}(\omega)$. Assembling all this and using (3.3) and (3.18) yield

$$\mathbb{P}\Big(F_i \cap F_j \cap \big\{M_1(H_2(r_i)) < r_j\big\}\Big) \le \mathbb{P}\Big(F_i \cap \big\{\widetilde{H}_2(r_j) > t_j/2\big\}\Big) \le C_9 \mathbb{P}\Big(F_i\Big)\mathbb{P}\Big(F_j\Big). \tag{3.20}$$

It remains to consider the case $(M_1(H_2(r_i))(\omega) \ge r_j)$, which means that there exists exactly one excursion living in the time interval $[d_{H_2(r_i)}, H_2(r_j)](\omega)$, whose height is larger than r_j; hence $H_2(r_j)(\omega) = \widetilde{H}_1(r_j)(\omega) + d_{H_2(r_i)}(\omega)$. It turns out that

$$\begin{aligned} \mathbb{P}\Big(F_i \cap F_j \cap \{M_1(H_2(r_i)) > r_j\}\Big) &\le \mathbb{P}\Big(F_i \cap \{\widetilde{H}_1(r_j) > t_j/2\}\Big) \\ &= \mathbb{P}\Big(F_i\Big)\, \mathbb{P}\Big(\widetilde{H}_1(1) > \frac{\tilde{f}^2(r_j)}{2}\Big) \\ &\le \mathbb{P}\Big(F_i\Big)\, \frac{2}{\tilde{f}^2(r_j)}, \end{aligned} \tag{3.21}$$

where the last inequality is due to Chebychev's inequality and to the fact that $\mathbb{E}\widetilde{H}_1(1) = 1$, since $\widetilde{H}_1(1)$ is the first hitting time at 1 by $|B|$ (in fact $\widetilde{H}_1(1)$ has a tail of exponential decay (cf. [4] for all Bessel hitting times), but here the rough estimate is sufficient for us). From (3.18), (3.20) and (3.21), it is elementary to show that

$$\liminf_{n\to\infty} \frac{\sum_{i_0 \le i,j \le n} \mathbb{P}\Big(F_i \cap F_j\Big)}{\Big(\sum_{i_0 \le i \le n} \mathbb{P}\Big(F_i\Big)\Big)^2} \le C_9,$$

which in view of (3.19) and a version of the Borel-Cantelli lemma (cf. [17]), yield that $\mathbb{P}\Big(F_i, \text{ i.o.}\Big) \ge 1/C_9 > 0$; hence we have proven (3.16) and finished the whole proof. □

4. LOW EXCURSIONS

In this Section, we shall study the asymptotic behavior of the height $M_{[n(t)]}(t)$ with $n(t) \uparrow \infty$ being a nondecreasing function. Our first preliminary result concerns the tail asymptotics:

Lemma 4.1. *Fix $r > 0$ and assume that $\lim_{t\to\infty} n(t)/(\log\log t) = c$ with $c \in [0, \infty]$. We have*

$$\log \mathbb{P}\Big(n(t) M_{[n(t)]}(1) > r\sqrt{\log\log t}\Big) \sim -r^2 \mu\left(\frac{c}{r^2}\right) \log\log t, \qquad t \to \infty, \tag{4.1}$$

where $\mu(0) = 2$, $\mu(\infty) = 1/2$ and for $0 < x < \infty$ $\mu(x)$ is defined in Theorem 1.2.

Proof of Lemma 4.1. Define $T(r) \overset{\text{def}}{=} \inf\{t > 0 : B(t) = r\}$ for $r \in \mathbb{R}$. Recall (3.1). It follows from (2.12) that

$$H_n(1) \overset{\text{law}}{=} \Sigma^{(n)} + T(n-1), \tag{4.2}$$

where $\Sigma^{(n)}$ is independent of $T(n-1)$, and $\mathbb{E}\exp\left(-\frac{\lambda^2}{2}\Sigma^{(n)}\right) = \cosh^{-n}(\lambda)$. It follows from the well-known Gaussian tail that

$$\mathbb{P}\Big(T(r) < \epsilon r^2\Big) = \mathbb{P}\Big(\sup_{0\le s\le 1} B(s) > \frac{1}{\sqrt{\epsilon}}\Big) \asymp \sqrt{\epsilon}\exp\left(-\frac{1}{2\epsilon}\right), \qquad \epsilon \to 0. \tag{4.3}$$

Let now $x = x(t) = n(t)/(r^2 \log\log t) \to c/r^2 \in [0, \infty]$, as $t \to \infty$. Then it is easy to see that (4.1) is equivalent to

$$\log \mathbb{P}\left(H_n(1) < x\,n\right) \sim -n\frac{\mu(x)}{x}, \qquad n \to \infty, \tag{4.4}$$

with three possible cases in (4.4), namely: $x \in (0, \infty)$ being fixed; $x \to 0$; and $x \to \infty$, as $n \to \infty$.

For fixed $0 < x < \infty$, since from (4.2), $H_n(1)$ is stochastically greater than $\Sigma^{(n-1)} + T(n-1)$, and smaller than $\Sigma^{(n)} + T(n)$, (4.4) follows from Cramér's theorem (cf. Dembo and Zeitouni [9, Theorem 2.2.3]). In the case when $x \to 0$ or $x \to \infty$ as $n \to \infty$, first estimate $\mathbb{P}\Big(H_n(1) < x\,n\Big)$ with $x > 0$ depending on n. The Laplace transform of $H_n(1)$ is given by (2.13). Using Chebychev's inequality

$$\mathbb{P}\Big(H_n(1) < xn\Big) \le e^{\frac{\lambda^2}{2}xn}\, \mathbb{E}e^{-\frac{\lambda^2}{2}H_n(1)} = e^{\frac{\lambda^2}{2}xn}\left(\cosh\lambda\right)^{-n} e^{-\lambda(n-1)}, \tag{4.5}$$

for all $\lambda > 0$. By putting $\lambda = \frac{2n-1}{xn}$ in (4.5) we get

$$\mathbb{P}\left(H_n(1) < xn\right) \le 2^n \exp\left(-\frac{(2n-1)^2}{2xn}\right), \tag{4.6}$$

giving an upper bound in (4.4) when $x \to 0$.

On the other hand, by taking $\lambda = (n-1)/(xn)$ in (4.5) we get

$$\mathbb{P}\left(H_n(1) < xn\right) \le \exp\left(-\frac{(n-1)^2}{2xn}\right) \tag{4.7}$$

showing an upper bound in the case $x \to \infty$.

To obtain a lower bound in the case when $x \to 0$ notice that $\Sigma^{(n)}$ is stochastically dominated by $T(n)$, which implies in view of (4.2) and (4.3) that

$$\begin{aligned}\mathbb{P}\Big(H_n(1) < xn\Big) &\ge \mathbb{P}\Big(\widehat{T}(n) + T(n-1) < xn\Big)\\ &= \mathbb{P}\Big(T(2n-1) < xn\Big)\\ &\ge \left(C_{10} \wedge \sqrt{\frac{x}{n}}\right) e^{-\frac{(2n-1)^2}{2xn}},\end{aligned} \tag{4.8}$$

where $C_{10} > 0$ is some universal constant, and $\widehat{T}(r)$ is an independent copy of $T(r)$. (4.6) and (4.8) together imply (4.4) in the case $x \to 0$.

Since $\Sigma^{(n)}$ can be written as the sum of n i.i.d. variables, of common law of $\Sigma^{(1)}$, with expectation 1, using the law of large numbers gives that for $x \gg 1$,

$$\begin{aligned}\mathbb{P}\Big(H_n(1) < xn\Big) &\geq \mathbb{P}\Big(\Sigma^{(n)} < 2n\Big)\,\mathbb{P}\Big(T(n-1) < (x-2)n\Big)\\ &\geq C_{11}\,\mathbb{P}\Big(T(n-1) < (x-2)n\Big)\\ &\geq C_{12}\sqrt{\frac{x}{n}}\,\exp\Big(-\frac{n}{2(x-2)}\Big),\end{aligned}$$

which, together with (4.7), implies (4.4) in the case $x \to \infty$. □

Proof of Theorem 1.2: First we show that $x \to \mu(x)$ is decreasing for $0 < x < \infty$. It is elementary to see that

$$\mu'(x) = -\lambda_0^2 x + \log\cosh\lambda_0 + \lambda_0 = -\lambda_0 \tanh\lambda_0 + \log\cosh\lambda_0 < 0, \tag{4.9}$$

for $\log\cosh\lambda < \lambda\tanh\lambda$ for all $\lambda > 0$, hence μ is decreasing and r_c is well defined in Theorem 1.2.

Now we prove the upper bound in (1.8) for fixed $0 < c < \infty$. Fix a small $\epsilon > 0$. Define $t_i = (1+\epsilon)^i$ for $i \geq 1$, then $\log\log t_i \sim \log i$. From (4.1) and using the monotonicity of μ, we have for all large i,

$$\begin{aligned}&\mathbb{P}\Big(n(t_i)\, M_{[n(t_i)]}(t_{i+1}) > (\epsilon+1) r_c \sqrt{t_i \log\log t_i}\Big)\\ &= \mathbb{P}\Big(n(t_i) M_{[n(t_i)]}(1) > r_c \sqrt{\epsilon+1}\,\sqrt{\log\log t_i}\Big)\\ &\leq \exp\Big(-(1+\frac{\epsilon}{2})\, r_c^2\, \mu\big(\frac{c}{(1+\epsilon) r_c^2}\big)\, \log\log t_i\Big)\\ &\leq \exp\Big(-(1+\frac{\epsilon}{2}) \log\log t_i\Big)\\ &\leq i^{-1-\frac{\epsilon}{3}},\end{aligned}$$

being summable, which according to the convergent part of Borel-Cantelli lemma, implies that almost surely for all large i, we have $M_{[n(t_i)]}(t_{i+1}) \leq (\epsilon+1) r_c \sqrt{t_i \log\log t_i}\,/n(t_i)$; hence for all large t, $t \in [t_i, t_{i+1})$, and by monotonicity $M_{[n(t)]}(t) \leq M_{[n(t_i)]}(t_{i+1}) \leq (\epsilon+1) r_c \sqrt{t_i \log\log t_i}\,/n(t_i) \leq r_c(\epsilon+1)\sqrt{t\log\log t}\,/n(t)$, i.e.

$$\limsup_{t\to\infty} \frac{n(t)}{\sqrt{t\log\log t}}\, M_{[n(t)]}(t) \leq (1+\epsilon) r_c, \qquad \text{a.s.,}$$

yielding the upper bound of (1.8) by letting $\epsilon \to 0$ along a countable sequence.

To obtain the lower bound of (1.8), fix a small $\delta > 0$ and let $t_i \stackrel{\text{def}}{=} i^i$. Consider the event

$$G_i \stackrel{\text{def}}{=} \Big\{ n(t_i) M_{[n(t_i)]}(t_i) > (1-\delta) r_c \sqrt{t_i \log\log t_i} \Big\}. \tag{4.10}$$

Write $(\mathcal{F}_t, t \geq 0)$ for the natural σ-fields generated by B. Obviously, G_i is $\mathcal{F}_{t_i}$ measurable. If we would have proven that

$$\sum_i \mathbb{P}\Big(G_i \,\big|\, \mathcal{F}_{t_{i-1}} \Big) = \infty, \qquad \text{a.s.}, \tag{4.11}$$

then, according to Lévy's version of Borel-Cantelli's lemma (cf. [25]), we would obtain that $\mathbb{P}\Big(G_i, \text{ i.o.}\Big) = 1$, which shows $\limsup_{t\to\infty} \frac{n(t)}{\sqrt{t \log\log t}} M_{[n(t)]}(t) \geq (1-\delta) r_c$, a.s., ending the proof of (1.8) by letting $\delta \to 0$.

To arrive at (4.11), define $d_{t_{i-1}} = \inf\{t > t_{i-1} : B(t) = 0\}$ the first return time to 0 of B after time t_{i-1}. Then $\widehat{B}(t) \stackrel{\text{def}}{=} B(t + d_{t_{i-1}}), t \geq 0$, is a Brownian motion independent of $\mathcal{F}_{d_{t_{i-1}}}$. Define $\widehat{M}_n(r)$ in terms of $\widehat{B}$ in the same way as $M_n(r)$ was defined in terms of B. Observe that conditioning on $|B(t_{i-1})| = r > 0$ and $\mathcal{F}_{t_{i-1}}$, $d_{t_{i-1}} - t_{i-1} \stackrel{\text{law}}{=} r^2 |\mathcal{N}|^{-2}$ with $\mathcal{N}$ a centered reduced Gaussian variable. Notice that $\frac{t_{i-1}}{t_i} \leq \frac{1}{i}$. It follows that for all large i, $\mathbb{P}\Big(\delta t_i |\mathcal{N}|^{-2} < \delta t_i - t_{i-1}\Big) \geq C_{13}$, and we have

$$\begin{aligned}
\mathbb{P}\Big(G_i \,\big|\, \mathcal{F}_{t_{i-1}} \Big) &\geq \mathbb{P}\Big(\Big\{ d_{t_{i-1}} < \delta t_i \Big\} \cap \Big\{ \widehat{M}_{[n(t_i)]}(t_i - d_{t_{i-1}}) > \frac{(1-\delta) r_c}{n(t_i)} \sqrt{t_i \log\log t_i} \Big\} \,\Big|\, \mathcal{F}_{t_{i-1}} \Big) \\
&\geq C_{13}\, \mathbb{1}_{(|B(t_{i-1})| \leq \sqrt{\delta t_i})}\, \mathbb{P}\Big(\widehat{M}_{[n(t_i)]}((1-\delta) t_i) > \frac{(1-\delta) r_c}{n(t_i)} \sqrt{t_i \log\log t_i} \Big) \\
&\geq C_{13}\, \mathbb{1}_{(|B(t_{i-1})| \leq \sqrt{\delta t_i})} \exp\Big(-(1 - \frac{\delta}{2})\, r_c^2 \mu\big(\frac{c}{(1-\delta) r_c^2}\big) \log\log t_i \Big) \\
&\geq C_{13}\, \mathbb{1}_{(|B(t_{i-1})| \leq \sqrt{\delta t_i})}\, i^{-1+\delta/3},
\end{aligned} \tag{4.12}$$

by virtue of (4.1) and monotonicity of μ. Using the classical LIL for $|B(t)|$ shows that almost surely for all but finite i, $|B(t_{i-1})| \leq \sqrt{3 t_{i-1} \log\log t_{i-1}} \leq \sqrt{\delta t_i}$, which, combining with (4.12) implies (4.11), proving (1.8) in the case $0 < c < \infty$.

The proof of (1.8) in other cases ($c = 0$ and $c = \infty$) is similar and therefore omitted. This completes the proof of Theorem 1.2. $\square$

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