Upper and Lower limits of doubly perturbed Brownian motion

by

Loïc Chaumont[1], Ronald A. Doney[2] and Yueyun Hu[1]

Summary. A doubly perturbed Brownian motion (DPBM) behaves as a Brownian motion between its minimum and maximum, and is perturbed at its extrema. We study here how these perturbations influence the asymptotic behaviours of the extrema by characterizing all upper and lower limits of DPBM for permissible perturbation parameters.

Keywords. Doubly perturbed Brownian motion, Upper and Lower limits, Passage time, Exit time, Law of iterated logarithm.

1991 Mathematics Subject Classification. 60F15, 60E07.

1. Introduction.

Let $(B_t, t \geq 0)$ be a real Brownian motion starting from 0. Fix $\alpha, \beta < 1$, and consider the doubly perturbed Brownian motion $(X_t, t \geq 0)$ (DPBM in short) defined as the (pathwise unique) solution of the following equation:

$$X_t = B_t + \alpha M_t - \beta I_t, \tag{1.1}$$

[1] Laboratoire de Probabilités et Modèles Aléatoires, CNRS UMR–7599, Université Paris VI, 4 Place Jussieu, F–75252 Paris cedex 05, France. E-mails: loc@ccr.jussieu.fr hu@ccr.jussieu.fr

[2] Department of Mathematics, University of Manchester, Oxford road, M13 9PL, Manchester, England. E-mail: rad@ma.man.ac.uk

with $X_0 = 0$, $M_t \stackrel{\text{def}}{=} \sup_{0\le s\le t} X_s$, and $I_t \stackrel{\text{def}}{=} \sup_{0\le s\le t}(-X_s)$. Le Gall [16] showed that the DPBM can be obtained as a limit process from a "weak" polymers model of Norris, Rogers and Williams [19]. A time changed version of (1.1), the so–called perturbed reflecting equation, appears also in the studies of the asymptotics of planar Brownian motion (cf. Le Gall and Yor [17]). The DPBM also arises as the scaling limit of some self-interacting random walks (see Tóth [25] and [26]). Recently, the equation (1.1) has attracted much interest from several directions: see e.g. Petit's thesis [21] and Yor [29] for motivations from Lévy's arc sine laws; Le Gall and Yor [17], Carmona et al. [2], Davis [9,10], Perman and Werner [20], Chaumont and Doney [4] for the existence and unicity of the solutions of (1.1); Carmona et al. [2,3], Werner [28], Chaumont and Doney [5], Doney [11] for related Ray-Knight theorems and calculations of laws; Shi and Werner [24] for the almost sure study of occupation time; Doney et al. [12] for the generalizations to perturbed Bessel processes. Let us mention that only in the case $\beta = 0$ (or similarly for $\alpha = 0$), has the process X an explicit form in term of Brownian functionals, i.e:

$$X_t = B_t + \frac{\alpha}{1-\alpha} \sup_{0\le s\le t} B_s, \qquad \text{for} \quad \beta = 0, \tag{1.2}$$

which, according to Lévy's identity in law for $\sup_{s\le t} B_s$ and the Brownian local time L_t at 0, is equivalent to the process $-|B_t| + \mu L_t$ with $\mu = 1/(1-\alpha)$ (cf. Petit [21], Yor [29] for μ-process). We also point out that $\alpha, \beta < 1$ is the necessary and sufficient condition for the equation (1.1) to have a (pathwise) unique solution (cf. [4]).

In this paper, we study the asymptotic behaviors of the extrema of X_t. First, we state an Erdős-Feller-Kolmogorov-Petrowsky (EFKP) type result:

Theorem 1.1. *Recall (1.1). Let $f > 0$ be a nondecreasing function; we have*

$$\mathbb{P}\Big(\sup_{0\le s\le t} X_s > \sqrt{t} f(t), \text{ i.o.}\Big) = \begin{cases} 0 \\ 1 \end{cases} \iff \int^{\infty} \frac{dt}{t} f(t) \exp\Big(-\frac{(1-\alpha)^2 f^2(t)}{2}\Big) \begin{cases} < \infty \\ = \infty \end{cases}, \tag{1.3}$$

where, here and in the sequel, "i.o." means "infinitely often" as the relevant index goes to infinity. Consequently, we have

$$\limsup_{t\to\infty} \frac{X_t}{\sqrt{2t \log\log t}} = \frac{1}{1-\alpha}, \qquad \text{a.s..} \tag{1.4}$$

Remark 1.1. Theorem 1.1 is not surprising. Indeed, intuitively, the extraordinarily large values of X_t should only depend on the perturbation at the maxima of X, and so the upper limits of X_t with α, β-double perturbations should agree with those of the α-simply perturbed Brownian motion given by (1.2). We also point out that in certain cases (for instance, $0 \le \alpha, \beta < 1$), the LIL (1.4) can be derived from that of Brownian motion and Skorokhod's reflection lemma.

The main results of this paper are the following two forthcoming theorems. The first one is a Hirsch-type integral test for the lower limits of $\sup_{s\le t} X_s$:

Theorem 1.2. *Recall (1.1). Let $f > 0$ be a nondecreasing function; we have*

$$(1.5)\qquad \liminf_{t\to\infty} \frac{f(t)}{\sqrt{t}} \sup_{0\le s\le t} X_s = \begin{cases} 0 \\ \infty \end{cases}, \qquad \text{a.s.} \iff \int^{\infty} \frac{dt}{t\, f(t)^{(1-\beta)}} \begin{cases} = \infty \\ < \infty \end{cases}.$$

In particular, we have almost surely

$$(1.6)\qquad \liminf_{t\to\infty} \frac{\left(\log t\right)^{\frac{1}{1-\beta}+\epsilon}}{\sqrt{t}} \sup_{0\le s\le t} X_s = \begin{cases} 0 & \text{if} \quad \epsilon \le 0, \\ \infty & \text{if} \quad \epsilon > 0. \end{cases}$$

It is noteworthy that the above integral test does not depend on α, i.e. the small values of $\sup_{s\le t} X_s$ only involves the perturbation at minima of X.

Remark 1.2. Denoting by $X^{\alpha,\beta}$ the solution of (1.1), it follows from the Brownian symmetry that $X^{\alpha,\beta} \overset{\text{law}}{=} -X^{\beta,\alpha}$. Then, the above results give corresponding versions for $-\inf_{0\le s\le t} X_s$ by interchanging α and β.

The lower functions of $\sup_{0\le s\le t} |X_s|$ are characterized as follows:

Theorem 1.3. *Recall (1.1). Let $f > 0$ be a non decreasing function; we have*

$$(1.7)\quad \mathbb{P}\Big(\sup_{0\le s\le t} |X_s| < \frac{\sqrt{t}}{f(t)}, \text{i.o.}\Big) = \begin{cases} 0 \\ 1 \end{cases} \iff \int^{\infty} \frac{dt}{t} f(t)^{2(1-\alpha-\beta)} \exp\Big(-\frac{\pi^2 f^2(t)}{8}\Big) \begin{cases} < \infty \\ = \infty \end{cases}.$$

Consequently, the following Chung-type LIL holds

$$(1.8)\qquad \liminf_{t\to\infty} \Big(\frac{\log\log t}{t}\Big)^{1/2} \sup_{0\le s\le t} |X_s| = \frac{\pi}{\sqrt{8}}, \qquad \text{a.s..}$$

Although we don't state it explicitly, all the above results admit corresponding versions as t goes to 0.

Let us point out that among these, Theorem 1.3 is more intrinsic, even though DPBM and a standard Brownian motion enjoy the same LIL. In a sense, this Chung–type integral test shows how the two perturbations at maximum and at minimum cancel or strengthen themselves.

Taking $\alpha = \beta = 0$ in Theorems 1.1–1.3, we obtain respectively the usual EFKP, Hirsch and Chung type integral tests for Brownian motion. We refer to Csörgő and Révész [8], and Révész [23] for detailed discussions of the almost sure behaviors of Brownian motion and random walk, and to Csáki [6] for the generalized Chung and Hirsch-type result.

This paper is organized as follows: In Section 2, we will state a Ray-Knight theorem for a general DPBM at its first hitting time, and give an estimate for the density functions of some infinitely divisible laws. The behaviors of tail probabilities are given in Section 3, which imply immediately the convergence parts of our integral tests, whereas the divergence parts need some uniform estimates which are given in Section 4. Finally, all theorems are proven in Section 5.

Throughout this paper, $\alpha < 1, \beta < 1$ will be considered as two universal constants. We write $f(x) \sim g(x)$ as $x \to x_0$ if $\lim_{x\to x_0} f(x)/g(x) = 1$. Unless stated otherwise, the constants ($C_i = C_i(\alpha, \beta), 1 \le i \le 25$) only depend on α and β.

Acknowledgements. We are grateful to Frédérique Petit and Marc Yor for helpful discussions on Ray-Knight theorems for perturbed Brownian motion. We thank an anonymous referee for her/his insightful comments.

2. PRELIMINARIES.

Firstly, let us recall a Ray-Knight type theorem for the DPBM with non zero initial values for its maximum and minimum. Fix $m_0 \ge 0$ and $i_0 \ge 0$. Consider the equation

$$\begin{cases} Y_t = B_t + \alpha(M_t^Y - m_0)^+ - \beta(I_t^Y - i_0)^+, & t > 0, \\ Y_0 = 0, \end{cases} \tag{2.1}$$

with $x^+ \stackrel{\text{def}}{=} x \vee 0$, $M_t^Y \stackrel{\text{def}}{=} \sup_{0\le s\le t} Y_s \vee 0$, and $I_t^Y \stackrel{\text{def}}{=} \sup_{0\le s\le t}(-Y_s) \vee 0$. We denote by $\{L_Y(t,x), t \ge 0, x \in \mathbb{R}\}$ the family of local times of the continuous semimartingale Y defined by the occupation time formula. Write

$$T_Y(b) \stackrel{\text{def}}{=} \inf\{t > 0 : Y_t > b\}, \qquad b > 0. \tag{2.2}$$

Throughout this paper, we write

$$\overline{\alpha} \equiv 1 - \alpha > 0, \tag{2.3}$$

$$\overline{\beta} \equiv 1 - \beta > 0, \tag{2.4}$$

$$BESQ_r^\delta \equiv \text{a process having the same law as the square of Bessel processes of dimension } \delta \text{ starting from } r \ge 0, \tag{2.5}$$

see [22, Chap. XI] for detailed studies on Bessel processes.

Proposition 2.1. *Fix $b > 0$. The process $\{L_Y(T_Y(b), b-t), t \ge 0\}$ has the same law as $(Z(t\wedge\zeta), t \ge 0)$, where Z is the unique solution of*

$$Z_t = 2\int_0^t \sqrt{Z_s}\, dB_s + \int_0^t \Big(2\overline{\alpha}\mathbb{1}_{(0\le s\le (b-m_0)^+)} + 2\mathbb{1}_{((b-m_0)^+\le s\le b)} + 2\beta\mathbb{1}_{(s\ge b+i_0)}\Big) ds,$$

and $\zeta \stackrel{\text{def}}{=} \inf\{t > b : Z_t = 0\}$. *In words, the process* $\{L_Y(T_Y(b), b-t), t \geq 0\}$ *is an inhomogeneous Markov process which is a* $BESQ_0^{2\overline{\alpha}}$ *on* $[0, (b-m_0)^+]$, *a* $BESQ^2$ *on* $((b-m_0)^+, b]$, *a* $BESQ^0$ *on* $(b, b+i_0]$ *and a* $BESQ^{2\beta}$ *on* $[b+i_0, \infty)$, *absorbed at its first zero after time* b.

Remark 2.2. The case of $m_0 = i_0 = 0$ of the above proposition has been stated in Carmona, Petit and Yor [3, Proposition 3.4]. See also their Ray-Knight theorem at the inverse of local time at 0 ([3, Theorem 3.3]). For the case $\beta = 0$ (i.e. the μ-process) see [2, Theorem 3.2].

Proof of Proposition 2.1. This result, probably not new, may have already been obtained by the experts of μ-process or perturbed Brownian motion. Its proof can be achieved by a method of studying the filtration generated by the excursions of Y below levels. This method, developed by McGill [18] and Jeulin [14] for the classical Ray–Knight theorems for Brownian local times, works in fact with more general diffusion processes or semimartingales (cf. [19]), and also has been used in [3] to obtain their Ray–Knight type results. Here, for the sake of completeness, we sketch the main steps, and the interested reader is referred to [18,14,19,3]. Applying Tanaka's formula to (2.1) gives that for $x \in \mathbb{R}$,

$$(Y_t - x)^+ = (-x)^+ + \int_0^t \mathbb{1}_{(Y_s > x)} dB_s + \alpha\Big(M_t^Y - m_0 \vee x\Big)^+ - \beta\Big(I_t^Y \wedge (-x) - i_0\Big)^+ + \frac{1}{2} L_Y(t, x).$$

Define $Z_y \stackrel{\text{def}}{=} L_Y(T_Y(b), b-y)$ for $y \geq 0$ and let $\zeta \equiv \inf\{t > b : Z_t = 0\}$. Observing $\zeta = b + I^Y_{T_Y(b)}$, we have for $y \geq 0$

$$Z_y = -2\int_0^{T_Y(b)} \mathbb{1}_{(Y_s > b-y)} dB_s + 2y \wedge b - 2\alpha y \wedge (b - m_0)^+ + 2\beta\Big(y \wedge \zeta - (b+i_0)\Big)^+.$$

It suffices to show that $y \in [0,\infty) \to \int_0^{T_Y(b)} \mathbb{1}_{(Y_s > b-y)} dB_s$ is a continuous martingale with respect to the natural filtration $(\mathcal{F}_t^Z, t \geq 0)$ of Z, with increasing process $y \to \int_0^y Z_x dx$. The key point is to show that for every $H \in L^2(\mathcal{F}_x^Z)$, there exists a process (h_s) predictable with respect to the filtration $(\sigma(B_s, s \leq t), t \geq 0)$ such that

$$H = \mathbb{E}\Big(H\Big) + \int_0^{T_Y(b)} h_s \, \mathbb{1}_{(Y_s > b-x)} dB_s.$$

To prove this representation, observe that the time-change arguments of [18,14,19] work in the present case once we use the facts that the equation (2.1) has a unique solution adapted to the natural filtration of the Brownian motion B, and that $T_Y(b)$ is also a stopping time with respect to the filtration $(\sigma(B_s, s \leq t), t \geq 0)$. The details are omitted. □

We shall make use of the following

Lemma 2.3. *Recall (2.3)–(2.5). Let $V_r^{(\delta)}$ be a process with law $BESQ_r^\delta$. For $\delta > 0$, denote by $\mathcal{Z}(\delta)$ the gamma distribution on $\mathbb{R}_+$ with density $x^{\delta-1}e^{-x}/\Gamma(\delta)$. We have for $u, t, \lambda, \mu > 0$ that*

$$\int_0^{H_0} ds V_r^{(2\beta)}(s) \overset{\text{law}}{=} \frac{r^2}{8\,\mathcal{Z}(\overline{\beta}/2)}, \tag{2.6}$$

$$\mathbb{E}\exp\Big(-\mu V_0^{(\delta)}(t) - \frac{\lambda^2}{2}\int_0^t ds V_0^{(\delta)}(s)\Big) = \Big(\cosh(\lambda t) + \frac{2\mu}{\lambda}\sinh(\lambda t)\Big)^{-\delta/2}, \tag{2.7}$$

$$\mathbb{E}\Big(\exp\big(-\frac{\lambda^2}{2}\int_0^\infty ds V_r^{(0)}(s)\big)\,\mathbb{1}_{(H_0<u)}\Big) = \exp\Big(-\frac{r}{2}\lambda\coth(\lambda u)\Big), \tag{2.8}$$

where in (2.6) and (2.8), H_0 denotes the respective first hitting time of the processes $V_r^{(2\beta)}$ and $V_r^{(0)}$. Furthermore, we have for all $x, r, t > 0$,

$$\mathbb{P}\Big(\int_0^{H_0} ds V_r^{(2\beta)}(s) > x\Big) \le 8^{-\overline{\beta}/2}\Big(\Gamma(1+\overline{\beta}/2)\Big)^{-1}\Big(\frac{r^2}{x}\Big)^{\overline{\beta}/2}, \tag{2.9}$$

$$\mathbb{P}\Big(\int_0^r ds V_0^{(2\overline{\alpha})}(s) > x\Big) \le 2^{\overline{\alpha}/2}\exp\Big(-\frac{\pi^2 x}{32\,r^2}\Big), \tag{2.10}$$

$$\mathbb{P}\Big(\int_0^r ds V_0^{(2\overline{\alpha})}(s) < \epsilon\Big) \sim \frac{2^{\overline{\alpha}+1/2}}{\overline{\alpha}\sqrt{\pi}}\frac{\sqrt{\epsilon}}{r}\exp\Big(-\frac{\overline{\alpha}^2 r^2}{2\epsilon}\Big), \qquad \epsilon/r^2 \to 0, \tag{2.11}$$

$$\mathbb{E}\Big(V_r^{(0)}(t)\Big)^{\overline{\beta}} \le \sqrt{e}\Big(1+\overline{\beta}\Big)^{\overline{\beta}}\Big(r \vee t\Big)^{\overline{\beta}}. \tag{2.12}$$

Proof of Lemma 2.3. By scaling, it suffices to treat the case $r \equiv 1$ in Lemma 2.3. See Yor [29, pp.16] for (2.7). To see (2.6), by Bessel time reversal,

$$\int_0^{H_0} ds V_1^{(2\beta)}(s) \overset{\text{law}}{=} \int_0^{\mathcal{L}_1} V_0^{(4-2\beta)}(s),$$

with $\mathcal{L}_1 \equiv \sup\{t \ge 0 : V_0^{(4-2\beta)}(t) \le 1\}$ being the last exit time at 1 of the transient Bessel square process $V_0^{(4-2\beta)}$. From this, (2.6) follows from Yor [29, pp.119]. To prove (2.8), again by time reversal we have

$$\{V_1^{(0)}(H_0 - s), 0 \le s \le H_0\} \overset{\text{law}}{=} \{V_0^{(4)}(s), 0 \le s \le \mathcal{L}_1\},$$

with $\mathcal{L}_1 \overset{\text{def}}{=} \sup\{s > 0 : V_0^{(4)}(s) \le 1\}$. Using the density function of $\mathcal{L}_1$ evaluated by Getoor [13] and conditioning on $\mathcal{L}_1$ give that the expectation term of (2.8) equals

$$= \mathbb{E}\Big(\exp\big(-\frac{\lambda^2}{2}\int_0^{\mathcal{L}_1} ds V_0^{(4)}(s)\big)\mathbb{1}_{(\mathcal{L}_1<u)}\Big)$$

$$
\begin{aligned}
&= \int_0^u \mathbb{P}\Big(\mathcal{L}_1 \in dx\Big)\, \mathbb{E}\Big[\exp\Big(-\frac{\lambda^2}{2}\int_0^x ds V_0^{(4)}(s)\Big)\Big|\mathcal{L}_1 = x\Big] \\
&= \int_0^u \mathbb{P}\Big(\mathcal{L}_1 \in dx\Big)\, \mathbb{E}\Big[\exp\Big(-\frac{\lambda^2}{2}\int_0^x ds V_0^{(4)}(s)\Big)\Big|V_0^{(4)}(x) = 1\Big] \\
&= \int_0^u \frac{dx}{2x^2}\exp\Big(-\frac{1}{2x}\Big)\Big(\frac{\lambda x}{\sinh(\lambda x)}\Big)^2 \exp\Big(\frac{1}{2x}(1-\lambda x\coth(\lambda x))\Big) \\
&= \exp\Big(-\frac{\lambda\coth(\lambda u)}{2}\Big),
\end{aligned}
$$

where the third equality is due to [30, pp. 53], and the fourth to [22, pp. 443]. (2.8) is thus proved. (2.9) follows immediately from (2.6) by bounding the density of $\mathcal{Z}(\overline{\beta}/2)$ by $x^{\overline{\beta}/2-1}/\Gamma(\overline{\beta}/2)$. To obtain (2.10), use of analytical continuation yields

$$
\mathbb{E}\exp\Big(\lambda\int_0^1 ds V_0^{(2\overline{\alpha})}(s)\Big) = \Big(\cos(\sqrt{2\lambda})\Big)^{-\overline{\alpha}}, \qquad 0<\lambda<\frac{\pi^2}{8},
$$

implying (2.10) by Chebychev's inequality at $\lambda = \pi^2/32$. Take $\mu = 0$ in (2.7). By inverting the Laplace transform, we get the density function

$$
\mathbb{P}\Big(\int_0^1 ds V_0^{(2\overline{\alpha})}(s) \in dt\Big)/dt = \sum_{n\ge 0}\frac{2^{\overline{\alpha}}(-1)^n\,\Gamma(n+\overline{\alpha})\,(2n+\overline{\alpha})}{\Gamma(\overline{\alpha})\,\Gamma(n+1)\,\sqrt{2\pi t^3}}\, e^{-\frac{(2n+\overline{\alpha})^2}{2t}}, \qquad t>0,
$$

implying (2.11). Finally, we use the following Laplace transform for the $BESQ_r^0$ process $V_r^{(0)}$ (cf. [22, Chap. XI])

$$
\mathbb{E}\exp\Big(-\lambda V_r^{(0)}(t)\Big) = \exp\Big(-\frac{\lambda r}{1+2\lambda t}\Big),
$$

which implies

$$
\mathbb{E}\exp\Big(\lambda V_r^{(0)}(t)\Big) = \exp\Big(\frac{\lambda r}{1-2\lambda t}\Big), \qquad \lambda<\frac{1}{2t}.
$$

Let $k \stackrel{\text{def}}{=} 1+[\overline{\beta}]$ the smallest integer greater than $\overline{\beta}$. Taking $\lambda = 1/4(r\vee t)$ in the above transform and using the elementary relation $x^k \le k! e^{\lambda x}\lambda^{-k}$ gives

$$
\mathbb{E}\Big(V_r^{(0)}(t)\Big)^{\overline{\beta}} \le \Big(\mathbb{E}\Big(V_r^{(0)}(t)\Big)^k\Big)^{\overline{\beta}/k} \le \Big(k!\Big)^{\overline{\beta}/k}\lambda^{-\overline{\beta}}\sqrt{e},
$$

gives (2.12) by means of $k! \le k^k \le (1+\overline{\beta})^k$. □

The following result shows the relation between the asymptotic behaviour of the density function of an infinitely divisible distribution and that of its Lévy measure in circumstances that do not seem to have been considered before, and may be of independent interest.

Lemma 2.4. *Let Ξ be an infinitely divisible random variable on $[0,\infty)$ whose Lévy measure has density function $\pi(x)$ such that*

$$\mathbb{E}\, e^{-\lambda\Xi} = \exp\Big(-\int_0^\infty (1-e^{-\lambda x})\pi(x)dx\Big). \tag{2.13}$$

Assume furthermore that $\sup_{0<x<\infty} x\pi(x) < \infty$ and there exist two constants $c>0$ and $\rho>0$ such that

$$\pi(x) \sim \rho\, x^{-1} e^{-cx} \qquad x \to \infty.$$

Then Ξ has at most a Dirac mass at 0 and

$$\mathbb{P}\Big(\Xi \in dt\Big)/dt = t^{\rho-1}\,\ell(t) e^{-ct}, \qquad t>0, \tag{2.14}$$

with some function $\ell(t)$ which is slowly varying at ∞.

Proof of Lemma 2.4. First, let us show that Ξ has at most a Dirac mass at 0. In fact, either $\int_0^\infty \pi(x)dx = \infty$, and Tucker [27] says that in this case the distribution of Ξ is absolutely continuous; or $\eta \stackrel{\text{def}}{=} \int_0^\infty \pi(x)dx < \infty$, in which case Ξ can be realized as a compound Poisson variable, i.e. $\Xi \stackrel{\text{law}}{=} \xi_1 + \xi_2 + ... + \xi_N$, where $(\xi_i, i=1,2,...)$ are i.i.d., with common distribution $\mathbb{P}\Big(\xi_1 \in dx\Big)/dx = \pi(x)/\eta$, $x>0$, and $(\xi_i)_i \geq 1$ are independent of N, which has the Poisson distribution of parameter η. Therefore $\mathbb{P}\Big(\Xi = 0\Big) = e^{-\eta}$. In terms of the density function $\pi(x)/\eta$ of ξ_i, it is easy to obtain that Ξ has a density function on $(0,\infty)$.

To prove (2.14), write $f(t) \stackrel{\text{def}}{=} \mathbb{P}\Big(\Xi \in dt\Big)/dt$ for $t>0$ and $\kappa \stackrel{\text{def}}{=} \mathbb{P}\Big(\Xi = 0\Big) \geq 0$. Differentiating (2.13) with respect to λ gives

$$\mathbb{E}\Big(\Xi e^{-\lambda\Xi}\Big) = \mathbb{E}\Big(e^{-\lambda\Xi}\Big)\int_0^\infty e^{-\lambda x}\, x\pi(x)dx, \qquad \lambda>0,$$

which implies in terms of f and κ that

$$tf(t) = \kappa\, t\pi(t) + \int_0^t f(t-s)\, s\pi(s)\, ds, \qquad t>0.$$

Define $f^*(t) \stackrel{\text{def}}{=} e^{ct} f(t)$, and $\pi^*(t) \stackrel{\text{def}}{=} e^{ct}\pi(t)$, so that

$$tf^*(t) = \kappa\, t\pi^*(t) + \int_0^t f^*(t-s)\, s\pi^*(s)\, ds, \qquad t>0. \tag{2.15}$$

Since $s\pi^*(s) \to \rho$ as $s\to\infty$, it is easy to show

$$\lim_{t\to\infty} \frac{\int_0^t f^*(t-s)\, s\pi^*(s)\, ds}{\int_0^t f^*(s)\, ds} = \rho. \tag{2.16}$$

In fact, notice that $\int_0^\infty f^*(t)dt = \infty$ (otherwise, applying dominated convergence to (2.15), we would have that $f^*(t) \sim \big(\kappa\rho + \rho\int_0^\infty f^*(s)ds\big)/t$, as $t \to \infty$, leading to a contradiction). Since $\sup_{s\geq 0} s\pi^*(s) = K < \infty$, (2.15) yields $f^*(t) \leq \frac{\kappa K}{t} + \frac{K}{t}\int_0^t f^*(s)\,ds$, and therefore for any fixed $A > 0$, since $\int_0^A s\pi^*(s)ds < \infty$, we have

$$\int_0^A f^*(t-s)s\pi^*(s)ds = o\Big(\int_0^t f^*(s)\,ds\Big), \qquad t \to \infty,$$

implying (2.16) in view of (2.15). Combining (2.15) and (2.16), we have

$$\lim_{t\to\infty} \frac{t\,f^*(t)}{\int_0^t f^*(s)\,ds} = \rho > 0,$$

so that, according to a result of Karamata (cf. [1, pp.30]), $f^*(t) = t^{\rho-1}\ell(t)$ which completes the proof since $f(t) = e^{-ct}f^*(t)$. □

3. Tails.

Consider the DPBM X of (1.1) and define

$$(3.1) \qquad T_X(a) \stackrel{\text{def}}{=} \begin{cases} \inf\{t > 0 : X_t > a\} & \text{if } a \geq 0, \\ \inf\{t > 0 : X_t < a\} & \text{if } a < 0. \end{cases}$$

Recall (2.3) and (2.4). The explicit form of the density function of $T_X(1)$ has been given in [3], and this yields

$$(3.2) \qquad \mathbb{P}\Big(T_X(1) < x\Big) \sim C_1 x^{1/2}\exp(-\frac{\overline{\alpha}^2}{2x}), \qquad x \to 0,$$

with $C_1 = 2^{\beta+1/2}\Gamma(\overline{\alpha}+\overline{\beta})/\overline{\alpha}\Gamma(3-\alpha-\beta/2)\Gamma(\overline{\beta}/2)$. The goal of this section is to get the behavior of the tail probabilities of $T_X(1)$ and of the exit time $T_X(-a) \wedge T_X(b)$ from the interval $[-a, b]$.

Lemma 3.1. *Recall (2.3)–(2.4) and (3.1). We have*

$$\mathbb{P}\Big(\,T_X(r) > t\,\Big) \;\sim\; C_2\Big(t/r^2\Big)^{-\overline{\beta}/2}, \qquad \frac{t}{r^2} \to \infty,$$

where $C_2 = 2^{(1+\beta)/2}\Gamma(\overline{\alpha}+\overline{\beta})/\Big(\overline{\beta}\,\Gamma(\overline{\beta}/2)\Gamma(\overline{\alpha})\Big)$.

Remark 3.2. The Laplace transform of $T_X(r)$ is given explicitly in [3], from which it is also possible to get the above tail behaviour by using a Tauberian theorem (cf. [1, pp. 333, Theorem 8.1.6]). Intuitively, the reason why the asymptotic behaviour of the tail of the distribution of $T_X(r)$ depends on β and *not* α is that if we write $T_X(r) = A^+(T_X(r)) + A^-(T_X(r))$, where $A^{+/-}(t)$ denotes the time spent positive/negative by X up to time t, then it is $A^-(T_X(r))$, which obviously depends only on β, which dominates.

Proof. By scaling, we need only consider $r = 1$. Applying Proposition 2.1 with $m_0 = i_0 = 0$ yields

$$\mathbb{P}\Big(T_X(1) > t\Big) = \mathbb{P}\Big(\int_0^1 Z(s)ds + \int_0^{H_0} V(s)ds > t\Big), \tag{3.3}$$

where $(Z(s), 0 \le s \le 1)$ is a $BESQ_0^{2\overline{\alpha}}$ and $(V(s), s \ge 0)$ denotes a $BESQ^{2\beta}$ starting from $Z(1)$, $H_0 = \inf\{t > 0 : V(t) = 0\}$. It follows that

$$\mathbb{P}\Big(\int_0^{H_0} V(s)ds > t\Big) \le \mathbb{P}\Big(T_X(1) > t\Big) \tag{3.4}$$

$$\le \mathbb{P}\Big(\int_0^{H_0} V(s)ds > t - \sqrt{t}\Big) + \mathbb{P}\Big(\int_0^1 Z(s)ds > \sqrt{t}\Big)$$

$$\le \mathbb{P}\Big(\int_0^{H_0} V(s)ds > t - \sqrt{t}\Big) + 2^{\overline{\alpha}/2} e^{-\pi^2\sqrt{t}/32}, \tag{3.5}$$

by applying (2.10) to Z. Using (2.6), we have

$$\mathbb{P}\Big(\int_0^{H_0} V(s)ds > t\Big) \sim \frac{2\,8^{-\overline{\beta}/2}\mathbb{E}Z(1)^{\overline{\beta}}}{\overline{\beta}\,\Gamma(\overline{\beta}/2)} t^{-\overline{\beta}/2} = C_2 t^{-\overline{\beta}/2}, \qquad t \to \infty,$$

which yields the desired estimate in view of (3.4) and (3.5). □

The main result of this section is the following tail behaviour of $\mathbb{P}\Big(T_X(-a) \wedge T_X(b) > t\Big)$, for fixed $a, b > 0$.

Proposition 3.3. *Recall (2.3), (2.4) and (3.1). Fix $a, b > 0$. We have*

$$\mathbb{P}\Big(T_X(-a) > T_X(b) > t\Big) \sim C_3 t^{-\alpha-\beta} \exp\Big(-\frac{\pi^2}{2(a+b)^2}t\Big), \qquad t \to \infty,$$

with $C_3 = \frac{2\pi^{\overline{\alpha}+\overline{\beta}-3}}{\Gamma(\overline{\alpha})\Gamma(\overline{\beta})}(a+b)^{2(\alpha+\beta)}\Big(\sin\frac{\pi a}{a+b}\Big)^{\overline{\alpha}+\overline{\beta}-1}$.

By exchanging α and β in the above result, we obtain the tail of $\mathbb{P}\Big(T_X(b) > T_X(-a) > t\Big)$ and therefore

$$\mathbb{P}\Big(T_{-a} \wedge T_b > t\Big) \sim 2\,C_3\,t^{-\alpha-\beta} \exp\Big(-\frac{\pi^2}{2(a+b)^2}t\Big), \qquad t \to \infty. \tag{3.6}$$

Our proof of Proposition 3.3 relies on the following Laplace transform obtained in [5]: for $\lambda > 0$,

$$\mathbb{E}\Big(e^{-\frac{\lambda^2}{2} T_X(b)}\, \mathbb{1}_{(T_X(b)<T_X(-a))}\Big) = \frac{\Gamma(\overline{\alpha}+\overline{\beta})}{\Gamma(\overline{\alpha})\,\Gamma(\overline{\beta})} \int_0^a du\, \frac{\lambda\,\big(\sinh \lambda b\big)^{\overline{\beta}} \,\big(\sinh \lambda u\big)^{\overline{\alpha}-1}}{\big(\sinh \lambda (b+u)\big)^{\overline{\alpha}+\overline{\beta}}}. \tag{3.7}$$

It seems difficult to directly invert the above Laplace transform. We shall write (3.7) in an equivalent form. For $\gamma > 0$ and $0 \le a_1 < a_2$, denote by $\Delta_\gamma(a_1, a_2)$ a r.v. having the following Laplace transform

$$\mathbb{E}\exp\Big(-\frac{\lambda^2}{2}\,\Delta_\gamma(a_1,a_2)\Big) = \begin{cases} \Big(\frac{a_2 \sinh(\lambda a_1)}{a_1\sinh(\lambda a_2)}\Big)^{\gamma}, & \text{if } a_1>0,\\ \Big(\frac{\lambda a_2}{\sinh(\lambda a_2)}\Big)^{\gamma}, & \text{if } a_1 = 0, \end{cases} \qquad \lambda > 0. \tag{3.8}$$

Write $\Delta_\gamma(a_2) \equiv \Delta_\gamma(0, a_2)$ for brevity. Observe the following monotonicity and scaling property of Δ_γ variables:

$$\Delta_\gamma(a_1,a_2) \overset{\text{law}}{=} a_2^2\, \Delta_\gamma(a_1/a_2, 1), \qquad 0 \le a_1 < a_2, \tag{3.9}$$

$$\mathbb{P}\Big(\Delta_\gamma(a_1,a_2) > t\Big) \le \mathbb{P}\Big(\Delta_\gamma(a_3,a_2) > t\Big), \quad 0 \le a_3 \le a_1 < a_2, \quad t \ge 0, \tag{3.10}$$

(A quick way to obtain (3.10) is to notice that $\Delta_\gamma(a_3, a_2) \overset{\text{law}}{=} \Delta_\gamma(a_1, a_2) + \Delta_\gamma(a_3, a_1)$, the sum of two independent variables). We can rewrite the RHS of (3.7) as

$$\frac{\Gamma(\overline{\alpha}+\overline{\beta})}{\Gamma(\overline{\alpha})\,\Gamma(\overline{\beta})} \int_0^a du\, \frac{u^{-\alpha} b^{\overline{\beta}}}{(b+u)^{\overline{\alpha}+\overline{\beta}}} \mathbb{E}\exp\Big(-\frac{\lambda^2}{2}\Big(\Delta_1(u) + \Delta_{\overline{\alpha}}(u, b+u) + \Delta_{\overline{\beta}}(b, b+u)\Big)\Big),$$

where the three random variables are assumed to be mutually independent. This implies that

$$\mathbb{P}\Big(T_X(b) > t; T_X(b) < T_X(-a)\Big) = \frac{\Gamma(\overline{\alpha}+\overline{\beta})}{\Gamma(\overline{\alpha})\,\Gamma(\overline{\beta})} \int_0^a du\, \frac{u^{-\alpha} b^{\overline{\beta}}}{(b+u)^{\overline{\alpha}+\overline{\beta}}} \mathbb{P}\Big(\Sigma_u > t\Big), \tag{3.11}$$

with $\Sigma_u \overset{\text{def}}{=} \Delta_1(u) + \Delta_{\overline{\alpha}}(u, b+u) + \Delta_{\overline{\beta}}(b, b+u)$.

Remark 3.4. Since $(T_X(b) < T_X(-a)) = (I_{T_X(b)} < a)$, we get the density function of $I_{T_X(b)}$ by differentiating (3.11) with respect to a. Furthermore, (3.11) tells us that conditionally to $(I_{T_X(b)} = u)$, $T_X(b) = \Delta_1(u)+\Delta_{\overline{\alpha}}(u, b+u)+\Delta_{\overline{\beta}}(b, b+u)$ is a sum of three independent hitting times which correspond respectively to $BES(3)$ (the three-dimensional Bessel process), to $BES(3, \alpha)$ (the α-perturbed three-dimensional Bessel process) and to $BES(3, \beta)$. It remains an open question to find a path transformation explaining this decomposition. For studies on perturbed Bessel processes, we refer to [12].

Lemma 3.5. *Recall (3.8). Fix* $0 \le z_1, z_2, z_3 < 1$. *Let* $\Theta \equiv \Delta_1(z_1) + \Delta_{\overline{\alpha}}(z_2, 1) + \Delta_{\overline{\beta}}(z_3, 1)$, *where the three* Δ*-random variables are assumed to be independent. We have*

$$\mathbb{P}\Big(\Theta \in dt\Big)/dt \sim \frac{z_1 \sin^{\overline{\alpha}}(z_2\pi) \sin^{\overline{\beta}}(z_3\pi)\, \pi^{1+\overline{\alpha}+\overline{\beta}}}{\Gamma(\overline{\alpha}+\overline{\beta})\, z_2^{\overline{\alpha}}\, z_3^{\overline{\beta}} \sin(z_1\pi)}\, t^{\overline{\alpha}+\overline{\beta}-1}\, e^{-\frac{\pi^2}{2}t}, \qquad t \to \infty,$$

where for $i = 1, 2, 3$, *the above constant should be understood as its limit when* $z_i \to 0$ *if* $z_i = 0$.

Proof of Lemma 3.5. Observe that the infinitely divisible random variable Θ has a continuous density function $f(t)$ on $(0, \infty)$, and its Lévy measure $\pi(x)dx$ is given by

$$\pi(x) = \frac{1}{x}\sum_{k\ge 1} e^{-\frac{k^2\pi^2}{2z_1^2}x} + \frac{\overline{\alpha}}{x}\sum_{k\ge 1}\Big(e^{-\frac{k^2\pi^2}{2}x} - e^{-\frac{k^2\pi^2}{2z_2^2}x}\Big) + \frac{\overline{\beta}}{x}\sum_{k\ge 1}\Big(e^{-\frac{k^2\pi^2}{2}x} - e^{-\frac{k^2\pi^2}{2z_3^2}x}\Big),$$

with the convention $\frac{1}{0} \equiv \infty$. Applying Lemma 2.4 with $c = \pi^2/2, \rho = \overline{\alpha} + \overline{\beta} > 0$ yields

$$f(t) = t^{\overline{\alpha}+\overline{\beta}-1}\, \ell(t)\, e^{-\frac{\pi^2}{2}t}, \qquad t \ge 0,$$

with a function $\ell(t)$ which is slowly varying at ∞. It remains to show that $\ell(t)$ is equivalent to the desired constant as $t \to \infty$. To this end, using the above expression for the density function $f(t)$ and writing the (positive) Laplace transform of Θ at $\frac{\pi^2}{2} - \epsilon$ for a small ϵ (by taking the limit in the following expression as the appropriate index goes to 0 if z_1, z_2, or z_3 equals to 0), use of (3.8) gives

$$\begin{aligned}
&\int_0^\infty e^{-\epsilon t}\, t^{\overline{\alpha}+\overline{\beta}-1}\, \ell(t)\, dt \\
&\quad = \mathbb{E}\Big(e^{(\frac{\pi^2}{2}-\epsilon)\,\Theta}\Big) \\
&\quad = \Big(\frac{z_1\sqrt{2(\frac{\pi^2}{2}-\epsilon)}}{\sin\big(z_1\sqrt{2(\frac{\pi^2}{2}-\epsilon)}\big)}\Big)\Big(\frac{\sin\big(z_2\sqrt{2(\frac{\pi^2}{2}-\epsilon)}\big)}{z_2\,\sin\sqrt{2(\frac{\pi^2}{2}-\epsilon)}}\Big)^{\overline{\alpha}}\Big(\frac{\sin\big(z_3\sqrt{2(\frac{\pi^2}{2}-\epsilon)}\big)}{z_3\,\sin\sqrt{2(\frac{\pi^2}{2}-\epsilon)}}\Big)^{\overline{\beta}} \\
&\quad \sim \frac{z_1 \sin^{\overline{\alpha}}(z_2\pi) \sin^{\overline{\beta}}(z_3\pi)\, \pi^{1+\overline{\alpha}+\overline{\beta}}}{z_2^{\overline{\alpha}}\, z_3^{\overline{\beta}} \sin(z_1\pi)}\, \epsilon^{-\overline{\alpha}+\overline{\beta}}, \qquad \epsilon \to 0,
\end{aligned}$$

which implies by a Tauberian theorem (cf. [1, pp.43, Theorem 1.7.6]) that

$$\ell(t) \sim \frac{1}{\Gamma(\overline{\alpha}+\overline{\beta})}\, \frac{z_1 \sin^{\overline{\alpha}}(z_2\pi) \sin^{\overline{\beta}}(z_3\pi)\, \pi^{1+\overline{\alpha}+\overline{\beta}}}{z_2^{\overline{\alpha}}\, z_3^{\overline{\beta}} \sin(z_1\pi)}, \qquad t \to \infty,$$

as desired. □

Proof of Proposition 3.3. We are going to show that

$$\limsup_{t\to\infty} \mathbb{P}\Big(T_X(-a)\wedge T_X(b) > t\Big)\, t^{\alpha+\beta}\, e^{-\frac{\pi^2}{2(a+b)^2}\,t} \le C_3, \tag{3.12}$$

$$\liminf_{t\to\infty} \mathbb{P}\Big(T_X(-a)\wedge T_X(b) > t\Big)\, t^{\alpha+\beta}\, e^{-\frac{\pi^2}{2(a+b)^2}\,t} \ge C_3. \tag{3.13}$$

First, let us show (3.12). Fix $0<\epsilon<a$. We rewrite (3.11) as

$$\begin{aligned}\mathbb{P}\Big(T_X(b) > t; T_X(b) < T_X(-a)\Big) &= \frac{\Gamma(\overline{\alpha}+\overline{\beta})}{\Gamma(\overline{\alpha})\,\Gamma(\overline{\beta})}\Big(\int_0^{a-\epsilon} + \int_{a-\epsilon}^{a}\Big)\, du\, \frac{u^{-\alpha}b^{\overline{\beta}}}{(b+u)^{\overline{\alpha}+\overline{\beta}}}\mathbb{P}\Big(\Sigma_u > t\Big),\\ &\equiv I_1(t)+I_2(t),\end{aligned} \tag{3.14}$$

with the obvious meaning for I_1 and I_2. Recall that $\Sigma_u \equiv \Delta_1(u)+\Delta_{\overline{\alpha}}(u,b+u)+\Delta_{\overline{\beta}}(b,b+b+u)$ is the sum of three independent random variables, and the variables have the properties (3.8)–(3.10). For $u\le a-\epsilon$, using (3.10), (3.9) and applying Lemma 3.5 with $z_1=(a-\epsilon)/(b+a-\epsilon)$, $z_2=z_3=0$ we see that

$$\begin{aligned}\mathbb{P}\Big(\Sigma_u > t\Big) &\le \mathbb{P}\Big(\Delta_1(a-\epsilon)+\Delta_{\overline{\alpha}}(b+a-\epsilon)+\Delta_{\overline{\beta}}(b+a-\epsilon) > t\Big)\\ &= \mathbb{P}\Big(\Delta_1(\frac{a-\epsilon}{b+a-\epsilon})+\Delta_{\overline{\alpha}}(1)+\Delta_{\overline{\beta}}(1) > \frac{t}{(b+a-\epsilon)^2}\Big)\\ &\le K\Big(\frac{t}{(b+a-\epsilon)^2}\Big)^{\overline{\alpha}+\overline{\beta}-1} e^{-\frac{\pi^2}{2(b+a-\epsilon)^2}\,t}, \qquad t\ge (a+b)^2/2,\end{aligned}$$

for some constant $K=K(\epsilon,a,b)>0$. It follows that

$$\limsup_{t\to\infty} I_1(t)\, t^{\alpha+\beta}\, e^{-\frac{\pi^2}{2(a+b)^2}\,t} = 0. \tag{3.15}$$

It remains to estimate $I_2(t)$. For $u\in[a-\epsilon,a]$, using (3.9) and (3.10) yields

$$\begin{aligned}\mathbb{P}\Big(\Sigma_u > t\Big) &= \mathbb{P}\Big(\Delta_1(\frac{u}{b+u})+\Delta_{\overline{\alpha}}(\frac{u}{b+u},1)+\Delta_{\overline{\beta}}(\frac{b}{b+u},1) > \frac{t}{(b+u)^2}\Big)\\ &\le \mathbb{P}\Big(\Delta_1(\frac{a}{b+a})+\Delta_{\overline{\alpha}}(\frac{a-\epsilon}{b+a-\epsilon},1)+\Delta_{\overline{\beta}}(\frac{b}{b+a},1) > \frac{t}{(b+u)^2}\Big) \qquad (3.16)\\ &\sim C_4(\epsilon)\Big(\frac{t}{(b+u)^2}\Big)^{\overline{\alpha}+\overline{\beta}-1} e^{-\frac{\pi^2}{2(b+u)^2}\,t}, \qquad t/(b+a)^2\to\infty \qquad (3.17)\end{aligned}$$

where the last equivalence is obtained by applying Lemma 3.5 with $z_1=a/(a+b)$, $z_2=(a-\epsilon)/(a+b-\epsilon)$, and $z_3=b/(b+a)$. It is easy to see

$$\lim_{\epsilon\to 0} C_4(\epsilon) = 2\Big((a+b)\pi\sin\big(\pi a/(a+b)\big)\Big)^{\overline{\alpha}+\overline{\beta}-1} a^{1-\overline{\alpha}}b^{-\overline{\beta}}\,/\Gamma(\overline{\alpha}+\overline{\beta}). \tag{3.18}$$

Applying (3.16)–(3.18) to $I_2(t)$ of (3.14), some lines of elementary calculations imply

$$\limsup_{t\to\infty} I_2(t)\, t^{\alpha+\beta}\, e^{-\frac{\pi^2}{2(a+b)^2}\,t} \le \frac{2\pi^{\overline{\alpha}+\overline{\beta}-3}}{\Gamma(\overline{\alpha})\Gamma(\overline{\beta})}\,(a+b)^{2(\alpha+\beta)}\Big(\sin\frac{\pi a}{a+b}\Big)^{\overline{\alpha}+\overline{\beta}-1}\Big(1+\eta(\epsilon)\Big),$$

where $\eta(\epsilon)\to 0$ as $\epsilon\to 0$, which, in view of (3.15), gives the desired upper bound (3.12) by letting $\epsilon\to 0$. For the lower bound, we use in lieu of (3.16) the following observation that for $u\in[a-\epsilon,a]$,

$$\mathbb{P}\Big(\Sigma_u > t\Big) \ge \mathbb{P}\Big(\Delta_1(\frac{a-\epsilon}{b+a-\epsilon}) + \Delta_{\overline{\alpha}}(\frac{a}{b+a},1) + \Delta_{\overline{\beta}}(\frac{b}{b+a-\epsilon},1) > \frac{t}{(b+u)^2}\Big),$$

and the lower bound (3.13) follows exactly in the same way from (3.14) and Lemma 3.5. □

We also need to bound uniformly the probability $\mathbb{P}\Big(T_X(-a) > T_X(b) > t\Big)$ for $a,b>0$.

Lemma 3.6. *Recall (2.3), (2.4) and (3.1). There exists a constant $C_5 = C_5(\alpha,\beta) > 0$ only depending on α,β such that for all $0<b\le a$ and $t>0$,*

$$\mathbb{P}\Big(T_X(-a) > T_X(b) > t\Big) \le C_5\,\exp\Big(-\frac{\pi^2\,t}{8(a+b)^2}\Big). \tag{3.19}$$

Moreover, for all $b\le a<2b$ and $t\ge (a+b)^2$, we have

$$\mathbb{P}\Big(T_X(-a) > T_X(b) > t\Big) \le C_5\, b^{2(\alpha+\beta)}\, t^{-\alpha-\beta}\,\exp\Big(-\frac{\pi^2\,t}{2(a+b)^2}\Big). \tag{3.20}$$

Proof of Lemma 3.6. It follows from (3.7) that for all $0<\lambda<\frac{\pi}{(a+b)}$,

$$\mathbb{E}\Big(e^{\frac{\lambda^2}{2}T_X(b)}\,\mathbb{1}_{(T_X(b)<T_X(-a))}\Big) = \frac{\Gamma(\overline{\alpha}+\overline{\beta})}{\Gamma(\overline{\alpha})\,\Gamma(\overline{\beta})}\int_0^a du\,\frac{\lambda\,\big(\sin\lambda b\big)^{\overline{\beta}}\,\big(\sin\lambda u\big)^{\overline{\alpha}-1}}{\big(\sin\lambda(b+u)\big)^{\overline{\alpha}+\overline{\beta}}}. \tag{3.21}$$

Taking $\lambda\equiv\frac{\pi}{2(a+b)}$ in (3.21) and using the elementary relation that $x\ge\sin x\ge 2x/\pi$ for $0\le x\le\pi/2$ yields that the RHS of (3.21) is bounded by

$$C_6(\alpha,\beta)\int_0^a du\frac{b^{\overline{\beta}}u^{\overline{\alpha}-1}}{(b+u)^{\overline{\alpha}+\overline{\beta}}} = C_6\int_0^{a/b} dx\frac{x^{\overline{\alpha}-1}}{(1+x)^{\overline{\alpha}+\overline{\beta}}} \le C_6\int_0^{\infty} dx\frac{x^{\overline{\alpha}-1}}{(1+x)^{\overline{\alpha}+\overline{\beta}}},$$

which, by applying Chebychev's inequality to (3.21), implies (3.19).

Now, we consider the case $b \le a \le 2a$. Recall (3.8)–(3.11). For $0 \le u \le a$, $u/(b+u) \le a/(b+a) \le 2/3$. Using (3.9)–(3.10) yields

$$\begin{aligned}\mathbb{P}\Big(\Sigma_u > t\Big) &= \mathbb{P}\Big(\Delta_1(\frac{u}{b+u}) + \Delta_{\overline{\alpha}}(\frac{u}{b+u}, 1) + \Delta_{\overline{\beta}}(\frac{b}{b+u}, 1) > \frac{t}{(b+u)^2}\Big)\\ &\le \mathbb{P}\Big(\Delta_1(\frac{2}{3}) + \Delta_{\overline{\alpha}}(1) + \Delta_{\overline{\beta}}(1) > \frac{t}{(b+u)^2}\Big)\\ &\le C_7(\alpha, \beta)\,\Big(\frac{t}{(b+u)^2}\Big)^{\overline{\alpha}+\overline{\beta}-1} \exp\Big(-\frac{\pi^2 t}{2(b+u)^2}\Big), \qquad t \ge (a+b)^2,\end{aligned}$$

where the last inequality is obtained by applying Lemma 3.5 with $z_1 = 2/3, z_2 = z_3 = 1$. Using the above estimate in (3.11), (3.20) follows from some elementary computations (with possibly a larger constant C_5). □

4. Main estimates.

This section gives the main estimates needed to prove the theorems 1.1–3. We consider a special case of the equation (2.1) with $m_0 = 0$ and $i_0 = v$,

$$\begin{cases} Y_t = B_t + \alpha M_t^Y - \beta\Big(I_t^Y - v\Big)^+, \\ Y_0 = 0, \end{cases} \tag{4.1}$$

with some constant $v \ge 0$ being given. Write throughout this section

$$T_Y(x) \overset{\text{def}}{=} \inf\{t > 0 : Y_t = x\}, \qquad x \in \mathbb{R}.$$

Lemma 4.1. *Recall (2.3)–(2.4). There exists a constant $C_8 = C_8(\alpha, \beta) > 0$ only depending on α and β such that for all $r, t > 0$*

$$\mathbb{P}\Big(T_Y(r) < t\Big) \le C_8\,\frac{\sqrt{t}}{r}\,\exp\Big(-\frac{\overline{\alpha}^2\, r^2}{2\,t}\Big), \tag{4.2}$$

$$\mathbb{P}\Big(T_Y(r) > t\Big) \le C_8 \exp\Big(-\frac{\pi^2\, t}{96\, r^2}\Big) + C_8 \frac{v\, r}{t} + C_8 \Big(v \vee r\Big)^{\overline{\beta}}\, t^{-\overline{\beta}/2}. \tag{4.3}$$

Proof of Lemma 4.1. Applying Proposition 2.1 with $b = r, m_0 = 0, i_0 = v$ yields

$$T_Y(r) \overset{\text{law}}{=} \int_0^r ds Z_s + \int_0^v ds U_s + \int_0^{H_0} ds V_s, \tag{4.4}$$

where $(Z_s, 0 \le s \le r)$ is a $BESQ_0^{2\overline{\alpha}}$ (cf. (2.5)), $(U_s, 0 \le s \le v)$ is a $BESQ^0$ starting from Z_r, and $(V_s, s \ge 0)$ is a $BESQ^{2\beta}$ starting from $U_v \ge 0$, with $H_0 \overset{\text{def}}{=} \inf\{t \ge 0 : V_t = 0\}$

(so $H_0 = 0$ if $U_v = 0$). By applying (2.11) to $V_0^{(2\overline{\alpha})} = Z$, (4.2) follows from the fact that $\mathbb{P}\Big(T_Y(r) < t\Big) \le \mathbb{P}\Big(\int_0^r ds Z_s < t\Big)$. Applying (2.10) to $V_0^{(2\overline{\alpha})} = Z$ and (2.9) to $V^{(2\beta)} = V$ (recall that $V_0 = U_v$) yield

$$\mathbb{P}\Big(T_Y(r) > t\Big) \le \mathbb{P}\Big(\int_0^r dsZ_s > \frac{t}{3}\Big) + \mathbb{P}\Big(\int_0^v dsU_s > \frac{t}{3}\Big) + \mathbb{P}\Big(\int_0^{H_0} dsV_s > \frac{t}{3}\Big)$$
$$\le 2^{\overline{\alpha}/2} \exp\Big(-\frac{\pi^2 t}{96\, r^2}\Big) + \frac{3}{t}\mathbb{E}\int_0^v dsU_s + \Big(\frac{3}{8}\Big)^{\overline{\beta}/2} \Big(\Gamma(1+\overline{\beta}/2)\Big)^{-1} \mathbb{E}\Big(U_v^{\overline{\beta}}\Big)\, t^{-\overline{\beta}/2}.$$

Using the fact that (U_s) is a martingale starting from Z_r gives $\mathbb{E}U_s = \mathbb{E}U_0 = \mathbb{E}Z_r = r\mathbb{E}Z_1$, by scaling. Finally, applying (2.12) to $V^{(0)} = U, t = v$ with $U_0 = Z_r$ gives $\mathbb{E}\Big(U_v^{\overline{\beta}}\Big) \le \sqrt{e}\,(1+\overline{\beta})^{\overline{\beta}}\Big(v^{\overline{\beta}} + \mathbb{E}\big(Z_r^{\overline{\beta}}\big)\Big) \le C_9(\alpha,\beta)\Big(v \vee r\Big)^{\overline{\beta}}$, implying the desired estimate (maybe with a larger constant C_8). □

Recall (4.1). The rest of this section is devoted to estimating the tail probability of the exit time of Y from an interval $[-a, b]$ with $b > 0$ and $a > v \ge 0$. In the case $v = 0$ in (4.1), recall that by Lemma 3.6, we know how to estimate this tail. The idea here consists of reducing the case of $v > 0$ to that of $v = 0$. Write

$$\phi_v\Big(a, b; t\Big) \equiv \mathbb{P}\Big(T_Y(-a) > T_Y(b) > t\Big). \tag{4.5}$$

Recall that $T_Y(x)$ is defined as the hitting time at x by Y the solution of equation (4.1) with initial value for the minimum of v (so the probability term of (4.5) depends implicitly on v). For $v = 0$, we have from Lemma 3.6 that there exists $C_5 = C_5(\alpha,\beta) > 0$ such that

(4.6)

$$\phi_0\Big(a, b; t\Big) \le \begin{cases} C_5 \exp\Big(-\frac{\pi^2 t}{8(a+b)^2}\Big), & \text{for all } a \ge b > 0,\ \ t > 0. \\ C_5\, b^{2(\alpha+\beta)}\, t^{-\alpha-\beta} \exp\Big(-\frac{\pi^2 t}{2(a+b)^2}\Big), & \text{for all } b \le a < 2b,\ \ t \ge (a+b)^2. \end{cases}$$

We distinguish the two cases, $\beta \ge 0$ and $\beta < 0$, in bounding $\phi_v\Big(a, b; t\Big)$.

Lemma 4.2. *Recall (4.5). If $\beta \ge 0$, we have for all $0 \le v < a$ and $b, t > 0$*

$$\phi_v\Big(a, b; t\Big) \le \begin{cases} \Big(\frac{a}{a-v}\Big)^{\beta} \phi_0\Big(a, b; t\Big), \\ 2^{\beta}\, \phi_0\Big(2a, b; t\Big). \end{cases} \tag{4.7}$$

Lemma 4.2 together with (4.6) give a uniform estimate for $\phi_v(a, b; t)$ in the case $\beta \ge 0$. In the case that $a/v \gg 1$, the first estimate of Lemma 4.2 is sharper, whereas the second deals with the case that v is nearby to a.

Proof of Lemma 4.2. We prove the two estimates in the same way. Let (Z_t) be the solution of the following equation:

$$(4.8)\qquad \begin{cases} dZ_t = 2\sqrt{Z_t}\,dW_t + \Big(2\overline{\alpha}\mathbb{1}_{(0\le t\le b)} + 2\beta\mathbb{1}_{(t\ge b+v)}\Big)\,dt, \\ Z_0 = 0, \\ Z_t \equiv 0, \qquad t \ge \zeta_Z \stackrel{\text{def}}{=} \inf\{t > b : Z_t = 0\}, \end{cases}$$

where (W_t) is a real valued Brownian motion. It follows from Proposition 2.1 that

$$(4.9)\qquad \phi_v\Big(a, b; t\Big) = \mathbb{P}\Big(\int_0^\infty Z_s ds > t;\ \zeta_Z < a + b\Big).$$

Let $(\widehat{W}_t)$ be an independent Brownian motion and consider a process (Θ_t) which is the solution of

$$(4.10)\qquad \begin{cases} d\Theta_t = 2\sqrt{\Theta_t}\,d\widehat{W}_t + 2\beta\mathbb{1}_{(b\le t\le b+v)}\,dt, \\ \Theta_s = 0, \ \ 0 \le s \le b, \\ \Theta_t \equiv 0, \qquad t \ge \zeta_\Theta \stackrel{\text{def}}{=} \inf\{t > b : \Theta_t = 0\}. \end{cases}$$

Define

$$(4.11)\qquad V_t \stackrel{\text{def}}{=} Z_t + \Theta_t, \qquad t \ge 0.$$

Applying the additivity of the squared Bessel processes (cf. [22, Chap. XI]) to the two independent processes Z and Θ given respectively by (4.8) and (4.10), we have for some Brownian motion $(\gamma(t), t \ge 0)$

$$(4.12)\qquad \begin{cases} dV_t = 2\sqrt{V_t}\,d\gamma_t + \Big(2\overline{\alpha}\mathbb{1}_{(0\le t\le b)} + 2\beta\mathbb{1}_{(t\ge b)}\Big)\,dt, \\ V_0 = 0, \\ V_t \equiv 0, \qquad t \ge \zeta_V \stackrel{\text{def}}{=} \inf\{t > b : V_t = 0\}, \end{cases}$$

so that the law of the process V does not depend on v. Observe that (4.9) is also valid for $v = 0$ by replacing the process Z by the process V, which means for all $x > 0$,

$$(4.13)\qquad \phi_0\Big(x, b; t\Big) = \mathbb{P}\Big(\int_0^\infty V_s ds > t;\ \zeta_V < x + b\Big).$$

Now, let $x \ge a$. Use of (4.11) shows that the probability term of (4.13) is

$$\begin{aligned} &\ge \mathbb{P}\Big(\int_0^\infty Z_s ds > t;\ \zeta_Z < a + b;\ \zeta_\Theta < x + b\Big) \\ &= \mathbb{P}\Big(\int_0^\infty Z_s ds > t;\ \zeta_Z < a + b\Big)\,\mathbb{P}\Big(\zeta_\Theta < x + b\Big) \\ (4.14)\qquad &= \phi_v\Big(a, b; t\Big)\,\mathbb{P}\Big(\zeta_\Theta < x + b\Big). \end{aligned}$$

It remains to compute the probability term in (4.14). Since $(\widehat{\Theta}_s \equiv \Theta_{s+b}, 0 \le s \le \zeta_\Theta - b)$ is a process of law $BESQ_0^{2\beta}$ on $[0, v]$ and of law $BESQ^0$ on (v, ∞) till its first hitting time at 0, we see that

$$\begin{aligned}
\mathbb{P}\Big(\zeta_\Theta < x + b\Big) &= \mathbb{E}\Big(\mathbb{P}\Big(BESQ_r^0 \text{ hits } 0 \text{ before time } x - v \,\Big|\, r = \widehat{\Theta}_v\Big)\Big) \\
&= \mathbb{E} \exp\Big(- \frac{\widehat{\Theta}_v}{2(x-v)}\Big) \\
&= \Big(\frac{x}{x-v}\Big)^{-\beta},
\end{aligned}$$

where the second equality is due to the fact that the $BESQ_r^0$ hits 0 before time u with probability $\exp(-\frac{r}{2u})$ (this can be seen, e.g., from (2.8) by letting $\lambda \to 0$ there), and the third follows from (2.7) by taking $\delta = 2\beta$ and by letting $\lambda \to 0$. This, by taking $x = a$ and $x = 2a$, combining with (4.14) yields the two estimates of (4.7) and completes the proof of Lemma. □

For the case $\beta < 0$, we have

Lemma 4.3. *Let $\beta < 0$. Recall (4.5). There exists a constant $C_9 = C_9(\alpha, \beta) > 0$ such that for all $a \ge b \vee v$, $v \ge 0$ and $b, t > 0$ that*

$$\phi_v\Big(a, b; t\Big) \le C_9 \exp\Big(- \frac{\pi^2 t}{16(a+b)^2}\Big). \tag{4.15}$$

Moreover, for all $b \le a \le 2b, 0 \le v < a$ and $t \ge (a+b)^2$, we have

$$\phi_v\Big(a, b; t\Big) \le C_9 \Big(\frac{t}{(a+b)^2}\Big)^{-(\alpha+\beta)} \exp\Big(- \frac{\pi^2 t}{2(a+b)^2}\Big). \tag{4.16}$$

Proof of Lemma 4.3. We use the same idea as in the previous proof, but the details are a little more complicated. Recall (4.12)–(4.13) for the process V. Let $(\widehat{\gamma}_t)$ be an independent Brownian motion and define in this proof the process (Θ_t) as the solution of (recalling $-\beta$ is positive)

$$\begin{cases} d\Theta_t = 2\sqrt{\Theta_t}\, d\widehat{\gamma}_t - 2\beta \mathbb{1}_{(b \le t \le b+v)} dt, \\ \Theta_s = 0, \ \ 0 \le s \le b, \\ \Theta_t \equiv 0, \qquad t \ge \zeta_\Theta \stackrel{\text{def}}{=} \inf\{t > b : \Theta_t = 0\}. \end{cases} \tag{4.17}$$

Therefore the two process Θ and V are independent. Define in this proof

$$Z_t \stackrel{\text{def}}{=} V_t + \Theta_t, \qquad t \ge 0. \tag{4.18}$$

Use of the additivity of $BESQ$ for (4.12) and (4.17) implies that the process Z verifies the equation (4.8) with some Brownian motion W, therefore (4.9) again holds. It follows from (4.13) that, with $0 < \sigma < 1$ being a constant whose value will be given ultimately,

$$\begin{aligned}
\phi_v\Big(a,b;t\Big) &= \mathbb{P}\Big[\int_0^\infty V_s\,ds + \int_0^\infty \Theta_s\,ds > t; \zeta_V < a+b; \zeta_\Theta < a+b\Big] \\
&\le \mathbb{P}\Big[\int_0^\infty \Theta_s\,ds > \sigma t; \zeta_\Theta < a+b\Big] + \mathbb{P}\Big[\int_0^\infty V_s\,ds > t - \int_0^\infty \Theta_s\,ds; \\
&\qquad\qquad \zeta_V < a+b;\ \int_0^\infty \Theta_s\,ds \le \sigma t;\ \zeta_\Theta < a+b\Big] \\
&= \mathbb{P}\Big[\int_0^\infty \Theta_s\,ds > \sigma t; \zeta_\Theta < a+b\Big] + \mathbb{E}\Big[\phi_0\big(a,b;t - \int_0^\infty ds\Theta_s\big)\mathbb{1}_{(\int_0^\infty \Theta_s\,ds \le \sigma t;\, \zeta_\Theta < a+b)}\Big]
\end{aligned}$$

$$\equiv I_3 + I_4, \tag{4.19}$$

with the obvious notation. Let $\widehat{\Theta}_s \stackrel{\text{def}}{=} \Theta_{s+b}, s \ge 0$. Then $\widehat{\Theta}$ is a $BESQ_0^{2|\beta|}$ on $[0,v]$ and a $BESQ^0$ on (v,∞), absorbed at its hitting time at 0. Using successively the Markov property of $\widehat{\Theta}$ at v, (2.8) and (2.7) gives the following equalities

$$\begin{aligned}
&\mathbb{E}\Big(e^{-\frac{\lambda^2}{2}\int_0^\infty \Theta_s ds}\ \mathbb{1}_{(\zeta_\Theta < a+b)}\Big) \\
&= \mathbb{E}\Big(e^{-\frac{\lambda^2}{2}\int_0^v \widehat{\Theta}_s ds}\ \mathbb{E}\Big[e^{-\frac{\lambda^2}{2}\int_0^\infty BESQ_r^0(s)ds}\ \mathbb{1}_{(BESQ_r^0(a-v)=0)}\Big|r = \widehat{\Theta}_v\Big]\Big) \\
&= \mathbb{E}\Big(\exp\Big(-\frac{\lambda^2}{2}\int_0^v \widehat{\Theta}_s ds - \frac{\widehat{\Theta}_v}{2}\lambda\coth(\lambda(a-v))\Big)\Big) \\
&= \Big(\frac{\sinh\lambda(a-v)}{\sinh\lambda a}\Big)^{|\beta|}.
\end{aligned}$$

It follows that

$$\mathbb{E}\Big(e^{\frac{\lambda^2}{2}\int_0^\infty \Theta_s ds}\ \mathbb{1}_{(\zeta_\Theta < a+b)}\Big) = \Big(\frac{\sin\lambda(a-v)}{\sin\lambda a}\Big)^{|\beta|}, \qquad 0 < \lambda < \pi/a. \tag{4.20}$$

Now, we are going to show (4.15). Take $\sigma = 1/2$ in (4.19). Applying Chebychev's inequality to (4.20) with $\lambda = \frac{\pi}{2a}$ gives

$$I_3 \le \exp\Big(-\frac{\pi^2 t}{16a^2}\Big),$$

and by (4.6),

$$I_4 \le \phi_0\Big(a,b;t/2\Big) \le C_5\ \exp\Big(-\frac{\pi^2 t}{16(a+b)^2}\Big),$$

implying (4.15) if $C_9 \ge 1 + C_5$.

It remains to consider the case $b \le a \le 2b, t \ge (a+b)^2$. Let $\sigma = \frac{a+b/2}{a+2b/3} \in [9/10, 15/16]$. Again applying Chebychev's inequality to (4.20) with $\lambda = \pi/(a+b/2)$ gives

$$I_3 \le \Big(\sin\frac{\pi a}{(a+b/2)}\Big)^{-|\beta|}\exp\Big(-\frac{\pi^2\,t}{2(a+2b/3)^2}\Big) \le (5/2)^{|\beta|}\exp\Big(-\frac{\pi^2\,t}{2(a+2b/3)^2}\Big). \tag{4.21}$$

Applying (4.6) to I_4 shows that

$$\begin{aligned} I_4 &\le C_5 b^{2(\alpha+\beta)} \max_{1/16\le x\le 1/10} x^{-\alpha-\beta} t^{-\alpha-\beta} e^{-\frac{\pi^2 t}{2(a+b)^2}} \mathbb{E}\Big[e^{-\frac{\pi^2}{2(a+b)^2}\int_0^\infty \Theta_s ds} \mathbb{1}_{(\zeta_\Theta < a+b)}\Big] \\ &\le C_{11}(\alpha,\beta)\Big(\frac{t}{(a+b)^2}\Big)^{-\alpha-\beta} \exp\Big(-\frac{\pi^2 t}{2(a+b)^2}\Big), \end{aligned}$$

which, in view of (4.19) and (4.21) implies (4.16), and we end the proof of Lemma 4.3 by taking $C_9 = (1+C_5)\vee(C_{10}+C_{11})$. □

Combining (4.6) and Lemmas 4.2 and 4.3 gives the following

Corollary 4.4. *Recall (4.1). There exists a constant $C_{12} = C_{12}(\alpha,\beta)$ only depending on α,β such that for all $a\ge b>0, 0\le v<a, t\ge (a+b)^2$*

$$\mathbb{P}\Big(T_Y(-a) > T_Y(b) > t\Big) \le C_{12}\exp\Big(-\frac{\pi^2 t}{32(a+b)^2}\Big). \tag{4.22}$$

Furthermore, if $0 < v \le a/2 \le b$, we have

$$\mathbb{P}\Big(T_Y(-a) > T_Y(b) > t\Big) \le C_{12} b^{2(\alpha+\beta)} t^{-(\alpha+\beta)} \exp\Big(-\frac{\pi^2 t}{2(a+b)^2}\Big). \tag{4.23}$$

5. Proofs of Theorems 1.1–3.

Recall (1.1). Let us at first establish a zero-one law:

Lemma 5.1. *Let $f>0$ be a nondecreasing function. The events $\Big\{\sup_{0\le s\le t} X_s > \sqrt{t}\, f(t),\ i.o.\Big\}$, $\Big\{\sup_{0\le s\le t} X_s < \frac{\sqrt{t}}{f(t)},\ i.o.\Big\}$ and $\Big\{\sup_{0\le s\le t} |X_s| < \frac{\sqrt{t}}{f(t)},\ i.o.\Big\}$ have probabilities 0 or 1.*

Proof of Lemma 5.1. The proof relies on the ergodicity of the Brownian scaling transformation. Precisely, for fixed $c>0$, define the processes $B^{(c)}$ and $X^{(c)}$ by $B^{(c)}_t \overset{\text{def}}{=} \frac{1}{\sqrt{c}} B_{ct}$ and $X^{(c)}_t \overset{\text{def}}{=} \frac{1}{\sqrt{c}} X_{ct}$ for $t\ge 0$. Therefore, we have that (see e.g. [22, Exercise XIII.1.17])

$$\big(B^{(c)},\, B\big) \xrightarrow{(d)} \big(\widehat{B},\, B\big), \qquad c\to\infty, \tag{5.1}$$

where $\widehat{B}$ denotes an independent Brownian motion, and $\xrightarrow{(d)}$ means convergence in law in the space of continuous functions $\Omega = \mathcal{C}(\mathbb{R}_+,\mathbb{R})$, endowed with the topology of the uniform convergence on every compact set. Let A be an event determined by $\mathbb{1}_A = F(B_t, t \geq 0)$ with $F : \Omega \to \{0,1\}$ a measurable function. Define A_c by $\mathbb{1}_{A_c} = F(B_t^{(c)}, t \geq 0)$. By approximating $F(B_t, t \geq 0)$ by bounded continuous functions in $L^1\big(\Omega, \mathbb{P}, \sigma(B)\big)$, we deduce from (5.1) that

$$\lim_{c\to\infty} \mathbb{P}\Big(A \cap A_c\Big) = \mathbb{P}\Big(A\Big)^2. \tag{5.2}$$

Now, we can prove Lemma 5.1 by using (5.2) and the fact that $\sigma\{X_t, t \geq 0\} = \sigma\{B_t, t \geq 0\}$ (which follows from the pathwise uniqueness of (1.1), see [4]) as follows: consider for example $A = \Big\{ \sup_{0\leq s\leq t} X_s > \sqrt{t}\, f(t), \text{ i.o.}\Big\}$ (the other two events can be treated in the same way), and $A_c \overset{\text{def}}{=} \Big\{ \sup_{0\leq s\leq t} X_s^{(c)} > \sqrt{t}\, f(t), \text{ i.o.}\Big\}$. Using the monotonicity of f, we have that

$$A \subset A_c\,, \qquad c \geq 1, \tag{5.3}$$

which in view of (5.2) implies that

$$\mathbb{P}\Big(A\Big) = \lim_{c\to\infty} \mathbb{P}\Big(A \cap A_c\Big) = \mathbb{P}\Big(A\Big)^2,$$

yielding that $\mathbb{P}\Big(A\Big) = 0$ or 1, as desired. □

Proof of Theorem 1.1. The convergence part of this test can be proven in a standard way. Let $t_n \overset{\text{def}}{=} \exp\Big(n/\log n\Big)$ for large n. It is well-known (cf. Csáki [7] for a rigorous justification) that we can limit our attention to the "critical" case

$$\frac{1}{2\overline{\alpha}}\sqrt{\log\log t} \leq f(t) \leq \frac{2}{\overline{\alpha}}\sqrt{\log\log t}, \qquad t \geq t_0. \tag{5.4}$$

Therefore it is easy to see that

$$\int^\infty \frac{dt}{t} f(t) \exp\Big(- \frac{\overline{\alpha}^2 f^2(t)}{2}\Big) < \infty \Longrightarrow \sum_n \frac{1}{f(t_n)} \exp\Big(- \frac{\overline{\alpha}^2 f^2(t_n)}{2}\Big) < \infty. \tag{5.5}$$

Recall (3.1). Using scaling and (3.2), we have

$$\mathbb{P}\Big(\sup_{0\leq s\leq t_{n+1}} X_s > \sqrt{t_n}\, f(t_n)\Big) = \mathbb{P}\Big(T_X(1) < \frac{t_{n+1}}{t_n f^2(t_n)}\Big) \leq \frac{C_{13}}{f(t_n)} \exp\Big(- \frac{\overline{\alpha}^2 f^2(t_n)}{2}\Big),$$

which is summable by (5.5). It follows that almost surely for all large n, $\sup_{0\leq s\leq t_{n+1}} X_s \leq \sqrt{t_n}\, f(t_n)$. In view of the monotonicity, we have for all $t \in [t_n, t_{n+1})$, $\sup_{0\leq s\leq t} X_s \leq \sup_{0\leq s\leq t_{n+1}} X_s \leq \sqrt{t_n}\, f(t_n) \leq \sqrt{t}\, f(t)$, proving the convergence part of Theorem 1.1.

To treat the divergence part of Theorem 1.1, we again assume (5.4). Let $\tilde f(t) \overset{\text{def}}{=} f(t^2)$. Define for $i \geq 2$, $r_i \overset{\text{def}}{=} \exp\Big(i/\log i\Big)$ and

$$A_i \overset{\text{def}}{=} \Big\{ \frac{r_{i-1}^2}{\tilde f^2(r_i)} < T_X(r_i) < \frac{r_i^2}{\tilde f^2(r_i)} \Big\}.$$

Observe that $r_i/r_{r-1} - 1 \sim 1/\log i$ and $\frac{1}{3\overline{\alpha}} \log i \leq \tilde f(r_i) \leq \frac{3}{\overline{\alpha}} \log i$ for large $i \geq i_0$. It follows from (3.2) that

$$\begin{aligned} \mathbb{P}\Big(A_i\Big) &= \mathbb{P}\Big(T_X(1) < \tilde f^{-2}(r_i)\Big) - \mathbb{P}\Big(T_X(1) < \frac{r_{i-1}^2}{r_i^2}\tilde f^{-2}(r_i)\Big) \\ &\sim \frac{C_1}{\tilde f(r_i)} \exp\Big(- \frac{\overline{\alpha}^2 \tilde f^2(r_i)}{2}\Big) - \frac{C_1}{\tilde f(r_i)} \exp\Big(- \frac{\overline{\alpha}^2 \tilde f^2(r_i) r_i^2}{2 r_{i-1}^2}\Big) \\ &\geq C_1(1 - e^{-1/3}) \frac{1}{\tilde f(r_i)} \exp\Big(- \frac{\overline{\alpha}^2 \tilde f^2(r_i)}{2}\Big), \qquad i \geq i_0. \end{aligned} \tag{5.6}$$

It then follows that

$$\sum_i \mathbb{P}\Big(A_i\Big) = \infty. \tag{5.7}$$

We shall apply the Borel-Cantelli lemma to show that

$$\mathbb{P}\Big(A_i;\ \text{i.o.}\ \Big) > 0. \tag{5.8}$$

To this end, let us estimate the second moment term of $\mathbb{P}\Big(A_i \cap A_j\Big)$ for $i_0 \leq i < j$. Recall (1.1). Applying the strong Markov property for the Brownian motion B at the stopping time $T_X(r_i)$ gives

$$\widehat X_t \overset{\text{def}}{=} X_{t+T_X(r_i)} - r_i = \widehat B_t + \alpha \widehat M_t - \beta\Big(\widehat I_t - (r_i + I_{T_X(r_i)})\Big)^+, \qquad t \geq 0, \tag{5.9}$$

where $\widehat B$ is a Brownian motion starting from 0, independent of $\mathcal{F}^X_{T_X(r_i)}$ ($(\mathcal{F}^X_t, t \geq 0)$ being the natural filtration of X), and $\widehat M_t$ and $\widehat I_t$ are respectively the past maximums of $\widehat X_t$ and of $-\widehat X_t$. Define similarly $\widehat T(r)$ for $r > 0$. Conditionally on $I_{T_X(r_i)}$, $\widehat T(r)$ is independent of $\mathcal{F}^X_{T_X(r_i)}$.

Notice that $T(r_j) = T_X(r_i) + \widehat T(r_j - r_i)$. Applying (4.2) to $\widehat T(r_j - r_i)$ with $r = r_j - r_i; t = \frac{r_j^2}{\tilde f^2(r_j)}; v = r_i + I_{T_X(r_i)}$ gives

$$\begin{aligned} \mathbb{P}\Big(A_i \cap A_j\Big) &\leq \mathbb{E}\Big[\mathbb{1}_{A_i} \mathbb{P}\Big(\widehat T(r_j - r_i) < \frac{r_j^2}{\tilde f^2(r_j)} \mid \mathcal{F}^X_{T_X(r_i)}\Big)\Big] \\ &\leq \mathbb{P}\Big(A_i\Big) \frac{C_8\, r_j}{(r_j - r_i)\tilde f(r_j)} \exp\Big(- \frac{\overline{\alpha}^2\, (r_j - r_i)^2\, \tilde f^2(r_j)}{2\, r_j^2}\Big). \end{aligned} \tag{5.10}$$

On the other hand, we have from (3.2)

$$\mathbb{P}\Big(A_i\Big) \le \frac{2C_1}{\tilde{f}(r_i)} \exp\Big(-\frac{\overline{\alpha}^2\tilde{f}^2(r_i)}{2}\Big), \qquad i \ge i_0. \tag{5.11}$$

In view of (5.6), (5.10)–(5.11), several lines of elementary calculations show that

$$\mathbb{P}\Big(A_i \cap A_j\Big) \le \begin{cases} C_{14}\mathbb{P}\Big(A_i\Big)\,\mathbb{P}\Big(A_j\Big), & \text{if} \quad j-i \ge \log^2 i, \\ C_{15}\mathbb{P}\Big(A_i\Big) j^{-C_{16}}, & \text{if} \quad \log i \le j-i < \log^2 i, \\ C_{17}\mathbb{P}\Big(A_i\Big)\, e^{-C_{18}(j-i)}, & \text{if} \quad 2 \le j-i < \log i. \end{cases} \tag{5.12}$$

It follows from (5.7) and (5.12) that

$$\liminf_{n\to\infty} \frac{\sum_{2\le i,j\le n} \mathbb{P}\Big(A_i \cap A_j\Big)}{\Big(\sum_{2\le i\le n} \mathbb{P}\Big(A_i\Big)\Big)^2} \le C_{14},$$

which in view of (5.7), according to Kochen and Stone's version of the Borel-Cantelli lemma (cf. [15]) implies $\mathbb{P}\Big(A_i; \text{i.o.}\Big) \ge 1/C_{14} > 0$. This probability in fact equals 1 according to Lemma 5.1. Finally, write $t_i = r_i^2/\tilde{f}^2(r_i)$. Recall $\tilde{f}(t) = f(t^2)$. Observe that on A_i, we have $\sup_{0\le s\le t_i} X_s > r_i = \sqrt{t_i}\,\tilde{f}(r_i) = \sqrt{t_i} f(r_i^2) \ge \sqrt{t_i}\, f(t_i)$. This completes the proof since we have shown $\mathbb{P}\Big(\ A_i;\ \text{i.o.}\ \Big) = 1$. □

Proof of Theorem 1.2. Since the proof is similar to the above one, we just sketch the main steps. First, the convergence part follows from Lemma 3.1 and the monotonicity, and the details are omitted. To prove the divergence part, we only have to treat the critical case

$$\Big(\log t\Big)^{1/(2\overline{\beta})} \le f(t) \le \Big(\log t\Big)^{2/\overline{\beta}}, \qquad t \ge t_0. \tag{5.13}$$

Define in this case

$$F_i \stackrel{\text{def}}{=} \Big\{t_i < T_X(r_i) < t_{i+1}; I(T_X(r_i)) < r_i\, \widehat{f}^2(r_i)\Big\}, \tag{5.14}$$

with $r_i \stackrel{\text{def}}{=} 2^i$, $t_i \stackrel{\text{def}}{=} r_i^2 \widehat{f}^2(r_i)$ and $\widehat{f}(x) \stackrel{\text{def}}{=} f(x^3)$. By changing X to $-X$ and interchanging α and β, we have from (3.2) that

$$\mathbb{P}\Big(T_X(-1) < \epsilon\Big) \sim \frac{2^{\alpha+1/2}\Gamma(\overline{\alpha}+\overline{\beta})}{\overline{\beta}\Gamma(3-\beta-\alpha/2)\Gamma(\overline{\alpha}/2)}\,\epsilon^{1/2} \exp\Big(-\frac{\overline{\beta}^2}{2\epsilon}\Big), \qquad \epsilon \to 0. \tag{5.15}$$

Applying Lemma 3.1 and (5.15) gives

$$\begin{aligned}\mathbb{P}\Big(F_i\Big) &\geq \mathbb{P}\Big(T_X(r_i) > t_i\Big) - \mathbb{P}\Big(T_X(r_i) > t_{i+1}\Big) - \mathbb{P}\Big(I(t_{i+1}) \geq r_i \widehat{f}^2(r_i)\Big)\\ &\geq C_{19}\widehat{f}(r_i)^{-\overline{\beta}} - \mathbb{P}\Big(T_X(-1) < \frac{t_{i+1}}{r_i^2\, \widehat{f}^4(r_i)}\Big)\\ &\geq \frac{C_{19}}{2}\widehat{f}(r_i)^{-\overline{\beta}}. \qquad (5.16)\end{aligned}$$

On the other hand, we have from Lemma 3.1 that

$$\mathbb{P}\Big(F_i\Big) \leq \mathbb{P}\Big(T_X(r_i) > t_i\Big) \leq C_{20}\widehat{f}(r_i)^{-\overline{\beta}}. \qquad (5.17)$$

For $j \geq i+2$, we recall (5.9) to bound $\mathbb{P}\Big(F_i \cap F_j\Big)$ in a similar way as to the proof of Theorem 1.1, by using (4.3) instead of (4.2). It can be shown that

$$\mathbb{P}\Big(F_i \cap F_j\Big) \leq C_{21}\mathbb{P}\Big(F_i\Big)\Big(e^{-C_{22}\widehat{f}^2(r_j)} + \frac{r_i}{r_j} + \Big(\frac{r_i}{r_j} + \frac{1}{\widehat{f}(r_j)}\Big)^{\overline{\beta}}\Big). \qquad (5.17)$$

From (5.16)–(5.17), the proof of the divergence part of Theorem 1.2 can be completed in a similar way to that of Theorem 1.1. The details are omitted. □

Proof of Theorem 1.3. Similarly, we only treat the divergence part. We can assume without any loss of generality that

$$\frac{2}{\pi}\sqrt{\log\log t} \leq f(t) \leq \frac{4}{\pi}\sqrt{\log\log t}, \qquad t \geq t_0. \qquad (5.18)$$

Define

$$G_i \stackrel{\text{def}}{=} \Big\{T_X(-r_i) > T_X(r_i) > t_i; T_X(r_i) < t_{i+1}\Big\}, \qquad (5.19)$$

with $r_i \stackrel{\text{def}}{=} \exp\Big(i/\log i\Big)$, and $t_i \stackrel{\text{def}}{=} r_i^2\widehat{f}^2(r_i)$, and $\widehat{f}(x) \stackrel{\text{def}}{=} f(x^3)$. It follows from Proposition 3.3 and our choices of r_i, t_i that

$$\mathbb{P}\Big(G_i\Big) \asymp \widehat{f}(r_i)^{-2(\alpha+\beta)} \exp\Big(-\frac{\pi^2\widehat{f}^2(r_i)}{8}\Big), \qquad (5.20)$$

where $f(x) \asymp g(x)$ means that $0 < \liminf_{x\to\infty} f(x)/g(x) \leq \limsup_{x\to\infty} f(x)/g(x) < \infty$. To estimate the second moment $\mathbb{P}\Big(G_i \cap G_j\Big)$, we recall (5.9). Use of the hitting time $\widehat{T}(x)$ at x by the process $\widehat{X}$ gives

$$\mathbb{P}\Big(G_i \cap G_j\Big) \leq \mathbb{E}\Big[\mathbb{1}_{G_i}\,\mathbb{P}\Big(\widehat{T}(-r_j - r_i) > \widehat{T}(r_j - r_i) > t_j - t_{i+1} \,\Big|\, \mathcal{F}^X_{T_X(r_i)}\Big)\Big],$$

which, by applying Corollary 4.4 to $T_Y(x) = \widehat{T}(x)$ with $a = r_j + r_i$, $b = r_j - r_i$, $t = t_j - t_{i+1}$ and $v = r_i + I(T_X(r_i)) \in (r_i, 2r_i)$ leads to the following estimate

$$\mathbb{P}\Big(G_i \cap G_j\Big) \leq \begin{cases} C_{23}\mathbb{P}\Big(G_i\Big)\,\mathbb{P}\Big(G_j\Big), & \text{if } j-i \geq \log i; \\ C_{24}\mathbb{P}\Big(G_i\Big)e^{-C_{25}(j-i)}, & \text{if } 2 \leq j-i \leq \log i. \end{cases} \tag{5.21}$$

From (5.20)–(5.21), the proof of the divergence part of Theorem 1.3 can be completed in exactly the same way as in the proof of Theorem 1.1. We omit the details. □

References

[1] Bingham, N.N., Goldie, C.M. and Teugels, J.L.: *Regular Variation.* Cambridge University Press (1987).

[2] Carmona, Ph., Petit, P. and Yor, M.: Some extensions of the arc sine law as partial consequences of the scaling property for Brownian motion. *Probab. Th. Rel. Fields* 100 (1994) 1-29.

[3] Carmona, Ph., Petit, P. and Yor, M.: Beta variables as times spent in $[0, \infty[$ by certain perturbed Brownian motions. *J. London Math. Soc.* 58 (1998) 239–256.

[4] Chaumont, L. and Doney, R.A.: Pathwise uniqueness for perturbed versions of Brownian motion and reflected Brownian motion. *Probab. Th. Rel. Fields* 113 (1999) 519–534.

[5] Chaumont, L. and Doney, R.A.: Some calculations for doubly perturbed Brownian motion. *Stoch. Proc. Appl.* (to appear, 1999).

[6] Csáki, E.: On the lower limits of maxima and minima of Wiener process and partial sums. *Z. Wahrsch. verw. Gebiete.* 43 (1978) 205–221.

[7] Csáki, E.: An integral test for the supremum of Wiener local time. *Probab. Th. Rel. Fields.* 83 (1989) 207–217.

[8] Csörgő, M. and Révész, P.: *Strong Approximations in Probability and Statistics.* (1981) Akadémiai Kiadó, Budapest and Academic press, New York.

[9] Davis, B.: Weak limits of perturbed Brownian motion and the equation $Y_t = B_t + \sigma \sup\{Y_s : s \leq t\} + \beta \inf\{Y_s : s \leq t\}$. *Ann. Probab.* 24 (1996) 2007–2017.

[10] Davis, B.: Brownian motion and random walk perturbed at extrema. *Probab. Th. Rel. Fields* 113 (1999) 501–518.

[11] Doney, R.A.: Some calculations for perturbed Brownian motion. *Sém. Probab. XXXII* (Eds: J. Azéma, M. Émery, M. Ledoux and M. Yor) Lect. Notes Math. 1686 (1998) pp. 231–236. Springer Berlin.

[12] Doney, R.A., Warren, J. and Yor, M.: Perturbed Bessel processes. (Ibid pp. 237–249)

[13] Getoor, R.K.: The Brownian escape process. *Ann. Probab.* 7 (1979) 864–867.

[14] Jeulin, Th.: Ray-Knight's theorem on Brownian local times and Tanaka's formula. *Sem. Stoch. Proc.* (Eds: Çinlar, E., Chung, K.L. and Getoor, R.K.) (1983) pp 131–142, Boston, Birkhauser 1984.

[15] Kochen, S.B. and Stone, C.J.: A note on the Borel–Cantelli lemma. *Illinois J. Math.* 8 (1964) 248–251.

[16] Le Gall, J.F.: L'équation stochastique $Y_t = B_t + \alpha M_t^Y + \beta I_t^Y$ comme limite des équations de Norris-Rogers-Williams. (unpublished notes, 1986)

[17] Le Gall, J.F. and Yor, M.: Enlacement du mouvement brownien autour des courbes de l'espace. *Trans. Amer. Math. Soc.* 317 (1990) 687-722.

[18] McGill, P.: Markov properties of diffusion local times: a martingale approach. *Adv. Appl. Probab.* 14 (1982) 789–810.

[19] Norris, J.R., Rogers, L.C.G. and Williams, D.: Self-avoiding random walk: a Brownian motion model with local time drift. *Probab. Th. Rel. Fields* 74 (1987) 271-287.

[20] Perman, M. and Werner, W.: Perturbed Brownian motions. *Probab. Th. Rel. Fields* 108 (1997) 357–383.

[21] Petit, F.: *Sur le temps passé par le mouvement brownien au-dessus d'un multiple de son supremum et quelques extensions de la loi de l'arc sinus.* Part of a Thèse de Doctorat, Université Paris VII (1992).

[22] Revuz, D. and Yor, M.: *Continuous Martingales and Brownian Motion.* (2nd edition) Springer, Berlin (1994).

[23] Révész, P.: *Random Walk in Random and Non-Random Environment.* World Scientific press. Singapore, London (1990).

[24] Shi, Z. and Werner, W.: Asymptotics for occupations times of half-lines by stable processes and perturbed reflecting Brownian motion. *Stochastics* 55 (1995) 71-85.

[25] Tóth, B.: The "true" self-avoiding walk with bond repulsion in $\mathbb{Z}$: limit theorems. *Ann. Probab.* 23 (1995) 1523–1556.

[26] Tóth, B.: "True" self-avoiding walk with generalized bond repulsion in $\mathbb{Z}$. *J. Stat. Phys.* 77 (1994) 17–33.

[27] Tucker, H.G.: On a necessary and sufficient condition that an infinitely divisible distribution be absolutely continuous. *Amer. Math. Soc. Trans.* 118 (1965) 316-330.

[28] Werner, W.: Some remarks on perturbed reflecting Brownian motion. *Sém. Probab. XXIX* (eds: J. Azéma, M. Émery, P.A. Meyer and M. Yor) Lect. Notes Math. 1613 (1995) 37-43 Springer, Berlin.

[29] Yor, M.: *Some aspects of Brownian motion, Part I: Some Special Functionals.* Lect. Notes. ETH Zürich, Birkhäuser, Basel (1992).

[30] Yor, M.: *Local Times and Excursions for Brownian Motion: A Concise Introduction.* Lecciones en Metemáticas. Número I. Universidao Central de Venezuela. Caracas (1995).