Moderate deviations for diffusions with Brownian potentials

Yueyun Hu[1] and Zhan Shi[1]

Abstract. We present precise moderate deviation probabilities, in both quenched and annealed settings, for a recurrent diffusion process with a Brownian potential. Our method relies on fine tools in stochastic calculus, including Kotani's lemma and Lamperti's representation for exponential functionals. In particular, our result for quenched moderate deviations is in agreement with a recent theorem of Comets and Popov [3] who studied the corresponding problem for Sinai's random walk in random environment.

Keywords. Moderate deviation, diffusion with random potential, Brownian valley.

AMS Classification 2000. 60K37, 60F10.

1 Introduction

Let $W := (W(x), x \in \mathbb{R})$ be a one-dimensional Brownian motion defined on $\mathbb{R}$ with $W(0) = 0$. Let $(\beta(t), t \geq 0)$ be another one-dimensional Brownian motion independent of $W(\cdot)$. Following Brox [2] and Schumacher [20], we consider the equation: $X(0) = 0$,

$$\mathrm{d}X(t) = \mathrm{d}\beta(t) - \frac{1}{2} W'(X(t))\, \mathrm{d}t, \qquad t \geq 0. \tag{1.1}$$

The solution X of (1.1) is called a diffusion with random potential W. The rigorous meaning of (1.1) can be given in terms of infinitesimal generator: Conditioning on each realization $\{W(x), x \in \mathbb{R}\}$, the process X is a real-valued diffusion with generator

$$\frac{1}{2} \mathrm{e}^{W(x)} \frac{\mathrm{d}}{\mathrm{d}x} \left(\mathrm{e}^{-W(x)} \frac{\mathrm{d}}{\mathrm{d}x} \right).$$

[1]Laboratoire de Probabilités et Modèles Aléatoires (CNRS UMR–7599), Université Paris VI, 4 Place Jussieu, F–75252 Paris Cedex 05, France. E-mails: hu@ccr.jussieu.fr, zhan@proba.jussieu.fr

Another representation of X by time change is given in Section 4.

The process X has been used in modelling some random phenomena in physics ([11]). It is also related to random walk in random environment ([18], [7], [27]). See [25] and [21] for recent surveys.

We denote by P and E the probability and the expectation with respect to the potential W, and by $\mathbf{P}_\omega$ and $\mathbf{E}_\omega$ the quenched probability and the quenched expectation ("quenched" means the conditioning with respect to the potential W). The total (or annealed) probability is $\mathbb{P} \stackrel{\text{def}}{=} P(\mathrm{d}\omega) \otimes \mathbf{P}_\omega$.

The typical long-time behaviour of $X(t)$ is described by a result of Brox [2], which is the continuous-time analogue of Sinai [22]'s well known theorem for recurrent random walk in random environment: under the total probability $\mathbb{P}$,

$$\frac{X(t)}{\log^2 t} \xrightarrow{(d)} b(1), \qquad t \to \infty,$$

where $\xrightarrow{(d)}$ denotes convergence in distribution, and $b(1)$ is a non-degenerate random variable whose distribution is known.

It is interesting to study the deviation probabilities:

$$\mathbf{P}_\omega \left\{ X(t) > v \right\} \quad \text{and} \quad \mathbb{P} \left\{ X(t) > v \right\}, \qquad t, v \to \infty, \ \ v \gg \log^2 t, \tag{1.2}$$

where here and in the sequel, we write $v \gg f(t)$ with some positive nondecreasing function f to mean that $\lim_{t,v\to\infty} \frac{v}{f(t)} = +\infty$. When v/t converges to a positive constant, this is a large deviation problem, and is solved by Taleb [24] (who actually studies the problem for all drifted Brownian potentials). In particular, it is showed that in this case both probabilities in (1.2) have exponential decays.

We focus on moderate deviation probabilities, i.e., when (v, t) is such that $\log^2 t \ll v \ll t$.

Our first result, which concerns the quenched setting, is in agreement with Theorem 1.2 of Comets and Popov [3] for random walk in random environment. This was indeed the original motivation of the present work.

Theorem 1.1 *We have,*

$$\frac{2\log(t/v)}{v} \log \mathbf{P}_\omega \left\{ X(t) > v \right\} \to -1, \qquad P\text{-a.s.}, \tag{1.3}$$

whenever $v, t \to \infty$ *such that* $v \gg (\log^2 t) \log\log\log t$ *and* $\log\log t = o(\log(t/v))$. *The same result holds for* $\sup_{0\le s\le t} X(s)$ *instead of* $X(t)$.

Loosely speaking, Theorem 1.1 says that in a typical potential W, $\mathbf{P}_\omega \left\{ X(t) > v \right\}$ behaves like $\exp[\,-(1+o(1))\frac{v}{2\log(t/v)}\,]$. However, if we take average over all the realizations of W (i.e., in the annealed setting), the deviation probability will become considerably larger. This is confirmed in our second result stated as follows.

Theorem 1.2 *We have,*

$$\frac{\log^2 t}{v} \log \mathbb{P}\{X(t) > v\} \to -\frac{\pi^2}{8}, \tag{1.4}$$

whenever $v, t \to \infty$ *such that* $v \gg \log^2 t$ *and* $\log v = o(\log t)$. *The same result holds for* $\sup_{0\le s\le t} X(s)$ *instead of* $X(t)$.

When $\log^2 t \le v \le (\log^2 t)(\log\log t)^{1/2}$, the convergence (1.4) has already been obtained in ([9]) by means of the Laplace method. This method, however, fails when v goes to infinity too quickly, for example if $v \gg \log^3 t$.

We say a few words about the proofs of Theorems 1.1 and 1.2. Although the methods adopted in the three parts (Theorem 1.1, upper bound and lower bound in Theorem 1.2) find all their roots in stochastic calculus, they rely on completely different ingredients.

In the proof of Theorem 1.1, we exploit Kotani's lemma as well as other fine tools in the theory of one-dimensional diffusion.

The proof of the upper bound in Theorem 1.2 relies on Lamperti's representation for exponential functionals and on Warren and Yor [26]'s skew-product theorem for Bessel processes. The proof of the lower bound, on the other hand, is based on a bare-hand analysis of pseudo-valleys where the diffusion X spends much time.

The rest of the paper is organized as follows. Section 2 is devoted to some preliminary results for local times of Brownian motion. In Section 3, we introduce Kotani's lemma and prove Theorem 1.1. The main result in Section 4, Theorem 4.1, is a joint arcsine type law for the occupation times of X, which may be of independent interest. This result will be used to prove Theorem 1.2 in Section 5.

Throughout this paper, we write f^{-1} for the inverse of any continuous and (strictly) increasing function f. Unless stated otherwise, for any continuous process ξ, we denote by $T_\xi(x) = \inf\{t \ge 0 : \xi(t) = x\}$, $x \in \mathbb{R}$, the first hitting time of ξ at x.

2 Preliminaries on local times

In this section, we collect a few preliminary results for the local times of Brownian motion. These results will be of use in the rest of the paper.

Let B be a one-dimensional Brownian motion starting from 0. Let $(L(t,x), t \ge 0, x \in \mathbb{R})$ be the family of the local times of B, i.e., for any Borel function $f : \mathbb{R} \to \mathbb{R}_+$, $\int_0^t f(B(s))\,\mathrm{d}s = \int_{-\infty}^{\infty} f(x)L(t,x)\,\mathrm{d}x$. We define

$$\tau(r) \stackrel{\text{def}}{=} \inf\{t > 0 : L(t,0) > r\}, \qquad r \ge 0, \tag{2.1}$$

$$\sigma(x) \stackrel{\text{def}}{=} \inf\{t > 0 : B(t) > x\}, \qquad x \ge 0. \tag{2.2}$$

Denote by BES(δ) (resp. BESQ(δ)) the Bessel process (resp. the squared Bessel process) of dimension δ. We recall that a δ-dimensional squared Bessel process has generator of form

$2x\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \delta\frac{\mathrm{d}}{\mathrm{d}x}$. When δ is an integer, a Bessel process can be realized as the Euclidean norm of an $\mathbb{R}^\delta$-valued Brownian motion. We refer to Revuz and Yor ([19], Chap. XI) for a detailed account of general properties of Bessel and squared Bessel processes, together with the proof of the following result.

Fact 2.1 (First Ray–Knight theorem) *Fix $a > 0$. The process $\{L(\sigma(a), a-x), x \ge 0\}$ is an inhomogeneous strong Markov process starting from 0, which is a* BESQ(2) *on $[0,a]$ and a* BESQ(0) *on (a, ∞).*

The rest of the section is devoted to a few preliminary results for local times of Brownian motion.

Lemma 2.2 *For $b > a \ge 0$ and $v > 0$,*

$$\frac{2}{\pi}\exp\left(-\frac{\pi^2}{8}\frac{v}{(b-a)^2}\right) \le \mathbb{P}\left(\int_a^b L(\sigma(b), x)\,\mathrm{d}x > v\right) \le \frac{4}{\pi}\exp\left(-\frac{\pi^2}{8}\frac{v}{(b-a)^2}\right). \tag{2.3}$$

Proof. By means of the strong Markov property, $\int_a^b L(\sigma(b), x)\,\mathrm{d}x$ is distributed as $\int_0^{b-a} L(\sigma(b-a), x)\,\mathrm{d}x$. According to Fact 2.1 and Revuz and Yor ([19], Corollary XI.1.8), for any $\lambda > 0$, $\mathbb{E}\mathrm{e}^{-\frac{\lambda^2}{2}\int_0^{b-a} L(\sigma(b-a),x)\,\mathrm{d}x} = 1/\cosh(\lambda(b-a))$, which is also the Laplace transform at $\frac{\lambda^2}{2}$ of $T_{|B|}(b-a)$ (the first hitting time of $b-a$ by $|B|$). Thus $\int_a^b L(\sigma(b), x)\,\mathrm{d}x \overset{\text{law}}{=} T_{|B|}(b-a)$. The lemma now follows from the exact distribution of Brownian exit time (Feller [6], p. 342). $\Box$

Lemma 2.3 *Let $b > a > 0$ and $\kappa > 0$, we have*

$$\int_0^{\sigma(a)} (b - B(s))^{(1/\kappa)-2}\,\mathrm{d}s \overset{\text{law}}{=} \Upsilon_{2-2\kappa}\left(2\kappa b^{1/(2\kappa)} \rightsquigarrow 2\kappa(b-a)^{1/(2\kappa)}\right),$$

where $\Upsilon_{2-2\kappa}(x \rightsquigarrow y)$ means the first hitting time of y by a BES$(2-2\kappa)$ *starting from x.*

Proof. Write

$$\int_0^{\sigma(b)} (b - B(s))^{(1/\kappa)-2}\,\mathrm{d}s = \int_0^{\sigma(a)} (b - B(s))^{(1/\kappa)-2}\,\mathrm{d}s + \int_{\sigma(a)}^{\sigma(b)} (b - B(s))^{(1/\kappa)-2}\,\mathrm{d}s.$$

By the strong Markov property, the integrals on the right hand side are independent random variables. Moreover, the second integral is distributed as $(b-a)^{1/\kappa}\int_0^{\sigma(1)} (1 - B(s))^{(1/\kappa)-2}\,\mathrm{d}s$.

On the other hand, according to Getoor and Sharpe ([8], Proposition 5.14), for any $\lambda > 0$,

$$\mathbb{E}\exp\left(-\frac{\lambda^2}{2}\int_0^{\sigma(1)} (1 - B(s))^{(1/\kappa)-2}\,\mathrm{d}s\right) = \frac{2}{\Gamma(\kappa)}(\lambda\kappa)^\kappa K_\kappa(2\kappa\lambda),$$

where K_κ denotes the modified Bessel function of index κ.

Assembling these pieces yields that

$$\mathbb{E}\exp\left(-\frac{\lambda^2}{2}\int_0^{\sigma(a)}(b-B(s))^{(1/\kappa)-2}\,\mathrm{d}s\right)=\left(\frac{b}{b-a}\right)^{1/2}\frac{K_\kappa(2\kappa\lambda b^{1/(2\kappa)})}{K_\kappa(2\kappa\lambda(b-a)^{1/(2\kappa)})}.$$

According to Kent [14], the expression on the right hand side is exactly the Laplace transform at $\lambda^2/2$ of $\Upsilon_{2-2\kappa}\left(2\kappa b^{1/(2\kappa)}\rightsquigarrow 2\kappa(b-a)^{1/(2\kappa)}\right)$. □

Lemma 2.4 *Almost surely,*

$$\frac{1}{r}\sup_{|x|\le u}(L(\tau(r),x)-r)\;\to\;0,$$

whenever $u\to\infty$ *and* $r\gg u\log\log u$.

Proof. By symmetry, we only need to treat the case $0\le x\le u$. It is proved by Csáki and Földes ([4], Lemma 2.1) that for any $\varepsilon>0$,

$$\lim_{r\to\infty}\frac{1}{r}\sup_{0\le x\le r/(\log\log r)^{1+\varepsilon}}(L(\tau(r),x)-r)\;=\;0,\qquad \text{a.s.}$$

Thus, we only have to deal with the case $r\gg u\log\log u$ and $r\le u\log u$.

Let $\varepsilon>0$. We shall prove that almost surely for all $u,r\to\infty$ such that $r\gg u\log\log u$ and $r\le u\log u$,

$$\frac{1}{r}\sup_{0\le x\le u}(L(\tau(r),x)-r)\le\varepsilon. \tag{2.4}$$

To this end, we consider the events

$$A_{k,j}\;\stackrel{\text{def}}{=}\;\left\{\exists\,(u,r)\in[u_k,u_{k+1}]\times[r_j,r_{j+1}]:\frac{256}{\varepsilon^2}u\log\log u\le r\le u\log u,\right.$$
$$\left.\sup_{0\le x\le u}(L(\tau(r),x)-r)>\varepsilon r\right\},$$

with $u_k\stackrel{\text{def}}{=}2^k$, $r_j\stackrel{\text{def}}{=}2^j$ and large $k,j\ge 100$. The desired conclusion (2.4) will follow from the Borel–Cantelli lemma once we show that

$$\sum_{j,\,k\ge 100}\mathbb{P}\,(A_{k,j})<\infty.$$

To prove $\sum_{j,\,k\ge100}\mathbb{P}\,(A_{k,j})<\infty$, we recall a result of Bass and Griffin ([1], Lemma 3.4) saying that there exists a constant $c>0$ such that for all $0<h,x<1$,

$$\mathbb{P}\left\{\sup_{1\le r\le 2}\sup_{0\le z\le h}(L(\tau(r),z)-r)>x\right\}\le c\,\frac{x}{h}\exp\left(-\frac{x^2}{32h}\right).$$

Therefore, by the monotonicity and the scaling property,

$$\begin{aligned}
\mathbb{P}\left(A_{k,j}\right) &\leq \mathbb{P}\left(\sup_{0\le x\le u_{k+1}} \sup_{r_j\le r\le r_{j+1}} (L(\tau(r),x)-r) \geq \varepsilon r_j\right) \\
&= \mathbb{P}\left(\sup_{0\le x\le u_{k+1}/r_j} \sup_{1\le r\le 2} (L(\tau(r),x)-r) \geq \varepsilon\right) \\
&\leq c\frac{\varepsilon r_j}{u_{k+1}} \exp\left(-\frac{\varepsilon^2 r_j}{32 u_{k+1}}\right) \\
&\leq c\frac{\varepsilon r_j}{u_{k+1}} (k\log 2)^{-4}.
\end{aligned}$$

Since $u_k \log\log u_k < r_{j+1} \leq u_{k+1}\log\log u_{k+1}$, this implies that $k\log\log k \leq j \leq k\log k$. Hence

$$\sum_{j,\, k\geq 100} \mathbb{P}\left(A_{k,j}\right) \leq c' \sum_{k\geq 100} k^{-2} \log k < \infty,$$

as desired. □

Lemma 2.5 *Let $r>0$. We have*

$$\mathbb{E}\exp\left(-\lambda\int_0^\infty \mathrm{e}^{-s} L(\sigma(r),-s)\,\mathrm{d}s\right) = \frac{1}{1+r\sqrt{2\lambda}\, I_1(\sqrt{8\lambda})}, \qquad \lambda>0, \tag{2.5}$$

where $I_1(\cdot)$ denotes the modified Bessel function of index 1. Consequently, there exists some constant $c>0$ such that for all $r>c$, $\lambda>0$ and $0<a\leq r$, we have

$$\mathbb{P}\left(\int_{-\infty}^a \mathrm{e}^{-|x|} L(\sigma(r),x)\,\mathrm{d}x > \lambda\right) \leq 3\exp\left(-\frac{\lambda}{8r}\right) + 2\exp\left(-\frac{\lambda}{4ar}\right). \tag{2.6}$$

Proof. By Fact 2.1, $s\mapsto L(\sigma(r),-s)$ for $s\geq 0$ is a BESQ(0) starting from $L(\sigma(r),0) \overset{\text{law}}{=} 2r\mathbf{e}$, where $\mathbf{e}$ is a standard exponential variable with mean 1. Let U_a be a BESQ(0) starting from $a>0$. The Laplace transform of $\int_0^\infty U_a(s)\mathrm{e}^{-s}\,\mathrm{d}s$ is given by the solution of some Sturm–Liouville equation, see Pitman and Yor [17]: We have for all $\lambda>0$,

$$\mathbb{E}\exp\left(-\lambda\int_0^\infty \mathrm{e}^{-s} U_a(s)\,\mathrm{d}s\right) = \exp\left(\frac{a}{2}\psi'_+(0)\right), \tag{2.7}$$

where $\psi'_+(0)$ is the right-derivative of the convex function ψ at 0, and ψ is the unique solution, decreasing, non-negative, of the Sturm–Liouville equation: $\psi(0)=1$,

$$\frac{1}{2}\psi''(x) = \lambda \mathrm{e}^{-x}\psi(x), x>0.$$

Elementary computations (Pitman and Yor [17], p. 435) show that

$$\psi(x) = I_0(\sqrt{8\lambda}\, \mathrm{e}^{-x/2}), \qquad x \geq 0,$$

which implies that $\psi'_+(0) = -\sqrt{2\lambda} I_1(\sqrt{8\lambda})$, where I_1 denotes the modified Bessel function of first kind of index 1. Plug this into (2.7) and integrate with respect to $\mathbf{e}$ give the Laplace transform (2.5). By the analytic continuation, we have for all sufficiently small $\lambda > 0$ (depending on r),

$$\mathbb{E} \exp\left(\lambda \int_0^\infty \mathrm{e}^{-s} L(\sigma(r), -s)\, \mathrm{d}s\right) = \frac{1}{1 - r\sqrt{2\lambda} J_1(\sqrt{8\lambda})},$$

where $J_1(\cdot)$ is the Bessel function of index 1. Since $J_1(x) \sim x/2$ when $x \to 0$, there exists some large $c > 0$ such that for all $r > c$, we have

$$\mathbb{E} \exp\left(\frac{1}{4r} \int_0^\infty \mathrm{e}^{-s} L(\sigma(r), -s)\, \mathrm{d}s\right) = \frac{1}{1 - \sqrt{\frac{r}{2}} J_1(\sqrt{\frac{2}{r}})} < 3.$$

This implies that for all $r > c$ and $\lambda > 0$, we have

$$\mathbb{P}\left(\int_0^\infty \mathrm{e}^{-s} L(\sigma(r), -s)\, \mathrm{d}s > \lambda\right) \leq 3 \exp\left(-\frac{\lambda}{4r}\right). \tag{2.8}$$

On the other hand, $\sup_{0 \leq x \leq a} L(\sigma(r), x)$ is the maximum of a BESQ(2) over $[r - a, r]$. It follows from reflection principle that

$$\begin{aligned} \mathbb{P}\left(\int_0^a L(\sigma(r), x) \mathrm{e}^{-|x|}\, \mathrm{d}x \geq \lambda\right) &\leq \mathbb{P}\left(\sup_{0 \leq x \leq a} L(\sigma(r), x) \geq \frac{\lambda}{a}\right) \\ &\leq 2\mathbb{P}\left(L(\sigma(r), 0) \geq \frac{\lambda}{a}\right) \\ &= 2 \exp\left(-\frac{\lambda}{2ar}\right), \end{aligned}$$

since $L(\sigma(r), 0) \overset{\text{law}}{=} 2r\mathbf{e}$. This together with (2.8) show (2.6) by the simple triangular inequality. □

3 The quenched case: proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1, by means of the so-called Kotani's lemma. Let X be the diffusion process in a Brownian potential W as in (1.1). We define

$$H(v) \overset{\text{def}}{=} \inf\{t > 0 : X(t) > v\}, \qquad v \geq 0.$$

Kotani's lemma (see[13]) gives the Laplace transform of $H(v)$ under the quenched probability $\mathbf{P}_\omega$.

Fact 3.1 (Kotani's lemma) *For $\lambda > 0$ and $v \ge 0$, we have*

$$\mathbf{E}_\omega \left(\mathrm{e}^{-\lambda H(v)} \right) = \exp\left(-2\,\lambda \int_0^v Z(s)\,\mathrm{d}s \right),$$

where $Z(\cdot) = Z_\lambda(\cdot)$ is the unique stationary and positive solution of the equation

$$\mathrm{d}Z(t) = Z(t)\,\mathrm{d}W(t) + \left(1 + \frac{1}{2} Z(t) - 2\lambda\, Z^2(t) \right)\,\mathrm{d}t.$$

Before starting the proof of Theorem 1.1, we study the almost sure behaviors of $\int_0^v Z(s)\,\mathrm{d}s$ as $v \to \infty$ and $\lambda \to 0$. Let us define

$$\begin{aligned} S(x) &\stackrel{\text{def}}{=} \int_1^x \mathrm{e}^{(2/y)+4\lambda y}\,\frac{\mathrm{d}y}{y}, \qquad x > 0, \\ h(x) &\stackrel{\text{def}}{=} S'(S^{-1}(x))S^{-1}(x) = \exp\left(\frac{2}{S^{-1}(x)} + 4\lambda S^{-1}(x) \right), \qquad x \in \mathbb{R}, \end{aligned}$$

where S^{-1} is the inverse function of S. Let B denote, as before, a standard Brownian motion, and let

$$\Phi(t) \stackrel{\text{def}}{=} \int_0^t \frac{\mathrm{d}s}{h^2(B(s))}, \qquad t \ge 0. \tag{3.1}$$

Recall $\tau(\cdot)$ from (2.1).

Lemma 3.2 *We have*

$$\Phi^{-1}(v) = \tau\left((1+o(1)) \frac{v}{\log(1/\lambda)} \right), \qquad \text{a.s.}, \tag{3.2}$$

$$\int_0^v Z(s)\,\mathrm{d}s = (1+o(1))\,\frac{v}{4\lambda\,\log(1/\lambda)}, \qquad \text{a.s.}, \tag{3.3}$$

whenever $\lambda \to 0$ and $v \to \infty$ such that $v \gg \log^2(1/\lambda)\,\log\log\log(1/\lambda)$.

Proof of Lemma 3.2. Without loss of generality, we assume $Z(0) = 1$. The function S being a scale function of the diffusion Z, Feller's time change representation says that there exists a standard one-dimensional Brownian motion B (which we have taken as the Brownian motion B figuring in (3.1)) such that

$$Z(t) = S^{-1}(B(\Phi^{-1}(t))), \qquad t \ge 0,$$

where Φ^{-1} is the inverse of $\Phi(\cdot)$ defined (3.1), associated to the same Brownian motion B.

We start with the proof of (3.2), which is equivalent to:

$$\Phi(\tau(r)) = (1+o(1))\,r\log(1/\lambda), \qquad \text{a.s.}, \tag{3.4}$$

whenever $\lambda \to 0$ and $r \gg \log(1/\lambda)\log\log\log(1/\lambda)$.

We shall make use of the following simple consequence of law of large numbers: if $f : \mathbb{R} \to \mathbb{R}$ such that $\int_{\mathbb{R}} |f(x)|\,\mathrm{d}x < \infty$, then

$$\lim_{r\to\infty} \frac{1}{r} \int_{\mathbb{R}} f(x)\, L(\tau(r), x)\,\mathrm{d}x = \int_{\mathbb{R}} f(x)\,\mathrm{d}x, \qquad \text{a.s..} \tag{3.5}$$

By occupation time formula and a change of variables, we have

$$\begin{aligned}
\Phi(\tau(r)) &= \int_{-\infty}^{\infty} \exp\left(-\frac{4}{S^{-1}(x)} - 8\lambda S^{-1}(x)\right) L(\tau(r), x)\,\mathrm{d}x \\
&\equiv \left(\int_{-\infty}^{0} + \int_{0}^{S(a)} + \int_{S(a)}^{\infty}\right) \exp\left(-\frac{4}{S^{-1}(x)} - 8\lambda S^{-1}(x)\right) L(\tau(r), x)\,\mathrm{d}x \\
&\stackrel{\text{def}}{=} \Delta_1 + \Delta_2 + \Delta_3,
\end{aligned}$$

where $a = a(r, \lambda) > 0$ is chosen such that

$$\begin{cases} \frac{\mathrm{e}^{4\lambda a}}{4\lambda a} = \log(1/\lambda)\ \log\log(1/\lambda) & \text{if } r \geq \log^2(1/\lambda) \\ a = \frac{1}{4\lambda}\log\log(1/\lambda) & \text{if } r < \log^2(1/\lambda) \end{cases}. \tag{3.6}$$

For $x < 0$, let $S^{-1}(x) = y \in (0, 1)$ and $-x = \int_y^1 \mathrm{e}^{4\lambda z + 2/z}\,\frac{\mathrm{d}z}{z} \leq \mathrm{e}\int_y^1 \mathrm{e}^{2/z}\,\frac{\mathrm{d}z}{z}$ since $\lambda < 1/4$. This implies that there exits some constant $c > 0$ such that for all $x < -c$ and $\lambda < 1/4$, we have $\frac{2}{S^{-1}(x)} \geq \log|x| - 2\log\log|x|$. Hence, by means of (3.5),

$$\Delta_1 \leq \int_{-\infty}^{0} \left(\mathbf{1}_{(x\geq -c)} + \mathbf{1}_{(x<-c)}\frac{\log^4|x|}{|x|^2}\right) L(\tau(r), x)\,\mathrm{d}x = O(r), \qquad \text{a.s.} \tag{3.7}$$

To treat Δ_3, we observe that for $y \geq a$:

$$\begin{aligned}
S(y) &= \int_1^y \mathrm{e}^{(2/z)+4\lambda z}\,\frac{\mathrm{d}z}{z} \\
&= (1+o(1)) \int_1^y \mathrm{e}^{4\lambda z}\,\frac{\mathrm{d}z}{z} \\
&= (1+o(1)) \left(\int_{4\lambda}^1 \mathrm{e}^x\,\frac{\mathrm{d}x}{x} + \int_1^{4y\lambda} \mathrm{e}^x\,\frac{\mathrm{d}x}{x}\right) \\
&= (1+o(1)) \left(\log(1/\lambda) + O(1) + \frac{1+o(1)}{4\lambda y}\,\mathrm{e}^{4\lambda y}\right).
\end{aligned} \tag{3.8}$$

Let us distinguish two cases: Firstly $r \geq \log^2(1/\lambda)$, then $\frac{1}{4a\lambda}\mathrm{e}^{4a\lambda} \gg \log(1/\lambda)$ hence $S(y) = (1+o(1))\frac{1}{4\lambda y}\mathrm{e}^{4\lambda y}$ for $y \geq a$. Then for $x \geq S(a)$, we have $\mathrm{e}^{4\lambda S^{-1}(x)} > x$ and $S^{-1}(x) \sim \frac{\log x}{4\lambda}$. It follows that

$$\Delta_3 \leq 2\int_{S(a)}^{\infty} L(\tau(r), x)\,\frac{\mathrm{d}x}{x^2} \leq 2\int_{10}^{\infty} L(\tau(r), x)\,\frac{\mathrm{d}x}{x^2} = O(r), \qquad \text{a.s.,}$$

uniformly in small λ and large r.

For the case $\log(1/\lambda)\log\log\log(1/\lambda) \ll r < \log^2(1/\lambda)$, we have

$$\Delta_3 \le L^*(\tau(r)) \int_{S(a)}^{\infty} e^{-\frac{4}{S^{-1}(x)} - 8S^{-1}(x)}\, dx = L^*(\tau(r)) \int_a^{\infty} e^{-\frac{2}{y} - 4\lambda y} \frac{dy}{y} < L^*(\tau(r)),$$

where

$$L^*(\tau(r)) \stackrel{\text{def}}{=} \sup_{x\in\mathbb{R}} L(\tau(r), x) \le r(\log r)^{1+\varepsilon}, \qquad \text{a.s.},$$

for any $\varepsilon > 0$, by using the usual law of iterated logarithm for $L^*(\cdot)$ (Kesten [15]). Since $r < \log^2(1/\lambda)$, this implies that $\Delta_3 < r(\log\log(1/\lambda))^2$, a.s. Therefore, in both situations, we have

$$\Delta_3 = O\left(r\,(\log\log(1/\lambda))^2\right), \qquad \text{a.s.} \tag{3.9}$$

To deal with Δ_2, we note that in both cases,

$$S(a) = o\left(\frac{r}{\log\log r}\right).$$

In fact, if $r \ge \log^2(1/\lambda)$ then $S(a) \le \frac{2}{4a\lambda} e^{4a\lambda} = 2\log(1/\lambda)\log\log(1/\lambda) = o\left(r/\log\log r\right)$; otherwise, $a = \frac{\log\log(1/\lambda)}{4\lambda}$ and by means of (3.8), we have $S(a) \le 2\,\log(1/\lambda) = o(r/\log\log r)$. Hence we may apply Lemma 2.4 to see that

$$\Delta_2 = r(1+o(1)) \int_0^{S(a)} \exp\left(-\frac{4}{S^{-1}(x)} - 8\lambda S^{-1}(x)\right) dx, \qquad \text{a.s.}$$

By a change of variables, the integral on the right hand side is, when $\lambda \to 0$,

$$= \int_1^a e^{-\frac{2}{y} - 4\lambda y} \frac{dy}{y} = \int_1^a e^{-4\lambda y} \frac{dy}{y} + O(1) = \log(1/\lambda) + O(1).$$

Accordingly,

$$\Delta_2 = (1+o(1))\, r\log(1/\lambda), \qquad \text{a.s.} \tag{3.10}$$

Since $\Phi(\tau(r)) = \Delta_1 + \Delta_2 + \Delta_3$, assembling (3.7), (3.9) and (3.10) readily yields (3.4), thus also (3.2).

It remains to check (3.3). By the occupation time formula,

$$\begin{aligned}
\int_0^v Z(s)\, ds &= \int_0^v S^{-1}(B(\Phi^{-1}(s)))\, ds \\
&= \int_0^{\Phi^{-1}(v)} \frac{S^{-1}(B(r))}{h^2(B(r))}\, dr \\
&= \int_{-\infty}^{\infty} \frac{S^{-1}(x)}{h^2(x)} L(\Phi^{-1}(v), x)\, dx \\
&\stackrel{\text{def}}{=} \Psi(\Phi^{-1}(v)),
\end{aligned}$$

with obvious definition of $\Psi(\cdot)$. Replacing $\Phi^{-1}(v)$ by $\tau(r)$, this leads to:

$$\begin{aligned}\Psi(\tau(r)) &= \int_{-\infty}^{\infty} S^{-1}(x)\,\exp\left(-\frac{4}{S^{-1}(x)} - 8\lambda S^{-1}(x)\right) L(\tau(r), x)\,\mathrm{d}x \\ &= \left(\int_{-\infty}^{0} + \int_{0}^{S(a)} + \int_{S(a)}^{\infty}\right)\cdots\,\mathrm{d}x \\ &\overset{\text{def}}{=} \Delta_4 + \Delta_5 + \Delta_6,\end{aligned}$$

where $a = a(r, \lambda)$ is given in (3.6).

Since $0 < S^{-1}(x) < 1$ for $x < 0$, we have

$$\Delta_4 \le \Delta_1 = O(r), \qquad \text{a.s.}$$

Similarly, we have

$$\begin{aligned}\Delta_5 &= r(1+o(1)) \int_0^{S(a)} S^{-1}(x)\exp\left(-\frac{4}{S^{-1}(x)} - 8\lambda S^{-1}(x)\right) \mathrm{d}x \\ &= r(1+o(1)) \int_1^a \mathrm{e}^{\frac{2}{y} - 4\lambda y}\,\mathrm{d}y \\ &= (1+o(1))\,\frac{r}{4\lambda}, \qquad \lambda \to 0.\end{aligned}$$

For Δ_6, we again discuss two cases. First case: $r \ge \log^2(1/\lambda)$. For $x \ge S(a)$, $S^{-1}(x) \sim (\log x)/(4\lambda)$. Hence

$$\Delta_6 \le \frac{1}{2\lambda}\int_{S(a)}^{\infty} \frac{\log x}{x^2} L(\tau(r), x)\,\mathrm{d}x = o\left(\frac{r}{\lambda}\right), \qquad \text{a.s.},$$

by means of (3.5) and the fact that $S(a) \to \infty$. Second case: $r < \log^2(1/\lambda)$. We have

$$\Delta_6 \le L^*(\tau(r)) \int_{S(a)}^{\infty} S^{-1}(x)\,\mathrm{e}^{-\frac{4}{S^{-1}(x)} - 8S^{-1}(x)}\,\mathrm{d}x = L^*(\tau(r)) \int_a^{\infty} \mathrm{e}^{-(2/y) - 4\lambda y}\,\mathrm{d}y,$$

which yields

$$\Delta_6 \le \frac{\mathrm{e}^{-4\lambda a}}{4\lambda} L^*(\tau(r)) = \frac{L^*(\tau(r))}{4\lambda\log(1/\lambda)} \le \frac{r}{\lambda}\,\frac{(\log\log(1/\lambda))^2}{\log(1/\lambda)} = o\left(\frac{r}{\lambda}\right).$$

Therefore, almost surely,

$$\Psi(\tau(r)) = (1+o(1))\,\frac{r}{4\lambda}, \qquad r \gg \log(1/\lambda)\log\log\log(1/\lambda).$$

Since $\int_0^v Z(s)\,\mathrm{d}s = \Psi(\Phi^{-1}(v))$, this, with the aid of (3.2), yields (3.3). Lemma 3.2 is proved. $\square$

We have now all the ingredients for the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $t, v \to \infty$ satisfying the conditions in Theorem 1.1.

First, we prove (1.3) for $\sup_{0\le s\le t} X(s)$ in place of $X(t)$: P-almost surely,

$$\mathbf{P}_\omega\left(\sup_{0\le s\le t} X(s) > v\right) = \mathbf{P}_\omega\left(H(v) < t\right) = \exp\left(-\left(1+o(1)\right)\frac{v}{2\log(t/v)}\right). \tag{3.11}$$

To this end, applying Chebyshev's inequality to Fact 3.1 and Lemma 3.2, we have that for almost surely all potentials W and for $\lambda \to 0$ satisfying $v \gg \log^2(1/\lambda)\log\log\log(1/\lambda)$,

$$\mathbf{P}_\omega\left\{H(v) < t\right\} \le \mathrm{e}^{\lambda t}\,\mathbf{E}_\omega\left[\mathrm{e}^{-\lambda H(v)}\right] = \exp\left(\lambda t - \left(1+o(1)\right)\frac{v}{2\log(1/\lambda)}\right). \tag{3.12}$$

By choosing $\lambda = \frac{v}{t\log^2(t/v)}$ (this is possible since $v \gg \log^2(t/v)\log\log\log(t/v)$), we have

$$\mathbf{P}_\omega\left\{H(v) < t\right\} \le \exp\left(-(1+o(1))\frac{v}{2\log(t/v)}\right), \qquad \text{a.s.}$$

This implies the upper bound in (3.11).

To get the lower bound, we keep the choice of λ and use the simple relation

$$\mathbf{P}_\omega\left\{H(v) < t\log^2 t\right\} \ge \mathbf{E}_\omega\left[\mathrm{e}^{-\lambda H(v)}\right] - \mathrm{e}^{-\lambda t\log^2 t} \ge \exp\left(-(1+o(1))\frac{v}{2\log(t/v)}\right),$$

by means of Lemma 3.2. Write $s = t\log^2 t$. Since $\log\log t = o(\log(t/v))$, we have $\log(s/v) \sim \log(t/v)$, which yields the lower bound in (3.11).

To prove (1.3), since $\mathbf{P}_\omega\left\{X(t) > v\right\} \le \mathbf{P}_\omega\left(H(v) < t\right)$, it remains to show the lower bound. Let $c_1, c_2 > 0$ and define

$$G(c_1, c_2) \stackrel{\text{def}}{=} \left\{(v,t) : t, v \ge 3^{81},\ v \ge c_1\log^2 t\,\log\log\log t,\ \log(t/v) \ge c_2\log\log t\right\}.$$

It suffices to show that for any small $\varepsilon > 0$ and almost surely all ω, there exist $c_1(\varepsilon), c_2(\varepsilon) > 0$ and $t_0(\omega) > 0$ such that for all $t, v \ge t_0$ and $(v,t) \in G\left(c_1(\varepsilon), c_2(\varepsilon)\right)$, we have

$$\log \mathbf{P}_\omega\left\{X(t) > (1-\varepsilon)v\right\} \ge -\frac{1+\varepsilon}{2}\,\frac{v}{\log(t/v)}. \tag{3.13}$$

Write $\mathbf{P}_{x,\omega}$ for the law of the diffusion X defined in (1.1) starting from $X(0) = x$. According to the lower bound in (3.11), there exist $c_1 \ge \frac{512}{(1-2\varepsilon)\pi^2\varepsilon}$ and $c_2 > 0$ such that for P-almost all ω and all $(v,t) \in G(\frac{c_1}{2}, \frac{c_2}{2})$, we have

$$\mathbf{P}_{0,\omega}\left\{H(v) < t\right\} \ge \exp\left(-\frac{1+\varepsilon/2}{2\log(t/v)}\right). \tag{3.14}$$

Let $j,\, k \geq j_0$ be sufficiently large and define $v_j = \mathrm{e}^{j/\log j}$ and $\log t_k = \mathrm{e}^{k/\log k}$. The strong Markov property implies that for any $0 < \varepsilon < 1$, $v_{j-1} \leq v \leq v_j$ and $t_{k-1} \leq t \leq t_k$,

$$\mathbf{P}_{0,\omega}\left\{X(t) > (1-\varepsilon)v\right\} \geq \mathbf{P}_{0,\omega}\left\{H(v_j) < t_{k-1}\right\} \mathbf{P}_{v_j,\omega}\left\{H((1-\varepsilon)v_j) \geq t_k\right\}. \tag{3.15}$$

It is elementary to show that

$$(v,t) \in G(c_1,c_2) \cap \{[v_{j-1}, v_j] \times [t_{k-1}, t_k]\} \Longrightarrow (v_j, t_{k-1}),\, (v_j, t_k) \in G\left(\frac{c_1}{2}, \frac{c_2}{2}\right).$$

Admitting for the moment that we have proved that

$$\sum_{j,k \geq j_0;\, (v_j,t_k)\in G(\frac{c_1}{2},\frac{c_2}{2})} P\left\{\mathbf{P}_{v_j,\omega}\left\{H((1-\varepsilon)v_j) < t_k\right\} > \frac{1}{2}\right\} < \infty, \tag{3.16}$$

then we will be able to prove the lower bound (3.13). Indeed, by (3.16), P-a.s. for all ω and for all large v, j satisfying $(v_j, t_k) \in G(\frac{c_1}{2}, \frac{c_2}{2})$, we have

$$\mathbf{P}_{v_j,\omega}\left\{H((1-\varepsilon)v_j) \geq t_k\right\} \geq \frac{1}{2}.$$

In view of (3.15), this estimate and (3.14) imply that almost surely for ω, for all large t, v such that $(v,t) \in G(c_1,c_2) \times \{[v_{j-1}, v_j] \times [t_{k-1}, t_k]\}$,

$$\begin{aligned}
\mathbf{P}_{0,\omega}\left\{X(t) > (1-\varepsilon)v\right\} &\geq \exp\left(-\frac{(1+\varepsilon/2)v_j}{2\log(t_{k-1}/v_j)}\right) \mathbf{P}_{v_j,\omega}\left\{H((1-\varepsilon)v_j) \geq t_k\right\} \\
&\geq \frac{1}{2}\exp\left(-\frac{(1+\varepsilon/2)v_j}{2\log(t_{k-1}/v_j)}\right) \\
&\geq \exp\left(-\frac{(1+\varepsilon)v}{2\log(t/v)}\right),
\end{aligned}$$

since $v_j/v_{j-1} \to 1$ and $\log t_k / \log t_{k-1} \to 1$. This will yield the lower bound (3.13).

It remains to prove (3.16). We remark by Brownian symmetry, $\mathbf{P}_{v_j,\omega}\left\{H((1-\varepsilon)v_j) < t_k\right\}$ is distributed as $\mathbf{P}_{0,\omega}\left\{H(\varepsilon\, v_j) < t_k\right\}$, so that by means of (5.8) in Section 5, for all large j and k,

$$\begin{aligned}
& E\left(\mathbf{P}_{v_j,\omega}\left\{H((1-\varepsilon)v_j) < t_k\right\}\right) \\
=\ & \mathbb{P}\left\{H(\varepsilon\, v_j) < t_k\right\} \\
\leq\ & c\,\varepsilon^2\, v_j^2 \mathrm{e}^{-\varepsilon\, v_j} + 9\exp\left(-(1-2\varepsilon)\frac{\pi^2}{8}\,\frac{\varepsilon v_j}{\log^2(\varepsilon v_j t_k)}\right) \\
\leq\ & (c+9)\exp\left(-\frac{(1-2\varepsilon)\pi^2\varepsilon}{32}\,\frac{v_j}{\log^2 t_k}\right).
\end{aligned} \tag{3.17}$$

We have used in the last inequality the fact that $v_j \leq t_k$.

By Chebyshev's inequality,

$$\begin{aligned}
&\sum_{j,k\geq j_0;\ (v_j,t_k)\in G(\frac{c_1}{2},\frac{c_2}{2})} P\left\{\mathbf{P}_{v_j,\omega}\left\{H((1-\varepsilon)v_j)<t_k\right\}>\frac12\right\}\\
\leq\ & 2\sum_{j,k\geq j_0;\ (v_j,t_k)\in G(\frac{c_1}{2},\frac{c_2}{2})} E\left(\mathbf{P}_{v_j,\omega}\left\{H((1-\varepsilon)v_j)<t_k\right\}\right)\\
\leq\ & 2(c+9)\sum_{j,k\geq j_0;\ (v_j,t_k)\in G(\frac{c_1}{2},\frac{c_2}{2})} \exp\left(-\frac{(1-2\varepsilon)\pi^2\varepsilon}{32}\,\frac{v_j}{\log^2 t_k}\right).
\end{aligned}$$

Several elementary computations show that the above sum is finite, in fact, we can decompose this sum into $\sum_{j\geq k^4}$ and $\sum_{j\leq k^4}$. Note that for $j\geq k^4$, $\frac{v_j}{\log^2 t_k} = \mathrm{e}^{\frac{j}{\log j}-2\frac{k}{\log k}} \geq \mathrm{e}^{j/(2\log j)}$. Hence $\sum_{j\geq k^4} \leq \sum_j j^{1/4}\exp(-\frac{(1-2\varepsilon)\pi^2\varepsilon}{32}\mathrm{e}^{j/(2\log j)}) < \infty$. For the case $j\leq k^4$, we use the definition of $G(\frac{c_1}{2},\frac{c_2}{2})$ which says that $v_j/\log^2 t_k \geq \frac{c_1}{2}\log\log\log t_k \geq \frac{256}{(1-2\varepsilon)\pi^2\varepsilon}\log\log\log t_k$, $\sum_{j\leq k^4} \leq \sum_k k^4\mathrm{e}^{-8\log\log\log t_k} \leq \sum_k k^{-4} < \infty$. This completes the proof of Theorem 1.1. $\Box$

4 The annealed case: a joint arcsine law

Let $\kappa\in\mathbb{R}$ and let

$$W_\kappa(x) = W(x) - \frac{\kappa}{2}x, \qquad x\in\mathbb{R},$$

where W is, as before, a Brownian motion defined on $\mathbb{R}$ with $W(0)=0$. In this section, we shall study the diffusion X with potential W_κ (i.e., replacing W by W_κ in (1.1)). Plainly, when $\kappa=0$, we recover the case of Brownian potential and X is recurrent, whereas $X(t)\to+\infty$ $\mathbb{P}$-a.s. if $\kappa>0$.

Let $\kappa\geq 0$ and we recall the time change representation of X (cf. Brox [2] for $\kappa=0$ and [10] for $\kappa>0$):

$$X(t) = A_\kappa^{-1}(B(T_\kappa^{-1}(t))), \qquad t\geq 0,$$

where B is a one-dimensional Brownian motion stating from 0, independent of W, and

$$\begin{aligned}
A_\kappa(x) &= \int_0^x \mathrm{e}^{W_\kappa(y)}\,\mathrm{d}y, \qquad x\in\mathbb{R},\\
T_\kappa(t) &= \int_0^t \mathrm{e}^{-2W_\kappa(A_\kappa^{-1}(B(s)))}\,\mathrm{d}s, \qquad t\geq 0.
\end{aligned}$$

(Recall that A_κ^{-1} and T_κ^{-1} denote the inverses of A_κ and T_κ, respectively.) Here, we stress the fact that the process B, a Brownian motion independent of W, is not the same B as in Section 3. There is no risk of confusion since we always separate the quenched and the annealed cases. Recall that $(L(t,x), t \geq 0, x \in \mathbb{R})$ denote the local times of B and $\sigma(\cdot)$ is the process of first hitting times of B:

$$\sigma(x) \stackrel{\text{def}}{=} \inf\{t > 0 : B(t) > x\}, \qquad x \geq 0.$$

Therefore $H(v)$, the first hitting time of X at v, can be represented as follows:

$$\begin{aligned} H(v) &= \inf\{t \geq 0 : X(t) = v\} \\ &= T_\kappa(\sigma(A_\kappa(v))) \qquad (4.1) \\ &= \int_0^{\sigma(A_\kappa(v))} \mathrm{e}^{-2W_\kappa(A_\kappa^{-1}(B(s))}\, \mathrm{d}s \\ &= \int_{-\infty}^{A_\kappa(v)} \mathrm{e}^{-2W_\kappa(A_\kappa^{-1}(x))}\, L(\sigma(A_\kappa(v)), x)\, \mathrm{d}x \\ &= \Theta_1(v) + \Theta_2(v), \end{aligned}$$

with

$$\begin{aligned} \Theta_1(v) &= \int_0^{A_\kappa(v)} \mathrm{e}^{-2W_\kappa(A_\kappa^{-1}(x))}\, L(\sigma(A_\kappa(v)), x)\, \mathrm{d}x = \int_0^{H(v)} \mathbf{1}_{\{X(s) \geq 0\}}\, \mathrm{d}s, \\ \Theta_2(v) &= \int_{-\infty}^{0} \mathrm{e}^{-2W_\kappa(A_\kappa^{-1}(x))}\, L(\sigma(A_\kappa(v)), x)\, \mathrm{d}x = \int_0^{H(v)} \mathbf{1}_{\{X(s) < 0\}}\, \mathrm{d}s. \end{aligned}$$

The main result in this section is an identity in law, considered in the annealed case, which relates $(\Theta_1(v), \Theta_2(v))$ to a functional of the Jacobi process.

Theorem 4.1 *Let $\kappa \geq 0$ and let $v > 0$. Under $\mathbb{P}$, we have*

$$(\Theta_1(v), \Theta_2(v)) \stackrel{\text{law}}{=} \left(4 \int_0^v \left(\mathrm{e}^{\Xi_\kappa(s)} - 1\right)\, \mathrm{d}s,\ 16\, \Upsilon_{2-2\kappa}\left(\mathrm{e}^{\Xi_\kappa(v)/2} \rightsquigarrow 1\right)\right),$$

where $\Upsilon_{2-2\kappa}(x \rightsquigarrow y)$ denotes the first hitting time of y by a BES$(2-2\kappa)$ *starting from x, independent of the diffusion Ξ_κ which is the unique non-negative solution of*

$$\Xi_\kappa(t) = \int_0^t \sqrt{1 - \mathrm{e}^{-\Xi_\kappa(s)}}\, \mathrm{d}\beta(s) + \int_0^t \left(-\frac{\kappa}{2} + \frac{1+\kappa}{2} \mathrm{e}^{-\Xi_\kappa(s)}\right) \mathrm{d}s, \qquad t \geq 0. \qquad (4.2)$$

The proof of Theorem 4.1 involves some deep known results. Let us first recall Lamperti's representation theorem for the exponential functionals ([16]).

Fact 4.2 (Lamperti's representation) *Let $\kappa \in \mathbb{R}$. There exists a* BES$(2+2\kappa)$*, denoted by $\widetilde{R}$ and starting from $\widetilde{R}(0) = 2$, such that*

$$\exp\left(W(x) + \frac{\kappa}{2} x\right) = \frac{1}{2} \widetilde{R}\left(\int_0^x \mathrm{e}^{W(y) + \kappa y/2}\, \mathrm{d}y\right), \qquad x \geq 0. \qquad (4.3)$$

Let d_1, $d_2 \geq 0$, $a \in [0,1]$, and consider the equation

$$\begin{cases} \mathrm{d}Y(t) = 2\sqrt{Y(t)(1-Y(t))}\,\mathrm{d}\beta(t) + (d_1 - (d_1+d_2)Y(t))\ \mathrm{d}t \\ Y(0) = a \end{cases}, \tag{4.4}$$

where β is a standard one-dimensional Brownian motion. The solution Y of the above equation is called a Jacobi process of dimension (d_1, d_2), starting from a (Karlin and Taylor [12]). We mention that almost surely, $0 \leq Y(t) \leq 1$ for all $t \geq 0$.

The following result gives the skew-product representation of two independent Bessel processes in terms of the Jacobi process.

Fact 4.3 (Warren and Yor [26]) *Let R_1 and R_2 be two independent Bessel processes of dimensions d_1 and d_2 respectively. We assume $d_1 + d_2 \geq 2$, $R_1(0) = r_1 \geq 0$ and $R_2(0) = r_2 > 0$. Then there exists a Jacobi process Y of dimension (d_1, d_2) starting from $\frac{r_1^2}{r_1^2+r_2^2}$, independent of the process $(R_1^2(t) + R_2^2(t),\ t \geq 0)$, such that*

$$\frac{R_1^2(t)}{R_1^2(t)+R_2^2(t)} = Y\left(\int_0^t \frac{\mathrm{d}s}{R_1^2(s)+R_2^2(s)}\right), \qquad t \geq 0.$$

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. Using Fact 2.1 and the independence of B and W, the process $\{\frac{1}{A_\kappa(v)}\, L(\sigma(A_\kappa(v)), (1-x)A_\kappa(v)), x \geq 0\}$ is a strong Markov process starting from 0, independent of W; it is a BESQ(2) for $x \in [0,1]$, and is a BESQ(0) for $x \geq 1$. By scaling,

$$(\Theta_1(v), \Theta_2(v)) \overset{\text{law}}{=} \left(\int_0^v \mathrm{e}^{-W_\kappa(x)}\, R^2\left(A_\kappa(v) - A_\kappa(x)\right)\, \mathrm{d}x,\ \int_{-\infty}^0 \mathrm{e}^{-2W_\kappa(A_\kappa^{-1}(x))}\, U(|x|)\, \mathrm{d}x\right),$$

where R is a BES(2) starting from 0, independent of W, and conditionally on (R, W), U is a BESQ(0) starting from $R^2(A_\kappa(v))$.

By time reversal, $(\widehat{W}_\kappa(y) \overset{\text{def}}{=} W_\kappa(v-y) - W_\kappa(v), 0 \leq y \leq v)$ has the same law as $(W_{-\kappa}(y) = W(y) + \frac{\kappa}{2}y, 0 \leq y \leq v)$. Observe that

$$\begin{aligned} \int_0^v \mathrm{e}^{-W_\kappa(x)}\, R^2\left(A_\kappa(v) - A_\kappa(x)\right)\, \mathrm{d}x &= \int_0^v \mathrm{e}^{-\widehat{W}_\kappa(y)} \mathrm{e}^{\widehat{W}_\kappa(v)} R^2 \left(\mathrm{e}^{-\widehat{W}_\kappa(v)} \int_0^y \mathrm{e}^{\widehat{W}_\kappa(z)}\, \mathrm{d}z\right)\, \mathrm{d}y, \\ R^2\left(A_\kappa(v)\right) &= R^2\left(\mathrm{e}^{-\widehat{W}_\kappa(v)} \int_0^v \mathrm{e}^{\widehat{W}_\kappa(z)}\, \mathrm{d}z\right). \end{aligned}$$

By scaling and independence of R and W, the process $x \mapsto \mathrm{e}^{\widehat{W}_\kappa(v)}\, R^2\left(x\mathrm{e}^{-\widehat{W}_\kappa(v)}\right)$, has the same law as R and is independent of W. It follows that

$$(\Theta_1(v), \Theta_2(v)) \overset{\text{law}}{=} \left(\int_0^v \mathrm{e}^{-W_{-\kappa}(x)}\, R^2(A_{-\kappa}(x))\, \mathrm{d}x,\ \int_{-\infty}^0 \mathrm{e}^{-2W_\kappa(A_\kappa^{-1}(x))}\, U(|x|)\, \mathrm{d}x\right), \tag{4.5}$$

where conditionally on (R,W), U has the same law as a BESQ(0) starting from $U(0) = \mathrm{e}^{-W_{-\kappa}(v)} R^2(A_{-\kappa}(v))$.

Let us first treat the part $(W_{-\kappa}(x), x\ge 0)$. By means of Fact 4.2, there exists a Bessel process $\widetilde{R}$ of dimension $2+2\kappa \ge 2$, starting from 2, such that

$$\begin{aligned}\int_0^v \mathrm{e}^{-W_{-\kappa}(x)}\, R^2\left(A_{-\kappa}(x)\right)\mathrm{d}x &= \int_0^{A_{-\kappa}(v)} \mathrm{e}^{-2W_{-\kappa}(A^{-1}_{-\kappa}(y))}\, R^2(y)\,\mathrm{d}y \\ &= 16\int_0^{A_{-\kappa}(v)} \frac{R^2(y)}{\widetilde{R}^4(y)}\,\mathrm{d}y, \qquad (4.6)\end{aligned}$$

where we stress the independence of the two Bessel processes R and $\widetilde{R}$. Observe that

$$A^{-1}_{-\kappa}(x) = 4\int_0^x \frac{\mathrm{d}u}{\widetilde{R}^2(u)}, \qquad x\ge 0.$$

We apply Fact 4.3 to R and $\widetilde{R}$, to see that there exists a Jacobi process Y of dimension $(2, 2+2\kappa)$ starting from 0 such that

$$\frac{R^2(x)}{R^2(x)+\widetilde{R}^2(x)} = Y\left(\int_0^x \frac{\mathrm{d}s}{R^2(s)+\widetilde{R}^2(s)}\right) \stackrel{\mathrm{def}}{=} Y(\Lambda(x)), \qquad x\ge 0,$$

where $\Lambda(x) \stackrel{\mathrm{def}}{=} \int_0^x \frac{\mathrm{d}s}{R^2(s)+\widetilde{R}^2(s)}$, and $\Lambda(\cdot)$ is independent of Y. Note that $Y(0)=0$ and $0 < Y(t) < 1$ for all $t>0$.

This representation together with (4.6) imply that

$$\begin{aligned}\int_0^v \mathrm{e}^{-W_{-\kappa}(x)}\, R^2\left(A_{-\kappa}(x)\right)\mathrm{d}x &= 16\int_0^{\Lambda(A_{-\kappa}(v))} \frac{Y(u)}{(1-Y(u))^2}\,\mathrm{d}u \\ &= 16\int_0^{\rho^{-1}(v)} \frac{Y(u)}{(1-Y(u))^2}\,\mathrm{d}u, \qquad (4.7)\end{aligned}$$

where

$$\rho(x) \stackrel{\mathrm{def}}{=} A^{-1}_{-\kappa}(\Lambda^{-1}(x)) = 4\int_0^{\Lambda^{-1}(x)} \frac{\mathrm{d}u}{\widetilde{R}^2(u)} = 4\int_0^x \frac{\mathrm{d}y}{1-Y(y)},$$

by a change of variables $y=\Lambda(u)$. Going back to (4.5),

$$U(0) = \mathrm{e}^{-W_{-\kappa}(v)}\, R^2(A_{-\kappa}(v)) = \frac{4R^2(A_{-\kappa}(v))}{\widetilde{R}^2(A_{-\kappa}(v))} = \frac{4Y(\Lambda(A_{-\kappa}(v)))}{1-Y(\Lambda(A_{-\kappa}(v)))} = \frac{4Y(\rho^{-1}(v))}{1-Y(\rho^{-1}(v))}.$$

Assume for the moment that for any fixed $r>0$, if U_r denotes a BESQ(0) starting from $U_r(0)=r$, independent of W, then

$$\int_{-\infty}^0 \mathrm{e}^{-2W_\kappa(A^{-1}_\kappa(x))}\, U_r(|x|)\,\mathrm{d}x \stackrel{\mathrm{law}}{=} 16\,\Upsilon_{2-2\kappa}\left(\sqrt{1+\frac{4}{r}} \rightsquigarrow 1\right), \qquad \kappa\ge 0. \qquad (4.8)$$

By admitting (4.8), it follows from (4.5), (4.6) and (4.7) that under the total probability $\mathbb{P}$,

$$(\Theta_1(v), \Theta_2(v)) \stackrel{\text{law}}{=} \left(16 \int_0^{\rho^{-1}(v)} \frac{Y(u)}{(1-Y(u))^2} \, \mathrm{d}u, \, 16 \, \Upsilon_{2-2\kappa}(\sqrt{1+\frac{4}{U(0)}} \rightsquigarrow 1) \right). \qquad (4.9)$$

Since $\mathrm{d}\rho^{-1}(x) = \frac{1-Y(\rho^{-1}(x))}{4} \, \mathrm{d}x$ and

$$\int_0^{\rho^{-1}(v)} \frac{Y(u)}{(1-Y(u))^2} \, \mathrm{d}u = \frac{1}{4} \int_0^v \frac{\mathrm{d}x}{1-Y(\rho^{-1}(x))} - \frac{v}{4}, \qquad v > 0,$$

it follows from (4.4) that $\Xi_\kappa(t) \stackrel{\text{def}}{=} -\log\{1 - Y(\rho^{-1}(t))\}$ satisfies the stochastic integral equation (4.2). Theorem 4.1 will then follow from the identity in law (4.9).

It remains to show (4.8). Note that $(W_\kappa(-x), x \geq 0)$ is distributed as $(W_{-\kappa}(x), x \geq 0)$. Thus

$$\int_{-\infty}^0 \mathrm{e}^{-2W_\kappa(A_\kappa^{-1}(x))} \, U_r(|x|) \, \mathrm{d}x \stackrel{\text{law}}{=} \int_0^\infty \mathrm{e}^{-2W_{-\kappa}(A_{-\kappa}^{-1}(x))} \, U_r(x) \, \mathrm{d}x = 16 \int_0^\infty \frac{U_r(x)}{\widetilde{R}^4(x)} \, \mathrm{d}x, \qquad (4.10)$$

by using again the BES$(2+2\kappa)$ process $\widetilde{R}$ defined in (4.6).

Applying Fact 4.3 to the two independent squared Bessel processes U_r and $\widetilde{R}^2$, we get a Jacobi process $\widehat{Y}$ of dimension $(0, 2+2\kappa)$ starting from $\frac{r}{r+4}$, such that

$$\frac{U_r(t)}{U_r(t)+\widetilde{R}^2(t)} = \widehat{Y} \left(\int_0^t \frac{\mathrm{d}s}{U_r(s)+\widetilde{R}^2(s)} \right) \stackrel{\text{def}}{=} \widehat{Y} \left(\widehat{\Lambda}(t) \right), \qquad t \geq 0,$$

with $\widehat{\Lambda}(t) \stackrel{\text{def}}{=} \int_0^t \frac{\mathrm{d}s}{U_r(s)+\widetilde{R}^2(s)}$, $t \geq 0$, independent of $\widehat{Y}$. Observe that $\widehat{Y}$ is absorbed at 0.

Then $\widehat{\Lambda}(T_{U_r}(0)) = T_{\widehat{Y}}(0) < \infty$. Using the change of variables $t = \widehat{\Lambda}(x)$, we obtain that

$$16 \int_0^\infty \frac{U_r(x)}{\widetilde{R}^4(x)} \, \mathrm{d}x = 16 \int_0^{T_{\widehat{Y}}(0)} \frac{\widehat{Y}(t)}{(1-\widehat{Y}(t))^2} \, \mathrm{d}t. \qquad (4.11)$$

By computing the scale function and using the Dubins–Schwarz theorem (Revuz and Yor [19], Theorem V.1.6) for continuous local martingales, we see that there exists some one-dimensional Brownian motion β starting from 0 such that

$$s(\widehat{Y}(t)) = \beta(\phi(t)), \qquad t \geq 0,$$

with

$$\begin{aligned} y_0 &\stackrel{\text{def}}{=} \frac{r}{r+4}, \\ s(y) &\stackrel{\text{def}}{=} \begin{cases} \log \frac{1-y_0}{1-y}, & \text{if } \kappa = 0 \\ \frac{1}{\kappa} \{(1-y)^{-\kappa} - (1-y_0)^{-\kappa}\} & \text{if } \kappa > 0 \end{cases}, \qquad 0 \leq y < 1, \\ \phi(t) &\stackrel{\text{def}}{=} 4 \int_0^t \frac{\widehat{Y}(s)}{(1-\widehat{Y}(s))^{1+2\kappa}} \, \mathrm{d}s, \qquad t \geq 0. \end{aligned}$$

Note that $\phi(T_{\widehat{Y}}(0)) = \inf\{t > 0 : \beta(t) = s(0)\} = T_\beta(s(0))$. Hence, for $\kappa > 0$,

$$\begin{aligned} 16 \int_0^{T_{\widehat{Y}}(0)} \frac{\widehat{Y}(t)}{(1-\widehat{Y}(t))^2}\,\mathrm{d}t &= 4 \int_0^{T_\beta(s(0))} (1 - s^{-1}(\beta(u)))^{2\kappa-1}\,\mathrm{d}u \\ &= 4 \int_0^{T_\beta(s(0))} (1 + \kappa(\beta(u) - s(0)))^{(1/\kappa)-2}\,\mathrm{d}u. \end{aligned} \tag{4.12}$$

When $\kappa = 0$, we get

$$\begin{aligned} 16 \int_0^{T_{\widehat{Y}}(0)} \frac{\widehat{Y}(t)}{(1-\widehat{Y}(t))^2}\,\mathrm{d}t &= \frac{4}{1-y_0} \int_0^{T_\beta(s(0))} \mathrm{e}^{\beta(u)}\,\mathrm{d}u \\ &\stackrel{\text{law}}{=} \frac{4}{1-y_0} \inf\left\{s > 0 : \widetilde{R}(s) = 2\mathrm{e}^{s(0)/2} = 2\sqrt{1-y_0}\right\}, \end{aligned}$$

where the last equality in law follows from (4.3) by replacing W by β (recalling $\widetilde{R}(0) = 2$). This together with scaling property yield (4.8) in the case $\kappa = 0$.

For $\kappa > 0$, it follows from symmetry and Lemma 2.3 that the expression on the right hand side of (4.12) is equal in law to

$$4\kappa^{(1/\kappa)-2} \int_0^{T_\beta(|s(0)|)} \left(\frac{1}{\kappa} + |s(0)| - \beta(u)\right)^{(1/\kappa)-2} \mathrm{d}u \stackrel{\text{law}}{=} 16\,\Upsilon_{2-2\kappa}\left(\sqrt{1+\frac{4}{r}} \rightsquigarrow 1\right),$$

completing the proof of (4.8). Theorem 4.1 is proved. □

5 Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. We prove the upper and lower bounds with different approaches.

5.1 Theorem 1.2: the upper bound

The proof of the upper bound is based on an analysis of the diffusion process $\Xi_0(\cdot)$ introduced in (4.2) (with $\kappa = 0$).

For notational convenience, we write $\Xi \stackrel{\text{def}}{=} \Xi_0$. Let us start with a couple of lemmas.

Lemma 5.1 *There exists a numerical constant $c > 0$ such that for all $t > 100$ and $0 < a < x < \sqrt{t}$, we have*

$$\mathbb{P}\left(\sup_{0\le t_1\le t_2<t;\, t_2-t_1<a} |\Xi(t_2) - \Xi(t_1)| > x\right) \le c\,\frac{t}{a}\,\exp\left(-\frac{x^2}{9a}\right), \tag{5.1}$$

$$\mathbb{P}\left(\sup_{0\le s\le t} \Xi(s) < 1\right) \le 2\,\exp\left(-\frac{t}{50}\right). \tag{5.2}$$

Proof of Lemma 5.1. By definition of Ξ in (4.2) (with $\kappa = 0$), $\Xi(t) = \int_0^t \sqrt{1-\mathrm{e}^{-\Xi(s)}}\,\mathrm{d}\beta(s) + \frac{1}{2}\int_0^t \mathrm{e}^{-\Xi(s)}\,\mathrm{d}s$. It follows from the Dubins–Schwarz theorem (Revuz and Yor [19], Theorem V.1.6) that

$$\Xi(t) = \gamma\left(\int_0^t (1-\mathrm{e}^{-\Xi(s)})\,\mathrm{d}s\right) + \frac{1}{2}\int_0^t \mathrm{e}^{-\Xi(s)}\,\mathrm{d}s, \qquad t \geq 0, \tag{5.3}$$

where $\gamma(\cdot)$ denotes a one-dimensional Brownian motion. Since $\Xi(s) \geq 0$ for all $s \geq 0$, we have

$$\sup_{0\leq t_1\leq t_2<t;\, t_2-t_1<a} |\Xi(t_2)-\Xi(t_1)| \leq \sup_{0\leq s_1<s_2<t;\, s_2-s_1<a} |\gamma(s_2)-\gamma(s_1)| + \frac{a}{2}.$$

According to Lemma 1.1.1 of Csörgő and Révész [5],

$$\mathbb{P}\left(\sup_{0\leq s_1<s_2<t;\, s_2-s_1<a} |\gamma(s_2)-\gamma(s_1)| > \frac{x}{2}\right) \leq c\,\frac{t}{a}\exp\left(-\frac{x^2}{9a}\right).$$

This implies (5.1).

To prove (5.2), we note that on $\{\sup_{0\leq s\leq t}\Xi(s) < 1\}$, we have $\int_0^t \mathrm{e}^{-\Xi(s)}\,\mathrm{d}s \geq t/\mathrm{e}$, hence for all $t > 100$,

$$\left\{\sup_{0\leq s\leq t}\Xi(s) < 1\right\} \subset \left\{\gamma\left(\int_0^t (1-\mathrm{e}^{-\Xi(s)})\,\mathrm{d}s\right) \leq 1 - \frac{t}{2\mathrm{e}} < -\frac{t}{5}\right\} \subset \left\{\inf_{0\leq u\leq t}\gamma(u) < -\frac{t}{5}\right\}.$$

The estimate (5.2) now follows from the usual estimate for Brownian tails. □

Lemma 5.2 *For any $\varepsilon \in (0,1]$, there exists some $v_0 = v_0(\varepsilon) > 0$ such that for all $x, v > v_0$, we have*

$$\frac{2}{\pi}\exp\left(-(1+\varepsilon)\frac{\pi^2}{8}\frac{v}{x^2}\right) \leq \mathbb{P}\left(\sup_{0\leq s\leq v}\Xi(s) < x\right) \leq 9\exp\left(-(1-\varepsilon)\frac{\pi^2}{8}\frac{v}{x^2}\right).$$

Proof of Lemma 5.2. Assume for the moment that Ξ starts from $\Xi(0) = 1$. Let

$$f(x) \stackrel{\mathrm{def}}{=} \int_1^x \frac{\mathrm{d}y}{1-\mathrm{e}^{-y}}, \qquad 0 < x < \infty,$$

be the scale function of Ξ. Since $t \mapsto f(\Xi(t))$ is a continuous local martingale, it follows from the Dubins–Schwarz theorem (Revuz and Yor [19], Theorem V.1.6) that

$$f(\Xi(t)) = B\left(\int_0^t \frac{\mathrm{d}s}{1-\mathrm{e}^{-\Xi(s)}}\right), \qquad t \geq 0,$$

for some one-dimensional Brownian motion B starting from 0. Therefore, by writing $T_\Xi(x) = \inf\{s > 0 : \Xi(s) > x\}$, we have,

$$T_\Xi(x) = \int_0^{\sigma(f(x))} (1-\mathrm{e}^{-f^{-1}(B(s))})\,\mathrm{d}B(s) = \int_{-\infty}^{f(x)} (1-\mathrm{e}^{-f^{-1}(y)})L(\sigma(f(x)),y)\,\mathrm{d}y,$$

where f^{-1} is the inverse of the increasing function f, $\sigma(x) \stackrel{\text{def}}{=} \inf\{s: \ B(s) = x\}$ for $x \in \mathbb{R}$, and L is the local time of B.

Observe that $f^{-1}(y) \sim y$ as $y \to \infty$, and $f^{-1}(y) \sim \mathrm{e}^{-|y|}$ as $y \to -\infty$. Let $y_0 = y_0(\varepsilon) > 0$ be sufficiently large such that $\mathrm{e}^{-f^{-1}(y)} < \varepsilon/2$ for all $y \geq y_0$. Denote by $b(\varepsilon) \stackrel{\text{def}}{=} \sup_{-\infty < y \leq y_0} (1 - \mathrm{e}^{-f^{-1}(y)})\mathrm{e}^{|y|} < \infty$. Then for all large x,

$$T_\Xi(x) \ \geq \ (1 - \frac{\varepsilon}{2}) \int_{y_0}^{f(x)} L(\sigma(f(x)), y) \, \mathrm{d}y, \tag{5.4}$$

$$T_\Xi(x) \ \leq \ \int_{y_0}^{f(x)} L(\sigma(f(x)), y) \, \mathrm{d}y + b(\varepsilon) \int_{-\infty}^{y_0} L(\sigma(f(x)), y) \, \mathrm{e}^{-|y|} \, \mathrm{d}y. \tag{5.5}$$

For the lower bound in Lemma 5.2, we note that by (5.4) and (2.3),

$$\begin{aligned} \mathbb{P}\{T_\Xi(x) > v\} \ &\geq \ \mathbb{P}\left\{ \int_{y_0}^{f(x)} L(\sigma(f(x)), y) \, \mathrm{d}y > \frac{v}{1 - \varepsilon/2} \right\} \\ &\geq \ \frac{2}{\pi} \exp\left(-\frac{\pi^2}{8} \frac{v}{(1 - \varepsilon/2)(f(x) - y_0)^2} \right). \end{aligned}$$

Since $f(x) \sim x$, $x \to \infty$, this yields the lower bound in Lemma 5.2 in the case when Ξ starts from $\Xi(0) = 1$, and a fortiori, in the case when Ξ starts from $\Xi(0) = 0$ by a comparison theorem for diffusion processes (Revuz and Yor [19], Theorem IX.3.7).

For the upper bound in Lemma 5.2, we again assume $\Xi(0) = 1$ for the moment. By (5.5) and triangular inequality, for $r \geq v_0$,

$$\begin{aligned} \mathbb{P}\left\{ \sup_{0 \leq s \leq r} \Xi(s) < x \right\} \ &= \ \mathbb{P}\{T_\Xi(x) > r\} \\ &\leq \ \mathbb{P}\left\{ \int_{y_0}^{f(x)} L(\sigma(f(x)), y) \, \mathrm{d}y \geq (1 - \varepsilon/2) r \right\} \\ &\quad + \mathbb{P}\left\{ \int_{-\infty}^{y_0} L(\sigma(f(x)), y) \, \mathrm{e}^{-|y|} \, \mathrm{d}y > \frac{\varepsilon}{2b(\varepsilon)} r \right\}. \end{aligned}$$

The first probability expression on the right hand side is $\leq \frac{4}{\pi} \exp(-\frac{\pi^2 (1-\varepsilon/2) r}{8(f(x)-y_0)^2})$ (see (2.3)), whereas the second is $\leq 3\exp(-\frac{\varepsilon}{16 b(\varepsilon) f(x)} r) + 2\exp(-\frac{\varepsilon}{8 y_0 b(\varepsilon) f(x)} r)$ in light of (2.6). Therefore, if $\Xi(0) = 1$, then for all $r, x \geq v_0$,

$$\mathbb{P}\left\{ \sup_{0 \leq s \leq r} \Xi(s) < x \right\} \leq \left(\frac{4}{\pi} + 5 \right) \exp\left(-\frac{\pi^2 (1 - \varepsilon) r}{8x^2} \right). \tag{5.6}$$

We are now back to the case $\Xi(0) = 0$ we were studying. By (5.2), for any $v > 100/\varepsilon$,

$$\mathbb{P}\{T_\Xi(1) > \varepsilon v\} \leq 2\exp\left(-\frac{\varepsilon v}{50} \right),$$

which, in view of (5.6) (taking $r \stackrel{\text{def}}{=} (1-\varepsilon)v$ there), yields that

$$\mathbb{P}\left\{ \sup_{0\le s\le v} \Xi(s) < x \right\} \le \left(\frac{4}{\pi}+5\right) \exp\left(-\frac{\pi^2(1-\varepsilon)^2 v}{8x^2}\right) + 2\exp\left(-\frac{\varepsilon v}{50}\right),$$

which yields the upper bound in Lemma 5.2. □

We are now ready to give the proof of the upper bound in Theorem 1.2.

Proof of Theorem 1.2: the upper bound. Observe that

$$\mathbb{P}\left\{ \sup_{0\le s\le t} X(s) > v \right\} = \mathbb{P}\left(H(v) < t\right) \le \mathbb{P}\left\{\Theta_1(v) < t\right\}.$$

By Theorem 4.1, $\Theta_1(v)$ is distributed as $4\int_0^v \mathrm{e}^{\Xi(s)}\,\mathrm{d}s - 4v$. Thus

$$\mathbb{P}\left\{ \sup_{0\le s\le t} X(s) > v \right\} \le \mathbb{P}\left\{ \int_0^v \mathrm{e}^{\Xi(s)}\,\mathrm{d}s < \frac{t}{4}+v \right\}. \tag{5.7}$$

According to (5.1) (taking $x \stackrel{\text{def}}{=} 3$ and $a \stackrel{\text{def}}{=} \frac{1}{v}$ there), we have

$$\mathbb{P}\left\{ \sup_{0\le t_1\le t_2\le v;\, t_2-t_1<1/v} |\Xi(t_2)-\Xi(t_1)| > 3 \right\} \le cv^2\mathrm{e}^{-v},$$

where c is the numerical constant in (5.1). On the event $\{\sup_{0\le t_1\le t_2\le v;\, t_2-t_1<1/v} |\Xi(t_2) - \Xi(t_1)| \le 3\}$, we have $\int_0^v \mathrm{e}^{\Xi(s)}\,\mathrm{d}s \ge \frac{1}{v}\mathrm{e}^{\sup_{0\le s\le v}\Xi(s)-3}$. Plugging this into (5.7) yields that for all sufficiently large v and t,

$$\begin{aligned}\mathbb{P}\left\{ \sup_{0\le s\le t} X(s) > v \right\} &\le cv^2\mathrm{e}^{-v} + \mathbb{P}\left\{ \sup_{0\le s\le v} \Xi(s) - 3 < \log(\frac{tv}{4}+v^2) \right\} \\ &\le cv^2\mathrm{e}^{-v} + 9\exp\left(-(1-2\varepsilon)\frac{\pi^2}{8}\frac{v}{\log^2(tv)}\right),\end{aligned} \tag{5.8}$$

the last inequality being a consequence of the upper bound in Lemma 5.2. We mention that (5.8) was already used in Section 3 to prove the estimate (3.17).

Since $\log v = o(\log t)$, (5.8) yields the upper bound in Theorem 1.2. □

5.2 Theorem 1.2: the lower bound

The ideas in this subsection essentially go back to Brox [2]. For $a, b \in \mathbb{R}$, we define

$$\begin{aligned}\overline{W}(a,b) &\stackrel{\text{def}}{=} \sup_{0\le s\le 1} W(a+s(b-a)), \\ \underline{W}(a,b) &\stackrel{\text{def}}{=} \inf_{0\le s\le 1} W(a+s(b-a)), \\ W^{\#}(a,b) &\stackrel{\text{def}}{=} \sup_{0\le s\le t\le 1} \left(W(a+t(b-a)) - W(a+s(b-a))\right).\end{aligned}$$

Note that $W^{\#}(a,b) \neq W^{\#}(b,a)$ in general. Let $\mathbf{P}_{x,\omega}$ be the quenched probability under which the diffusion X starts from x.

Recall that $H(y) = \inf\{t \geq 0 : X(t) = y\}$.

We start with the following lemma. We mention that $(W(y),\ a \leq y \leq c)$ is *not* necessarily a valley in the sense of Brox.

Lemma 5.3 *Let $a < x < c$.*

(1) *For any $\lambda > 0$, we have*

$$\mathbf{P}_{x,\omega}\left\{H(a) \wedge H(c) > \lambda\,(c-a)^2\,\mathrm{e}^{\min(W^{\#}(a,c),W^{\#}(c,a))}\right\} \;\leq\; \frac{36}{\lambda}. \tag{5.9}$$

(2) *There exists a non-decreasing deterministic function $\psi:\ \mathbb{R}_+ \to \mathbb{R}_+$, independent of (a,x,c), with $\lim_{u\to 0}\psi(u) = 0$ such that if*

$$a' = \sup\{y \leq x : W(x) - W(x) > \lambda\}, \tag{5.10}$$

$$c' = \inf\{y > x : W(y) - W(x) > \lambda\}, \tag{5.11}$$

and if

$$(c'-a')\mathrm{e}^{\lambda} \;\leq\; \frac{1}{2}\min\left\{\int_a^x \mathrm{e}^{W(y)-W(x)}\,\mathrm{d}y,\ \int_x^c \mathrm{e}^{W(y)-W(x)}\,\mathrm{d}y\right\} \overset{\text{def}}{=} \frac{1}{2}\,\Gamma(a,c), \tag{5.12}$$

then for all $0 < \varepsilon < 1$,

$$\mathbf{P}_{x,\omega}\left\{H(a)\wedge H(c) \;\leq\; \varepsilon\,(c'-a')\,\mathrm{e}^{-\lambda}\,\Gamma(a,c)\right\} \;\leq\; \psi(\varepsilon). \tag{5.13}$$

Proof of Lemma 5.3. (1) According to Brox ([2], pp. 1213–1214, proof of (i)),

$$H(a)\wedge H(c) \leq (c-a)^2\,\mathrm{e}^{W^{\#}(a,c)}\left(\sup_{-\infty<y\leq 1} L(\sigma(1),y) + \Delta\right),$$

with $\Delta \geq 0$ and

$$\mathbf{E}_{x,\omega}\left(\Delta^2\right) \leq 12.$$

The same estimate holds with $W^{\#}(c,a)$ instead of $W^{\#}(a,c)$. Therefore, (5.9) will follow from Chebyshev's inequality once we can show that for all $\lambda > 0$,

$$\mathbb{P}\left\{\sup_{-\infty<y\leq 1} L(\sigma(1),y) > \lambda\right\} \leq \frac{6}{\lambda}. \tag{5.14}$$

To prove (5.14), we note that by Fact 2.1, $y \in [0,1] \mapsto L(\sigma(1),y)$ is a BESQ(2) starting from 0, and $y \in [0,\infty) \mapsto L(\sigma(1),-y)$ is a BESQ(0) starting from $L(\sigma(1),0)$. Using successively the triangular inequality, the reflection principle for BESQ(2) and the martingale property of BESQ(0), we obtain that

$$\begin{aligned}
&\mathbb{P}\left\{\sup_{-\infty<y\le 1} L(\sigma(1),y)>\lambda\right\}\\
\le\ & \mathbb{P}\left\{\sup_{0\le y\le 1} L(\sigma(1),y)>\lambda\right\}+\mathbb{E}\left(\mathbf{1}_{\{L(\sigma(1),0)<\lambda\}}\mathbf{1}_{\{\sup_{-\infty<y\le 0} L(\sigma(1),y)>\lambda\}}\right)\\
\le\ & 2\mathbb{P}\left\{L(\sigma(1),0)>\lambda\right\}+\mathbb{E}\left(\mathbf{1}_{\{L(\sigma(1),0)<\lambda\}}\frac{L(\sigma(1),0)}{\lambda}\right).
\end{aligned}$$

Since $L(\sigma(1),0)$ has the exponential distribution of mean 2, this leads to

$$\mathbb{P}\left\{\sup_{-\infty<y\le 1} L(\sigma(1),y)>\lambda\right\}\le 2\mathrm{e}^{-\lambda/2}+\frac{2}{\lambda}\int_0^{\lambda/2} y\mathrm{e}^{-y}\,\mathrm{d}y\le \frac{6}{\lambda},$$

proving (5.14) and thus (5.9).

(2) The proof of (5.13) is essentially from Brox ([2], page 1215, line -11). Without loss of generality, we assume $x=0$. Under (5.12), we have $a<a'<0<c'<c$. In view of (4.1), we have, by occupation time formula,

$$\begin{aligned}
H(a)\wedge H(c) &= T\left(\sigma(A(a))\wedge\sigma(A(c))\right)\\
&= \int_a^c \mathrm{e}^{-W(y)}\, L(\sigma(A(a))\wedge\sigma(A(c)),A(y))\,\mathrm{d}y\\
&\ge (c'-a')\,\mathrm{e}^{-\lambda}\inf_{a'\le y\le c'} L(\sigma(A(a))\wedge\sigma(A(c)),A(y))\\
&\ge (c'-a')\,\mathrm{e}^{-\lambda}\inf_{|z|\le (c'-a')\mathrm{e}^{\lambda}} L(\sigma(A(a))\wedge\sigma(A(c)),z),
\end{aligned}$$

since $A(c')\le c'\mathrm{e}^{\lambda}$ and $A(a')\ge -|a'|\mathrm{e}^{\lambda}$. In view of (5.12) and scaling,

$$\begin{aligned}
\mathbf{P}_{x,\omega}\left\{H(a)\wedge H(c)\ \le\ \varepsilon\,(c'-a')\,\mathrm{e}^{-\lambda}\,\Gamma(a,c)\right\} &\le \mathbb{P}\left\{\inf_{|y|\le 1/2} L(\sigma(1)\wedge\sigma(-1),y)<\varepsilon\right\}\\
&\stackrel{\mathrm{def}}{=} \psi(\varepsilon),
\end{aligned}$$

proving (5.13).

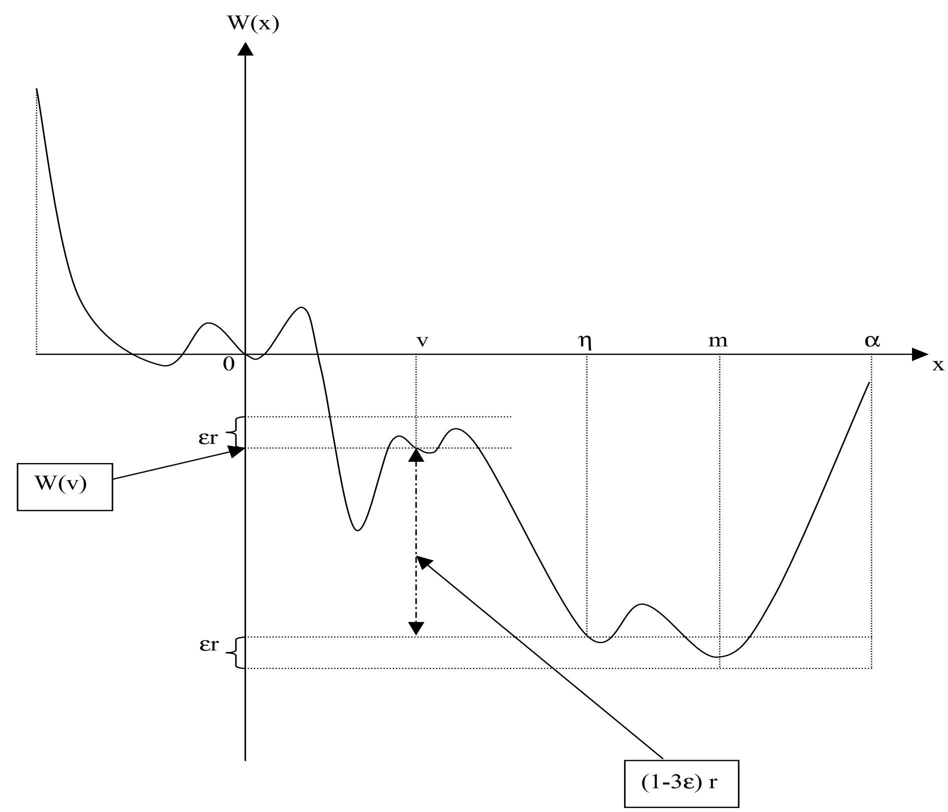
We mention that it is possible to use a theorem of Spitzer [23] to prove that $\psi(\varepsilon)\le C/\log(1/\varepsilon)$, for some constant $C>0$ and all $\varepsilon\in(0,1/2]$. □

We have now all the ingredients to prove the lower bound in Theorem 1.2.

Proof of Theorem 1.2: the lower bound. Fix a small $\varepsilon>0$. Consider large t and v such that $t^{\varepsilon}\ge v\ge \log^2 t$. Let $r=\frac{\log t}{1-10\varepsilon}$. For $r>0$, define $d_-(r)\stackrel{\mathrm{def}}{=}\sup\{t<0: W^{\#}(t,0)>r\}$.

Define three random times $v < \eta < m < \alpha$ by

$$\begin{aligned} \eta &= \inf\{s > v : W(s) - W(v) = -(1-3\varepsilon)r\}, \\ \alpha &= \inf\{s > \eta : W(s) - W(\eta) > r\}, \\ m &= \inf\{s > v : W(s) = \underline{\mathrm{W}}(v,\alpha)\}. \end{aligned}$$



We consider the following events:

$$\begin{aligned} F_1 &\stackrel{\text{def}}{=} \left\{|d_-(r)| < r^2;\ |\underline{\mathrm{W}}(d_-(r),0)| \le \varepsilon r;\ \int_{d_-(r)}^0 \mathrm{e}^{W(z)}\,\mathrm{d}z > \mathrm{e}^{r/2}\right\}, \\ F_2 &\stackrel{\text{def}}{=} \left\{W^\#(0,v) < (1-20\varepsilon)r;\ \overline{W}(0,v) < \frac{r}{3}\right\}, \\ F_3 &\stackrel{\text{def}}{=} \left\{\eta - v \le r^{5/2};\ \sup_{v\le s\le \eta}(W(s)-W(v)) < \varepsilon r;\ W^\#(v,\eta) < \frac{r}{3};\right. \\ &\qquad \left.\int_v^\eta \mathrm{e}^{W(x)-W(\eta)}\,\mathrm{d}x > \mathrm{e}^{(1-4\varepsilon)r}\right\}, \\ F_4 &\stackrel{\text{def}}{=} \left\{\alpha - \eta \le r^{5/2};\ \inf_{\eta\le s\le\alpha}(W(s)-W(\eta)) > -\varepsilon r;\ W^\#(\eta,m) < \frac{r}{3};\ W^\#(\alpha,\eta) < \frac{r}{3};\right. \\ &\qquad \left.\sup_{m\le x\le m+1}(W(x)-W(m)) \le r^{2/3};\ \int_m^\alpha \mathrm{e}^{W(x)-W(m)}\,\mathrm{d}x > \mathrm{e}^{(1-\varepsilon)r}\right\}. \end{aligned}$$

Observe that by the strong Markov property, the events $(F_j)_{1\le j\le 4}$ are independent. Moreover, $P\{F_3\}$ and $P\{F_4\}$ do not depend on v.

Under $\cap_{j=1}^4 F_j$, we have

$$\begin{aligned}\mathbf{P}_{0,\omega}\{X(t)>v\} &\ge \mathbf{P}_{0,\omega}\{H(m)<t;\ X(t)>v\}\\ &\ge \mathbf{P}_{0,\omega}\{H(m)<t\}\,\mathbf{P}_{m,\omega}\{H(v)\wedge H(\alpha)>t\}.\end{aligned} \tag{5.15}$$

Applying (5.9) to $(d_-(r),0,m)$ and using the scaling function $A(\cdot)$ of X, we have

$$\begin{aligned}&\mathbf{P}_{0,\omega}\{H(m)\ge t\}\\ \le\ &\mathbf{P}_{0,\omega}\{H(d_-(r))<H(m)\}+\mathbf{P}_{0,\omega}\{H(d_-(r))\wedge H(m)\ge t\}\\ =\ &\frac{A(m)}{A(m)-A(d_-(r))}+\mathbf{P}_{0,\omega}\{H(d_-(r))\wedge H(m)\ge t\}\\ \le\ &\frac{m\mathrm{e}^{r/3}}{\mathrm{e}^{r/2}}+\frac{36}{t}(m-d_-(r))^2\mathrm{e}^{(1-19\varepsilon)r},\end{aligned}$$

since $A(m)\le m\mathrm{e}^{\overline{W}(m)}\le m\mathrm{e}^{r/3}$, $|A(d_-(r))|\ge \mathrm{e}^{r/2}$ by definition of F_1 and $W^{\#}(d_-(r),m)\le \max(W^{\#}(0,v)-\underline{W}(d_-(r),0),W^{\#}(v,m)+\varepsilon r)\le(1-19\varepsilon)r$.

Note that $m\le\alpha\le r^3+v\le 2t^{\varepsilon}$ and $t=\mathrm{e}^{(1-10\varepsilon)r}$, we obtain that

$$\mathbf{P}_{0,\omega}\{H(m)\ge t\}\le \mathrm{e}^{-4\varepsilon r},\qquad \text{if } \omega\in\bigcap_{j=1}^4 F_j. \tag{5.16}$$

To apply (5.13) to (v,m,α), we choose $\lambda=r^{2/3}$ and verify that the assumption (5.12) is satisfied on $\cap_1^4 F_j$, because

$$\begin{aligned}\Gamma(v,\alpha) &= \min\left\{\int_v^m \mathrm{e}^{W(y)-W(m)}\,\mathrm{d}y,\ \int_m^\alpha \mathrm{e}^{W(y)-W(m)}\,\mathrm{d}y\right\}\\ &\ge \min\left\{\int_v^\eta \mathrm{e}^{W(y)-W(\eta)}\,\mathrm{d}y,\ \mathrm{e}^{(1-\varepsilon)r}\right\}\\ &\ge \mathrm{e}^{(1-4\varepsilon)r}\ge 2(c'-a')\mathrm{e}^{\lambda}.\end{aligned}$$

It follows from (5.13) that

$$\mathbf{P}_{m,\omega}\{H(v)\wedge H(\alpha)\le t\}\le\psi\left(\frac{t\mathrm{e}^{\lambda}}{(c'-a')\Gamma(v,\alpha)}\right).$$

Since $c'-a'\ge c'-m\ge 1$ by definition of F_4, $\psi(\frac{t\mathrm{e}^{\lambda}}{(c'-a')\Gamma(v,\alpha)})\le\psi(\mathrm{e}^{(1-10\varepsilon)r+r^{2/3}-(1-4\varepsilon)r})\le\psi(\mathrm{e}^{-5\varepsilon r})$. Therefore for all large $r\ge r_0(\varepsilon)$, we get from (5.15) that

$$\mathbf{P}_{0,\omega}\{X(t)>v\}\ge(1-\mathrm{e}^{-r/6})\left(1-\psi(\mathrm{e}^{-5\varepsilon r})\right)\ge\frac{1}{2},\qquad \text{if } \omega\in\bigcap_{j=1}^4 F_j.$$

Hence

$$\mathbb{P}\left\{X(t) > v\right\} = E\left(\mathbf{P}_{0,\omega}\left\{X(t) > v\right\}\right) \geq \frac{1}{2}\, P\left\{\bigcap_{j=1}^{4} F_j\right\} = \frac{1}{2}\prod_{j=1}^{4} P\left\{F_j\right\}, \tag{5.17}$$

by the independence of F_j. When $r \to \infty$, $P\{\eta_v - v > r^{5/2}\} \to 0$, and $\frac{1}{r}\log\int_v^\eta \mathrm{e}^{W(x)-W(\eta)}\,\mathrm{d}x \sim \frac{1}{r}\sup_{v\leq x\leq\eta}(W(x) - W(\eta)) \geq (1-3\varepsilon)$. It follows that

$$\liminf_{r\to\infty}\; P\{F_3\} \geq \liminf_{r\to\infty}\; P\left\{\sup_{v\leq s\leq\eta}(W(s)-W(v)) < \varepsilon r;\; W^{\#}(v,\eta) < \frac{r}{3}\right\} = C(\varepsilon) > 0,$$

for some constant C depending only on ε (by scaling). The same holds for $P\{F_4\}$ and $P\{F_1\}$. It follows that for all large $r \geq r_0$, we have

$$P\{F_1\}\, P\{F_3\}\, P\{F_4\} \geq C'(\varepsilon) > 0. \tag{5.18}$$

Finally, we recall the asymptotic expansion of distribution of $(W^{\#}(0,v), \overline{W}(0,v))$ ([9], Theorem 2.1): for any fixed $0 < a \leq 1$, when $\delta \to 0+$,

$$P\left\{W^{\#}(0,1) < \delta;\; \overline{W}(0,1) < a\delta\right\} \sim \frac{4\sin(a\pi/2)}{\pi}\exp\left(-\frac{\pi^2}{8\delta^2}\right).$$

It follows from scaling that when $r \to \infty$,

$$P\{F_2\} \sim \frac{4\sin(\pi/6(1-20\varepsilon))}{\pi}\exp\left(-\frac{\pi^2}{8(1-20\varepsilon)^2}\,\frac{v}{r^2}\right).$$

Plugging this into (5.17) and (5.18) implies

$$\liminf_{t,v\to\infty,\; v\gg\log^2 t,\; \log v = o(\log t)} \frac{\log^2 t}{v}\log\mathbb{P}\left\{\, X(s) > v\right\} \geq -\frac{\pi^2}{8}.$$

The lower bound in Theorem 1.2 is proved. □

References

[1] Bass, R.F. and Griffin, P.S.: The most visited site of Brownian motion and simple random walk. *Z. Wahrsch. Verw. Gebiete* 70 (1985) 417–436.

[2] Brox, T.: A one–dimensional diffusion process in a Wiener medium. *Ann. Probab.* 14 (1986) 1206–1218.

[3] Comets, F. and Popov, S.: Limit law for transition probabilities and moderate deviations for Sinai's random walk in random environment. *Probab. Theory Related Fields* (to appear).

[4] Csáki, E. and Földes, A.: A note on the stability of the local time of a Wiener process. *Stochastic Process. Appl.* 25 (1987) 203–213.

[5] Csörgő, M. and Révész, P.: *Strong Approximations in Probability and Statistics.* Academic Press, New York, 1981.

[6] Feller, W.: *An Introduction to Probability Theory and its Applications.* Vol. II. Second edition. Wiley, New York, 1971.

[7] Gantert, N. and Zeitouni, O.: Large deviations for one-dimensional random walk in a random environment — a survey. *Random walks (Budapest, 1998)* (Eds.: P. Révész & B. Tóth) 127–165, Bolyai Soc. Math. Stud. 9 Bolyai Math. Soc., Budapest, 1999.

[8] Getoor, R.K. and Sharpe, M.J.: Excursions of Brownian motion and Bessel processes. *Z. Wahrsch. Verw. Gebiete* 47 (1979) 83–106.

[9] Hu, Y. and Shi, Z.: The limits of Sinai's simple random walk in random environment. *Ann. Probab.* 26 (1998) 1477–1521.

[10] Hu, Y., Shi, Z. and Yor, M.: Rates of convergence of diffusions with drifted Brownian potentials. *Trans. Amer. Math. Soc.* 351 (1999) 3915–3934.

[11] Le Doussal, P., Monthus, C. and Fisher, D.S.: Random walkers in one-dimensional random environments: exact renormalization group analysis. *Phys. Rev. E* (3) 59 (1999) 4795–4840.

[12] Karlin, S. and Taylor, H.M.: *A Second Course in Stochastic Processes.* Academic Press, New York, 1981.

[13] Kawazu, K. and Tanaka, H.: A diffusion process in a Brownian environment with drift. *J. Math. Soc. Japan* 49 (1997) 189–211.

[14] Kent, J.: Some probabilistic properties of Bessel functions. *Ann. Probab.* 6 (1978) 760–770.

[15] Kesten, H.: An iterated logarithm law for local time. *Duke Math. J.* 32 (1965) 447–456.

[16] Lamperti, J.: Semi-stable Markov processes, I. *Z. Wahrsch. Verw. Gebiete* 22 (1972) 205–225.

[17] Pitman, J. and Yor, M.: A decomposition of Bessel bridges. *Z. Wahrsch. Verw. Gebiete* 59 (1982) 425–457.

[18] Révész, P.: *Random Walk in Random and Non-Random Environments.* World Scientific, Singapore, 1990.

[19] Revuz, D. and Yor, M.: *Continuous Martingales and Brownian Motion.* Third edition. Springer, Berlin, 1999.

[20] Schumacher, S.: Diffusions with random coefficients. *Contemp. Math.* 41 (1985) 351–356.

[21] Shi, Z.: Sinai's walk via stochastic calculus. *Panoramas et Synthèses* (Eds: F. Comets & E. Pardoux) 12 (2001) Société Mathématique de France.

[22] Sinai, Ya.G.: The limiting behaviours of a one-dimensional random walk in a random medium. *Theory Probab. Appl.* 27 (1982) 256–268.

[23] Spitzer, F.: Some theorems concerning 2-dimensional Brownian motion. *Trans. Amer. Math. Soc.* 87 (1958) 187–197.

[24] Taleb, M.: Large deviations for a Brownian motion in a drifted Brownian potential. *Ann. Probab.* 29 (2001) 1173–1204.

[25] Tanaka, H.: Diffusion processes in random environments. *Proc. ICM* (Ed.: S.D. Chatterji) 1047–1054. Birkhäuser, Basel, 1995.

[26] Warren, J. and Yor, M.: The Brownian burglar: conditioning Brownian motion by its local time process. *Séminaire de Probabilités XXXII* pp. 328–342, Lecture Notes in Mathematics 1686. Springer, Berlin, 1998.

[27] Zeitouni, O.: *Lecture Notes on Random Walks in Random Environment.* (Saint Flour lecture notes, 2001)